∞-CATEGORIES AND HOMOTOPICAL ALGEBRA

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These notes are based on several sources : especially, D.-C. Cisinski's thesis [3] and lecture notes [4], M. Hovey's treatise [8], and D. Dugger and D.I. Spivak's article [5].

Contents

1. Prelude

1.1. **Some set-theoretical and categorical preliminaries.** Throughout this text, *we adopt Bernays-Gödel's axiomatic system for set theory,* augmented with *the axiom of global choice* ([13, §1.5.1]). We denote the cardinality of any set S by :

$$|S|.$$

Definition 1.1.1. Let κ be a cardinal.

(i) We denote by κ^+ the *successor cardinal* of κ, *i.e.* the smallest cardinal $> \kappa$.

(ii) We say that κ is *regular,* if κ is infinite and for every family $(S_i \mid i \in I)$ of subsets of κ with $|S_i| < \kappa$ for every $i \in I$ and with $|I| < \kappa$, we have $|\bigcup_{i \in I} S_i| < \kappa$.

Example 1.1.2. (i) The smallest infinite cardinal $\aleph_0$ (*i.e.* the set $\mathbb{N}$ of natural numbers) is clearly regular.

(ii) If $\kappa \geq \aleph_0$, then κ^+ is regular. Indeed, let $(S_i \mid i \in I)$ be a family of subsets of κ^+ with $|S_i| \leq \kappa$ for every $i \in I$ and with $|I| \leq \kappa$; then $|\bigcup_{i \in I} S_i| \leq \kappa^2 = \kappa$.

(iii) Define inductively $\aleph_{i+1} := \aleph_i^+$ for every $i \in \mathbb{N}$; then $\aleph_\omega := \bigcup_{i \in \mathbb{N}} \aleph_i$ is not regular, since $|\mathbb{N}| < \aleph_\omega$ and $|\aleph_i| = \aleph_i < \aleph_\omega$ for every $i \in \mathbb{N}$.

(iv) A cardinal is said to be *weakly inaccessible*, if it is regular and it is neither $\aleph_0$, nor a successor cardinal. The existence of such large cardinals is unprovable within our set-theoretic framework.

1.1.3. Recall that a *relation* between two classes X, Y is a subclass of $X \times Y$. Especially, a *map* $f : X \to Y$ *from* X *to* Y is a relation $\Gamma_f \subset X \times Y$ such that for every $x \in X$ there exists a unique $y \in Y$ with $(x, y) \in \Gamma_f$, and as usual one writes $y = f(x)$ for this element. Thus, *we encode relations and maps between classes by their graphs.* Hence, *if X is a set, then every map $f : X \to Y$ is a set as well.*

Example 1.1.4. Let X be a class, and $\mathscr{R} \subset X \times X$ a relation on X. Then there exists a smallest equivalence relation $\overline{\mathscr{R}}$ containing $\mathscr{R}$. Indeed, let first $\mathscr{R}' := \mathscr{R} \cup \Delta_X$, where $\Delta_X := \{(x, x) \mid x \in X\}$ is *the diagonal* of $X \times X$. Then we let $\overline{\mathscr{R}} \subset X \times X$ be the subclass of all pairs $(x, x') \in X \times X$ such that there exists $k \in \mathbb{N} \setminus \{0\}$ and a sequence $x_0, \dots, x_k$ of elements of X with $x_0 = x$, $x_k = x'$, and such that either $(x_i, x_{i+1}) \in \mathscr{R}'$ or $(x_{i+1}, x_i) \in \mathscr{R}'$ for every $i = 0, \dots, k-1$. It is easily seen that $\overline{\mathscr{R}}$ is an equivalence relation on X, and obviously every equivalence relation on X containing $\mathscr{R}$ must also contain $\overline{\mathscr{R}}$. We call $\overline{\mathscr{R}}$ *the equivalence relation on X generated by $\mathscr{R}$.*

- The *disjoint union* and the *(cartesian) product* of a family of sets $(X_i \mid i \in I)$ indexed by a set I shall be denoted respectively :

$$\boxed{\coprod_{i \in I} X_i := \bigcup_{i \in I} \{i\} \times X_i \qquad \text{and} \qquad \prod_{i \in I} X_i.}$$

We will discuss in §1.2 the extension of these operations to arbitrary classes; for now, we only define the disjoint union of any two given classes X_0 and X_1 : namely

$$\boxed{X_0 \sqcup X_1 := (\{0\} \times X_0) \cup (\{1\} \times X_1).}$$

Moreover, *the pair* (X_0, X_1) is the map $X_0 \sqcup X_1 \to \{0, 1\}$ that sends $\{i\} \times X_i$ to i, for $i = 0, 1$.

- For every pair of maps between classes $f : X \to Y$, $g : X' \to Y$ we let

$$\boxed{X \times_{(f,g)} X' := \{(x, x') \in X \times X' \mid f(x) = g(x')\}}$$

and we call this class *the fibre product of X and X' over f and g* (or simply *over Y*); often the same class is just denoted by $X \times_Y X'$, unless the notation gives rise to ambiguities.

1.1.5. Let $\mathscr{A}$ be any category; for every pair (X, Y) of objects of $\mathscr{A}$, we shall write

$$\mathscr{A}(X, Y)$$

for the *set* of morphisms $X \to Y$ in $\mathscr{A}$. The *identity* of an object X will be denoted:

$$\mathbf{1}_X.$$

We shall denote the *classes* of objects and of morphisms of $\mathscr{A}$ respectively by :

$$\boxed{\mathrm{Ob}(\mathscr{A}) \qquad \text{and} \qquad \mathrm{Mor}(\mathscr{A}) := \coprod_{(X,Y)\in \mathrm{Ob}(\mathscr{A})^2} \mathscr{A}(X, Y).}$$

We have obvious *source* and *target* maps :

$$\boxed{\mathrm{Ob}(\mathscr{A}) \xleftarrow{s} \mathrm{Mor}(\mathscr{A}) \xrightarrow{t} \mathrm{Ob}(\mathscr{A}) \qquad X \leftarrow\!\shortmid ((X, Y), f : X \to Y) \mapsto Y.}$$

- The opposite category of $\mathscr{A}$ will be denoted :

$$\mathscr{A}^{\mathrm{op}}$$

and for every morphism $f : X \to Y$ of $\mathscr{A}$, we shall sometimes write $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ (or $f^{\mathrm{op}} : Y \to X$) to denote f, regarded as a morphism of $\mathscr{A}^{\mathrm{op}}$. Likewise, we write:

$$\mathscr{F}^{\mathrm{op}} := \{f^{\mathrm{op}} \mid f \in \mathscr{F}\} \qquad \text{for every subclass } \mathscr{F} \subset \mathrm{Mor}(\mathscr{C}).$$

With this notation, every functor $F : \mathscr{A} \to \mathscr{B}$ induces a functor :

$$\boxed{F^{\mathrm{op}} : \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}} \qquad A^{\mathrm{op}} \mapsto (FA)^{\mathrm{op}} \qquad f^{\mathrm{op}} \mapsto (Ff)^{\mathrm{op}}}$$

and every natural transformation $\tau_\bullet : F \Rightarrow G$ induces a natural transformation :

$$\boxed{\tau_\bullet^{\mathrm{op}} : G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}} \qquad A \mapsto (\tau_A^{\mathrm{op}} : (GA)^{\mathrm{op}} \to (FA)^{\mathrm{op}}).}$$

- Let $F, G, H : \mathscr{A} \to \mathscr{B}$ be three functors, and consider two natural transformations $\tau_\bullet : F \Rightarrow G$ and $\eta_\bullet : G \Rightarrow H$; then *the composition* of $\tau_\bullet$ and $\eta_\bullet$ is the natural transformation (see [13, Rem.1.127(ii)]) :

$$\boxed{\eta_\bullet \circ \tau_\bullet : F \Rightarrow H \qquad A \mapsto \eta_A \circ \tau_A.}$$

Also, we associate with every pair of natural transformations :

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\omega_\bullet"] & \mathscr{A}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\omega'_\bullet"] & \mathscr{A}'' \end{tikzcd}$$

the *Godement product* of $\omega_\bullet$ and $\omega'_\bullet$, defined as the natural transformation :

$$\boxed{\omega'_\bullet \star \omega_\bullet : F' \circ F \Rightarrow G' \circ G \qquad A \mapsto \omega'_{GA} \circ F'(\omega_A)}$$

(see [13, Exerc.1.129]). For $\omega_\bullet = \mathbf{1}_F$ (resp. for $\omega'_\bullet = \mathbf{1}_{F'}$), we usually write $\omega'_\bullet \star F$ (resp. $F' \star \omega_\bullet$) in lieu of $\omega'_\bullet \star \mathbf{1}_F$ (resp. in lieu of $\mathbf{1}_{F'} \star \omega_\bullet$); hence :

$$(\omega'_\bullet \star F)_A = \omega'_{FA} \quad \text{and} \quad (F' \star \omega_\bullet)_A = F'(\omega_A) \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

- For a pair $(\tau_\bullet, f)$ consisting of a natural transformation $\tau_\bullet : F \Rightarrow G$ between functors $F, G : \mathscr{A} \to \mathscr{B}$, and $f \in \mathscr{A}(X, Y)$, we shall also use the notation :

$$\boxed{\tau_\bullet \otimes X := \tau_X \qquad \tau_\bullet \otimes f := \tau_Y \circ Ff = Gf \circ \tau_X : FX \to GY.}$$

Notice that for every composable pair of natural transformations $F \stackrel{\tau_\bullet}{\Rightarrow} G \stackrel{\eta_\bullet}{\Rightarrow} H$ and every composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathscr{A}$, we have the identity :

$$\boxed{(\eta_\bullet \circ \tau_\bullet) \otimes (g \circ f) = (\eta_\bullet \otimes g) \circ (\tau_\bullet \otimes f).}$$

Definition 1.1.6. (i) We say that the category $\mathscr{A}$ is *small* (resp. *finite*) if $\mathrm{Ob}(\mathscr{A})$ and $\mathrm{Mor}(\mathscr{A})$ are sets (resp. are finite sets).

(ii) If $|\mathrm{Mor}(\mathscr{A})| < \kappa$ for a cardinal κ, we say that $\mathscr{A}$ *is κ-small.*

(iii) We say that $\mathscr{A}$ is *connected*, if $\mathrm{Ob}(\mathscr{A}) \neq \varnothing$, and every pair of objects $A, B \in \mathrm{Ob}(\mathscr{A})$ can be joined by a finite chain of morphisms of $\mathscr{A}$:

$$A \to C_1 \leftarrow C_2 \to \cdots \leftarrow C_k \to B.$$

(iv) We say that $\mathscr{A}$ is *directed,* if for every $A, B \in \mathrm{Ob}(\mathscr{C})$ there exists $C \in \mathrm{Ob}(\mathscr{A})$ with morphisms $A \to C \leftarrow B$. We say that $\mathscr{A}$ is *codirected*, if $\mathscr{A}^{\mathrm{op}}$ is directed.

(v) We say that $\mathscr{A}$ is *filtered*, if $\mathscr{A}$ is directed, $\mathrm{Ob}(\mathscr{A}) \neq \varnothing$, and for every $A, B \in \mathrm{Ob}(\mathscr{A})$ and $f, g \in \mathscr{A}(A, B)$ there exists $C \in \mathrm{Ob}(\mathscr{A})$ and $h \in \mathscr{C}(B, C)$ that *coequalizes* f and g, *i.e.* $h \circ f = h \circ g$. We say that $\mathscr{A}$ is *cofiltered*, if $\mathscr{A}^{\mathrm{op}}$ is filtered.

(vi) We say that $\mathscr{A}$ is *discrete*, if $\mathscr{A}(X, Y) = \varnothing$ for every pair (X, Y) of distinct objects of $\mathscr{A}$, and $\mathscr{A}(X, X) = \{\mathbf{1}_X\}$ for every $X \in \mathrm{Ob}(\mathscr{A})$.

Remark 1.1.7. Notice that if κ is a regular cardinal, then the category $\mathscr{A}$ is κ-small if and only if we have both $|\mathrm{Ob}(\mathscr{A})| < \kappa$ and $|\mathscr{A}(A, B)| < \kappa$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$.

1.1.8. *Limits and colimits.* Let $A \in \mathrm{Ob}(\mathscr{A})$; for every category I, we shall denote

$$c_A : I \to \mathscr{A}$$

the constant functor with value A, *i.e.* the functor such that $c_A(i) := A$ for every $i \in \mathrm{Ob}(I)$ and $c_A(\phi) := \mathbf{1}_A$ for every morphism ϕ of I. Every morphism $f : A \to B$ induces an obvious *constant natural transformation*

$$c_f : c_A \Rightarrow c_B \qquad i \mapsto f.$$

- Let $F : I \to \mathscr{A}$ be any other functor; *a cone with basis F and vertex A* is a natural transformation $c_A \Rightarrow F$. Dually *a co-cone with basis F and vertex A* is a cone $c_{A^{\mathrm{op}}} \Rightarrow F^{\mathrm{op}}$, *i.e.* a natural transformation $F \Rightarrow c_A$. We say that a cone $\tau_\bullet : c_A \Rightarrow F$ is *universal*, if for every other cone $\eta_\bullet : c_X \Rightarrow F$ there exists a unique morphism $f : X \to A$ of $\mathscr{A}$ such that

$$\eta_\bullet = \tau_\bullet \circ c_f$$

and in this case we say that A *represents the limit of F in $\mathscr{A}$*. Likewise, a co-cone $\tau_\bullet : F \Rightarrow c_A$ is *universal* if $\tau_\bullet^{\mathrm{op}} : c_{A^{\mathrm{op}}} \Rightarrow F^{\mathrm{op}}$ is a universal cone, *i.e.* if for every other co-cone $\eta_\bullet : F \Rightarrow c_X$ there exists a unique $f \in \mathscr{A}(A, X)$ such that $\eta_\bullet = c_f \circ \tau_\bullet$; in which case, we say that A *represents the colimit of F in $\mathscr{A}$* (see [13, §2.2]). The limit (resp. the colimit) of F is not necessarily representable in $\mathscr{A}$, but if it is, then clearly the object of $\mathscr{A}$ representing the limit (resp. the colimit) of F is determined up to isomorphism; we shall use the standard notation :

$$\lim_I F \qquad \text{and} \qquad \operatorname*{colim}_I F$$

to signify given choices of representing objects for the limit and colimit of F.

- Consider a sequence of functors :

$$J \xrightarrow{\phi} I \xrightarrow{F} \mathscr{A} \xrightarrow{H} \mathscr{B}$$

such that the limits of F and of $HF\phi$ are representable in $\mathscr{A}$ and respectively $\mathscr{B}$, and choose universal cones $\tau_\bullet : c_{\lim_I F} \Rightarrow F$ and $\eta_\bullet : c_{\lim_J HF\phi} \Rightarrow HF\phi$; then there exists a unique morphism of $\mathscr{B}$

$$\boxed{\lim_\phi H : H(\lim_I F) \to \lim_J HF\phi \quad \text{such that} \quad \eta_\bullet \circ c_{\lim_\phi H} = H \star \tau_\bullet \star \phi.}$$

In other words, this is the unique morphism that makes commute the diagrams :

$$\begin{tikzcd} H(\lim_I F) \arrow[rr, "\lim_\phi H"] \arrow[dr, "H(\tau_{\phi j})"'] & & \lim_J HF\phi \arrow[dl, "\eta_j"] \\ & HF\phi(j) & \end{tikzcd} \qquad \forall j \in \mathrm{Ob}(J).$$

Dually, if the colimits of F and of $HF\phi$ are representable, then for any given choice of universal co-cones $\tau'_\bullet : F \Rightarrow c_{\mathrm{colim}_I F}$ and $\eta'_\bullet : HF\phi \Rightarrow c_{\mathrm{colim}_J HF\phi}$, we have a unique morphism

$$\boxed{\operatorname*{colim}_\phi H : \operatorname*{colim}_J HF\phi \to H(\operatorname*{colim}_I F) \quad \text{such that} \quad c_{\mathrm{colim}_\phi H} \circ \eta'_\bullet = H \star \tau'_\bullet \star \phi.}$$

As usual, $\lim_\phi H$ (resp. $\mathrm{colim}_\phi H$) depends on the choice of universal cones (resp. of universal co-cones), but its categorical properties are *intrinsic.*

• For any category I, we say that $\mathscr{A}$ is *I-complete* (resp. *I-cocomplete*) if the limit (resp. the colimit) of every functor $I \to \mathscr{A}$ is representable in $\mathscr{A}$. Moreover, we say that *$\mathscr{A}$ is complete* (resp. *cocomplete*) if $\mathscr{A}$ is I-complete (resp. I-cocomplete) for every small category I; we shall say that $\mathscr{A}$ is *finitely complete* (resp. *finitely cocomplete*) if $\mathscr{A}$ is I-complete (resp. I-cocomplete) for every finite category I.

Example 1.1.9. (i) We denote by :

$$\mathrm{Set} \qquad \text{and} \qquad \mathrm{Cat}$$

respectively the category of sets and the category of small categories; the morphisms of Set (resp. of Cat) are the maps of sets (resp. the functors between small categories) with the natural composition law. We have an obvious functor :

$$\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set} \qquad \mathscr{C} \mapsto \mathrm{Ob}(\mathscr{C})$$

sending every functor $F : \mathscr{C} \to \mathscr{C}'$ to the underlying map $\mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}') : X \mapsto FX$ for every $X \in \mathrm{Ob}(\mathscr{C})$. The category Set is complete and cocomplete ([13, Prob.2.51(i)]); we shall show later that the same holds for the category Cat (proposition 1.10.4).

(ii) Recall that to every class Λ we can naturally attach the discrete category $\mathscr{C}_\Lambda$ with $\mathrm{Ob}(\mathscr{C}_\Lambda) = \Lambda$ (see [13, Exemp.1.114(vi,vii)]), and the data of Λ and $\mathscr{C}_\Lambda$ are essentially equivalent; especially, we have a fully faithful functor :

$$\mathrm{dis} : \mathrm{Set} \to \mathrm{Cat} \qquad \Lambda \mapsto \mathscr{C}_\Lambda.$$

So, we won't usually distinguish between a class and its associated category.

(iii) For every class Λ, the limit (resp. the colimit) of a functor $F : \mathscr{C}_\Lambda \to \mathscr{A}$ (also denoted $F : \Lambda \to \mathscr{A}$) is called *the product* (resp. *the coproduct*) of the family $(F\lambda \mid \lambda \in \Lambda)$ of objects of $\mathscr{A}$, and if it is representable in $\mathscr{A}$, any choice of representative is denoted :

$$\prod_{\lambda\in\Lambda} F\lambda \qquad (\text{resp. } \coprod_{\lambda\in\Lambda} F\lambda).$$

If Λ is a finite set of cardinality n, the datum of a functor $\Lambda \to \mathscr{A}$ amounts to that of a sequence $A_\bullet := (A_1, \dots, A_n)$ of n objects of $\mathscr{A}$, and any representative in $\mathscr{A}$ for the product of $A_\bullet$ is denoted as usual by $A_1 \times \cdots \times A_n$.

Definition 1.1.10. Let I be a category, and $\phi : I \to \mathscr{A}$, $F : \mathscr{A} \to \mathscr{B}$ two functors.

(i) We say that F *preserves the limit* of ϕ, if there exists a universal cone $\tau_\bullet : c_L \Rightarrow \phi$ such that $F \star \tau_\bullet : c_{FL} \Rightarrow F \circ \phi$ is universal. We say that F *preserves I-limits*, if it preserves the limit of all functors $I \to \mathscr{A}$ (so, $\mathscr{A}$ is I-complete in this case). We say that F *preserves small* (resp. *connected*, resp. *cofiltered*) *limits*, if it preserves I-limits, for every small (resp. connected, resp. cofiltered) category I.

(ii) We say that F *preserves the colimit of* ϕ, if F^{op} preserves the limit of ϕ^{op}. We say that F *preserves I-colimits,* if F^{op} preserves I^{op}-limits. We say that F *preserves small* (resp. *connected,* resp. *filtered) colimits*, if it preserves I-colimits, for every small (resp. connected, resp. filtered) category I.

(iii) We say that F *reflects the limit of* ϕ, if for every non-universal cone $\tau_\bullet : c_L \Rightarrow \phi$, the cone $F \star \tau_\bullet : c_{FL} \Rightarrow F \circ \phi$ is non-universal. We say that F *reflects I-limits*, if it reflects the limit of every functor $I \to \mathscr{A}$. We say that F *reflects the colimit of* ϕ (resp. *I-colimits*) if F^{op} reflects the limit of ϕ^{op} (resp. I^{op}-limits). We say that F *reflects small* (resp. *finite*, resp. *connected*, resp. *cofiltered*) limits, if it preserves I-limits, for every small (resp. finite, resp. cofiltered) category I. We say that F *reflects small* (resp. *finite,* resp. *connected*, resp. *filtered*) *colimits*, if F^{op} reflects small (resp. finite, resp. connected, resp. cofiltered) limits.

(iv) We say that F is *left* (resp. *right*) *exact*, if it preserves finite limits (resp. finite colimits). We say that F is *exact*, if it is both left and right exact.

(v) Let $\boldsymbol{P}(f)$ be a property of morphisms such as *e.g.* "f is a monomorphism" or "f is an epimorphism". We say that F *preserves* (resp. *reflects*) $\boldsymbol{P}$, if for every morphism f of $\mathscr{A}$ we have : $\boldsymbol{P}(f) \Rightarrow \boldsymbol{P}(Ff)$ (resp. $\boldsymbol{P}(Ff) \Rightarrow \boldsymbol{P}(f)$).

Remark 1.1.11. (i) Clearly $\mathscr{A}$ is I-complete $\Leftrightarrow$ $\mathscr{A}^{\mathrm{op}}$ is I^{op}-cocomplete. Likewise $\mathscr{A}$ is complete (resp. finitely complete) $\Leftrightarrow$ $\mathscr{A}^{\mathrm{op}}$ is cocomplete (resp. finitely cocomplete). Also, a functor F preserves (resp. reflects) the limit of a functor ϕ $\Leftrightarrow$ F^{op} preserves (resp. reflects) the colimit of ϕ^{op}. Hence, F preserves (resp. reflects) I-limits $\Leftrightarrow$ F^{op} preserves (resp. reflects) I^{op}-colimits, and F is left exact $\Leftrightarrow$ F^{op} is right exact.

(ii) The limit (resp. colimit) of the empty functor $\varnothing \to \mathscr{A}$ is represented by any final (resp. initial) object of $\mathscr{A}$, when such an object exists. Recall that an object $A \in \mathrm{Ob}(\mathscr{A})$ is *final* (resp. *initial*) in $\mathscr{A}$ if for every $X \in \mathrm{Ob}(\mathscr{A})$ there exists a unique morphism $X \to A$ (resp. $A \to X$) (see [13, Rem.2.27]).

(iii) Every faithful functor F reflects both monomorphisms and epimorphisms. If the functor F preserves fibre products, then it preserves monomorphisms. Dually, if F preserves amalgamated sums, then it preserves epimorphisms : see [13, Exerc.2.66(ii,iv)]. Trivially, every functor preserves isomorphisms; we say that F is *conservative*, if it reflects isomorphisms.

(iv) If F is conservative and preserves fibre products, then F both preserves and reflects monomorphisms. Indeed, let $f \in \mathscr{A}(A, A')$ and $\Delta_{A/A'} : A \to A \times_{A'} A$ the diagonal morphism of f; since F preserves fibre products, $F(\Delta_{A/A'})$ is the diagonal morphism of Ff, and the latter is an isomorphism if and only if Ff is a monomorphism ([13, Exerc.2.66(i)]). But since F is conservative, $F(\Delta_{A/A'})$ is an isomorphism if and only if the same holds for $\Delta_{A/A'}$, and again the latter holds if and only if f is a monomorphism. Dually, if F is conservative and preserves amalgamated sums, then F both preserves and reflects epimorphisms.

Definition 1.1.12. Let $\mathscr{A}$ be a category, and $f, g : X \rightrightarrows Y$ two morphisms of $\mathscr{A}$; notice that the equalizer $\mathrm{Equal}(f, g) \to X$ of f and g is a monomorphism, and the coequalizer $Y \to \mathrm{Coequal}(f, g)$ is an epimorphism. We say that a morphism of $\mathscr{A}$ is a *regular monomorphism* (resp. a *regular epimorphism*), if it is the equalizer (resp. the coequalizer) of a pair of morphisms of $\mathscr{A}$.

Remark 1.1.13. (i) Clearly $i : X' \to X$ is a regular monomorphism in $\mathscr{A}$ if and only if $i^{\mathrm{op}} : X^{\mathrm{op}} \to X'^{\mathrm{op}}$ is a regular epimorphism in $\mathscr{A}^{\mathrm{op}}$. Also, every functor $\mathscr{A} \to \mathscr{B}$ that preserves equalizers (resp. coequalizers), also preserves regular monomorphisms (resp. regular epimorphisms).

(ii) Let $i : X' \to X$ be a monomorphism of $\mathscr{A}$, and suppose that the amalgamated sum $X \sqcup_{X'} X$ is representable. Then i is regular if and only if it represents the equalizer of the two natural morphisms $e_1, e_2 : X \rightrightarrows X \sqcup_{X'} X$. Indeed, the condition is obviously sufficient. Conversely, suppose that i represents the equalizer of a pair of morphisms $f, g : X \rightrightarrows Y$, and let $h : Z \to X$ be a morphism of $\mathscr{A}$ such that $e_1 \circ h = e_2 \circ h$; we have $e_1 \circ i = e_2 \circ i$, and since $f \circ i = g \circ i$, we have moreover a unique morphism $k : X \sqcup_{X'} X \to Y$ such that $f = k \circ e_1$ and $g = k \circ e_2$. Therefore :

$$f \circ h = k \circ e_1 \circ h = k \circ e_2 \circ h = g \circ h$$

so $h = h' \circ i$ for a unique $h' \in \mathscr{A}(Z, X')$, and thus, i is the equalizer of e_1 and e_2.

(iii) Dually, if $p : Y \to Y'$ is a an epimorphism of $\mathscr{A}$, and if the fibre product $Y \times_{Y'} Y$ is representable in $\mathscr{A}$, then p is regular if and only if it represents the coequalizer of the two natural projections $Y \times_{Y'} Y \rightrightarrows Y$.

Example 1.1.14. (i) It is easily seen that every monomorphism and every epimorphism of the category Set is regular : the details are left to the reader.

(ii) Let $(S_i \mid i \in I)$ be a family of subsets of a given set S; by (i), the epimorphism $T' := \bigsqcup_{i \in I} S_i \to T := \bigcup_{i \in I} S_i$ is regular, and notice that $T' \times_T T'$ is represented by the set $\bigsqcup_{(i,j) \in I^2} S_i \cap S_j$; we then get a diagram :

$$(*) \qquad \boxed{\bigsqcup_{(i,j) \in I^2} S_i \cap S_j \underset{j_2}{\overset{j_1}{\rightrightarrows}} \bigsqcup_{i \in I} S_i \xrightarrow{p} S}$$

where p is the map whose restriction to S_i is the inclusion $S_i \to S$ for every $i \in I$, and j_1 (resp. j_2) is the map whose restriction to $S_i \cap S_j$ is the inclusion $S_i \cap S_j \to S_i$ (resp. $S_i \cap S_j \to S_j$) for every $(i, j) \in I^2$. With this notation, remark 1.1.13(iii) implies that $(*)$ is *exact, i.e.* identifies the image of p with the coequalizer in Set of the pair of maps (j_1, j_2). This boils down to the obvious assertion that for every set X, the datum of a map $f : \bigcup_{i \in I} S_i \to X$ is equivalent to that of a family of maps $(f_i : S_i \to X \mid i \in I)$ such that $f_{i|S_i \cap S_j} = f_{j|S_i \cap S_j}$ for every $(i, j) \in I^2$.

(iii) In the situation of (ii), set $J := \{(i, j) \in I^2 \mid i \neq j\}$; clearly the exactness of $(*)$ holds as well if we replace $T' \times_T T'$ by its subset $\bigsqcup_{(i,j) \in J} S_i \cap S_j$. Moreover, if we endow I with an arbitrary total ordering, we may also replace J by its subset $J' := \{(i, j) \in I^2 \mid i < j\}$ and still get an exact diagram $(*)$.

1.2. **Families of classes and of categories.**

We shall deal often with families of objects of various kind. A *family of sets*

$$(S_i \mid i \in I)$$

indexed by a set or a class I can be defined as a map $S_\bullet : I \to \mathrm{Ob}(\mathrm{Set})$, or equivalently, a functor $S_\bullet : I \to \mathrm{Set}$, where, as usual, we identify I with its associated discrete category.

However, clearly families of proper classes cannot be defined in this manner, not even when $I \neq \varnothing$ is a finite set. Instead, we define a *family of classes* $C_\bullet := (C_i \mid i \in I)$ indexed by a class I as a pair (C, p) of a class C and a map

$$p : C \to I.$$

For every $i \in I$, the *i-th member C_i of the family* (C, p) is just the fibre $C_i := p^{-1}(i)$. Likewise, *a family of maps* $f_\bullet : C_\bullet \to D_\bullet$ between two given families of classes $C_\bullet := (C, p : C \to I)$ and $D_\bullet := (D, q : D \to J)$ is the datum of a pair of maps

$$f : C \to D \qquad \phi : I \to J \qquad \text{with} \qquad q \circ f = \phi \circ p.$$

Hence, for every $i \in I$, the map f yields by restriction a map of classes $f_i : C_i \to D_{\phi(i)}$.

- If each member C_i is a set, this definition is equivalent to the previously considered notion : namely, to such (C, p) we may attach the functor $C_\bullet : I \to \mathsf{Set}$ given by the rule : $i \mapsto C_i$, and conversely, to every functor $S_\bullet : I \to \mathsf{Set}$ we may attach the pair (S, p_S) with $S := \coprod_{i \in I} S_i$, with its obvious projection $p_S : S \to I$. These two constructions are mutually inverse to each other, up to canonical bijections of classes and canonical isomorphisms of functors.
- Every such family (C, p) admits a *tautological disjoint union*, namely

$$\coprod_{i \in I} C_i := C$$

and *if I is a set*, the class of all sections of p yields a *product*

$$\prod_{i \in I} C_i := \{s : I \to C \mid p \circ s = \mathbf{1}_I\}.$$

Even though there is no category of classes, these constructions *enjoy the universal properties of categorical sums and products* : namely, we have a canonical family of maps $(j_i : C_i \to \coprod_{i \in I} C_i \mid i \in I)$ such that for every class X and every family of maps $(f_i : C_i \to X \mid i \in I)$ there exists a unique map $f : \coprod_{i \in I} C_i \to X$ with $f \circ j_i = f_i$ for every $i \in I$. However, these families of maps must themselves be taken in the foregoing sense : so we consider the constant families $(C, u : C \to \{\varnothing\})$, $(X, v : X \to \{\varnothing\})$, and $j_\bullet$ and $f_\bullet$ are regarded as the pairs

$$j_\bullet := (\mathbf{1}_C, \phi) : (C, p) \to (C, u) \qquad f_\bullet := (f, \phi) : (C, p) \to (X, v)$$

where ϕ is the unique map $\phi : I \to \{\varnothing\}$, and f is a given map $C \to X$. Then the universal property for $\coprod_{i \in I} C_i$, actually comes down to the trivial identity $v \circ f = u$.

- Likewise, *if I is a set*, we have a canonical family of maps $(p_i : \prod_{i \in I} C_i \to C_i \mid i \in I)$ such that for every class X and every family of maps $(g_i : X \to C_i \mid i \in I)$ there exists a unique map $g : X \to \prod_{i \in I} C_i$ with $p_i \circ g = g_i$ for every $i \in I$. Namely, set $P := \prod_{i \in I} C_i$; then $p_\bullet$ and $g_\bullet$ are regarded as the pairs

$$p_\bullet := (q, \mathbf{1}_I) : (I \times P, \pi_P) \to (C, p) \qquad g_\bullet := (h, \mathbf{1}_I) : (I \times X, \pi_X) \to (C, p)$$

where $q : I \times P \to C$ is given by the rule : $(i, s) \mapsto s(i)$ for every $(i, s) \in I \times P$, and $\pi_P : I \times P \to I$, $\pi_X : I \times X \to I$ are the projections. Then, the map $g : X \to P$ is given by the rule : $x \mapsto (i \mapsto h(i, x))$ for every $x \in X$ and $i \in I$.

Remark 1.2.1. (i) Let C be a class, and $\mathscr{R} \subset C \times C$ an equivalence relation on C; we may associate with $\mathscr{R}$ a family of classes : namely, the members of the family are the $\mathscr{R}$-equivalence classes, indexed by the *quotient class* $C/\mathscr{R}$. If C is a set, the family corresponding to a given equivalence relation $\mathscr{R}$ on C is the same as the datum of a *partition of*

C, *i.e.* a subset P of the power set $\mathscr{P}(C)$ (of all subsets of C) such that $\varnothing \notin P$, $C = \bigcup_{X\in P} X$, and $X \cap Y = \varnothing$ for every pair of distinct elements X, Y of P. Indeed, to $\mathscr{R}$ one attaches the partition whose elements are the $\mathscr{R}$-equivalence classes, and conversely, every such partition P of C defines a unique equivalence relation $\mathscr{R}_P$ on C such that $(x, y) \in \mathscr{R}_P$ if and only if there exists $X \in P$ such that $x, y \in X$. Then, the quotient $C/\mathscr{R}_P$ is just P.

(ii) In particular, if C is a set, then clearly *the class of all quotients of* C is a subclass of $\mathscr{P}(\mathscr{P}(C))$, and it is therefore a *set*. On the other hand, if C is a proper class, then the $\mathscr{R}$-equivalence classes may be proper classes as well, in which case they cannot correspond to any partition of C in the sense of (i). However, a construction due to the logician Dana Scott still allows us to attach to $\mathscr{R}$ a quotient class $C/\mathscr{R}$ and a projection $p : C \to C/\mathscr{R}$ such that the fibres of p are precisely the $\mathscr{R}$-equivalence classes, so that p describes again C as the disjoint union of the family of its $\mathscr{R}$-equivalence classes : see the discussion of *Scott's trick* in [13, Rem.2.9(ii)].

1.2.2. *Wide categories.* For some discussions, the usual framework of (large) categories is still not general enough, since for certain constructions one would like to drop the condition that the morphisms between pairs of objects are given by sets, and just allow them to be arbitrary classes. We are thus led to the following definition :

Definition 1.2.3. (i) A *wide pair* $\mathscr{C}$ is the datum of :

- a class $\mathrm{Ob}(\mathscr{C})$, whose elements are called *the objects of* $\mathscr{C}$
- a family of classes $p : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C})$ indexed by $\mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C})$, whose elements are called *the morphisms of* $\mathscr{C}$.

For a wide pair $(\mathrm{Ob}(\mathscr{C}), p)$, we let $\mathrm{Ob}(\mathscr{C}) \xleftarrow{\mathrm{s}} \mathrm{Mor}(\mathscr{C}) \xrightarrow{\mathrm{t}} \mathrm{Ob}(\mathscr{C})$ be the compositions of p with the projections $\mathrm{Ob}(\mathscr{C}) \leftarrow \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$, so that $p(f) = (\mathrm{s}(f), \mathrm{t}(f))$ for every morphism f of $\mathscr{C}$, and we call $\mathrm{s}(f)$ and $\mathrm{t}(f)$ the *source* and *target* of f. As usual, we also write $f : A \to B$ if the source and target of f are A and B, and set

$$\mathscr{C}(A, B) := p^{-1}(A, B) \qquad \forall A, B \in \mathrm{Ob}(\mathscr{C}).$$

(ii) A *wide category* $\mathscr{C}$ is the datum of a wide pair $(\mathrm{Ob}(\mathscr{C}), p)$ together with :

- a *composition law, i.e.* a family of maps of classes indexed by $\mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{C})$:
$$\mathrm{Mor}(\mathscr{C}) \times_{(\mathrm{t},\mathrm{s})} \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{C}) \qquad \big(A \xrightarrow{f} B, B \xrightarrow{g} C\big) \mapsto \big(A \xrightarrow{g\circ f} C\big)$$
such that $(h\circ g)\circ f = h\circ(g\circ f)$ for every sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$
- a family of *identity morphisms, i.e.* a map of classes
$$\mathrm{Ob}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{C}) \qquad A \mapsto (\mathbf{1}_A : A \to A)$$
such that $\mathbf{1}_B \circ f = f \circ \mathbf{1}_A$ for every morphism $f : A \to B$ of $\mathscr{C}$.

Remark 1.2.4. (i) Let $\mathscr{C}$ be any wide category. Clearly, if $\mathscr{C}(A, B)$ is a *set* for every $A, B \in \mathrm{Ob}(\mathscr{C})$, then $\mathscr{C}$ is the same as a (usual) category. Moreover, most standard categorical notions, such as monomorphisms, epimorphisms, and isomorphisms extend *verbatim* to wide categories. Also, one can define connected, directed, codirected, and filtered wide categories, just as in definition 1.1.6, as well as the opposite wide category $\mathscr{C}^{\mathrm{op}}$, just as for usual categories. Our terminology is perhaps less than ideal, since every category is a wide category, but not every wide category is a category.

(ii) We define a *functor* $F : \mathscr{C} \to \mathscr{C}'$ between wide categories as a pair of maps

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}) &\to \mathrm{Ob}(\mathscr{C}') & A &\mapsto FA \\ \mathrm{Mor}(\mathscr{C}) &\to \mathrm{Mor}(\mathscr{C}') & (f : A \to B) &\mapsto (Ff : FA \to FB) \end{aligned}$$

such that $F\mathbf{1}_A = \mathbf{1}_{FA}$ for every $A \in \mathrm{Ob}(\mathscr{C})$ and $F(g \circ f) = Fg \circ Ff$ for every pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$. So, if $\mathscr{C}$ and $\mathscr{C}'$ are usual categories, F is an ordinary functor. Clearly, functors between wide categories can be composed just as ordinary functors between usual categories; also, every functor F of wide categories induces an opposite functor $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}'^{\mathrm{op}}$, just as for usual categories.

(iii) Likewise, *a natural transformation* $\tau_\bullet : F \Rightarrow G$ of two functors $F, G : \mathscr{C} \rightrightarrows \mathscr{C}'$ between wide categories, is the datum of a map

$$\mathrm{Ob}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{C}') \qquad A \mapsto (\tau_A : FA \to GA)$$

such that $Gf \circ \tau_A = \tau_B \circ Ff$ for every morphism $f : A \to B$ of $\mathscr{C}$. Clearly, if $\mathscr{C}$ and $\mathscr{C}'$ are usual categories, $\tau_\bullet$ is an ordinary natural transformation of ordinary functors. Also, the composition laws for natural transformations detailed in §1.1.5 extend essentially *verbatim* to the context of wide categories, and likewise for the usual definitions of cones and co-cones : the details shall be left to the reader. Lastly, one can define faithful functors, full functors, and equivalences of wide categories, just as for usual categories.

(iv) In the same vein as in the foregoing discussion of families of classes, *a family of wide categories* $(\mathscr{C}_i \mid i \in I)$ *indexed by a class* I shall be defined as a pair $(\mathscr{C}, \pi)$ consisting of a wide category $\mathscr{C}$ and a functor $\pi : \mathscr{C} \to I$ from $\mathscr{C}$ to the discrete category I. Hence, with every $i \in I$ one may associate a wide category $\mathscr{C}_i$ with $\mathrm{Ob}(\mathscr{C}_i) := \pi^{-1}(i)$ and $\mathrm{Mor}(\mathscr{C}_i) := \{f \in \mathrm{Mor}(\mathscr{C}) \mid \pi(f) = i\}$, and with the composition law and identities induced from $\mathscr{C}$, in the obvious fashion. Especially, every such family induces the family of classes

$$(\mathrm{Ob}(\mathscr{C}_i) \mid i \in I) := (\mathrm{Ob}(\mathscr{C}), \pi).$$

If every $\mathscr{C}_i$ is a usual (resp. small) category, we say that $(\mathscr{C}_i \mid i \in I)$ is *a family of categories* (resp. *of small categories*). Then, for given families of wide categories $\mathscr{C}_\bullet := (\mathscr{C}, \pi)$ and $\mathscr{C}'_\bullet := (\mathscr{C}', \pi')$ indexed by classes I and I', we define a *family of functors* $F_\bullet : \mathscr{C}_\bullet \to \mathscr{C}'_\bullet$ as a pair consisting of a functor $F : \mathscr{C} \to \mathscr{C}'$ and a map $\phi : I \to I'$ such that $\pi' \circ F = \phi \circ \pi$. Again, any family of wide categories $(\mathscr{C}, p)$ admits a *tautological disjoint union* :

$$\coprod_{i \in I} \mathscr{C}_i := \mathscr{C}$$

and *if I is a set*, we have as well a *product* $\prod_{i \in I} \mathscr{C}_i$ whose class of objects is $\prod_{i \in I} \mathrm{Ob}(\mathscr{C}_i)$ and with morphisms :

$$\prod_{i \in I} \mathscr{C}_i(s, s') := \prod_{i \in I} \mathscr{C}_i(s(i), s'(i)) \qquad \forall s, s' \in \mathrm{Ob}\Big(\prod_{i \in I} \mathscr{C}_i\Big).$$

The reader is invited to spell out the universal properties of these constructions.

Example 1.2.5. For every wide category $\mathscr{C}$, we let $\sim_{\mathscr{C}}$ be the equivalence relation on $\mathrm{Ob}(\mathscr{C})$ such that $X \sim_{\mathscr{C}} Y$ if and only if there exists an isomorphism $X \xrightarrow{\sim} Y$ in $\mathscr{C}$. With Scott's trick (remark 1.2.1(ii)), we may form the quotient :

$$\overline{\mathrm{Ob}}(\mathscr{C}) := \mathrm{Ob}(\mathscr{C})/{\sim_{\mathscr{C}}} .$$

Every functor $F : \mathscr{C} \to \mathscr{C}'$ of wide categories induces a map $\overline{F} : \overline{\mathrm{Ob}}(\mathscr{C}) \to \overline{\mathrm{Ob}}(\mathscr{C}')$, and we say that F *is essentially injective* (resp. *essentially surjective*) if $\overline{F}$ is injective (resp. surjective).

Example 1.2.6. (i) For every wide category $\mathscr{C}$ we let $\sim_{\mathscr{C}}$ be the relation on $\mathrm{Ob}(\mathscr{C})$ such that $X \sim_{\mathscr{C}} Y$ if and only if there exists $k \in \mathbb{N}$ and a chain of morphisms of $\mathscr{C}$:

$$X \to X_1 \leftarrow X_2 \to \cdots \leftarrow X_k \to Y.$$

It is easily seen that $\sim_{\mathscr{C}}$ is an equivalence relation on $\mathrm{Ob}(\mathscr{C})$, and by applying Scott's trick (remark 1.2.1(ii)), we can form the quotient map

$$p_{\mathscr{C}} : \mathrm{Ob}(\mathscr{C}) \to \pi_0(\mathscr{C}) := \mathrm{Ob}(\mathscr{C})/\!\sim_{\mathscr{C}} .$$

Obviously, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ we have $\mathscr{C}(X, Y) \neq \varnothing$ if and only $p(X) = p(Y)$; hence, we may regard $p_{\mathscr{C}}$ as a functor $p_{\mathscr{C}} : \mathscr{C} \to \pi_0(\mathscr{C})$, *i.e.* a family of wide categories

$$(\mathscr{C}_i \mid i \in \pi_0(\mathscr{C}))$$

and every fibre $\mathscr{C}_i$ is a connected full wide subcategory of $\mathscr{C}$. We call $(\mathscr{C}_i \mid i \in \pi_0(\mathscr{C}))$ *the decomposition* of $\mathscr{C}$ as the union of its connected components; by remark 1.2.4(iv), we get a tautological identity :

$$\boxed{\mathscr{C} = \coprod_{i \in \pi_0(\mathscr{C})} \mathscr{C}_i.}$$

(ii) It is easily seen that every functor $F : \mathscr{B} \to \mathscr{C}$ induces a unique map

$$\boxed{\pi_0(F) : \pi_0(\mathscr{B}) \to \pi_0(\mathscr{C}) \qquad \text{such that} \qquad \pi_0(F) \circ p_{\mathscr{B}} = p_{\mathscr{C}} \circ F.}$$

(iii) Especially, the rules : $\mathscr{C} \mapsto \pi_0(\mathscr{C})$ and $(F : \mathscr{B} \to \mathscr{C}) \mapsto \pi_0(F)$ yield a functor :

$$\pi_0 : \mathsf{Cat} \to \mathsf{Set}.$$

Remark 1.2.7. We have moreover, three adjoint pairs of functors :

$$\boxed{\pi_0 : \mathsf{Cat} \rightleftarrows \mathsf{Set} : \mathsf{dis} : \mathsf{Set} \rightleftarrows \mathsf{Cat} : \mathsf{Ob} : \mathsf{Cat} \rightleftarrows \mathsf{Set} : \mathsf{ch}}$$

where dis is the functor of example 1.1.9(ii). Indeed, clearly every functor $\mathscr{C} \to \mathsf{dis}(S)$ factors through $p_{\mathscr{C}}$ and $\mathsf{dis}(f)$, for a unique map $f : \pi_0(\mathscr{C}) \to S$; conversely, any such f induces a functor $\mathscr{C} \to \mathsf{dis}(S)$. Hence, the rule : $\mathscr{C} \mapsto p_{\mathscr{C}}$ yields the unit for the first adjunction. Notice that $[0] := \mathsf{dis}(\{\varnothing\})$ is the final object of Cat. The unit for the second adjunction attaches to every set S the identity map $\mathbf{1}_S : S \to S = \mathrm{Ob}(\mathsf{dis}(S))$. The functor ch attaches to every set S the *chaotic category structure on* S, defined as the unique category $\mathsf{ch}(S)$ that is equivalent to the final category $[0]$, and whose set of objects is S; *i.e.* $\mathsf{ch}(S)(x, y) := \{\varnothing\}$ for every $x, y \in S$. Notice that the datum of a functor $\mathscr{C} \to \mathsf{ch}(S)$ is equivalent to that of a map $\mathrm{Ob}(\mathscr{C}) \to S$, whence the last stated adjunction.

1.2.8. *Wide limits and wide colimits.* Just as for sets, a *family of small categories* can also be defined as a functor from a discrete category I

$$\mathscr{C}_\bullet : I \to \mathsf{Cat} \qquad i \mapsto \mathscr{C}_i.$$

To such a functor we may attach the family of categories $(\mathscr{C}, p)$ such that $\mathrm{Ob}(\mathscr{C}) := \coprod_{i \in I} \mathrm{Ob}(\mathscr{C}_i)$; we set $\mathscr{C}((i, X), (i, Y)) := \mathscr{C}_i(X, Y)$ for every $i \in I$ and every $X, Y \in \mathrm{Ob}(\mathscr{C}_i)$, and $\mathscr{C}((i, X), (j, Y)) = \varnothing$ whenever $i \neq j$, and we endow $\mathscr{C}$ with the obvious composition law, so that the rules : $X \mapsto (i, X)$ for every $X \in \mathrm{Ob}(\mathscr{C}_i)$ and $f \mapsto f$ for every morphism f of $\mathscr{C}_i$, yield a functor $j_j : \mathscr{C}_i \to \mathscr{C}$ for every $i \in I$. The functor $p : \mathscr{C} \to I$ shall just be the projection : $(i, X) \mapsto i$ for every $(i, X) \in \mathrm{Ob}(\mathscr{C})$. *If* $\mathscr{C}$ *is small*, it represents the direct sum in Cat of the family of small categories $\mathscr{C}_\bullet$, and $j_\bullet := (j_i \mid i \in I)$ is a universal co-cone. This leads to the following definition :

Definition 1.2.9. (i) Let I be a category, $F : I \to \mathsf{Cat}$ a functor, and $\mathscr{C}$ a wide category. A *co-cone with basis F and vertex* $\mathscr{C}$ is a family of functors $\tau_\bullet := (\tau_i : Fi \to \mathscr{C} \mid i \in \mathrm{Ob}(I))$ such that $F\phi \circ \tau_i = \tau_i$ for every morphism $\phi : i \to j$ of I. We say that $\tau_\bullet$ is *a global cone*, if for every wide category $\mathscr{B}$ and every cone $\eta_\bullet$ with basis F and vertex $\mathscr{B}$, there exists a unique functor $G : \mathscr{B} \to \mathscr{C}$ such that $\eta_i = G \circ \tau_i$ for every $i \in \mathrm{Ob}(I)$. In this case, we also say that $\mathscr{C}$ *represents the wide colimit of* F.

(ii) Dually, A *cone with basis* F *and vertex* $\mathscr{C}$ is a family of functors $\tau_\bullet := (\tau_i : \mathscr{C} \to Fi \,|\, i \in \mathrm{Ob}(I))$ such that $\tau_j \circ F\phi = \tau_i$ for every morphism $\phi : i \to j$ of I. We say that $\tau_\bullet$ is *a global cone*, if for every wide category $\mathscr{B}$ and every cone $\eta_\bullet$ with basis F and vertex $\mathscr{B}$, there exists a unique functor $G : \mathscr{C} \to \mathscr{B}$ such that $\eta_i = \tau_i \circ G$ for every $i \in \mathrm{Ob}(I)$. In this case, we also say that $\mathscr{C}$ *represents the wide limit of* F.

Lemma 1.2.10. *(i) Let* $\mathscr{C}$ *be a wide category, and* $S \subset \mathrm{Mor}(\mathscr{C})$ *any subclass. Then there exists a unique minimal wide subcategory* $\mathscr{C}_S$ *with* $S \subset \mathrm{Mor}(\mathscr{C}_S)$.

(ii) If S *is a set, then* $\mathscr{C}_S$ *is a small category.*

(iii) Let $(\mathscr{P}(\mathscr{C}), \leq)$ *be the class of all subsets of* $\mathrm{Mor}(\mathscr{C})$*, filtered by inclusion of subsets. Then* $\mathscr{C}$ *represents the wide colimit of the functor*

$$\mathscr{C}_\bullet : (\mathscr{P}(\mathscr{C}), \leq) \to \mathsf{Cat} \qquad S \mapsto \mathscr{C}_S.$$

Proof. (i): Let $\mathrm{s}, \mathrm{t} : \mathrm{Mor}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ the source and target maps (see definition 1.2.3(i)), and set $X := \mathrm{s}(S) \cup \mathrm{t}(S)$. Without loss of generality, we may assume that $\{\mathbf{1}_A \,|\, A \in X\} \subset S$. For every $A, B \in X$, define inductively the family of classes $(H_n(A, B) \,|\, n \in \mathbb{N})$ as follows :

- $H_0(A, B) := \mathscr{C}(A, B) \cap S$
- $H_{n+1}(A, B) := \bigcup_{C \in X} \{g \circ f \,|\, f \in H_n(A, C), g \in H_n(C, B)\}$ for every $n \in \mathbb{N}$.

It is easily seen that there exists a unique wide subcategory $\mathscr{C}_S$ of $\mathscr{C}$ such that $\mathrm{Ob}(\mathscr{C}_S) := X$ and $\mathscr{C}_S(A, B) := \bigcup_{n \in \mathbb{N}} H_n(A, B)$ for every $A, B \in X$, and clearly $\mathscr{C}_S$ will do.

Assertion (ii) follows by direct inspection. To check (iii), it suffices to notice that $\bigcup_{S \in \mathscr{P}(\mathscr{C})} \mathrm{Ob}(\mathscr{C}_S) = \mathrm{Ob}(\mathscr{C})$ and $\bigcup_{S \in \mathscr{P}(\mathscr{C})} \mathrm{Mor}(\mathscr{C}_S) = \mathrm{Mor}(\mathscr{C})$. □

1.2.11. *Class-valued functors.* Even though there cannot be any category (nor any wide category) whose objects are all the classes, we can still define a workable replacement for the would-be functors taking values in the inexistent category of classes. Namely, we make the following :

Definition 1.2.12. (i) A *class-valued functor* on a wide category I is the datum of :

(a) a family of classes $p : \mathscr{F} \to \mathrm{Ob}(I)$ indexed by $\mathrm{Ob}(I)$

(b) a family of maps indexed by $\mathrm{Mor}(I)$ (notation of §1.1.3 and §1.1.5) :

$$F : \mathscr{F} \times_{(p,s)} \mathrm{Mor}(I) \to \mathscr{F} \times_{(p,t)} \mathrm{Mor}(I)$$

where $\mathscr{F} \times_{(p,s)} \mathrm{Mor}(I)$ is regarded as a family of classes indexed by $\mathrm{Mor}(I)$, via the natural projection $\mathscr{F} \times_{(p,s)} \mathrm{Mor}(I) \to \mathrm{Mor}(I)$, and likewise for $\mathscr{F} \times_{(p,t)} \mathrm{Mor}(I)$. We write $Fi := p^{-1}(i)$ for every $i \in \mathrm{Ob}(I)$, and denote by $F\phi : Fi \to Fj$ the restriction of F to the fibres over any morphism $\phi : i \to j$ of I. The datum $(\mathscr{F}_\bullet; F)$ must moreover fulfill the usual identities, *i.e.* $F\mathbf{1}_i = \mathbf{1}_{Fi}$ for every $i \in \mathrm{Ob}(I)$, and $F(\psi \circ \phi) = F(\psi) \circ F(\phi)$ for every composable pair of morphisms $\phi : i \to j$, $\psi : j \to k$ of I. Clearly, if each fibre Fi is a set, such a datum is equivalent to that of a functor $F : I \to \mathsf{Set}$.

(ii) Let F, G be two class-valued functors on I. A natural transformation $\tau_\bullet : F \Rightarrow G$ is the datum of a family of maps $(\tau_i : FA \to GA \,|\, i \in \mathrm{Ob}(I))$ such that $G\phi \circ \tau_i = \tau_j \circ F\phi$ for every morphism $\phi : i \to j$ of I. Likewise, we define cones $c_S \Rightarrow F$ and co-cones $F \Rightarrow c_S$, for a given class-valued functor F and a given class S, just as in §1.1.8.

Example 1.2.13. (i) Let X, Y be two classes, and $f, g : X \rightrightarrows Y$ two maps; according to example 1.1.4 and remark 1.2.1(ii) we may form first the smallest equivalence relation $\sim$ on Y such that $f(x) \sim g(x)$ for every $x \in X$, and then construct the quotient map

$p : Y \to Q := Y/\sim$. If X and Y are sets, (Q, p) represents the coequalizer of f and g in the category Set. In case X and Y are proper classes, (Q, p) still *enjoys the universal property of a coequalizer* : for every class Z and every map $h : Y \to Z$ such that $h \circ f = h \circ g$, there exists a unique map $k : Q \to Z$ such that $h = k \circ p$.

(ii) Recall also that for every family of classes $(C_i \mid i \in I)$, the (tautological) disjoint union $\bigsqcup_{i\in I} C_i$ enjoys the universal property of a direct sum; then, arguing as in the proof of [13, Prop.2.40], we may attach more generally to any class-valued functor F (on an arbitrary wide category I), a class C and a co-cone $\tau_\bullet := (\tau_i : Fi \to C \mid i \in \mathrm{Ob}(I))$ (meant as in definition 1.2.12(ii)) that *enjoys the universal property of a universal co-cone* : for every class D and every co-cone $\eta_\bullet := (\eta_i : Fi \to D \mid i \in \mathrm{Ob}(I))$ there exists a unique map $f : C \to D$ such that $\eta_i = f \circ \tau_i$ for every $i \in \mathrm{Ob}(I)$: the details are left to the reader. We call C the *global colimit* and $\tau_\bullet$ the *global co-cone* of F.

(iii) If I is a filtered wide category, a more explicit construction for the colimit or global colimit of any class-valued functor F on I is available; indeed, set

$$S := \coprod_{i\in\mathrm{Ob}(I)} Fi.$$

We have an equivalence relation $\sim$ on S such that $(i, x) \sim (i', x')$ if and only if there exists $i'' \in \mathrm{Ob}(I)$ and morphisms $i \xrightarrow{\phi} i'' \xleftarrow{\psi} i'$ of I such that $F\phi(x) = F\psi(x')$ (see [13, Exerc.2.31(i)]). The quotient class $Q := S/\sim$ is well defined as in remark 1.2.1(ii), and for every $i \in \mathrm{Ob}(I)$ we have a map $\tau_i : Fi \to Q$ that assigns with every $x \in Fi$ the class $[i, x]$ of the pair $(i, x) \in S$. As explained in [13, Exerc.2.31(ii)], if F is actually a set-valued functor on a usual category I, and if Q is a set, then Q represents the colimit of F, and $\tau_\bullet := (\tau_i \mid i \in \mathrm{Ob}(I))$ is a universal co-cone $F \Rightarrow c_Q$; for a general F, the class Q represents the global colimit of F, and the family of maps $\tau_\bullet$ yields a global co-cone.

(iv) Likewise, to any *small* category I and any class-valued functor F on I we can associate a *global limit* L and a *global cone* $\mu_\bullet := (\mu_i : L \to Fi \mid i \in \mathrm{Ob}(I))$: namely, L is the subclass of the product $P := \prod_{i\in\mathrm{Ob}(I)} Fi$ (defined as in §1.2) consisting of all *coherent sequences,* defined as in [13, Exemp.2.22(i)], and each μ_i is the restriction of the projection $P \to Fi$. The reader is invited to spell out the universal property of $(L, \mu_\bullet)$.

1.2.14. *Powers and copowers.* For any category $\mathscr{C}$, any $X \in \mathrm{Ob}(\mathscr{C})$ and any class S, we denote by

$$X^{(S)} \qquad (\text{resp. } X^S)$$

the coproduct (resp. the product) in $\mathscr{C}$ of the family $(X_s \mid s \in S)$ with $X_s := X$ for every $s \in S$, *i.e.* the disjoint union (resp. the product) of copies of X indexed by S, when such coproduct (resp. product) is representable in $\mathscr{C}$, in which case we sometimes call it the S-*copower* (resp. the S-*power*) of X. Notice that every map of classes $f : S \to T$ induces well-defined morphisms

$$X^{(f)} : X^{(S)} \to X^{(T)} \qquad \text{and} \qquad X^f : X^T \to X^S$$

when $X^{(S)}$ and $X^{(T)}$ (resp. X^T and X^S) are representable. Namely, $X^{(f)}$ and X^f are the unique morphisms of $\mathscr{C}$ that make commute the diagrams :

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "j_s^S"'] & X \arrow[d, "j^T_{f(s)}"] \\ X^{(S)} \arrow[r, "X^{(f)}"] & X^{(T)} \end{tikzcd} \qquad \begin{tikzcd} X^T \arrow[r, "X^f"] \arrow[d, "\pi^T_{f(s)}"'] & X^S \arrow[d, "\pi^S_s"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad \forall s \in S$$

where $j_\bullet^S := (j_s^S : X \to X^{(S)} \mid s \in S)$ and $j_\bullet^T := (j_t^T : X \to X^{(T)} \mid t \in T)$ are chosen universal co-cones, and likewise, $\pi_\bullet^S$ and $\pi_\bullet^T$ are universal cones. The morphism $X^{(f)}$ depends on the choice of $j_\bullet^S$ and $j_\bullet^T$, but a different choice alters $X^{(f)}$ by left and right compositions with isomorphisms of $\mathscr{C}$, so the categorical properties of $X^{(f)}$ are *intrinsic*. The same applies to X^f. If $X^{(S)}$ is representable for every set S, then we get a functor

$$X^{(-)} : \mathsf{Set} \to \mathscr{C} \qquad S \mapsto X^{(S)} \qquad (S \xrightarrow{f} T) \mapsto X^{(f)}$$

and if X^S is representable for every set S, we get a functor

$$X^{-} : \mathsf{Set}^{\mathrm{op}} \to \mathscr{C} \qquad S \mapsto X^S \qquad (S \xrightarrow{f} T) \mapsto X^f.$$

1.3. **Categories of functors.** To any small category I we attach a category :

$$\mathsf{Fun}(I, \mathscr{A}) \qquad \text{or sometimes} \qquad \mathscr{A}^I$$

whose objects are the functors $I \to \mathscr{A}$ (notice that, since I is small, every such functor is encoded by a *set*), and whose morphisms are the natural transformations between such functors, with the standard composition law explicited in §1.1.5. Notice the natural isomorphism of categories ([13, Rem.1.127(iv)]) :

$$\boxed{\mathsf{Fun}(I, \mathscr{A})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Fun}(I^{\mathrm{op}}, \mathscr{A}^{\mathrm{op}}) \quad F \mapsto F^{\mathrm{op}} \quad (\tau_\bullet : F \Rightarrow G) \mapsto (\tau_\bullet^{\mathrm{op}} : G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}).}$$

- Every functor $\phi : I \to J$ between small categories induces a functor

$$\boxed{\mathscr{A}^\phi : \mathscr{A}^J \to \mathscr{A}^I \qquad (J \xrightarrow{F} \mathscr{A}) \mapsto (I \xrightarrow{F \circ \phi} \mathscr{A}) \qquad (\tau_\bullet : F \Rightarrow G) \mapsto \tau_\bullet \star \phi}$$

also denoted $\mathsf{Fun}(\phi, \mathscr{A})$. Likewise, every functor $H : \mathscr{A} \to \mathscr{B}$ induces a functor

$$\boxed{\mathsf{Fun}(I, H) = H^I : \mathscr{A}^I \to \mathscr{B}^I \qquad (I \xrightarrow{F} \mathscr{A}) \mapsto H \circ F \qquad (\tau_\bullet : F \Rightarrow G) \mapsto H \star \tau_\bullet.}$$

Example 1.3.1. (i) For every set S, we may form the category $\mathscr{A}^S$ (here S is regarded as a discrete category : see example 1.1.9(ii)); if $\mathscr{A}$ is small, then $\mathscr{A}^S$ represents the S-power of $\mathscr{A}$ in Cat (as in §1.2.14), with universal cone given by the system

$$\mathscr{A}^{j_s} : \mathscr{A}^S \to \mathscr{A}^{\{s\}} \xrightarrow{\sim} \mathscr{A} \qquad \forall s \in S$$

where $j_s : \{s\} \to S$ denotes the inclusion of $\{s\}$ in S, regarded as an inclusion functor of discrete categories. When $\mathscr{A}$ is not small, the system $\mathscr{A}^{j_\bullet} := (\mathscr{A}^{j_s} : \mathscr{A}^S \to \mathscr{A} \mid s \in S)$ still yields a well-defined family of functors indexed by S (in the sense of remark 1.2.4(iv)), providing a *global cone*, and $\mathscr{A}^S$ still enjoys the universal property of a categorical power. Explicitly, the objects of $\mathscr{A}^S$ are the families $A_\bullet := (A_s \mid s \in S)$ of objects of $\mathscr{A}$, and the morphisms $f_\bullet : A_\bullet \to B_\bullet$ are the systems $(f_s : A_s \to B_s \mid s \in S)$ of morphisms of $\mathscr{A}$; it is then easily seen that such a morphism $f_\bullet$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if the same holds for f_s, for every $s \in S$.

(ii) A functor $F : I \to \mathscr{A}^S$ is the just datum of a family $(F_s : I \to \mathscr{A} \mid s \in S)$ of functors, and a natural transformation $\tau_\bullet : F \Rightarrow G$ between functors $F, G : I \rightrightarrows \mathscr{A}^S$ is just a family of natural transformations $(\tau_\bullet^s : F_s \Rightarrow G_s \mid s \in S)$. Especially, for every $A_\bullet \in \mathrm{Ob}(\mathscr{A}^S)$, a cone $\tau_\bullet : c_{A_\bullet} \Rightarrow F$ is the datum of a family of cones $(\tau_\bullet^s : c_{A_s} \Rightarrow F_s \mid s \in S)$; hence, such a cone $\tau_\bullet$ is universal if and only if the same holds for every $\tau_\bullet^s$, and likewise for universal co-cones : *i.e. each functor $\mathscr{A}^{j_s}$ preserves all representable limits and colimits.*

1.3.2. Let $j_I : \mathrm{Ob}(I) \to I$ be the unique functor from the discrete category $\mathrm{Ob}(I)$ to I that is the identity map on objects; the associated *evaluation functor*

$$\boxed{\mathsf{e}_I := \mathscr{A}^{j_I} : \mathscr{A}^I \to \mathscr{A}^{\mathrm{Ob}(I)} \qquad F \mapsto (Fi \mid i \in \mathrm{Ob}(I))}$$

is conservative and preserves and reflects every representable limit and colimit of $\mathscr{A}^I$: in other words, *limits and colimits in* $\mathscr{A}^I$ *are computed termwise.* As a consequence, if $\mathscr{A}$ is J-complete (resp. J-cocomplete) for some category J, then the same holds for $\mathscr{A}^I$ (see [13, Lemma 2.55, Rem.2.56 and 2.58(ii)]). Also, if all fibre products (resp. all amalgamated sums) are representable in $\mathscr{A}$, then e_I preserves and reflects monomorphisms (resp. epimorphisms), by virtue of remark 1.1.11(iv).

- For every $i \in \mathrm{Ob}(I)$, the composition of e_I with $\mathscr{A}^{j_i} : \mathscr{A}^{\mathrm{Ob}(I)} \to \mathscr{A}$ is the *evaluation functor at i :*

$$\boxed{\mathsf{e}_i : \mathscr{A}^I \to \mathscr{A} \qquad F \mapsto Fi \qquad (\tau_\bullet : F \Rightarrow G) \mapsto (\tau_i : Fi \to Gi)}$$

that preserves all representable limits and colimits, in light of example 1.3.1(ii).

- The category $\mathscr{A}$ is I-complete (resp. I-cocomplete) if and only if the functor

$$\boxed{\mathsf{c}_I : \mathscr{A} \to \mathscr{A}^I \qquad X \mapsto c_X \qquad (f : X \to Y) \mapsto (c_f : c_X \to c_Y)}$$

(notation of §1.1.8) admits a right adjoint (resp. a left adjoint)

$$\boxed{\underset{I}{\mathsf{Lim}} : \mathscr{A}^I \to \mathscr{A} \qquad (\text{resp. } \underset{I}{\mathsf{Colim}} : \mathscr{A}^I \to \mathscr{A}).}$$

Namely, for every functor $F : I \to \mathscr{A}$ fix a universal cone $\tau^F_\bullet : c_{L(F)} \Rightarrow F$ (resp. a universal co-cone $\eta^F_\bullet : F \Rightarrow c_{C(F)}$); then Lim_I (resp. Colim_I) assigns to F the object $L(F)$ (resp. $C(F)$) of $\mathscr{A}$, and to every natural transformation $\mu_\bullet : F \Rightarrow G$, the unique morphism

$$\lim_I \mu_\bullet : L(F) \to L(G) \qquad (\text{resp. } \operatorname*{colim}_I \tau_\bullet : C(F) \to C(G))$$

such that $\tau^G_\bullet \circ c_{\lim_I \tau_\bullet} = \mu_\bullet \circ \tau^F_\bullet$ (resp. such that $c_{\mathrm{colim}_I \tau_\bullet} \circ \eta^F_\bullet = \eta^G_\bullet \circ \mu_\bullet$) : see [13, §2.3.3].

1.3.3. Let J be a small category, and $\mathscr{C}, \mathscr{D}$ two categories. The datum of a functor

$$G : J \times \mathscr{C} \to \mathscr{D}$$

is equivalent to that of a functor

$$G' : \mathscr{C} \to \mathscr{D}^J.$$

Namely, for such G and every $X \in \mathrm{Ob}(\mathscr{C})$ we let $G'_X : J \to \mathscr{D}$ be the functor given by the rules : $j \mapsto G(j, X)$ and $u \mapsto G(u, \mathbf{1}_X)$ for every $j \in \mathrm{Ob}(J)$ and every $u \in \mathrm{Mor}(J)$; then, to every morphism $f : X \to Y$ of $\mathscr{C}$, we attach the natural transformation $G'_{f,\bullet} : G'_X \to G'_Y$ such that $j \mapsto (G(\mathbf{1}_j, f) : G'_X(j) \to G'_Y(j))$ for every $j \in \mathrm{Ob}(J)$. Conversely, from every such G' we can recover easily the corresponding functor G : see [13, Lemma 2.55].

- Suppose moreover that $\mathscr{D}$ is J-complete (resp. J-cocomplete); we then set

$$\boxed{\lim_J{}' G := \underset{J}{\mathsf{Lim}} \circ G' : \mathscr{C} \to \mathscr{D} \quad (\text{resp. } \operatorname*{colim}_J{}' G := \underset{J}{\mathsf{Colim}} \circ G' : \mathscr{C} \to \mathscr{D}).}$$

For every $j \in \mathrm{Ob}(J)$, let also $G''_j : \mathscr{C} \to \mathscr{D}$ be the functor given by the rules : $X \mapsto G(j, X)$ and $f \mapsto G(\mathbf{1}_j, f)$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every morphism f of $\mathscr{C}$.

Lemma 1.3.4. *In the situation of §1.3.3, let I be a category such that G''_j preserves I-limits (resp. I-colimits) for every $j \in \mathrm{Ob}(J)$. Then $\lim'_J G$ (resp. $\mathrm{colim}'_J G$) preserves I-limits (resp. I-colimits).*

Proof. To construct $\lim'_J G$ we fix a universal cone $\tau^X_\bullet : c_{\lim'_J GX} \Rightarrow G'_X$ for every $X \in \mathrm{Ob}(\mathscr{C})$; for every $f \in \mathscr{C}(X, Y)$, the morphism $\lim'_J Gf : \lim'_J GX \to \lim'_J GY$ is then characterized as the unique morphism of $\mathscr{D}$ that makes commute the diagram

$$\begin{tikzcd} \lim'_J GX \arrow[r, "\lim'_J Gf"] \arrow[d, "\tau^X_j"'] & \lim'_J GY \arrow[d, "\tau^Y_j"] \\ G(j, X) \arrow[r, "G(\mathbf{1}_j, f)"] & G(j, Y) \end{tikzcd} \qquad \forall j \in \mathrm{Ob}(J).$$

Hence, for every $j \in \mathrm{Ob}(J)$ we get a natural transformation

$$\tau^\bullet_j : \lim_J{}' G \Rightarrow G''_j \qquad X \mapsto \tau^X_j.$$

Now, let $F : I \to \mathscr{C}$ be any functor, and $\eta_\bullet : c_L \Rightarrow F$ a universal cone; by assumption, $G''_j \star \eta_\bullet : c_{G(j,L)} \Rightarrow G''_j \circ F$ is a universal cone for every $j \in \mathrm{Ob}(J)$, and we need to check that the same holds for $\lim'_J G \star \eta_\bullet : c_{\lim'_J GL} \Rightarrow (\lim'_J G) \circ F$.

Hence, let $\omega_\bullet : c_D \Rightarrow (\lim'_J G) \circ F$ be any cone; then for every $j \in \mathrm{Ob}(J)$ there exists a unique morphism $\phi_j : D \to G(j, L)$ such that :

$$(G''_j \star \eta_\bullet) \circ c_{\phi_j} = (\tau^\bullet_j \star F) \circ \omega_\bullet.$$

Claim 1.3.5. The rule : $j \mapsto \phi_j$ defines a cone $\phi_\bullet : c_D \Rightarrow G'_L$.

Proof : We need to check that $G'_L(u) \circ \phi_j = \phi_k$ for every morphism $u : j \to k$ of J. To this aim, consider the commutative diagram :

$$\begin{tikzcd} D \arrow[r, "\phi_j"] \arrow[d, "\omega_i"'] & G(j, L) \arrow[r, "G(u, \mathbf{1}_L)"] \arrow[d, "G(\mathbf{1}_j, \eta_i)"] & G(k, L) \arrow[d, "G(\mathbf{1}_k, \eta_i)"] \\ \lim'_J G(Fi) \arrow[r, "\tau^{Fi}_j"] & G(j, Fi) \arrow[r, "G(u, \mathbf{1}_{Fi})"] & G(k, Fi) \end{tikzcd} \qquad \forall i \in \mathrm{Ob}(I)$$

and notice that the composition of the bottom horizontal arrows is τ^{Fi}_k; hence the composition of the top horizontal arrows must be ϕ_k, by the uniqueness property of the latter morphism. ◇

By claim 1.3.5 we get a unique morphism

$$g : D \to \lim_J{}' GL \qquad \text{such that} \qquad \tau^L_\bullet \circ c_g = \phi_\bullet.$$

Let us check that $(\lim'_J G \star \eta_\bullet) \circ c_g = \omega_\bullet$. The assertion means that $\omega_i = \lim'_J G(\eta_i) \circ g$ for every $i \in \mathrm{Ob}(I)$, and the latter is equivalent to the system of identities :

$$\tau^{Fi}_j \circ \omega_i = \tau^{Fi}_j \circ \lim_J{}' G(\eta_i) \circ g \qquad \forall j \in \mathrm{Ob}(J)$$

in view of the universality of $\tau^{Fi}_\bullet$. However :

$$\tau^{Fi}_j \circ \omega_i = G(\mathbf{1}_j, \eta_i) \circ \phi_j = G(\mathbf{1}_j, \eta_i) \circ \tau^L_j \circ g = \tau^{Fi}_j \circ \lim_J{}' G(\eta_i) \circ g$$

as required. Conversely, if $h : D \to \lim'_J GL$ is another morphism of $\mathscr{D}$ with $(\lim'_J G \star \eta_\bullet) \circ c_h = \omega_\bullet$, then by arguing as in the foregoing we see that $G(\mathbf{1}_j, \eta_i) \circ \phi_j = G(\mathbf{1}_j, \eta_i) \circ \tau^L_j \circ h$

for every $i \in \mathrm{Ob}(I)$ and $j \in \mathrm{Ob}(J)$, whence $\tau^L_j \circ h = \phi_j$ for every such j, by universality of $G''_j \star \eta_\bullet$, and finally, $h = g$, by universality of $\tau^L_\bullet$.

The assertion for $\mathrm{colim}'_J\, G$ follows by duality with remark 1.1.11(i), recalling that we have $\mathrm{colim}'_J\, G = (\lim'_{J^{\mathrm{op}}} G^{\mathrm{op}})^{\mathrm{op}}$ and $(G^{\mathrm{op}})''_j = (G''_j)^{\mathrm{op}}$ for every $j \in \mathrm{Ob}(J)$. □

1.3.6. *Permutation of limits and colimits.* Let I, J be two small categories, and $\mathscr{C}$ any category. To every functor $G : J \times I \to \mathscr{C}$ we may then attach $G' : I \to \mathscr{C}^J$ as in §1.3.3; moreover, by composing G with the obvious isomorphisms of categories $I \times J \xrightarrow{\sim} J \times I$ we get a functor $I \times J \to \mathscr{C}$, whence a functor as in §1.3.3, that we denote $G'' : J \to \mathscr{C}^I$. These associations define isomorphisms of categories ([13, Lemma 2.55(i)]) :

$$\boxed{(\mathscr{C}^J)^I \xleftarrow{\sim} \mathscr{C}^{I\times J} \xrightarrow{\sim} (\mathscr{C}^I)^J \qquad G' \leftarrow\!\shortmid G \mapsto G''.}$$

If $\mathscr{C}$ is I-complete (resp. I-cocomplete), we may set as well :

$$\boxed{\lim_I{}'' G := \mathop{\mathsf{Lim}}_I \circ G'' : J \to \mathscr{C} \quad (\text{resp. } \mathop{\mathrm{colim}}_I{}'' G := \mathop{\mathsf{Colim}}_I \circ G'' : J \to \mathscr{C}\,).}$$

Lastly, suppose that $\mathscr{C}$ is *both I-complete and J-complete* (resp. *both I-cocomplete and J-cocomplete*); then $\mathscr{C}$ is also *$(I \times J)$-complete* (resp. *$(I \times J)$-cocomplete*), and the rules : $G \mapsto \lim'_J G$ and $G \mapsto \lim''_I G$ (resp. the rules : $G \mapsto \mathrm{colim}'_J\, G$ and $G \mapsto \mathrm{colim}''_I\, G$) define functors :

$$\mathscr{C}^I \xleftarrow{\mathsf{Lim}'_J} \mathscr{C}^{I\times J} \xrightarrow{\mathsf{Lim}''_I} \mathscr{C}^J \qquad (\text{resp. } \mathscr{C}^I \xleftarrow{\mathsf{Colim}'_J} \mathscr{C}^{I\times J} \xrightarrow{\mathsf{Colim}''_I} \mathscr{C}^J).$$

Moreover, we have natural isomorphisms of functors ([13, Prop.2.57, Rem.2.58]) :

$$\boxed{\mathop{\mathsf{Lim}}_I \circ \mathop{\mathsf{Lim}'}_J \xleftarrow{\sim} \mathop{\mathsf{Lim}}_{I\times J} \xrightarrow{\sim} \mathop{\mathsf{Lim}}_J \circ \mathop{\mathsf{Lim}''}_I \quad (\text{resp. } \mathop{\mathsf{Colim}}_I \circ \mathop{\mathsf{Colim}'}_J \xleftarrow{\sim} \mathop{\mathsf{Colim}}_{I\times J} \xrightarrow{\sim} \mathop{\mathsf{Colim}}_J \circ \mathop{\mathsf{Colim}''}_I).}$$

Summing up, we may say that *the I-limits in $\mathscr{C}$ commute with the J-limits* (resp. *the I-colimits in $\mathscr{C}$ commute with the J-colimits*). On the other hand, suppose that $\mathscr{C}$ is *both I-complete and J-cocomplete*; then we have a natural transformation :

$$\boxed{\omega^{I,J}_\bullet : \mathop{\mathsf{Colim}}_J \circ \mathop{\mathsf{Lim}''}_I \to \mathop{\mathsf{Lim}}_I \circ \mathop{\mathsf{Colim}'}_J}$$

which however is not necessarily an isomorphism of functors ([13, Prob.2.59]), but in case it is, we say that *the I-limits in $\mathscr{C}$ commute with the J-colimits.* As usual, the construction of $\omega^{I,J}_\bullet$ requires several choices, but is independent of such choices, up to composition with isomorphisms, so its categorical properties are *intrinsic.*

Example 1.3.7. (i) According to [13, Prob.2.59(ii)], *the finite limits in the category* Set *commute with all small filtered colimits, i.e.* the I-limits in Set commute with the J-colimits, for every finite category I and every small filtered category J.

(ii) Let $\mathscr{A}$ be any small category. Since the limits and colimits in $\mathsf{Set}^{\mathscr{A}}$ are computed termwise (see §1.3), it follows from (i) that *the finite limits in the category* $\mathsf{Set}^{\mathscr{A}}$ *commute with small filtered colimits.* See lemma 4.2.3 for a generalization.

(iii) From (i) it follows that the functor $\mathsf{Colim}_J : \mathsf{Set}^J \to \mathsf{Set}$ *is exact for every filtered set J*; combining with §1.3.2 and remark 1.1.11(iii), we deduce that Colim_J *preserves monomorphisms* : explicitly, for every $F, G \in \mathrm{Ob}(\mathsf{Set}^J)$ and every natural transformation $\tau_\bullet : F \Rightarrow G$ such that $\tau_j : Fj \to Gj$ is injective for every $j \in J$, the induced map $\mathrm{colim}_J\, \tau_\bullet : \mathrm{colim}_J\, F \to \mathrm{colim}_J\, G$ is injective. More generally, by the same token, *the colimit of any filtered system of monomorphisms in* $\mathsf{Set}^{\mathscr{A}}$ *is a monomorphism.*

(iv) It is also easily seen that *the fibre products in the category* Set *commute with all small direct sums* : *i.e.*, for every set S, every family of maps of sets :

$$(X_s \to Z_s \leftarrow Y_s \mid s \in S)$$

induces a natural bijection :

$$\coprod_{s\in S} X_s \times_{Z_s} Y_s \to \big(\coprod_{s\in S} X_s\big) \times_{\coprod_{s\in S} Z_s} \big(\coprod_{s\in S} Y_s\big).$$

Again, it follows that the same assertion holds more generally in $\mathrm{Set}^{\mathscr{A}}$.

Remark 1.3.8. (i) The foregoing discussion extends partially to global limits and global colimits : let I, J be two categories, $F_{\bullet\bullet}$ a class-valued functor on $I \times J$, and recall that $F_{\bullet\bullet}$ is defined by a family of classes $(F_{i,j} \mid (i,j) \in \mathrm{Ob}(I \times J))$ and a family of maps (see §1.2):

$$(F_{\phi,\psi} : F_{i,j} \to F_{i',j'} \mid ((\phi,\psi) : (i,j) \to (i',j')) \in \mathrm{Mor}(I \times J)).$$

For each $i \in \mathrm{Ob}(I)$, we deduce by restriction a class-valued functor $F_{i,\bullet}$ on J, given by the family of classes $(F_{i,j} \mid j \in \mathrm{Ob}(J))$ and the family of maps $(F_{i,\psi} := F_{\mathbf{1}_i,\psi} \mid \psi \in \mathrm{Mor}(J))$. Then, for every such i, let C_i and $\eta_{i,\bullet}$ be respectively the global colimit and the global co-cone of $F_{i,\bullet}$, constructed as in example 1.2.13(ii); we thus obtain a well-defined family of classes $(C_i \mid i \in \mathrm{Ob}(I))$, and with the universal properties of the pairs $(C_i, \eta_{i,\bullet})$ we may assign to every morphism $\phi : i \to i'$ of I a unique map $C_\phi : C_i \to C_j$ such that $C_\phi \circ \eta_{i,j} = \eta_{i',j} \circ F_{\phi,\mathbf{1}_j}$ for every $j \in \mathrm{Ob}(J)$. Clearly, we get in this way a family of maps $(C_\phi \mid \phi \in \mathrm{Mor}(I))$ which yields a class-valued functor $C_\bullet$, whose global colimit and global co-cone we denote by C and respectively $\eta_\bullet := (\eta_i : C_i \to C \mid i \in \mathrm{Ob}(I))$; one checks easily that C also represents the global colimit for the class-valued functor $F_{\bullet\bullet}$, with global co-cone given by the family of maps $(\eta_i \circ \eta_{i,j} : F_{i,j} \to C \mid (i,j) \in \mathrm{Ob}(I \times J))$.

(ii) Likewise, *if I and J are small categories*, we may form the global limit L_j of each class-valued functor $F_{\bullet,j}$ on I (defined as the obvious restriction of $F_{\bullet\bullet}$), and together with the corresponding system of global cones $\varepsilon_{\bullet,j}$, the system $(L_j \mid j \in \mathrm{Ob}(J))$ yields a well-defined class-valued functor $L_\bullet$ on J, whose global limit also represents the global limit of $F_{\bullet,\bullet}$: we leave the details to the reader.

(iii) Lastly, *if I is small*, we may first form the class-valued functors $C_\bullet$ on I and $L_\bullet$ on J as in (i) and (ii), together with the respective systems of global co-cones $(\eta_{i,\bullet} \mid i \in \mathrm{Ob}(I))$ and global cones $(\varepsilon_{\bullet,j} \mid j \in \mathrm{Ob}(J))$; next, let $\mathrm{colim}_J L_\bullet$ and $\lim_I C_\bullet$ be respectively the global colimit of $L_\bullet$ and the global limit of $C_\bullet$, with their global co-cone and global cone

$$\eta_\bullet := (\eta_j : L_j \to \operatorname*{colim}_J L_\bullet \mid j \in \mathrm{Ob}(J)) \qquad \varepsilon_\bullet := (\varepsilon_i : \lim_I C_\bullet \to C_i \mid i \in \mathrm{Ob}(I)).$$

Then, following [13, Prob.2.59], we get a natural map of classes

$$\tau : \operatorname*{colim}_J L_\bullet \to \lim_I C_\bullet \qquad \text{such that} \qquad \varepsilon_i \circ \tau \circ \eta_j = \eta_{i,j} \circ \varepsilon_{i,j} \qquad \forall (i,j) \in \mathrm{Ob}(I \times J).$$

The map τ is not always a bijection, but arguing as in the solution of [13, Prob.2.59(ii)] one can check that *τ is bijective in case I is a finite category and J is a filtered category.*

1.4. **Slice and comma categories.** Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, and $F : \mathscr{A} \to \mathscr{B}$ a functor; to every $B \in \mathrm{Ob}(\mathscr{B})$ we attach the *slice category of F over B*, denoted :

$$F\mathscr{A}/B$$

whose objects are the pairs $(A, FA \xrightarrow{u} B)$ for all $A \in \mathrm{Ob}(\mathscr{A})$ and all $u \in \mathscr{B}(FA, B)$; the morphisms $w/B : (A,u) \to (A',u')$ of $F\mathscr{A}/B$ are the morphisms $w : A \to A'$ of $\mathscr{A}$ such that $u' \circ Fw = u$, *i.e.* the commutative triangles

$$\begin{tikzcd} FA \arrow[rr, "Fw"] \arrow[dr, "u"'] & & FA' \arrow[dl, "u'"] \\ & B & \end{tikzcd}$$

with the composition law inherited from $\mathscr{A}$. We have an obvious *source functor* :

$$\boxed{\mathrm{s}_B : F\mathscr{A}/B \to \mathscr{A} \qquad (A,u) \mapsto A \qquad w/B \mapsto w.}$$

- Dually, we define *the slice category of F under B* as :

$$B/F\mathscr{A} := (F^{\mathrm{op}}\mathscr{A}^{\mathrm{op}}/B^{\mathrm{op}})^{\mathrm{op}}.$$

Namely, the objects of $B/F\mathscr{A}$ are the pairs $(A, B \xrightarrow{u} FA)$, ranging over all $A \in \mathrm{Ob}(\mathscr{A})$ and all $u \in \mathscr{A}(B, FA)$, and the morphisms $B/w : (A,u) \to (A',u')$ are the commutative triangles of $\mathscr{B}$:

$$\begin{tikzcd} & B \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ FA \arrow[rr, "Fw"] & & FA'. \end{tikzcd}$$

Then the correponding *target functor* is :

$$\boxed{\mathrm{t}_B := (\mathrm{s}_{B^{\mathrm{op}}})^{\mathrm{op}} : B/F\mathscr{A} \to \mathscr{A} \qquad (A,u) \mapsto A \qquad B/w \mapsto w.}$$

- In the special case where $F := \mathbf{1}_{\mathscr{A}}$, we get, for every $A \in \mathrm{Ob}(\mathscr{A})$, *the slice categories of objects of $\mathscr{A}$ over A* and respectively *under A* :

$$\mathscr{A}/A := \mathbf{1}_{\mathscr{A}}\mathscr{A}/A \qquad A/\mathscr{A} := A/\mathbf{1}_{\mathscr{A}}\mathscr{A}.$$

So $\mathrm{Ob}(\mathscr{A}/A)$ consists of the pairs $(A', u : A' \to A)$ with $A' \in \mathrm{Ob}(\mathscr{A})$ and $u \in \mathscr{A}(A', A)$, and the morphisms $w/A : (A',u) \to (A'',v)$ are the morphisms $w : A' \to A''$ of $\mathscr{A}$ such that $v \circ w = u$, and likewise for the objects and morphisms of $A/\mathscr{A}$.

- The family $(FA/\mathscr{B} \mid A \in \mathrm{Ob}(\mathscr{A}))$ can be combined into a single *comma category*

$$F\mathscr{A}/\mathscr{B}$$

whose objects are the triples (A, B, f) with $A \in \mathrm{Ob}(\mathscr{A})$, $B \in \mathrm{Ob}(\mathscr{B})$ and $f \in \mathscr{B}(FA, B)$. The morphisms $(h,k) : (A,B,f) \to (A',B',f')$ are the commutative diagrams :

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ FA' \arrow[r, "f'"] & B' \end{tikzcd}$$

with $h \in \mathscr{A}(A,A')$ and $k \in \mathscr{B}(B,B')$, with the obvious composition law : $(h',k') \circ (h,k) := (h' \circ h, k' \circ k)$ for every composable pair $(h,k), (h',k')$ of morphisms.

Likewise, the family $(B/F\mathscr{A} \mid B \in \mathrm{Ob}(\mathscr{B}))$ can be combined into the *comma category*

$$\mathscr{B}/F\mathscr{A} := (F^{\mathrm{op}}A^{\mathrm{op}}/\mathscr{B}^{\mathrm{op}})^{\mathrm{op}}$$

and we have obvious source and target functors :

$$\boxed{\begin{array}{lll} \mathscr{A} \xleftarrow{\mathrm{s}} F\mathscr{A}/\mathscr{B} \xrightarrow{\mathrm{t}} \mathscr{B} & A \leftarrow\!\shortmid (A,B,f) \mapsto B & h \leftarrow\!\shortmid (h,k) \mapsto k \\ \mathscr{B} \xleftarrow{\mathrm{s}} \mathscr{B}/F\mathscr{A} \xrightarrow{\mathrm{t}} \mathscr{A} & B \leftarrow\!\shortmid (B,A,f) \mapsto A & k \leftarrow\!\shortmid (k,h) \mapsto h. \end{array}}$$

1.4.1. Every functor $F : \mathscr{A} \to \mathscr{B}$ and every morphism $f : B \to B'$ of $\mathscr{B}$ induce a functor

$$\boxed{f^* : F\mathscr{A}/B \to F\mathscr{A}/B' \qquad (FA \xrightarrow{u} B) \mapsto (FA \xrightarrow{f\circ u} B') \qquad w/B \mapsto w/B'}$$

and by dualizing, we also get a corresponding functor $f_! := ((f^{\mathrm{op}})^*)^{\mathrm{op}}$; explicitly :

$$\boxed{f_! : B'/F\mathscr{A} \to B/F\mathscr{A} \quad (B' \xrightarrow{u'} FA) \mapsto (B \xrightarrow{u'\circ f} FA) \quad B'/w' \mapsto B/w'.}$$

- With this notation, F also induces functors :

$$\boxed{F/B : F\mathscr{A}/B \to \mathscr{B}/B \qquad \text{and} \qquad B/F : B/F\mathscr{A} \to B/\mathscr{B} \qquad (A,u) \mapsto (FA,u)}$$

such that $w/B \mapsto F(w)/B$ (resp. $B/w \mapsto B/F(w)$) for every $w/B \in \mathrm{Mor}(F\mathscr{A}/B)$ (resp. for every $B/w \in \mathrm{Mor}(B/F\mathscr{A})$). Moreover, any $A \in \mathrm{Ob}(\mathscr{A})$ induces functors :

$$\boxed{\begin{array}{lll} F_{/A} : \mathscr{A}/A \to \mathscr{B}/FA & (A',u) \mapsto (FA',Fu) & w/A \mapsto Fw/FA \\ {}_{A/}F : A/\mathscr{A} \to FA/\mathscr{B} & (A',u) \mapsto (FA',Fu) & A/w \mapsto FA/Fw. \end{array}}$$

Remark 1.4.2. (i) Let $f : A \to A'$ be any morphism of $\mathscr{A}$. *If all fibre products are representable in* $\mathscr{A}$, then $f^* : \mathscr{A}/A \to \mathscr{A}/A'$ admits a right adjoint

$$\boxed{f_* : \mathscr{A}/A' \to \mathscr{A}/A \quad (B' \xrightarrow{u'} A') \mapsto (B' \times_{A'} A \xrightarrow{u'\times_{A'}A} A) \quad w'/A' \mapsto (w' \times_{A'} A)/A}$$

where $B' \times_{A'} A$ denotes a chosen representative in $\mathscr{A}$ for the fibre product of the diagram $B' \xrightarrow{u'} A' \xleftarrow{f} A$, and where for every morphism $w' : B' \to C'$ of $\mathscr{A}$ we let $w' \times_{A'} A : B' \times_{A'} A \to C' \times_{A'} A$ be the induced morphism between these chosen fibre products. Namely, the adjunction is given by the associations :

$$\begin{tikzcd} B \arrow[rr, "w"] \arrow[dr, "u"'] & & B' \times_{A'} A \arrow[dl, "u'\times_{A'}A"] \\ & A & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} B \arrow[rr, "(B'\times_{A'}f)\circ w"] \arrow[dr, "f\circ u"'] & & B' \arrow[dl, "u'"] \\ & A'. & \end{tikzcd}$$

(ii) Dually, *if every amalgamated sum is representable in* $\mathscr{A}$, then the functor $f_! : A'/\mathscr{A} \to A/\mathscr{A}$ admits a left adjoint

$$\boxed{f^! : A/\mathscr{A} \to A'/\mathscr{A} \quad (A \xrightarrow{u} B) \mapsto (A' \xrightarrow{u\sqcup_A A'} B \sqcup_A A') \quad A/w \mapsto A'/(w \sqcup_A A').}$$

Proposition 1.4.3. *Let $\mathscr{A}, \mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a functor, and $B \in \mathrm{Ob}(\mathscr{B})$. Then we have :*

(i) The source functor $\mathrm{s}_B : F\mathscr{A}/B \to \mathscr{A}$ reflects epimorphisms, is conservative, and preserves and reflects monomorphisms. Dually, the target functor $\mathrm{t}_B : B/F\mathscr{A} \to \mathscr{A}$ reflects monomorphisms, is conservative, and preserves and reflects epimorphisms.

(ii) If $\mathscr{A}$ is finitely complete and F is left exact, then $B/F\mathscr{A}$ is cofiltered.

(iii) If $\mathscr{A}$ is finitely cocomplete and F is right exact, then $F\mathscr{A}/B$ is filtered.

(iv) s_B both preserves and reflects all representable connected limits.

(v) Dually, t_B both preserves and reflects all representable connected colimits.

Proof. (i): Clearly s_B is conservative, and reflects epimorphisms and monomorphisms. Then, suppose that $w/B : (A,u) \to (A',u')$ is a monomorphism of $F\mathscr{A}/B$, and consider two morphisms $f,g : A'' \rightrightarrows A$ of $\mathscr{A}$ such that $w \circ f = w \circ g$; hence $u'' := u \circ Ff = u' \circ Fw \circ Ff = u' \circ Fw \circ Fg = u \circ Fg$, so f and g yield morphisms $f/B, g/B : (A'',u'') \rightrightarrows (A,u)$

with $w/B \circ f/B = w/B \circ g/B$, whence $f = g$, which shows that w is a monomorphism. The assertions about t_B follow by duality.

(ii) Indeed, if $(\phi_i : B \to FA_i \mid i = 1, 2)$ are any two objects of $B/F\mathscr{A}$, the product $A_1 \times A_2$ is representable by some $A \in \mathrm{Ob}(\mathscr{A})$, and a universal cone is given by a pair $(p_i : A \to A_i \mid i = 1, 2)$ of morphisms of $\mathscr{A}$; by assumption, the pair $(Fp_i : FA \to FA_i \mid i = 1, 2)$ is still a universal cone for the product $FA_1 \times FA_2$, whence a unique morphism $\phi : B \to FA$ such that $F(p_i) \circ \phi = \phi_i$ for $i = 1, 2$. This shows that $B/F\mathscr{A}$ is codirected. Moreover, if A is any final object of $\mathscr{A}$, then FA is a final object of $\mathscr{B}$ (remark 1.1.11(ii)), hence $\mathrm{Ob}(B/F\mathscr{A})$ is non-empty. It then remains only to check that $B/F\mathscr{A}$ satisfies the equalizing condition dual to that of definition 1.1.6(v). Namely, let ϕ_1 and ϕ_2 be as in the foregoing, and suppose we have a pair of morphisms $\psi, \psi' : A_1 \rightrightarrows A_2$ such that $F(\psi) \circ \phi_1 = \phi_2 = F(\psi') \circ \phi_1$. The equalizer of ψ and ψ' is representable by some $E \in \mathrm{Ob}(\mathscr{A})$, and a universal cone for E is given by a morphism $\beta : E \to A_1$ such that $\psi \circ \beta = \psi' \circ \beta$; then FE represents the equalizer of $F\psi$ and $F\psi'$, and $F\beta : FE \to FA_1$ still yields a universal cone. There follows a unique morphism $\gamma : B \to FE$ in $\mathscr{B}$ such that $F(\beta) \circ \gamma = \phi_1$, whence the claim. Assertion (iii) follows from (ii) by duality.

(iv): Let $\phi : I \to F\mathscr{A}/B$ be a functor with I connected, and say $\phi(i) := (A_i, FA_i \xrightarrow{u_i} B)$ for every $i \in \mathrm{Ob}(I)$; let also $\tau_\bullet : c_A \Rightarrow \mathrm{s}_B \circ \phi$ be any cone. We observe :

Claim 1.4.4. There exists $g \in \mathscr{B}(FA, B)$ such that $g = u_i \circ F\tau_i$ for every $i \in \mathrm{Ob}(I)$.

Proof : By assumption $\mathrm{Ob}(I) \neq \varnothing$, so we may pick any $i_0 \in \mathrm{Ob}(I)$ and set $g := u_{i_0} \circ F\tau_{i_0}$. Then, since I is connected, a simple induction reduces to checking that $u_i \circ F\tau_i = u_j \circ F\tau_j$ for every morphism $t : i \to j$ of I. But we have : $u_i \circ F\tau_i = u_j \circ F\phi t \circ F\tau_i = u_j \circ F\tau_j$. ◇

By claim 1.4.4, we may regard $\tau_\bullet$ as a cone $\tau^*_\bullet : c_{(A,g)} \Rightarrow \phi$, and thus obtain a bijection $\phi_\bullet \mapsto \phi^*_\bullet$ between the cones with basis $\mathrm{s}_B \circ \phi$ and the cones with basis ϕ : indeed $\tau_\bullet = \mathrm{s}_B \star \tau^*_\bullet$ for every such $\tau_\bullet$. Clearly we have $\tau^*_\bullet \circ c_{h/B} = (\tau_\bullet \circ c_h)^*$ for every morphism $h/B : (A', g \circ Fh) \to (A, g)$ of $F\mathscr{A}/B$; it follows easily that $\tau_\bullet$ is universal if and only if the same holds for $\tau^*_\bullet$, so s_B preserves and reflects the limit of ϕ. Assertion (v) follows from (iv) by duality. ◻

Proposition 1.4.5. *(i) Let $\mathscr{A}, \mathscr{B}, I$ be three categories, $F : \mathscr{A} \to \mathscr{B}$ a functor that preserves all representable I-colimits, and $B \in \mathrm{Ob}(\mathscr{B})$. Then we have :*

- *(a) The source functor $\mathrm{s}_B : F\mathscr{A}/B \to \mathscr{A}$ reflects all I-colimits.*
- *(b) If $\mathscr{A}$ is I-cocomplete, the same holds for $F\mathscr{A}/B$, and s_B preserves I-colimits.*

(ii) Dually, if F preserves all representable I-limits, we have :

- *(a) the target functor $\mathrm{t}_B : B/F\mathscr{A} \to \mathscr{A}$ reflects all I-limits.*
- *(b) If $\mathscr{A}$ is I-complete, the same holds for $B/F\mathscr{A}$, and t_B preserves I-limits.*

Proof. By duality, it suffices to show (i). To prove (i.a), consider a functor

$$\phi : I \to F\mathscr{A}/B \qquad i \mapsto (A_i, FA_i \xrightarrow{u_i} B)$$

and a co-cone $\tau_\bullet : \phi \Rightarrow c_{(L,f)}$. Suppose that $\mathrm{s}_B \star \tau_\bullet$ is universal, and let $\eta_\bullet : \phi \Rightarrow c_{(A,g)}$ be any other co-cone; then there exists a unique $h \in \mathscr{A}(L, A)$ such that

$$c_h \circ (\mathrm{s}_B \star \tau_\bullet) = \mathrm{s}_B \star \eta_\bullet$$

and we need to show that $c_{h/B} \circ \tau_\bullet = \eta_\bullet$, *i.e.* that $g \circ Fh = f$. But we have :

$$g \circ Fh \circ F\tau_i = g \circ F\eta_i = u_i = f \circ F\tau_i \qquad \forall i \in \mathrm{Ob}(I)$$

whence the sought identity, as by assumption $F \star \mathrm{s}_B \star \tau_\bullet$ is still a universal co-cone.

(i.b): Let ϕ be as in the proof of (i.a); by assumption, there exists a universal co-cone $\tau_\bullet : \mathrm{s}_B \circ \phi \Rightarrow c_L$. On the other hand, we have a co-cone

$$\eta^F_\bullet : F \circ \mathrm{s}_B \Rightarrow c_B \qquad (A, FA \xrightarrow{u} B) \mapsto u.$$

Since by assumption $F \star \tau_\bullet$ is still universal, we then get a unique :

$$f \in \mathscr{B}(FL, B) \qquad \text{such that} \qquad \eta^F_\bullet \star \phi = c_f \circ (F \star \tau_\bullet).$$

Hence, the rule : $i \mapsto \tau_i/B$ defines the unique co-cone

$$\tau_{\bullet/B} : \phi \Rightarrow c_{(L,f)} \qquad \text{such that} \qquad \mathrm{s}_B \star \tau_{\bullet/B} = \tau_\bullet$$

and it suffices to check that $\tau_{\bullet/B}$ is universal. Thus, let $\lambda_\bullet : \phi \Rightarrow c_{(A,g)}$ be any other co-cone; then we get a unique $h \in \mathscr{A}(L, A)$ with $\mathrm{s}_B \star \lambda_\bullet = c_h \circ \tau_\bullet$, and arguing as in the proof of (i.a) we see that $\lambda_\bullet = c_{h/B} \circ \tau_{\bullet/B}$, whence the assertion. □

Corollary 1.4.6. *Let $\mathscr{A}$ and I be two categories, and $A \in \mathrm{Ob}(\mathscr{A})$. We have :*

(i) The source functor $\mathrm{s}_A : \mathscr{A}/A \to \mathscr{A}$ reflects epimorphisms, is conservative, and preserves and reflects monomorphisms. Moreover, s_A preserves all representable connected limits, and reflects all colimits and all connected limits.

(ii) Dually, the target functor $\mathrm{t}_A : A/\mathscr{A} \to \mathscr{A}$ reflects monomorphisms, is conservative, and preserves and reflects epimorphisms. Moreover, t_A preserves all representable connected colimits, and reflects all limits and all connected colimits.

(iii) If $\mathscr{A}$ is I-cocomplete, the same holds for $\mathscr{A}/A$, and s_A preserves I-colimits. Dually, if $\mathscr{A}$ is I-complete, the same holds for $A/\mathscr{A}$, and t_A preserves I-limits.

Proof. This is a special case of propositions 1.4.3 and 1.4.5. □

Lemma 1.4.7. *Let $\mathscr{A}$, $\mathscr{B}$ and I be three categories, $G : \mathscr{A} \to \mathscr{B}$ a functor, and $A \in \mathrm{Ob}(\mathscr{A})$.*

(i) Denote by $I_\circ$ the category with $\mathrm{Ob}(I_\circ) := I \sqcup \{\varnothing\}$, such that :

(a) I is a full subcategory of $I_\circ$, and $\varnothing$ is a final object of $I_\circ$

(b) $I_\circ(\varnothing, i) = \varnothing$ for every $i \in \mathrm{Ob}(I)$.

Then, if $\mathscr{A}$ is $I_\circ$-complete, the slice category $\mathscr{A}/A$ is I-complete.

(ii) Dually, set $I^\circ := (I^{\mathrm{op}})^{\mathrm{op}}_\circ$; then if $\mathscr{A}$ is I°-cocomplete, $A/\mathscr{A}$ is I-cocomplete.

(iii) Moreover, if G preserves $I_\circ$-limits, then $G_{/A} : \mathscr{A}/A \to \mathscr{B}/GA$ preserves I-limits.

(iv) Dually, if G preserves I°-colimits, then ${}_{A/}G : A/\mathscr{A} \to GA/\mathscr{B}$ preserves I-colimits.

Proof. (i): For every $i \in \mathrm{Ob}(I)$, let $\{p_i\} := I_\circ(i, \varnothing)$. Every functor

$$F : I \to \mathscr{A}/A \qquad i \mapsto Fi := (B_i, u_i : B_i \to A)$$

induces a functor $F_\circ : I_\circ \to \mathscr{A}$ whose restriction to I agrees with $\mathrm{s}_A \circ F$, such that $F_\circ(\varnothing) := A$ and $F_\circ(p_i) := u_i$ for every $i \in \mathrm{Ob}(I)$. Moreover, every cone $\tau_\bullet : c_{(B,u)} \Rightarrow F$ induces a cone $\tau_{\circ\bullet} : c_B \Rightarrow F_\circ$ with $\tau_{\circ,i} := \mathrm{s}_A(\tau_i)$ for every $i \in \mathrm{Ob}(I)$ and $\tau_{\circ,\varnothing} := u$; conversely, every cone $\eta_\bullet : c_B \Rightarrow F_\circ$ induces a cone $\eta_{\bullet/A} : c_{(B,\eta_\varnothing)} \Rightarrow F$ with $\eta_{i/A} := \eta_i/A$ for every $i \in \mathrm{Ob}(I)$. Clearly the rules $\tau_\bullet \mapsto \tau_{\circ\bullet}$ and $\eta_\bullet \mapsto \eta_{\bullet/A}$ are mutually inverse, and

$$(\eta_\bullet \circ c_f)_{/A} = \eta_{\bullet/A} \circ c_{f/A} \qquad (\tau_\bullet \circ c_{g/A})_\circ = \tau_{\circ\bullet} \circ c_g$$

for every $f \in \mathscr{A}(B', B)$ and every morphism $g/A : (B', u') \to (B, u)$ of $\mathscr{A}/A$, where $c_f : c_{B'} \Rightarrow c_B$ and $c_{g/A} : c_{(B',u')} \Rightarrow c_{(B,u)}$ denote the constant natural transformations with values f and respectively g/A, and likewise for $c_{f/A}$ and c_g. It follows easily that if $\eta_\bullet$ is universal, the same holds for $\eta_{\bullet/A}$, whence the assertion.

(ii) follows from (i), by duality, and likewise, (iv) follows from (iii).

(iii): Let $\eta_\bullet : c_B \Rightarrow F_\circ$ be a universal cone, so that, by assumption, the cone $G \star \eta_\bullet : c_{GB} \Rightarrow G \circ F_\circ$ is again universal; notice that $(G \star \eta_\bullet)_{/A} = G_{/A} \star \eta_{\bullet/A}$, so both $\eta_{\bullet/A}$ and $G_{/A} \star \eta_{\bullet/A}$ are universal, by (i), whence the contention. □

1.4.8. Let $\mathscr{A}, \mathscr{B}$ be two categories, $F : \mathscr{A} \to \mathscr{B}$ a functor, $G : \mathscr{B} \to \mathscr{A}$ a left adjoint for F, and ϑ an adjunction for the pair (G, F). Then, for every $A \in \mathrm{Ob}(\mathscr{A})$ the functor $F_{/A} : \mathscr{A}/A \to \mathscr{B}/FA$ of §1.4.1 admits a left adjoint that we denote

$$G_{/A} : \mathscr{B}/FA \to \mathscr{A}/A \qquad (B \xrightarrow{f} FA) \mapsto (GB \xrightarrow{\vartheta^{-1}_{AB}(f)} A)$$

that assigns to every morphism $h/FA : (B, f) \to (B', f')$ of $\mathscr{B}/FA$ the morphism $Gh/A : (GB, \vartheta^{-1}_{AB}(f)) \to (GB', \vartheta^{-1}_{AB'}(f'))$ in $\mathscr{A}/A$. Indeed, ϑ induces an adjunction $\vartheta_{/A}$ for the pair $(G_{/A}, F_{/A})$: to every $(f : B \to FA) \in \mathrm{Ob}(\mathscr{B}/FA)$ and $(g : A' \to A) \in \mathrm{Ob}(\mathscr{A}/A)$ we assign the bijection

$$(\vartheta_{/A})_{g,f} : \mathscr{A}/A(\vartheta^{-1}_{AB}(f), g) \xrightarrow{\sim} \mathscr{B}/FA(f, Fg) \qquad h/A \mapsto \vartheta_{A'B}(h)/FA.$$

• Dually, since F^{op} is left adjoint to G^{op} ([13, Exerc.2.14(iii)]), we see that for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor ${}_{B/}G : B/\mathscr{B} \to GB/\mathscr{A}$ of §1.4.1 admits the right adjoint

$${}_{B/}F : GB/\mathscr{A} \to B/\mathscr{B} \qquad (GB \xrightarrow{f} A) \mapsto (B \xrightarrow{\vartheta_{AB}(f)} FA) \qquad GB/h \mapsto B/Fh.$$

The detailed verifications shall be left to the reader. See also remark 1.6.9(i).

Remark 1.4.9. In the situation of §1.4.8, the adjunction ϑ also induces an isomorphism of categories :

$$G\mathscr{A}/\mathscr{B} \xrightarrow{\sim} \mathscr{A}/F\mathscr{B} \qquad (GB \xrightarrow{f} A) \mapsto (B \xrightarrow{\vartheta_{AB}(f)} FA)$$

and conversely, every adjunction for the pair (G, F) arises from such an isomorphism. Indeed, this construction – that allows to describe adjunctions without involving sets – motivated the introduction of comma categories by Lawvere.

1.4.10. In the situation of §1.4.8, let $\eta_\bullet : \mathbf{1}_\mathscr{B} \Rightarrow FG$ and $\varepsilon_\bullet : GF \Rightarrow \mathbf{1}_\mathscr{A}$ be the unit and counit of the adjunction ϑ, and suppose that *all the fibre products of $\mathscr{B}$ are representable*; then for every $B \in \mathrm{Ob}(\mathscr{B})$ the functor $G_{/B} : \mathscr{B}/B \to \mathscr{A}/GB$ admits a right adjoint

$$F_{/B} : \mathscr{A}/GB \to \mathscr{B}/B.$$

Indeed, for every object $(g : A \to GB)$ of $\mathscr{A}/GB$ let us fix a cartesian diagram :

$$\begin{tikzcd} F^*A \arrow[r, "F^*g"] \arrow[d, "\eta^*_A"'] & B \arrow[d, "\eta_B"] \\ FA \arrow[r, "Fg"] & FGB. \end{tikzcd}$$

If $h/GB : (A, g) \to (A', g')$ is a morphism of $\mathscr{A}/GB$, the universal property of the fibre product yields a unique morphism $F^*h : F^*A \to F^*A'$ of $\mathscr{B}$ such that $F^*g' \circ F^*h = F^*g$ and $\eta^*_{A'} \circ F^*h = Fh \circ \eta^*_A$. With this notation, we set

$$F_{/B}(A, g) := (F^*A, F^*g) \qquad \text{and} \qquad F_{/B}(h/GB) := F^*h/B : F_{/B}(A, g) \to F_{/B}(A', g')$$

for every such (A, g) and h/GB. It is easily seen that these rules define a functor as sought. In order to check that $F_{/B}$ is right adjoint to $G_{/B}$, consider any morphism

$$G_{/B}(B' \xrightarrow{f} B) \xrightarrow{h/GB} (A \xrightarrow{g} GB) \qquad \text{in } \mathscr{A}/GB.$$

Hence, $h : GB' \to A$ is a morphism in $\mathscr{A}$ with $g \circ h = Gf$, and we notice that :

$$Fg \circ \vartheta_{AB'}(h) = Fg \circ Fh \circ \eta_{B'} = FGf \circ \eta_{B'} = \eta_B \circ f.$$

It follows that there exists a unique morphism $k : B' \to F^*A$ of $\mathscr{B}$ such that $\eta^*_A \circ k = \vartheta_{AB'}(h)$ and $F^*g \circ k = f$. With this notation, we set

$$(\vartheta_{/B})_{g,f}(h/GB) := k/B : (B', f) \to F_{/B}(A, g) \qquad \text{in } \mathscr{B}/B.$$

Conversely, to every morphism $k/B : (B', f) \to F_{/B}(A, g)$ of $\mathscr{B}/B$ we attach the morphism $h := \vartheta^{-1}_{AB'}(\eta^*_A \circ k) : GB' \to A$. Let us show that $h/GB : G_{/B}(B', f) \to (A, g)$ is a morphism in $\mathscr{A}/GB$; recalling that $F^*g \circ k = f$, it suffices to compute :

$$\begin{aligned} g \circ h = g \circ \varepsilon_A \circ G(\eta^*_A \circ k) = \varepsilon_{GB} \circ GFg \circ G(\eta^*_A \circ k) &= \varepsilon_{GB} \circ G(\eta_B \circ F^*g \circ k) \\ &= \varepsilon_{GB} \circ G(\eta_B \circ f) = Gf \end{aligned}$$

where the last equality follows from the triangular identities for the unit and counit $(\eta_\bullet, \varepsilon_\bullet)$ (see [13, Prob.2.13(iii,iv)]). Hence the map

$$(\vartheta_{/B})_{g,f} : \mathscr{A}/GB(G_{/B}(B', f), (A, g)) \to \mathscr{B}/B((B', f), F_{/B}(A, g))$$

is a bijection with inverse given by the rule : $k/B \mapsto \vartheta^{-1}_{AB'}(\eta^*_A \circ k)/GB$. Indeed, by definition $(\vartheta_{/B})_{g,f}(\vartheta^{-1}_{AB'}(\eta^*_A \circ k)/GB)$ is the morphism $(B', f) \to F_{/B}(A, g)$ of $\mathscr{B}/B$ determined by the pair $(\vartheta_{AB'}\vartheta^{-1}_{AB'}(\eta^*_A \circ k), f) = (\eta^*_A \circ k, f)$, which is just k/B, and on the other hand, $\vartheta^{-1}_{AB'}(\eta^*_A \circ (\vartheta_{/B})_{g,f}(h/GB)) = \vartheta^{-1}_{AB'} \circ \vartheta_{AB'}(h) = h$, whence the contention. The naturality of $(\vartheta_{/B})_{g,f}$ with respect to g and f follows by a simple inspection : details left to the reader.

• Dually, we see that *if all the amalgamated sums of $\mathscr{A}$ are representable*, then for every $A \in \mathrm{Ob}(\mathscr{A})$ the functor ${}_{A/}F : A/\mathscr{A} \to FA/\mathscr{B}$ admits a left adjoint

$${}_{A/}G : FA/\mathscr{B} \to A/\mathscr{A}$$

which the reader is invited to spell out. See also remark 1.6.9(ii).

1.5. **Coinitial and cofinal functors.** Let I and $\mathscr{C}$ be two categories, and $F : I \to \mathscr{C}$ any functor. For the computation of the limit or colimit of F, it may sometimes be desirable to replace the indexing category I by simpler ones. That is, we would like to be able to detect whether a given functor $\phi : J \to I$ induces an isomorphism from the colimit of F to that of $F \circ \phi$, and if possible, to construct useful functors of this type, to aid with the calculation of limits or colimits. Concerning the first aim, one has a general criterion, for which we shall need the following :

Definition 1.5.1. Let I, J be two categories, and $\phi : J \to I$ a functor.

(i) We say that ϕ is *cofinal* if the slice category $i/\phi J$ is connected, for every $i \in \mathrm{Ob}(I)$ (see definition 1.1.6(iii)).

(ii) We say that ϕ is *coinitial* if $\phi^{\mathrm{op}} : J^{\mathrm{op}} \to I^{\mathrm{op}}$ is cofinal.

(iii) If ϕ is the inclusion functor of a subcategory J of I, and ϕ is cofinal (resp. coinitial) we also say that *J is cofinal in I* (resp. that *J is coinitial in I*).

(iv) We say that a category $\mathscr{C}$ is *cofinally small* (resp. *coinitially small*) if there exists a small category $\mathscr{B}$ with a cofinal (resp. coinitial) functor $\mathscr{B} \to \mathscr{C}$.

Proposition 1.5.2. *With the notation of definition 1.5.1, we have :*

(i) ϕ is a cofinal functor $\Leftrightarrow$ for every category $\mathscr{A}$, every $A \in \mathrm{Ob}(\mathscr{A})$, every $F \in \mathrm{Ob}(\mathscr{A}^I)$ and every co-cone $\tau_\bullet : F \circ \phi \Rightarrow c_A$, there exists a unique co-cone $\eta_\bullet : F \Rightarrow c_A$ with $\tau_\bullet = \eta_\bullet \star \phi$.

(ii) ϕ is a coinitial functor $\Leftrightarrow$ for every $\mathscr{A}, A, F$ as in (i), and every cone $\tau_\bullet : c_A \Rightarrow F \circ \phi$, there exists a unique cone $\eta_\bullet : c_A \Rightarrow F$ such that $\tau_\bullet = \eta_\bullet \star \phi$.

Proof. By duality, it suffices to prove (i). Then, suppose first that the condition of (i) holds for every $\mathscr{A}, A, F$ and $\tau_\bullet$; for every $i \in \mathrm{Ob}(I)$, consider the functor

$$I(i, -) : I \to \mathrm{Set} \qquad i' \mapsto I(i, i') \qquad (i' \xrightarrow{u} i'') \mapsto (I(i, i') \xrightarrow{u_*} I(i, i''))$$

where we define $u_*(v) := u \circ v$ for every $v \in I(i, i')$. To every set S and every co-cone $\tau_\bullet : I(i, -) \circ \phi \Rightarrow c_S$ we attach the map

$$g_\tau : \mathrm{Ob}(i/\phi J) \to S \qquad (j, i \xrightarrow{v} \phi j) \mapsto \tau_j(v).$$

Claim 1.5.3. (i) The map g_τ factors through a (unique) map $\overline{g}_\tau : \pi_0(i/\phi J) \to S$.

(ii) Conversely, for every map $f : \pi_0(i/\phi J) \to S$ there exists a unique co-cone $\tau_\bullet : I(i, -) \circ \phi \Rightarrow c_S$ such that $f = \overline{g}_\tau$ (notation of example 1.2.6).

Proof : (i): Clearly it suffices to check that $g_\tau(v) = g_\tau(v')$ for every object $(j, i \xrightarrow{v} \phi j)$ and every morphism $i/u : (j, v) \to (j', v')$ of $i/\phi J$. But since $v' = \phi(u)_*(v)$ we have $\tau_{j'}(v') = \tau_{j'}(\phi(u)_*(v)) = \tau_j(v)$, whence the assertion.

(ii): Let $p : \mathrm{Ob}(i/\phi J) \to \pi_0(i/\phi J)$ be the projection; for every $j \in \mathrm{Ob}(J)$ we let $\tau_j : I(i, \phi j) \to S$ be the map such that $\tau_j(v) := f \circ p(v)$ for every $v \in I(i, \phi j)$. It is easily seen that the system $(\tau_j \mid j \in \mathrm{Ob}(J))$ yields the sought co-cone $\tau_\bullet$, and the uniqueness of $\tau_\bullet$ is clear. ◇

Now, by applying claim 1.5.3 with $\phi := \mathbf{1}_I$, we get as well, for every set S, a bijective correspondence between the co-cones $\eta_\bullet : I(i, -) \Rightarrow c_S$ and the associated maps $\overline{g}_\eta : \pi_0(i/I) \to S$; moreover, by inspection of the constructions we see that :

$$\overline{g}_{\eta_\bullet \star \phi} = \overline{g}_\eta \circ \pi_0(i/\phi).$$

Summing up, under our assumption for ϕ, claim 1.5.3 implies that $\pi_0(i/\phi)$ is a bijection $\pi_0(i/\phi J) \xrightarrow{\sim} \pi_0(i/I)$. But $\pi_0(i/I)$ is a set of cardinality one, since i/I has an initial object; on the other hand, $i/\phi J$ is connected if and only if $\pi_0(i/\phi J)$ is a set of cardinality one, so the condition for ϕ in (i), with $\mathscr{A} := \mathrm{Set}$ and every functor $F := I(i, -)$, implies that ϕ is cofinal, as stated.

Conversely, suppose that ϕ is cofinal, let $F \in \mathrm{Ob}(\mathscr{A}^I)$, and consider two co-cones $\eta_\bullet, \eta'_\bullet : F \Rightarrow c_A$ such that $\eta_\bullet \star \phi = \eta'_\bullet \star \phi$; by assumption, for every $i \in \mathrm{Ob}(I)$ the category $i/\phi J$ is non-empty, so pick any morphism $u : i \to \phi j$ in I. We then have :

$$\eta_i = \eta_{\phi j} \circ Fu = \eta'_{\phi j} \circ Fu = \eta'_i$$

so $\eta_\bullet = \eta'_\bullet$. It remains to check that every co-cone $\tau_\bullet : F \circ \phi \Rightarrow c_A$ equals $\eta_\bullet \star \phi$ for some co-cone $\eta_\bullet : F \Rightarrow c_A$. To this aim, we claim that the map

$$\widehat{\tau}_\bullet : \mathrm{Ob}(i/\phi J) \to \mathrm{Mor}(\mathscr{A}) \qquad (i \xrightarrow{u} \phi j) \mapsto (Fi \xrightarrow{\tau_j \circ Fu} A)$$

is constant for every $i \in \mathrm{Ob}(I)$. Indeed, say that $\phi j' \xleftarrow{u'} i \xrightarrow{u} \phi j$ are two morphisms of $\mathscr{A}$; we need to check that $\tau_j \circ Fu = \tau_{j'} \circ Fu'$, and since $i/\phi J$ is connected, we may assume that there exists $v \in J(j, j')$ such that $\phi(v) \circ u = u'$. Then :

$$\tau_{j'} \circ Fu' = \tau_{j'} \circ F(\phi v) \circ Fu = \tau_j \circ Fu$$

as required. We then obtain a well-defined map $\eta_\bullet : \mathrm{Ob}(I) \to \mathrm{Mor}(\mathscr{A})$ such that $\tau_j \circ Fu = \eta_i$ for every $i \in \mathrm{Ob}(I)$ and every morphism $u : i \to \phi j$ of I; to conclude, it suffices to check that $\eta_\bullet$ is a co-cone $F \Rightarrow c_A$. Indeed, let $w \in I(i, i')$, and pick $(i \xrightarrow{u} \phi j) \in \mathrm{Ob}(i/\phi J)$ and $(i' \xrightarrow{u'} \phi j') \in \mathrm{Ob}(i'/\phi J)$; we have just seen that $\widehat{\tau}(i, u) = \widehat{\tau}(i, u' \circ w) = \widehat{\tau}(i', u') \circ Fw$, *i.e.* $\eta_i = \eta_{i'} \circ Fw$, whence the contention. □

Corollary 1.5.4. *Let $I, J, \mathscr{A}$ be three categories, $\phi : J \to I$ and $F : I \to \mathscr{A}$ two functors, $A \in \mathrm{Ob}(\mathscr{A})$, $\eta_\bullet : c_A \Rightarrow F$ a cone, and $\tau_\bullet : F \Rightarrow c_A$ a co-cone. We have :*

(i) Suppose that ϕ is cofinal; then $\tau_\bullet$ is universal $\Leftrightarrow$ the same holds for $\tau_\bullet \star \phi$.

(ii) Suppose that ϕ is coinitial; then $\eta_\bullet$ is universal $\Leftrightarrow$ the same holds for $\eta_\bullet \star \phi$.

Proof. By duality, it suffices to prove (i). Hence, suppose first that $\tau_\bullet$ is universal, and let $\beta_\bullet : F \star \phi \Rightarrow c_{A'}$ be another co-cone; by virtue of proposition 1.5.2(i), there exists a unique co-cone $\alpha_\bullet : F \Rightarrow c_{A'}$ such that $\beta_\bullet = \alpha_\bullet \star \phi$, and then $\alpha_\bullet = c_f \circ \tau_\bullet$ for a unique $f \in \mathscr{A}(A, A')$, so that $\beta_\bullet = c_f \circ (\tau_\bullet \star \phi)$. Suppose next that $\beta_\bullet = c_g \circ (\tau_\bullet \star \phi)$ for some other $g \in \mathscr{A}(A, A')$; then $(c_f \circ \tau_\bullet) \star \phi = (c_g \circ \tau_\bullet) \star \phi$, so $c_f \circ \tau_\bullet = c_g \circ \tau_\bullet$, again by virtue of proposition 1.5.2(i), and finally, $f = g$, by the universality of $\tau_\bullet$. Hence, $\tau_\bullet \star \phi$ is universal.

The proof of the converse assertion is similar, and shall be left to the reader. □

Remark 1.5.5. (i) A direct inspection shows that the proof of proposition 1.5.2 also gives the following criterion. A functor $\phi : J \to I$ is cofinal $\Leftrightarrow$ for every class-valued functor F on I, every class A, and every co-cone $\tau_\bullet : F \circ \phi \Rightarrow c_A$, there exists a unique co-cone $\eta_\bullet : F \Rightarrow c_A$ with $\tau_\bullet = \eta_\bullet \star \phi$ (see definition 1.2.12).

(ii) Dually, a functor $\phi : J \to I$ is coinitial $\Leftrightarrow$ for every F and A as in (i) and every cone $\tau_\bullet : c_A \Rightarrow F \circ \phi$, there exists a unique cone $\eta_\bullet : c_A \Rightarrow F$ with $\tau_\bullet = \eta_\bullet \star \phi$.

(iii) Likewise, corollary 1.5.4 admits the following variant. Let $\phi : J \to I$ be a cofinal functor, F a class-valued functor on I, and A a class; then a co-cone $\tau_\bullet : F \Rightarrow c_A$ is a global if and only if the same holds for $\tau_\bullet \star \phi$. Dually, if ϕ is coinitial, then a cone $\eta_\bullet : c_A \Rightarrow F$ is global if and only if the same holds for $\eta_\bullet \star \phi$ (see example 1.2.13(ii,iv)).

Corollary 1.5.6. *(i) Let $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ be two functors, and suppose that ϕ is cofinal (resp. coinitial). Then ψ is cofinal (resp. coinitial) if and only if the same holds for $\psi \circ \phi$.*

(ii) Let $I \xrightarrow{\phi} J \xrightarrow{F} \mathscr{A} \xrightarrow{G} \mathscr{B}$ be three functors, and suppose that ϕ is cofinal (resp. conitial). Then G preserves the colimit (resp. limit) of $F \Leftrightarrow G$ preserves the colimit (resp. limit) of $F \circ \phi$. Also, G reflects the colimit (resp. limit) of $F \Leftrightarrow G$ reflects the colimit (resp. limit) of $F \circ \phi$.

Proof. This follows straighforwardly from proposition 1.5.2 and corollary 1.5.4. □

Proposition 1.5.7. *Let I, J be two categories, and $\phi : J \to I$ a functor. We have :*

(i) If J is filtered, then ϕ is cofinal if and only if the following conditions hold :

  (a) For every $i \in \mathrm{Ob}(I)$ there exist $j \in \mathrm{Ob}(J)$ and a morphism $i \to \phi(j)$ in I.

  (b) For every $i \in \mathrm{Ob}(I)$, every $j \in \mathrm{Ob}(J)$ and every pair of morphisms $f, g : i \rightrightarrows \phi(j)$ in I, there exists $h \in J(j, j')$ such that $\phi(h) \circ f = \phi(h) \circ g$.

(ii) If J is filtered and ϕ is cofinal, then I is filtered.

(iii) If I is filtered, and condition (i.a) is fulfilled, the following holds :

  (a) If ϕ is full, then ϕ is cofinal and J is directed.

  (b) If ϕ is fully faithful, then J is filtered.

Proof. (i): Suppose first that $i/\phi J$ is connected for every $i \in \mathrm{Ob}(I)$; then clearly (i.a) holds, and we notice :

Claim 1.5.8. The category $i/\phi J$ is *directed.*

Proof : Indeed, let $\phi j \xleftarrow{u} i \xrightarrow{u'} \phi j'$ be two objects of $i/\phi J$; we need to exhibit $(i \xrightarrow{u''} \phi j'') \in \mathrm{Ob}(i/\phi J)$ and morphisms $j \xrightarrow{v} j'' \xleftarrow{v'} j'$ in J such that $\phi(v) \circ u = u'' = \phi(v') \circ u'$. Now, by assumption there exists a commutative diagram in I :

$$
\begin{tikzcd}
 & & i \arrow[dll, "u"'] \arrow[dl, "u_1"'] \arrow[d, "u_2"] \arrow[drr, "u_k"] \arrow[drrr, "u'"] & & & \\
\phi j \arrow[r, "\phi(v_0)"'] & \phi j_1 & \phi j_2 \arrow[l, "\phi(v_1)"] \arrow[r, "\phi(v_2)"'] & \cdots & \phi j_k \arrow[l, "\phi(v_{k-1})"] \arrow[r, "\phi(v_k)"'] & \phi j'
\end{tikzcd}
$$

and we argue by induction on the length k of the horizontal chain of morphisms. Notice that the length k is always even, since the direction of each arrow v_r is rightward if r is even and leftward if r is odd. If $k = 0$, the diagram consists of a unique morphism $j \xrightarrow{v_0} j'$, so we can take $j'' := j'$, $v := v_0$ and $v' := \mathbf{1}_{j'}$. Next, suppose that $k > 1$, and that suitable v, v' have already been exhibited for every pair of objects of $i/\phi J$ linked by a chain of morphisms of length $k - 2$. Since J is filtered, we may find $j'_{k-1} \in \mathrm{Ob}(J)$ and morphisms $j_{k-1} \xrightarrow{w} j'_{k-1} \xleftarrow{w'} j'$ such that $w \circ v_{k-1} = w' \circ v_k$; then we get a commutative diagram of length $k - 2$:

$$
\begin{tikzcd}
 & & i \arrow[dll, "u"'] \arrow[dl, "u_1"'] \arrow[d, "u_2"] \arrow[drr, "u_{k-2}"] \arrow[drrr, "t"] & & & \\
\phi j \arrow[r, "v_0"'] & \phi j_1 & \phi j_2 \arrow[l, "\phi(v_1)"] \arrow[r, "\phi(v_2)"'] & \cdots & \phi j_{k-2} \arrow[l, "\phi(v_{k-3})"] \arrow[r, "\phi(w \circ v_{k-2})"'] & \phi j'_{k-1}
\end{tikzcd}
$$

with $t := \phi(w) \circ u_{k-1}$. By induction, we then get morphisms $j \xrightarrow{s} l \xleftarrow{s'} j'_{k-1}$ in J with $\phi(s) \circ u = \phi(s') \circ t$; then we can take $j'' := l$, $v := s$ and $v' := s' \circ w'$. ◇

Claim 1.5.8 implies that, for f, g as in (i.b), there exist $j' \in \mathrm{Ob}(J)$ and $h_1, h_2 \in J(j, j')$ such that $\phi(h_1) \circ f = \phi(h_2) \circ g$. But since J is filtered, we may then find $h' \in J(j', j'')$ such that $h' \circ h_1 = h' \circ h_2$, so (i.b) holds with $h := h' \circ h_1$.

Conversely, if (i.a) holds, then $i/\phi J$ is non-empty for every $i \in \mathrm{Ob}(I)$. Next, let $g : i \to \phi(k)$ and $g' : i \to \phi(k')$ be any two objects of $i/\phi J$; since J is directed we may find morphisms $h : k \to j$ and $h' : k' \to j$ for some $j \in \mathrm{Ob}(J)$, whence the pair of objects $\phi(h) \circ g, \phi(h') \circ g' : i \rightrightarrows \phi(j)$ of $i/\phi J$, and using (i.b) we find an object $g'' : i \to \phi(j')$ and morphisms $g \to g''$, $g' \to g''$ in $i/\phi J$, i.e. $i/\phi J$ is directed.

(ii): Let us check that I is directed : if $i, i' \in \mathrm{Ob}(I)$, by condition (i.a) we may find $j, j' \in \mathrm{Ob}(J)$ and morphisms $f : i \to \phi(j)$, $f' : i' \to \phi(j')$ in I, and since J is directed, we have as well morphisms $j \xrightarrow{g} j'' \xleftarrow{g'} j'$ in J, for some $j'' \in \mathrm{Ob}(J)$, whence morphisms $\phi(g) \circ f : i \to \phi(j'')$ and $\phi(g') \circ f' : i' \to \phi(j'')$ in I. It remains to check the coequalizing condition of definition 1.1.6(v), but the latter is an immediate consequence of conditions (i.a) and (i.b).

(iii.a): It has already been remarked that condition (i.a) says that the category $i/\phi J$ is non-empty for every $i \in \mathrm{Ob}(I)$. Next, let $g : i \to \phi(k)$ and $g' : i \to \phi(k')$ be two objects of $i/\phi J$. Since I is filtered, we may find $i' \in \mathrm{Ob}(I)$ and morphisms $\phi(k) \xrightarrow{h} i' \xleftarrow{h'} \phi(k')$ of I such that $f := h \circ g = h' \circ g'$, and by (i.a) we find $k'' \in \mathrm{Ob}(J)$ with a morphism $l : i' \to \phi(k'')$ in I; then, since ϕ is full, there exist morphisms $k \xrightarrow{t} k'' \xleftarrow{t'} k'$ in J with $\phi(t) = l \circ h$ and $\phi(t') = l \circ h'$. Then $l \circ f : i \to \phi(k'')$ is an object of $i/\phi J$ with morphisms $(k, g) \xrightarrow{i/t} (k'', l \circ f) \xleftarrow{i/t'} (k', g')$, so $i/\phi J$ is directed; this shows that ϕ is cofinal. Next, let

$j, j' \in \mathrm{Ob}(J)$; since I is filtered, there exists $i \in \mathrm{Ob}(I)$ with morphisms $\phi(j) \xrightarrow{g} i \xleftarrow{g'} \phi(j')$ in I. Since (i.a) holds, there exist $k \in \mathrm{Ob}(J)$ and $h \in I(i, \phi(k))$, and since ϕ is full, there exist morphisms $j \xrightarrow{t} k \xleftarrow{t'} j'$ in J with $\phi(t) = h \circ g$ and $\phi(t') = h \circ g'$; so, J is directed.

(iii.b): Clearly the assumptions show that $\mathrm{Ob}(J) \neq \varnothing$, and we know already that J is directed, by (iii.a). Next, let $g, g' : j \rightrightarrows j'$ be two morphisms in J; since I is filtered, we have a morphism $h : \phi(j') \to i$ in I such that $h \circ \phi(g) = h \circ \phi(g')$, and since (i.a) holds, there exists $j'' \in \mathrm{Ob}(J)$ with a morphism $l : i \to \phi(j'')$ in I. Since ϕ is full, there exists $t \in J(j', j'')$ such that $\phi(t) = l \circ h$, so that $\phi(t \circ g) = \phi(t \circ g')$, and since ϕ is faithful, we must then have $t \circ g = t \circ g'$, so J is filtered. □

Example 1.5.9. (i) If I is a filtered category, and $i \in \mathrm{Ob}(I)$, it is easily seen that the category i/I is again filtered; moreover, the target functor $\mathrm{t}_i : i/I \to I$ fulfills conditions (a) and (b) of proposition 1.5.7(i), so it is cofinal. Dually, if I is cofiltered, then the same holds for I/i, and the source functor $\mathrm{s}_i : I/i \to I$ is coinitial.

(ii) If I admits a final object i_0, then the inclusion functor $J \to I$ of the (full) subcategory J of I with $\mathrm{Ob}(J) = \{i_0\}$ fulfills conditions (a) and (b) of proposition 1.5.7(i), so it is cofinal (and I is trivially filtered). Then, let $F : I \to \mathscr{C}$ be any functor, and $\tau_\bullet : F \Rightarrow c_L$ a universal co-cone; with corollary 1.5.4(i) we see that Fi_0 represents the colimit of F, and $\tau_{i_0} : Fi_0 \to L$ is an isomorphism in $\mathscr{C}$. Dually, if i'_0 is any initial object of I, and $\tau'_\bullet : c_{L'} \Rightarrow F$ any universal cone, then Fi'_0 represents the limit of F and $\tau'_{i'_0} : L' \to Fi'_0$ is an isomorphism.

Remark 1.5.10. (i) With the notation of lemma 1.4.7, suppose that I has a final object e; then it is easily seen that the inclusion functor $j : I \to I_\circ$ has a left inverse $p : I_\circ \to I$ with $p(\varnothing) := e$, and both j and p are coinitial. With corollary 1.5.4(i) we deduce that $\mathscr{A}$ is I-complete $\Leftrightarrow$ $\mathscr{A}$ is $I_\circ$-complete. Summing up, in light of lemma 1.4.7(i,iii) we conclude that for every such I and every $A \in \mathrm{Ob}(\mathscr{A})$, if $\mathscr{A}$ is I-complete, the same holds for $\mathscr{A}/A$; moreover, if the functor $G : \mathscr{A} \to \mathscr{B}$ preserves I-limits, the same holds for $G_{/A} : \mathscr{A}/A \to \mathscr{B}/GA$ (corollary 1.5.6(ii)). Dually, if I has an initial object and $\mathscr{A}$ is I-cocomplete, the same holds for $A/\mathscr{A}$, and if moreover G preserves I-colimits, the same holds for ${}_{A/}G$.

(ii) Let $\mathscr{A}$ be a category with a final object E, and $G : \mathscr{A} \to \mathscr{B}$ a functor that preserves all small (resp. finite) connected limits, and such that GE is a final object of $\mathscr{B}$; then G preserves all small (resp. finite) limits. Indeed, lemma 1.4.7(iii) implies that for any small (resp. finite) category I, the functor $G_{/E} : \mathscr{A}/E \to \mathscr{B}/GE$ preserves I-limits; but on the other hand, the source functors $\mathscr{A}/E \to \mathscr{A}$ and $\mathscr{B}/GE \to \mathscr{B}$ are isomorphisms of categories, since E and GE are final, and these isomorphisms identify $G_{/E}$ with G, whence the assertion. Dually, if $\mathscr{A}$ has an initial object, and if G preserves initial objects and all small (resp. finite) connected colimits, then G preserves all small (resp. finite) colimits.

Example 1.5.11. As a last application, let us show that *all filtered wide colimits are representable* (definition 1.2.9(i)). Indeed, let I be a filtered category, and consider a functor

$$\mathscr{C}_\bullet : I \to \mathsf{Cat} \qquad i \mapsto \mathscr{C}_i \qquad (\phi : i \to j) \mapsto (\mathscr{C}_\phi : \mathscr{C}_i \to \mathscr{C}_j).$$

We deduce a functor $\mathrm{Ob}(\mathscr{C}_\bullet) : I \to \mathsf{Set}$ that assigns to every $i \in \mathrm{Ob}(I)$ the set $\mathrm{Ob}(\mathscr{C}_i)$, and to every morphism $\phi : i \to j$ of I the map $\mathrm{Ob}(\mathscr{C}_i) \to \mathrm{Ob}(\mathscr{C}_j)$ defined by $\mathscr{C}_\phi$. We set

$$L := \operatorname*{colim}_I \mathrm{Ob}(\mathscr{C}_\bullet)$$

where colim_I denotes the global colimit. According to example 1.2.13(iii), the elements of L are the equivalence classes $[i, X]$ of pairs (i, X), where $i \in \mathrm{Ob}(I)$ and $X \in \mathrm{Ob}(\mathscr{C}_i)$, for the equivalence relation $\sim$ such that $(i, X) \sim (j, Y)$ if and only if there exist $k \in \mathrm{Ob}(I)$ and morphisms $i \xrightarrow{\phi_1} k \xleftarrow{\phi_2} j$ such that $\mathscr{C}_{\phi_1}X = \mathscr{C}_{\phi_2}Y$. For every $i \in \mathrm{Ob}(I)$, let $\mathrm{t}_i : i/I \to I$ be the target functor (see §1.4); notice that for every $i, j \in \mathrm{Ob}(I)$, the objects of the category

$$(i,j)/I := i/I \times_{(\mathrm{t}_i,\mathrm{t}_j)} j/I$$

are the pairs $i \xrightarrow{\phi_1} k \xleftarrow{\phi_2} j$ of morphisms of I, and the morphisms

$$(*) \qquad (i,j)/\nu : \left(i \xrightarrow{\phi_1} k \xleftarrow{\phi_2} j\right) \to \left(i \xrightarrow{\phi'_1} k' \xleftarrow{\phi'_2} j\right)$$

are the morphisms $\nu : k \to k'$ of I such that $\nu \circ \phi_r = \phi'_r$ for $r = 1, 2$. For every couple of pairs $(i, X), (j, Y)$ as in the foregoing, we get a functor

$$h(i,X,j,Y) : (i,j)/I \to \mathrm{Set} \qquad \left(i \xrightarrow{\phi_1} k \xleftarrow{\phi_2} j\right) \mapsto \mathscr{C}_k(\mathscr{C}_{\phi_1}X, \mathscr{C}_{\phi_2}Y)$$

that assigns to every morphism $(i, j)/\nu$ as in $(*)$ the induced map

$$\mathscr{C}_k(\mathscr{C}_{\phi_1}X, \mathscr{C}_{\phi_2}Y) \to \mathscr{C}_{k'}(\mathscr{C}_{\phi'_1}X, \mathscr{C}_{\phi'_2}Y) \qquad f \mapsto \mathscr{C}_\nu(f)$$

and we consider the global colimit

$$H(i,X,j,Y) := \operatorname*{colim}_{(i,j)/I} h(i,X,j,Y).$$

Let also $\mathrm{t}_{i,j} : (i,j)/I \to I$ be the *target functor* given by the rules : $(i \to k \leftarrow j) \mapsto k$, and $(i,j)/\nu \mapsto \nu$; for every $i, j, k \in \mathrm{Ob}(I)$ we set as well

$$(i,j,k)/I := (i,j)/I \times_{(\mathrm{t}_{i,j},\mathrm{t}_k)} k/I.$$

The objects of this category are the systems $\phi_\bullet := (\phi_1 : i \to t, \phi_2 : j \to t, \phi_3 : k \to t)$ of morphisms of I with a common target t that we call *the target of* $\phi_\bullet$. The morphisms

$$(i,j,k)/\nu : \phi_\bullet \to \phi'_\bullet$$

are the morphisms ν of I with $\nu \circ \phi_r = \phi'_r$ for $r = 1, 2, 3$. With this notation, for every third pair (k, Z) as in the foregoing, we get a system of composition maps

$$h(i,X,j,Y)(\phi_\bullet) \times h(j,Y,k,Z)(\phi_\bullet) \to h(i,X,k,Z)(\phi_\bullet) \qquad \forall \phi_\bullet \in \mathrm{Ob}((i,j,k)/I)$$

given by the composition law of the category $\mathscr{C}_t$, where t is the target of $\phi_\bullet$. Clearly this system of maps is natural with respect to morphisms of $(i,j,k)/I$; moreover it is easily seen that $(i,j,k)/I$ is a filtered category, and then, by the criterion of proposition 1.5.7(i,ii), the same holds for $(i,j)/I$, $(j,k)/I$ and $(i,k)/I$, and the obvious projection functors

$$\begin{tikzcd} & (i,j,k)/I \arrow[dl] \arrow[d] \arrow[dr] & \\ (i,j)/I & (j,k)/I & (i,k)/I \end{tikzcd}$$

are cofinal. Therefore, in light of remark 1.5.5(iii), taking global colimits over $(i,j,k)/I$ of the foregoing compatible system of composition maps yields a well-defined map

$$H(i,X,j,Y) \times H(j,Y,k,Z) \to H(i,X,k,Z) \qquad (f,g) \mapsto g \circ f$$

and a simple inspection shows that this composition law is associative : if $h \in H(k,Z,t,W)$ is any other element, we have $h \circ (g \circ f) = (h \circ g) \circ f$ (details left to the reader); likewise, the class of $\mathbf{1}_X$ in $H(i,X,i,X)$ yields a left and right unit for this composition law.

Hence, let us choose for every $A \in L$ a representative (i_A, X_A); we obtain a well-defined category $\mathscr{L}$ with $\mathrm{Ob}(\mathscr{L}) := L$ and with $\mathscr{L}(A,B) := H(i_A, X_A, i_B, X_B)$ for every $A, B \in L$, with the composition law thus defined. Moreover, for every $i \in \mathrm{Ob}(I)$ we obtain a well-defined functor

$$G_i : \mathscr{C}_i \to \mathscr{L} \qquad X \mapsto [i, X]$$

that assigns to every morphism $X \to Y$ of $\mathscr{C}_i$ its class in $\mathscr{L}([i, X], [i, Y])$.

Lastly, let $F_\bullet := (F_i : \mathscr{C}_i \to \mathscr{D} \mid i \in \mathrm{Ob}(I))$ be a co-cone with basis $\mathscr{C}_\bullet$ and whose vertex is any wide category $\mathscr{D}$. The induced co-cone $(\mathrm{Ob}(\mathscr{C}_i) \to \mathrm{Ob}(\mathscr{D}) \mid i \in \mathrm{Ob}(I))$ yields a well-defined map $L \to \mathrm{Ob}(\mathscr{D})$. Moreover, for every two pairs (i, X), (j, Y) and every object $(i \xrightarrow{\phi_1} k \xleftarrow{\phi_2} j)$ of $(i, j)/I$, the functor F_k yields a map

$$\mathscr{C}_k(\mathscr{C}_{\phi_1}X, \mathscr{C}_{\phi_2}Y) \to \mathscr{D}(F_k \circ \mathscr{C}_{\phi_1}X, F_k \circ \mathscr{C}_{\phi_2}Y) = \mathscr{D}(F_iX, F_jY)$$

which defines a co-cone with vertex $\mathscr{D}(F_iX, F_jY)$ and for basis the functor $h(i, X, j, Y)$, whence a well-defined map

$$F_{i,X,j,Y} : H(i, X, j, Y) \to \mathscr{D}(F_iX, F_jY).$$

A simple inspection shows that $F_{i,X,k,Z}(g \circ f) = F_{j,Y,k,Z}(g) \circ F_{i,X,j,Y}(f)$ for every f, g as in the foregoing (details left to the reader). We obtain therefore a well-defined functor

$$F : \mathscr{L} \to \mathscr{D} \qquad A \mapsto F_{i_A}(X_A) \qquad (A \xrightarrow{f} B) \mapsto F_{i_A,X_A,i_B,X_B}(f).$$

By construction, we have $F \circ G_i = F_i$ for every $i \in \mathrm{Ob}(I)$, and F is clearly the unique functor fulfilling this system of identities, so $\mathscr{L}$ represents the wide colimit of $\mathscr{C}_\bullet$, and $G_\bullet := (G_i \mid i \in \mathrm{Ob}(I))$ is a global co-cone with basis $\mathscr{C}_\bullet$ and vertex $\mathscr{L}$.

Remark 1.5.12. In the situation of example 1.5.11, a simple inspection shows that if I is small, then the same holds for the global colimit $\mathscr{L}$ of $\mathscr{C}_\bullet$; hence, in this case $\mathscr{L}$ also represents the (usual) colimit of $\mathscr{C}_\bullet$, and the global co-cone $G_\bullet$ is also a universal co-cone.

1.6. **Presheaves.**

Definition 1.6.1. (i) Let $\mathscr{A}$ be a category. A *presheaf over* $\mathscr{A}$ is a functor

$$F : \mathscr{A}^{\mathrm{op}} \to \mathsf{Set}.$$

The natural transformations $F \Rightarrow G$ between presheaves over $\mathscr{A}$ shall also be called *morphisms of presheaves,* and denoted $F \to G$.

(ii) For every $A \in \mathrm{Ob}(\mathscr{A})$, the *fibre* of F at A, denoted

$$F_A := F(A)$$

is the evaluation of F at A. Its elements are also called the *sections* of F over A.

(iii) For $u \in \mathscr{A}(A, B)$, the induced map from F_B to F_A will often be denoted

$$u^* := F(u) : F_B \to F_A.$$

(iv) If $\mathscr{A}$ is a small category, the presheaves over $\mathscr{A}$ form a category :

$$\boxed{\widehat{\mathscr{A}} := \mathsf{Fun}(\mathscr{A}^{\mathrm{op}}, \mathsf{Set}).}$$

Remark 1.6.2. (i) Let $\mathscr{A}$ be a small category. By §1.3.2, the category $\widehat{\mathscr{A}}$ is complete and cocomplete, and the limits and colimits in $\widehat{\mathscr{A}}$ are computed termwise; moreover, the finite limits in $\widehat{\mathscr{A}}$ commute with all small filtered colimits, and the fibre products in $\widehat{\mathscr{A}}$ commute with all small direct sums (example 1.3.7(ii,iv)).

(ii) A morphism $f : F \to G$ of $\widehat{\mathscr{A}}$ is a monomorphism (resp. an epimorphism, resp. an isomorphism) $\Leftrightarrow$ the map $f_A : F_A \to G_A$ is injective (resp. surjective, resp. bijective) for every $A \in \mathrm{Ob}(\mathscr{A})$, by virtue of §1.3.2 and example 1.3.1(i). Furthermore, in light of (i) and example 1.1.14(i), every monomorphism and every epimorphism of $\widehat{\mathscr{A}}$ is regular.

Definition 1.6.3. (i) With every $A \in \mathrm{Ob}(\mathscr{A})$ we associate the presheaf

$$h_A : \mathscr{A}^{\mathrm{op}} \to \mathrm{Set} \qquad B \mapsto \mathscr{A}(B, A) \qquad (B \xrightarrow{u} C) \mapsto (\mathscr{A}(C, A) \xrightarrow{u^*} \mathscr{A}(B, A))$$

where $u^*(s) := s \circ u$ for every morphism $u : C \to A$ of $\mathscr{A}$. We say that a presheaf F on $\mathscr{A}$ is *representable by A,* if there exists an isomorphism of presheaves $h_A \xrightarrow{\sim} F$. (The latter means that there is an invertible natural transformation $h_A \Rightarrow F$, which can be asserted legitimately even if $\mathscr{A}$ is not small, in which case $\widehat{\mathscr{A}}$ is not well defined.)

(ii) Every morphism $f : A \to A'$ of $\mathscr{A}$ induces a morphism of presheaves

$$h_f : h_A \to h_{A'}$$

such that $h_f(B) : h_A(B) \to h_{A'}(B)$ is the map :

$$(B \xrightarrow{u} A) \mapsto (B \xrightarrow{f \circ u} A') \qquad \forall B \in \mathrm{Ob}(\mathscr{A}), \forall u \in h_A(B).$$

(iii) If $\mathscr{A}$ is small, it is easily seen that the rules : $A \mapsto h_A$ and $f \mapsto h_f$ for every $a \in \mathrm{Ob}(\mathscr{A})$ and every morphism f of $\mathscr{A}$, define a functor :

$$\boxed{h^{\mathscr{A}} : \mathscr{A} \to \widehat{\mathscr{A}}}$$

that we call the *Yoneda embedding* of $\mathscr{A}$ into $\widehat{\mathscr{A}}$.

Remark 1.6.4. (i) Notice that for every $A, B \in \mathrm{Ob}(\mathscr{A})$ and every set S such that $A^{(S)}$ is representable in $\mathscr{A}$, we have natural bijections (notation of §1.2.14) :

$$\mathscr{A}(A^{(S)}, B) \xrightarrow{\sim} \mathrm{Set}(S, \mathscr{A}(A, B)) \qquad (A^{(S)} \xrightarrow{u} B) \mapsto (A \xrightarrow{u \circ j_s} B \mid s \in S)$$

where $(j_s : A \to A^{(S)} \mid s \in S)$ is a fixed universal co-cone. Thus, if $A^{(S)}$ is representable for every set S, then the functor $A^{(-)}$ is left adjoint to $h_{A^{\mathrm{op}}} : \mathscr{A} \to \mathrm{Set}$, and especially, *it preserves all representable colimits* ([13, Prop.2.49(ii)]).

(ii) Notice that $h_f(A) = h_{A^{\mathrm{op}}}(f^{\mathrm{op}})$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and $f \in \mathrm{Mor}(\mathscr{A})$. Also, it is easily seen that f is a monomorphism (resp. a split epimorphism) $\Leftrightarrow h_f(A)$ is injective (resp. surjective) for every $A \in \mathrm{Ob}(\mathscr{A})$.

(iii) Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, X a presheaf on $\mathscr{A}$, and Y a presheaf on $\mathscr{B}$. Since $(\mathscr{A} \times \mathscr{B})^{\mathrm{op}} = \mathscr{A}^{\mathrm{op}} \times \mathscr{B}^{\mathrm{op}}$, we can define a presheaf on $\mathscr{A} \times \mathscr{B}$, denoted

$$\boxed{X \boxtimes Y : (\mathscr{A} \times \mathscr{B})^{\mathrm{op}} \to \mathrm{Set} \qquad (A, B) \mapsto X_A \times Y_B.}$$

For every pair of morphisms $f : A \to A'$ of $\mathscr{A}$ and $g : B \to B'$ of $\mathscr{B}$, the induced map $(f, g)^* : (X \boxtimes Y)_{A',B'} \to (X \boxtimes Y)_{A,B}$ is of course $f^* \times g^* : X_{A'} \times Y_{B'} \to X_A \times Y_B$.

Notice that this product operation preserves representable presheaves : indeed, we have natural isomorphisms

$$\boxed{h_A \boxtimes h_B \xrightarrow{\sim} h_{(A,B)} \qquad \forall A \in \mathrm{Ob}(\mathscr{A}), \forall B \in \mathrm{Ob}(\mathscr{B}).}$$

Moreover, every pair of morphisms of presheaves $\phi : X \to X'$ over $\mathscr{A}$ and $\psi : Y \to Y'$ over $\mathscr{B}$ induces an obvious morphism of presheaves over $\mathscr{A} \times \mathscr{B}$

$$\phi \boxtimes \psi : X \boxtimes Y \to X' \boxtimes Y' \qquad (A, B) \mapsto \phi_A \times \psi_B.$$

(iv) Especially, if $\mathscr{A}$ and $\mathscr{B}$ are small, we get a natural functor :

$$\boxed{- \boxtimes - : \widehat{\mathscr{A}} \times \widehat{\mathscr{B}} \to \widehat{\mathscr{A} \times \mathscr{B}}}$$

making commute the diagram :

$$\begin{tikzcd} & \mathscr{A}\times\mathscr{B} \arrow[dl, "h^{\mathscr{A}}\times h^{\mathscr{B}}"'] \arrow[dr, "h^{\mathscr{A}\times\mathscr{B}}"] & \\ \widehat{\mathscr{A}}\times\widehat{\mathscr{B}} \arrow[rr, "-\boxtimes-"] & & \widehat{\mathscr{A}\times\mathscr{B}}. \end{tikzcd}$$

Theorem 1.6.5. *Let $\mathscr{A}$ be a category, and F a presheaf on $\mathscr{A}$.*

(i) (Yoneda's lemma) *For every $A \in \mathrm{Ob}(\mathscr{A})$ and every $s \in F_A$ there exists a unique natural transformation $\omega_{s,\bullet} : h_A \Rightarrow F$ such that $\omega_{s,A}(\mathbf{1}_A) = s$.*

(ii) Especially, if $\mathscr{A}$ is small, we have a natural bijection :

$$\gamma : \widehat{\mathscr{A}}(h_A, F) \xrightarrow{\sim} F_A \qquad (h_A \xrightarrow{f_\bullet} F) \mapsto f_A(\mathbf{1}_A).$$

(iii) In particular, if $\mathscr{A}$ is small, the Yoneda embedding of $\mathscr{A}$ is fully faithful.

Proof. (i,ii): With every $s \in F_A$ and every $B \in \mathrm{Ob}(\mathscr{A})$ we associate the map

$$\omega_{s,B} : h_A(B) \to F_B \qquad (B \xrightarrow{u} A) \mapsto u^*(s).$$

Let us check that the rule $B \mapsto \omega_{s,B}$ yields a natural transformation $\omega_{s,\bullet} : h_A \Rightarrow F$. Indeed, let $v : C \to B$ be any morphism of $\mathscr{A}$; the assertion comes down to the commutativity of the diagram :

$$\begin{tikzcd} \mathscr{A}(B,A) \arrow[r, "\omega_{s,B}"] \arrow[d, "h_v"'] & F_B \arrow[d, "v^*"] \\ \mathscr{A}(C,A) \arrow[r, "\omega_{s,C}"] & F_C \end{tikzcd}$$

which amounts to the identity : $v^* \circ u^* = (u \circ v)^*$ for every $u \in \mathscr{A}(B,A)$.

By construction, $\omega_{s,A}(\mathbf{1}_A) = s$; conversely, if $\eta_\bullet : h_A \Rightarrow F$ is any natural transformation with $\eta_A(\mathbf{1}_A) = s$, then we must have $\eta_\bullet = \omega_{s,\bullet}$, since :

$$\eta_B(u) = \eta_B \circ h_u(\mathbf{1}_A) = u^* \circ \eta_A(\mathbf{1}_A) = u^*(s) \qquad \forall B \in \mathrm{Ob}(\mathscr{A}), \forall u \in \mathscr{A}(B,A).$$

(iii): In case $F = h_B$ for a given $B \in \mathrm{Ob}(\mathscr{A})$, the map $\gamma : \widehat{\mathscr{A}}(h_A, h_B) \to \mathscr{A}(A,B)$ precisely assigns to every morphism of presheaves $h_u : h_A \to h_B$ the morphism $u \in \mathscr{A}(A,B)$. But γ is bijective by (ii), whence the assertion. □

Remark 1.6.6. (i) We often abuse notation, to denote $s : A \to F$ the morphism of presheaves $\omega_{s,\bullet} : h_A \to F$ attached, via theorem 1.6.5(i), to $A \in \mathrm{Ob}(\mathscr{A})$ and $s \in F_A$.

(ii) Every functor $H : \mathscr{A} \to \mathscr{B}$ between small categories induces a functor

$$\boxed{H^* : \widehat{\mathscr{B}} \to \widehat{\mathscr{A}} \qquad F \mapsto F \circ H^{\mathrm{op}} \qquad (F \xrightarrow{f} G) \mapsto (F \circ H^{\mathrm{op}} \xrightarrow{f \star H^{\mathrm{op}}} G \circ H^{\mathrm{op}})}$$

(notation of §1.1.5). After composing with the Yoneda embedding $\mathscr{B} \to \widehat{\mathscr{B}}$, we deduce as well a functor that, by a slight abuse of notation, we denote also by :

$$H^* : \mathscr{B} \to \widehat{\mathscr{A}}.$$

Explicitly, the latter attaches to every $B \in \mathrm{Ob}(\mathscr{B})$ the presheaf $H^*(B)$ on $\mathscr{A}$ with

$$H^*(B)_A := \mathscr{B}(H(A), B) \qquad \forall A \in \mathrm{Ob}(\mathscr{A})$$

which assigns to any morphism $u : A' \to A$ of $\mathscr{A}$ the map

$$u^* : H^*(B)_A \to H^*(B)_{A'} \quad \text{given by the rule :} \quad (H(A) \xrightarrow{v} B) \mapsto (H(A') \xrightarrow{v \circ H(u)} B).$$

(iii) Notice that the construction of the functor $H^* : \mathscr{B} \to \widehat{\mathscr{A}}$ of (ii) makes sense more generally, for any functor H from a small category $\mathscr{A}$ to any category $\mathscr{B}$.

(iv) Moreover, every natural transformation $\tau_\bullet : H \Rightarrow K$ between functors $H, K : \mathscr{A} \to \mathscr{B}$ induces a natural transformation

$$\tau_\bullet^* : K^* \Rightarrow H^* \qquad F \mapsto (F \circ K^{\mathrm{op}} \xrightarrow{F \star \tau_\bullet^{\mathrm{op}}} F \circ H^{\mathrm{op}})$$

where H^* and K^* may denote either the functors $\widehat{\mathscr{B}} \to \widehat{\mathscr{A}}$ induced by H and K, or else their respective compositions $\mathscr{B} \to \widehat{\mathscr{A}}$ with Yoneda embeddings, as in (ii).

1.6.7. *Uniqueness of adjoints.* Let $\mathscr{A}, \mathscr{B}$ be two categories, and

$$F : \mathscr{A} \rightleftarrows \mathscr{B} : G$$

a pair of functors. An *adjunction* for the pair (F, G) is equivalent to the datum of a family of isomorphisms of presheaves :

$$(\dagger) \qquad \vartheta_{B,\bullet} : h_B \circ F^{\mathrm{op}} \xrightarrow{\sim} h_{GB} \qquad \forall B \in \mathrm{Ob}(\mathscr{B})$$

such that the following diagram commutes (see [13, Prob.2.10]) :

$$(*) \qquad \begin{tikzcd} h_B \circ F^{\mathrm{op}} \arrow[r, "\vartheta_{B,\bullet}"] \arrow[d, "h_v \star F^{\mathrm{op}}"'] & h_{GB} \arrow[d, "h_{Gv}"] \\ h_{B'} \circ F^{\mathrm{op}} \arrow[r, "\vartheta_{B',\bullet}"] & h_{GB'}. \end{tikzcd} \qquad \forall B, B' \in \mathrm{Ob}(\mathscr{B}), \forall v \in \mathscr{B}(B, B')$$

It follows that a right adjoint for F is equivalent to the datum of a map $G : \mathrm{Ob}(\mathscr{B}) \to \mathrm{Ob}(\mathscr{A})$, together with a system of isomorphisms $(\dagger)$: indeed, for every morphism $v : B \to B'$ of $\mathscr{B}$, a unique morphism $Gv : GB \to GB'$ of $\mathscr{A}$ is then determined by requiring that the diagram $(*)$ commutes, by virtue of Yoneda's lemma (theorem 1.6.5(i)), and the uniqueness property of Gv easily implies that $G\mathbf{1}_B = \mathbf{1}_{GB}$ for every $B \in \mathrm{Ob}(\mathscr{B})$, and $Gu \circ Gv = G(u \circ v)$ for every composable pair $B \xrightarrow{v} B' \xrightarrow{u} B''$ of morphisms of $\mathscr{B}$. As an application, we get the following criterion :

Proposition 1.6.8. *A functor $F : \mathscr{A} \to \mathscr{B}$ has a left (resp. right) adjoint if and only if $B/F\mathscr{A}$ has an initial object (resp. if and only if $F\mathscr{A}/B$ has a final object) for all $B \in \mathrm{Ob}(\mathscr{B})$.*

Proof. Suppose that $F\mathscr{A}/B$ has a final object $(A_0, FA_0 \xrightarrow{u_0} B)$, and set $GB := A_0$; then for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $v \in \mathscr{B}(FA, B)$ there exists a unique $v^* \in \mathscr{A}(A, GB)$ such that $u_0 \circ Fv^* = v$. Clearly the rule : $v \mapsto v^*$ is natural with respect to morphisms $A \to A'$ of $\mathscr{A}$, so it yields a system of isomorphisms as $(\dagger)$. But we have just seen that $(\dagger)$ is equivalent to the datum of a right adjoint G for F together with an adjunction for the pair (F, G). Conversely, if G is a right adjoint for F, and $\varepsilon_\bullet : FG \Rightarrow \mathbf{1}_{\mathscr{B}}$ is the counit of an adjunction for the pair (F, G), then it is easily seen that (GB, ε_B) is a final object of $F\mathscr{A}/B$, for every $B \in \mathrm{Ob}(\mathscr{B})$. The assertion for initial objects in $B/F\mathscr{A}$ follows by duality. □

• It follows as well that *any two right adjoints G, G' of F are isomorphic* : more precisely, given adjunctions $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$ respectively for (F, G) and (F, G'), Yoneda's lemma yields for every $B \in \mathrm{Ob}(\mathscr{B})$ a unique isomorphism

$$\omega_B \in \mathscr{A}(GB, G'B) \qquad \text{such that} \qquad h_{\omega_B} \circ \vartheta_{B,\bullet} = \vartheta'_{B,\bullet}$$

and we claim that the rule : $B \mapsto \omega_B$ defines an isomorphism of functors $\omega_\bullet : G \xrightarrow{\sim} G'$. Indeed, the assertion means that for every $u \in \mathscr{B}(B, B')$ we have $G'u \circ \omega_B = \omega_{B'} \circ Gu$; the latter is equivalent to : $h_{G'u} \circ h_{\omega_B} = h_{\omega_{B'}} \circ h_{Gu}$, which follows easily from $(*)$ and the corresponding diagram for the adjunction $\vartheta'_{\bullet\bullet}$.

• Arguing with the opposite adjoint pair $(G^{\mathrm{op}}, F^{\mathrm{op}})$ (see [13, Exerc.2.14(iii)]), one sees that, dually, *any two left adjoints of G are isomorphic* as well. Explicitly, for every $(A, B) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B})$ we may regard $\vartheta_{B,A}$ as a bijection :

$$\vartheta_{B,A} : \mathscr{B}^{\mathrm{op}}(B^{\mathrm{op}}, F^{\mathrm{op}}A^{\mathrm{op}}) \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}}(G^{\mathrm{op}}B^{\mathrm{op}}, A^{\mathrm{op}})$$

and it is easily seen that the system of inverse maps :

$$(\vartheta^{-1}_{\bullet,A} : h_{A^{\mathrm{op}}} \circ G^{\mathrm{op}} \xrightarrow{\sim} h_{F^{\mathrm{op}}A} \,|\, A^{\mathrm{op}} \in \mathrm{Ob}(\mathscr{A}^{\mathrm{op}}))$$

is an adjunction for the pair $(G^{\mathrm{op}}, F^{\mathrm{op}})$ that we call *the opposite adjunction of $\vartheta_{\bullet\bullet}$*.

• Given any two adjunctions $\vartheta_{\bullet\bullet}, \lambda_{\bullet\bullet}$ for the pair (F, G), there exist unique automorphisms $\tau_\bullet : G \xrightarrow{\sim} G$ and $\mu_\bullet : F \xrightarrow{\sim} F$ such that :

$$\lambda_{B,\bullet} = h_{\tau_B} \circ \vartheta_{B,\bullet} \qquad \text{and} \qquad \vartheta_{B,\bullet} \circ (h_B \star \mu_\bullet^{\mathrm{op}}) = \lambda_{B,\bullet} \qquad \forall B \in \mathrm{Ob}(\mathscr{B}).$$

Indeed, the existence and uniqueness of $\tau_\bullet$ is a special case of the foregoing discussion on the existence and uniqueness of $\omega_\bullet$. Then, the existence and uniqueness of $\mu_\bullet$ is deduced, by arguing with the opposite adjunctions $\vartheta^{-1}_{\bullet\bullet}$ and $\lambda^{-1}_{\bullet\bullet}$ (details left to the reader).

• Moreover, consider a second adjoint pair of functors :

$$F' : \mathscr{B} \rightleftarrows \mathscr{C} : G'$$

with its adjunction $\vartheta'_{\bullet\bullet}$. It is easily seen that the system of isomorphisms

$$\vartheta'_{\bullet\bullet} \circ \vartheta_{\bullet\bullet} := (\vartheta_{G'C,\bullet} \circ (\vartheta'_{\bullet,C} \star F^{\mathrm{op}}) \,|\, C \in \mathrm{Ob}(\mathscr{C}))$$

is an adjunction for the pair $(F'F, GG')$. We call $\vartheta'_{\bullet\bullet} \circ \vartheta_{\bullet\bullet}$ *the composition of $\vartheta_\bullet$ and $\vartheta'_{\bullet\bullet}$*.

Remark 1.6.9. (i) As an application, we easily deduce from §1.6.7 that the adjoint pairs

$$(G_{/A} : \mathscr{B}/FA \rightleftarrows \mathscr{A}/A : F_{/A}) \qquad \text{and} \qquad ({}_{B/}G : B/\mathscr{B} \rightleftarrows GB/\mathscr{A} : {}_{B/}F)$$

associated with an adjoint pair of functors $(G : \mathscr{B} \rightleftarrows \mathscr{A} : F)$ and any $A \in \mathrm{Ob}(\mathscr{A}), B \in \mathrm{Ob}(\mathscr{B})$ as in §1.4.8, are well defined up to natural isomorphisms.

(ii) Likewise, the adjoint pairs

$$(G_{/B} : \mathscr{B}/B \rightleftarrows \mathscr{A}/GB : F_{/B}) \qquad \text{and} \qquad ({}_{A/}G : FA/\mathscr{B} \to A/\mathscr{A} : {}_{A/}F)$$

of §1.4.10 are well unique up to natural isomorphisms, whenever they exist.

1.6.10. *Adjoint natural transformations.* Next, consider two adjoint pairs of functors

$$F : \mathscr{A} \rightleftarrows \mathscr{B} : G \qquad F' : \mathscr{A} \rightleftarrows \mathscr{B} : G'$$

with respective adjunctions $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$, and let $\tau_\bullet : F \Rightarrow F'$ be a given natural transformation. By invoking yet again Yoneda's lemma, we obtain for every $B \in \mathrm{Ob}(\mathscr{B})$ a unique morphism $\tau^\vee_B : G'B \to GB$ making commute the diagram :

$$\begin{tikzcd} h_B \circ F'^{\mathrm{op}} \arrow[r, "\vartheta'_{B,\bullet}"] \arrow[d, "h_B \star \tau_\bullet^{\mathrm{op}}"'] & h_{G'B} \arrow[d, "h_{\tau^\vee_B}"] \\ h_B \circ F^{\mathrm{op}} \arrow[r, "\vartheta_{B,\bullet}"] & h_{GB} \end{tikzcd}$$

and we claim that the rule : $B \mapsto \tau^\vee_B$ yields a natural transformation

$$\boxed{(\tau_\bullet, \vartheta, \vartheta')^\vee : G' \Rightarrow G}$$

that we often denote just by $\tau^\vee_\bullet$, and that we call *the adjoint of $\tau_\bullet$ relative to $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$*. Indeed, the assertion means that for every morphism $u : B \to B'$ of $\mathscr{B}$ we have

$Gu \circ \tau^\vee_B = \tau^\vee_{B'} \circ G'u$, or equivalently : $h_{Gu} \circ h_{\tau^\vee_B} = h_{\tau^\vee_{B'}} \circ h_{G'u}$. But in light of $(*)$ and the corresponding diagram for $\vartheta'_{\bullet\bullet}$, the latter identity is reduced to :

$$(**) \qquad (h_{B'} \star \tau^{\mathrm{op}}_\bullet) \circ (h_u \star F'^{\mathrm{op}}) = (h_u \star F^{\mathrm{op}}) \circ (h_B \star \tau^{\mathrm{op}}_\bullet)$$

and the *interchange law for the Godement product* (see [13, Exerc.1.129(iii)]) shows that both sides of $(**)$ equal $h_u \star \tau^{\mathrm{op}}_\bullet$. The previous discussion shows that $\tau^\vee_\bullet$ *is independent of the choice of* $\vartheta_{\bullet\bullet}$ *and* $\vartheta'_{\bullet\bullet}$, up to composition with automorphisms of G and G'; especially, *the categorical properties of* $\tau^\vee_\bullet$ *are intrinsic.*

Notice that the commutativity of the foregoing diagram amounts to the identity :

$$\boxed{\vartheta_{B,A}(f \circ \tau_A) = \tau^\vee_B \circ \vartheta'_{B,A}(f) \qquad \forall (A,B) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B}), \forall f \in \mathscr{B}(F'A,B).}$$

- In the foregoing situation, consider as well a third adjoint pair

$$F'' : \mathscr{A} \rightleftarrows \mathscr{B} : G''$$

with its adjunction $\vartheta''_\bullet$, and a second natural transformation $\tau'_\bullet : F' \Rightarrow F''$. With this notation, it is easily seen that :

$$\boxed{(\tau'_\bullet \circ \tau_\bullet, \vartheta, \vartheta'')^\vee = (\tau_\bullet, \vartheta, \vartheta')^\vee \circ (\tau'_\bullet, \vartheta', \vartheta'')^\vee.}$$

- On the other hand, for a natural transformation $\mu_\bullet : G' \Rightarrow G$, we consider $\mu^{\mathrm{op}}_\bullet : G^{\mathrm{op}} \Rightarrow G'^{\mathrm{op}}$ and recall that $(G^{\mathrm{op}}, F^{\mathrm{op}})$ and $(G'^{\mathrm{op}}, F'^{\mathrm{op}})$ are again two adjoint pairs; hence, we define *the adjoint of* $\mu_\bullet$ *relative to* $\vartheta_\bullet$ *and* $\vartheta'_\bullet$ as :

$$(\mu_\bullet, \vartheta, \vartheta')^\vee := ((\mu^{\mathrm{op}}_\bullet)^\vee)^{\mathrm{op}} : F \Rightarrow F'$$

oftend denoted just by $\mu^\vee_\bullet$, where $(\mu^{\mathrm{op}}_\bullet)^\vee : F'^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$ denotes the adjoint of $\mu^{\mathrm{op}}_\bullet$, relative to the opposite adjunctions $\vartheta^{-1}_{\bullet\bullet}$ and $\vartheta'^{-1}_{\bullet\bullet}$ detailed in §1.6.7. Hence, $\mu^\vee_\bullet$ is characterized by the identity :

$$\vartheta^{-1}_{B,A}(\mu_B \circ g) = \vartheta'^{-1}_{B,A}(g) \circ \mu^\vee_A \qquad \forall (A,B) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B}), \forall g \in \mathscr{A}(A, GB).$$

Comparing with the corresponding identity for $\tau^\vee_\bullet$, we easily deduce that :

$$\boxed{(\tau^\vee_\bullet)^\vee = \tau_\bullet \qquad \text{and} \qquad (\mu^\vee_\bullet)^\vee = \mu_\bullet}$$

for every natural transformation $\tau_\bullet : F \Rightarrow F'$ and $\mu_\bullet : G' \Rightarrow G$.

- Lastly, a direct inspection of the definitions shows that :

$$\boxed{(\mathbf{1}_F, \vartheta, \vartheta)^\vee = \mathbf{1}_G \qquad \text{and} \qquad (\mathbf{1}_G, \vartheta, \vartheta)^\vee = \mathbf{1}_F.}$$

Remark 1.6.11. (i) Let $G, G' : \mathscr{B} \rightrightarrows \mathscr{A}$ be two right adjoints for a given functor $F : \mathscr{A} \to \mathscr{B}$, and $\vartheta_{\bullet\bullet}$ (resp. $\vartheta'_{\bullet\bullet}$) an adjunction for the pair (F, G) (resp. for the pair (F, G')). Then, it is easily seen that the isomorphism $\omega_\bullet : G \xrightarrow{\sim} G'$ of §1.6.7 is precisely $(\mathbf{1}_F, \vartheta', \vartheta)^\vee$ (and its inverse is $(\mathbf{1}, \vartheta, \vartheta')^\vee$). By the foregoing, $\omega_\bullet$ is then characterized by the identities : $\vartheta'_{B,A}(f) = \omega_B \circ \vartheta_{B,A}(f)$ for every $(A,B) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B})$ and every $f \in \mathscr{B}(FA, B)$. Letting $A := GB$ and $f := \vartheta^{-1}_{B,GB}(\mathbf{1}_{GB})$, we get : $\omega_B = \vartheta'_{B,GB}(\vartheta^{-1}_{B,GB}(\mathbf{1}_{GB}))$. If $(\eta_\bullet, \varepsilon_\bullet)$ (resp. $(\eta'_\bullet, \varepsilon'_\bullet)$) are the unit and counit of $\vartheta_{\bullet\bullet}$ (resp. of $\vartheta'_{\bullet\bullet}$), we have $\varepsilon_B = \vartheta^{-1}_{B,GB}(\mathbf{1}_{GB})$, so we conclude that $\omega_\bullet$ is given by the composition :

$$\boxed{\omega_B : GB \xrightarrow{\eta'_{GB}} G'FGB \xrightarrow{G'\varepsilon_B} G'B \qquad \forall B \in \mathrm{Ob}(\mathscr{B}).}$$

(ii) Dually, if $F, F' : \mathscr{A} \rightrightarrows \mathscr{B}$ are two left adjoints for a functor $G : \mathscr{B} \to \mathscr{A}$, , and $(\eta_\bullet, \varepsilon_\bullet)$ (resp. $(\eta'_\bullet, \varepsilon'_\bullet)$) are the unit and counit of a given adjunction for the pair (F, G)

(resp. for the pair (F', G)), then we have a natural isomorphism $F \xrightarrow{\sim} F'$, given by the system of compositions :

$$\boxed{FA \xrightarrow{F\eta'_A} FGF'A \xrightarrow{\varepsilon_{F'A}} F'A \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).}$$

Proposition 1.6.12. *In the situation of §1.6.10, the following are equivalent :*

(a) τ_A *is an epimorphism (resp. a split monomorphism) for every* $A \in \mathrm{Ob}(\mathscr{A})$

(b) $\tau^\vee_B$ *is a monomorphism (resp. a split epimorphism) for every* $B \in \mathrm{Ob}(\mathscr{B})$.

Proof. Condition (a) is equivalent to the injectivity (resp. surjectivity) of the map

$$(h_B \star \tau^{\mathrm{op}}_\bullet)_A : \mathscr{B}(F'A, B) \to \mathscr{B}(FA, B) \qquad \phi \mapsto \phi \circ \tau_A$$

for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $B \in \mathrm{Ob}(\mathscr{B})$ (remark 1.6.4(ii)), which is the same as the injectivity (resp. surjectivity) of

$$(h_{\tau^\vee_B})_A : \mathscr{A}(A, G'B) \to \mathscr{A}(A, GB) \qquad \psi \mapsto \tau^\vee_B \circ \psi$$

for every such A and B. In turn, the latter condition is equivalent to (b). □

Example 1.6.13. (i) Suppose that the *binary product* $X \times Y$ is representable in $\mathscr{A}$, for every $X, Y \in \mathrm{Ob}(\mathscr{A})$, and fix a universal cone (*i.e.* projections) :

$$X \xleftarrow{p^{X,Y}} X \times Y \xrightarrow{q^{X,Y}} Y$$

for every such pair (X, Y). Then, every $Y \in \mathrm{Ob}(\mathscr{A})$ induces a functor

$$\boxed{(-) \times Y : \mathscr{A} \to \mathscr{A} \qquad X \mapsto X \times Y.}$$

Namely, with every $f \in \mathscr{A}(X, X')$ one associates the unique morphism

$$X \times Y \xrightarrow{f \times Y} X' \times Y \quad \text{with} \quad p^{X',Y} \circ (f \times Y) = f \circ p^{X,Y} \quad \text{and} \quad q^{X',Y} \circ (f \times Y) = q^{X,Y}.$$

The functors $(-) \times Y$ as usual are independent, up to unique isomorphisms, on the choices of representatives and projections for the binary products.

(ii) Moreover, every morphism $g : Y \to Y'$ induces a natural transformation

$$(-) \times g : (-) \times Y \Rightarrow (-) \times Y'.$$

Namely, to every $X \in \mathrm{Ob}(\mathscr{A})$ one attaches the unique morphism

$$X \times Y \xrightarrow{X \times g} X \times Y' \quad \text{with} \quad p^{X,Y'} \circ (X \times g) = p^{X,Y} \quad \text{and} \quad q^{X,Y'} \circ (f \times Y) = g \circ q^{X,Y}.$$

Clearly, for every pair $Y \xrightarrow{g} Y' \xrightarrow{h} Y''$ of morphisms of $\mathscr{A}$ we have :

$$((-) \times h) \circ ((-) \times g) = (-) \times (h \circ g).$$

(iii) In the situation of (i), we say that the category $\mathscr{A}$ is *cartesian closed* if furthermore, for every $Y \in \mathrm{Ob}(\mathscr{A})$, the functor $(-) \times Y$ admits a right adjoint

$$\boxed{\mathscr{H}om(Y, -) : \mathscr{A} \to \mathscr{A} \qquad Z \mapsto \mathscr{H}om(Y, Z)}$$

so that have a system of bijections

$$(*) \qquad \mathscr{A}(X \times Y, Z) \xrightarrow{\sim} \mathscr{A}(X, \mathscr{H}om(Y, Z)) \qquad \forall X, Y, Z \in \mathrm{Ob}(\mathscr{A})$$

natural in X and Z. The discussion of §1.6.10 then shows that, after fixing adjunctions for each pair $((-) \times Y, \mathscr{H}om(Y, -))$, every morphism $g : Y \to Y'$ of $\mathscr{A}$ induces a well-defined adjoint transformation

$$\mathscr{H}om(g, -) := ((-) \times g)^\vee : \mathscr{H}om(Y', -) \Rightarrow \mathscr{H}om(Y, -)$$

whence the naturality with respect to Y of the bijections $(*)$. Also, for every pair $Y \xrightarrow{g} Y' \xrightarrow{h} Y''$ of morphisms of $\mathscr{A}$ we have the identities :

$$\mathscr{H}om(g,-) \circ \mathscr{H}om(h,-) = \mathscr{H}om(h \circ g,-).$$

Again, the construction of the *internal Hom functors* $\mathscr{H}om(Y,-)$ involves several auxiliary choices, but is independent of such choices, up to unique isomorphisms.

(iv) Summing up, we see that in every cartesian closed category $\mathscr{A}$, the binary products and the internal Hom functors define bifunctors

$$\boxed{(-)\times(-) : \mathscr{A} \times \mathscr{A} \to \mathscr{A}} \qquad \text{and} \qquad \boxed{\mathscr{H}om(-,-) : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{A}.}$$

Lemma 1.6.14. *Let $\mathscr{C}$ be a cartesian closed category, and $\mathscr{H}om(-,-)$ its internal Hom functor. Then, for every $Y \in \mathrm{Ob}(\mathscr{C})$, the functors :*

$$\mathscr{H}om(Y,-) : \mathscr{C} \to \mathscr{C} \qquad \textit{and} \qquad \mathscr{H}om(-,Y) : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$$

preserve all representable limits.

Proof. The assertion concerning $\mathscr{H}om(Y,-)$ is clear, since the latter is a right adjoint ([13, Prop.2.49(i)]); by the same token, for every $X \in \mathrm{Ob}(\mathscr{C})$ the functor $X \times (-) : \mathscr{C} \to \mathscr{C}$ preserves all representable colimits, since it is isomorphic to the functor $(-) \times X$, which is a left adjoint. Next, let $F : I \to \mathscr{C}$ be any functor, and $\tau_\bullet := (\tau_i : Fi \to L \mid i \in \mathrm{Ob}(I))$ a universal cocone; we deduce natural bijections for every $X \in \mathrm{Ob}(\mathscr{C})$:

$$\begin{aligned}\mathscr{C}(X, \mathscr{H}om(L,Y)) \xrightarrow{\sim} \mathscr{C}(X \times L, Y) \xrightarrow{\sim} \mathscr{C}(\varinjlim_{i \in \mathrm{Ob}(I)} X \times Fi, Y) &\xrightarrow{\sim} \varprojlim_{i \in \mathrm{Ob}(I)} \mathscr{C}(X \times Fi, Y) \\ &\xrightarrow{\sim} \varprojlim_{i \in \mathrm{Ob}(I)} \mathscr{C}(X, \mathscr{H}om(Fi,Y)) \\ &\xrightarrow{\sim} \mathscr{C}(X, \varprojlim_{i \in \mathrm{Ob}(I)} \mathscr{H}om(Fi,Y))\end{aligned}$$

inducing, by Yoneda's lemma (theorem 1.6.5(i)), an isomorphism of $\mathscr{C}$:

$$\varprojlim_{i \in \mathrm{Ob}(I)} \mathscr{H}om(Fi,Y) \xrightarrow{\sim} \mathscr{H}om(L,Y)$$

whose composition with the cone

$$\mathscr{H}om(\tau_\bullet, L) := (\mathscr{H}om(\tau_i, Y) : \mathscr{H}om(L,Y) \to \mathscr{H}om(Fi,Y) \mid i \in \mathrm{Ob}(I))$$

is a universal cone, so also $\mathscr{H}om(\tau_\bullet, Y)$ is universal, whence the claim for $\mathscr{H}om(-,Y)$. □

1.6.15. *Universal colimits.* Let $I, \mathscr{A}$ be two categories, $F : I \to \mathscr{A}$ a functor, $A \in \mathrm{Ob}(\mathscr{A})$. Notice that the datum of a co-cone $\tau_\bullet : F \Rightarrow c_A$ is equivalent to that of the functor

$$\Phi^\tau : I \to \mathscr{C}/A \qquad i \mapsto (Fi \xrightarrow{\tau_i} A) \qquad (i \xrightarrow{\phi} j) \mapsto (\Phi^\tau_i \xrightarrow{F\phi/A} \Phi^\tau_j).$$

Suppose now that *all the fibre products are representable in $\mathscr{A}$*. Then, for every morphism $f : A' \to A$ of $\mathscr{A}$ we get an associated functor $f_* : \mathscr{C}/A \to \mathscr{C}/A'$ as in remark 1.4.2(i), and there exists a unique pair $(f_*F, f_*\tau_\bullet)$ consisting of a functor $f_*F : I \to \mathscr{C}$ and a co-cone $f_*\tau_\bullet : f_*F \Rightarrow c_{A'}$ such that :

$$\Phi^{f_*\tau} = f_* \circ \Phi^\tau.$$

Moreover, let $(G, \eta_\bullet)$ be any other pair consisting of a functor $G : I \to \mathscr{A}$ and a co-cone $\eta_\bullet : G \Rightarrow c_A$; then the datum of a natural transformation $\mu_\bullet : F \Rightarrow G$ with $\eta_\bullet \circ \mu_\bullet = \tau_\bullet$ is equivalent to that of the natural transformation

$$\Phi^{\mu/A} : \Phi^\tau \Rightarrow \Phi^\eta \qquad i \mapsto \mu_i/A.$$

Clearly we can recover F from Φ^τ, and $\mu_\bullet$ from $\Phi^{\mu/A}$, via the identities :

$$\mathrm{s}_A \circ \Phi^\tau = F \qquad \text{and} \qquad \mathrm{s}_A \star \Phi^{\mu/A} = \mu_\bullet.$$

Notice that the pair $(f_*F, f_*\tau_\bullet)$ is independent, up to isomorphisms, of the choice of the functor f_* : indeed, for any other right adjoint f'_* of f^*, the discussion of §1.6.7 yields an isomorphism $\omega_\bullet : f_* \xrightarrow{\sim} f'_*$, whence a unique isomorphism

$$\widetilde{\omega}_\bullet = \mathrm{s}_A \star \omega_\bullet \star \Phi^\tau : f_*F \xrightarrow{\sim} f'_*F \quad \text{with} \quad (f'_*\tau_\bullet) \circ \widetilde{\omega}_\bullet = f_*\tau_\bullet \quad \text{and} \quad \Phi^{\widetilde{\omega}/A'} = \omega_\bullet \star \Phi^\tau.$$

Definition 1.6.16. In the situation of §1.6.15, suppose that the co-cone $\tau_\bullet : F \Rightarrow c_A$ is universal; then we say that *the colimit of F is universal*, if for every $A' \in \mathrm{Ob}(\mathscr{A})$ and every morphism $f : A' \to A$ of $\mathscr{A}$, the co-cone $f_*\tau_\bullet$ is also universal.

Remark 1.6.17. Notice that the universality of the colimit of F is an *intrinsic* property of F, *i.e.* it is independent of the choices of $\tau_\bullet$ and of the functors f_*. Indeed, the independence of the choice of f_* is clear from the foregoing discussion, since the co-cone $f_*\tau_\bullet$ is universal if and only if the same holds for it composition with the isomorphism $\widetilde{\omega}_\bullet$. Likewise, if $\tau'_\bullet : F \Rightarrow c_B$ is another universal co-cone, then there exists a unique isomorphism $g : B \xrightarrow{\sim} A$ with $\tau_\bullet = c_g \circ \tau'_\bullet$, whence, for every $f \in \mathscr{A}(A', A)$, a commutative diagram :

$$\begin{tikzcd}
Fi \times_B B' \arrow[r, "\sim"] \arrow[dr] \arrow[rr, bend left, "f'_*\tau'_i"] & Fi \times_A A' \arrow[r] \arrow[d] \arrow[rr, bend left, "f_*\tau_i"] & B' \arrow[r, "g'"] \arrow[d, "f'"] & A' \arrow[d, "f"] \\
 & Fi \arrow[r, "\tau'_i"] & B \arrow[r, "g"] & A
\end{tikzcd}$$

whose two square subdiagrams are cartesian; we get then an isomorphism

$$\omega_\bullet : f'_*F \xrightarrow{\sim} f_*F \qquad \text{such that} \qquad f_*\tau_\bullet \circ \omega_\bullet = c_{g'} \circ f'_*\tau'_\bullet$$

which easily implies that $f_*\tau_\bullet$ is universal if and only if the same holds for $f'_*\tau'_\bullet$.

Lemma 1.6.18. *Let $\mathscr{A}$ be a cocomplete (resp. finitely cocomplete) category with representable fibre products, and suppose that all coequalizers and all small (resp. finite) direct sums of $\mathscr{A}$ are universal. Then all small (resp. finite) colimits of $\mathscr{A}$ are universal.*

Proof. If $\mathscr{A}$ is cocomplete, let $F : I \to \mathscr{A}$ be a functor from a small category I, and $\tau_\bullet : F \Rightarrow c_L$ a co-cone. Let also J be the category with $\mathrm{Ob}(J) = \{0, 1\}$, and whose only non-identity morphisms are $a, b : 0 \rightrightarrows 1$; we consider the functor $F^\dagger : J \to \mathscr{A}$ such that

$$F^\dagger(0) := \coprod_{(\phi : i \to j) \in \mathrm{Mor}(I)} Fi \qquad F^\dagger(1) := \coprod_{i \in \mathrm{Ob}(I)} Fi$$

and where $F^\dagger(a), F^\dagger(b) : F^\dagger(0) \rightrightarrows F^\dagger(1)$ are the unique morphisms that make commute the diagrams :

$$\begin{tikzcd}
Fj \arrow[d, "\eta'_j"'] & Fi \arrow[l, "F\phi"'] \arrow[r, equal] \arrow[d, "\eta_\phi"] & Fi \arrow[d, "\eta'_i"] \\
F^\dagger(1) & F^\dagger(0) \arrow[l, "F^\dagger(b)"'] \arrow[r, "F^\dagger(a)"] & F^\dagger(1)
\end{tikzcd} \qquad \forall (\phi : i \to j) \in \mathrm{Mor}(I)$$

where $(\eta_\phi : Fi \to F^\dagger(0) \mid (\phi : i \to j) \in \mathrm{Mor}(I))$ and $(\eta'_i : Fi \to F^\dagger(1) \mid i \in \mathrm{Ob}(I))$ are universal co-cones for these direct sums. Let also $g : F^\dagger(0) \to L$ and $g' : F^\dagger(1) \to L$ be the unique morphisms of $\mathscr{A}$ such that $g \circ \eta_\phi = \tau_i = g' \circ \eta'_i$ for every morphism $\phi : i \to j$ of I and every $i \in \mathrm{Ob}(I)$. We attach to $\tau_\bullet$ the unique co-cone

$$\tau^\dagger_\bullet : F^\dagger \Rightarrow c_L \qquad \text{such that} \qquad \tau^\dagger_1 \circ \eta'_i = \tau_i \qquad \forall i \in \mathrm{Ob}(I).$$

Recall that $\tau_\bullet$ is universal $\Leftrightarrow$ the same holds for $\tau^\dagger_\bullet$ (see the proof of [13, Prop.2.40]). Now, let $f : A \to L$ be any morphism of $\mathscr{A}$, and consider the cartesian diagrams :

$$\begin{tikzcd} A'_0 \arrow[r] \arrow[d, "h_0"'] & A \arrow[d, "f"] & A'_1 \arrow[l] \arrow[d, "h_1"] \\ F^\dagger(0) \arrow[r, "g"] & L & F^\dagger(1). \arrow[l, "g'"'] \end{tikzcd}$$

By assumption, the co-cone $f_*\tau^\dagger_\bullet$ is universal, if the same holds for $\tau^\dagger_\bullet$. By the same token, $h_{*0}\eta_\bullet$ and $h_{1*}\eta'_\bullet$ are universal co-cones as well, so they induce isomorphisms :

$$(*) \qquad \bigsqcup_{(\phi:i\to j)\in\mathrm{Mor}(I)} Fi \times_L A \xrightarrow{\sim} A'_0 \qquad \bigsqcup_{i\in\mathrm{Ob}(I)} Fi \times_L A \xrightarrow{\sim} A'_1.$$

But it is easily seen that the isomorphisms $(*)$ identify the functor $f_*F^\dagger$ with $(f_*F)^\dagger$, and the natural transformation $f_*\tau^\dagger_\bullet$ with $(f_*\tau_\bullet)^\dagger$; summing up, if $\tau_\bullet$ is universal, the same holds for $(f_*\tau_\bullet)^\dagger$, and then also for $f_*\tau_\bullet$, as stated. If $\mathscr{A}$ is finitely cocomplete, the same argument still applies, provided I is a finite category, and concludes the proof. □

Example 1.6.19. (i) Let us show that *all small colimits of* Set *are universal.* To this aim, the lemma reduces to checking that the coequalizers and the small disjoint unions in Set are universal. Hence, let $(S_i \mid i \in I)$ be any small family of sets, $S := \bigsqcup_{i\in I} S_i$, and $T \to S$ any map of sets; the assertion comes down to the bijectivity of the map :

$$\bigsqcup_{i\in I}(S_i \times_S T) \to T$$

induced by the projections $(S_i \times_S T \to T \mid i \in I)$, which holds by simple inspection.

Next, let $f, g : X \rightrightarrows Y$ be two maps of sets; recall that the coequalizer of f and g is represented by the quotient $C := Y/\!\sim$, where $\sim$ is the smallest equivalence relation on Y such that $f(x) \sim g(x)$ for every $x \in X$ ([13, Prob.2.51(i)]). We need to check that for every map $u : C' \to C$, the projection $p : Y \times_C C' \to C'$ identifies C' with the coequalizer of $f \times_C C', g \times_C C' : X \times_C C' \rightrightarrows Y \times_C C'$. However, since the projection $q : Y \to C$ is surjective, the same holds for p; it remains then only to check that for every $c' \in C'$, the preimage $p^{-1}(c')$ is an equivalence class for the smallest equivalence relation $\sim$ on $Y' \times_C C'$ such that $(f(x), c') \sim (g(x), c')$ for every $(x, c') \in X \times_C C'$. But it is clear that the equivalence class of any $(y, c') \in Y \times_C C'$ is $q^{-1}q(y) \times \{c'\} = p^{-1}(c')$, as required.

(ii) Let $\mathscr{A}$ be any small category; since $\widehat{\mathscr{A}}$ is complete and cocomplete, and since limits and colimits in $\widehat{\mathscr{A}}$ are computed termwise (see §1.3.2), it follows from (i) that *all small colimits of* $\widehat{\mathscr{A}}$ *are universal.*

1.7. **The category of elements of a presheaf.** Let $\mathscr{A}$ be a category, and F a presheaf over $\mathscr{A}$. The *category of elements of* F is

$$\mathscr{A}/F := (\{\varnothing\}/F\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}.$$

Since a map of sets $\{\varnothing\} \to F_A$ is just a section $s \in F_A$, the objects of $\mathscr{A}/F$ are the pairs (A, s) with $A \in \mathrm{Ob}(\mathscr{A})$ and $s \in F_A$; the morphisms $(A, s) \to (B, t)$ in $\mathscr{A}/F$ are the morphisms $u : A \to B$ in $\mathscr{A}$ such that $u^*(t) = s$. With the notation of remark 1.6.6(i), this amounts to the commutativity of the diagram of presheaves :

$$(*) \qquad \begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "s"'] & & B \arrow[dl, "t"] \\ & F. & \end{tikzcd}$$

Remark 1.7.1. (i) With the notation of §1.7, we have a faithful functor

$$\psi_F := \mathsf{t}^{\mathrm{op}}_{\{\varnothing\}} : \mathscr{A}/F \to \mathscr{A} \qquad (A, s) \mapsto A$$

which is the identity on morphisms (where $\mathsf{t}_{\{\varnothing\}}$ is the target functor of §1.4).

(ii) Suppose that $\mathscr{A}$ is small; then clearly the same holds for $\mathscr{A}/F$; moreover, in this case we have a natural identification :

$$h\mathscr{A}/F \xrightarrow{\sim} \mathscr{A}/F \qquad (A, f : h_A \to F) \mapsto (A, f_A(\mathbf{1}_A))$$

where $h : \mathscr{A} \to \widehat{\mathscr{A}}$ is the Yoneda embedding. Then, we have as well a functor

$$h \circ \psi_F : \mathscr{A}/F \to \widehat{\mathscr{A}} \qquad (A, s) \mapsto h_A \qquad ((A, s) \xrightarrow{u} (B, t)) \mapsto h_u.$$

Furthermore, from the commutativity of the diagrams $(*)$ we get a natural co-cone of basis $h \circ \psi_F$ and vertex F :

$$\boxed{\tau_F : h \circ \psi_F \Rightarrow c_F \qquad (A, s) \mapsto (s : A \to F).}$$

(iii) Every morphism $f : F \to G$ of presheaves on $\mathscr{A}$ induces a functor

$$\mathscr{A}/f : \mathscr{A}/F \to \mathscr{A}/G \qquad (A, s) \mapsto (A, f_A(s))$$

which is the identity map on morphisms. Indeed we have :

$$f_A(s) = f_A(u^*(t)) = u^*(f_B(t)) \qquad \forall u : (A, s) \to (B, t) \quad \text{in } \mathscr{A}/F$$

whence the assertion. Clearly there results a commutative diagram of functors :

$$\begin{tikzcd} \mathscr{A}/F \arrow[rr, "\mathscr{A}/f"] \arrow[dr, "\psi_F"'] & & \mathscr{A}/G \arrow[dl, "\psi_G"] \\ & \mathscr{A}. & \end{tikzcd}$$

(iv) A direct inspection of the definitions shows that :

$$\mathscr{A}/h_A = \mathscr{A}/A \qquad \forall A \in \mathrm{Ob}(\mathscr{A})$$

where $\mathscr{A}/A$ is the slice category over A, as in §1.4. Moreover, under this identification, $\psi_{h_A} : \mathscr{A}/h_A \to \mathscr{A}$ corresponds to the source functor $\mathsf{s}_A : \mathscr{A}/A \to \mathscr{A}$.

(v) Let I be any category; according to proposition 1.4.5(ii.b), if $\mathscr{A}$ is I-cocomplete and $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ preserves I-limits, then $\mathscr{A}/F$ is I-cocomplete. Moreover, *F is a representable presheaf if and only if $\mathscr{A}/F$ admits a final object* : indeed, (A, s) is a final object of $\mathscr{A}/F \Leftrightarrow$ for every $B \in \mathrm{Ob}(\mathscr{A})$ and every $t \in F_B$ there exists a unique morphism $u \in \mathscr{A}(B, A)$ such that $t = u^*(s)$, and the latter condition says precisely that (A, s) is a universal pair for the functor F, whence the assertion ().[13, §2.1.1]

Lemma 1.7.2. *(i) If $\mathscr{A}$ is a small category, then for every presheaf F on $\mathscr{A}$ we have a natural equivalence of categories :*

$$\boxed{\widehat{\mathscr{A}/F} \xrightarrow{\sim} \widehat{\mathscr{A}}/F}$$

where $\widehat{\mathscr{A}/F}$ denotes the slice category of $\widehat{\mathscr{A}}$ over F.

(ii) For every $(A,s) \in \mathrm{Ob}(\mathscr{A}/F)$, the natural equivalence of (i) identifies the representable presheaf $h_{(A,s)}$ on $\mathscr{A}/F$ with the object (h_A, s) of $\widehat{\mathscr{A}}/F$ (see remark 1.6.6(i)).

Proof. (i): To every $(G, f : G \to F) \in \mathrm{Ob}(\widehat{\mathscr{A}}/F)$ we attach the presheaf $G^{\triangleright}$ on $\mathscr{A}/F$ with

$$G^{\triangleright}_{(A,s)} := f_A^{-1}(s) \qquad \forall (A,s) \in \mathrm{Ob}(\mathscr{A}/F).$$

The map $u^* : G^{\triangleright}_{(B,t)} \to G^{\triangleright}_{(A,s)}$ associated with any morphism $u : (A,s) \to (B,t)$ of $\mathscr{A}/F$ is the restriction of $u^* : G_B \to G_A$.

Moreover, every morphism $(f : G \to F) \xrightarrow{v} (f' : G' \to F)$ of $\widehat{\mathscr{A}}/F$ induces a morphism of presheaves $v^{\triangleright} : G^{\triangleright} \to G'^{\triangleright}$: namely, $v_{(A,s)} : G^{\triangleright}_{(A,s)} \to G'^{\triangleright}_{(A,s)}$ is the restriction of $v_A : G_A \to G'_A$, for every $(A,s) \in \mathrm{Ob}(\mathscr{A})$.

It is easily seen that the rules : $G \mapsto G^{\triangleright}$ and $v \mapsto v^{\triangleright}$ yield a well-defined functor $(-)^{\triangleright} : \widehat{\mathscr{A}}/F \to \widehat{\mathscr{A}/F}$; conversely, to every presheaf T on $\mathscr{A}/F$ we attach the presheaf $T^{\triangleleft}$ on $\mathscr{A}$ such that

$$T^{\triangleleft}_A := \coprod_{s \in F_A} T_{(A,s)} := \bigcup_{s \in F_A} \{s\} \times T_{(A,s)} \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

The map $u^* : T^{\triangleleft}_B \to T^{\triangleleft}_A$ associated with any morphism $u : A \to B$ of $\mathscr{A}$ is the unique one whose restriction to $\{t\} \times T_{(B,t)} \xrightarrow{\sim} T_{(B,t)}$ coincides with $u^* : T_{(B,t)} \to T_{(A,u^*(t))}$, for every $t \in F_B$. Likewise, every morphism $w : T \to T'$ of presheaves on $\mathscr{A}/F$ induces a morphism of presheaves $w^{\triangleleft} : T^{\triangleleft} \to T'^{\triangleleft}$: namely, $w^{\triangleleft}_A : T^{\triangleleft}_A \to T'^{\triangleleft}_A$ is the unique map whose restriction to $T_{(A,s)}$ agrees with $w_{(A,s)}$, for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $s \in F_A$. For every presheaf T on $\mathscr{A}/F$, we get furthermore an obvious morphism of presheaves $T^{\triangleleft} \to F$ given, for every $A \in \mathrm{Ob}(\mathscr{A})$, by the disjoint union of the system of projections $(T_{(A,s)} \to \{s\} \mid s \in F_A)$. Hence $T^{\triangleleft}$ is naturally an object of $\widehat{\mathscr{A}}/F$, and clearly $w^{\triangleleft}$ is a morphism of $\widehat{\mathscr{A}}/F$, for every morphism w of presheaves on $\mathscr{A}/F$; summing up, we have a well-defined functor $(-)^{\triangleleft} : \widehat{\mathscr{A}/F} \to \widehat{\mathscr{A}}/F$, and a simple inspection shows that both $(-)^{\triangleright} \circ (-)^{\triangleleft}$ and $(-)^{\triangleleft} \circ (-)^{\triangleright}$ are isomorphic to the respective identity functors, whence (i).

(ii) follows by inspecting the construction of $(-)^{\triangleleft}$: details left to the reader. □

Proposition 1.7.3. *For all $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$, the co-cone τ_F of remark 1.7.1(ii) is universal in $\widehat{\mathscr{A}}$.*

Proof. Let $G \in \mathrm{Ob}(\widehat{\mathscr{A}})$ and $\mu : h \circ \psi_F \Rightarrow c_G$ a co-cone. We need to exhibit a unique morphism of presheaves $f : F \to G$ with $\mu = c_f \circ \tau_F$, *i.e.* such that

$$(\dagger) \qquad \mu_{(A,s)} = (f_A(s) : A \to G) \qquad \forall (A,s) \in \mathrm{Ob}(\mathscr{A}/F).$$

Now, μ amounts to the datum of a morphism of presheaves $\mu_{(A,s)} : h_A \to G$ for every object (A,s) of $\mathscr{A}/F$, such that the diagram :

$$(\dagger\dagger) \qquad \begin{tikzcd} h_A \arrow[rr, "h_u"] \arrow[dr, "\mu_{(A,s)}"'] & & h_B \arrow[dl, "\mu_{(B,t)}"] \\ & G & \end{tikzcd}$$

commutes, for every morphism $u : (A,s) \to (B,t)$ of $\mathscr{A}/F$. To such $\mu_{(A,s)}$, the natural bijection of Yoneda's lemma attaches the section

$$f_A(s) := \mu_{(A,s),A}(\mathbf{1}_A) \in G_A$$

and the commutativity of (††) amounts to the identity

$$\begin{aligned} f_A(u^*(t)) = f_A(s) &= \mu_{(B,t),A} \circ h_{u,A}(\mathbf{1}_A) \\ &= \mu_{(B,t),A}(u) \\ &= \mu_{(B,t),A} \circ h_{B,u}(\mathbf{1}_B) \\ &= u^*(\mu_{(B,t),B}(\mathbf{1}_B)) = u^*(f_B(t)) \end{aligned}$$

(here $h_{u,A} : h_A(A) \to h_B(A)$ is the map induced by $h_u : h_A \to h_B$, and $h_{B,u} : h_B(B) \to h_B(A)$ is the map induced by $u : A \to B$). Thus, for every $A \in \mathrm{Ob}(\mathscr{A})$ the rule :

$$s \mapsto f_A(s) \qquad \forall s \in F_A$$

yields a map $f_A : F_A \to G_A$, and the above calculation shows that the system of maps $(f_A \,|\, A \in \mathrm{Ob}(\mathscr{A}))$ is a natural transformation $f : F \to G$. By construction, f is clearly the unique morphism of presheaves verifying (†). □

1.7.4. *Extensions of functors by colimits.* Proposition 1.7.3 implies the following result, a variant of the construction of the *left Kan extension* of a functor :

Theorem 1.7.5. (Kan) *Let $\mathscr{A}$ be a small category, $\mathscr{C}$ a cocomplete category, and $u : \mathscr{A} \to \mathscr{C}$ any functor. Then we have :*

(i) The induced functor $u^* : \mathscr{C} \to \widehat{\mathscr{A}}$ (see remark 1.6.6(iii)) admits a left adjoint

$$\boxed{u_! : \widehat{\mathscr{A}} \to \mathscr{C}.}$$

(ii) Moreover, the diagram :

$$\begin{tikzcd} \mathscr{A} \arrow[r, "u"] \arrow[d, "h^{\mathscr{A}}"'] & \mathscr{C} \\ \widehat{\mathscr{A}} \arrow[ru, "u_!"'] & \end{tikzcd}$$

is essentially commutative, i.e. *we have a natural isomorphism in $\mathscr{C}$:*

$$u(A) \xrightarrow{\sim} u_!(h_A) \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

We call $u_!$ the extension of u to $\widehat{\mathscr{A}}$ by colimits.

(iii) For a functor $w : \widehat{\mathscr{A}} \to \mathscr{C}$, the following conditions are equivalent :

(a) w preserves all small colimits

(b) The extension of $w \circ h^{\mathscr{A}}$ by colimits is isomorphic to w

(c) w admits a right adjoint.

Proof. (i): For every $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$, define $\psi_F : \mathscr{A}/F \to \mathscr{A}$ as in remark 1.7.1(i); pick $u_!(F) \in \mathrm{Ob}(\mathscr{C})$ representing the colimit of $u \circ \psi_F$, and a universal co-cone :

$$\tau_\bullet^F := (\tau_{(A,s)}^F : u \circ \psi_F(A,s) = u(A) \to u_!(F) \,|\, (A,s) \in \mathrm{Ob}(\mathscr{A}/F)).$$

Let $f : F \to G$ be any morphism of $\widehat{\mathscr{A}}$, and recall that $\psi_G \circ (\mathscr{A}/f) = \psi_F$ (see remark 1.7.1(iii)); then, the universality of $\tau_\bullet^F$ gives a unique morphism in $\mathscr{C}$

$$u_!(f) : u_!(F) \to u_!(G) \quad \text{such that} \quad \tau^G_{(A,f_A(s))} = u_!(f) \circ \tau^F_{(A,s)} \quad \forall (A,s) \in \mathrm{Ob}(\mathscr{A}/F).$$

It follows easily that the rules : $F \mapsto u_!(F)$ and $f \mapsto u_!(f)$ for every presheaf F and every morphism f of presheaves, yield a well-defined functor $u_! : \widehat{\mathscr{A}} \to \mathscr{C}$.

We need to check that $u_!$ is left adjoint to u^*. To this aim, let $X \in \mathrm{Ob}(\mathscr{C})$ and $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$; again by the universality of $\tau_\bullet^F$, we have a natural identification of $\mathscr{C}(u_!(F), X)$

with the set of co-cones $(u(A) \to X \mid (A,s) \in \mathrm{Ob}(\mathscr{A}/F))$ in $\mathscr{C}$ of basis $u \circ \psi_F$ and vertex X, and on the other hand, by virtue of proposition 1.7.3, the set $\widehat{\mathscr{A}}(F, u^*(X))$ is naturally identified with the set of co-cones $(h_A \to u^*(X) \mid (A,s) \in \mathrm{Ob}(\mathscr{A}/F))$ of basis $h \circ \psi_F$ and vertex $u^*(X)$. Moreover, Yoneda's lemma (theorem 1.6.5(ii)) yields natural bijections :

$$(*) \qquad \widehat{\mathscr{A}}(h_A, u^*(X)) \xrightarrow{\sim} u^*(X)_A = \mathscr{C}(u(A), X) \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

Claim 1.7.6. The bijections $(*)$ induce a bijection between the set of co-cones $\nu_\bullet : u \circ \psi_F \Rightarrow c_X$ and the set of co-cones $\mu_\bullet : h \circ \psi_F \Rightarrow c_{u^*(X)}$.

Proof : Such a $\mu_\bullet$ is a system $(\mu_{(A,s)} : h_A \to u^*(X) \mid (A,s) \in \mathrm{Ob}(\mathscr{A}))$ of morphisms of presheaves that makes commute the diagram :

$$(\dagger) \qquad \begin{tikzcd} h_A \arrow[rr, "h_f"] \arrow[dr, "\mu_{(A,s)}"'] & & h_B \arrow[dl, "\mu_{(B,t)}"] \\ & u^*(X) & \end{tikzcd}$$

for every morphism $f : (A,s) \to (B,t)$ of $\mathscr{A}/F$. The bijection $(*)$ attaches to $\mu_{(A,s)}$ the morphism $\nu_{(A,s)} := \mu_{(A,s)}(\mathbf{1}_A) : u(A) \to X$ of $\mathscr{C}$, and we need to check the commutativity of the induced diagram :

$$(\dagger\dagger) \qquad \begin{tikzcd} u(A) \arrow[rr, "u(f)"] \arrow[dr, "\nu_{(A,s)}"'] & & u(B) \arrow[dl, "\nu_{(B,t)}"] \\ & X. & \end{tikzcd}$$

However, the naturality of $\mu_{(B,t)}$ yields the commutative diagram :

$$\begin{tikzcd} h_B(B) \arrow[r, "\mu_{(B,t),B}"] \arrow[d, "h_B(f)"'] & \mathscr{C}(u(B), X) \arrow[d, "\mathscr{C}(u(f),X)"] \\ h_B(A) \arrow[r, "\mu_{(B,t),A}"] & \mathscr{C}(u(A), X) \end{tikzcd}$$

which implies the identity : $\mu_{(B,t),A}(f) = \nu_{(B,t)} \circ u(f)$, and on the other hand, from $(\dagger)$ we get : $\mu_{(B,t),A}(f) = \nu_{(A,s)}$, whence $(\dagger\dagger)$. Conversely, to a given co-cone $\nu_\bullet : u \circ \psi_F \Rightarrow c_X$, the bijections $(*)$ attach a system of morphisms $(\mu_{(A,s)} : h_A \to u^*(X) \mid (A,s) \in \mathrm{Ob}(\mathscr{A}))$, and arguing similarly one checks the commutativity of the resulting diagrams $(\dagger)$: the details shall be left to the reader. ◇

Summing up, with claim 1.7.6 we have obtained bijections :

$$(**) \qquad \mathscr{C}(u_!(F), X) \xrightarrow{\sim} \widehat{\mathscr{A}}(F, u^*(X)) \qquad \forall F \in \mathrm{Ob}(\widehat{\mathscr{A}}), \forall X \in \mathrm{Ob}(\mathscr{C}).$$

Lastly, let us check that the bijections $(**)$ are natural with respect to morphisms $\beta : G \to F$ of $\widehat{\mathscr{A}}$ and $f : X \to Y$ of $\mathscr{C}$. Explicitly, $(**)$ assigns to every morphism of presheaves $\phi : F \to u^*(X)$ the unique morphism $\phi^\vee : u_!(F) \to X$ of $\mathscr{C}$ such that $\phi^\vee \circ \tau^F_{(A,s)} = \phi_A(s)$ for every $(A,s) \in \mathrm{Ob}(\mathscr{A}/F)$. Hence, $\phi^\vee \circ \tau^F_{(A,\beta_A(s))} = \phi_A \circ \beta_A(s)$ for every such (A,s); but $\tau^F_{(A,\beta_A(s))} = u_!(\beta) \circ \tau^G_{(A,s)}$, so $\phi^\vee \circ u_!(\beta) = (\phi \circ \beta)^\vee$, which shows the naturality with respect to β. Likewise, $f \circ \phi^\vee \circ \tau^F_{(A,s)} = f \circ \phi_A(s) = (u^*(f) \circ \phi)_A(s)$ for every $(A,s) \in \mathrm{Ob}(\mathscr{A}/F)$, whence $(u^*(f) \circ \phi)^\vee = f \circ \phi^\vee$, which gives the naturality of $(**)$ with respect to f.

(ii): Notice that $(A, \mathbf{1}_A)$ is a final object in the category $\mathscr{A}/h_A$: indeed, for every $(B,t) \in \mathrm{Ob}(\mathscr{A}/h_A)$, the unique morphism $(B,t) \to (A, \mathbf{1}_A)$ in $\mathscr{A}/h_A$ is $t : B \to A$. But then, the colimit of $u \circ \psi_F$ is represented by $u \circ \psi_F(A, \mathbf{1}_A) = u(A)$, *i.e.*

$$\eta_A := \tau^{h_A}_{(A,\mathbf{1}_A)} : u(A) \to u_!(h_A)$$

is an isomorphism, and it remains only to check that the rule : $A \mapsto \eta_A$ is natural with respect to morphisms $t : B \to A$ of $\mathscr{A}$. Now, such t induces a morphism of presheaves $h_t : h_B \to h_A$, and by construction we know that

$$u_!(h_t) \circ \eta_B = \tau^{h_A}_{(B,t)} = \eta_A \circ u(t)$$

whence the assertion.

(iii): From the construction of $u_!$, it is clear that (a)⇒(b), and on the other hand, (i) implies that (b)⇒(c). The implication (c)⇒(a) is a general property of left adjoint functors (see [13, Prop.2.49(ii)]). □

Proposition 1.7.7. *Let $\mathscr{A}$ be a small category, and F a presheaf on $\widehat{\mathscr{A}}$. We have :*

(i) F is representable if and only if it preserves all small limits of $\widehat{\mathscr{A}}$.

(ii) If F is representable, then it is represented by the presheaf $F_{|\mathscr{A}} := F \circ (h^{\mathscr{A}})^{\mathrm{op}}$ on $\mathscr{A}$.

(iii) More precisely, we have, for every representable presheaf F on $\widehat{\mathscr{A}}$, an isomorphism

$$\beta^F : F \xrightarrow{\sim} h_{F_{|\mathscr{A}}}$$

of presheaves on $\widehat{\mathscr{A}}$, that is natural with respect to morphisms of representable presheaves.

Proof. (Here $h^{\mathscr{A}} : \mathscr{A} \to \widehat{\mathscr{A}}$ is the Yoneda embedding). If F is representable, then it preserves small limits, by [13, Exemp.2.39(ii)]. Conversely, suppose that F preserves small limits, and let X be any presheaf on $\mathscr{A}$; recall that the objects of $\mathscr{A}/X$ are the pairs (A, s) with $A \in \mathrm{Ob}(\mathscr{A})$ and $s \in X_A$, and the morphisms $(A, s) \to (A', s')$ are the $f \in \mathscr{A}(A, A')$ such that $f^*s' = s$, where as usual we write $f^* := X_f : X_{A'} \to X_A$. To every such pair (A, s), Yoneda's lemma attaches a unique morphism $\omega_s : h_A \to X$ of $\widehat{\mathscr{A}}$, and then the condition on f can be written as $\omega_{s'} \circ h_f = \omega_s$, for the morphism of presheaves $h_f : h_A \to h_{A'}$ induced by f. Our assumption on F and proposition 1.7.3 imply that

$$(\omega^*_s : FX \to Fh_A \mid (A, s) \in \mathrm{Ob}(\mathscr{A}/X))$$

is a universal cone $c_{FX} \Rightarrow F_{|\mathscr{A}} \circ \psi^{\mathrm{op}}_X$. Hence, the rule : $t \mapsto (\omega^*_s(t) \mid (A, s) \in \mathrm{Ob}(\mathscr{A}/X))$ establishes a bijection between FX and the set of all families

$$\sigma_\bullet := (\sigma_{(A,s)} \mid (A, s) \in \mathrm{Ob}(\mathscr{A}/X))$$

with $\sigma_{(A,s)} \in Fh_A$ for every such (A, s), and where $h^*_f(\sigma_{(A',s')}) = \sigma_{(A,s)}$ for every morphism $f : (A, s) \to (A', s)$ of $\mathscr{A}/X$, where $h^*_f : Fh_{A'} \to Fh_A$ is the map induced by h_f. But such a datum $\sigma_\bullet$ is just a morphism $\widehat{\sigma} : X \to F_{|\mathscr{A}}$ of $\widehat{\mathscr{A}}$: namely $\widehat{\sigma}_A(s) := \sigma_{(A,s)}$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and every $s \in X_A$. We have thus obtained a bijection :

$$\beta^F_X : FX \xrightarrow{\sim} \widehat{\mathscr{A}}(X, F_{|\mathscr{A}}) \qquad t \mapsto (\omega^*_s(t) \mid (A, s) \in \mathrm{Ob}(\mathscr{A}/X))$$

and we need to check its naturality, relative to morphisms $\phi : X' \to X$ of $\widehat{\mathscr{A}}$. Now, for every $t \in FX$ we have $\beta^F_{X'}(\phi^*t) = (\omega^*_{s'}(\phi^*t) \mid (A, s') \in \mathrm{Ob}(\mathscr{A}/X'))$, where $\phi^* : FX \to FX'$ is the map induced by ϕ; however, $\phi \circ \omega_{s'} = \omega_{\phi(s')}$ for every $(A, s') \in \mathrm{Ob}(\mathscr{A}/X')$, so

$$\beta^F_{X'}(\phi^*t) = (\omega^*_{\phi(s')}(t) \mid (A, s') \in \mathrm{Ob}(\mathscr{A}/X')) = \beta^F_X(t) \circ \phi : X' \to F_{|\mathscr{A}}$$

which is the required naturality. This completes the proof of (i) and (ii). For (iii), we need to check that every morphism $\gamma : F \to F'$ of representable presheaves on $\widehat{\mathscr{A}}$ and every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$ induce a commutative diagram :

$$\begin{tikzcd} FX \arrow[r, "\beta_X^F"] \arrow[d, "\gamma_X"'] & \widehat{\mathscr{A}}(X, F_{|\mathscr{A}}) \arrow[d, "\widehat{\mathscr{A}}(X,\gamma_{|\mathscr{A}})"] \\ F'X \arrow[r, "\beta_X^{F'}"] & \widehat{\mathscr{A}}(X, F'_{|\mathscr{A}}) \end{tikzcd} \qquad \text{with} \quad \gamma_{|\mathscr{A}} := \gamma \star (h^{\mathscr{A}})^{\mathrm{op}} : F_{|\mathscr{A}} \to F'_{|\mathscr{A}}.$$

However, for every $t \in FX$ we have $\beta_X^{F'}(\gamma_X t) = (\omega_s^*(\gamma_X t) \mid (A,s) \in \mathrm{Ob}(\mathscr{A}/X))$, and the naturality of γ yields, for every $(A,s) \in \mathrm{Ob}(\mathscr{A}/X)$, the commutative diagram :

$$\begin{tikzcd} FX \arrow[r, "\omega_s^*"] \arrow[d, "\gamma_X"'] & Fh_A \arrow[d, "(\gamma_{|\mathscr{A}})_A"] \\ F'X \arrow[r, "\omega_s^*"] & F'h_A \end{tikzcd}$$

which easily implies that $\beta_X^{F'}(\gamma_X t) = \gamma_{|\mathscr{A}} \circ \beta_X^F(t)$, as required. □

Remark 1.7.8. (i) With the notation of theorem 1.7.5, notice that assertion (iii) of the theorem implies that $h_!^{\mathscr{A}}$ is isomorphic to the identity endofunctor of $\widehat{\mathscr{A}}$.

(ii) Theorem 1.7.5 also implies that, for any small category $\mathscr{A}$, the category $\widehat{\mathscr{A}}$ is *cartesian closed* (see example 1.6.13(iii)). Indeed, in light of theorem 1.7.5(iii) it suffices to check that for every $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$, the functor $(-)\times F$ preserves small colimits, and since colimits are computed termwise in $\widehat{\mathscr{A}}$, we come down to showing that for every set S, the functor

$$(-) \times S : \mathsf{Set} \to \mathsf{Set} \qquad T \mapsto T \times S \qquad (T' \xrightarrow{f} T) \mapsto (f \times S)$$

preserves small colimits, which holds by example 1.6.19(i). Moreover, by proposition 1.7.7(ii), for the internal Hom functor we can take

$$\boxed{\mathscr{H}\!om(F,G)_A := \widehat{\mathscr{A}}(h_A \times F, G) \qquad \forall A \in \mathrm{Ob}(\mathscr{A})}$$

and proposition 1.7.7(iii) says that for every $u \in \widehat{\mathscr{A}}(G,G')$, $w \in \widehat{\mathscr{A}}(F',F)$, $A \in \mathrm{Ob}(\mathscr{A})$, the associated map $\mathscr{H}\!om(w,u)_A : \mathscr{H}\!om(F,G)_A \to \mathscr{H}\!om(F',G')_A$ is given by the rule :

$$(h_A \times F \xrightarrow{v} G) \mapsto (h_A \times F' \xrightarrow{h_A \times w} h_A \times F \xrightarrow{v} G \xrightarrow{u} G').$$

(iii) In the situation of theorem 1.7.5, let $v : \mathscr{A} \to \mathscr{C}$ be another functor, and $\tau_\bullet : u \Rightarrow v$ a natural transformation; by remark 1.6.6(iv), we get an induced natural transformation $\tau_\bullet^* : v^* \Rightarrow u^*$, whence an adjoint natural transformation (see §1.6.10)

$$\tau_{!\bullet} := (\tau_\bullet^*)^\vee : u_! \Rightarrow v_!.$$

Corollary 1.7.9. *(i) In the situation of remark 1.7.8(iii), we have a commutative diagram of natural transformations :*

$$\begin{tikzcd} u \arrow[rr, Rightarrow, "\tau_\bullet"] \arrow[d, Rightarrow, "\omega_\bullet^u"'] & & v \arrow[d, Rightarrow, "\omega_\bullet^v"] \\ u_! \circ h^{\mathscr{A}} \arrow[rr, Rightarrow, "\tau_{!\bullet} \star h"] & & v_! \circ h^{\mathscr{A}} \end{tikzcd}$$

whose vertical arrows are the natural isomorphisms of theorem 1.7.5(ii).

(ii) Suppose that τ_A is an epimorphism for every $A \in \mathrm{Ob}(\mathscr{A})$; then $\tau_{!F}$ is an epimorphism for every $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$.

Proof. (i): The proof of theorem 1.7.5(i,ii) exhibits an explicit adjunction $\vartheta^u_{\bullet\bullet}$ (resp. $\vartheta^v_{\bullet\bullet}$) for the pair $(u_!, u^*)$ (resp. for the pair $(v_!, v^*)$); for given $A \in \mathrm{Ob}(\mathscr{A})$ and $X \in \mathrm{Ob}(\mathscr{C})$, consider then the diagram :

$$\begin{tikzcd} \widehat{\mathscr{A}}(h_A, v^*X) \arrow[r, "\vartheta^v_{h_A,X}"] \arrow[d, "\widehat{\mathscr{A}}(h_A,\tau^*_X)"'] & \mathscr{C}(v_!(h_A), X) \arrow[r, "\mathscr{C}(\omega^v_A,X)"] \arrow[d, "\mathscr{C}(\tau_{!h_A},X)"'] & \mathscr{C}(v(A), X) \arrow[d, "\mathscr{C}(\tau_A,X)"] \\ \widehat{\mathscr{A}}(h_A, u^*X) \arrow[r, "\vartheta^u_{h_A,X}"] & \mathscr{C}(u_!(h_A), X) \arrow[r, "\mathscr{C}(\omega^u_A,X)"] & \mathscr{C}(u(A), X) \end{tikzcd}$$

whose left square commutes, by definition of $\tau_{!\bullet}$, and notice that the assertion is equivalent to the commutativity of the right square for every such A and X. However, a direct inspection of the constructions shows that the composition of the top horizontal arrows is the canonical bijection

$$\widehat{\mathscr{A}}(h_A, v^*X) \xrightarrow{\sim} v^*X(A) = \mathscr{C}(v(A), X) \qquad (h_A \xrightarrow{\phi_\bullet} v^*X) \mapsto \phi_A(\mathbf{1}_A)$$

provided by Yoneda's lemma, and likewise for the composition of the bottom horizontal arrows. The assertion follows straighforwardly.

(ii): By proposition 1.7.3, there exists a small category I and a functor $\phi : I \to \mathscr{A}$ such that F represents the colimit of $h^{\mathscr{A}} \circ \phi : I \to \widehat{\mathscr{A}}$; then $u_!F$ and $v_!F$ represent the colimits of $u_! \circ h^{\mathscr{A}} \circ \phi$ and respectively $v_! \circ h^{\mathscr{A}} \circ \phi$ ([13, Prop.2.49(ii)]), and under these identifications, $\tau_{!F}$ correspond to the colimit of $\tau_! \circ h^{\mathscr{A}} \circ \phi$. In light of (i), it then suffices to check that if τ_A is an epimorphism for every $A \in \mathrm{Ob}(\mathscr{A})$, the colimit of $\tau_\bullet \circ \phi$ is an epimorphism as well, and the latter follows from [13, Exerc.2.34(ii)]. □

1.8. **Subobjects and quotients.** Let $\mathscr{C}$ be a category, and $X \in \mathrm{Ob}(\mathscr{C})$; we denote

$$\mathsf{Sub}_{\mathscr{C}}(X)$$

the *class of subobjects* of X (see [13, Def.2.8 and Rem.2.9(ii)]). If all fibre products are representable in $\mathscr{C}$, every morphism $f : Y \to X$ induces a map ([13, Rem.2.48(iii)])

$$\mathsf{Sub}_{\mathscr{C}}(f) : \mathsf{Sub}_{\mathscr{C}}(X) \to \mathsf{Sub}_{\mathscr{C}}(Y) \qquad (Z \hookrightarrow X) \mapsto (Y \times_X Z \hookrightarrow Y).$$

Definition 1.8.1. Let $\mathscr{C}$ be any category.

(i) We say that $\mathscr{C}$ is *well powered*, if $\mathsf{Sub}_{\mathscr{C}}(X)$ is a set for every $X \in \mathrm{Ob}(\mathscr{C})$.

(ii) Suppose that $\mathscr{C}$ is well powered, and that all fibre products are representable in $\mathscr{C}$. Then evidently we get a well-defined presheaf :

$$\mathsf{Sub}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set} \qquad X \mapsto \mathsf{Sub}_{\mathscr{C}}(X) \qquad (f : Y \to X) \mapsto \mathsf{Sub}_{\mathscr{C}}(f).$$

We call *subobject classifier for* $\mathscr{C}$ any object of $\mathscr{C}$ that represents the presheaf $\mathsf{Sub}_{\mathscr{C}}$.

Remark 1.8.2. (i) A well powered category $\mathscr{C}$ with fibre products does not necessarily admit a subobject classifier, but as usual, by Yoneda's lemma, if such an object exists, it is unique up to isomorphism.

(ii) Notice also that, for every category $\mathscr{C}$ with fibre products, every monomorphism $f : X \to Y$ induces a surjection $\mathsf{Sub}_{\mathscr{C}}(f)$: indeed, every subobject $i : Z \hookrightarrow X$ of X

induces a commutative diagram of $\mathscr{C}$ where the projection p_Z is an isomorphism :

$$\begin{tikzcd} Z \times_Y X \arrow[rr, "p_Z"] \arrow[dr, "p_X"'] & & Z \arrow[dl, "i"] \\ & X. & \end{tikzcd}$$

Also, every split epimorphism g induces an injection $\mathrm{Sub}_{\mathscr{C}}(g)$, since every contravariant functor sends split epimorphisms to split monomorphisms.

Example 1.8.3. (i) The set $[1] = \{0, 1\}$ is a subobject classifier for the category Set : indeed, for every set S, the set $\mathrm{Sub}_{\mathrm{Set}}(S)$ is of course the *power set* $\mathscr{P}(S)$ of all subsets of S, and we have a natural bijection $\mathscr{P}(S) \xrightarrow{\sim} \mathrm{Set}(S, [1])$ that assigns to every subset $T \subset S$ its *characteristic function* $\chi_T : S \to [1]$ such that $\chi_T^{-1}(1) = T$.

(ii) The map $\mathrm{Sub}_{\mathrm{Set}}(f)$ associated with a map of sets $f : S' \to S$ corresponds, under the identifications of (i), to the map $\mathrm{Set}(f, [1]) : \mathrm{Set}(S, [1]) \to \mathrm{Set}(S', [1])$ that assigns to every characteristic function $\chi_T : S \to [1]$ the characteristic function $\chi_T \circ f = \chi_{f^{-1}(T)}$.

(iii) Notice that the map f is injective (resp. surjective) if and only if $\mathrm{Set}(f, [1])$ is surjective (resp. injective). Indeed, let $x', y' \in S'$ be two distinct elements; if $\mathrm{Set}(f, [1])$ is surjective, then $\{x'\} = f^{-1}\{f(x')\}$ and $\{y'\} = f^{-1}\{f(y')\}$, so $f(x') \neq f(y')$. If $\mathrm{Set}(f, [1])$ is injective, then for every $x \in S$ we must have $f^{-1}(x) \neq f^{-1}(\varnothing) = \varnothing$, *i.e.* f is surjective. The converse assertions follow from remark 1.8.2(ii).

(iv) Assertion (iii) holds more generally for any map $f : X \to Y$ of *classes* : such f is injective (resp. surjective) $\Leftrightarrow$ every subclass of X is of the form $f^{-1}(S)$ for a subclass S of Y (resp. $\Leftrightarrow$ for any two distinct subclasses $T_1, T_2 \subset Y$ we have $f^{-1}T_1 \neq f^{-1}T_2$).

(v) Let I be a category, $F : I \to \mathrm{Set}$ a functor, and $(Q, \tau_\bullet)$ the canonical pair consisting of the global colimit and the global co-cone (see example 1.2.13(ii)). Suppose moreover that F admits a colimit C (this holds, *e.g.* if I is cofinally small), and let $\eta_\bullet : F \Rightarrow c_C$ be a universal co-cone. Then we get a unique map $f : Q \to C$ such that $\eta_i = f \circ \tau_i$ for every $i \in \mathrm{Ob}(I)$; furthermore, by the universal properties of $\tau_\bullet$ and $\eta_\bullet$, the map f induces a bijection between the set $\mathrm{Set}(C, [1])$ and the set of all maps $Q \to [1]$ (the fact that the maps $Q \to [1]$ form a set follows from the existence of the bijection). By (iv), it follows that f is a bijection, so Q is a set, and $\tau_\bullet$ is a universal co-cone.

Proposition 1.8.4. *Let $\mathscr{A}$ be any small category; we have :*

(i) The category $\widehat{\mathscr{A}}$ is well powered and admits a suboboject classifier.

(ii) $\mathrm{Sub}_{\widehat{\mathscr{A}}} \circ (h^{\mathscr{A}})^{\mathrm{op}}$ is a subobject classifier for $\widehat{\mathscr{A}}$ (notation of definition 1.6.3).

Proof. For every $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$ we have a natural injection :

$$\mathrm{Sub}_{\widehat{\mathscr{A}}}(F) \to \prod_{A \in \mathrm{Ob}(\mathscr{A})} \mathrm{Sub}_{\mathrm{Set}}(FA)$$

whence the first assertion of (i). For (ii) and the second assertion of (i), proposition 1.7.7(i) reduces to checking that $\mathrm{Sub}_{\widehat{\mathscr{A}}}$ preserves small limits, and then it suffices to show that $\mathrm{Sub}^{\mathrm{op}}_{\widehat{\mathscr{A}}} : \widehat{\mathscr{A}} \to \mathrm{Set}^{\mathrm{op}}$ preserves small direct sums and coequalizers ([13, Exerc.2.42]).

• Hence, let $(F_\lambda \mid \lambda \in \Lambda)$ be any small family of presheaves on $\mathscr{A}$, and set $F := \bigsqcup_{\lambda \in \Lambda} F_\lambda$; we need to check the bijectivity of the natural map :

$$(*) \qquad \mathrm{Sub}_{\widehat{\mathscr{A}}}(F) \to \prod_{\lambda \in \Lambda} \mathrm{Sub}_{\widehat{\mathscr{A}}}(F_\lambda) \qquad G \mapsto (G \cap F_\lambda \mid \lambda \in \Lambda).$$

For the injectivity, it suffices to notice that G represents the direct sum of the family $(G \cap F_\lambda \mid \lambda \in \Lambda)$, by example 1.6.19(ii). Next, let $(G_\lambda \mid \lambda \in \Lambda)$ be a family of presheaves on $\mathscr{A}$ with $G_\lambda \subset F_\lambda$ for every $\lambda \in \Lambda$; after evaluating on each object of $\mathscr{A}$, we get $(\bigsqcup_{\lambda\in\Lambda} G_\lambda) \cap F_\lambda = G_\lambda$ for every $\lambda \in \Lambda$, whence the surjectivity of $(*)$.

- Lastly, let $f, g : F \rightrightarrows G$ be two morphisms of $\widehat{\mathscr{A}}$, and $p : G \to E$ the epimorphism onto the coequalizer E of the pair (f, g). We need to check that $\mathsf{Sub}_{\widehat{\mathscr{A}}}(p) : \mathsf{Sub}_{\widehat{\mathscr{A}}}(E) \to \mathsf{Sub}_{\widehat{\mathscr{A}}}(G)$ identifies $\mathsf{Sub}_{\widehat{\mathscr{A}}}(E)$ with the equalizer of the maps

$$\mathsf{Sub}_{\widehat{\mathscr{A}}}(f), \mathsf{Sub}_{\widehat{\mathscr{A}}}(g) : \mathsf{Sub}_{\widehat{\mathscr{A}}}(G) \rightrightarrows \mathsf{Sub}_{\widehat{\mathscr{A}}}(F).$$

The injectivity of $\mathsf{Sub}_{\widehat{\mathscr{A}}}(p)$ is clear, again by virtue of example 1.6.19(ii). Then, consider a subobject $j : G' \hookrightarrow G$ such that $f^{-1}G' := F \times_{(f,j)} G' = g^{-1}G' := F \times_{(g,j)} G'$, and let $E' \subset E$ be the image of G'; we need to check that $G' = G'' := G \times_E E'$. Now, by construction we have $G' \subset G''$; for the converse inclusion, we have to show that $G''A \subset G'A$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and since limits and colimits in $\widehat{\mathscr{A}}$ are computed termwise (see 1.3), we have $E'A = p_A(G'A)$ and $G''A = GA \times_{EA} E'A \xrightarrow{\sim} p_A^{-1}(E'A)$. By the same token, EA is the coequalizer of the two maps $f_A, g_A : FA \rightrightarrows GA$, *i.e.* $EA = GA/\!\sim$, where $\sim$ is the smallest equivalence relation on GA such that $f_A(x) \sim g_A(x)$ for every $x \in FA$. Hence, let $(y, z) \in G''$, so that $y \in GA$, $z \in E'A$ and $p_A(y) = z$; we come down to checking that $y \in G'A$. To this aim, define the relation $\approx$ on GA by :

$$y_1 \approx y_2 \quad \Leftrightarrow \quad \text{either } \{y_1, y_2\} \subset G'A \text{ or } \{y_1, y_2\} \cap G'A = \varnothing.$$

It is easily seen that $\approx$ is an equivalence relation. On the other hand, it is easily seen that $f_A(x) \approx g_A(x)$ for every $x \in FA$, since $f_A^{-1}(G'A) = g_A^{-1}(G'A)$; but then we have : $y_1 \sim y_2 \Rightarrow y_1 \approx y_2$ for every $y_1, y_2 \in GA$. By assumption, there exists $y' \in G'A$ with $p(y') = z$, so $y \sim y'$, and finally $y \approx y'$, *i.e.* $y \in G'A$. □

1.8.5. Let $\mathscr{C}$ be a well powered and cartesian closed category (see example 1.6.13(iii)), with representable fibre products, and let Ω be a subobject classifier for $\mathscr{C}$. We set

$$\Gamma(X) := \mathscr{H}om(X, \Omega) \qquad \forall X \in \mathrm{Ob}(\mathscr{C})$$

where $\mathscr{H}om$ denotes a given internal Hom functor for $\mathscr{C}$ (see example 1.6.13(iii)). We then get natural bijections :

$$\mathscr{C}(Y, \Gamma(X)) \xrightarrow{\sim} \mathscr{C}(Y \times X, \Omega) \xrightarrow{\sim} \mathsf{Sub}_{\mathscr{C}}(Y \times X) \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C})$$

and we denote by

$$\delta_X : X \to \Gamma(X)$$

the unique morphism of $\mathscr{C}$ corresponding, under these bijections, to the *diagonal subobject* $\Delta_X : X \to X \times X$. The following lemma shall be used in remark 4.2.8, in order to provide an alternative proof for corollary 4.2.7.

Lemma 1.8.6. *In the situation of §1.8.5, for every $X \in \mathrm{Ob}(\mathscr{C})$ we have :*

(i) δ_X is a monomorphism of $\mathscr{C}$.

(ii) $\Gamma(X)$ is an injective object of $\mathscr{C}$.

(iii) Especially, if $\mathscr{C}$ admits a final object, Ω is an injective object of $\mathscr{C}$.

Proof. (i): Consider two morphisms $f, g : Y \rightrightarrows X$ of $\mathscr{C}$ such that $\delta_X \circ f = \delta_X \circ g$. Notice that every morphism $h : Z \to Y$ of $\mathscr{C}$ induces a commutative diagram :

$$(*) \qquad \begin{tikzcd} \mathscr{C}(Y, \Gamma(X)) \arrow[r, "\sim"] \arrow[d, "h^*"'] & \mathrm{Sub}_{\mathscr{C}}(Y \times X) \arrow[d, "\mathrm{Sub}_{\mathscr{C}}(h \times X)"] \\ \mathscr{C}(Z, \Gamma(X)) \arrow[r, "\sim"] & \mathrm{Sub}_{\mathscr{C}}(Z \times X) \end{tikzcd}$$

whose horizontal arrows are the bijection of §1.8.5. As $f^*(\delta_X) = g^*(\delta_X)$, we may apply $(*)$ with h replaced by either f or g, to deduce :

$$(**) \qquad [(Y \times X) \times_{(f \times X, \Delta_X)} X] = [(Y \times X) \times_{(g \times X, \Delta_X)} X] \qquad \text{in } \mathrm{Sub}_{\mathscr{C}}(Y \times X)$$

where, for every monomorphism $j : T \to S$ of $\mathscr{C}$, we let $[j]$ denote the class of j in $\mathrm{Sub}_{\mathscr{C}}(S)$. Notice moreover that f yields the commutative diagram :

$$\begin{tikzcd} Y \times_{(f, \mathbf{1}_X)} X \arrow[r, "\sim"] \arrow[d] & Y \arrow[r, "\Gamma_f"] & Y \times X \arrow[r, "q_1"] \arrow[d, "f \times X"] & Y \arrow[d, "f"] \\ X \arrow[rr, "\Delta_X"] & & X \times X \arrow[r, "p_1"] & X \end{tikzcd}$$

whose two square subdiagrams are cartesian, and where we let $X \xleftarrow{p_1} X \times X \xrightarrow{p_2} X$ and $Y \xleftarrow{q_1} Y \times X \xrightarrow{q_2} X$ be the natural projections. Likewise, g induces a corresponding morphism $\Gamma_g : Y \to Y \times X$, and by construction, we have :

$$q_1 \circ \Gamma_f = q_1 \circ \Gamma_g = \mathbf{1}_Y \qquad q_2 \circ \Gamma_f = f \qquad q_2 \circ \Gamma_g = g.$$

In other words, Γ_f and Γ_g are the *graphs* of the morphisms f and g, and the identities $(**)$ come down to :

$$[\Gamma_f] = [\Gamma_g] \qquad \text{in } \mathrm{Sub}_{\mathscr{C}}(Y \times X).$$

The latter means that $\Gamma_f = \Gamma_g \circ \omega$ for an automorphism $\omega : Y \xrightarrow{\sim} Y$; but then $\mathbf{1}_Y = q_1 \circ \Gamma_f = q_1 \circ \Gamma_g \circ \omega = \omega$, *i.e.* $\Gamma_f = \Gamma_g$, and finally, $f = q_2 \circ \Gamma_f = q_2 \circ \Gamma_g = g$, whence the contention.

(ii): Let $h : Z \to Y$ be a monomorphism, and $f : Z \to \Gamma(X)$ any morphism. Under the bijections of §1.8.5, f corresponds to the class in $\mathrm{Sub}_{\mathscr{C}}(Z \times X)$ of some monomorphism $i : S_f \to Z \times X$; notice that $h \times X : Z \times X \to Y \times X$ is also a monomorphism ([13, Exemp.2.23(iii)]), so the same holds for $j := (h \times X) \circ i : S_f \to Y \times X$, and the class $[j]$ of j is an element of $\mathrm{Sub}_{\mathscr{C}}(Y \times X)$, corresponding to a unique morphism $g : Y \to \Gamma(X)$. It is then easily seen that $\mathrm{Sub}_{\mathscr{C}}(h \times X)([j]) = [i]$ (the details are left to the reader), and in view of $(*)$, we get $h^*(g) = f$, *i.e.* $f = g \circ h$, whence the assertion.

(iii): Let E be a final object of $\mathscr{C}$, and notice that the natural projection $X \times E \to X$ is an isomorphism for every $X \in \mathrm{Ob}(\mathscr{C})$. We deduce a natural bijection

$$\mathscr{C}(X, \Omega) \xrightarrow{\sim} \mathscr{C}(X \times E, \Omega) \xrightarrow{\sim} \mathscr{C}(X, \mathscr{H}om(E, \Omega)) \qquad \forall X \in \mathrm{Ob}(\mathscr{C})$$

whence, by Yoneda's lemma (theorem 1.6.5(i)), an isomorphism $\Omega \xrightarrow{\sim} \mathscr{H}om(E, \Omega)$, so the assertion follows from (ii). □

1.8.7. *Images and coimages.* Let $\mathscr{C}$ be a category, and $f : X \to Y$ a morphism of $\mathscr{C}$. We let $\mathscr{S}_f$ be the full subcategory of $\mathscr{C}/Y$ whose objects are the *monomorphisms* $j : Y' \to Y$ in $\mathscr{C}$ such that f factors (necessarily uniquely) through j. The *image* of f is defined as the initial object of $\mathscr{S}_f$, if such an initial object exist, in which case it is denoted :

$$j_f : \mathrm{Im}(f) \to Y$$

and it is easily seen that j_f represents in $\mathscr{C}/Y$ the product of all objects of $\mathscr{S}_f$, so it is well defined up to unique isomorphism. We have also a natural map :

$$(*) \qquad \mathrm{Ob}(\mathscr{S}_f) \to \mathrm{Sub}_{\mathscr{C}}(Y) \qquad (Y' \xrightarrow{j} Y) \mapsto [j]$$

where $[j]$ denotes the equivalence class of j. Let $S_f \subset \mathrm{Sub}_{\mathscr{C}}(Y)$ be the image of $(*)$, and pick a section $\iota_f : S_f \to \mathrm{Ob}(\mathscr{S}_f)$ of $(*)$; then the image of ι_f is the set of objects of a (cofinal and) coinitial full subcategory $\mathscr{S}'_f$ of $\mathscr{S}_f$, so in order to represent the imag*e of f, it suffices to be able to form the product in $\mathscr{C}/Y$ of all the objects of $\mathscr{S}'_f$* (corollary 1.5.4(ii)). Especially, if *$\mathscr{C}$ is complete and well-powered*, then $\mathscr{C}/Y$ is complete for every $Y \in \mathrm{Ob}(\mathscr{C})$ (lemma 1.4.7(i)), and the image of every morphism $X \to Y$ of $\mathscr{C}$ is representable in $\mathscr{C}/Y$.

- Suppose that the image of f exists in $\mathscr{C}/Y$, and that *all equalizers in $\mathscr{C}$ are representable*. Then the resulting morphism $\alpha_f : X \to \mathrm{Im}(f)$ (such that $f = j_f \circ \alpha_f$) *is an epimorphism of $\mathscr{C}$*. Indeed, let $g, h \in \mathscr{C}(\mathrm{Im}(f), Z)$ such that $g \circ \alpha_f = h \circ \alpha_f$, and denote by $i : E \to \mathrm{Im}(f)$ the equalizer of g and h; then $\alpha_f = i \circ \phi$ for a unique morphism $\phi : X \to E$, whence $f = j_f \circ i \circ \phi$, and i is a monomorphism of $\mathscr{C}$, so the same holds for $j_f \circ i$. But then $i/Y : (E, j_f \circ i) \to (\mathrm{Im}(f), j_f)$ is a morphism in $\mathscr{S}_f$, which must be an isomorphism, since $(\mathrm{Im}(f), j_f)$ is initial in $\mathscr{S}_f$; so i is an isomorphism in $\mathscr{C}$, whence $g = h$, QED.
- Dually, we define the *coimage* of f as the image of $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ in $\mathscr{C}^{\mathrm{op}}$. If the latter exists, it is then an epimorphism :

$$\pi_f : X \to \mathrm{Coim}(f)$$

well defined up to unique isomorphism, such that f factors uniquely through π_f. *If $\mathscr{C}$ is cocomplete and co-well-powered* (*i.e.* such that $\mathscr{C}^{\mathrm{op}}$ is well-powered), then the coimage of every morphism $f : X \to Y$ of $\mathscr{C}$ exists in $X/\mathscr{C}$. Moreover, if the coimage of f exists, and if *all coequalizers in $\mathscr{C}$ are representable*, the resulting morphism $\beta_f : \mathrm{Coim}(f) \to Y$ *is a monomorphism of $\mathscr{C}$*.
- If the image and coimage of $f : X \to Y$ exist in $\mathscr{C}$, and if either all the equalizers or all the coequalizers in $\mathscr{C}$ are representable, then we have a unique factorization of f as :

$$\boxed{X \xrightarrow{\alpha_f} \mathrm{Im}(f) \xrightarrow{\omega_f} \mathrm{Coim}(f) \xrightarrow{\beta_f} Y \quad \text{with} \quad \pi_f = \omega_f \circ \alpha_f \quad \text{and} \quad j_f = \beta_f \circ \omega_f}$$

where ω_f is both a monomorphism and an epimorphism (details left to the reader). Simple examples show that ω_f is not always an isomorphism, in this generality.

Example 1.8.8. (i) Let us show that the monomorphisms of Cat are the functors $F : \mathscr{A} \to \mathscr{B}$ between small categories, that factor through the inclusion of a subcategory $\mathscr{B}' \hookrightarrow \mathscr{B}$ and an isomorphism $\mathscr{A} \xrightarrow{\sim} \mathscr{B}'$. Indeed, such functors are clearly monomorphisms of Cat; the converse amounts to checking that every such monomorphism induces an injection $\mathrm{Ob}(F) : \mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{B})$, and injections $F_{XY} : \mathscr{A}(X, Y) \to \mathscr{B}(FX, FY)$ for every $X, Y \in \mathrm{Ob}(\mathscr{A})$. However, suppose that $FX_1 = FX_2$ for some $X_1, X_2 \in \mathrm{Ob}(\mathscr{A})$, and let $[0]$ be the discrete category with $\mathrm{Ob}([0]) = \{0\}$; then we have two functors

$$G_1, G_2 : [0] \to \mathscr{A} \qquad G_i(0) := X_i \qquad (i = 1, 2)$$

with $F \circ G_1 = F \circ G_2$, whence $G_1 = G_2$, *i.e.* $X_1 = X_2$. This shows that $\mathrm{Ob}(F)$ is an injection. Likewise, suppose that $f_1, f_2 \in \mathscr{A}(X, Y)$ verify $Ff_1 = Ff_2$, and let $[1]$ be the category such that $\mathrm{Ob}([1]) = \{0, 1\}$ and $\mathrm{Mor}([1]) = \{\mathbf{1}_0, \mathbf{1}_1, \overrightarrow{01} : 0 \to 1\}$; then we have two functors

$$H_1, H_2 : [1] \to \mathscr{B} \qquad \text{such that} \qquad H_i(\overrightarrow{01}) := f_i \qquad (i = 1, 2)$$

with $F \circ H_1 = F \circ H_2$, and this implies that $f_1 = f_2$, as stated.

(ii) From (i) we see that Cat *is well powered*, and more precisely, for every $\mathscr{A} \in \mathrm{Ob}(\mathrm{Cat})$, the set $\mathrm{Sub}_{\mathrm{Cat}}(\mathscr{A})$ is (in natural bijection with) the set of all subcategories of $\mathscr{A}$. Moreover, for every family $(\mathscr{A}_i \,|\, i \in I)$ of subcategories of $\mathscr{A}$, the *intersection* $\mathscr{A} := \bigcap_{i\in I} \mathscr{A}_i$ is well defined : namely, it is the subcategory $\mathscr{B}$ of $\mathscr{A}$ with $\mathrm{Ob}(\mathscr{B}) := \bigcap_{i\in I} \mathrm{Ob}(\mathscr{A}_i)$ and $\mathscr{B}(X,Y) := \bigcap_{i\in I} \mathscr{A}_i(X,Y)$ for every $X, Y \in \mathrm{Ob}(\mathscr{B})$. Hence, the discussion of §1.8.7 applies to Cat, and shows that *every functor $F : \mathscr{A} \to \mathscr{B}$ between small categories has an image* $\mathrm{Im}(F) \subset \mathscr{B}$, and $\mathrm{Ob}(\mathrm{Im}(F))$ is $\{FA \,|\, A \in \mathrm{Ob}(\mathscr{A})\}$.

(iii) Consider next a category $\mathscr{A}$, a wide category $\mathscr{B}$ (see definition 1.2.3(ii)), and a functor $F : \mathscr{A} \to \mathscr{B}$. By lemma 1.2.10(i,ii), there exists a minimal wide subcategory $\mathscr{B}' \subset \mathscr{B}$ with $\{Ff \,|\, f \in \mathrm{Mor}(\mathscr{A})\} \subset \mathrm{Mor}(\mathscr{B}')$ and $\mathrm{Ob}(\mathscr{B}') = \{FA \,|\, A \in \mathrm{Ob}(\mathscr{A})\}$; also, if $\mathscr{A}$ is small, the same holds for $\mathscr{B}'$. Clearly F factors uniquely through $\mathscr{B}'$, and it is easily seen that $\mathscr{B}'$ enjoys the categorical properties of an image for the functor F : *i.e.* F factors through a wide subcategory $\mathscr{C}$ of $\mathscr{B}$ if and only if $\mathscr{B}' \subset \mathscr{C}$. By a small abuse of terminology, we may then call $\mathscr{B}'$ *the image of the functor F*.

(iv) As an application, we deduce that *a category $\mathscr{C}$ is cofinally small if and only if it has a small cofinal subcategory* (see definition 1.5.1(iii,iv)). Indeed, the condition is trivially sufficient; conversely, let $F : \mathscr{A} \to \mathscr{C}$ be a given functor from a small category $\mathscr{A}$; the explicit construction of (iii) shows that the image $\mathscr{C}'$ of F is a small subcategory of $\mathscr{C}$, and by a direct inspection of the definitions we easily check that if F is cofinal, then $\mathscr{C}'$ is a cofinal subcategory of $\mathscr{C}$. Moreover, the full subcategory $\mathscr{C}''$ of $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}'') = \mathrm{Ob}(\mathscr{C}')$ is also small, and by the same token it is again cofinal in $\mathscr{C}$; taking into account proposition 1.5.7(ii,iii.b) we deduce that *$\mathscr{C}$ is filtered and cofinally small if and only if it has a small filtered cofinal subcategory.*

Remark 1.8.9. (i) Let $\mathscr{C}$ be a category, and $f : X \to Y$ a morphism of $\mathscr{C}$. Let $\mathscr{R}_f$ be the full subcategory of $\mathscr{S}_f$ whose objects are the *regular monomorphisms* $j : Y' \to Y$ (see definition 1.1.12). The *regular image* of f is defined as the initial object of $\mathscr{R}_f$, denoted

$$j^*_f : \mathrm{Im}^*(f) \to Y$$

if such an object exists, in which case it is unique up to unique isomorphism. So, j^*_f is a regular monomorphism and f is the composition of j^*_f and a unique morphism $\alpha^*_f : X \to \mathrm{Im}^*(f)$. If every monomorphism of $\mathscr{C}$ is regular, then $\mathscr{R}_f = \mathscr{S}_f$ and $\mathrm{Im}(f) = \mathrm{Im}^*(f)$.

(ii) Suppose that *all the equalizers and all the amalgamated sums are representable in $\mathscr{C}$*. Then we claim that the regular image of f is represented by the equalizer $l : E \to Y$ of the two natural morphisms

$$e_1, e_2 : Y \rightrightarrows Y \sqcup_X Y.$$

Indeed, let $i : Z \to Y$ be a regular monomorphism such that f factors through i and a morphism $g : X \to Z$; then i is the equalizer of the two natural morphisms $e'_1, e'_2 : Y \rightrightarrows Y \sqcup_Z Y$ (remark 1.1.13(ii)), and since $e'_1 \circ f = e'_1 \circ i \circ g = e'_2 \circ i \circ g = e'_2 \circ f$, we have a unique morphism $h : Y \sqcup_X Y \to Y \sqcup_Z Y$ such that $h \circ e_1 = e'_1$ and $h \circ e_2 = e'_2$. Then we get :

$$e'_1 \circ l = h \circ e_1 \circ l = h \circ e_2 \circ l = e'_2 \circ l$$

so that l factors through i, necessarily uniquely, since i is a monomorphism.

(iii) Dually, *the regular coimage* of f, denoted

$$\pi^*_f : X \to \mathrm{Coim}^*(f)$$

is the regular image of f^{op} in $\mathscr{C}^{\mathrm{op}}$. So π^*_f is a regular epimorphism such that :

$$f = \beta^*_f \circ \pi^*_f \qquad \text{with} \qquad \beta^*_f := \alpha^*_{f^{\mathrm{op}}} : \mathrm{Coim}^*(f) \to Y$$

and is a final object of the subcategory of $\mathscr{S}_{f^{\mathrm{op}}}$ whose objects are the regular epimorphisms through which f factors. If *all the coequalizers and all the fibre products are representable in* $\mathscr{C}$, then the regular coimage of f is represented by the coequalizer of the two natural morphisms

$$p_1, p_2 : X \times_Y X \rightrightarrows X.$$

(iv) A direct inspection of the definitions yields a unique monomorphism of $\mathscr{C}/Y$ (resp. epimorphism of $X/\mathscr{C}$) :

$$(*) \qquad\qquad \mathrm{Im}(f) \to \mathrm{Im}^*(f) \qquad \mathrm{Coim}^*(f) \to \mathrm{Coim}(f)$$

whenever these images (resp. coimages) are both defined. Also, whenever the regular image and the regular coimage of f both exist, we have a unique factorization of f:

$$\boxed{X \xrightarrow{\pi^*_f} \mathrm{Coim}^*(f) \xrightarrow{\omega^*_f} \mathrm{Im}^*(f) \xrightarrow{j^*_f} Y.}$$

Indeed, say that j^*_f is the equalizer of a pair of morphisms $u, v : Y \rightrightarrows Z$; then

$$u \circ \beta^*_f \circ \pi^*_f = u \circ f = v \circ f = v \circ \beta^*_f \circ \pi^*_f$$

whence $u \circ \beta^*_f = v \circ \beta^*_f$, since π^*_f is an epimorphism, and therefore β^*_f factors through $\mathrm{Im}^*(f)$. Neither ω^*_f, nor the morphisms $(*)$ are necessarily isomorphisms. Moreover, whereas the class of monomorphisms is stable under compositions, a composition of regular monomorphisms is not necessarily regular. However, we have :

Lemma 1.8.10. *(i) Let $\mathscr{C}$ be a category as in remark 1.8.9(ii), and $X \xrightarrow{f} Y \xrightarrow{g} Z$ two regular monomorphisms of $\mathscr{C}$. Suppose moreover that every push-out of every regular monomorphism of $\mathscr{C}$* is a monomorphism*; then $g \circ f$ is regular.*

(ii) Let $\mathscr{C}$ be a category, $f : X \to Y$ a morphism of $\mathscr{C}$ whose regular image exists in $\mathscr{C}$. Then $\alpha^*_f : X \to \mathrm{Im}^*(f)$ is the coimage of f $\Leftrightarrow$ α^*_f is an epimorphism.

(iii) Let $\mathscr{C}$ be as in remark 1.8.9(ii). Suppose moreover that the class of regular monomorphisms of $\mathscr{C}$ is stable under compositions. Then $\alpha^*_f : X \to \mathrm{Im}^*(f)$ is an epimorphism for every morphism f of $\mathscr{C}$ whose regular image exists in $\mathscr{C}$.

Proof. (i): The condition means that for every cocartesian square in $\mathscr{C}$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "\alpha'"] & B' \end{tikzcd}$$

such that α is a regular monomorphism, the same holds for α'. For the proof, denote by $e_1, e_2 : Z \rightrightarrows Z \sqcup_X Z$ and $e'_1, e'_2 : Z \rightrightarrows Z \sqcup_Y Z$ the two pairs of natural morphisms; since $e'_1 \circ gf = e'_2 \circ gf$, we have a unique morphism $h : Z \sqcup_X Z \to Z \sqcup_Y Z$ such that $h \circ e_i = e'_i$ for $i = 1, 2$. Let also $j : E \to Z$ be the equalizer of e_1 and e_2; then $e'_1 \circ j = h \circ e_1 \circ j = h \circ e_2 \circ j = e'_2 \circ j$, and since, by remark 1.1.13(ii), $g : Y \to Z$ represents the equalizer of e'_1 and e'_2, we deduce a unique morphism $k : E \to Y$ such that $j = g \circ k$. Since j is a monomorphism, the same holds for k, and it follows easily that k represents the equalizer of $e_1 \circ g$ and $e_2 \circ g$: the details are left to the reader.

On the other hand, let $\varepsilon_1, \varepsilon_2 : Y \rightrightarrows Y \sqcup_X Y$ and $Y \xrightarrow{\varepsilon'_1} Y \sqcup_X Z \xleftarrow{\varepsilon'_2} Z$ be the two pairs of natural morphisms, and consider the cocartesian squares :

$$\begin{tikzcd} Y \sqcup_X Y \arrow[d, "Y \sqcup_X g"'] & Y \arrow[l, "\varepsilon_2"'] \arrow[r, "\varepsilon'_1"] \arrow[d, "g"] & Y \sqcup_X Z \arrow[d, "g \sqcup_X Z"] \\ Y \sqcup_X Z & Z \arrow[l, "\varepsilon'_2"'] \arrow[r, "e_1"] & Z \sqcup_X Z. \end{tikzcd}$$

By assumption, both $Y \sqcup_X g$ and $g \sqcup_X Z$ are monomorphisms, so the same holds for their composition $g \sqcup_X g : Y \sqcup_X Y \to Z \sqcup_X Z$. Lastly, notice that $(g \sqcup_X Z) \circ \varepsilon'_2 = e_2$ and $(Y \sqcup_X g) \circ \varepsilon_1 = \varepsilon'_1$, whence $e_i \circ g = (g \sqcup_X g) \circ \varepsilon_i$ for $i = 1, 2$, so the equalizer of $e_1 \circ g$ and $e_2 \circ g$ is also the equalizer of ε_1 and ε_2; but the latter is represented by f, according to remark 1.1.13(ii). Summing up, we get a unique isomorphism $l : X \xrightarrow{\sim} E$ of $\mathscr{C}$ such that $f = k \circ l$, whence $g \circ f = j \circ l$, and this proves that $g \circ f$ is regular.

(ii): The condition is obviously necessary. Conversely, let $p : X \to Z$ be an epimorphism of $\mathscr{C}$ such that $f = h \circ p$ for some morphism $h : Z \to Y$; arguing as in remark 1.8.9(iv), we see that h factors though j^*_f and a unique morphism $Z \to \mathrm{Im}^*(f)$. Hence, if α^*_f is an epimorphism, it is the coimage of f.

(iii): Consider morphisms $u, v : \mathrm{Im}^*(f) \rightrightarrows Z$ such that $u \circ \alpha^*_f = v \circ \alpha^*_f$, and let $i : E \to \mathrm{Im}^*(f)$ be the equalizer of u and v. Then i is a regular monomorphism, and $\alpha^*_f = i \circ q$ for some morphism $q : X \to E$. By assumption, $j^*_f \circ i : E \to Y$ is a regular monomorphism as well, so there exists a morphism $i' : \mathrm{Im}^*(f) \to E$ of $\mathscr{C}$ such that $j^*_f \circ i \circ i' = j^*_f$, *i.e.* $i \circ i' = \mathbf{1}_{\mathrm{Im}^*(f)}$, and then i is an isomorphism ([13, Exerc.1.119(ii)]), so $u = v$, whence the assertion. □

Example 1.8.11. (i) Let $\mathscr{A}$ be a small category; then every morphism $f : X \to Y$ of $\widehat{\mathscr{A}}$ admits an image : namely, $\mathrm{Im}(f)$ is the subpresheaf of Y such that

$$\mathrm{Im}(f)_A := f_A(X_A) \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

By remark 1.6.2(ii), $\mathrm{Im}(f)$ is also the regular image of f.

(ii) Let $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, and $X_\bullet := (X_i \mid i \in I)$ a small family of subobjects of X; we define the *union* $\bigcup_{i \in I} X_i$ *of the family* $X_\bullet$, as the image of the induced morphism $p : \coprod_{i \in I} X_i \to X$, whose restriction to each X_i is the inclusion $X_i \to X$. Hence, the set of sections of $\bigcup_{i \in I} X_i$ over any $A \in \mathrm{Ob}(\mathscr{A})$ is just $\bigcup_{i \in I} X_i(A)$. Moreover, we have a natural diagram of presheaves :

$$\boxed{\coprod_{(i,j) \in J} X_i \cap X_j \overset{j_1}{\underset{j_2}{\rightrightarrows}} \coprod_{i \in I} X_i \xrightarrow{p} X}$$

with $J := \{(i, j) \in I^2 \mid i \neq j\}$ and $X_i \cap X_j := X_i \times_X X_j$ for every $(i, j) \in J$, where j_1 (resp. j_2) is the unique morphism of $\widehat{\mathscr{A}}$ whose restriction to $X_i \cap X_j$ agrees with the inclusion $X_i \cap X_j \to X_i$ (resp. $X_i \cap X_j \to X_j$). Arguing as in example 1.1.14(ii,iii), we see that p identifies $\bigcup_{i \in I} X_i$ with the coequalizer of the pair of morphisms (j_1, j_2). The same holds if we choose an arbitrary total ordering of I, and replace J by $\{(i, j) \in I^2 \mid i < j\}$.

1.9. **Preordered and partially ordered sets.** Recall that a *preorder* on a class C is a binary relation $\leq$ on C that is reflexive and transitive; then $\leq$ is a *partial order* if it is also anti-symmetric, *i.e.* if $x \leq y \leq x \Rightarrow x = y$. A *total order* on C is a partial order $\leq$ such that for every $x, y \in C$ we have either $x \leq y$ or $y \leq x$. A *preordered* (resp. *partially ordered*, resp. *totally ordered*) *class* is the datum $(C, \leq)$ of a class and a preorder (resp. a partial order, resp. a total order) $\leq$ on C; if C is a set, naturally we say that $(C, \leq)$ is a *preordered* (resp. *partially ordered*, resp. *totally ordered*) *set.*

Remark 1.9.1. (i) Let S be a given *set*; the set $\mathrm{Pr}(S)$ of preorders on S admits a natural partial order : namely, each preorder is a subset of $S \times S$, and we endow $\mathrm{Pr}(S)$ with the ordering induced by the inclusion of subsets of $S \times S$.

(ii) Every family $(\leq_\lambda \mid \lambda \in \Lambda)$ of preorders on S admits an *infimum* $\leq$ in $\mathrm{Pr}(S)$: namely, each $\leq_\lambda$ can be seen as a subset of $S \times S$, and $\leq$ is then just the intersection of these subsets. Clearly, if every $\leq_\lambda$ is a partial order, the same holds for $\leq$.

Moreover, every such family admits a *supremum* $\leq'$ in $\mathrm{Pr}(S)$, namely the *minimal transitive relation* containing the union of the preorders $\leq_\lambda$. The supremum of a family of partial orders is not necessarily a partial order; however, if $(\leq_\lambda \mid \lambda \in \Lambda)$ is a *filtered family of partial orders* (that is, it is filtered by inclusion, when regarded as a family of subsets of $S \times S$), it is easily seen that its supremum is again a partial order.

- A *morphism of preordered classes* $f : (C, \leq) \to (C', \leq')$ is a map of classes $f : C \to C'$ such that $x \leq y \Rightarrow f(x) \leq' f(y)$ for every $x, y \in C$. If $(C, \leq)$ and $(C', \leq')$ are partially ordered classes, naturally we say that f is a *morphism of partially ordered classes.*

Clearly, if $g : (C', \leq') \to (C'', \leq'')$ is another morphism of preordered classes, then the same holds for $g \circ f : (C \leq) \to (C'', \leq'')$. Hence, the classes of preordered *sets* and of partially ordered *set*s, with their morphisms, form two categories

$$\mathsf{prSet} \qquad \text{and} \qquad \mathsf{poSet}.$$

- The fully faithful inclusion functor $\mathsf{poSet} \to \mathsf{prSet}$ admits a left adjoint

$$\mathsf{prSet} \to \mathsf{poSet} \qquad (\Lambda, \leq) \mapsto (\overline{\Lambda}, \leq^a)$$

assigning to every preordered set $(\Lambda, \leq)$ its *anti-symmetric quotient* $(\overline{\Lambda}, \leq^a)$, with $\overline{\Lambda} := \Lambda/\!\sim$, where $\sim$ is the equivalence relation on Λ such that $x \sim y \Leftrightarrow x \leq y \leq x$; then $\leq^a$ is the *minimal* partial order on $\overline{\Lambda}$ (for the ordering on $\mathrm{Pr}(\overline{\Lambda})$ given by remark 1.9.1(i)) such that the projection $\pi : \Lambda \to \overline{\Lambda}$ is a morphism of preordered sets $(\Lambda, \leq) \to (\overline{\Lambda}, \leq^a)$. This amounts to declaring that $\pi(x) \leq^a \pi(y) \Leftrightarrow x \leq y$, for every $x, y \in \Lambda$.

1.9.2. We have moreover a fully faithful functor

$$\mathscr{C}_{(-,-)} : \mathsf{prSet} \to \mathsf{Cat} \qquad (\Lambda, \leq) \mapsto \mathscr{C}_{(\Lambda,\leq)}$$

that assigns to every preordered set $(\Lambda, \leq)$ the category $\mathscr{C}_{(\Lambda,\leq)}$ with $\mathrm{Ob}(\mathscr{C}_{(\Lambda,\leq)}) := \Lambda$, such that $\mathscr{C}_{(\Lambda,\leq)}(\lambda, \mu) = \varnothing$ if $\lambda \nleq \mu$, and otherwise $\mathscr{C}_{(\Lambda,\leq)}(\lambda, \mu)$ contains exactly one morphism, that we denote sometimes :

$$\overrightarrow{\lambda\mu} : \lambda \to \mu$$

(so, $\overrightarrow{\lambda\lambda} = \mathbf{1}_\lambda$, and if $\lambda, \mu, \nu \in \Lambda$ with $\lambda \leq \mu \leq \nu$, then necessarily $\overrightarrow{\mu\nu} \circ \overrightarrow{\lambda\mu} = \overrightarrow{\lambda\nu}$). Every morphism $f : (\Lambda, \leq) \to (\Lambda', \leq')$ of preordered sets induces the functor

$$\mathscr{C}_f : \mathscr{C}_{(\Lambda,\leq)} \to \mathscr{C}_{(\Lambda',\leq')} \qquad \lambda \mapsto f(\lambda) \qquad \overrightarrow{\lambda\mu} \mapsto \overrightarrow{f(\lambda)f(\mu)}.$$

The functor $\mathscr{C}_{(-,-)}$ admits a left adjoint

$$|-| : \mathsf{Cat} \to \mathsf{prSet} \qquad \mathscr{C} \mapsto |\mathscr{C}|$$

that assigns to every small category $\mathscr{C}$ the preordered set $|\mathscr{C}| := (\mathrm{Ob}(\mathscr{C}), \leq_\mathscr{C})$, where $\leq_\mathscr{C}$ is the preorder on $\mathrm{Ob}(\mathscr{C})$ such that $X \leq_\mathscr{C} Y \Leftrightarrow \mathscr{C}(X, Y) \neq \varnothing$. Then, every functor $F : \mathscr{C} \to \mathscr{C}'$ between small categories induces the morphism $|F| : |\mathscr{C}| \to |\mathscr{C}'|$ such that $X \mapsto FX$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Remark 1.9.3. (i) The constructions of the category $\mathscr{C}_{(\Lambda,\leq)}$ and of the functor $\mathscr{C}_f$ make sense, more generally, for every preordered *class* $(\Lambda,\leq)$ and every map f of *classes.* Likewise, using Scott's trick (see remark 1.2.1(ii)), the foregoing construction of the anti-symmetric quotient can be extended to every preordered class.

(ii) Just as in example 1.1.9(ii), since the data of $(\Lambda,\leq)$ and its associated category $\mathscr{C}_{(\Lambda,\leq)}$ are essentially equivalent, we won't usually distinguish between the two of them, and often the notation $(\Lambda,\leq)$ shall refer to either of them; especially, we shall write usually $(\Lambda,\leq)$*-complete* in lieu of $\mathscr{C}_{(\Lambda,\leq)}$-complete, and likewise for $(\Lambda,\leq)$*-cocomplete.*

(iii) For every integer $n\in\mathbb{N}$ we define the totally ordered set

$$[n] := \{0,1,\dots,n\}$$

whose ordering is induced by the standard ordering of the natural numbers. In other words, $[n]$ is the ordinal number $n{+}1$, for every $n\in\mathbb{N}$. As explained in (ii), we regard often every such $[n]$ as a category, denoted again by $[n]$; notice that the functors $[n]\to[m]$ are the morphisms of totally ordered sets, *i.e.* the non-decreasing maps.

1.9.4. Furthermore, the forgetful functor

$$\mathsf{prSet}\to\mathsf{Set}\qquad (\Lambda,\leq)\mapsto\Lambda$$

admits a left adjoint

$$\mathsf{Set}\to\mathsf{prSet}\qquad S\mapsto(S,\leq^d_S)$$

that endows every set S with its *discrete ordering* $\leq^d_S$ such that $x\leq^d_S y\Leftrightarrow x=y$. Clearly, every map of sets $g:S\to T$ is a morphism of partially ordered sets $g:(S,\leq^d_S)\to(T,\leq^d_T)$.

The forgetful functor admits as well a right adjoint

$$\mathsf{Set}\to\mathsf{prSet}\qquad S\mapsto(S,\leq^c_S)$$

that endows every set S with its *chaotic preorder* $\leq^c_S$ such that $x\leq^c_S y$ for every $x,y\in S$. Again, every map $g:S\to T$ is a morphism $g:(S,\leq^c_S)\to(T,\leq^c_T)$ of preordered sets.

Proposition 1.9.5. *The categories* prSet *and* poSet *are complete and cocomplete.*

Proof. The direct sum of every small family $E_\bullet := ((E_\lambda,\leq_\lambda)\,|\,\lambda\in\Lambda)$ of preordered sets is representable in prSet as follows : we endow the disjoint union $E:=\bigsqcup_{\lambda\in\Lambda}E_\lambda$ with the unique preorder such that for every $(\lambda,x),(\mu,y)\in E$ we have $(\lambda,x)\leq(\mu,y)\Leftrightarrow\lambda=\mu$ and $x\leq_\lambda y$. The universal co-cone is then given by the system of natural injections $(E_\lambda\to E\,|\,\lambda\in\Lambda)$ (details left to the reader). Next, let $f,g:(E,\leq_E)\rightrightarrows(F,\leq_F)$ be a pair of morphisms of prSet; we let Q be the coequalizer of the pair of underlying maps of sets $f,g:E\rightrightarrows F$, we denote by $p:F\to Q$ the natural projection, and we endow Q with the minimal transitive relation $\leq$ such that $p(x)\leq p(y)$ whenever $x\leq y$. Then $(Q,\leq)$ represents the coequalizer of f and g in prSet, and $p:(F,\leq_F)\to(Q,\leq)$ is the universal co-cone. By [13, Prop.2.40], this shows that prSet is cocomplete.

In order to check that prSet is complete, consider any small category I and any functor $F:I\to\mathsf{prSet}$; the composition F with the forgetful functor $\mathsf{prSet}\to\mathsf{Set}$ yields a functor $I\to\mathsf{Set}$, whose limit we denote by Λ. Hence, Λ is the subset of $\prod_{i\in\mathrm{Ob}(I)}Fi$ consisting of all *coherent sequences* $x_\bullet := (x_i\,|\,i\in I)$, *i.e.* all the elements $x_\bullet$ such that $F\phi(x_i)=x_j$ for every morphism $\phi:i\to j$ of I. Then, for given $x_\bullet,y_\bullet\in\Lambda$ we declare that $x_\bullet\leq y_\bullet\Leftrightarrow x_i\leq y_i$ for every $i\in I$; it is easily seen that $(\Lambda,\leq)$ represents the limit of F in prSet, with universal cone given by the system of natural projections $(\Lambda\to Fi\,|\,i\in\mathrm{Ob}(I))$.

Since the inclusion functor $j:\mathsf{poSet}\to\mathsf{prSet}$ admits a left adjoint, it follows that poSet is a cocomplete category ([13, Prop.2.49(ii)]) : indeed, for any given functor $G:I\to\mathsf{poSet}$, let $(\Lambda, \leq)$ be the colimit of $j \circ G : I \to \mathsf{prSet}$; then the anti-symmetric quotient $(\overline{\Lambda}, \leq^a)$ of $(\Lambda, \leq)$ represents the colimit of G (with the obvious universal co-cone deduced from a universal co-cone for $(\Lambda, \leq)$). Lastly, from the foregoing explicit description of limits in prSet we see that the limit of $j \circ G$ is represented by a partially ordered set; then it follows that this object also represents already the limit of G ([13, Lemma 2.52(i)]), and this shows that poSet is a complete category. □

Corollary 1.9.6. *The monomorphisms (resp. the epimorphisms) of* prSet *and of* poSet *are the injective (resp. surjective) order-preserving maps.*

Proof. Since the forgetful functor prSet → Set admits both a left and a right adjoint, it is both left and right exact ([13, Prop.2.49]). Then, since prSet is complete and cocomplete (proposition 1.9.5), for every monomorphism (resp. epimorphism) $f : (\Lambda, \leq) \to (\Lambda', \leq')$ of prSet, the underlying map $f : \Lambda \to \Lambda'$ is a monomorphism (resp. an epimorphism) of Set (remark 1.1.11(iii)), *i.e.* it is an injection (resp. a surjection); the converse assertion is clear (details left to the reader).

Likewise, the inclusion poSet → prSet is left exact, since it admits a left adjoint; then, since poSet is complete, every monomorphism of poSet is also a monomorphism of prSet, *i.e.* it is an injective map, and conversely, every injective order-preserving map of partially ordered sets is clearly a monomorphism of poSet.

Lastly, it is clear that every surjective order-preserving map is an epimorphism of poSet. Conversely, let $f : (S, \leq_S) \to (T, \leq_T)$ be a morphism of poSet, and $t \in T \setminus f(S)$; we endow $T' := T \setminus \{t\}$ with the partial order $\leq_{T'}$ induced by the inclusion into T, and we let $(T^*, \leq^*) := (T, \leq_T) \sqcup_{(T', \leq_{T'})} (T, \leq_T)$. Explicitly, the set T^* is the amalgamated sum $T \sqcup_{T'} T$, *i.e.* $T^* = T \sqcup \{t^*\}$ and the universal co-cone $T \xrightarrow{e_1} T^* \xleftarrow{e_2} T$ is given by the natural inclusion map e_1, and by the map e_2 such that $e_2(x) := x$ for every $x \neq t$, and $e_2(t) := t^*$. Then $\leq^*$ is the minimal transitive relation on T^* such that $e_1, e_2 : (T, \leq_T) \to (T^*, \leq^*)$ are morphisms of poSet. Clearly $e_1 \circ f = e_2 \circ f$, but $e_1 \neq e_2$, so f is not an epimorphism. □

Remark 1.9.7. The existence of the left adjoint $|-|$ in §1.9.2 implies that *the fully faithful functor* $\mathscr{C}_{(-,-)} : \mathsf{prSet} \to \mathsf{Cat}$ *preserves all small limits.* Also, the explicit construction of the coproduct of a small family of categories explicited in remark 1.2.4(v), combined with the description of coproducts in prSet given by proposition 1.9.5, imply that $\mathscr{C}_{(-,-)}$ *preserves and reflects all small coproducts*, and the same holds for its restriction $\Phi : \mathsf{poSet} \to \mathsf{Cat}$. However, the latter does not in general preserve coequalizers. Indeed, consider two morphisms $f, g : (E, \leq_E) \rightrightarrows (F, \leq_F)$ of poSet, and denote by $p : (F, \leq_F) \to (Q, \leq_Q)$ the projection onto the coequalizer of f and g in prSet; we have seen that the coequalizer of f and g in poSet is the anti-symmetric quotient $(\overline{Q}, \leq)$ of $(Q, \leq_Q)$, with universal co-cone $(F, \leq_F) \to (\overline{Q}, \leq)$ given by the composition of p with the projection $\pi : (Q, \leq_Q) \to (\overline{Q}, \leq)$. Hence, if Φ preserved coequalizers, then in particular the projection $p : (F, \leq) \to (Q, \leq)$ would factor through $\pi \circ p$ and a unique morphism $(\overline{Q}, \leq) \to (Q, \leq)$ of prSet; but since p is surjective, π would then have to be a bijection, and this fails in easy examples.

Example 1.9.8. (i) Not many epimorphisms of poSet are regular (see definition 1.1.12), but some regular epimorphisms of partially ordered sets will play a special role in later chapters. To describe them, consider any partially ordered set $(E, \leq)$, denote by $\xi(E)$ the set of all finite totally ordered non-empty chains of elements of E, endow every $C \in \xi(E)$ with the total ordering induced by the inclusion $C \subset E$, and endow $\xi(E)$ with the ordering given by inclusions of chains; then we claim that *the resulting system of inclusions*

$$j_\bullet := (j_C : C \hookrightarrow E \mid C \in \xi(E))$$

is a universal co-cone in the category Cat. Indeed, let $\mathscr{C}$ be any small category, and $F_\bullet := (F_C : C \to \mathscr{C} \mid C \in \xi(E))$ any co-cone; so, every F_C is a functor $C \to \mathscr{C}$, and for every $C' \in \xi(E)$ with $C' \subset C$, the functor $F_{C'}$ is the restriction of F_C. We associate with $F_\bullet$ a functor $F : E \to \mathscr{C}$ as follows; for every $x \in E$ pick any $C \in \xi(E)$ with $x \in C$ (*e.g.* we can choose $C := \{x\}$), and set $Fx := F_C(x)$. We notice that Fx is independent of the choice of C : indeed, if $x \in C'$, then $C \cap C' \in \xi(E)$ and both F_C and $F_{C'}$ agree with $F_{C\cap C'}$ on $C \cap C'$, whence the claim. Next, for every $x, y \in E$ with $x \leq y$, choose $C \in \xi(E)$ with $x, y \in C$ (*e.g.* $C := \{x, y\}$) and set $F(\overrightarrow{xy}) := F_C(\overrightarrow{xy})$; again, it is easily seen that this definition is independent of the choice of C, and clearly $F(\mathbf{1}_x) = \mathbf{1}_{Fx}$ for every $x \in E$. Lastly, for every $x, y, z \in E$ with $x \leq y \leq z$ we need to check that $F(\overrightarrow{yz}) \circ F(\overrightarrow{xy}) = F(\overrightarrow{xz})$; then, pick any $C \in \xi(E)$ with $x, y, z \in C$; by the foregoing we have $F(\overrightarrow{xy}) = F_C(\overrightarrow{xy})$, $F(\overrightarrow{yz}) = F_C(\overrightarrow{yz})$ and $F(\overrightarrow{xz}) = F_C(\overrightarrow{xz})$, and on the other hand $F_C(\overrightarrow{yz}) \circ F_C(\overrightarrow{xy}) = F_C(\overrightarrow{xz})$, whence the assertion. Obviously F is the unique functor $E \to \mathscr{C}$ that restricts to F_C over C, for every $C \in \xi(E)$, whence the stated universality of $j_\bullet$.

(ii) Now, let $E' := \bigsqcup_{C\in\xi(E)} C \xrightarrow{p} E$ be the unique epimorphism of poSet whose restriction to C agrees with j_C, for every $C \in \xi(E)$; notice that $E' \times_E E' = \bigsqcup_{(C,C')\in\xi(E)^2} C \cap C'$; then, the universality of $j_\bullet$ easily implies that p identifies $(E, \leq)$ with the coequalizer in Cat of the two natural projections $E' \times_E E' \rightrightarrows E$; in other words, *$p$ is a regular epimorphism* in Cat, and therefore also in poSet.

(iii) Let $(E, \leq)$ be a *finite* partially ordered set, and denote by $\xi_0(E)$ the set of all *maximal* totally ordered chains of E; with this notation, we get the following variant of the discussion of (i,ii). We have a diagram of morphisms of poSet :

$$\boxed{\bigsqcup_{(C,C')\in\xi_0^2(E)} C \cap C' \underset{j'}{\overset{j}{\rightrightarrows}} \bigsqcup_{C\in\xi_0(E)} C \xrightarrow{p} E}$$

where p is the unique morphism of poSet whose restriction to C is the inclusion into E, for every $C \in \xi_0(E)$, and where j (resp. j') is the unique morphism whose restriction to $C \cap C'$ is the inclusion map $C \cap C' \to C$ (resp. $C \cap C' \to C'$) for every $C, C' \in \xi_0(E)$. We claim that the diagram is *exact in the category* Cat, *i.e.* p identifies E with the coequalizer in Cat of the pair of functors (j, j') : the proof is similar to that of (i), and shall be left to the reader. In particular, *p is a regular epimorphism* in both Cat and poSet.

1.9.9. It follows easily from corollary 1.9.6 that the categories prSet and poSet are both well-powered and co-well-powered, so the image and coimage of every morphism $f : (\Lambda, \leq) \to (\Lambda', \leq')$ in either of these categories is well defined. Moreover, since prSet and poSet are also complete and cocomplete, the natural morphisms $\alpha_f : (\Lambda, \leq) \to \mathrm{Im}(f)$ and $\beta_f : \mathrm{Coim}(f) \to (\Lambda', \leq')$ are respectively an epimorphism and a monomorphism (see the discussion of §1.8.7). We deduce that, in either category, $\mathrm{Im}(f) = (f(\Lambda), \leq_f)$, where $\leq_f$ is the minimal transitive relation on $f(\Lambda)$ such that $\lambda \leq \mu \Rightarrow f(\lambda) \leq_f f(\mu)$, for every $\lambda, \mu \in \Lambda$ (notice that if $(\Lambda', \leq')$ is partially ordered, then also $\leq_f$ is a partial order, since the inclusion map $f(\Lambda) \to \Lambda'$ is a monomorphism $(f(\Lambda), \leq_f) \to (\Lambda', \leq')$ in prSet). Likewise, $\mathrm{Coim}(f) = (f(\Lambda), \leq'_f)$, where $\leq'_f$ is the restriction to $f(\Lambda)$ of the preorder $\leq'$ of Λ'; again, if $(\Lambda', \leq')$ is partially ordered, clearly the same holds for $(f(\Lambda), \leq'_f)$. The details shall be left to the reader.

Remark 1.9.10. (i) With the notation of §1.9.9, notice that the canonical morphism $\omega_f : \mathrm{Im}(f) \to \mathrm{Coim}(f)$ is not necessarily an isomorphism, though it is always a bijection.

(ii) However, if $(\Lambda, \leq)$ is a *total order,* then the same holds for both $\mathrm{Im}(f)$ and $\mathrm{Coim}(f)$, and therefore ω_f is an isomorphism, since any order-preserving bijection between total orders is obviously an isomorphism of poSet.

1.10. **Freyd's adjoint functor theorem and applications.** In this section we state and prove Freyd's adjoint functor theorem and Freyd's representability theorem, that we will use later to construct some interesting categories and functors. Both of Freyd's theorems are corollaries of the following proposition :

Proposition 1.10.1. *Let $\mathscr{C}$ be a complete category. Then $\mathscr{C}$ has an initial object if and only if there exists a subset $S \subset \mathrm{Ob}(\mathscr{C})$ such that :*

$$(*) \qquad \bigcup_{X \in S} \mathscr{C}(X, Y) \neq \varnothing \qquad \forall Y \in \mathrm{Ob}(\mathscr{C}).$$

Proof. If X is an initial object of $\mathscr{C}$, then clearly $S := \{X\}$ will do. Conversely, suppose that $S \subset \mathrm{Ob}(\mathscr{C})$ verifies $(*)$, and notice that the product $U := \prod_{X \in S} X$ is representable in $\mathscr{C}$, since $\mathscr{C}$ is complete. We have $\mathscr{C}(U, Y) \neq \varnothing$ for every $Y \in \mathrm{Ob}(\mathscr{C})$, since by assumption for every such Y there exists $X \in S$ and $f \in \mathscr{C}(X, Y)$, so $\mathscr{C}(U, Y)$ contains the composition of f with the projection $U \to X$.

Next, let $e : I \to U$ be the equalizer of all the endomorphisms of U in $\mathscr{C}$ (again, this is representable in $\mathscr{C}$); thus, $g \circ e = h \circ e$ for every $g, h \in \mathscr{C}(U, U)$, and every morphism $f : Y \to U$ such that $g \circ f = h \circ f$ for every such g and h, factors uniquely through e. Obviously $\mathscr{C}(I, Y) \neq \varnothing$ for every $Y \in \mathrm{Ob}(\mathscr{C})$; suppose that $u, v \in \mathscr{C}(I, Y)$, and let $e' : I' \to I$ be the equalizer of u and v. Pick also any $w \in \mathscr{C}(U, I')$; then $e \circ e' \circ w \in \mathscr{C}(U, U)$, and since e equalizes all the endomorphisms of U, we have :

$$e \circ e' \circ w \circ e = \mathbf{1}_U \circ e = e \circ \mathbf{1}_I.$$

Since e is a monomorphism, we deduce that $e' \circ w \circ e = \mathbf{1}_I$, so e' is both a monomorphism and a split epimorphism, *i.e.* e' is an isomorphism of $\mathscr{C}$ ([13, Exerc.1.119(ii)]). But then $u = v$, and this shows that I is an initial object of $\mathscr{C}$. □

Theorem 1.10.2. (Freyd's adjoint functor theorem) *Let $\mathscr{A}$ be a complete category, and $F : \mathscr{A} \to \mathscr{B}$ a functor. The following conditions are equivalent :*

(a) F admits a left adjoint.

(b) F preserves all small limits, and every $B \in \mathrm{Ob}(\mathscr{B})$ has a solution set, i.e. *a subset $S_B \subset \mathrm{Ob}(\mathscr{A})$ such that, for every $A \in \mathrm{Ob}(\mathscr{A})$, every $f \in \mathscr{B}(B, FA)$ factors as*

$$f = F(h) \circ g \qquad \textit{with} \qquad A' \in S_B \quad h \in \mathscr{A}(A', A) \quad g \in \mathscr{B}(B, FA').$$

Proof. (a)⇒(b) : Indeed, if F admits a left adjoint $G : \mathscr{B} \to \mathscr{A}$, then F preserves all representable limits of $\mathscr{A}$ ([13, Prop.2.49(i)]), and if $(\eta_\bullet, \varepsilon_\bullet)$ are the unit and counit of an adjunction for the pair (G, F), we get the sought identity with $h := \varepsilon_A \circ Gf : GB \to A$ and $g := \eta_B$, so that $\{GB\}$ is a solution set for every $B \in \mathrm{Ob}(\mathscr{B})$.

(b)⇒(a): Recall that F has a left adjoint if and only if the category $\mathscr{C}_B := B/F\mathscr{A}$ has an initial object for every $B \in \mathrm{Ob}(\mathscr{B})$ (proposition 1.6.8), and by virtue of propositions 1.4.5(ii.b) and 1.10.1, the latter holds if and only if there exists a subset $\mathscr{S}_B \subset \mathrm{Ob}(\mathscr{C}_B)$ such that $\bigcup_{X \in \mathscr{S}_B} \mathscr{C}_B(X, Y) \neq \varnothing$ for every $Y \in \mathrm{Ob}(\mathscr{C}_B)$. But condition (b) says that $\mathscr{S}_B := \mathrm{t}_B^{-1}(S_B)$ will do, for every such B. □

The following result generalizes proposition 1.7.7 :

Theorem 1.10.3. (Freyd's representability theorem) *Let $\mathscr{A}$ be a cocomplete category, and F a presheaf on $\mathscr{A}$. Then the following conditions are equivalent :*

(a) *F is representable*

(b) *$F : \mathscr{A}^{\mathrm{op}} \to \mathrm{Set}$ preserves all small limits, and there exists a* solution set *:*

$$S \subset \mathrm{Ob}(\mathscr{A}) \qquad \textit{such that} \qquad F_A = \bigcup_{A' \in S} \bigcup_{f \in \mathscr{A}(A,A')} f^*(F_{A'}) \qquad \forall A \in \mathrm{Ob}(\mathscr{A}).$$

Proof. (a)⇒(b): Indeed, if F is representable by some $A' \in \mathrm{Ob}(\mathscr{A})$, then F preserves all small limits ([13, Exemp.2.39(ii)]), and it is easily seen that $S := \{A'\}$ fulfills condition (b).

(b)⇒(a): According to remark 1.7.1(v), since $\mathscr{A}^{\mathrm{op}}$ is complete, the same holds for $\mathscr{B} := (\mathscr{A}/F)^{\mathrm{op}} = \{\varnothing\}/F\mathscr{A}^{\mathrm{op}}$, and F is representable ⇔ $\mathscr{B}$ has an initial object. But proposition 1.10.1 says that $\mathscr{B}$ has an initial object ⇔ there exists a subset $\mathscr{S} \subset \mathrm{Ob}(\mathscr{B})$ such that $\bigcup_{X \in \mathscr{S}} \mathscr{B}(Y,X) \neq \varnothing$ for every $Y \in \mathrm{Ob}(\mathscr{B})$. It is easily seen that $\mathscr{S} := \mathrm{t}^{-1}_{\{\varnothing\}}(S)$ will do. □

As an application, let us show :

Proposition 1.10.4. *The category* Cat *is complete and cocomplete.*

Proof. Consider any functor from a small category I :

$$\mathscr{C}_\bullet : I \to \mathrm{Cat} \qquad i \mapsto \mathscr{C}_i \qquad (i \xrightarrow{\phi} j) \mapsto (\mathscr{C}_i \xrightarrow{\mathscr{C}_\phi} \mathscr{C}_j).$$

Then the limit of $\mathscr{C}_\bullet$ is represented by the category $\mathscr{C}$ with $\mathrm{Ob}(\mathscr{C}) := \lim_I \mathrm{Ob} \circ \mathscr{C}_\bullet$, where $\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set}$ is defined as in example 1.1.9(i). Hence, an object of $\mathscr{C}$ is a family $(X_i \mid i \in \mathrm{Ob}(I))$, where $X_i \in \mathrm{Ob}(\mathscr{C}_i)$ for every $i \in \mathrm{Ob}(I)$, and $\mathscr{C}_\phi(X_i) = X_j$ for every morphism $\phi : i \to j$ in I. For any two objects $X_\bullet := (X_i \mid i \in \mathrm{Ob}(I))$ and $Y_\bullet := (Y_i \mid i \in \mathrm{Ob}(I))$, notice that the rule : $i \mapsto \mathscr{C}_i(X_i, Y_i)$ for every $i \in \mathrm{Ob}(I)$ defines a functor $H_{X_\bullet,Y_\bullet} : I \to \mathrm{Set}$; namely, to every morphism $\phi : i \to j$ in I we assign the map $H_{X_\bullet,Y_\bullet}(i) \to H_{X_\bullet,Y_\bullet}(j)$ given by : $(X_i \xrightarrow{f} Y_i) \mapsto (X_j \xrightarrow{\mathscr{C}_\phi(f)} Y_j)$. Then we set

$$\mathscr{C}(X_\bullet, Y_\bullet) := \lim_I H_{X_\bullet,Y_\bullet} \qquad \forall X_\bullet, Y_\bullet \in \mathrm{Ob}(\mathscr{C}).$$

The composition of morphisms in $\mathscr{C}$ is induced by the composition laws of the categories $\mathscr{C}_i$, in the obvious way. The obvious projection functors $\mathscr{C} \to \mathscr{C}_i$ yield a universal cone $c_{\mathscr{C}} \Rightarrow \mathscr{C}_\bullet$: the details shall be left to the reader.

Next, in order to check that the colimit of $\mathscr{C}_\bullet$ is representable in Cat, let us consider the functor $L : \mathrm{Cat} \to \mathrm{Set}$ that attaches to every small category $\mathscr{A}$ the set $L(\mathscr{A})$ of all co-cones $\eta_\bullet : \mathscr{C}_\bullet \Rightarrow c_{\mathscr{A}}$, and to every functor $G : \mathscr{A} \to \mathscr{A}'$ the map $L(G) : L(\mathscr{A}) \to L(\mathscr{A}')$: $\eta_\bullet \mapsto c_G \circ \eta_\bullet$. Then L is a presheaf on $\mathrm{Cat}^{\mathrm{op}}$, and it is easily seen that L is a representable presheaf if and only if the colimit of $\mathscr{C}_\bullet$ is representable in Cat : more precisely, $\mathrm{colim}_I \mathscr{C}_\bullet$ is represented by any small category representing the presheaf L. However, we have just shown that $\mathrm{Cat}^{\mathrm{op}}$ is cocomplete, and it is easily seen that L preserves all small limits of Cat, so by theorem 1.10.3 we are reduced to exhibiting a solution set for L. To this aim, let $\mathscr{D} := \coprod_{i \in \mathrm{Ob}(I)} \mathscr{C}_i$ be the coproduct of the family of categories $(\mathscr{C}_i \mid i \in \mathrm{Ob}(I))$ (remark 1.2.4(v)); for every small category $\mathscr{B}$ and every co-cone $\tau_\bullet : \mathscr{C}_\bullet \Rightarrow c_{\mathscr{B}}$, let $F_\tau : \mathscr{D} \to \mathscr{B}$ be the unique functor whose restriction to $\mathscr{C}_i$ agrees with τ_i for every $i \in \mathrm{Ob}(I)$, and denote by $\mathscr{B}_\tau \subset \mathscr{B}$ the image of F_τ (see example 1.8.8(ii)). The system of categories

$$\mathscr{F} := (\mathscr{B}_\tau \mid \mathscr{B} \in \mathrm{Ob}(\mathrm{Cat}), \tau_\bullet : \mathscr{C}_\bullet \Rightarrow c_{\mathscr{B}})$$

is not necessarily a set, but by remark 1.2.1(ii) we have a quotient $\mathscr{F}/\!\sim$ for the equivalence relation $\sim$ such that $\mathscr{B}_\tau \sim \mathscr{B}_\mu \Leftrightarrow \mathscr{B}_\tau$ and $\mathscr{B}_\mu$ are isomorphic categories.

Claim 1.10.5. The quotient $\mathscr{F}/{\sim}$ is a *set.*

Proof : For every equivalence relation $\mathscr{R}$ on $\mathrm{Ob}(\mathscr{D})$ and every $X \in \mathrm{Ob}(\mathscr{D})$ we denote by $[X]_{\mathscr{R}} \in \mathrm{Ob}(\mathscr{D})/\mathscr{R}$ the $\mathscr{R}$-equivalence class of X. For every $X, Y \in \mathrm{Ob}(\mathscr{D})$ let also $\mathscr{H}_{\mathscr{R}}([X]_{\mathscr{R}}, [Y]_{\mathscr{R}})$ be the set of all sequences of morphisms of $\mathscr{D}$

$$(f_i : Z_i \to Z'_i \mid i = 1, \dots, n)$$

of arbitrary finite length n, such that $[Z_1]_{\mathscr{R}} = [X]_{\mathscr{R}}$, $[Z'_n]_{\mathscr{R}} = [Y]_{\mathscr{R}}$ and $[Z_{i+1}]_{\mathscr{R}} = [Z'_i]_{\mathscr{R}}$ for every $i = 1, \dots, n-1$. Next, let S be the set of all pairs $(\mathscr{R}, \mathscr{L})$, where $\mathscr{R}$ is an equivalence relation $\mathscr{R}$ on $\mathrm{Ob}(\mathscr{D})$, and $\mathscr{L}$ is the datum of :

(a) a quotient $\overline{\mathscr{H}}_{\mathscr{R}}([X]_{\mathscr{R}}, [Y]_{\mathscr{R}})$ of $\mathscr{H}_{\mathscr{R}}([X]_{\mathscr{R}}, [Y]_{\mathscr{R}})$, for every $X, Y \in \mathrm{Ob}(\mathscr{D})$

(b) for every $X, Y, Z \in \mathrm{Ob}(\mathscr{D})$, a *composition map*

$$\overline{\mathscr{H}}_{\mathscr{R}}([X]_{\mathscr{R}}, [Y]_{\mathscr{R}}) \times \overline{\mathscr{H}}_{\mathscr{R}}([Y]_{\mathscr{R}}, [Z]_{\mathscr{R}}) \to \overline{\mathscr{H}}_{\mathscr{R}}([X]_{\mathscr{R}}, [Z]_{\mathscr{R}}).$$

Now, let $\tau_\bullet : \mathscr{C}_\bullet \Rightarrow c_{\mathscr{B}}$ be any co-cone; we attach to $\tau_\bullet$ the equivalence relation $\mathscr{R}_\tau$ on $\mathrm{Ob}(\mathscr{D})$ such that $(X, Y) \in \mathscr{R}_\tau$ if and only if $F_\tau X = F_\tau Y$, and to ease notation we set $[X]_\tau := [X]_{\mathscr{R}_\tau}$ for every $X \in \mathrm{Ob}(\mathscr{D})$. For every $X, Y \in \mathrm{Ob}(\mathscr{D})$, we then have an obvious surjection

$$\mathscr{H}_{\mathscr{R}_\tau}([X]_\tau, [Y]_\tau) \to \mathscr{B}_\tau(F_\tau X, F_\tau Y) \qquad (f_i \mid i = 1, \dots, n) \mapsto F_\tau f_n \circ \cdots \circ F_\tau f_1$$

that induces a bijection of $\mathscr{B}_\tau(F_\tau X, F_\tau Y)$ with a unique quotient $\overline{\mathscr{H}}_\tau([X]_\tau, [Y]_\tau)$ of the set $\mathscr{H}_{\mathscr{R}_\tau}([X]_\tau, [Y]_\tau)$, and the composition law of $\mathscr{B}_\tau$ yields a system of composition maps $\overline{\mathscr{H}}_\tau([X]_\tau, [Y]_\tau) \times \overline{\mathscr{H}}_\tau([Y]_\tau, [Z]_\tau) \to \overline{\mathscr{H}}_\tau([X]_\tau, [Z]_\tau)$, for all $X, Y, Z \in \mathrm{Ob}(\mathscr{D})$.

The datum $\mathscr{L}_\tau$ formed by such quotients and their composition maps then yields an element $(\mathscr{R}_\tau, \mathscr{L}_\tau) \in S$. By construction, if we have $(\mathscr{R}_\tau, \mathscr{L}_\tau) = (\mathscr{R}_\mu, \mathscr{L}_\mu)$ for two co-cones $\tau_\bullet, \mu_\bullet : \mathscr{C}_\bullet \Rightarrow c_{\mathscr{C}}$, then clearly $\mathscr{B}_\tau \sim \mathscr{B}_\mu$. This shows that $\mathscr{F}/{\sim}$ admits a natural bijection with a quotient of a subset of the set S, whence the claim. ◇

By the axiom of global choice we may pick in $\mathscr{F}$ a representative for each isomorphism class; by claim 1.10.5, the set of such representatives finally yields a solution set for L. □

Remark 1.10.6. Recall that the product and coproduct of any small family of categories admit also more explicit descriptions, independent of the constructions proposed in the proof of proposition 1.10.4 : see remark 1.2.4(iv,v). Likewise, see remark 1.5.12 for an explicit construction of the colimit of any *small filtered family* of categories. Moreover, the discussion of example 1.5.11 can now be completed as follows :

Proposition 1.10.7. *Let I be a category, and $F : I \to \mathsf{Cat}$ a functor. We have :*

(i) There exists a global co-cone $\tau_\bullet := (\tau_i : Fi \to \mathscr{C} \mid i \in \mathrm{Ob}(I))$ (see definition 1.2.9(i)).

(ii) If I is small, the same holds for the wide colimit $\mathscr{C}$, and $\tau_\bullet$ is also a universal co-cone.

Proof. (ii): So, suppose that I is small, and let $\tau_\bullet := (\tau_i : Fi \to \mathscr{C} \mid i \in \mathrm{Ob}(I))$ be a universal co-cone, which exists by virtue of proposition 1.10.4; it suffices to check that $\tau_\bullet$ is a global co-cone. Hence, let $\mathscr{B}$ be any wide category, $\eta_\bullet := (\eta_i : Fi \to \mathscr{B} \mid i \in \mathrm{Ob}(I))$ a co-cone, and for every $i \in \mathrm{Ob}(I)$ denote by $\mathscr{B}_i \subset \mathscr{B}$ the image of τ_i (example 1.8.8(iii)). By lemma 1.2.10(i,ii) there exists a minimal subcategory $\mathscr{B}' \subset \mathscr{B}$ with $\bigcup_{i \in \mathrm{Ob}(I)} \mathrm{Mor}(\mathscr{B}_i) \subset \mathscr{B}'$, and $\mathscr{B}'$ is a small category. Let also $j' : \mathscr{B}' \to \mathscr{B}$ be the inclusion functor, and $\eta'_\bullet := (\eta'_i : Fi \to \mathscr{B}' \mid i \in \mathrm{Ob}(I))$ the unique co-cone with $\eta_i = j' \circ \eta'_i$ for every $i \in \mathrm{Ob}(I)$; by the universality of $\tau_\bullet$, there exists a unique functor $G' : \mathscr{C} \to \mathscr{B}'$ such that $G' \circ \tau_i = \eta'_i$ for every $i \in \mathrm{Ob}(I)$, and if we set $G := j' \circ G' : \mathscr{C} \to \mathscr{B}$, we get $G \circ \tau_i = \eta_i$ for every $i \in \mathrm{Ob}(I)$. Next, suppose that $H : \mathscr{C} \to \mathscr{B}$ is another functor with $H \circ \tau_i = \eta_i$ for every $i \in \mathrm{Ob}(I)$, and let $j'' : \mathscr{B}'' \to \mathscr{C}$ be the inclusion of the minimal subcategory of $\mathscr{B}$ containing the images of G and H; then $\mathscr{B}''$ is small (lemma 1.2.10), and both H and G factor through j'' and unique functors $H'', G'' : \mathscr{C} \rightrightarrows \mathscr{B}''$. Clearly, $H'' \circ \tau_i = G'' \circ \tau_i$ for every $i \in \mathrm{Ob}(I)$, whence $H'' = G''$ by the universality of $\tau_\bullet$, and finally $H = G$, as required.

(i): Let $\mathscr{P}(I)$ be the class of all small subcategories of I; it follows easily from lemma 1.2.10(i,ii) that $\mathscr{P}(I)$ is filtered by inclusion. By (ii), for every $J \in \mathscr{P}(I)$ we may find a global co-cone $\tau^J_\bullet := (\tau^J_j : Fj \to \mathscr{C}_J \mid j \in \mathrm{Ob}(J))$ for the restriction $F_{|J} : J \to \mathsf{Cat}$ of the functor F, and $\mathscr{C}_J$ is a small category. Then, every inclusion $J \subset K$ of small subcategories of I induces a unique functor

$$\mathscr{C}_{JK} : \mathscr{C}_J \to \mathscr{C}_K \qquad \text{such that} \qquad \tau^K_j \circ \mathscr{C}_{JK} = \tau^J_j \qquad \forall k \in \mathrm{Ob}(J).$$

It follows easily that the rules : $J \mapsto \mathscr{C}_J$ and $(J \subset K) \mapsto \mathscr{C}_{JK}$ yield a functor

$$\mathscr{C}_\bullet : \mathscr{P}(I) \to \mathsf{Cat}$$

whose global colimit is representable by a wide category $\mathscr{C}$, by example 1.5.11, and we fix as well a global co-cone $\mu_\bullet := (\mu_J : \mathscr{C}_J \to \mathscr{C} \mid J \in \mathscr{P}(I))$. Next, for every $i \in \mathrm{Ob}(I)$ and every $J \in \mathscr{P}(I)$ with $i \in \mathrm{Ob}(J)$, let us set

$$\tau_i := \mu_J \circ \tau^J_i : Fi \to \mathscr{C}.$$

It is easily seen that τ_i is independent on the choice of J, and that the rule : $i \mapsto \tau_i$ yields a co-cone $\tau_\bullet$ with basis F and vertex $\mathscr{C}$: the details shall be left to the reader. Now, let $\mathscr{B}$ be another wide category, and $\nu_\bullet := (\nu_i : Fi \to \mathscr{B} \mid i \in \mathrm{Ob}(I))$ a co-cone; for every $J \in \mathscr{P}(I)$, the restriction $\nu^J_\bullet := (\nu_j \mid j \in \mathrm{Ob}(J))$ is a co-cone with basis $F_{|J}$, whence a unique functor $G_J : \mathscr{C}_J \to \mathscr{B}$ such that $\nu_j = G_J \circ \tau^J_j$ for every $j \in \mathrm{Ob}(J)$. The uniqueness of G_J easily implies that for every inclusion $J \subset K$ of elements of $\mathscr{P}(I)$, the functor G_J equals $G_K \circ \mathscr{C}_{JK}$, whence a co-cone $(G_J \mid J \in \mathscr{P}(I))$, which in turns determines a unique functor $G : \mathscr{C} \to \mathscr{B}$ such that $G_J = G \circ \mu_J$ for every $J \in \mathscr{P}(I)$. It follows easily that $G \circ \tau_i = \nu_i$ for every $i \in \mathrm{Ob}(I)$. Lastly, let $H : \mathscr{C} \to \mathscr{B}$ be another functor such that $H \circ \tau_i = \nu_i$ for every $i \in \mathrm{Ob}(I)$; then for every $J \in \mathscr{P}(I)$ and every $i \in \mathrm{Ob}(J)$ we get $H \circ \mu_J \circ \tau^J_i = G \circ \mu_J \circ \tau^J_i$, whence $H \circ \mu_J = G \circ \mu_J$ for every such J, so finally $H = G$. ◻

1.11. **Localizations of categories.** As an application of proposition 1.10.4, we now wish to explain how to formally invert given morphisms of a category : this problem – framed as a suitable universal factorization property – is solved by the *localization of a category along a class of its morphisms*; later we also explain the technique of *calculus of fractions in a category*, that enables us to describe explicitly such localizations, under suitable assumptions. The original source for most of this material is [6, Ch.1].

Definition 1.11.1. Let $\mathscr{C}$ be a category (resp. a wide category), and $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ a given subclass. For every wide category $\mathscr{D}$, let also $\mathrm{Isom}(\mathscr{D}) \subset \mathrm{Mor}(\mathscr{D})$ be the class of isomorphisms of $\mathscr{D}$. A *localization* (resp. a *wide localization*) of $\mathscr{C}$ *along* Σ is a pair :

$$(\mathscr{C}[\Sigma^{-1}], \gamma) \qquad (\text{resp. } (\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma))$$

where $\mathscr{C}[\Sigma^{-1}]$ is a category and $\gamma : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ a functor with $\gamma(\Sigma) \subset \mathrm{Isom}(\mathscr{C}[\Sigma^{-1}])$ (resp. where $\mathscr{C}\langle\Sigma^{-1}\rangle$ is a wide category, and $\gamma : \mathscr{C} \to \mathscr{C}\langle\Sigma^{-1}\rangle$ a functor with $\gamma(\Sigma) \subset \mathrm{Isom}(\mathscr{C}\langle\Sigma^{-1}\rangle)$), and such that for every category (resp. every wide category) $\mathscr{D}$ and every functor $F : \mathscr{C} \to \mathscr{D}$ such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, there exists a unique functor $G : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$ (resp. a unique functor $G : \mathscr{C}\langle\Sigma^{-1}\rangle \to \mathscr{D}$) with $F = G \circ \gamma$.

Example 1.11.2. (i) Let $\mathscr{C}$ be a category that admits either a final or an initial object, and take $\Sigma := \mathrm{Mor}(\mathscr{C})$. Then $\mathscr{C}[\Sigma^{-1}]$ is (isomorphic to) the category $\overline{\mathscr{C}}$ whose class of objects is $\mathrm{Ob}(\mathscr{C})$ and such that for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, the set $\overline{\mathscr{C}}(X, Y)$ contains exactly one element ϕ_{XY}. Indeed, there exists a unique functor $\gamma : \mathscr{C} \to \overline{\mathscr{C}}$ which is the identity on objects, and obviously $\gamma(\Sigma) \subset \mathrm{Isom}(\overline{\mathscr{C}})$. Next, let $F : \mathscr{C} \to \mathscr{D}$ be a functor such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$; we need to check that F factors uniquely through γ. To this aim, say that X_0 is a final object for $\mathscr{C}$, and for every $X \in \mathrm{Ob}(\mathscr{C})$, let $t_X : X \to X_0$ be the unique morphism; then, for every morphism $f : X \to Y$ of $\mathscr{C}$ we have $t_Y \circ f = t_X$, whence $Ff = (Ft_Y)^{-1} \circ Ft_X$, and thus we have $F = G \circ \gamma$ if and only if $G : \overline{\mathscr{C}} \to \mathscr{D}$ is the unique functor such that $GX := FX$ and $G\phi_{XY} := (Ft_Y)^{-1} \circ Ft_X$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$. One argues likewise in case $\mathscr{C}$ admits an initial object.

(ii) If $\mathscr{C}$ is a wide category that admits either an initial or a final object, the argument of (i) also yields the same description for the wide localization $\mathscr{C}\langle\Sigma^{-1}\rangle$ along $\Sigma := \mathrm{Mor}(\mathscr{C})$. Especially, if $\mathscr{C}$ is a category, the localization and the wide localization of $\mathscr{C}$ along $\mathrm{Mor}(\mathscr{C})$ coincide (as usual, up to unique isomorphism), under either of these assumptions.

Remark 1.11.3. (i) If the localization $(\mathscr{C}[\Sigma^{-1}], \gamma)$ exists, *the functor γ is a bijection on objects*; *i.e.* we may always take $\mathrm{Ob}(\mathscr{C}[\Sigma^{-1}]) = \mathrm{Ob}(\mathscr{C})$, and then γ will be the identity on objects. Indeed, consider the category $\mathscr{D}$ such that $\mathrm{Ob}(\mathscr{D}) = \mathrm{Ob}(\mathscr{C})$ and $\mathscr{D}(X, Y) := \mathscr{C}[\Sigma^{-1}](\gamma X, \gamma Y)$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, with the composition law deduced from the composition law for morphisms of $\mathscr{C}[\Sigma^{-1}]$. We have a functor $\lambda : \mathscr{C} \to \mathscr{D}$ that is the identity on objects, and such that $\lambda(f) := \gamma(f)$ for every morphism f of $\mathscr{C}$. Clearly $\lambda(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, so λ factors through γ and a unique functor $\alpha : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$; on the other hand, γ factors through λ and a unique functor $\beta : \mathscr{D} \to \mathscr{C}[\Sigma^{-1}]$: namely, $\beta X := \gamma X$ for every $X \in \mathrm{Ob}(\mathscr{D})$, and β is the identity map on morphisms. The identity $\lambda = \alpha \circ \gamma$ implies that α is surjective on objects. Next, since $\gamma = \beta \circ \lambda = \beta \circ \alpha \circ \gamma$, we must have $\beta \circ \alpha = \mathbf{1}_{\mathscr{C}[\Sigma^{-1}]}$ by the universal property of γ; this implies that α is injective on objects, *i.e.* α is bijective on objects, so the same holds for γ.

(ii) Moreover, every morphism of $\mathscr{C}[\Sigma^{-1}]$ can be written as a composition :

$$(*) \qquad X_0 \xrightarrow{\gamma(f_0)} Y_0 \xrightarrow{\gamma(g_0)^{-1}} X_1 \xrightarrow{\gamma(f_1)} Y_1 \xrightarrow{\gamma(g_1)^{-1}} \cdots \xrightarrow{\gamma(f_n)} Y_n$$

of some arbitrary length n, where $f_0, \dots, f_n$ are morphisms of $\mathscr{C}$, and $g_0, \dots, g_{n-1}$ are elements of Σ. Indeed, clearly any composition of morphisms of the type $(*)$ is again of the same type, so let $\mathscr{B} \subset \mathscr{C}[\Sigma^{-1}]$ be the subcategory such that $\mathrm{Ob}(\mathscr{B}) = \mathrm{Ob}(\mathscr{C})$, and whose morphisms are of the form $(*)$. Obviously γ factors through the inclusion $\mathscr{B} \to \mathscr{C}[\Sigma^{-1}]$ and a unique functor $\gamma' : \mathscr{C} \to \mathscr{B}$, and clearly $\gamma'(\Sigma) \subset \mathrm{Isom}(\mathscr{B})$. Moreover, if $F : \mathscr{C} \to \mathscr{D}$ is any functor with $F(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, then the unique functor $G : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$ with $F = G \circ \gamma$ restricts to a functor $G' : \mathscr{B} \to \mathscr{D}$ with $F = G' \circ \gamma'$, and it is easily seen that there can be at most one such functor G'. Hence, $(\mathscr{B}, \gamma')$ is another localization of $\mathscr{C}$ along Σ, and one deduces easily that $\mathscr{B} = \mathscr{C}[\Sigma^{-1}]$.

(iii) The discussion of (i) and (ii) also applies *verbatim* to the wide localization $\mathscr{C}\langle\Sigma^{-1}\rangle$ of any wide category $\mathscr{C}$ along any subclass $\Sigma \subset \mathrm{Mor}(\mathscr{C})$.

(iv) Suppose that $\mathscr{C}$ is a *small* category, and let $(\mathscr{L}, \gamma)$ be a pair consisting of a *small* category $\mathscr{L}$ and a functor $\gamma : \mathscr{C} \to \mathscr{L}$ with $\gamma(\Sigma) \subset \mathrm{Isom}(\mathscr{L})$, such that for every *small* category $\mathscr{C}'$, the functor γ induces a bijection :

$$\mathsf{Cat}(\mathscr{L}, \mathscr{C}') \xrightarrow{\sim} \{F \in \mathsf{Cat}(\mathscr{C}, \mathscr{C}') \mid F(\Sigma) \subset \mathrm{Isom}(\mathscr{C}')\} \qquad G \mapsto G \circ \gamma.$$

Then $(\mathscr{L}, \gamma)$ is a localization of $\mathscr{C}$ along Σ. Indeed, let $F : \mathscr{C} \to \mathscr{D}$ be any functor such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, and let $j : \mathscr{D}' \to \mathscr{D}$ be the inclusion functor of the minimal subcategory $\mathscr{D}' \subset \mathscr{D}$ such that $F(\mathrm{Mor}(\mathscr{C})) \cup \{(Ff)^{-1} \mid f \in \Sigma\} \subset \mathrm{Mor}(\mathscr{D}')$. Then F is the composition of j and a unique functor $F' : \mathscr{C} \to \mathscr{D}'$. Clearly $F'(\Sigma) \subset \mathrm{Isom}(\mathscr{D}')$; moreover, $\mathscr{D}'$ is a small category (lemma 1.2.10(i,ii)), so by assumption F' is the composition of γ with a unique functor $G' : \mathscr{L} \to \mathscr{D}'$. With $G := j \circ G'$, we get $F = G \circ \gamma$. Lastly, suppose that $H : \mathscr{L} \to \mathscr{D}$ is another functor such that $F = H \circ \gamma$, and let $i : \mathscr{D}'' \to \mathscr{D}$ be the inclusion functor of the minimal subcategory $\mathscr{D}'' \subset \mathscr{D}$ with $G(\mathrm{Mor}(\mathscr{L})) \cup H(\mathrm{Mor}(\mathscr{L})) \subset \mathrm{Mor}(\mathscr{D}'')$; then G (resp. H) is the composition of i with a unique functor $G'' : \mathscr{L} \to \mathscr{D}''$ (resp. $H'' : \mathscr{L} \to \mathscr{D}''$), and clearly $G'' \circ \gamma = H'' \circ \gamma$, whence $G'' = H''$, by our assumption on the pair $(\mathscr{L}, \gamma)$, so finally $G = H$, as required. The same argument shows that, *in this situation, $(\mathscr{L}, \gamma)$ also represents the wide localization of $\mathscr{C}$ along Σ.*

Proposition 1.11.4. *(i) For every wide category $\mathscr{C}$ and every subclass $\Sigma \subset \mathrm{Mor}(\mathscr{C})$, the wide localization $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ of $\mathscr{C}$ along Σ exists.*

(ii) If $\mathscr{C}$ is a small category, the same holds for $\mathscr{C}\langle\Sigma^{-1}\rangle$.

(iii) If $\mathscr{C}$ is a category and Σ is a set, $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ is also a localization of $\mathscr{C}$ along Σ.

Proof. Suppose first that $\mathscr{C}$ is small, and that $\Sigma = \{f\}$ for some morphism $f : X \to Y$ of $\mathscr{C}$. We let I be the category with $\mathrm{Ob}(I) := \{0, 1\}$ and such that the inclusion functor $[0] \to I$ is an equivalence of categories (so $\mathrm{Mor}(I) = \{\mathbf{1}_0, \mathbf{1}_1, \overrightarrow{01} : 0 \to 1, \overrightarrow{10} : 1 \to 0\}$). Clearly the totally ordered set $[1]$ is a subcategory of I, and we let $j : [1] \to I$ be the inclusion; since Cat is cocomplete (proposition 1.10.4) we may then consider the cocartesian diagram :

$$\begin{tikzcd} {[1]} \arrow[r, "F"] \arrow[d, "j"'] & \mathscr{C} \arrow[d, "\gamma"] \\ I \arrow[r] & \mathscr{D} \end{tikzcd}$$

where F is the unique functor such that $F(\overrightarrow{01}) := f$. Hence, for every small category $\mathscr{A}$, the datum of a functor $\mathscr{D} \to \mathscr{A}$ is equivalent to that of a pair of functors $\mathscr{C} \xrightarrow{G} \mathscr{A} \xleftarrow{H} I$ with $G \circ F = H \circ j$. However, notice that H is the same as the datum of an isomorphism of $\mathscr{A}$, and the foregoing identity then means that Gf is an isomorphism in $\mathscr{A}$. Taking into account remark 1.11.3(iv), this shows that $(\mathscr{D}, \gamma)$ is a localization $\mathscr{C}[f^{-1}]$ of $\mathscr{C}$ along $\{f\}$.

- Next, let $\Sigma = \{f_1, \dots, f_n\}$ be any finite subset of $\mathrm{Mor}(\mathscr{C})$; we set

$$\mathscr{D}_0 := \mathscr{C} \qquad \text{and} \qquad \mathscr{D}_i := \mathscr{D}_{i-1}[\overline{f}_i^{-1}] \qquad \forall i = 1, \dots, n$$

where $\overline{f}_i$ denotes the image of f_i in $\mathscr{D}_{i-1}$, for every $i = 1, \dots, n$. It is easily seen that $\mathscr{D}_n$ represents the localization $\mathscr{C}[\Sigma^{-1}]$, and the localization functor $\gamma : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ is the composition of the localizations $(\mathscr{D}_{i-1} \to \mathscr{D}_i \mid i = 1, \dots, n)$.

- Lastly, let $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ be an arbitrary subset, and denote by $\mathscr{P}_0(\Sigma)$ the set of all finite subsets of Σ, partially ordered by inclusion of subsets. By the foregoing, for every $\Delta \in \mathscr{P}_0(\Sigma)$ we get a localization $\gamma_\Delta : \mathscr{C} \to \mathscr{C}[\Delta^{-1}]$, and clearly, for every $\Delta' \in \mathscr{P}_0(\Sigma)$ with $\Delta \subset \Delta'$ there exists a unique functor $\gamma_{\Delta\Delta'} : \mathscr{C}[\Delta^{-1}] \to \mathscr{C}[\Delta'^{-1}]$ such that $\gamma_{\Delta\Delta'} \circ \gamma_\Delta = \gamma_{\Delta'}$. We then get a well-defined functor

$$\mathscr{C}[-] : \mathscr{P}_0(\Sigma) \to \mathrm{Cat} \qquad \Delta \mapsto \mathscr{C}[\Delta^{-1}] \qquad (\Delta \subset \Delta') \mapsto \gamma_{\Delta\Delta'}$$

and again, in light of remark 1.11.3(iv), it is easily seen that the colimit $\mathscr{L}$ of $\mathscr{C}[-]$ represents the localization $\mathscr{C}[\Sigma^{-1}]$; indeed, if $(\delta_\Delta : \mathscr{C}[\Delta^{-1}] \to \mathscr{L} \mid \Delta \in \mathscr{P}_0(\Sigma))$ is a universal co-cone, then $\delta_\varnothing : \mathscr{C} = \mathscr{C}[\varnothing^{-1}] \to \mathscr{L}$ is the localization functor. Moreover, $(\mathscr{C}[\Sigma^{-1}], \delta_\varnothing)$ is also a wide localization of $\mathscr{C}$ along Σ, so the proof of (ii) is complete.

- To prove (i), denote by $\mathscr{P}(\mathscr{C})$ the class of all subsets of $\mathrm{Mor}(\mathscr{C})$, and recall that for every $S \in \mathscr{P}(\mathscr{C})$ there exists a minimal wide subcategory $\mathscr{C}_S$ of $\mathscr{C}$ with $S \subset \mathrm{Mor}(\mathscr{C}_S)$, and $\mathscr{C}_S$ is a small category; moreover, $\mathscr{C}$ is the wide colimit of the filtered family $(\mathscr{C}_S \mid S \in \mathscr{P}(\mathscr{C}))$ (lemma 1.2.10). For every $S, T \in \mathscr{P}(\mathscr{C})$ with $S \subset T$, let $j_S : \mathscr{C}_S \to \mathscr{C}$ and $\mathscr{C}_{ST} : \mathscr{C}_S \to \mathscr{C}_T$ be the inclusion functors, set $\Sigma_S := S \cap \Sigma$, and let $(\mathscr{C}_S\langle\Sigma_S^{-1}\rangle, \gamma_S)$ be the wide localization of $\mathscr{C}_S$ along S; by (ii), we get an induced functor

$$F : \mathscr{P}(\mathscr{C}) \to \mathsf{Cat} \qquad S \mapsto \mathscr{C}_S\langle\Sigma_S^{-1}\rangle \qquad (S \subset T) \mapsto (F_{ST} : \mathscr{C}_S\langle\Sigma_S^{-1}\rangle \to \mathscr{C}_T\langle\Sigma_T^{-1}\rangle)$$

where F_{ST} is the unique functor such that $F_{ST} \circ \gamma_S = \gamma_T \circ \mathscr{C}_{ST}$. According to example 1.5.11, the wide colimit of F is represented by a wide category $\mathscr{L}$, and we choose a global co-cone $\tau_\bullet := (\tau_S : \mathscr{C}_S\langle\Sigma_S^{-1}\rangle \to \mathscr{L} \mid S \in \mathscr{P}(\mathscr{C}))$. Let then $\gamma : \mathscr{C} \to \mathscr{L}$ be the unique functor such that $\gamma \circ j_S = \tau_S \circ \gamma_S$ for every $S \in \mathscr{P}(\mathscr{C})$; we claim that $(\mathscr{L}, \gamma)$ is the sought wide localization of $\mathscr{C}$ along Σ. Indeed, it is clear that $\gamma(\Sigma) \subset \mathrm{Isom}(\mathscr{L})$; next, let $G : \mathscr{C} \to \mathscr{D}$ be any functor of wide categories such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, and for every $S \in \mathscr{P}(\mathscr{C})$ let $G_S : \mathscr{C}_S\langle\Sigma_S^{-1}\rangle \to \mathscr{D}$ be the unique functor such that $G_S \circ \gamma_S = G \circ j_S$. It follows easily that $G_T \circ F_{ST} \circ \gamma_S = G_S \circ \gamma_S$ for every $S, T \in \mathscr{P}(\mathscr{C})$ with $S \subset T$, whence $G_T \circ F_{ST} = G_S$, so that we get a unique functor

$$H : \mathscr{L} \to \mathscr{D} \qquad \text{such that} \qquad H \circ \tau_S = G_S \qquad \forall S \in \mathscr{P}(\mathscr{C}).$$

It follows that $G \circ j_S = G_S \circ \gamma_S = H \circ \tau_S \circ \gamma_S = H \circ \gamma \circ j_S$ for every $S \in \mathscr{P}(\mathscr{C})$, so that $G = H \circ \gamma$. Lastly, let $K : \mathscr{L} \to \mathscr{D}$ be another functor with $K \circ \gamma = G$; it follows that $K \circ \tau_S \circ \gamma_S = K \circ \gamma \circ j_S = G \circ j_S = H \circ \tau_S \circ \gamma_S$ for every $S \in \mathscr{P}(\mathscr{C})$, whence $K \circ \tau_S = H \circ \tau_S$ for every such S, and finally $K = H$, as required.

(iii): Denote by S the class of all objects of $\mathscr{C}$ that are either source or target of elements of Σ, and for every $X, Y \in S$ set $\Sigma(X, Y) := \mathscr{C}(X, Y) \cap \Sigma$; clearly, if Σ is a set, the same holds for S. For every $n \in \mathbb{N}$ and every $X, Y \in \mathrm{Ob}(\mathscr{C})$, let $T_n(X, Y)$ be the set of all sequences $(X_\bullet, Y_\bullet) := ((X_0, Y_0), \cdots, (X_n, Y_n))$ with $X_0 = X$, $Y_n = Y$ and $X_i, Y_{i-1} \in S$ for every $i = 1, \dots, n$. Remark 1.11.3(ii) yields a surjection

$$\bigcup_{n\in\mathbb{N}} \bigcup_{(X_\bullet, Y_\bullet)\in T_n(X,Y)} \prod_{i=0}^{n} \mathscr{C}(X_i, Y_i) \times \prod_{i=0}^{n-1} \Sigma(X_{i+1}, Y_i) \to \mathscr{C}\langle\Sigma^{-1}\rangle(X, Y).$$

Thus, if Σ is a set, the same holds for $\mathscr{C}\langle\Sigma^{-1}\rangle(X, Y)$, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$. □

Remark 1.11.5. (i) When $\mathscr{C}$ is a (usual) category and Σ is is a proper class, the existence of the localization $\mathscr{C}[\Sigma^{-1}]$ is in general a delicate question, and depends on the chosen set-theoretic framework. On the other hand, it is clear that *if the wide localization $\mathscr{C}\langle\Sigma^{-1}\rangle$ is a (usual) category, then $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ is also a localization of $\mathscr{C}$ along* Σ. Moreover, standard arguments show that the pair $(\mathscr{C}[\Sigma^{-1}], \gamma)$, when it exists, is determined by its universal property : if $(\mathscr{L}, \gamma')$ is another localization of $\mathscr{C}$ along Σ, there exists a unique isomorphism of categories $\omega : \mathscr{C}[\Sigma^{-1}] \xrightarrow{\sim} \mathscr{L}$ such that $\gamma' \circ \omega = \gamma$; the same of course applies, *mutatis mutandis,* to wide localizations.

(ii) Let $\mathscr{C}$ be a category, $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ a subclass, $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ the wide localization of $\mathscr{C}$ along Σ, and suppose as well that the localization $(\mathscr{C}[\Sigma^{-1}], \gamma')$ exists; by remark 1.11.3(i,iii) we may suppose that γ and γ' induce identifications : $\mathrm{Ob}(\mathscr{C}\langle\Sigma^{-1}\rangle) =$

$\mathrm{Ob}(\mathscr{C}[\Sigma^{-1}]) = \mathrm{Ob}(\mathscr{C})$, and we have a unique functor

$$\phi : \mathscr{C}\langle\Sigma^{-1}\rangle \to \mathscr{C}[\Sigma^{-1}] \qquad \text{such that} \qquad \phi \circ \gamma = \gamma\text{'}.$$

Especially, ϕ is the identity map on objects. It follows easily that the rule :

$$(X, Y) \mapsto \mathscr{D}(X, Y) := \phi(\mathscr{C}\langle\Sigma^{-1}\rangle(X, Y)) \qquad \forall(X, Y) \in \mathrm{Ob}(\mathscr{C})^2$$

defines a subcategory $\mathscr{D} \subset \mathscr{C}[\Sigma^{-1}]$ with $\mathrm{Ob}(\mathscr{D}) = \mathrm{Ob}(\mathscr{C})$, such that γ' factors uniquely through a functor $\gamma'' : \mathscr{C} \to \mathscr{D}$, with $\gamma''(\Sigma) \subset \mathrm{Isom}(\mathscr{D})$, and ϕ factors through a unique functor $\phi' : \mathscr{C}\langle\Sigma^{-1}\rangle \to \mathscr{D}$. Now, let $F : \mathscr{C} \to \mathscr{A}$ be any functor of categories such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{A})$, so that F factors uniquely through γ' and a functor $\overline{F} : \mathscr{C}[\Sigma^{-1}] \to \mathscr{A}$, and denote by $G : \mathscr{D} \to \mathscr{A}$ the restriction of $\overline{F}$. Clearly $G \circ \gamma'' = F$; moreover, suppose that $G' : \mathscr{D} \to \mathscr{A}$ is another functor with $G' \circ \gamma'' = F$. It follows that $G \circ \phi' \circ \gamma = G' \circ \phi' \circ \gamma$, whence $G \circ \phi' = G' \circ \phi'$; but ϕ' is the identity on objects, and is surjective on morphisms, whence $G = G'$. This shows that $(\mathscr{D}, \gamma'')$ is a localization of $\mathscr{C}$ along Σ, and then the inclusion $\mathscr{D} \to \mathscr{C}[\Sigma^{-1}]$ must be an isomorphism of categories, by (i). Summing this proves that ϕ *is a full functor.* However, will shall later show that ϕ *is not necessarily an isomorphism of (wide) categories* : see proposition 1.12.14.

(iii) It is clear from the definition, that the localization $(\mathscr{C}[\Sigma^{-1}], \gamma)$ exists if and only if the same holds for the localization $(\mathscr{C}^{\mathrm{op}}[\Sigma^{\mathrm{op}-1}], \gamma')$, and if these categories exist, we have a unique isomorphism of categories :

$$\boxed{\omega : \mathscr{C}^{\mathrm{op}}[\Sigma^{\mathrm{op}-1}] \xrightarrow{\sim} \mathscr{C}[\Sigma^{-1}]^{\mathrm{op}} \qquad \text{such that} \qquad \gamma^{\mathrm{op}} = \omega \circ \gamma'}$$

and again, the same applies to wide localizations.

Proposition 1.11.6. *Let $\mathscr{C}, \mathscr{D}$ be two categories, $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ two functors and*

$$\Sigma_F := \{f \in \mathrm{Mor}(\mathscr{C}) \mid Ff \text{ is an isomorphism}\}.$$

Suppose that G is fully faithful, and is either right or left adjoint to F. Then :

(i) The localization $(\mathscr{C}[\Sigma_F^{-1}], \gamma)$ exists.

(ii) F factors uniquely through γ and an equivalence $\overline{F} : \mathscr{C}[\Sigma_F^{-1}] \xrightarrow{\sim} \mathscr{D}$.

Proof. Obviously we have $\Sigma_F^{\mathrm{op}} = \Sigma_{F^{\mathrm{op}}}$, and (F, G) is an adjoint pair if and only if the same holds for $(G^{\mathrm{op}}, F^{\mathrm{op}})$ ([13, Rem.2.15(i)]); taking into account remark 1.11.5(iii), we may therefore assume that G is right adjoint to F.

Now, let $\overline{\mathscr{C}}$ be the category with $\mathrm{Ob}(\overline{\mathscr{C}}) := \mathrm{Ob}(\mathscr{C})$, and

$$\overline{\mathscr{C}}(X, Y) := \mathscr{D}(FX, FY) \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C})$$

with composition law deduced from that of $\mathscr{D}$, in the obvious way. We have a functor $\gamma : \mathscr{C} \to \overline{\mathscr{C}}$ that is the identity on objects, and such that $\gamma(f) := Ff$ for every morphism f of $\mathscr{C}$; then F is the composition of γ and a unique functor $\overline{F} : \overline{\mathscr{C}} \to \mathscr{D}$ such that $\overline{F}X := FX$ and $\overline{F}g := g$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $g \in \mathrm{Mor}(\overline{\mathscr{C}})$. By construction, $\overline{F}$ is fully faithful. Moreover, let $\eta_\bullet : \mathbf{1}_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon_\bullet : FG \Rightarrow \mathbf{1}_{\mathscr{D}}$ the unit and respectively the counit of an adjunction for (F, G); since G is fully faithful, $\varepsilon_\bullet$ is an isomorphism of functors ([13, Prop.2.16(iii)]), *i.e.* $\varepsilon_A : FGA \to A$ is an isomorphism for every $A \in \mathrm{Ob}(\mathscr{D})$, and especially, $\overline{F}$ is essentially surjective, so it is an equivalence of categories, as stated.

It remains only to check that $(\overline{\mathscr{C}}, \gamma)$ is a localization of $\mathscr{C}$ along Σ_F. Obviously, $\gamma(f)$ is an isomorphism of $\overline{\mathscr{C}}$, for every $f \in \Sigma_F$. Next, let $H : \mathscr{C} \to \mathscr{C}'$ be any functor such that Hf is an isomorphism of $\mathscr{C}'$ for every $f \in \Sigma_F$; we must exhibit a unique functor

$\overline{H} : \overline{\mathscr{C}} \to \mathscr{C}'$ such that $H = \overline{H} \circ \gamma$. Clearly, we must then set $\overline{H}X := HX$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Notice moreover that for every such X, we have

$$\varepsilon_{FX} \circ F\eta_X = \mathbf{1}_{FX} \qquad \text{in } \mathscr{D}$$

by the triangular identities for $(\eta_\bullet, \varepsilon_\bullet)$ ([13, Prob.2.13(ii)]). We have already noticed that ε_{FX} is an isomorphism of $\mathscr{D}$, so the same holds for $F\eta_X$, *i.e.* $\eta_X \in \Sigma_F$, and we get an isomorphism :

$$(*) \qquad \varepsilon_{FX} : GFX \xrightarrow{\sim} X \quad \text{in } \overline{\mathscr{C}} \quad \text{such that} \quad \varepsilon_{FX}^{-1} = \gamma(\eta_X).$$

In view of $(*)$, every morphism $g : X \to Y$ of $\overline{\mathscr{C}}$ (*i.e.* every $g \in \mathscr{D}(FX, FY)$) then yields the following commutative diagram in $\overline{\mathscr{C}}$:

$$(**) \qquad \begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle \gamma(\eta_X)}\downarrow & & \downarrow{\scriptstyle \gamma(\eta_Y)} \\ GFX & \xrightarrow{\gamma Gg} & GFY. \end{array}$$

Hence, for every such g we set $\overline{H}g := (H\eta_Y)^{-1} \circ HGg \circ H\eta_X : \overline{H}X \to \overline{H}Y$.

It is easily seen that $\overline{H}$ is a well-defined functor $\overline{\mathscr{C}} \to \mathscr{C}'$, and we get :

$$\overline{H} \circ \gamma(f) = (H\eta_Y)^{-1} \circ HGFf \circ H\eta_X = (H\eta_Y)^{-1} \circ H\eta_Y \circ Hf = Hf \qquad \forall f \in \mathscr{C}(X, Y)$$

so that $\overline{H} \circ \gamma = H$. Lastly, if $K : \overline{\mathscr{C}} \to \mathscr{C}'$ is any functor with $H = K \circ \gamma$, we apply K termwise to the diagram $(**)$ to easily deduce that $Kg = \overline{H}g$, and the proof is complete. □

Lemma 1.11.7. *Let $\mathscr{C}, \mathscr{D}$ be two categories, and $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ such that the localization $(\mathscr{C}[\Sigma^{-1}], \gamma)$ exists. Let also $F, G : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}$ be two functors, and $\tau_\bullet : F \circ \gamma \Rightarrow G \circ \gamma$ a natural transformation. Then we have :*

(i) There exists a unique natural transformation $\beta_\bullet : F \Rightarrow G$ such that $\tau_\bullet = \beta_\bullet \star \gamma$.

(ii) Especially, if $\mathscr{C}$ is small, γ induces a fully faithful functor

$$\mathsf{Fun}(\gamma, \mathscr{D}) : \mathsf{Fun}(\mathscr{C}[\Sigma^{-1}], \mathscr{D}) \to \mathsf{Fun}(\mathscr{C}, \mathscr{D}).$$

Proof. Clearly (ii) follows from (i). Now, the datum of $\tau_\bullet$ is equivalent to that of a functor

$$T : \mathscr{C} \to \mathscr{D}^{[1]} \qquad \text{such that} \qquad \mathrm{e}_0 \circ T = F \circ \gamma \quad \text{and} \quad \mathrm{e}_1 \circ T = G \circ \gamma$$

where $[1]$ is defined as in remark 1.9.3(iii), and $\mathrm{e}_0, \mathrm{e}_1 : \mathscr{D}^{[1]} \to \mathscr{D}$ are the evaluation functors, as in §1.3. Namely, we set $TX := (F\gamma X \xrightarrow{\tau_X} G\gamma X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Notice that a morphism g of $\mathscr{D}^{[1]}$ is an isomorphism if and only if both $\mathrm{e}_0(g)$ and $\mathrm{e}_1(g)$ are isomorphisms of $\mathscr{D}$; then clearly T maps every $f \in \Sigma$ to an isomorphism Tf of $\mathscr{D}^{[1]}$, so it factors through γ and a unique functor

$$T' : \mathscr{C}[\Sigma^{-1}] \to \mathscr{D}^{[1]} \qquad \text{such that} \qquad \mathrm{e}_0 \circ T' = F \quad \text{and} \quad \mathrm{e}_1 \circ T' = G.$$

In turns, T' is equivalent to the datum of a natural transformation $\beta_\bullet : F \Rightarrow G$ with the sought property. □

Proposition 1.11.8. *Let $F : \mathscr{A} \xrightarrow{\sim} \mathscr{B}$ be an equivalence of wide categories, $\Sigma \subset \mathrm{Mor}(\mathscr{A})$ a subclass, and $\Delta := F(\Sigma) \subset \mathrm{Mor}(\mathscr{B})$. We have :*

(i) F induces an equivalence of wide categories $\mathscr{A}\langle\Sigma^{-1}\rangle \xrightarrow{\sim} \mathscr{B}\langle\Delta^{-1}\rangle$.

(ii) If $\mathscr{A}$ and $\mathscr{B}$ are categories, and if the localization $(\mathscr{A}[\Sigma^{-1}], \gamma_{\mathscr{A}})$ exists, then the same holds for the localization $(\mathscr{B}[\Delta^{-1}], \gamma_{\mathscr{B}})$, and F induces an equivalence $\mathscr{A}[\Sigma^{-1}] \xrightarrow{\sim} \mathscr{B}[\Delta^{-1}]$.

Proof. (i): Let $G : \mathscr{B} \xrightarrow{\sim} \mathscr{A}$ be a quasi-inverse for F, and $\eta_\bullet : \mathbf{1}_\mathscr{B} \xrightarrow{\sim} FG$, $\varepsilon_\bullet : GF \xrightarrow{\sim} \mathbf{1}_\mathscr{A}$ the unit and counit of an adjunction for the pair (G, F) (so $\eta_\bullet$ and $\varepsilon_\bullet$ are isomorphisms of functors : see [13, Prop.2.16(iii)]). We consider the wide category $\overline{\mathscr{B}}$ such that $\mathrm{Ob}(\overline{\mathscr{B}}) := \mathrm{Ob}(\mathscr{B})$, and with $\overline{\mathscr{B}}(X, Y) := \mathscr{A}\langle\Sigma^{-1}\rangle(\gamma_\mathscr{A} GX, \gamma_\mathscr{A} GY)$ for every $X, Y \in \mathrm{Ob}(\mathscr{B})$; the composition law of $\overline{\mathscr{B}}$ is induced by that of $\mathscr{A}\langle\Sigma^{-1}\rangle$ in the obvious fashion. We have a functor

$$\gamma_\mathscr{B} : \mathscr{B} \to \overline{\mathscr{B}} \qquad X \mapsto X \qquad (f : X \to Y) \mapsto \gamma_\mathscr{A} Gf$$

and we claim that $\gamma_\mathscr{B}(\Delta) \subset \mathrm{Isom}(\overline{\mathscr{B}})$. Indeed, for every $(f : X \to Y) \in \mathrm{Mor}(\mathscr{A})$ we have

$$\gamma_\mathscr{B}(Ff) = \gamma_\mathscr{A}(\varepsilon_Y)^{-1} \circ \gamma_\mathscr{A}(f) \circ \gamma_\mathscr{A}(\varepsilon_X) : \gamma_\mathscr{A} GFX \to \gamma_\mathscr{A} GFY$$

so $\gamma_\mathscr{B}(Ff)$ is an isomorphism whenever $f \in \Sigma$, whence the assertion.

• Next, let $H : \mathscr{B} \to \mathscr{C}$ be any functor of wide categories such that $H(\Delta) \subset \mathrm{Isom}(\mathscr{C})$; hence $HF(\Sigma) \subset \mathrm{Isom}(\mathscr{C})$, so there exists a unique functor $K : \mathscr{A}\langle\Sigma^{-1}\rangle \to \mathscr{C}$ such that $K\gamma_\mathscr{A} = HF$. Especially, for the case $H = \gamma_\mathscr{B}$, we denote by $\overline{F} : \mathscr{A}\langle\Sigma^{-1}\rangle \to \overline{\mathscr{B}}$ the unique functor such that $\overline{F}\gamma_\mathscr{A} = \gamma_\mathscr{B} F$. We define a functor $\overline{H} : \overline{\mathscr{B}} \to \mathscr{C}$ as follows. For every $X, Y \in \mathrm{Ob}(\mathscr{B})$ we set $\overline{H}X := HX$, and for every $\phi \in \mathscr{A}\langle\Sigma^{-1}\rangle(\gamma_\mathscr{A} GX, \gamma_\mathscr{A} GY)$ we let

$$\overline{H}\phi := H(\eta_Y)^{-1} \circ K(\phi) \circ H(\eta_X) : HX \to HY.$$

We claim that $\overline{H}\gamma_\mathscr{B} = H$. Indeed, $\overline{H}\gamma_\mathscr{B}(X) = HX$ for every $X \in \mathrm{Ob}(\mathscr{B})$; then, if $\phi = \gamma_\mathscr{B}(f)$ for $f \in \mathscr{B}(X, Y)$, we get $K(\phi) = K\gamma_\mathscr{A} Gf = HFGf$, so that $\overline{H}\phi = Hf$, as stated.

• Lastly, let $\overline{H}' : \overline{\mathscr{B}} \to \mathscr{C}$ be another functor with $\overline{H}'\gamma_\mathscr{B} = H$. We need to show that $\overline{H}' = \overline{H}$, and since $\gamma_\mathscr{B}$ is the identity on objects, we see already that $\overline{H}'X = \overline{H}X$ for every $X \in \mathrm{Ob}(\mathscr{B})$; we are then reduced to checking :

Claim 1.11.9. For every $X, Y \in \mathrm{Ob}(\mathscr{B})$, every $\phi \in \overline{\mathscr{B}}(X, Y)$ can be written as a composition

$$X \xrightarrow{\gamma_\mathscr{B}(f_0)} B_0 \xrightarrow{\gamma_\mathscr{B}(g_0)^{-1}} C_1 \xrightarrow{\gamma_\mathscr{B}(f_1)} B_1 \xrightarrow{\gamma_\mathscr{B}(g_1)^{-1}} \cdots \xrightarrow{\gamma_\mathscr{B}(f_n)} Y$$

for some $n \in \mathbb{N}$, where $f_0, \dots, f_n$ are morphisms of $\mathscr{B}$ and $g_0, \dots, g_{n-1} \in \Delta$.

Proof : By definition, $\phi \in \mathscr{A}\langle\Sigma^{-1}\rangle(\gamma_\mathscr{A} GX, \gamma_\mathscr{A} GY)$, so it can be written as a composition

$$GX \xrightarrow{\gamma_\mathscr{A}(h_0)} Y_0 \xrightarrow{\gamma_\mathscr{A}(s_0)^{-1}} X_1 \xrightarrow{\gamma_\mathscr{A}(h_1)} Y_1 \xrightarrow{\gamma_\mathscr{A}(s_1)^{-1}} \cdots \xrightarrow{\gamma_\mathscr{A}(h_n)} GY$$

for some $n \in \mathbb{N}$, where $h_0, \dots, h_n$ are morphisms of $\mathscr{A}$ and $s_0, \dots, s_{n-1} \in \Sigma$ (remark 1.11.3(ii)). Recall that $G\eta_B = \varepsilon_{GB}^{-1}$ for every $B \in \mathrm{Ob}(\mathscr{B})$ ([13, Prob.2.13(ii)]); we deduce the commutative diagram of $\mathscr{A}$:

$$\begin{tikzcd}
GFGX \arrow[r, "GFh_0"] & GFY_0 \arrow[d, "\varepsilon_{Y_0}"] & GFX_1 \arrow[l, "GFs_0"'] \arrow[r, "GFh_1"] \arrow[d, "\varepsilon_{X_1}"] & GFY_1 \arrow[d, "\varepsilon_{Y_1}"] & \cdots \arrow[l, "GFs_1"'] \arrow[r, "GFh_n"] & GFGY \arrow[d, "G(\eta_Y^{-1})"] \\
GX \arrow[u, "G\eta_X"] \arrow[r, "h_0"] & Y_0 & X_1 \arrow[l, "s_0"'] \arrow[r, "h_1"] & Y_1 & \cdots \arrow[l, "s_1"'] \arrow[r, "h_n"] & GY
\end{tikzcd}$$

all whose vertical arrows are isomorphisms. Hence, we can take $B_i := FY_i$, $C_{i+1} := FX_{i+1}$ for $i = 0, \dots, n-1$, and $f_0 := Fs_0 \circ \eta_X$, $f_n := \eta_Y^{-1} \circ Fh_n$, $f_i := Fh_i$ for $i = 1, \dots, n-1$, and $g_i := Fs_i$ for $i = 0, \dots, n-1$. ◇

With claim 1.11.9 we conclude that $(\overline{\mathscr{B}}, \gamma_\mathscr{B})$ is the wide localization of $\mathscr{B}$ along Δ; the remaining assertion of (i) follows now from :

Claim 1.11.10. $\overline{F}$ is an equivalence of wide categories.

Proof : Clearly $\overline{F}\gamma_{\mathscr{A}}X = FX$ for every $X \in \mathrm{Ob}(\mathscr{A})$, and we show that :

$$\overline{F}\psi = \gamma_{\mathscr{A}}(\varepsilon_X) \circ \psi \circ \gamma_{\mathscr{A}}(\varepsilon_Y)^{-1} : \gamma_{\mathscr{A}}GFX \to \gamma_{\mathscr{A}}GFY \qquad \forall \psi \in \mathscr{A}\langle\Sigma^{-1}\rangle(\gamma_{\mathscr{A}}X, \gamma_{\mathscr{A}}Y).$$

Indeed, there exists a functor $\overline{F}' : \mathscr{A}\langle\Sigma^{-1}\rangle \to \overline{\mathscr{B}}$ such that $\overline{F}'\gamma_{\mathscr{A}}X := FX$ for every $X \in \mathrm{Ob}(\mathscr{A})$, and with $\overline{F}'\psi := \gamma_{\mathscr{A}}(\varepsilon_X) \circ \psi \circ \gamma_{\mathscr{A}}(\varepsilon_Y)^{-1}$ for every ψ as in the foregoing, and it is easily seen that $\overline{F}'(\gamma_{\mathscr{A}}f) = \gamma_{\mathscr{B}}Ff$ for every $f \in \mathscr{A}(X,Y)$, whence $\overline{F}' = \overline{F}$, by the uniqueness property of $\overline{F}$. It follows that $\overline{F}$ is fully faithful, and since F is essentially surjective, we also easily deduce that the same holds for $\overline{F}$, whence the assertion. ◇

(ii): If $\mathscr{A}$ and $\mathscr{B}$ are categories and if $\mathscr{A}[\Sigma^{-1}]$ exists, the proof of (i) yields, *mutatis mutandis,* the construction of $\mathscr{B}[\Delta^{-1}]$, whence the claim (details left to the reader). □

The following result, that I learned from O.Gabber, shows that every wide category can be realized, up to equivalence, as a wide localization of some (large) category.

Proposition 1.11.11. *For every wide category $\mathscr{C}$ there exists a category $\mathscr{B}$, a class $\Sigma \subset \mathrm{Mor}(\mathscr{B})$ and an equivalence of wide categories $\mathscr{C} \xrightarrow{\sim} \mathscr{B}\langle\Sigma^{-1}\rangle$ that is injective on objects.*

Proof. Let T be the class of all composable pairs (f,g) of morphisms of $\mathscr{C}$, *i.e.* such that the target of f equals the source of g; denote by $\mathscr{B}_0$ be the discrete category with

$$\mathrm{Ob}(\mathscr{B}_0) := \{X_c \mid c \in \mathrm{Ob}(\mathscr{C})\} \sqcup \{Y_f \mid f \in \mathrm{Mor}(\mathscr{C})\} \sqcup \{Z_{f,g} \mid (f,g) \in T\}.$$

- Let $\mathscr{B}_1$ be the category obtained by adding to $\mathscr{B}_0$ the family of morphisms :

$$X_{\mathsf{s}(f)} \xleftarrow{\sigma(f)} Y_f \xrightarrow{\tau(f)} X_{\mathsf{t}(f)} \qquad \forall f \in \mathrm{Mor}(\mathscr{C})$$

where $\mathsf{s}(f)$ and $\mathsf{t}(f)$ are respectively the source and target of f (see definition 1.2.3(i)), and we also require that $\sigma(f) = \tau(f)$ if and only if $f = \mathbf{1}_c$ for some $c \in \mathrm{Ob}(\mathscr{C})$. Notice that for every composable pair (ϕ,ψ) of morphisms of $\mathscr{B}_1$, either ϕ or ψ is an identity morphism, so there exists a unique (trivial) composition law on such composable pairs verifying the usual associativity and unit axioms.

- Next, let $\mathscr{B}$ be the category obtained by adding to $\mathscr{B}_1$ the family of morphisms :

$$\begin{tikzcd}
X_{\mathsf{s}(f)} & X_{\mathsf{t}(f)} = X_{\mathsf{s}(g)} & X_{\mathsf{t}(g)} \\
& Z_{f,g} \arrow[ul, "\lambda^*(f,g)"] \arrow[u, "\mu^*(f,g)"'] \arrow[ur, "\rho^*(f,g)"'] \arrow[dl, "\lambda(f,g)"'] \arrow[d, "\mu(f,g)"] \arrow[dr, "\rho(f,g)"] & \\
Y_f & Y_{gf} & Y_g
\end{tikzcd} \qquad \forall (f,g) \in T$$

where $\lambda(f,g)$, $\mu(f,g)$, $\rho(f,g)$ are all distinct, and we require the identities :

$$\sigma(f) \circ \lambda(f,g) = \lambda^*(f,g) = \sigma(gf) \circ \mu(f,g)$$
$$\tau(f) \circ \lambda(f,g) = \mu^*(f,g) = \sigma(g) \circ \rho(f,g)$$
$$\tau(gf) \circ \mu(f,g) = \rho^*(f,g) = \tau(g) \circ \rho(f,g)$$

so that $\lambda^*(f,g) = \mu^*(f,g)$ (resp. $\lambda^*(f,g) = \rho^*(f,g)$, resp. $\mu^*(f,g) = \rho^*(f,g)$) if and only if $f = \mathbf{1}_c$ (resp. if and only if $gf = \mathbf{1}_c$, resp. if and only if $g = \mathbf{1}_c$) for some $c \in \mathrm{Ob}(\mathscr{C})$. Notice that for every composable triple (ϕ,ψ,ξ) of morphisms of $\mathscr{B}$ (*i.e.* such that both (ϕ,ψ) and (ψ,ξ) are composable pairs), either ϕ, ψ or ξ is an identity morphism, so that the foregoing identities uniquely determine a composition law on $\mathscr{B}$ fulfilling the usual associativity and unit axioms (details left to the reader).

- Let $(\mathscr{B}\langle\Sigma^{-1}\rangle, \gamma)$ be the wide localization of $\mathscr{B}$ along the class

$$\Sigma := \{\sigma(f) \mid f \in \mathrm{Mor}(\mathscr{C})\} \cup \{\lambda(f,g), \mu(f,g), \rho(f,g) \mid (f,g) \in T\}.$$

We define a functor $F : \mathscr{C} \to \mathscr{B}\langle\Sigma^{-1}\rangle$ by the rules :

$$c \mapsto X_c \qquad f \mapsto \gamma(\tau(f)) \circ \gamma(\sigma(f))^{-1} \qquad \forall c \in \mathrm{Ob}(\mathscr{C}), \forall f \in \mathrm{Mor}(\mathscr{C}).$$

Let us check that $F\mathbf{1}_c = \mathbf{1}_{Fc}$ for every $c \in \mathrm{Ob}(\mathscr{C})$ and that $F(g \circ f) = Fg \circ Ff$ for every $(f,g) \in T$. The first identity is clear, as $\sigma(\mathbf{1}_c) = \tau(\mathbf{1}_c)$; for the second identity, we compute:

$$\begin{aligned}
F(g \circ f) &= \gamma(\tau(gf)) \circ \gamma(\sigma(gf))^{-1} \\
&= \gamma(\tau(gf)) \circ \gamma(\mu(f,g)) \circ \gamma(\mu(f,g))^{-1} \circ \gamma(\sigma(gf))^{-1} \\
&= \gamma(\tau(g)) \circ \gamma(\rho(f,g)) \circ \gamma(\lambda(f,g))^{-1} \circ \gamma(\sigma(f))^{-1} \\
&= \gamma(\tau(g)) \circ \gamma(\sigma(g))^{-1} \circ \gamma(\sigma(g)) \circ \gamma(\rho(f,g)) \circ \gamma(\lambda(f,g))^{-1} \circ \gamma(\sigma(f))^{-1} \\
&= Fg \circ \gamma(\tau(f)) \circ \gamma(\lambda(f,g)) \circ \gamma(\lambda(f,g))^{-1} \circ \gamma(\sigma(f))^{-1} \\
&= Fg \circ Ff.
\end{aligned}$$

- We define as well a functor $G : \mathscr{B} \to \mathscr{C}$ by the rules :

$$\begin{array}{rcl}
X_c \mapsto c & \qquad & Y_f, Z_{f,g} \mapsto \mathrm{s}(f) \\
\sigma(f), \lambda(f,g), \mu(f,g) \mapsto \mathbf{1}_{\mathrm{s}(f)} & \qquad & \tau(f), \rho(f,g) \mapsto f
\end{array}$$

for every $c \in \mathrm{Ob}(\mathscr{C})$, every $f \in \mathrm{Mor}(\mathscr{C})$ and every $(f,g) \in T$ (and of course, with $G\mathbf{1}_B := \mathbf{1}_{GB}$ for every $B \in \mathrm{Ob}(\mathscr{B})$). Clearly $G(\Sigma) \subset \mathrm{Isom}(\mathscr{C})$, so G factors uniquely through γ and a functor $\overline{G} : \mathscr{B}\langle\Sigma^{-1}\rangle \to \mathscr{C}$.

- Clearly $\overline{G} \circ F = \mathbf{1}_{\mathscr{C}}$. Moreover, we get a natural transformation $\omega_\bullet : F \circ G \Rightarrow \gamma$ by :

$$X_c \mapsto \gamma(\mathbf{1}_{X_c}) \qquad Y_f \mapsto \gamma(\sigma(f)) \qquad Z_{f,g} \mapsto \gamma(\lambda^*(f,g))$$

for every $c \in \mathrm{Ob}(\mathscr{C})$, every $f \in \mathrm{Mor}(\mathscr{C})$ and every $(f,g) \in T$ (details left to the reader). Then, by lemma 1.11.7, there exists a unique natural transformation $\overline{\omega}_\bullet : F \circ \overline{G} \to \mathbf{1}_{\mathscr{B}\langle\Sigma^{-1}\rangle}$ such that $\omega_\bullet = \overline{\gamma} \star \gamma$, and it follows easily that $\overline{\omega}_\bullet$ is an isomorphism of functors, so F and $\overline{G}$ are equivalences of wide categories, whence the proposition. □

1.11.12. A *groupoid* (resp. a *wide groupoid*) is a category (resp. a wide category) in which every morphism is an isomorphism. The class of *small groupoids* (that is, small categories that are groupoids) forms a full subcategory

$$\mathsf{Grp}$$

of the category Cat of small categories, and the inclusion functor $i : \mathsf{Grp} \to \mathsf{Cat}$ admits both a left adjoint $\pi_1 : \mathsf{Cat} \to \mathsf{Grp}$ and a right adjoint $k : \mathsf{Cat} \to \mathsf{Grp}$. Indeed, for any wide category $\mathscr{C}$, let $k(\mathscr{C}) \subset \mathscr{C}$ be the wide subcategory with $\mathrm{Ob}(k(\mathscr{C})) = \mathrm{Ob}(\mathscr{C})$ and such that the morphisms of $k(\mathscr{C})$ are the isomorphisms of $\mathscr{C}$. Hence $k(\mathscr{G})$ is the largest wide groupoid contained in $\mathscr{C}$. Clearly, for every wide groupoid $\mathscr{G}$, every functor $\mathscr{G} \to \mathscr{C}$ factors uniquely through the inclusion $k(\mathscr{C}) \to \mathscr{C}$; especially, every functor $F : \mathscr{B} \to \mathscr{C}$ induces by restriction a functor $k(F) : k(\mathscr{B}) \to k(\mathscr{C})$. Hence, by restricting the construction k to small categories, we do obtain a well-defined right adjoint for i.

- Next, for every small category $\mathscr{C}$ we set

$$\pi_1(\mathscr{C}) := \mathscr{C}[\mathrm{Mor}(\mathscr{C})^{-1}].$$

We see easily from remark 1.11.3(ii) that $\pi_1(\mathscr{C})$ is a groupoid, and clearly for every groupoid $\mathscr{G}$, every functor $\mathscr{C} \to \mathscr{G}$ factors uniquely through the localization $\gamma_{\mathscr{C}} : \mathscr{C} \to \pi_1(\mathscr{C})$; especially, every functor $F : \mathscr{B} \to \mathscr{C}$ between small categories induces a unique functor $\pi_1(F) : \pi_1(\mathscr{B}) \to \pi_1(\mathscr{C})$ such that $\pi_1(F) \circ \gamma_{\mathscr{B}} = \gamma_{\mathscr{C}} \circ F$. It follows easily that the rules $\mathscr{C} \mapsto \pi_1(\mathscr{C})$ and $F \mapsto \pi_1(F)$ yield the sought left adjoint for i.

1.12. **Calculus of fractions.** The following definition – inspired by analogous considerations arising in the study of non-commutative rings – yields natural conditions for the existence of localizations of categories, that can be applied in a variety of useful situations.

Definition 1.12.1. Let $\mathscr{C}$ be a wide category, and $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ a given subclass.

(i) We say that Σ *admits a right calculus of fractions*, if the following holds :

(CF1) $\mathbf{1}_X \in \Sigma$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

(CF2) For every $s : X \to Y$ and $t : Y \to Z$ with $s, t \in \Sigma$, we have $t \circ s \in \Sigma$.

(CF3) For every morphism $f : X \to Y$ in $\mathscr{C}$ and every $s : W \to Y$ in Σ, there exists $g : Z \to W$ in $\mathscr{C}$ and $t : Z \to X$ in Σ such that $f \circ t = s \circ g$.

(CF4) If $f, g : X \rightrightarrows Y$ are any two morphisms in $\mathscr{C}$ such that $s \circ f = s \circ g$ for some $s : Y \to Z$ in Σ, then there exists $t : W \to X$ in Σ such that $f \circ t = g \circ t$.

(ii) For every $X \in \mathrm{Ob}(\mathscr{C})$, let Σ_X be the full wide subcategory of $\mathscr{C}/X$ whose objects are the elements of Σ with target equal to X. We say that Σ *is right cofinally small*, if Σ_X^{op} is cofinally small for every $X \in \mathrm{Ob}(\mathscr{C})$ (see definition 1.5.1(iv)).

(iii) We say that Σ *admits a left calculus of fractions,* if the subclass Σ^{op} admits a right calculus of fractions in $\mathscr{C}^{\mathrm{op}}$. We say that Σ *is left cofinally small*, if $(\Sigma^{\mathrm{op}})_X$ is coinitially small for every $X \in \mathrm{Ob}(\mathscr{C})$.

Example 1.12.2. In the situation of proposition 1.11.6, suppose that G is a right (resp. left) adjoint to F. Then the class Σ_F admits a left (resp. right) calculus of fraction. For the proof, by duality, we may assume that G is right adjoint to F, and we let $(\eta_\bullet, \varepsilon_\bullet)$ be the unit and counit of an adjunction for the pair (F, G). Now, (CF1) and (CF2) trivially hold for Σ_F^{op}. Next, let $f : X \to Y$ be a morphism of $\mathscr{C}$ and $s : X \to W$ an element of Σ_F, so that Fs is an isomorphism of $\mathscr{D}$; we set $g := GF(f) \circ GF(s)^{-1} \circ \eta_W : W \to Z := GFY$, and $t := \eta_Y : Y \to Z$. By the triangular identities, $\varepsilon_{FY} \circ Ft = \mathbf{1}_{FY}$, and on the other hand, ε_{FY} is an isomorphism, since G is fully faithful; hence Ft is an isomorphism, *i.e.* $t \in \Sigma_F$. To see that Σ_F^{op} fulfills (FC3), it then suffices to check that $t \circ f = g \circ s$; we compute :

$$g \circ s = GF(f) \circ GF(s)^{-1} \circ \eta_W \circ s = GF(f) \circ GF(s)^{-1} \circ GF(s) \circ \eta_X = GF(f) \circ \eta_X = t \circ f$$

as required. Lastly, let $f, g : X \rightrightarrows Y$ be two morphisms of $\mathscr{C}$ such that $f \circ s = g \circ s$ for some $s \in \Sigma_F$; it follows easily that $Ff = Fg$, whence :

$$\eta_Y \circ f = GF(f) \circ \eta_X = GF(g) \circ \eta_X = \eta_Y \circ g$$

and arguing as in the foregoing we see that $\eta_Y \in \Sigma_F$; this shows that (CF4) holds for Σ_F^{op}.

1.12.3. Let $\mathscr{C}$ and Σ be as in definition 1.12.1(i); we consider the functors

$$H_{X,Y} : \Sigma_X^{\mathrm{op}} \to \mathrm{Set} \qquad (Z \xrightarrow{t} X) \mapsto \{t\} \times \mathscr{C}(Z, Y) \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C})$$

that assign to every morphism $h/X : (Z' \xrightarrow{t \circ h} X) \to (Z \xrightarrow{t} X)$ of Σ_X the map

$$H_{X,Y}(h) : H_{X,Y}(t) \to H_{X,Y}(t \circ h) \qquad (t, f) \mapsto (t \circ h, f \circ h).$$

We denote the global colimit of $H_{X,Y}$ by :

$$[X, Y]_r$$

and for every $(t : Z \to X) \in \Sigma$ and every $f \in \mathscr{C}(Z, Y)$ we let $[t, f]_r \in [X, Y]_r$ be the image of $(t, f) \in H_{X,Y}(t)$ under the global co-cone $H_{X,Y} \Rightarrow c_{[X,Y]_r}$.

Proposition 1.12.4. *Let $\mathscr{C}$ be a wide category, and $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ the wide localization of $\mathscr{C}$ along a subclass $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ that admits a right calculus of fraction. We have :*

(i) The wide category Σ_X is cofiltered, for every $X \in \mathrm{Ob}(\mathscr{C})$.

(ii) For every $X, Y \in \mathrm{Ob}(\mathscr{C})$ we have a natural bijection :

$$\boxed{[X, Y]_r \xrightarrow{\sim} \mathscr{C}\langle\Sigma^{-1}\rangle(\gamma X, \gamma Y) \qquad [t, f]_r \mapsto \gamma(f) \circ \gamma(t)^{-1}.}$$

Proof. (i): First, $\mathrm{Ob}(\Sigma_X) \neq \varnothing$, due to (CF1). Next, let $(Z \xrightarrow{t} X), (Z' \xrightarrow{t'} X) \in \mathrm{Ob}(\Sigma_X)$; by virtue of (CF3), we find $Z'' \in \mathrm{Ob}(\mathscr{C})$, $f \in \mathscr{C}(Z'', Z)$, and an element $g : Z'' \to Z'$ of Σ such that $t \circ f = t' \circ g$. Then, (CF2) says that $t' \circ g$ lies in Σ, so it defines an object t'' of Σ_X, with morphisms $f/X : t'' \to t$ and $g/X : t'' \to t'$ in Σ_X. Lastly, let $h/X, k/X : t' \rightrightarrows t$ be any two morphisms in Σ_X. By (CF4), we may find $s : Z'' \to Z$' in Σ such that $h \circ s = k \circ s$; clearly $s/X : (Z'' \xrightarrow{t' \circ s} X) \to (Z' \xrightarrow{t'} X)$ is a morphism in Σ_X equalizing h/X and k/X.

(ii): By virtue of (i) and example 1.2.13(iii), for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ the global colimit of $H_{X,Y}$ is represented by the class $[X, Y]_r$ of equivalence classes $[t, f]_r$ of all pairs

$$(t, f) \in T_{X,Y} := \bigcup_{t \in \mathrm{Ob}(\Sigma_X)} H_{X,Y}(t)$$

for the equivalence relation $\sim$ such that $(t, f) \sim (t', f') \Leftrightarrow$ there exist $(Z'' \xrightarrow{t''} X) \in \mathrm{Ob}(\Sigma_X)$ and morphisms $g/X : t'' \to t$ and $g'/X : t'' \to t'$ of Σ_X such that $f \circ g = f' \circ g'$. Let us then define a wide category $\mathscr{D}$ such that $\mathrm{Ob}(\mathscr{D}) = \mathrm{Ob}(\mathscr{C})$, and

$$\mathscr{D}(X, Y) := [X, Y]_r \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C}).$$

The composition law of $\mathscr{D}$ is as follows. Let $A, B, C \in \mathrm{Ob}(\mathscr{C})$, $(B \xleftarrow{f} I \xrightarrow{s} A) \in T_{A,B}$ and $(C \xleftarrow{g} J \xrightarrow{t} B) \in T_{B,C}$; by (CF3) we may find $K \xrightarrow{\tau} I$ in Σ and $K \xrightarrow{\phi} J$ in $\mathscr{C}$ with $f \circ \tau = t \circ \phi$, so that we get the commutative diagram :

$$\begin{tikzcd} & & K \arrow[dl, "\phi"'] \arrow[dr, "\tau"] & & \\ C & J \arrow[l, "g"'] \arrow[r, "t"] & B & I \arrow[l, "f"'] \arrow[r, "s"] & A \end{tikzcd}$$

where $s \circ \tau \in \Sigma$, by (CF2), and we set :

$$(t, g) \circ (s, f) := [s \circ \tau, g \circ \phi]_r \in [A, C]_r.$$

Claim 1.12.5. The class $(t, g) \circ (s, f)$ does not depend on the choices of K, τ and ϕ, and depends only on the classes $[s, f]_r$ and $[t, g]_r$.

Proof : Indeed, suppose that $K' \xrightarrow{\tau'} I$ is another element of Σ and $K' \xrightarrow{\phi'} J$ is another morphism of $\mathscr{C}$ such that $f \circ \tau' = t \circ \phi'$. By (CF3), there exist $K'' \xrightarrow{h} K$ in Σ and $K'' \xrightarrow{h'} K'$ in $\mathscr{C}$ with $\tau \circ h = \tau' \circ h'$, whence :

$$t \circ \phi \circ h = f \circ \tau \circ h = f \circ \tau' \circ h' = t \circ \phi' \circ h'.$$

Then, by (CF4) there exists $K''' \xrightarrow{u} K''$ in Σ such that $\phi \circ h \circ u = \phi' \circ h' \circ u$. By (CF2), we may then replace K'' by K''' and h, h' with $h \circ u$ and respectively $h' \circ u$, after which we may assume as well that $\phi' \circ h' = \phi \circ h$. Finally, we find :

$$[s \circ \tau, g \circ \phi]_r = [s \circ \tau \circ h, g \circ \phi \circ h]_r = [s \circ \tau' \circ h', g \circ \phi' \circ h']_r = [s \circ \tau', g \circ \phi']_r$$

as required. Next, let us check that $[s\circ\tau, g\circ\phi]_r$ depends only on $[s,f]_r$ and $[t,g]_r$; indeed, let $I' \xrightarrow{u} I$ be any morphism of $\mathscr{C}$ such that $s\circ u \in \Sigma$, and set $(s', f') := (s\circ u, f\circ u)$. Suppose also that $f'\circ\mu = t\circ\phi'$ for some $K' \xrightarrow{\mu} I'$ in Σ and $K' \xrightarrow{\phi'} J$ in $\mathscr{C}$; we need to show that $[s'\circ\mu, g\circ\phi']_r = [s\circ\tau, g\circ\phi]_r$. However, by (CF3) we find $K'' \xrightarrow{l} K$ in $\mathscr{C}$ and $K'' \xrightarrow{l'} K'$ in Σ such that $u\circ\mu\circ l' = \tau\circ l$; hence, $t\circ\phi'\circ l' = f\circ u\circ\mu\circ l' = f\circ\tau\circ l = t\circ\phi\circ l$, so that, by (CF4) there exists $K''' \xrightarrow{l''} K''$ in Σ such that $\phi'\circ l'\circ l'' = \phi\circ l\circ l''$. Since $[s'\circ\mu, g\circ\phi']_r = [s'\circ\mu\circ l'\circ l'', g\circ\phi'\circ l'\circ l'']_r$, we may then replace K', μ and ϕ respectively by K''', $\mu\circ l'\circ l''$ and $\phi'\circ l'\circ l''$; then we get the commutative diagram :

$$\begin{tikzcd}
& K' \arrow[r, "\mu"] \arrow[d, "u'"'] & I' \arrow[d, "u"] \arrow[dr, "s'"] & \\
& K \arrow[r, "\tau"] \arrow[d, "\phi"'] & I \arrow[r, "s"] \arrow[d, "f"] & A \\
C & J \arrow[l, "g"'] \arrow[r, "t"] & B &
\end{tikzcd}$$

with $u' := l\circ l''$ and $\phi' = \phi\circ u'$, whence $[s'\circ\mu, g\circ\phi']_r = [s\circ\tau\circ u', g\circ\phi\circ u']_r = [s\circ\tau, g\circ\phi]_r$ as required. Similarly, we check the independence on the representative for $[t,g]_r$: let $v : J' \to J$ be a morphism of $\mathscr{C}$ such that $t\circ v \in \Sigma$, let us set $(t', g') := (t\circ v, g\circ v)$, and pick also, by (CF3), $\mu : K' \to I$ in Σ and $\phi' : K' \to J'$ in $\mathscr{C}$ such that $f\circ\mu = t'\circ\phi'$; we need to show that $[s\circ\mu, g'\circ\phi']_r = [s\circ\tau, g\circ\phi]_r$. To this aim, we apply again (CF3) to find $l : K'' \to K$ in $\mathscr{C}$ and $l' : K'' \to K'$ in Σ such that $\mu\circ l' = \tau\circ l$; it follows that $t\circ v\circ\phi'\circ l' = f\circ\mu\circ l' = f\circ\tau\circ l = t\circ\phi\circ l$, so by (CF4) there exists $l'' : K''' \to K''$ such that $v\circ\phi'\circ l'\circ l'' = \phi\circ l\circ l''$. Set $v' := l\circ l''$ and $v'' := l'\circ l''$; then $[s\circ\mu, g'\circ\phi']_r = [s\circ\mu\circ v'', g'\circ\phi'\circ v'']_r$ and $[s\circ\tau, g\circ\phi]_r = [s\circ\tau\circ v', g\circ\phi\circ v']_r$, and on the other hand we have $s\circ\mu\circ v'' = s\circ\tau\circ l\circ l'' = s\circ\tau\circ v'$ and $g'\circ\phi'\circ v'' = g\circ v\circ\phi'\circ l'\circ l'' = g\circ\phi\circ v'$, whence the claim. ◇

By virtue of claim 1.12.5, for every $[s,f]_r$ and $[t,g]_r$ as in the foregoing, we may set

$$[t,g]_r \circ [s,f]_r := (t,g)\circ(s,f).$$

Next, in light of (CF1) we have $[\mathbf{1}_A, \mathbf{1}_A] \in [A,A]_r$ for every $A \in \mathrm{Ob}(\mathscr{C})$, and obviously $[\mathbf{1}_A, \mathbf{1}_A]_r \circ [s,f]_r = [s,f]_r = [s,f]_r \circ [\mathbf{1}_B, \mathbf{1}_B]_r$ for all $A, B \in \mathrm{Ob}(\mathscr{C})$ and all $[s,f]_r \in [B,A]_r$. Thus, it remains only to check the associativity of the composition law thus obtained. To this aim, say that $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$ are any four objects, and $(B \xleftarrow{f} I \xrightarrow{s} A) \in T_{A,B}$, $(C \xleftarrow{g} J \xrightarrow{t} B) \in T_{B,C}$, $(D \xleftarrow{h} K \xrightarrow{u} C) \in T_{C,D}$. Choose $L \in \mathrm{Ob}(\mathscr{C})$ with an element $L \xrightarrow{\tau} I$ of Σ and a morphism $L \xrightarrow{\phi} J$ of $\mathscr{C}$ with $f\circ\tau = t\circ\phi$, so that $[t,g]_r \circ [s,f]_r = [s\circ\tau, g\circ\phi]_r$. Then, choose $M \in \mathrm{Ob}(\mathscr{C})$, $M \xrightarrow{\nu} J$ in Σ and $M \xrightarrow{\psi} K$ in $\mathscr{C}$ with $g\circ\nu = u\circ\psi$, so that $[u,h]_r \circ [t,g]_r = [t\circ\nu, h\circ\psi]_r$. By (CF3) we may find $N \in \mathrm{Ob}(\mathscr{C})$, $N \xrightarrow{\mu} L$ in Σ and $N \xrightarrow{\lambda} M$ in $\mathscr{C}$ such that $\phi\circ\mu = \nu\circ\lambda$; summing up, we get the commutative diagram :

$$\begin{tikzcd}
& & & N \arrow[dl, "\lambda"'] \arrow[dr, "\mu"] & & & \\
& & M \arrow[dl, "\psi"'] \arrow[dr, "\nu"] & & L \arrow[dl, "\phi"'] \arrow[dr, "\tau"] & & \\
D & K \arrow[l, "h"'] \arrow[r, "u"] & C & J \arrow[l, "g"'] \arrow[r, "t"] & B & I \arrow[l, "f"'] \arrow[r, "s"] & A.
\end{tikzcd}$$

By (CF2), $\tau\circ\mu$ lies in Σ, and then it is easily seen that the pair $(s\circ\tau\circ\mu, h\circ\psi\circ\lambda)$ represents both $[t\circ v, h\circ\psi]_r\circ[s,f]_r$ and $[u,h]_r\circ[s\circ\tau, g\circ\phi]_r$, whence the assertion.

Next, we have a functor $\gamma:\mathscr{C}\to\mathscr{D}$ that is the identity on objects, and such that $\gamma(f):=[\mathbf{1}_X,f]_r$ for every morphism $f:X\to Y$ of $\mathscr{C}$. Notice that $[\mathbf{1}_Z,t]_r\circ[t,\mathbf{1}_Z]_r=[t,t]_r=[\mathbf{1}_X,\mathbf{1}_X]_r$ and $[t,\mathbf{1}_Z]_r\circ[\mathbf{1}_Z,t]_r=[\mathbf{1}_Z,\mathbf{1}_Z]_r$ for every $(Z\xrightarrow{t}X)\in\Sigma$, so $\gamma(t)$ is an isomorphism for every such t. Lastly, let $F:\mathscr{C}\to\mathscr{C}'$ be any functor of wide categories such that Ft is an isomorphism of $\mathscr{C}'$ for every $t\in\Sigma$; we set

$$F(t,f):=Ff\circ(Ft)^{-1}:FX\to FY \qquad \forall X,Y\in\mathrm{Ob}(\mathscr{C}),\forall(t,f)\in T_{X,Y}.$$

Claim 1.12.6. $F(t,f)$ depends only on the class $[t,f]_r$ of (t,f).

Proof: Indeed, say that $[t,f]_r=[t',f']_r$, for some $(Z\xrightarrow{t}X),(Z'\xrightarrow{t'}X)\in\Sigma$; then there exists $(Z''\xrightarrow{t''}X)\in\Sigma$ with morphisms $g/X:t''\to t$ and $g'/X:t''\to t'$ of Σ_X such that $f\circ g=f'\circ g'$, and notice that $Fg=(Ft)^{-1}\circ Ft''$ and $Fg'=(Ft')^{-1}\circ Ft''$ in $\mathscr{C}'$. It follows easily that $F(t,f)=F(f\circ g,t'')=F(f'\circ g',t'')=F(f',t')$, whence the assertion. ◇

In view of claim 1.12.6, we may set

$$\overline{F}[t,f]_r:=F(t,f) \qquad \forall A,B\in\mathrm{Ob}(\mathscr{D}),\forall[t,f]_r\in\mathscr{D}(A,B).$$

We claim that the system of maps thus obtained yelds a well-defined functor $\overline{F}:\mathscr{D}\to\mathscr{C}'$ with $\overline{F}X:=FX$ for every $X\in\mathrm{Ob}(\mathscr{D})$. Indeed, obviously $\overline{F}[\mathbf{1}_X,\mathbf{1}_X]_r=\mathbf{1}_{FX}$ for every $X\in\mathrm{Ob}(\mathscr{D})$, so it remains only to check that $\overline{F}([t,g]_r\circ[s,f]_r)=\overline{F}[t,g]_r\circ\overline{F}[s,f]_r$ for every $A,B,C\in\mathrm{Ob}(\mathscr{D})$, every $(B\xleftarrow{f}I\xrightarrow{s}A)\in T_{A,B}$, and every $(C\xleftarrow{g}J\xrightarrow{t}B)\in T_{B,C}$. Then, pick $K\xrightarrow{\tau}I$ in Σ and $K\xrightarrow{\phi}J$ in $\mathscr{C}$ with $f\circ\tau=t\circ\phi$, and notice that $(Ft)^{-1}\circ Ff=F\phi\circ(F\tau)^{-1}$ in $\mathscr{C}'$; consequently, $\overline{F}[s\circ\tau,g\circ\phi]_r=Fg\circ F\phi\circ(F\tau)^{-1}\circ(Fs)^{-1}=Fg\circ(Ft)^{-1}\circ Ff\circ(Fs)^{-1}$, whence the sought identity. Clearly $F=\overline{F}\circ\gamma$, and since $[t,f]_r=\gamma(f)\circ\gamma(t)^{-1}$ for every morphism $[t,f]_r$ of $\mathscr{D}$, it is easily seen that any functor $G:\mathscr{D}\to\mathscr{C}'$ such that $F=G\circ\gamma$ must coincide with $\overline{F}$. This shows that $(\mathscr{D},\gamma)$ is a wide localization of $\mathscr{C}$ along Σ; assertion (ii) is then clear from the construction of $\mathscr{D}$. □

Remark 1.12.7. (i) In the situation of proposition 1.12.4, if both $\mathscr{C}$ and $\mathscr{C}\langle\Sigma^{-1}\rangle$ are categories, then we know that $(\mathscr{C}\langle\Sigma^{-1}\rangle,\gamma)$ also represents the localization of $\mathscr{C}$ along Σ (remark 1.11.5(i)); the latter holds in particular, whenever $\Sigma\subset\mathrm{Mor}(\mathscr{C})$ is right cofinally small (see example 1.8.3(v)). The proposition then says that every morphism $X\to Y$ of $\mathscr{C}[\Sigma^{-1}]$ is the class $[t,f]_r$ of some pair $(X\xleftarrow{t}Z\xrightarrow{f}Y)$ with $t\in\mathrm{Ob}(\Sigma_X)$, and the localization $\gamma:\mathscr{C}\to\mathscr{C}[\Sigma^{-1}]$ is the identity on objects, and maps every morphism $f:X\to Y$ of $\mathscr{C}$ to $[\mathbf{1}_X,f]_r$. Then $[t,f]_r=\gamma(f)\circ\gamma(t)^{-1}$, which we may shorten as $f\circ t^{-1}$, by a tolerable abuse of notation; this exhibits the morphisms of $\mathscr{C}[\Sigma^{-1}]$ as *right fractions, with denominators in* Σ.

(ii) The dual of proposition 1.12.4 provides a corresponding computation of the morphisms in $\mathscr{C}\langle\Sigma^{-1}\rangle$ in terms of *left fractions.* Namely, suppose that Σ admits a left calculus of fractions; for every $Y\in\mathrm{Ob}(\mathscr{C})$ we let $\Sigma^Y:=(\Sigma^{\mathrm{op}})_Y^{\mathrm{op}}$, *i.e.* Σ^Y is the full wide subcategory of $Y/\mathscr{C}$ whose objects are the elements of Σ with source equal to Y. By the dual of proposition 1.12.4(i), Σ^Y is a filtered wide category. For every $X,Y\in\mathrm{Ob}(\mathscr{C})$ we consider the functor

$$H^{X,Y}:\Sigma^Y\to\mathsf{Set} \qquad (Y\xrightarrow{t}Z)\mapsto\{t\}\times\mathscr{C}(X,Z)$$

that assigns to every morphism $X/h : (X \xrightarrow{t} Z) \to (X \xrightarrow{h \circ t} Z')$ of Σ^X the map

$$H^{X,Y}(h) : H^{X,Y}(t) \to H^{X,Y}(h \circ t) \qquad (f, t) \mapsto (h \circ f, h \circ t).$$

We denote the global colimit of $H^{X,Y}$ by $[X, Y]_l$. Then we have natural identifications :

$$\boxed{[X, Y]_l \xrightarrow{\sim} \mathscr{C}\langle\Sigma^{-1}\rangle(\gamma X, \gamma Y) \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C}).}$$

So, every morphism $X \to Y$ of $\mathscr{C}[\Sigma^{-1}]$ is the class $[f, t]_l$ of a pair $(X \xrightarrow{f} Z \xleftarrow{t} Y)$ with $t \in \mathrm{Ob}(\Sigma^Y)$, and may regarded as the left fraction $t^{-1} \circ f$. If $\mathscr{C}$ and $\mathscr{C}\langle\Sigma^{-1}\rangle$ are categories (especially, if Σ is left cofinally small, *i.e.* if Σ^Y is cofinally small for every $Y \in \mathrm{Ob}(\mathscr{C})$), then $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ is also a localization of $\mathscr{C}$ along Σ.

(iii) In case the class Σ admits both a left and a right calculus of fractions, we have then two different constructions of the wide localization $\mathscr{C}\langle\Sigma^{-1}\rangle$, which must therefore produce isomorphic wide categories (remark 1.11.5(i)), *i.e.* we get natural bijections :

$$\boxed{[X, Y]_r \xrightarrow{\sim} [X, Y]_l \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C})}$$

compatible with the respective composition laws. Explicitly, for every pair (t, f) with $(t : Z \to X) \in \Sigma$ and $f \in \mathscr{C}(Z, Y)$, the left calculus of fractions ensures the existence of $(s : Y \to W) \in \Sigma$ and $g \in \mathscr{C}(X, W)$ with $s \circ f = g \circ t$, and the foregoing bijection is given by the rule : $[t, f]_r \mapsto [s, g]_l$. The inverse map is obtained likewise, by invoking the right calculus of fractions for Σ (details left to the reader).

Example 1.12.8. In the situation of proposition 1.11.6, suppose that G is a right (resp. left) adjoint to $F : \mathscr{C} \to \mathscr{D}$. Then Σ_F admits a left (resp. right) calculus of fraction, by example 1.12.2, and we claim that $[X, Y]_l$ (resp. $[X, Y]_r$) is a set for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, so the morphisms of $\mathscr{C}[\Sigma_F^{-1}]$ are the left (resp. right) fractions, as detailed in remark 1.12.7. For the proof, we may assume that G is left adjoint to F; recall that F factors through an equivalence $\overline{F} : \mathscr{C}[\Sigma_F^{-1}] \xrightarrow{\sim} \mathscr{D}$, inducing, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, a map :

$$[X, Y]_r \to \mathscr{D}(FX, FY) \qquad [t, f]_r \mapsto Ff \circ (Ft)^{-1}$$

so it suffices to check that every such map is injective. Hence, let $[t, f]_r, [t', f']_f \in [X, Y]_r$ with $Ff \circ (Ft)^{-1} = Ff' \circ (Ft')^{-1}$; since $(\Sigma_F)_X$ is a cofiltered category (proposition 1.12.4(i)), we may assume that $t = t' : Z \to X$, in which case $Ff = Ff'$. Now, let $(\eta_\bullet, \varepsilon_\bullet)$ be the unit and counit of an adjunction for the pair (G, F), and recall that $\eta_\bullet$ is an isomorphism ([13, Prop.2.16(iii)]); we get

$$f \circ \varepsilon_Z = \varepsilon_Y \circ GFf = \varepsilon_Y \circ GFf' = f' \circ \varepsilon_Z.$$

On the other hand, $F(\varepsilon_Z)$ is an isomorphism of $\mathscr{D}$, by the triangular identities for $(\eta_\bullet, \varepsilon_\bullet)$ ([13, Prob.2.13(ii)]), *i.e.* $\varepsilon_Z \in \Sigma_F$, and then $[t, f]_r = [t \circ \varepsilon_Z, f \circ \varepsilon_Z]_r = [t \circ \varepsilon_Z, f' \circ \varepsilon_Z]_r = [t, f']_r$, as required.

Corollary 1.12.9. *(i) In the situation of proposition 1.12.4, suppose that $\mathscr{C}$ is a finitely complete category and that $[X, Y]_r$ is a set for every $X, Y \in \mathrm{Ob}(\mathscr{C})$. Then $\mathscr{C}[\Sigma^{-1}]$ is finitely complete, and the localization $\gamma : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ is left exact.*

(ii) Dually, let $\mathscr{C}$ be a finitely cocomplete category, $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ a subclass admitting a left calculus of fractions, and suppose that $[X, Y]_l$ is a set for every $X, Y \in \mathrm{Ob}(\mathscr{C})$. Then $\mathscr{C}[\Sigma^{-1}]$ is finitely cocomplete and the localization $\gamma : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ is right exact.

Proof. By duality, it suffices to check (i). Hence, let $F : I \to \mathscr{C}$ be a functor from a finite category I, and pick a universal cone $\varepsilon_\bullet := (\varepsilon_i : L \to Fi \,|\, i \in \mathrm{Ob}(I))$. we need to check $\gamma \star \varepsilon_\bullet$ is still universal, *i.e.* that every $X \in \mathrm{Ob}(\mathscr{C})$ induces a bijection :

$$(*) \qquad \mathscr{C}[\Sigma^{-1}](X,L) \xrightarrow{\sim} \lim_I \mathscr{C}[\Sigma^{-1}](X,F)$$

where $\mathscr{C}[\Sigma^{-1}](X,F) : I \to \mathsf{Set}$ is the induced functor such that $i \mapsto \mathscr{C}[\Sigma^{-1}](X,Fi)$ for every $i \in \mathrm{Ob}(I)$. However, remark 1.12.7(i) identifies $(*)$ with the natural map

$$(**) \qquad \operatorname*{colim}_{\Sigma_X^{\mathrm{op}}} H_{X,L} \to \lim_I \operatorname*{colim}_{\Sigma_X^{\mathrm{op}}} G_{\bullet\bullet}$$

where $G_{\bullet\bullet} : I \times \Sigma_X^{\mathrm{op}} \to \mathsf{Set}$ is the functor such that $(i,t) \mapsto G_{t,i} := \{t\} \times \mathscr{C}(Z,Fi)$ for every $(t : Z \to X) \in \Sigma_X$ and every $i \in \mathrm{Ob}(I)$, and we denote by $\operatorname{colim}_{\Sigma_X^{\mathrm{op}}} G : I \to \mathsf{Set}$ the functor that assigns to every $i \in \mathrm{Ob}(I)$ the (global) colimit of the functor $G_{\bullet,i} : \Sigma_X^{\mathrm{op}} \to \mathsf{Set}$ given by the rule : $t \mapsto G_{t,i}$ for every $t \in \Sigma_X$. Notice that for every such t, the set $H_{X,L}(t)$ is in turns naturally identified – via the universal cone $\varepsilon_\bullet$ – with the limit of the functor $G_{t,\bullet} : I \to \mathsf{Set}$ such that $i \mapsto G_{t,i}$ for every $i \in \mathrm{Ob}(I)$, so we may regard $(**)$ as a map

$$\tau : \operatorname*{colim}_{\Sigma_X^{\mathrm{op}}} \lim_I G_{\bullet\bullet} \to \lim_I \operatorname*{colim}_{\Sigma_X^{\mathrm{op}}} G_{\bullet\bullet}.$$

A direct inspection shows that τ is precisely the map exhibited in remark 1.3.8(iii), which is therefore a bijection, since I is finite and Σ_X^{op} is filtered (proposition 1.12.4(i)).

Next, to show that $\mathscr{C}[\Sigma^{-1}]$ is finitely complete, it suffices to prove that the product of every finite family $X_\bullet := (X_i \,|\, i \in I)$ of objects of $\mathscr{C}[\Sigma^{-1}]$ is representable in $\mathscr{C}[\Sigma^{-1}]$, and that every pair of morphisms $\phi, \psi : X \to Y$ of $\mathscr{C}[\Sigma^{-1}]$ admits an equalizer ([13, Prop.2.40]). However, since γ is an identity map on objects, and since we have just shown that γ preserves finite products, it is clear that the product in $\mathscr{C}$ of the family $X_\bullet$ is also the product of $X_\bullet$ in $\mathscr{C}[\Sigma^{-1}]$. Lastly, by construction we have $(t : Z \to X), (t' : Z' \to X) \in \Sigma$ and $f \in \mathscr{C}(Z,X)$, $g \in \mathscr{C}(Z',X)$ such that $\gamma(f) = \phi \circ \gamma(t)$ and $\gamma(g) = \psi \circ \gamma(t')$; then, since Σ_X is cofiltered (proposition 1.12.4(i)), we find $(t'' : Z'' \to X) \in \Sigma$ with morphisms $s/X : t'' \to t$ and $s'/X : t'' \to t'$ of Σ_X. Set $f' := f \circ s$ and $g' := g \circ s$, and let $h : E \to Z''$ be the equalizer of f' and g' in $\mathscr{C}$; by the foregoing, $\gamma(h)$ is the equalizer of $\gamma(f')$ and $\gamma(g')$ in $\mathscr{C}[\Sigma^{-1}]$, and since $\gamma(f') = \phi \circ \gamma(t'')$ and $\gamma(g') = \psi \circ \gamma(t'')$, it follows easily that $\gamma(t'' \circ h) : E \to X$ is the equalizer of ϕ and ψ in $\mathscr{C}[\Sigma^{-1}]$. □

Corollary 1.12.10. *(i) Let $\mathscr{C}$ be a pre-additive category, $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ a subclass that admits a right (resp. left) calculus of fractions and suppose that $[X,Y]_r$ (resp. $[X,Y]_l$) is a set for every $X, Y \in \mathrm{Ob}(\mathscr{C})$. Then $\mathscr{C}[\Sigma^{-1}]$ is pre-additive, and the localization $\gamma : \mathscr{C} \to \mathscr{C}[\Sigma^{-1}]$ is an additive functor (see* [13, Def.2.71]*).*

(ii) If moreover, $\mathscr{C}$ is an additive category, the same holds for $\mathscr{C}[\Sigma^{-1}]$ (see [13, Def.2.80]*).*

(iii) Let $\mathscr{A}$ be an abelian category, and $\Sigma \subset \mathrm{Mor}(\mathscr{A})$ a subclass that admits both a right and a left calculus of fractions, and such that $[A,B]_r$ is a set for every $A, B \in \mathrm{Ob}(\mathscr{A})$ (so the same holds also for $[A,B]_l$, by remark 1.12.7(iii)). Then $\mathscr{A}[\Sigma^{-1}]$ is an abelian category, and the localization $\gamma : \mathscr{A} \to \mathscr{A}[\Sigma^{-1}]$ is an exact additive functor.

Proof. (i): By [13, Rem.2.72(i)], $\mathscr{C}$ is pre-additive if and only if the same holds for $\mathscr{C}^{\mathrm{op}}$, and γ is additive if and only if the same holds for γ^{op}, so we may assume that Σ admits a right calculus of fractions; then, by the pre-additive structure on $\mathscr{C}$, it is easily seen that each functor $H_{X,Y}$ is the composition of a functor

$$\mathsf{H}_{X,Y} : \Sigma_X^{\mathrm{op}} \to \mathbb{Z}-\mathsf{Mod}$$

and the forgetful functor $\Phi : \mathbb{Z}-\mathsf{Mod} \to \mathsf{Set}$ from abelian groups to sets. Moreover, since Φ preserves filtered colimits, $\mathscr{C}[\Sigma^{-1}](X,Y)$ inherits an abelian group structure, and the universal co-cone $H_{X,Y} \Rightarrow c_{[X,Y]_r}$ upgrades to a universal co-cone

$$\tau_\bullet^{X,Y} : \mathsf{H}_{X,Y} \Rightarrow c_{\mathscr{C}[\Sigma^{-1}](X,Y)}.$$

A simple inspection shows that γ induces group homorphisms $\mathscr{C}(X,Y) \to \mathscr{C}[\Sigma^{-1}](X,Y)$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, so it remains only to check that the composition maps

$$\mathscr{C}[\Sigma^{-1}](A,B) \times \mathscr{C}[\Sigma^{-1}](B,C) \to \mathscr{C}[\Sigma^{-1}](A,C)$$

are $\mathbb{Z}$-bilinear, for every $A, B, C \in \mathrm{Ob}(\mathscr{C})$. To this aim, let $\phi, \phi' : A \rightrightarrows B$ be two morphisms of $\mathscr{C}[\Sigma^{-1}]$; by proposition 1.12.4(i) we may assume that there exists $t : Z \to A$ in Σ and $f, f' \in \mathscr{C}(Z,B)$ such that ϕ and ϕ' are the classes $[t,f]_r$ and respectively $[t,f']_r$ of the pairs $(t,f), (t,f') \in \mathsf{H}_{A,B}(t)$, and then $\phi + \phi' = [t, f+f']_r$. Let also $\psi : B \to C$ be another morphism of $\mathscr{C}[\Sigma^{-1}]$ with $\psi = [t', g]$ for some $t' : Z' \to B$ in Σ and $g \in \mathscr{C}(Z', B)$; we pick $s : U \to Z$ in Σ and $h \in \mathscr{C}(U, Z')$ with $f \circ s = t' \circ h$, so that $\psi \circ \phi = [t \circ s, g \circ h]_r : A \to C$, and likewise we pick $s' : U' \to Z$ in Σ and $h' \in \mathscr{C}(U', Z')$ with $f' \circ s' = t' \circ h'$, so that $\psi \circ \phi' = [t \circ s', g \circ h']_r : A \to C$. Invoking again proposition 1.12.4(i), we then find $s'' : U'' \to Z$ in Σ and $v \in \mathscr{C}(U'', U), v \in \mathscr{C}(U'', U')$ with $s \circ v = s'' = s' \circ v'$; then $(f+f') \circ s'' = t' \circ (h \circ v + h' \circ v')$, so that $\psi \circ (\phi + \phi') = [t \circ s'', g \circ (h \circ v + h' \circ v')]_r$. On the other hand, we have :

$$\psi \circ \phi = [t \circ s \circ v, g \circ h \circ v]_r = [t \circ s'', g \circ h \circ v]_r \qquad \psi \circ \phi' = [t \circ s' \circ v', g \circ h' \circ v']_r = [t \circ s'', g \circ h' \circ v']_r$$

whence $\psi \circ (\phi + \phi') = \psi \circ \phi + \psi + \phi'$. Likewise, if $\psi' : B \to C$ is another morphism of $\mathscr{C}[\Sigma^{-1}]$ one checks that $(\psi + \psi') \circ \phi = \psi \circ \phi + \psi' \circ \phi$, whence the assertion.

(ii): Indeed, we know by (i) that $\mathscr{C}[\Sigma^{-1}]$ is preadditive, and the proof of corollary 1.12.9 shows that all biproducts are representable in $\mathscr{C}[\Sigma^{-1}]$. Hence, it remains only to check that $\mathscr{C}[\Sigma^{-1}]$ admits a zero object; but recall that an object of a pre-additive category is initial if and only if it is final ([13, Rem.2.72(ii)]), and γ preserves either initial objects or final objects, by corollary 1.12.9 and remark 1.1.11(ii), whence the assertion.

(iii): Indeed, since $\mathscr{A}$ is finitely complete and finitely cocomplete ([13, Th.2.89(ii)]), the same holds for $\mathscr{A}[\Sigma^{-1}]$, and γ is exact, by corollary 1.12.9; moreover, we know already that $\mathscr{A}[\Sigma^{-1}]$ is additive and that γ is an additive functor, by (i) and (ii). Hence, it remains only to check that every morphism $\phi : A \to B$ of $\mathscr{A}[\Sigma^{-1}]$ is strict. However, by construction there exists an element $s : A \to Z$ of Σ and $f \in \mathscr{A}(Z,B)$ such that $\phi = \gamma(f) \circ \gamma(s)^{-1}$, so it suffices to show that $\gamma(f)$ is strict; hence, we may assume that $\phi = \gamma(f)$ for some morphism $f : A \to B$ of $\mathscr{A}$. But then, the exactness of γ implies that the natural morphism $\beta_\phi : \mathrm{Coim}(\phi) \to \mathrm{Im}(\phi)$ coincides with $\gamma(\beta_f)$, where $\beta_f : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is the natural morphism; the latter is an isomorphism, so the same follows for β_ϕ. ◻

Example 1.12.11. (i) Let $\mathscr{A}$ be an abelian category, and $\mathscr{B} \subset \mathscr{A}$ a full non-empty subcategory; we say that $\mathscr{B}$ is *a Serre subcategory* (or *a thick subcategory*) *of* $\mathscr{A}$ if the following holds. For every short exact sequence $0 \to A' \to A \to A'' \to 0$ of $\mathscr{A}$ we have $A \in \mathrm{Ob}(\mathscr{B}) \Leftrightarrow A', A'' \in \mathrm{Ob}(\mathscr{A})$. We say that a morphism f of $\mathscr{A}$ is *a $\mathscr{B}$-isomorphism*, if $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathrm{Ob}(\mathscr{B})$. We wish to show that *the class Σ of $\mathscr{B}$-isomorphisms of $\mathscr{A}$ admits both a left and a right calculus of fractions.*

(ii) To this aim, notice first that $\mathscr{B}$ is a Serre subcategory of $\mathscr{A} \Leftrightarrow \mathscr{B}^{\mathrm{op}}$ is a Serre subcategory of $\mathscr{A}^{\mathrm{op}}$, and f is a $\mathscr{B}$-isomorphism of $\mathscr{A} \Leftrightarrow f^{\mathrm{op}}$ is a $\mathscr{B}^{\mathrm{op}}$-isomorphism of $\mathscr{A}^{\mathrm{op}}$ ([13, Rem.2.69(iii), Exerc.2.101(ii)]). Hence, it suffices to check that Σ admits a right calculus of fractions. Axiom (CF1) of definition 1.12.1(i) trivially holds, and (CF2) and

(CF3) follow respectively from [13, Prob.2.108(i)] and [13, Prob.2.108(ii)]. Lastly, let $f, g : X \rightrightarrows Y$ be two morphisms of $\mathscr{A}$, and $s : Y \to Z$ a $\mathscr{B}$-isomorphism such that $s \circ f = s \circ g$; then $f - g$ is the composition of a morphism $h : X \to \mathrm{Ker}(s)$ and the inclusion $\mathrm{Ker}(s) \to Y$. Let 0 denote the zero object of $\mathscr{A}$; since the unique morphism $0 \to \mathrm{Ker}(s)$ is a $\mathscr{B}$-isomorphism, the same holds for the inclusion $t : \mathrm{Ker}(h) \to X$, again by [13, Prob.2.108(ii)]. We get $f \circ t = g \circ t$, whence (CF4).

(iii) Next, for every $X \in \mathrm{Ob}(\mathscr{A})$, let $\mathrm{Sub}'_{\mathscr{A}}(X)$ (resp. $\mathrm{Sub}''_{\mathscr{A}}(X)$) be the subclass of all subobjects $(j : X' \hookrightarrow X) \in \mathrm{Sub}_{\mathscr{A}}(X)$ such that j (resp. the projection $p_{X'} : X \to X/X'$) is a $\mathscr{B}$-isomorphism (notation of §1.8). We claim that *if $\mathrm{Sub}'_{\mathscr{A}}(X) \cup \mathrm{Sub}''_{\mathscr{A}}(X)$ is a set for every $X \in \mathrm{Ob}(\mathscr{A})$, then the class $[X, Y]_r$ of right fractions with denominators in Σ is also a set, for every $X, Y \in \mathrm{Ob}(\mathscr{A})$*; hence, in this case the localization $\mathscr{A}[\Sigma^{-1}]$ exists, and its morphisms $X \to Y$ are given by the sets $[X, Y]_r$; also the localization functor $\mathscr{A} \to \mathscr{A}[\Sigma^{-1}]$ is an exact additive functor, by corollary 1.12.10(iii). For the proof, let $[X, Y]'_r$ be the subclass of all $[t, f]_r \in [X, Y]_r$ with $(t : Z \hookrightarrow X) \in \mathrm{Sub}'_{\mathscr{A}}(X)$, and consider any pair $X \xleftarrow{s} Z \xrightarrow{f} Y$ of morphisms of $\mathscr{A}$ with $s \in \Sigma$; let $Y' \to Y$ be the image of the restriction $\mathrm{Ker}(s) \to Y$ of f, and set $X' := \mathrm{Im}(s)$. We get a commutative diagram :

$$\begin{tikzcd} & & Z \arrow[dl, "q"'] \arrow[dr, "f"] & & \\ X & X' \arrow[l, "j"'] \arrow[r, "\overline{f}"] & Y/Y' & Y \arrow[l, "p_{Y'}"'] \arrow[r, equal] & Y \end{tikzcd}$$

where $p_{Y'}$ and q are the natural epimorphisms, j is the natural monomorphism with $j \circ q = s$, and $\overline{f}$ is induced by $p_{Y'} \circ f$. It follows that $[s, f]_r = [p_{Y'}, \mathbf{1}_Y]_r \circ [j, \overline{f}]_r$ in $\mathscr{A}\langle\Sigma^{-1}\rangle$; hence, we get a surjection :

$$\bigcup_{Y' \in \mathrm{Sub}''_{\mathscr{A}}(Y)} [X, Y/Y']'_r \to [X, Y]_r \qquad [j, g]_r \mapsto [p_{Y'}, \mathbf{1}_Y]_r \circ [j, g]_r.$$

Now, if $\mathrm{Sub}'_{\mathscr{A}}(X)$ is a set for every $X \in \mathrm{Ob}(\mathscr{A})$, the subclass $[X, Y]'_r$ is also a set, for every $X, Y \in \mathrm{Ob}(\mathscr{A})$, and the claim follows.

(iv) In the situation of (i), the wide localization $\mathscr{A}\langle\Sigma^{-1}\rangle$ is called *the quotient of $\mathscr{A}$ by the Serre subcategory $\mathscr{B}$*, and it is usually denoted $\mathscr{A}/\mathscr{B}$. As a special case of (iii), we see that *if $\mathscr{A}$ is well-powered, then the quotient $\mathscr{A}/\mathscr{B}$ is an abelian category.*

Proposition 1.12.12. *Let $\mathscr{C}$ be a category, $i : \mathscr{C}_0 \to \mathscr{C}$ the inclusion of a full subcategory, Σ a subclass of $\mathrm{Mor}(\mathscr{C})$, and set $\Sigma_0 := \Sigma \cap \mathrm{Mor}(\mathscr{C}_0)$. Suppose that :*

(a) For every $X \in \mathrm{Ob}(\mathscr{C})$, every $X_0 \in \mathrm{Ob}(\mathscr{C}_0)$ and every $f \in \mathscr{C}(X, iX_0) \cap \Sigma$ there exists $Y_0 \in \mathrm{Ob}(\mathscr{C}_0)$ and $g \in \mathscr{C}(iY_0, X)$ such that $f \circ g \in \Sigma_0$.

(b) Σ admits a right calculus of fractions.

Then we have :

(i) Σ_0 admits a right calculus of fractions.

(ii) If Σ is right cofinally small, the localizations $(\mathscr{C}[\Sigma^{-1}], \gamma)$ and $(\mathscr{C}_0[\Sigma_0^{-1}], \gamma_0)$ exist, and i induces a fully faithful functor

$$i[\Sigma_0^{-1}] : \mathscr{C}_0[\Sigma_0^{-1}] \to \mathscr{C}[\Sigma^{-1}] \qquad \textit{such that} \qquad i[\Sigma_0^{-1}] \circ \gamma_0 = \gamma \circ i.$$

Proof. (i): Axioms (CF1) and (CF2) obviously holds for Σ_0. Next, let $X, Y, W \in \mathrm{Ob}(\mathscr{C}_0)$ and $Z \in \mathrm{Ob}(\mathscr{C})$, and consider any commutative diagram of $\mathscr{C}$:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "t"'] & iW \arrow[d, "i(s)"] \\ iX \arrow[r, "i(f)"] & iY \end{tikzcd} \qquad \text{with } s, t \in \Sigma.$$

By (a), we may then find a morphism $u : iZ_0 \to Z$ of $\mathscr{C}$ for some $Z_0 \in \mathrm{Ob}(\mathscr{C}_0)$, such that $t' := t \circ u : iZ_0 \to iX$ lies in Σ_0, $g' := g \circ u : iZ_0 \to iW$ lies in $\mathscr{C}_0(Z_0, W)$, and clearly $f \circ t' = s \circ g'$, whence (CF3). Lastly, let $X, Y \in \mathrm{Ob}(\mathscr{C}_0)$, $Z \in \mathrm{Ob}(\mathscr{C})$, and suppose that we have $f, g \in \mathscr{C}_0(X, Y)$ and $t : Z \to iX$ in Σ such that $i(f) \circ t = i(g) \circ t$. Pick again $u : iZ_0 \to Z$ such that $t' := t \circ u$ lies in Σ_0; then $f \circ t' = g \circ t'$, whence (CF4).

(ii): We know already that $(\mathscr{C}[\Sigma^{-1}], \gamma)$ exists, by remark 1.12.7(i). Lastly, for every $X \in \mathrm{Ob}(\mathscr{C}_0)$ (resp. for every $X \in \mathrm{Ob}(\mathscr{C})$), define the subcategory $\Sigma_{0,X}$ of $\mathscr{C}_0/X$ (resp. the subcategory Σ_X of $\mathscr{C}/X$) as in definition 1.12.1(ii); taking into account proposition 1.12.4(ii), corollary 1.5.4(i) and example 1.8.3(v), in order to prove the existence of $(\mathscr{C}_0[\Sigma^{-1}], \gamma_0)$ and to show that $i[\Sigma^{-1}]$ is fully faithful, it suffices to check that $\Sigma_{0,X}^{\mathrm{op}}$ is a cofinal subcategory of $\Sigma_{iX}^{\mathrm{op}}$, for every $X \in \mathrm{Ob}(\mathscr{C}_0)$. However, we know already that $\Sigma_{0,X}^{\mathrm{op}}$ and $\Sigma_{iX}^{\mathrm{op}}$ are filtered categories, by proposition 1.12.4(i). Then, by proposition 1.5.7(iii.a), we are reduced to showing that for every object $Z \xrightarrow{t} iX$ of Σ_{iX} there exists an object $Z_0 \xrightarrow{t_0} X$ of $\Sigma_{0,X}$ and a morphism $u/X : i(t_0) \to t$ of Σ_{iX}. But this is precisely our condition (a). □

1.12.13. Suppose now that $\mathscr{C}$ is a category, and $\Sigma \subset \mathrm{Mor}(\mathscr{C})$ is a subclass admitting a (left or right) calculus of fraction, and such that the localization $(\mathscr{C}[\Sigma^{-1}], \gamma)$ exists. One might ask whether, in such a situation, the morphisms $X \to Y$ of $\mathscr{C}[\Sigma^{-1}]$ are always given by the class $[X, Y]$ of (left or right) fractions with denominators in Σ, as detailed in remark 1.12.7(i,ii). More precisely, we do know, by virtue of remark 1.11.5(ii), that the natural map $[X, Y] \to \mathscr{C}[\Sigma^{-1}](X, Y)$ is always surjective (this can also be established more directly from remark 1.11.3(ii)), but the injectivity of this map does not follow from our previous results, *unless we already know that* $[X, Y]$ *is a set for every* $X, Y \in \mathrm{Ob}(\mathscr{C})$. It turns out that the answer is negative, in general; indeed we have the following result, that I have learned from O.Gabber :

Proposition 1.12.14. *Let* $G_\bullet := (G_\alpha \mid \alpha \in \Omega)$ *be a totally ordered family of groups, indexed by the class* Ω *of ordinals, with injective transition homomorphisms* $G_\alpha \to G_\beta$ *for every* $\alpha, \beta \in \Omega$ *with* $\beta \geq \alpha$. *Suppose moreover that the wide group* $G := \bigcup_{\alpha \in \Omega} G_\alpha$ *does not admit any non-trivial quotient groups (*i.e. *for every surjective homomorphism of wide groups* $G \to G'$, *either* G' *is a proper class, or else* G' *is the trivial group* 0 *with one element).*

We consider the category $\mathscr{C}$ *with* $\mathrm{Ob}(\mathscr{C}) := \Omega$ *and with*

$$\mathscr{C}(\alpha, \beta) := \begin{cases} G_\beta & \text{if } \alpha \leq \beta \\ \varnothing & \text{if not} \end{cases}$$

with composition law induced by the group laws of the groups $G_\bullet$*, in the obvious fashion.*

(i) *Set* $\Sigma := \mathrm{Mor}(\mathscr{C})$*; the wide localization* $(\mathscr{C}\langle\Sigma^{-1}\rangle, \gamma)$ *is the wide groupoid with :*

$$\mathrm{Ob}(\mathscr{C}\langle\Sigma^{-1}\rangle) = \Omega \qquad \text{and} \qquad \mathscr{C}\langle\Sigma^{-1}\rangle(\alpha, \beta) = G \qquad \forall \alpha, \beta \in \Omega.$$

(ii) *On the other hand, the localization* $(\mathscr{C}[\Sigma^{-1}], \gamma')$ *is the groupoid with :*

$$\mathrm{Ob}(\mathscr{C}[\Sigma^{-1}]) = \Omega \qquad \text{and} \qquad \mathscr{C}[\Sigma^{-1}](\alpha, \beta) = 0 \qquad \forall \alpha, \beta \in \Omega.$$

(iii) Moreover, the class Σ admits a left calculus of fraction.

Proof. (iii): The duals of axioms (CF1) and (CF2) trivially hold for Σ. Next, let $\beta \xleftarrow{g} \alpha \xrightarrow{g'} \beta'$ be morphisms of $\mathscr{C}$, and set $\beta'' := \max(\beta, \beta')$; we get the commutative diagram of $\mathscr{C}$:

$$\begin{tikzcd} & \beta \arrow[dr, "g' \bullet g^{-1}"] & \\ \alpha \arrow[ur, "g"] \arrow[dr, "g'"'] & & \beta'' \\ & \beta' \arrow[ur, "1"'] & \end{tikzcd}$$

(where 1 denotes the neutral element of $G_{\beta''}$) whence the dual of (CF3). Lastly, let $g, g' \in \mathscr{C}(\alpha, \beta)$ and $h \in \mathscr{C}(\gamma, \alpha)$ with $g \circ h = g' \circ h$; the latter means that $g \bullet h = g' \bullet h$, where $- \bullet -$ denotes the group law of G_β. Hence $g = g'$, whence the dual of (CF4).

(i): According to remark 1.12.7(ii), for every $\alpha, \beta \in \Omega$ the morphisms $\alpha \to \beta$ in $\mathscr{C}\langle\Sigma^{-1}\rangle$ are represented by the class of left fractions $[\alpha, \beta]_l$; we have a well-defined map

$$\omega_{\alpha\beta} : [\alpha, \beta]_l \to G \qquad [g, h]_l \mapsto h^{-1} \bullet g$$

where h^{-1} is the inverse of h in the wide group G, and $- \bullet -$ denotes the group law of G. Recall that $[g, h]_l$ is the equivalence class of a pair $(\alpha \xrightarrow{g} \gamma \xleftarrow{h} \beta)$ of morphisms; if $(\beta \xrightarrow{g'} \gamma' \xleftarrow{h'} \delta)$ is another such pair, the composition $[g', h']_l \circ [g, h]_l$ is formed as follows. Set $\gamma'' := \max(\gamma, \gamma')$; as in the foregoing, we get morphisms $\gamma \xrightarrow{g' \bullet h^{-1}} \gamma'' \xleftarrow{1} \gamma'$, and then

$$[g', h']_l \circ [g, h]_l = [g' \bullet h^{-1} \bullet g, h']_l.$$

We see then that $\omega_{\alpha\delta}(\phi' \circ \phi) = \omega_{\beta\delta}(\phi') \bullet \omega_{\alpha\beta}(\phi)$ for every composable pair $\alpha \xrightarrow{\phi} \beta \xrightarrow{\phi'} \delta$ of morphisms of $\mathscr{C}\langle\Sigma^{-1}\rangle$. It is easily seen that $\omega_{\alpha\beta}$ is surjective for every $\alpha, \beta \in \Omega$. Lastly, suppose that $\omega_{\alpha\beta}([g, h]_l) = \omega_{\alpha\beta}([g', h']_l)$ for some pairs $(\alpha \xrightarrow{g} \gamma \xleftarrow{h} \beta), (\alpha \xrightarrow{g'} \gamma' \xleftarrow{h'} \beta)$, and say that $\gamma \geq \gamma'$; the identity means that $h^{-1} \bullet g = h'^{-1} \bullet g'$ in G_γ, so $k := g \bullet g'^{-1} = h \bullet h'^{-1}$ is a morphism $\gamma' \to \gamma$ of $\mathscr{C}$, and $[g', h']_l = [k \circ g', k \circ h']_l = [g, h]_l$, hence $\omega_{\alpha\beta}$ is injective.

(ii): Let $\mathscr{A}$ be any category, and $F : \mathscr{C} \to \mathscr{A}$ a functor such that $F(\Sigma) \subset \mathrm{Isom}(\mathscr{A})$; the latter means that F factors through a functor $F' : \mathscr{C} \to \mathscr{G}$ and the inclusion $\mathscr{G} \to \mathscr{A}$, where $\mathscr{G}$ is the largest groupoid contained in $\mathscr{A}$. Let $\overline{G} : \mathscr{C}\langle\Sigma^{-1}\rangle \to \mathscr{G}$ be the unique functor with $F' = \overline{F} \circ \gamma$; clearly $\overline{F}$ induces homomorphisms of wide groups

$$\overline{F}_{\alpha\alpha} : G = \mathscr{C}\langle\Sigma^{-1}\rangle(\alpha, \alpha) \to \mathscr{G}(G\alpha, G\alpha) \qquad \forall \alpha \in \Omega.$$

However, since by assumption G does not admit group quotients, we see that $\overline{F}_{\alpha\alpha}$ maps every element of G to $\mathbf{1}_{G\alpha}$, for every such α. Now, every morphism $g : \alpha \to \beta$ of $\mathscr{C}\langle\Sigma^{-1}\rangle$ can be written as a composition $\alpha \xrightarrow{g} \alpha \xrightarrow{1} \beta$, so $\overline{F}_{\alpha\beta}(g) = f_{\alpha\beta} := \overline{F}_{\alpha\beta}(1)$ for every $\alpha, \beta \in \Omega$. Summing up, this shows that the map $F_{\alpha\beta} : \mathscr{C}(\alpha, \beta) \to \mathscr{A}(F\alpha, F\beta)$ induced by F is the constant map with value $f_{\alpha\beta}$, for every $\alpha, \beta \in \Omega$; *i.e.* F factors uniquely through the category $\overline{\mathscr{C}}$ such that $\mathrm{Ob}(\overline{\mathscr{C}}) = \Omega$ and $\overline{\mathscr{C}}(\alpha, \beta) = 0$ for every $\alpha, \beta \in \Omega$, whence the assertion. ◻

Example 1.12.15. (i) To complete this discussion, we wish to exhibit a sequence of groups $(G_\alpha \,|\, \alpha \in \Omega)$ fulfilling the conditions of proposition 1.12.14. To this aim, let first Γ be any group, $\gamma_0 \in \Gamma$ an element of infinite order, and $S \subset \Gamma \setminus \{\gamma_0\}$ any subset of elements of infinite order; denote by $\Delta(S)$ the free group with basis S, by $j_S : S \to \Delta(S)$

the universal map, and by $\Gamma * \Delta(S)$ the free product of Γ and $\Delta(S)$. By [10, Ch.IV, lemma 1.1], we have injective group homomorphisms

$$\Gamma \xrightarrow{i_\Gamma} \Gamma * \Delta(S) \xleftarrow{i_S} \Delta(S).$$

We let $\overline{\gamma}_0 := i_\Gamma(\gamma_0)$, and we consider also the composition

$$\tau : S \xrightarrow{j_S} \Delta(S) \xrightarrow{i_S} \Gamma * \Delta(S) \qquad s \mapsto \tau_s.$$

We set

$$G(\Gamma, \gamma_0, S) := (\Gamma * \Delta(S))/\mathscr{R}$$

where $\mathscr{R}$ is the smallest normal subgroup of $\Gamma * \Delta(S)$ containing the system of elements.

$$(\tau_s^{-1} i_\Gamma(s) \tau_s \overline{\gamma}_0^{-1}) \mid s \in S).$$

I.e. the image in $G(\Gamma, \gamma_0, S)$ of every $s \in S$ is conjugate to $\overline{\gamma}_0$, via the inner automorphism of $G(\Gamma, \gamma_0, S)$ induced by τ_s. The construction of $G(\Gamma, \gamma_0, S)$ is an instance of an HNN extension (named after G.Higman, B.H.Neumann and H.Neumann : see [10, Ch.IV, § 2]).

(ii) We claim that *the natural map* $i_{\Gamma,S} : \Gamma \to G(\Gamma, \gamma_0, S)$ *is injective.* For the proof, let $\mathscr{P}_0(S)$ be the set of all finite subsets of S, and notice that $G(\Gamma, \gamma_0, S)$ is the filtered colimit of the induced system of groups $(G(\Gamma, \gamma_0, T) \mid T \in \mathscr{P}_0(S))$; hence, we are reduced to the case where S is a finite set. In this case, we argue by induction on the cardinality c of S. The assertion is trivial if $c = 0$, so suppose that $c > 0$, and that the assertion is already known for every subset T of cardinality $< c$. Pick $s_0 \in S$, let $\Gamma' := G(\Gamma, \gamma_0, S \setminus \{s_0\})$, and denote by $\overline{\gamma}_0$ and $\overline{s}_0$ the images of γ_0 and respectively s_0 in Γ'; then it is easily seen that $G(\Gamma, \gamma_0, S) = G(\Gamma', \overline{\gamma}_0, \{\overline{s}_0\})$. By inductive assumption, the natural map $\Gamma \to \Gamma'$ is injective, and the same holds for the natural map $\Gamma' \to G(\Gamma', \overline{\gamma}_0, \{\overline{s}_0\})$, by virtue of [10, Ch.IV, Th.2.1(i)], whence the assertion.

(iii) Next, we claim that *if* Γ *is a torsion-free group, the same holds for* $G(\Gamma, \gamma_0, S)$. Indeed, arguing as in (ii), we reduce first to the case where S is a subset of finite cardinality c, and then to the case where $c = 1$; in this case, the assertion follows from [10, Ch.IV, Th.2.4].

(iv) Let $f : \Gamma \to \Gamma'$ be an injective group homomorphism, set $\gamma'_0 := f(\gamma_0)$ and let $S' \subset \Gamma' \setminus \{\gamma'_0\}$ be a subset of elements of infinite order such that $f(S) \subset S'$. Then we claim that f *induces a cartesian square of injective homomorphisms in the category of groups :*

$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "i_{\Gamma,S}"'] & \Gamma' \arrow[d, "i_{\Gamma',S'}"] \\ G(\Gamma, \gamma_0, S) \arrow[r, "G(f,S,S')"] & G(\Gamma', \gamma'_0, S'). \end{tikzcd}$$

Indeed, clearly, f induces a group homomorphism $\Gamma * \Delta(S) \to \Gamma' * \Delta(S')$ that maps the normal subgroup $\mathscr{R}$ to the normal subgroup $\mathscr{R}' := \mathrm{Ker}(\Gamma' * \Delta(S') \to G(\Gamma', \gamma'_0, S'))$, whence the induced homomorphism $G(f, S, S')$. Now, set $\Gamma'' := G(\Gamma', \gamma'_0, f(S))$, and let γ''_0 and T be the images of γ'_0 and respectively $S' \setminus f(S)$ in Γ''; then $G(\Gamma', \gamma'_0, S') = G(\Gamma'', \gamma''_0, T)$, and $G(f, S, S')$ is the composition of $G(S, f(S)) : G(\Gamma, \gamma_0, S) \to \Gamma''$ and the natural group homomorphism $i : \Gamma'' \to G(\Gamma'', \gamma''_0, T)$. According to (ii), the map i is injective, so we are reduced to checking that the assertion holds for $G(S, f(S))$, *i.e. we may assume that* $S' = f(S)$, and in this case we just write $G(f, S)$ for the map $G(f, S, f(S))$. With this notation, it is easily seen that $G(f, S)$ is the filtered colimit of the system of maps $(G(f, T) \mid T \in \mathscr{P}_0(S))$, so we are reduced to the case where S is a set of finite cardinality c, and we argue by induction on c. The assertion is trivial if $c = 0$, so suppose that $c > 0$, and that the assertion is already known for every subset T of cardinality $< c$. Pick $s_0 \in S$, let

$s'_0 := f(s_0)$, set $\Delta := G(\Gamma, \gamma_0, S \setminus \{s_0\})$, $\Delta' := G(\Gamma', \gamma'_0, S' \setminus \{s'_0\})$, and denote by δ_0 and δ'_0 the images of γ_0 in Δ and of γ'_0 in Δ'; by inductive assumption, $g := G(f, S \setminus \{s_0\}) : \Delta \to \Delta'$ and f yield a cartesian square of injective maps, and on the other hand, the map $G(f, S)$ corresponds to $G(g, \{s_0\})$ under the natural identifications :

$$G(\Gamma, \gamma_0, S) \xrightarrow{\sim} G(\Delta, \delta_0, \{s_0\}) \qquad G(\Gamma', \gamma'_0, S') \xrightarrow{\sim} G(\Delta', \delta'_0, \{s'_0\}).$$

So we are further reduced to the case where $S = \{s_0\}$. In this case, set also $\tau := \tau_{s_0}$, and pick a set U (resp. V) of representive for the set $\Gamma/\langle\gamma_0\rangle$ (resp. $\Gamma/\langle s_0\rangle$) of right cosets in Γ of the subgroup $\langle\gamma_0\rangle$ generated by γ_0 (resp. of the subgroup $\langle s_0\rangle$ generated by s_0), with the neutral element 1 of Γ contained in both U and V; according to [10, Ch.IV, Th.2.1(ii)] every element of $G(\Gamma, \gamma_0, \{s_0\})$ can be written uniquely in (U, V)-*normal form* :

$$g_0\tau^{\varepsilon_1}g_1 \cdots \tau^{\varepsilon_n}g_n \qquad \text{with} \qquad \varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$$

where g_0 is an element of Γ, there is no consecutive sequence $\tau^\varepsilon 1 \tau^{-\varepsilon}$, and for $i = 1, \dots, n$, the following holds : if $\varepsilon_i = -1$ then $g_i \in U$, and if $\varepsilon_i = 1$ then $g_i \in V$. Notice now that the map f induces injective maps :

$$\Gamma/\langle\gamma_0\rangle \to \Gamma'/\langle\gamma'_0\rangle \qquad \Gamma/\langle s_0\rangle \to \Gamma'/\langle s'_0\rangle.$$

Hence, we may complete $f(U)$ and $f(V)$ by adding suitable elements, to obtain set of representatives U' and V' for the sets of right cosets $\Gamma'/\langle\gamma'_0\rangle$ and $\Gamma'/\langle s'_0\rangle$. Then clearly $G(f, \{s_0\})$ sends (U, V)-normal forms to (U', V')-normal forms, whence the assertion.

(v) Now, for every ordinal α we construct inductively a system of groups $(G^\alpha_n \mid n \in \mathbb{N})$ with injective transition maps, as follows. Let $[0, \alpha]$ be the *set* of all ordinals $\leq \alpha$, and let $G^\alpha_0 := \mathbb{Z}^{([0,\alpha])}$ (the free abelian group with basis $[0, \alpha]$). Especially, $G^0_0 = \mathbb{Z}$, and we let γ_0 be a generator of G^0_0; the inclusion $[0, \alpha] \subset [0, \beta]$ for every $\alpha \leq \beta$ induces an injective group homomorphism $G^\alpha_0 \to G^\beta_0$, so we may regard γ_0 as an element of G^α_0, for every ordinal α. Let as well $G^\alpha_{-1} := \{0, \gamma_0\}$, and denote by $i_0 : G^\alpha_{-1} \to G^\alpha_0$ the inclusion map; then, for every $n \geq 0$ set inductively

$$S^\alpha_n := G^\alpha_n \setminus i_n(G^\alpha_{n-1}) \qquad G^\alpha_{n+1} := G(G^\alpha_n, \gamma_0, S^\alpha_n)$$

and denote by $i_{n+1} : G^\alpha_n \to G^\alpha_{n+1}$ the natural injective group homomorphism of (ii). Notice that, since G^α_0 is torsion-free, from (iii) and an easy induction argument we see that G^α_n is torsion-free for every $n \in \mathbb{N}$, so G^α_{n+1} is well-defined.

(vi) We set $G_\alpha := \bigcup_{n\in\mathbb{N}} G^\alpha_n$ (where we identify each G^α_n to a subgroup of G^α_{n+1}, via i_{n+1}). Clearly G_α is torsion-free, and by construction, every element of G_α is conjugate to γ_0, hence to every other element. Every pair of ordinals $\alpha \leq \beta$ induces a map $j^{\alpha,\beta}_0 : G^\alpha_0 \to G^\beta_0$, and since $j^{\alpha,\beta}_0(S^\alpha_0) \subset S^\beta_0$, we get an induced group homomorphism $j^{\alpha,\beta}_1 : G^\alpha_1 \to G^\beta_1$, as in (iv). Next, invoking the cartesianity of the diagram of (iv), we get inductively a system of injective group homomorphisms $j^{\alpha,\beta}_\bullet := (j^{\alpha,\beta}_n : G^\alpha_n \to G^\beta_n \mid n \in \mathbb{N})$ such that $j^{\alpha,\beta}_n(S^\alpha_n) \subset S^\beta_n$ for every $n \in \mathbb{N}$ (details left to the reader). The colimit of the system $j^{\alpha,\beta}_\bullet$ is an injective group homomorphism $j_{\alpha,\beta} : G_\alpha \to G_\beta$; moreover, a direct inspection of the construction shows that for every ordinal $\gamma \geq \beta$ we have $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$. This completes the construction of our system of groups $(G_\alpha \mid \alpha \in \Omega)$. It is then clear that *every element of the wide group* $G := \bigcup_{\alpha\in\Omega} G_\alpha$ *is conjugate in* G *to every other element*, since the same holds for each G_α; hence, the only normal subgroups of G are 0 and G. Lastly, let H be any group (so, H is a *set*), and $f : G \to H$ any homomorphism of wide groups; since G is a proper class, f cannot be injective, and therefore its kernel is not 0, so it must be G, as required.

2. Simplicial objects

2.1. The category of simplicial sets. We shall denote by :

$$\mathbf{\Delta}$$

the full subcategory of Cat whose objects are the totally ordered finite sets :

$$[n] := \{0, 1, \dots, n\} \qquad \forall n \in \mathbb{N}.$$

As detailed in remark 1.9.3(iii), the morphisms of $\mathbf{\Delta}$ are the non-decreasing maps.

Definition 2.1.1. (i) A *simplicial set* is a presheaf over the category $\mathbf{\Delta}$. We write

$$\boxed{\mathrm{sSet} := \widehat{\mathbf{\Delta}}}$$

for the category of simplicial sets. For a simplicial set X and $n \in \mathbb{N}$, we shall usually write $X_n := X_{[n]}$, and we call X_n the *set of n-simplices of X*.

(ii) For every $n \in \mathbb{N}$, the *standard n-simplex* is the simplicial set

$$\Delta^n := h_{[n]}$$

representing the object $[n]$ of $\mathbf{\Delta}$. Yoneda's lemma yields a natural bijection :

$$X_n \xrightarrow{\sim} \mathrm{sSet}(\Delta^n, X) \qquad \forall n \in \mathbb{N}, \forall X \in \mathrm{Ob}(\mathrm{sSet}).$$

For every non-decreasing map $u : [m] \to [n]$, we shall also sometimes write

$$\Delta^u : \Delta^m \to \Delta^n$$

for the induced morphism $h_u : h_{[m]} \to h_{[n]}$ of sSet.

(iii) For all integers $n \geq 1$ and $0 \leq i \leq n$, the *i-th face morphism*

$$\partial_i^n : [n-1] \to [n]$$

is the unique injective morphism of $\mathbf{\Delta}$ whose image does not contain i.

(iv) For all integers $n \geq 0$ and $0 \leq i \leq n$, the *i-th degeneracy morphism*

$$\sigma_i^n : [n+1] \to [n]$$

is the unique surjective morphism of $\mathbf{\Delta}$ which takes the value i twice.

(v) By Yoneda's lemma, the face and degeneracy maps correspond naturally to morphisms of simplicial sets that we call by the same names, and we denote as well

$$\boxed{\partial_i^n : \Delta^{n-1} \to \Delta^n \qquad \text{and} \qquad \sigma_i^n : \Delta^{n+1} \to \Delta^n.}$$

Also, for any simplicial set X, we shall write :

$$\boxed{d_n^i := (\partial_i^n)^* : X_n \to X_{n-1} \qquad \text{and} \qquad s_n^i := (\sigma_i^n)^* : X_n \to X_{n+1}.}$$

Proposition 2.1.2. *(i) The following identities hold in $\mathbf{\Delta}$:*

$$\partial_j^{n+1} \circ \partial_i^n = \partial_i^{n+1} \circ \partial_{j-1}^n \qquad \forall 0 \leq i < j \leq n+1$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \qquad \forall 0 \leq i \leq j \leq n$$

$$\sigma_j^{n-1} \circ \partial_i^n = \begin{cases} \partial_i^{n-1} \circ \sigma_{j-1}^{n-2} & \forall 0 \leq i < j \leq n-1 \\ \mathbf{1}_{[n-1]} & \forall 0 \leq j \leq i \leq j+1 \leq n-1 \\ \partial_{i-1}^{n-1} \circ \sigma_j^{n-2} & \forall 0 \leq j < i-1 \leq n-1. \end{cases}$$

(ii) Any morphism $\Delta^m \to \Delta^n$ in $\mathbf{\Delta}$ admits a unique factorization $f = i \circ \pi$ into a split epimorphism $\pi : \Delta^m \to \Delta^p$ followed by a monomorphism $i : \Delta^p \to \Delta^n$.

Proof. (i) is an easy verification, by inspection of the definition.

(ii): It is easily seen that the epimorphisms (resp. monomorphisms) of $\mathbf{\Delta}$ are the surjective (resp. injective) maps of ordered sets $\Delta^p \to \Delta^q$, and clearly every epimorphism admits a right inverse in $\mathbf{\Delta}$, *i.e.* is split. Assertion (ii) is an immediate consequence. □

Example 2.1.3. For every $n \in \mathbb{N}$ we consider the *topological n-dimensional simplex*

$$|\Delta^n| := \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \dots + x_n = 1\}$$

which we endow with the topology induced by the inclusion into $\mathbb{R}^n_{\geq 0}$. To any morphism $f : \Delta^m \to \Delta^n$ of $\mathbf{\Delta}$ we attach the continuous map

$$|f| : |\Delta^m| \to |\Delta^n| \qquad (x_1, \dots, x_m) \mapsto (\textstyle\sum_{j \in f^{-1}(i)} x_j \mid i = 1, \dots, n).$$

The rules : $\Delta^n \mapsto |\Delta^n|$ and $f \mapsto |f|$ clearly define a functor

$$\boxed{|\cdot| : \mathbf{\Delta} \to \mathsf{Top}}$$

from $\mathbf{\Delta}$ to the category Top of topological spaces (whose morphisms are the continuous maps). As in remark 1.6.6(ii,iii), the functor $|\cdot|$ induces a functor

$$\boxed{\mathsf{Sing} := |\cdot|^* : \mathsf{Top} \to \mathsf{sSet} \qquad T \mapsto ([n] \mapsto \mathsf{Top}(|\Delta^n|, T))}$$

attaching to every topological space T its *singular (simplicial) complex* $\mathsf{Sing}(T)$. And by virtue of Kan's theorem 1.7.5, the functor Sing admits a left adjoint :

$$\boxed{|\cdot| : \mathsf{sSet} \to \mathsf{Top} \qquad X \mapsto |X|}$$

called the *realisation functor.*

2.1.4. *Front-to-back duality.* Let $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$ be the functor that is the identity on objects, and that associates with every morphism $f : [m] \to [n]$ of $\mathbf{\Delta}$ the morphism

$$f^{\mathrm{op}} : [m] \to [n] \qquad m - i \mapsto n - f(i).$$

Clearly $(-)^{\mathrm{op}}$ *is an involution, i.e.* $(-)^{\mathrm{op}} \circ (-)^{\mathrm{op}} = \mathbf{1}_{\mathbf{\Delta}}$. The functor $(-)^{\mathrm{op}}$ induces a functor

$$\mathsf{sSet} \to \mathsf{sSet} \qquad X \mapsto X^{\mathrm{op}} := X \circ (-)^{\mathrm{op}} \quad (u : X \to Y) \mapsto (u^{\mathrm{op}} := u \star (-)^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}).$$

For every simplicial set X, we call X^{op} the *front-to-back dual of* X; clearly :

$$(X^{\mathrm{op}})^{\mathrm{op}} = X \qquad \text{and} \qquad (X \times Y)^{\mathrm{op}} = X^{\mathrm{op}} \times Y^{\mathrm{op}} \qquad \forall X, Y \in \mathrm{Ob}(\mathsf{sSet}).$$

Example 2.1.5. (i) A direct calculation shows that :

$$(\partial^n_i)^{\mathrm{op}} = \partial^n_{n-i} \quad \forall n \geq 1, \forall 0 \leq i \leq n \qquad (\sigma^m_j)^{\mathrm{op}} = \sigma^m_{m-j} \quad \forall m \geq 0, \forall 0 \leq j \leq m.$$

(ii) For every $n \in \mathbb{N}$ we have a natural isomorphism :

$$\omega_n : (\Delta^n)^{\mathrm{op}} \xrightarrow{\sim} \Delta^n$$

that assigns to every $k \in \mathbb{N}$ the bijection :

$$(\Delta^n)^{\mathrm{op}}_k \xrightarrow{\sim} \Delta^n_k \qquad ([k] \xrightarrow{f} [n]) \mapsto ([k] \xrightarrow{f^{\mathrm{op}}} [n]).$$

(iii) In light of (i) and (ii), we deduce commutative diagrams :

$$\boxed{\begin{array}{ccccccc} (\Delta^{n-1})^{\mathrm{op}} & \xrightarrow{\omega_{n-1}} & \Delta^{n-1} & \qquad & (\Delta^{m+1})^{\mathrm{op}} & \xrightarrow{\omega_{m+1}} & \Delta^{m+1} \\ {\scriptstyle (\partial^n_i)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \partial^n_{n-i}} & & {\scriptstyle (\sigma^m_j)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \sigma^m_{m-j}} \\ (\Delta^n)^{\mathrm{op}} & \xrightarrow{\omega_n} & \Delta^n & & (\Delta^m)^{\mathrm{op}} & \xrightarrow{\omega_m} & \Delta^m \end{array}}$$

for every $n \geq 1$ and $0 \leq i \leq n$, and every $m \geq 0$ and $0 \leq j \leq m$.

2.1.6. As for any category of presheaves, sSet is complete and cocomplete, and its limits and colimits are computed termwise; moreover, the finite limits in sSet commute with all small filtered colimits. Also, all small colimits of sSet are universal (example 1.6.19(ii)), all its monomorphisms and epimorphisms are regular, and a morphism $f : X \to Y$ of sSet is a monomorphism (resp. an epimorphism) if and only if the map $f_n : X_n \to Y_n$ is injective (resp. surjective) for every $n \in \mathbb{N}$ (remark 1.6.2). Recall also that, by remark 1.7.8(ii), sSet admits an internal Hom-functor

$$\mathscr{H}om(-,-) : \mathsf{sSet}^{\mathrm{op}} \times \mathsf{sSet} \to \mathsf{sSet} \qquad (X, Y) \mapsto \mathscr{H}om(X, Y)$$

such that for every $X \in \mathrm{Ob}(\mathsf{sSet})$, the resulting functor $\mathscr{H}om(X, -) : \mathsf{sSet} \to \mathsf{sSet}$ is right adjoint to the functor $(-) \times X : \mathsf{sSet} \to \mathsf{sSet}$. Explicitly, we have :

$$\mathscr{H}om(X, Y)_n := \mathsf{sSet}(\Delta^n \times X, Y) \qquad \forall n \in \mathbb{N}$$

and every pair of morphisms $f : X' \to X$, $g : Y \to Y'$ of sSet induces the morphism $\mathscr{H}om(f, g) : \mathscr{H}om(X, Y) \to \mathscr{H}om(X', Y')$ given by the rule :

$$(\Delta^n \times X \xrightarrow{u} Y) \mapsto (\Delta^n \times X' \xrightarrow{g \circ u \circ (\Delta^n \times f)} \Delta^n \times Y') \qquad \forall n \in \mathbb{N}, \forall u \in \mathscr{H}om(X, Y)_n.$$

Remark 2.1.7. By example 2.1.5(ii), we get for all $X, Y \in \mathrm{Ob}(\mathsf{sSet})$ a natural isomorphism :

$$\mathscr{H}om(X, Y)^{\mathrm{op}} \xrightarrow{\sim} \mathscr{H}om(X^{\mathrm{op}}, Y^{\mathrm{op}}) \qquad (f : \Delta^n \times X \to Y) \mapsto (f^{\mathrm{op}} : \Delta^n \times X^{\mathrm{op}} \to Y^{\mathrm{op}}).$$

2.1.8. *Bisimplicial sets.* A *bisimplicial set* is a presheaf on the category $\mathbf{\Delta} \times \mathbf{\Delta}$. For a bisimplicial set X, we shall usually write

$$X_{m,n} := X([m], [n]) \qquad \forall m, n \in \mathbb{N}.$$

As usual, the bisimplicial sets form a category

$$\boxed{\mathsf{bSet} := \mathsf{Fun}((\mathbf{\Delta} \times \mathbf{\Delta})^{\mathrm{op}}, \mathsf{Set}).}$$

- According to remark 1.6.4(iv), we have a natural *product functor* :

$$\boxed{\boxtimes : \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{bSet} \qquad (X, Y) \mapsto X \boxtimes Y}$$

that preserves representable objects; namely, we have natural identifications :

$$\Delta^m \boxtimes \Delta^n \xrightarrow{\sim} h_{([m],[n])} \qquad \forall m, n \in \mathbb{N}.$$

- We shall also consider the obvious *diagonal functor*

$$\boxed{\mathrm{diag} := \mathsf{Fun}(\boldsymbol{\delta}^{\mathrm{op}}, \mathsf{Set}) : \mathsf{bSet} \to \mathsf{sSet} \qquad \mathrm{diag}(X)_n := X_{n,n} \qquad \forall n \in \mathbb{N}}$$

(notation of §1.3), for the diagonal embedding

$$\boldsymbol{\delta} : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta} \qquad [n] \mapsto ([n], [n]) \qquad \forall n \in \mathbb{N}$$

as well as the *flip functor*

$$\boxed{(-)^{\phi} := \mathsf{Fun}(\boldsymbol{\phi}^{\mathrm{op}}, \mathsf{Set}) : \mathsf{bSet} \xrightarrow{\sim} \mathsf{bSet} \qquad X^{\phi}_{m,n} := X_{n,m} \qquad \forall m, n \in \mathbb{N}}$$

induced by the *flip automorphism* of $\mathbf{\Delta} \times \mathbf{\Delta}$ that swaps the two factors

$$\boldsymbol{\phi} : \mathbf{\Delta} \times \mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta} \times \mathbf{\Delta} \qquad ([n], [m]) \mapsto ([m], [n]) \qquad \forall m, n \in \mathbb{N}.$$

• Moreover, a fourth type of functor will be useful for our discussion : let X be a bisimplicial set; since $(\mathbf{\Delta}\times\mathbf{\Delta})^{\mathrm{op}} = \mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}$, we may regard X as a functor $X : \mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}} \to$ Set, or equivalently, as a functor

$$X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{sSet} \qquad [n] \mapsto X_{\bullet,n}$$

via the natural identification of §1.3.3. Hence, $X_{\bullet,n}$ is the simplicial set such that $[m] \mapsto X_{m,n}$, for every $m, n \in \mathbb{N}$. Then, since $\mathsf{sSet}^{\mathrm{op}}$ is cocomplete (see §2.1.6), the functor $X_\bullet^{\mathrm{op}} : \mathbf{\Delta} \to \mathsf{sSet}^{\mathrm{op}}$ admits an extension by colimits (theorem 1.7.5(i)) which shall be denoted

$$\boxed{\langle -, X\rangle : \mathsf{sSet} \to \mathsf{sSet}^{\mathrm{op}} \qquad K \mapsto \langle K, X\rangle.}$$

Explicitly, for every $K \in \mathrm{Ob}(\mathsf{sSet})$, the simplicial set $\langle K, X\rangle$ represents the limit of the functor (notation of §1.7) :

$$(\mathbf{\Delta}/K)^{\mathrm{op}} \to \mathsf{sSet} \qquad ([n] \xrightarrow{s} K) \mapsto X_{\bullet,n}$$

and we fix for every such K a universal cone

$$\tau_s^K : \langle K, X\rangle \to X_{\bullet,n} \qquad \forall([n] \xrightarrow{s} K) \in \mathrm{Ob}(\mathbf{\Delta}/K).$$

Then, for every morphism $u : K \to L$ of simplicial sets, the induced morphism $\langle u, X\rangle : \langle L, X\rangle \to \langle K, X\rangle$ is the unique morphism of sSet such that the following diagram commutes for every $s \in \mathrm{Ob}(\mathbf{\Delta}/K)$:

$$\begin{tikzcd} \langle L, X\rangle \arrow[r, "\tau^L_{u\circ s}"] \arrow[d, "\langle u,X\rangle"'] & X_{\bullet,n} \arrow[d, equal] \\ \langle K, X\rangle \arrow[r, "\tau^K_s"] & X_{\bullet,n}. \end{tikzcd}$$

• Clearly, every morphism $\phi : X \to Y$ of bisimplicial sets induces a natural transformation $\phi_\bullet : X_\bullet \Rightarrow Y_\bullet$, whence an opposite natural transformation $\phi_\bullet^{\mathrm{op}} : Y_\bullet^{\mathrm{op}} \Rightarrow X_\bullet^{\mathrm{op}}$. Recall that the extension by colimits of $X_\bullet^{\mathrm{op}}$ is left adjoint to the functor $(X_\bullet^{\mathrm{op}})^* : \mathsf{sSet}^{\mathrm{op}} \to \mathsf{sSet}$ (notation of remark 1.6.6). Then $\phi_\bullet^{\mathrm{op}}$ induces first a natural transformation $(\phi_\bullet^{\mathrm{op}})^* : (X_\bullet^{\mathrm{op}})^* \Rightarrow (Y_\bullet^{\mathrm{op}})^*$ (remark 1.6.6(iv)), and then an adjoint transformation (relative to the *canonical adjunctions* for the pairs $(\langle -, X\rangle, (X_\bullet^{\mathrm{op}})^*)$ and $(\langle -, Y\rangle, (Y_\bullet^{\mathrm{op}})^*)$: see §1.6.10); we denote the latter by

$$\langle -, \phi\rangle := (\phi_\bullet^{\mathrm{op}})^{*\vee} : \langle -, Y\rangle \Rightarrow \langle -, X\rangle.$$

If $\psi : Y \to Z$ is a second morphism of bSet, then clearly $(\psi \circ \phi)_\bullet = \psi_\bullet \circ \phi_\bullet$, and a simple inspection yields : $((\psi \circ \phi)_\bullet^{\mathrm{op}})^* = (\psi_\bullet^{\mathrm{op}})^* \circ (\phi_\bullet^{\mathrm{op}})^*$, whence (see §1.6.10)

$$\langle -, \psi \circ \phi\rangle = \langle -, \phi\rangle \circ \langle -, \psi\rangle.$$

Summing up, we get a well-defined functor :

$$\boxed{\langle -, -\rangle : \mathsf{sSet}^{\mathrm{op}} \times \mathsf{bSet} \to \mathsf{sSet} \qquad (K, X) \mapsto \langle K, X\rangle}$$

that assigns to every pair of morphisms $u \in \mathsf{sSet}(K, L)$ and $\phi \in \mathsf{bSet}(X, Y)$, the morphism $\langle u, \phi\rangle := \langle K, \phi\rangle \circ \langle u, X\rangle = \langle u, Y\rangle \circ \langle L, \phi\rangle : \langle L, X\rangle \to \langle K, Y\rangle$ of sSet.

Remark 2.1.9. (i) By theorem 1.7.5(ii), we have natural identifications in sSet :

$$\boxed{\langle \Delta^n, X\rangle \xrightarrow{\sim} X_{\bullet,n} \qquad \forall X \in \mathrm{Ob}(\mathsf{bSet}), \forall n \in \mathbb{N}.}$$

(ii) A direct inspection yields natural identifications in bSet and sSet :

$$\boxed{\begin{array}{ll} (K \boxtimes L) \times (K' \boxtimes L') \xrightarrow{\sim} (K \times K') \boxtimes (L \times L') & \\ \qquad \mathsf{diag}(K \boxtimes L) = K \times L & \forall K, L, K', L' \in \mathrm{Ob}(\mathsf{sSet}) \\ \qquad (K \boxtimes L)^{\phi} = L \boxtimes K & \\ \qquad \mathsf{diag}(X^{\phi}) = \mathsf{diag}(X) & \forall X \in \mathrm{Ob}(\mathsf{bSet}). \end{array}}$$

Proposition 2.1.10. *Every $L \in \mathrm{Ob}(\mathsf{sSet})$ induces a natural adjoint pair of functors*

$$\boxed{- \boxtimes L : \mathsf{sSet} \rightleftarrows \mathsf{bSet} : \langle L, - \rangle.}$$

Proof. We need to exhibit bijections :

$$\mathsf{bSet}(K \boxtimes L, X) \xrightarrow{\sim} \mathsf{sSet}(K, \langle L, X \rangle) \qquad \forall K \in \mathrm{Ob}(\mathsf{sSet}), \forall X \in \mathrm{Ob}(\mathsf{bSet})$$

natural in K and X. Notice that we have natural bijections :

$$(*) \qquad \mathsf{sSet}(K, \langle L, X \rangle) = \mathsf{sSet}^{\mathrm{op}}(\langle L, X \rangle, K) \xrightarrow{\sim} \mathsf{sSet}(L, (X_\bullet^{\mathrm{op}})^*(K)).$$

On the other hand, to every $\phi \in \mathsf{bSet}(K \boxtimes L, X)$, every $n \in \mathbb{N}$ and every $s \in L_n$ we may assign the morphism

$$\phi_s : K \to X_{\bullet,n} \qquad \text{such that} \qquad t \mapsto \phi_{m,n}(t,s) \qquad \forall m \in \mathbb{N}, \forall t \in K_m.$$

It is easily seen that the rule : $s \mapsto \phi_s$ defines a morphism $\phi_\bullet : L \to (X_\bullet^{\mathrm{op}})^*(K)$ of sSet, and conversely, every morphism $L \to (X_\bullet^{\mathrm{op}})^*(K)$ is of the form $\phi_\bullet$ for a unique $\phi \in \mathsf{bSet}(K \boxtimes L, X)$ (the details shall be left to the reader). Hence, we get a system of bijections :

$$\mathsf{bSet}(K \boxtimes L, X) \xrightarrow{\sim} \mathsf{sSet}(L, (X_\bullet^{\mathrm{op}})^*(K)) \qquad \phi \mapsto \phi_\bullet$$

that are clearly natural in K and X. Combining with $(*)$ concludes the proof. □

2.1.11. *Connected components of a simplicial set.* Let S be any set; the *constant simplicial set* associated with S is the constant presheaf c_S on $\mathbf{\Delta}$ with value S (see §1.1.8); hence $c_{S,n} := S$ for every $n \in \mathbb{N}$, and $c_{S,\phi} := \mathbf{1}_S$ for every morphism ϕ of $\mathbf{\Delta}$, *i.e.* $c_S = (\Delta^0)^{(S)}$ (notation of §1.2.14). We thus obtain a fully faithful functor

$$\mathsf{c}_{\mathbf{\Delta}^{\mathrm{op}}} : \mathsf{Set} \to \mathsf{sSet} \qquad S \mapsto c_S$$

that assigns to every map of sets $f : S \to T$ the induced morphism of presheaves $c_f : c_S \to c_T$. Since Set is complete and cocomplete, according to §1.3 the functor $\mathsf{c}_{\mathbf{\Delta}^{\mathrm{op}}}$ admits left and right adjoints :

$$\mathsf{Colim}_{\mathbf{\Delta}^{\mathrm{op}}} : \mathsf{sSet} \to \mathsf{Set} \qquad \mathsf{Lim}_{\mathbf{\Delta}^{\mathrm{op}}} : \mathsf{sSet} \to \mathsf{Set}.$$

The left adjoint $\mathsf{Colim}_{\mathbf{\Delta}^{\mathrm{op}}}$ assigns to every $X \in \mathrm{Ob}(\mathsf{sSet})$ its *set of connected components,* and is traditionally denoted as well

$$\pi_0 : \mathsf{sSet} \to \mathsf{Set}.$$

Especially, we shall say that X *is connected* if $\pi_0(X)$ is a set of one element. This terminology will be justified by remark 2.3.1(iii). In order to describe more explicitly this functor, let $\mathbf{J} \subset \mathbf{\Delta}$ be the subcategory with $\mathrm{Ob}(\mathbf{J}) := \{[0], [1]\}$ and whose morphisms are $\mathbf{1}_{[0]}, \mathbf{1}_{[1]}$, and the face maps $\partial_0^1, \partial_1^1 : [0] \to [1]$. We observe :

Lemma 2.1.12. $\mathbf{J}$ *is coinitial in* $\mathbf{\Delta}$.

Proof. Let $i : \mathbf{J} \to \mathbf{\Delta}$ be the inclusion; we need to check that the category $i\mathbf{J}/[n]$ is connected for every $n \in \mathbb{N}$. To this aim, consider any two objects $[0] \xrightarrow{\phi} [n]$ and $[1] \xrightarrow{\psi} [n]$ of $i\mathbf{J}/[n]$; it suffices to exhibit a sequence of morphisms of $i\mathbf{J}/[n]$:

$$([0] \xrightarrow{\phi} [n]) \xrightarrow{\alpha/[n]} ([1] \xrightarrow{\tau} [n]) \xleftarrow{\beta/[n]} ([0] \xrightarrow{\eta} [n]) \xrightarrow{\gamma/[n]} ([1] \xrightarrow{\psi} [n]).$$

We let $\eta := \psi \circ \partial^1_1 : [0] \to [n]$, and $\gamma := \partial^1_1 : [0] \to [1]$; next, we let $\tau : [1] \to [n]$ be the unique map whose image is $\{\phi(0), \eta(0)\}$, and α (resp. β) any map $[0] \to [1]$ whose image lies in $\tau^{-1}(\phi(0))$ (resp. in $\tau^{-1}(\eta(0))$). □

From lemma 2.1.12 it follows that $\mathbf{J}^{\mathrm{op}}$ is cofinal in $\mathbf{\Delta}^{\mathrm{op}}$, and with corollary 1.5.4(i), we conclude that $\pi_0(X)$ is represented by the coequalizer of the face maps in degree 1, *i.e.* we have an *exact* diagram of sets, for every simplicial set X :

$$\boxed{X_1 \overset{d^0_1}{\underset{d^1_1}{\rightrightarrows}} X_0 \longrightarrow \pi_0(X).}$$

Example 2.1.13. For every $n \in \mathbb{N}$, the set $\Delta^n_0 := \mathbf{\Delta}([0], [n])$ is naturally identified with $[n]$; also, for every $i, j \in [n]$ there exists a unique $s \in \Delta^n_1 := \mathbf{\Delta}([1], [n])$ such that $\{d^0_1(s), d^1_1(s)\} = \{i, j\}$. In light of the foregoing discussion, we deduce that :

$$\boxed{\pi_0(\Delta^n) = \{\varnothing\} \qquad \forall n \in \mathbb{N}}$$

i.e. Δ^n *is connected* for every $n \in \mathbb{N}$.

2.1.14. *Pointed simplicial sets.* The *category of pointed simplicial sets* is the slice category

$$\mathsf{sSet}_\circ := \Delta^0/\mathsf{sSet}.$$

- Since sSet is complete and cocomplete (§2.1.6), the same holds for $\mathsf{sSet}_\circ$, and the target functor (notation of §1.4)

$$\mathrm{t} := \mathrm{t}_{\Delta^0} : \mathsf{sSet}_\circ \to \mathsf{sSet}$$

preserves and reflects all representable limits and all representable connected colimits; also, t is conservative, and preserves and reflects both monomorphisms and epimorphisms (corollary 1.4.6(ii,iii), lemma 1.4.7(ii) and [13, Exerc.2.66(ii)]). Moreover, t has a left adjoint

$$(-)_\circ : \mathsf{sSet} \to \mathsf{sSet}_\circ \qquad X \mapsto (j_X : \Delta^0 \to X_\circ := X \sqcup \Delta^0)$$

where j_X denotes the natural inclusion : namely, the adjunction for $((-)_\circ, \mathrm{t})$ assigns to every $(Y, y : \Delta^0 \to Y) \in \mathrm{Ob}(\mathsf{sSet}_\circ)$ and every morphism $f : X \to Y$ of sSet the unique morphism $(X_\circ, j_X) \to (Y, y)$ of $\mathsf{sSet}_\circ$ whose restriction to X agrees with f.

- We will also use the notation :

$$\underline{X}_n := X_n \qquad \forall \underline{X} := (X, x) \in \mathrm{Ob}(\mathsf{sSet}_\circ), \forall n \in \mathbb{N}.$$

Example 2.1.15. (i) For every $n \in \mathbb{N}$, the n-simplex Δ^n has a distinguished base point $0_n : \Delta^0 \to \Delta^n$, namely the morphism induced by the inclusion map $[0] \to [n]$, and clearly 0_n is also a base point for $\partial\Delta^n$, whence well-defined pointed simplicial sets

$$\underline{\Delta}^n := (\Delta^n, 0_n) \qquad \text{and} \qquad \partial\underline{\Delta}^n := (\partial\Delta^n, 0_n) \qquad \forall n \in \mathbb{N}.$$

(ii) For every $\underline{X} := (X, x), \underline{Y} := (Y, y) \in \mathrm{Ob}(\mathsf{sSet}_\circ)$, the simplicial set $\mathscr{H}om(X, Y)$ is endowed with a natural base point : namely, the 0-simplex $\Delta^0 \to \mathscr{H}om(X, Y)$ corresponding to the unique morphism $0_{XY} : X \to Y$ of sSet that factors through $y : \Delta^0 \to Y$. We have then a well-defined pointed simplicial set

$$\underline{\mathscr{H}om}(\underline{X}, \underline{Y}) := (\mathscr{H}om(X, Y), 0_{XY}).$$

Moreover, we define *the simplicial set of pointed morphisms from* $\underline{X}$ *to* $\underline{Y}$ as the fibre product in the cartesian square of $\mathsf{sSet}_\circ$:

$$\begin{tikzcd} \mathscr{H}om_\circ(\underline{X}, \underline{Y}) \arrow[r] \arrow[d] & \underline{\mathscr{H}om}(X, Y) \arrow[d, "x^*"] \\ \underline{\Delta}^0 \arrow[r, "y"] & \underline{Y} \end{tikzcd}$$

where $x^* := \mathscr{H}om(x, Y) : \underline{\mathscr{H}om}(X, Y) \to \underline{\mathscr{H}om}(\Delta^0, Y) \xrightarrow{\sim} \underline{Y}$ is *the evaluation at* x. Hence, we get a well-defined functor

$$\mathscr{H}om_\circ(-, -) : \mathsf{sSet}_\circ^{\mathrm{op}} \times \mathsf{sSet}_\circ \to \mathsf{sSet}_\circ.$$

(iii) Furthermore, the *wedge sum of* $\underline{X}$ *and* $\underline{Y}$ is defined as the pointed simplicial set

$$\underline{X} \vee \underline{Y} := (X \vee Y, (x, y)) \qquad \text{where} \qquad X \vee Y := X \times \{y\} \cup \{x\} \times Y$$

and *the smashed product of* $\underline{X}$ *and* $\underline{Y}$ is the amalgamated sum in the cocartesian square :

$$\begin{tikzcd} \underline{X} \vee \underline{Y} \arrow[r] \arrow[d] & \underline{X} \times \underline{Y} \arrow[d] \\ \underline{\Delta}^0 \arrow[r] & \underline{X} \wedge \underline{Y} \end{tikzcd}$$

and we let $X \wedge Y$ be the simplicial set underlying $\underline{X} \wedge \underline{Y}$. Clearly we get functors :

$$\mathsf{sSet}_\circ \times \mathsf{sSet}_\circ \xrightarrow{-\wedge-} \mathsf{sSet}_\circ \xleftarrow{-\vee-} \mathsf{sSet}_\circ \times \mathsf{sSet}_\circ.$$

Proposition 2.1.16. *(i) For every pointed simplicial set* $\underline{Y} := (Y, y)$*, the smashed product and the simplicial set of pointed morphisms yield by restriction an adjunction :*

$$\boxed{- \wedge \underline{Y} : \mathsf{sSet}_\circ \rightleftarrows \mathsf{sSet}_\circ : \mathscr{H}om_\circ(\underline{Y}, -).}$$

(ii) For every $\underline{X}, \underline{Y}, \underline{Z} \in \mathrm{Ob}(\mathsf{sSet}_\circ)$ *we have natural isomorphisms :*

$$\underline{X} \wedge \underline{Y} \xrightarrow{\sim} \underline{Y} \wedge \underline{X} \qquad (\underline{X} \wedge \underline{Y}) \wedge \underline{Z} \xrightarrow{\sim} \underline{X} \wedge (\underline{Y} \wedge \underline{Z}) \qquad \partial\underline{\Delta}^1 \wedge \underline{X} \xrightarrow{\sim} \underline{X} \qquad \underline{\Delta}^0 \wedge \underline{X} \xrightarrow{\sim} \underline{\Delta}^0$$
$$\mathscr{H}om_\circ(\underline{X}, \underline{\mathscr{H}om}(\underline{Y}, \underline{Z})) \xrightarrow{\sim} \underline{\mathscr{H}om}(\underline{Y}, \mathscr{H}om_\circ(\underline{X}, \underline{Z}))$$
$$\mathscr{H}om_\circ(\underline{X} \wedge \underline{Y}, \underline{Z}) \xrightarrow{\sim} \mathscr{H}om_\circ(\underline{X}, \mathscr{H}om_\circ(\underline{Y}, \underline{Z})).$$

(iii) For every $X, Y \in \mathrm{Ob}(\mathsf{sSet})$ *we have natural isomorphisms :*

$$(X \sqcup Y)_\circ \xrightarrow{\sim} X_\circ \vee Y_\circ \qquad (X \times Y)_\circ \xrightarrow{\sim} X_\circ \wedge Y_\circ.$$

Proof. (i): Let $\underline{X} := (X, x)$ and $\underline{Z} := (Z, z)$ be two pointed simplicial sets; the datum of a morphism $\underline{X} \to \mathscr{H}om_\circ(\underline{Y}, \underline{Z})$ of $\mathsf{sSet}_\circ$ is the same as that of a morphism $f : X \to \mathscr{H}om(Y, Z)$ of sSet whose composition with the evaluation $y^* : \mathscr{H}om(Y, Z) \to Z$ is the unique morphism $X \to Z$ that factors through $z : \Delta^0 \to Z$, and that maps the base point x of X to the base point of $\mathscr{H}om_\circ(\underline{Y},\underline{Z})$. The adjunction $\vartheta_{\bullet\bullet}$ of §2.1.6 for the pair $(-\times Y, \mathscr{H}om(Y,-))$ yields a commutative diagram whose vertical arrow are bijections :

$$\begin{tikzcd}
\mathrm{sSet}(\{x\},\mathscr{H}om(Y,Z)) \arrow[d] & \mathrm{sSet}(X,\mathscr{H}om(Y,Z)) \arrow[l] \arrow[r] \arrow[d] & \mathrm{sSet}(X,\mathscr{H}om(\{y\},Z)) \arrow[d] \\
\mathrm{sSet}(\{x\}\times Y,Z) & \mathrm{sSet}(X\times Y,Z) \arrow[l] \arrow[r] & \mathrm{sSet}(X\times\{y\},Z)
\end{tikzcd}$$

from which we see that f corresponds, under $\vartheta_{\bullet\bullet}$, to a morphism $X\times Y\to Z$ of sSet whose restriction to $\{x\}\times Y$ and $X\times\{y\}$ both factor through $z:\Delta^0\to Z$. Summing up, we see that, under $\vartheta_{\bullet\bullet}$, the datum of a morphism $\underline{X}\to\mathscr{H}om_\circ(\underline{Y},\underline{Z})$ of $\mathrm{sSet}_\circ$ is equivalent to that of a morphism $\underline{X}\wedge\underline{Y}\to\underline{Z}$. The naturality of this correspondence follows from the naturality of $\vartheta_{\bullet\bullet}$, whence (i).

(ii): The first four isomorphisms follow by a direct inspection of the constructions. Next, say that $\underline{X}=(X,x)$, $\underline{Y}=(Y,y)$, $\underline{Z}:=(Z,z)$; by virtue of lemma 1.6.14, we have cartesian squares of $\mathrm{sSet}_\circ$:

$$\begin{tikzcd}
\mathscr{H}om_\circ(\underline{X},\underline{\mathscr{H}om}(\underline{Y},\underline{Z})) \arrow[r] \arrow[d] & \underline{\Delta}^0 \arrow[d, "0_{YZ}"] \\
\underline{\mathscr{H}om}(\underline{X},\underline{\mathscr{H}om}(\underline{Y},\underline{Z})) \arrow[r, "\alpha"] & \underline{\mathscr{H}om}(\underline{Y},\underline{Z})
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{\mathscr{H}om}(\underline{Y},\mathscr{H}om_\circ(\underline{X},\underline{Z})) \arrow[r] \arrow[d] & \underline{\mathscr{H}om}(\underline{Y},\underline{\Delta}^0)=\underline{\Delta}^0 \arrow[d, "0_{YZ}"] \\
\underline{\mathscr{H}om}(\underline{Y},\underline{\mathscr{H}om}(\underline{X},\underline{Z})) \arrow[r, "\beta"] & \underline{\mathscr{H}om}(\underline{Y},\underline{Z})
\end{tikzcd}$$

with $\alpha:=\underline{\mathscr{H}om}(x,\underline{\mathscr{H}om}(\underline{Y},\underline{Z}))$ and $\beta:=\mathscr{H}om(\underline{Y},x^*)$. But the natural isomorphism

$$\underline{\mathscr{H}om}(\underline{X},\underline{\mathscr{H}om}(\underline{Y},\underline{Z}))\xrightarrow{\sim}\underline{\mathscr{H}om}(\underline{Y},\underline{\mathscr{H}om}(\underline{X},\underline{Z}))$$

identifies α with β, whence the fifth isomorphism of (ii).

For the last isomorphism, let us also write $\underline{X}\wedge\underline{Y}=(X\wedge Y,(x,y))$, and notice that, for every $n\in\mathbb{N}$, the n-simplices of $\mathscr{H}om_\circ(\underline{X}\wedge\underline{Y},\underline{Z})$ are naturally identified with the morphisms $\Delta^n\times(X\wedge Y)\to Z$ of simplicial sets whose composition with $\Delta^n\times(x,y)$: $\Delta^n\times\Delta^0\xrightarrow{\sim}\Delta^n\to\Delta^n\times(X\wedge Y)$ equals $\Delta^n\to\Delta^0\xrightarrow{z}Z$. By adjunction, these correspond to the morphisms $X\wedge Y\to\mathscr{H}om(\Delta^n,Z)$ whose composition with $(x,y):\Delta^0\to X\wedge Y$ equals the base point $\Delta^0\to\mathscr{H}om(\Delta^n,Z)$ of $\underline{\mathscr{H}om}(\underline{\Delta}^n,\underline{Z})$. So, by virtue of (i), the set of n-simplices of $\mathscr{H}om_\circ(\underline{X}\wedge\underline{Y},\underline{Z})$ is naturally identified with :

$$\mathrm{sSet}_\circ(\underline{X}\wedge\underline{Y},\underline{\mathscr{H}om}(\underline{\Delta}^n,\underline{Z}))\xrightarrow{\sim}A_n:=\mathrm{sSet}_\circ(\underline{X},\mathscr{H}om_\circ(\underline{Y},\underline{\mathscr{H}om}(\underline{\Delta}^n,\underline{Z}))).$$

On the other hand, the n-simplices of $\mathscr{H}om_\circ(\underline{X},\mathscr{H}om_\circ(\underline{Y},\underline{Z}))$ are naturally identified with the morphisms $\Delta^n\times X\to\mathscr{H}om_\circ(\underline{Y},\underline{Z})$ of simplicial sets whose composition with $\Delta^n\times x:\Delta^n\times\Delta^0\xrightarrow{\sim}\Delta^n\to\Delta^n\times X$ equals $\Delta^n\to\Delta^0\xrightarrow{0_{YZ}}\mathscr{H}om_\circ(\underline{Y},\underline{Z})$, and the adjunction for the pair $(-\times\Delta^n,\mathscr{H}om(\Delta^n,-))$ naturally identifies the latter with the set :

$$B_n:=\mathrm{sSet}_\circ(\underline{X},\underline{\mathscr{H}om}(\underline{\Delta}^n,\mathscr{H}om_\circ(\underline{Y},\underline{Z}))).$$

But by the foregoing, we have as well natural isomorphism of $\mathrm{sSet}_\circ$:

$$\omega_n:\mathscr{H}om_\circ(\underline{Y},\underline{\mathscr{H}om}(\underline{\Delta}^n,\underline{Z}))\xrightarrow{\sim}\underline{\mathscr{H}om}(\underline{\Delta}^n,\mathscr{H}om_\circ(\underline{Y},\underline{Z}))\qquad\forall n\in\mathbb{N}$$

and it is easily seen that the induced system $(\mathrm{sSet}_\circ(\underline{X},\omega_n):A_n\xrightarrow{\sim}B_n\mid n\in\mathbb{N})$ translates as the last sought isomorphism of (ii).

(iii) follows by inspecting the definitions. □

Corollary 2.1.17. *For every* $\underline{X},\underline{Y}\in\mathrm{Ob}(\mathrm{sSet}_\circ)$ *we have natural identifications :*

$$\boxed{\mathscr{H}om_\circ(\underline{X},\underline{Y})_n\xrightarrow{\sim}\mathrm{sSet}_\circ((\Delta^n)_\circ\wedge\underline{X},\underline{Y})\qquad\forall n\in\mathbb{N}.}$$

Proof. Indeed, the adjunction for the pair $((-)_\circ, \mathrm{t})$ yields natural identifications :

$$\begin{aligned}\mathscr{H}om_\circ(\underline{X},\underline{Y})_n \xrightarrow{\sim} \mathsf{sSet}(\Delta^n, \mathrm{t}(\mathscr{H}om_\circ(\underline{X},\underline{Y}))) &\xrightarrow{\sim} \mathsf{sSet}_\circ((\Delta^n)_\circ, \mathscr{H}om_\circ(\underline{X},\underline{Y})) \\ &\xrightarrow{\sim} \mathscr{H}om_\circ((\Delta^n)_\circ, \mathscr{H}om_\circ(\underline{X},\underline{Y}))_0\end{aligned}$$

and on the other hand, proposition 2.1.16(ii) yields the natural identifications :

$$\mathscr{H}om_\circ((\Delta^n)_\circ, \mathscr{H}om_\circ(\underline{X},\underline{Y}))_0 \xrightarrow{\sim} \mathscr{H}om_\circ((\Delta^n)_\circ \wedge \underline{X},\underline{Y})_0 \xrightarrow{\sim} \mathsf{sSet}_\circ((\Delta^n)_\circ \wedge \underline{X},\underline{Y})$$

whence the corollary. □

2.2. **Cellular filtrations.** This section presents an axiomatic treatment of some arguments from Gabriel-Zisman's classical book [6]. The interest of this axiomatic treatment is that it makes these arguments available in more general situations.

Definition 2.2.1. An *Eilenberg-Zilber category* is a datum

$$(\mathscr{A}_\star, d) := (\mathscr{A}, \mathscr{A}_+, \mathscr{A}_-, d)$$

where $\mathscr{A}$ is a small category, $\mathscr{A}_+, \mathscr{A}_- \subset \mathscr{A}$ are subcategories with $\mathrm{Ob}(\mathscr{A}_+) = \mathrm{Ob}(\mathscr{A}_-) = \mathrm{Ob}(\mathscr{A})$, and $d : \mathrm{Ob}(\mathscr{A}) \to \mathbb{N}$ is a map verifying the following conditions :

- **(EZ0)** Every isomorphism of $\mathscr{A}$ is an identity morphism.
- **(EZ1)** If $a \to b$ is a morphism in $\mathscr{A}_+$ (resp. in $\mathscr{A}_-$) that is not an identity, then $d(a) < d(b)$ (resp. $d(a) > d(b)$).
- **(EZ2)** Any morphism u of $\mathscr{A}$ has a factorization $u = i \circ p$ with p in $\mathscr{A}_-$ and i in $\mathscr{A}_+$.
- **(EZ3)** For every morphism $p : a \to b$ of $\mathscr{A}_-$ there exists $s \in \mathscr{A}(b, a)$ that is *a section of* p, *i.e.* such that $p \circ s = \mathbf{1}_b$; moreover, any two morphisms $p, p' : a \rightrightarrows b$ in $\mathscr{A}_-$ having the same set of sections in $\mathscr{A}$ are equal.

We shall say that an object a of $\mathscr{A}$ *is of dimension n* if $d(a) = n$.

Example 2.2.2. (i) $\mathbf{\Delta}$ is an Eilenberg-Zilber category, with $\mathbf{\Delta}_+$ (resp. $\mathbf{\Delta}_-$) the subcategory of monomorphisms (resp. epimorphisms), and $d([n]) := n$ for all $n \in \mathbb{N}$.

(ii) If $(\mathscr{A}_\star, d)$ is an Eilenberg-Zilber category, and if F is any presheaf on $\mathscr{A}$, we get an induced Eilenberg-Zilber structure on $\mathscr{A}/F$: namely, if the functor $\psi_F : \mathscr{A}/F \to \mathscr{A}$ is defined as in §1.7, we set $(\mathscr{A}/F)_+ := \mathscr{A}_+ \times_{\mathscr{A}} \mathscr{A}/F$ and $(\mathscr{A}/F)_- := \mathscr{A}_- \times_{\mathscr{A}} \mathscr{A}/F$, *i.e.* :

$$\mathrm{Mor}((\mathscr{A}/F)_+) := \psi_F^{-1}(\mathrm{Mor}(\mathscr{A}_+)) \qquad \text{and} \qquad \mathrm{Mor}((\mathscr{A}/F)_-) := \psi_F^{-1}(\mathrm{Mor}(\mathscr{A}_-)).$$

Then we set $d(a, s) := d(a)$ for every $(a, s) \in \mathrm{Ob}(\mathscr{A}/F)$. Indeed, axioms (EZ0), (EZ1) and (EZ2) trivially hold for $((\mathscr{A}/F)_\star, d)$. Lastly, let $p : (a, s) \to (b, t)$ be a morphism of $(\mathscr{A}/F)_-$ and $i : b \to a$ a morphism of $\mathscr{A}$ with $p \circ i = \mathbf{1}_b$; then $i^*(s) = i^*p^*(t) = t$, so $i : (b, t) \to (a, s)$ is a section of p in $\mathscr{A}/F$, whence (EZ3).

(iii) If $(\mathscr{A}_\star, d_{\mathscr{A}})$ and $(\mathscr{B}_\star, d_{\mathscr{B}})$ are two Eilenberg-Zilber categories, the product $\mathscr{A} \times \mathscr{B}$ carries a natural Eilenberg-Zilber structure, such that $(\mathscr{A} \times \mathscr{B})_\star := \mathscr{A}_\star \times \mathscr{B}_\star$ for all $\star \in \{+, -\}$, and with $d(a, b) := d_{\mathscr{A}}(a) + d_{\mathscr{B}}(b)$ for all $(a, b) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B})$.

Definition 2.2.3. (i) Let $(\mathscr{A}_\star, d)$ be an Eilenberg-Zilber category, X a presheaf on $\mathscr{A}$, and x a section of X over some $a \in \mathrm{Ob}(\mathscr{A})$. We say that x is *degenerate* if there exists a morphism $\sigma : a \to b$ of $\mathscr{A}$ with $d(a) > d(b)$, and $y \in X_b$ such that $x = \sigma^*(y)$. The pair (σ, y) shall be called a *decomposition of x*.

(ii) For every $n \in \mathbb{Z}$, we denote by :

$$Sk_n(X)$$

the maximal subpresheaf of X such that for every $b \in \mathrm{Ob}(\mathscr{A})$ with $d(b) > n$, all the sections of $Sk_n(X)$ over b are degenerate. Notice that $Sk_n(X) = \varnothing$ if $n < 0$. It is easily seen that every morphism of presheaves $f : X \to Y$ induces by restriction a morphism

$$Sk_n(f) : Sk_n(X) \to Sk_n(Y).$$

So, the rules : $X \mapsto Sk_n(X)$ and $f \mapsto Sk_n(f)$ yield a well-defined functor $Sk_n : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$.

(iii) For every $a \in \mathrm{Ob}(\mathscr{A})$, the *boundary* of the presheaf h_a represented by a is :

$$\partial h_a := Sk_{d(a)-1}(h_a).$$

Remark 2.2.4. Notice also that :

$$h_a = Sk_{d(a)}(h_a) \qquad \forall a \in \mathrm{Ob}(\mathscr{A}).$$

Indeed, for every $b \in \mathrm{Ob}(\mathscr{A})$ with $d(b) > d(a)$ and every $u \in h_a(b)$ we have $u = u^*(\mathbf{1}_a)$, so any such u is a degenerate section of h_a.

Lemma 2.2.5. (Eilenberg-Zilber) *Let $(\mathscr{A}_\star, d)$ be an Eilenberg-Zilber category, X a presheaf on $\mathscr{A}$, and x a degenerate section of X over some $a \in \mathrm{Ob}(\mathscr{A})$. Then there exists a unique decomposition (σ, y) of x such that σ is a morphism of $\mathscr{A}_-$ and y is non-degenerate.*

Proof. For every decomposition $(\tau : a \to b, z)$ of x, let us set $d(\tau, z) := d(b)$, and let us denote by Σ the set of decompositions (τ, z) of x such that τ lies in $\mathscr{A}_-$.

Claim 2.2.6. For every decomposition (τ, z) of x there exists $(\tau', z') \in \Sigma$ with $d(\tau', z') \leq d(\tau, z)$.

Proof: By (EZ2) we have $\tau = i \circ p$ for some morphism $p : a \to c$ of $\mathscr{A}_-$ and $i : c \to b$ of $\mathscr{A}_+$; notice that $p \neq \mathbf{1}_a$, since otherwise we would have $d(a) = d(c) < d(b)$. Then $(p, i^*(z)) \in \Sigma$, and $d(p, i^*(z)) = d(c) \leq d(\tau, z)$. ◇

Hence, let $(\sigma; a \to b, y) \in \Sigma$ such that $d(\sigma, y) = \min\{d(\tau, z) \mid (\tau, z) \in \Sigma\}$. Then y must be non-degenerate, due to claim 2.2.6 and the minimality of $d(\sigma, y)$. It remains to check the uniqueness property of (σ, y). However, let $(\sigma' : a \to b', y') \in \Sigma$ be another decomposition with $d(b) = d(b')$; by (EZ3) we have a morphism $i : b \to a$ with $\sigma \circ i = \mathbf{1}_b$, and we set $u := \sigma' \circ i : b \to b'$. Then :

$$u^*(y') = i^* \circ \sigma'^*(y') = i^*(x) = i^* \circ \sigma^*(y) = y.$$

Next, by (EZ2) we have a factorization $u = j \circ p$ with $p : b \to e$ in $\mathscr{A}_-$ and $j : e \to b'$ in $\mathscr{A}_+$; if $p \neq \mathbf{1}_b$, then $d(b) > d(e)$, by (EZ1); but since $(p \circ \sigma : a \to e, j^*(y')) \in \Sigma$, this would contradict the minimality of $d(b)$. Hence $p = \mathbf{1}_b$, and then $j = \mathbf{1}_b$ as well, again by (EZ1), since $d(b) = d(b')$. Thus, $u = \mathbf{1}_b$, so that $b = b'$ and $y = y'$. Morever, it follows that $\sigma' \circ i = \mathbf{1}_b$, *i.e.* i is a section of σ' as well; symmetrically, every section of σ' is also a section of σ, and then $\sigma = \sigma'$, by (EZ3). □

Example 2.2.7. (i) Let X be any simplicial set, *i.e.* a presheaf on the Eilenberg-Zilber category $(\mathbf{\Delta}, \mathbf{\Delta}_+, \mathbf{\Delta}_-, d)$ of example 2.2.2(i); it follows easily from lemma 2.2.5 that the degenerate sections of X in any degree $n \in \mathbb{N}$ are precisely the n-simplices of X that are in the image of some degeneracy map σ_i^n.

(ii) Let $(\mathscr{A}_\star, d_{\mathscr{A}})$ and $(\mathscr{B}_\star, d_{\mathscr{B}})$ be two Eilenberg-Zilber categories, $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, $Y \in \mathrm{Ob}(\widehat{\mathscr{B}})$, and endow $\mathscr{A} \times \mathscr{B}$ with the Eilenberg-Zilber category structure as in example 2.2.2(iii). With the notation of remark 1.6.4(iv) and example 1.8.11(ii), we have :

$$\boxed{Sk_n(X \boxtimes Y) = \bigcup_{i+j=n} Sk_i(X) \boxtimes Sk_j(Y) \qquad \forall n \in \mathbb{N}}$$

Especially, in view of remark 2.2.4, we deduce:

$$\boxed{\partial(h_{(a,b)}) = \partial h_a \boxtimes h_b \cup h_a \boxtimes \partial h_b \qquad \forall (a,b) \in \mathrm{Ob}(\mathscr{A} \times \mathscr{B}).}$$

Theorem 2.2.8. *Let $(\mathscr{A}_\star, d)$ be an Eilenberg-Zilber category, $X \subset Y$ two presheaves over $\mathscr{A}$, and $n \in \mathbb{N}$. We have a cocartesian diagram in $\widehat{\mathscr{A}}$:*

$$(*) \qquad \begin{tikzcd} \coprod_{a\in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)} \arrow[r] \arrow[d] & X \cup Sk_{n-1}(Y) \arrow[d] \\ \coprod_{a\in d^{-1}(n)} h_a^{(\Sigma_a)} \arrow[r] & X \cup Sk_n(Y) \end{tikzcd}$$

where $\Sigma_a := Y_a \setminus (Sk_{n-1}(Y)_a \cup X_a)$ for every $a \in d^{-1}(n)$, and with vertical arrows given by the natural inclusions (the copower $(-)^{(\Sigma_a)}$ is as in §1.2.14).

Proof. Explicitly, for every $a \in d^{-1}(n)$ and every $y \in \Sigma_a$ we have, by Yoneda's lemma, a unique morphism $f^y : h_a \to Y$ of $\widehat{\mathscr{A}}$ such that $\mathbf{1}_a \mapsto y$, and in light of remark 2.2.4, the image of f^y lands in the subpresheaf $Sk_n(Y)$; also, the restriction $g^y : \partial h_a \to Y$ of f^y obviously has image in $Sk_{n-1}(Y)$, and the upper (resp. lower) horizontal arrows of $(*)$ are the disjoint union of these morphisms g^y (resp. f^y). The commutativity of $(*)$ is then immediate. Now, let Z be any presheaf on $\mathscr{A}$, and $X \cup Sk_{n-1}(Y) \xrightarrow{u} Z \xleftarrow{v} \coprod_{a\in d^{-1}(n)} h_a^{(\Sigma_a)}$ two morphisms of $\widehat{\mathscr{A}}$ that agree on $\coprod_{a\in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)}$. Hence, v is equivalent to the datum of a system :

$$(z^y \mid a \in d^{-1}(n), y \in \Sigma_a) \quad \text{with} \quad z^y \in Z_a \quad \forall a \in d^{-1}(n), \forall y \in \Sigma_a$$

and every z^y corresponds to a unique morphism $v^y : h_a \to Z$. We define a morphism $w : X \cup Sk_n(Y) \to Z$ as follows. Let $a \in \mathrm{Ob}(\mathscr{A})$ and $y \in X_a \cup Sk_n(Y)_a$; if $y \in X_a \cup Sk_{n-1}(Y)_a$, we set $w_a(y) := u_a(y)$. If $y \notin X_a \cup Sk_{n-1}(Y)_a$, then we must have $d(a) \geq n$; if $d(a) = n$, then $y \in \Sigma_a$, and we set $w_a(y) := z^y$. Lastly, if $d(a) > n$, then y is a degenerate section of Y_a, so there exists a unique decomposition (σ, s) of y where $\sigma : a \to c$ is a morphism of $\mathscr{A}_-$ and s is non-degenerate (lemma 2.2.5); but notice that $d(c) = n$ and $s \notin X_c$, since otherwise we would have $y \in X_a \cup Sk_{n-1}(Y)_a$. Hence, in this case we let $w_a(y) := \sigma^*(z^s)$.

- We need to check that the rule : $a \mapsto w_a$ yields a morphism $X \cup Sk_n(Y) \to Z$ of presheaves. Hence, let $t : b \to a$ be any morphism in $\mathscr{A}$, and $y \in X_a \cup Sk_n(Y)_a$; if $y \in X_a \cup Sk_{n-1}(Y)_a$, then $t^*(y) \in X_b \cup Sk_{n-1}(Y)_b$, so $w_b(t^*(y)) = u_b(t^*(y)) = t^*u_a(y) = t^*(w_a(y))$, as required. In case $y \notin X_a \cup Sk_{n-1}(Y)_a$, in view of (EZ2), we may consider separately the cases where t lies in $\mathscr{A}_+$ and in $\mathscr{A}_-$, and clearly we may suppose as well that $t \neq \mathbf{1}_a$. Suppose first that $d(a) = n$ and that t lies in $\mathscr{A}_+$, so that $d(b) < n$ by (EZ1); then $w_a(y) = z^y$ and $t^*(y) \in Sk_{n-1}(Y)$, whence $w_b(t^*(y)) = u_b(t^*(y)) = t^*(u_a(y))$. On the other hand, we have :

$$t^*(w_a(y)) = t^*(v_a^y(\mathbf{1}_a)) = v_b^y(t^*(\mathbf{1}_a)) = u_b \circ g_b^y(t^*(\mathbf{1}_a)) = t^*(u_a \circ f_a^y(\mathbf{1}_a)) = t^*(u_a(y))$$

since $t^*(\mathbf{1}_a) \in (\partial h_a)_b$ and since u and v agree on $\coprod_{a\in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)}$.

- If $d(a) = n$ and t lies in $\mathscr{A}_-$, then $d(b) > n$ and (t, y) is the unique decomposition of $t^*(y)$ provided by lemma 2.2.5, so that the sought identity $t^*(w_a(y)) = w_b(t^*(y))$ holds by definition of w_b.

- Lastly, if $d(a) > n$, let $(\sigma : a \to c, s)$ be as in the foregoing; by (EZ2) we may write $\sigma \circ t = i \circ p$, where p lies in $\mathscr{A}_-$ and i lies in $\mathscr{A}_+$. By construction :

$$t^*(w_a(y)) = t^* \circ \sigma^*(z^s) = p^* \circ i^*(z^s) = p^* \circ i^*(w_c(s))$$

and $w_b(t^*(s)) = w_b(p^* \circ i^*(s))$, so we are reduced to checking the sought identity for $t = i$ and $t = p$; both these cases are already known, by the foregoing. It is clear that w is the unique morphism that restricts to u and v on $X \cup Sk_{n-1}(Y)$ and respectively $\bigsqcup_{a \in d^{-1}(n)} h_a^{(\Sigma_a)}$, so the proof is concluded. □

Definition 2.2.9. Let $\mathscr{A}$ be a small category, and $\Sigma \subset \mathrm{Ob}(\widehat{\mathscr{A}})$. We say that Σ is *saturated by monomorphisms*, if the following holds :

(a) For any small family $(F_i \mid i \in I)$ of elements of Σ, we have $\bigsqcup_{i \in I} F_i \in \Sigma$.

(b) For any cocartesian square of $\widehat{\mathscr{A}}$ whose vertical arrows are monomorphisms :

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \\ G \arrow[r] & G' \end{tikzcd}$$

and with $F, F', G \in \Sigma$, we have $G' \in \Sigma$.

(c) For any sequence of monomorphisms of $\widehat{\mathscr{A}}$:

$$F_0 \to F_1 \to F_2 \to \cdots$$

such that $F_i \in \Sigma$ for every $i \in \mathbb{N}$, we have $\bigcup_{i \in \mathbb{N}} F_i \in \Sigma$.

Corollary 2.2.10. *Let $(\mathscr{A}_\star, d)$ be an Eilenberg-Zilber category, and $\Sigma \subset \mathrm{Ob}(\widehat{\mathscr{A}})$ a class saturated by monomorphism. If Σ contains all representable presheaves, then $\Sigma = \mathrm{Ob}(\widehat{\mathscr{A}})$.*

Proof. Let Y be a presheaf on $\mathscr{A}$; we apply theorem 2.2.8 with $X = \varnothing$. Then, for every $n \in \mathbb{N}$ we have a cocartesian square :

$$\begin{tikzcd} \bigsqcup_{a \in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)} \arrow[r] \arrow[d] & Sk_{n-1}(Y) \arrow[d] \\ \bigsqcup_{a \in d^{-1}(n)} h_a^{(\Sigma_a)} \arrow[r] & Sk_n(Y) \end{tikzcd}$$

with $\Sigma_a := Y_a \setminus Sk_{n-1}(Y)_a$ for every $a \in d^{-1}(n)$. Let us check by induction on n that each $Sk_n(Y)$ lies in Σ. For $n = 0$, we have $\partial h_a = Sk_{-1}(Y) = \varnothing$, so $Sk_0(Y)$ is the coproduct of a small family of representable presheaves, so it lies in Σ.

Suppose next that $n > 0$, and that the assertion is already known for $n-1$; then the top row of the diagram consists of elements of Σ, since $\partial h_a = Sk_{n-1}(\partial h_a)$. The same holds for the lower left corner, so also for $Sk_n(Y)$. Now, $Y = \bigcup_{n \in \mathbb{N}} Sk_n(Y)$, so $Y \in \Sigma$. □

Corollary 2.2.11. *Let $(\mathscr{A}_\star, d)$ be an Eilenberg-Zilber category, such that each representable presheaf over $\mathscr{A}$ has finitely many non-degenerate sections. Then, for every presheaf F which has only finitely many non-degenerate sections, the functor*

$$\widehat{\mathscr{A}}(F, -) : \widehat{\mathscr{A}} \to \mathsf{Set} \qquad G \mapsto \widehat{\mathscr{A}}(F, G)$$

preserves all small filtered colimits.

Proof. Let $\Sigma \subset \mathrm{Ob}(\widehat{\mathscr{A}})$ be the class of all presheaves X such that the functor $\widehat{\mathscr{A}}(X, -)$ preserves filtered colimits. Let us check first that Σ is stable under finite colimits. Indeed, let $X_\bullet : I \to \widehat{\mathscr{A}}$ be a functor from a finite category I, such that $X_i \in \Sigma$ for all $i \in \mathrm{Ob}(I)$, and let $G_\bullet : J \to \widehat{\mathscr{A}}$ be a functor from a small filtered category J; we need to check that the natural map :

$$(*) \qquad \varinjlim_{j\in J} \widehat{\mathscr{A}}(\varinjlim_{i\in I} X_i, Y_j) \to \widehat{\mathscr{A}}(\varinjlim_{i\in I} X_i, \varinjlim_{j\in J} Y_j)$$

is bijective. However, it is easily seen that the natural map :

$$(\dagger) \qquad \widehat{\mathscr{A}}(\varinjlim_{i\in I} X_i, Y_j) \to \varprojlim_{i\in I} \widehat{\mathscr{A}}(X_i, Y_j)$$

is bijective for every $j \in \mathrm{Ob}(J)$; on the other hand, since filtered colimits commute with finite limits in the category of sets, the natural map :

$$(\dagger\dagger) \qquad \varinjlim_{j\in J} \varprojlim_{i\in I} \widehat{\mathscr{A}}(X_i, Y_j) \to \varprojlim_{i\in I} \varinjlim_{j\in J} \widehat{\mathscr{A}}(X_i, Y_j)$$

is bijective as well, and by assumption, the natural map :

$$(\dagger\dagger\dagger) \qquad \varinjlim_{j\in J} \widehat{\mathscr{A}}(X_i, Y_j) \to \widehat{\mathscr{A}}(X_i, \varinjlim_{j\in J} Y_j)$$

is bijective for every $i \in \mathrm{Ob}(I)$. Lastly, it is easily seen that by combining $(\dagger)$, $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$ we get the map $(*)$, so the latter is a bijection, as sought.

Now, let F be a presheaf on $\mathscr{A}$ with only finitely many non-degenerate sections; by applying theorem 2.2.8 with $Y = F$ and $X = \varnothing$, we deduce that $F = Sk_n(F)$ for some $n \in \mathbb{Z}$, and we argue by induction on n. If $n < 0$, then $F = \varnothing$, and the assertion is clear, since a filtered colimit of sets of one element is a set of one element. If $n \geq 0$, theorem 2.2.8 yields a cocartesian diagram in $\widehat{\mathscr{A}}$:

$$\begin{tikzcd}
\coprod_{a\in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)} \arrow[r] \arrow[d] & Sk_{n-1}(F) \arrow[d] \\
\coprod_{a\in d^{-1}(n)} h_a^{(\Sigma_a)} \arrow[r] & Sk_n(F)
\end{tikzcd}$$

where $\Sigma_a := F_a \setminus Sk_{n-1}(F)_a$ is a finite set for every $a \in d^{-1}(n)$. By inductive assumption, the presheaves $\coprod_{a\in d^{-1}(n)} (\partial h_a)^{(\Sigma_a)}$ and $Sk_{n-1}(F)$ lie in Σ, hence, by the foregoing, we are reduced to checking that the same holds for $\coprod_{a\in d^{-1}(n)} h_a^{(\Sigma_a)}$.

We are then easily reduced to checking that h_a lies in Σ for every $a \in \mathrm{Ob}(\mathscr{A})$; but Yoneda's lemma identifies the functor $\widehat{\mathscr{A}}(h_a, -)$ with the evaluation functor : $G \mapsto G_a$, and the latter commutes with all representable colimits. □

Example 2.2.12. (i) Let A be a simplicial set that has finitely many non-degenerate simplices. Then the functor

$$\mathscr{H}om(A, -) : \mathsf{sSet} \to \mathsf{sSet} \qquad X \mapsto \mathscr{H}om(A, X)$$

preserves all small filtered colimits. Indeed, since the colimits of sSet are computed termwise (see §2.1.6), it suffices to check that the functor

$$\mathsf{sSet}(\Delta^n \times A, -) : \mathsf{sSet} \to \mathsf{Set} \qquad X \mapsto \mathsf{sSet}(\Delta^n \times A, X)$$

preserves small filtered colimits for every $n \in \mathbb{N}$. Since $\Delta^n \times A$ has finitely many non-degenerate sections for every such n, this follows from corollary 2.2.11.

(ii) Since the finite limits in sSet commute with all small filtered colimits, we easily deduce from (i) and corollary 1.4.6(ii,iii) that for every pointed simplicial set $\underline{A} := (A, a)$ such that A has finitely many non-degenerate sections, the functor

$$\mathscr{H}om_\circ(\underline{A}, -) : \mathrm{sSet}_\circ \to \mathrm{sSet}_\circ \qquad \underline{X} \mapsto \mathscr{H}om_\circ(\underline{A}, \underline{X})$$

preserves all small filtered colimits.

2.3. **Nerves.** Recall that the category poSet of partially ordered sets can be regarded as a full subcategory of the category Cat of all small categories (see §1.9.2). By restricting such embedding to $\mathbf{\Delta}$, we get a fully faithful functor :

$$i : \mathbf{\Delta} \to \mathrm{Cat}.$$

The *nerve functor* is defined as the evaluation at i (notation of remark 1.6.6(iii)) :

$$\boxed{N := i^* : \mathrm{Cat} \to \mathrm{sSet} \qquad \mathscr{C} \mapsto ([n] \mapsto \mathrm{Cat}([n], \mathscr{C})).}$$

Hence, we have a natural bijection :

$$\eta^0_{\mathscr{C}} : \mathrm{Ob}(\mathscr{C}) \xrightarrow{\sim} N(\mathscr{C})_0$$

and for every $n \in \mathbb{N} \setminus \{0\}$, the n-simplexes of $N(\mathscr{C})$ are the strings :

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of n morphisms of $\mathscr{C}$. Especially, we have a natural identification :

$$\eta^1_{\mathscr{C}} : \mathrm{Mor}(\mathscr{C}) \xrightarrow{\sim} N(\mathscr{C})_1$$

such that :

$$d^1_1 \circ \eta^1_{\mathscr{C}}(u) = \eta^0_{\mathscr{C}}(x) \qquad d^0_1 \circ \eta^1_{\mathscr{C}}(u) = \eta^0_{\mathscr{C}}(y) \qquad \forall x, y \in \mathrm{Ob}(\mathscr{C}), \forall u \in \mathscr{C}(x, y).$$

Remark 2.3.1. (i) From the foregoing description, we get a natural identification :

$$N(\mathscr{C}^{\mathrm{op}}) \xrightarrow{\sim} N(\mathscr{C})^{\mathrm{op}}$$

where $N(\mathscr{C})^{\mathrm{op}}$ denotes the front-to-back dual of $N(\mathscr{C})$ as in §2.1.4.

(ii) Moreover, the composition law of the category $\mathscr{C}$ is determined by the datum of $Sk_2(N(\mathscr{C}))$, as follows. Let $x, y, z \in \mathrm{Ob}(\mathscr{C})$, $u \in \mathscr{C}(x, y)$ and $v \in \mathscr{C}(y, z)$. Then we get the 2-simplex of $N(\mathscr{C})$:

$$\eta^2_{\mathscr{C}}(u, v) := (x \xrightarrow{u} y \xrightarrow{v} z) \quad \text{with} \quad \eta^1_{\mathscr{C}}(u) = d^2_2(\eta^2_{\mathscr{C}}(u, v)) \quad \eta^1_{\mathscr{C}}(v) = d^0_2(\eta^2_{\mathscr{C}}(u, v))$$

and notice that in the category (*i.e.* in the partially ordered set) $[2]$, we have $\overrightarrow{12} \circ \overrightarrow{01} = \overrightarrow{02}$; it follows that :

$$(*) \qquad \eta^1_{\mathscr{C}}(v \circ u) = d^1_2(\eta^2_{\mathscr{C}}(u, v)).$$

(iii) A direct inspection of the definitions yields a natural identification :

$$\boxed{\pi_0(N\mathscr{C}) \xrightarrow{\sim} \pi_0(\mathscr{C}) \qquad \forall \mathscr{C} \in \mathrm{Ob}(\mathrm{Cat})}$$

where $\pi_0(N\mathscr{C})$ is the set of connected components of the nerve of $\mathscr{C}$, defined as in §2.1.11, and $\pi_0(\mathscr{C})$ is defined as in example 1.2.6.

Lemma 2.3.2. *The nerve functor is fully faithful.*

Proof. Let $F : \mathscr{C} \to \mathscr{C}'$ be any functor between small categories. By a simple inspection we get commutative diagrams of sets :

$$\begin{tikzcd} \mathrm{Ob}(\mathscr{C}) \arrow[r, "\mathrm{Ob}(F)"] \arrow[d, "\eta^0_{\mathscr{C}}"'] & \mathrm{Ob}(\mathscr{C}') \arrow[d, "\eta^0_{\mathscr{C}'}"] \\ N(\mathscr{C})_0 \arrow[r, "N(F)_0"] & N(\mathscr{C}') \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}(x,y) \arrow[r, "F_{x,y}"] \arrow[d, "\eta^1_{\mathscr{C}}"'] & \mathscr{C}'(Fx,Fy) \arrow[d, "\eta^1_{\mathscr{C}'}"] \\ N(\mathscr{C})_1 \arrow[r, "N(F)_1"] & N(\mathscr{C}')_1 \end{tikzcd} \qquad \forall x, y \in \mathrm{Ob}(\mathscr{C})$$

whence the faithfulness of N. Next, let $g : N(\mathscr{C}) \to N(\mathscr{C}')$ be any morphism of simplicial sets. We define a functor $G : \mathscr{C} \to \mathscr{C}'$ as follows. The map $\mathrm{Ob}(G) : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$ is induced by $g_0 : N(\mathscr{C})_0 \to N(\mathscr{C}')_0$ via the identifications $\eta^0_{\mathscr{C}}$ and $\eta^0_{\mathscr{C}'}$. Likewise, for every $x, y \in \mathrm{Ob}(\mathscr{C})$, the map $G_{x,y} : \mathscr{C}(x,y) \to \mathscr{C}'(f(x), f(y))$ is induced by $g_1 : N(\mathscr{C})_1 \to N(\mathscr{C}')_1$ via the identifications $\eta^1_{\mathscr{C}}$ and $\eta^1_{\mathscr{C}'}$.

We need to check that $G(\mathbf{1}_x) = \mathbf{1}_{Gx}$ for every $x \in \mathrm{Ob}(\mathscr{C})$ and $G(v \circ u) = Gv \circ Gu$ for every composable pair of morphisms $x \xrightarrow{u} y \xrightarrow{v} z$ of $\mathscr{C}$. However, notice that :

$$s^0_0(\eta^0_{\mathscr{C}}(x)) = \eta^1_{\mathscr{C}}(\mathbf{1}_x) \qquad \forall x \in \mathrm{Ob}(\mathscr{C}).$$

Since $g_1 \circ s^0_0 = s^0_0 \circ g_0$, we then get the first stated identity. For the second stated identity, in light of identity $(*)$ of remark 2.3.1(ii) it suffices to notice that :

$$g_2(\eta^2_{\mathscr{C}}(u,v)) = \eta^2_{\mathscr{C}'}(Gu, Gv)$$

and to recall that $d^1_2 \circ g_2 = g_1 \circ d^1_2$.

Lastly, let us check that $N(G) = g$. To this aim, for every $n \in \mathbb{N} \setminus \{0\}$ and every $i = 0, \dots, n-1$, let $t^i_n : [1] \to [n]$ be the unique morphism of $\mathbf{\Delta}$ whose image is $\{i, i+1\}$. Now, let $f_\bullet := (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n)$ be any n-simplex of $N(\mathscr{C})$; notice that $f_\bullet$ is characterized as the *unique* n-simplex of $N(\mathscr{C})$ such that $t^{i*}_n(f_\bullet) = f_{i+1}$ for every $i = 0, \dots, n-1$. Likewise, $Gf_\bullet := (Gx_0 \xrightarrow{Gf_1} Gx_1 \xrightarrow{Gf_2} \cdots \xrightarrow{Gf_n} Gx_n)$ is characterized as the *unique* n-simplex of $N(\mathscr{C}')$ with $t^{i*}_n(Gf_\bullet) = Gf_{i+1}$ for every $i = 0, \dots, n-1$. But we have :

$$t^{i*}_n \circ g_n(f_\bullet) = g_1 \circ t^{i*}_n(f_\bullet) = g_1(f_{i+1}) = Gf_{i+1} \qquad \forall i = 0, \dots, n-1$$

whence $g_n(f_\bullet) = Gf_\bullet$, QED. This proves that N is a full functor. □

2.3.3. Recall that the category Cat is cocomplete (proposition 1.10.4); by virtue of theorem 1.7.5(i), the functor N admits therefore a left adjoint

$$\tau := i_! : \mathsf{sSet} \to \mathsf{Cat}.$$

Then N preserves all representable limits of Cat ([13, Prop.2.49(i)]), and for every small category $\mathscr{C}$, the counit of adjunction is an isomorphism of categories :

$$\boxed{\tau \circ N(\mathscr{C}) \xrightarrow{\sim} \mathscr{C}}$$

in view of lemma 2.3.2 and [13, Prop.2.16(iii)]. Notice also that :

$$\boxed{N([n]) = \Delta^n \qquad \forall n \in \mathbb{N}.}$$

Lemma 2.3.4. *For every simplicial set X we have natural bijections :*

$$\boxed{X_0 \xrightarrow{\sim} \mathrm{Ob}(\tau X) \qquad \textit{and} \qquad \pi_0(X) \xrightarrow{\sim} \pi_0(\tau X).}$$

Proof. Recall that the functor $\mathrm{Ob} : \mathsf{Cat} \to \mathsf{Set}$ is left adjoint to the functor $\mathrm{ch} : \mathsf{Set} \to \mathsf{Cat}$ that assigns to every set S the chaotic category structure on S (see remark 1.2.7); hence $\mathrm{Ob} \circ \tau : \mathsf{sSet} \to \mathsf{Set}$ is left adjoint to the functor $N \circ \mathrm{ch} : \mathsf{Set} \to \mathsf{sSet}$. The latter assigns to every set S the simplicial set $S^\bullet$ whose set of n-simplices is $S^{n+1} = \mathsf{Set}([n], S)$, for every $n \in \mathbb{N}$, whose face and degeneracy maps are induced by the corresponding face and degeneracy morphisms ∂_i^n and σ_i^n in the category $\mathbf{\Delta}$, in the obvious way. Notice then that the n-simplices of $S^\bullet$ are determined uniquely by their images under the canonical projections $(\pi_i^n : S^{n+1} \to S \mid i = 0, \dots, n)$; moreover, each $\pi_i^n : \mathsf{Set}([n], S) \to \mathsf{Set}([0], S)$ is induced by the morphism $j_i^n : [0] \to [n]$ such that $0 \mapsto i$. It follows easily that every morphism $u : X \to S^\bullet$ of simplicial sets is of the form :

$$(*) \qquad x \mapsto (u_0 \circ j_0^{n*} x, \dots, u_0 \circ j_n^{n*} x) \qquad \forall n \in \mathbb{N}, \forall x \in X_n$$

where $j_i^{n*} : X_n \to X_0$ is the map induced by j_i^n, for every $i = 0, \dots, n$. Conversely, every map $u_0 : X_0 \to S$ determines a unique morphism $X \to S^\bullet$ of sSet, via the rule $(*)$: the details are left to the reader. Summing up, this shows that the evaluation functor

$$\mathrm{e}_0 : \mathsf{sSet} \to \mathsf{Set} \qquad X \mapsto X_0 \qquad (X \xrightarrow{v} Y) \mapsto (X_0 \xrightarrow{v_0} Y_0)$$

is another left adjoint to the functor $N \circ \mathrm{ch}$, and thus is isomorphic to $\mathrm{Ob} \circ \tau$ (see §1.6.7).

Next, recall that $\pi_0 : \mathsf{Cat} \to \mathsf{Set}$ is left adjoint to the functor $\mathrm{dis} : \mathsf{Set} \to \mathsf{Cat}$ that assigns to every set S the discrete category whose set of objects is S (remark 1.2.7). Hence $\pi_0 \circ \tau : \mathsf{sSet} \to \mathsf{Set}$ is left adjoint to $N \circ \mathrm{dis} : \mathsf{Set} \to \mathsf{sSet}$; but it is easily seen that $N \circ \mathrm{dis} = \mathrm{c}_{\mathbf{\Delta}^{\mathrm{op}}}$ (notation of §2.1.11), so $\pi_0 \circ \tau$ is isomorphic to $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$. □

Remark 2.3.5. Notice that the counit for the adjoint pair $(\mathrm{e}_0, N \circ \mathrm{ch})$ assigns to every set S the identity map $\mathbf{1}_S : (N \circ \mathrm{ch}(S))_0 \xrightarrow{\sim} S$. Also, let $\eta_\bullet^1$ (resp. $\eta_\bullet^2$) be the unit for the adjoint pair (τ, N) (resp. for the adjoint pair $(\mathrm{Ob}, \mathrm{ch})$); then the composition of the respective adjunctions is an adjunction for the pair $(\mathrm{Ob} \circ \tau, N \circ \mathrm{ch})$, whose unit is given by

$$\eta_X : X \xrightarrow{\eta_X^1} N \circ \tau(X) \xrightarrow{N(\eta_{\tau X}^2)} N \circ \mathrm{ch} \circ \mathrm{Ob} \circ \tau(X) \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet}).$$

By remark 1.6.11(ii), the isomorphism $\mathrm{e}_0 \xrightarrow{\sim} \mathrm{Ob} \circ \tau$ of lemma 2.3.4 is then given by

$$\eta_{X,0} := \mathrm{e}_0(\eta_X) : X_0 \xrightarrow{\sim} \mathrm{Ob} \circ \tau(X) \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet}).$$

However, notice as well that $\mathrm{e}_0 \circ N = \mathrm{Ob}$ and $\mathrm{Ob}(\eta_{\mathscr{C}}^2) = \mathbf{1}_{\mathrm{Ob}(\mathscr{C})}$ for every small category $\mathscr{C}$; hence the isomorphism $\eta_{X,0}$ coincides with

$$\boxed{\mathrm{e}_0 \star \eta_\bullet^1 : \mathrm{e}_0 \xrightarrow{\sim} \mathrm{Ob} \circ \tau.}$$

Example 2.3.6. As an application, we deduce a natural isomorphism of sSet :

$$\boxed{\mathscr{H}om(N\mathscr{B}, N\mathscr{C}) \xrightarrow{\sim} N(\mathscr{C}^{\mathscr{B}}) \qquad \forall \mathscr{B}, \mathscr{C} \in \mathrm{Ob}(\mathsf{Cat}).}$$

Indeed, for every $n \in \mathbb{N}$ we have natural identifications :

$$\mathscr{H}om(N\mathscr{B}, N\mathscr{C})_n = \mathsf{sSet}(\Delta^n \times N\mathscr{B}, N\mathscr{C}) \xrightarrow{\sim} \mathsf{sSet}(N([n] \times \mathscr{B}), N\mathscr{C}) \xrightarrow{\sim} \mathsf{Cat}([n] \times \mathscr{B}, \mathscr{C})$$

since N preserves products (§2.3.3), and by virtue of lemma 2.3.2. On the other hand, by §1.3.3 (and again lemma 2.3.2) we have natural identifications :

$$\mathsf{Cat}([n] \times \mathscr{B}, \mathscr{C}) \xrightarrow{\sim} \mathsf{Cat}([n], \mathscr{C}^{\mathscr{B}}) \xrightarrow{\sim} \mathsf{sSet}(\Delta^n, N(\mathscr{C}^{\mathscr{B}})) \xrightarrow{\sim} N(\mathscr{C}^{\mathscr{B}})_n$$

from which the sought isomorphism follows easily.

2.3.7. For every finite totally ordered set E, we let

$$\Delta^E := N(E).$$

Thus, $\Delta^{[n]} = \Delta^n$ for every $n \in \mathbb{N}$. Every morphism of finite totally ordered sets $\phi : E \to F$ induces a morphism of simplicial sets

$$\Delta^\phi := N(\phi) : \Delta^E \to \Delta^F.$$

In particular, every subset $E' \subset E$ is still totally ordered for the order induced by E, so we get an induced inclusion $\Delta^{E'} \subset \Delta^E$ of simplicial sets, and we say that $\Delta^{E'}$ is a *face* of Δ^E.

- With this notation, we define the *boundary* of Δ^E as the simplicial set :

$$\boxed{\partial\Delta^E := \bigcup_{E' \subsetneq E} \Delta^{E'} \subset \Delta^E}$$

(see example 1.8.11(ii) for the union of presheaves). If the cardinality of E is n, notice that $\partial\Delta^E = Sk_{n-1}(\Delta^E)$, so this notation agrees with that of definition 2.2.3(iii).

- For every $k, n \in \mathbb{N}$ with $k \leq n \neq 0$, we define the *k-th horn* of Δ^n as :

$$\boxed{\Lambda^n_k := \bigcup_{k \in E \subsetneq [n]} \Delta^E \subset \Delta^n = \bigcup_{i \in [n] \setminus \{k\}} \Delta^{[n]\setminus\{i\}}.}$$

If $0 < k < n$, we shall say that Λ^n_k is *an inner horn* of Δ^n.

- For every $n \in \mathbb{N} \setminus \{0\}$, the *spine* of Δ^n is the simplicial set :

$$\boxed{Sp^n := \bigcup_{0 \leq i < n} \Delta^{\{i,i+1\}} \subset \Delta^n.}$$

Remark 2.3.8. (i) Let $n \in \mathbb{N}\setminus\{0\}$; by virtue of example 2.1.5(ii,iii), the natural identification $(\Delta^n)^{\mathrm{op}} \xrightarrow{\sim} \Delta^n$ induces by restriction natural identifications :

$$\boxed{(\Lambda^n_k)^{\mathrm{op}} \xrightarrow{\sim} \Lambda^n_{n-k} \qquad \forall k = 0, \dots, n \qquad (Sp^n)^{\mathrm{op}} \xrightarrow{\sim} Sp^n.}$$

(ii) Also, for every finite totally ordered set E, we have natural identifications :

$$\boxed{(\Delta^E)^{\mathrm{op}} \xrightarrow{\sim} \Delta^{E^{\mathrm{op}}} \qquad (\partial\Delta^E)^{\mathrm{op}} \xrightarrow{\sim} \partial(\Delta^{E^{\mathrm{op}}})}$$

where E^{op} denotes the set E, endowed with the reverse of the order of E.

Example 2.3.9. (i) (Standard presentation of the boundary) For every $n \in \mathbb{N}$ we have a natural diagram of sSet :

$$(*) \qquad \boxed{\textstyle\coprod_{0 \leq i < j \leq n} \Delta^{[n]\setminus\{i,j\}} \overset{j_1}{\underset{j_2}{\rightrightarrows}} \coprod_{0 \leq i \leq n} \Delta^{[n]\setminus\{i\}} \xrightarrow{p} \partial\Delta^n}$$

where p is the morphism whose restriction to each face $\Delta^{[n]\setminus\{i\}}$ is the natural inclusion, and where j_1 (resp. j_2) is the morphism whose restriction to $\Delta^{[n]\setminus\{i,j\}}$ is the inclusion $\Delta^{[n]\setminus\{i,j\}} \to \Delta^{[n]\setminus\{i\}}$ (resp. $\Delta^{[n]\setminus\{i,j\}} \to \Delta^{[n]\setminus\{j\}}$). In light of example 1.8.11(ii), diagram $(*)$ is *exact, i.e.* identifies $\partial\Delta^n$ with the coequalizer of (j_1, j_2).

(ii) Likewise, for every $k, n \in \mathbb{N}$ with $k \leq n \neq 0$, we get the exact diagram :

$$\boxed{\textstyle\coprod_{i,j \in [n]\setminus\{k\}} \Delta^{[n]\setminus\{i,j\}} \overset{j_1}{\underset{j_2}{\rightrightarrows}} \coprod_{i \in [n]\setminus\{k\}} \Delta^{[n]\setminus\{i\}} \xrightarrow{p} \Lambda^n_k}$$

where p, j_1 and j_2 are defined as in (i) : details left to the reader.

Example 2.3.10. (i) (Canonical presentations) Let $(E, \leq)$ be any partially ordered set. For every $n \in \mathbb{N}$, the morphisms $f : [n] \to (E, \leq)$ of partially ordered sets are the non-decreasing maps $\{0, \dots, n\} \to E$, and example 2.2.7(i) says that such an f is a non-degenerate simplex of $N(E)$ if and only if it is a strictly increasing map, *i.e.* if and only if f is a totally ordered chain of elements of E. Suppose now that E is a *finite set*, , and denote by $\xi_0(E)$ the set of all *maximal totally ordered chains* of elements of E. We deduce a diagram of simplicial sets :

$$(*) \qquad \boxed{\coprod_{(C,C')\in\xi_0(E)^2} \Delta^{C\cap C'} \underset{N(j')}{\overset{N(j)}{\rightrightarrows}} \coprod_{C\in\xi_0(E)} \Delta^C \xrightarrow{N(p)} N(E)}$$

where p, j and j' are as in example 1.9.8(iii). Clearly, every non-degenerate simplex of $N(E)$ is of the form $\tau^*(C)$ for some morphism τ of $\mathbf{\Delta}$ and some $C \in \xi_0(E)$; then the same holds in fact for every simplex of $N(E)$, by Eilenberg-Zilber's lemma 2.2.5. This amounts to saying that $N(E)$ is the image of $N(p)$, and in light of example 1.8.11(ii), diagram $(*)$ is then *exact, i.e.* p identifies $N(E)$ with the coequalizer in sSet of $N(j)$ and $N(j')$. We call this diagram *the canonical presentation of* $N(E)$.

(ii) Let us take $p, q \in \mathbb{N}$ and $E := [p] \times [q]$, where the product is taken in the category of partially ordered sets. Then it is easily seen that every maximal totally ordered chain of E is of the form :

$$(a_0, b_0) := (0, 0) < (a_1, b_1) < \cdots < (a_{p+q}, b_{p+q}) := (p, q)$$

such that either $a_{i+1} = a_i$ and $b_{i+1} = b_i + 1$ or else $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i$ for every $i = 0, \dots, p + q$. Such a maximal chain C is determined by the subset $\Sigma_C := \{0 \leq i \leq p + q \,|\, a_{i+1} = a_i\}$ and we have $|\Sigma_C| = p$ for every such C. Hence $|\xi_0([p] \times [q])| = \binom{p+q}{p}$, and $\Delta^C = \Delta^{p+q}$ for every $C \in \xi_0([p] \times [q])$.

The following result will be useful in later sections :

Lemma 2.3.11. *(i) Let $\mathscr{C}$ be a category, and consider a commutative diagram :*

$$\mathscr{D} \quad : \quad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "l"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

in $\mathscr{C}$, such that j, k and l have left inverses r, q and respectively p, with $pk = ir$. Then the diagram $\mathscr{D}$ is cartesian.

(ii) For every $n \geq 2$, the commutative diagrams of sSet *:*

$$\mathscr{D}_{i,j} \quad : \quad \begin{tikzcd} \Delta^{[n]\setminus\{i,j\}} \arrow[r] \arrow[d] & \Delta^{[n]\setminus\{j\}} \arrow[d] \\ \Delta^{[n]\setminus\{i\}} \arrow[r] & \Delta^{[n]} \end{tikzcd} \qquad \forall 0 \leq i \leq j \leq n$$

whose arrows are the natural inclusions, fulfill the conditions of (i).

Proof. (i): Let $B \xleftarrow{v} X \xrightarrow{u} C$ be two morphisms of $\mathscr{C}$ with $ku = lv$. We must show that there exists a unique morphism $w : X \to A$ such that $jw = u$ and $iw = v$. For the uniqueness, notice that $ru = rjw = w$ for any such w. So, set $w := ru$; we get :

$$v = plv = pku = iru = iw$$
$$u = qku = qlv = qliw = qkjw = jw$$

whence the assertion.

(ii): Consider the commutative diagram of inclusions of totally ordered sets :

$$\mathscr{E}_{i,j} \quad : \quad \begin{tikzcd} {[n]} \setminus \{i,j\} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & {[n]} \setminus \{j\} \arrow[d, "\gamma"] \\ {[n]} \setminus \{i\} \arrow[r, "\delta"] & {[n]}. \end{tikzcd}$$

• If $i = j$, then α and β are identities, and $\gamma = \delta$, so we get a diagram $\mathscr{D}$ as in (i) by letting $q = p$ be any left inverse of δ.

• If $i < j < n$, we get a diagram $\mathscr{D}$ as in (i) with $i := \alpha$, $j := \beta$, $l := \gamma$ and $k := \delta$; indeed, we may take for r (resp. for p) the unique left inverse of β (resp. of γ) that maps j to $j + 1$, and for q the unique left inverse of δ that maps i to $i + 1$.

• If $j = n$ and $i < n - 1$, we get a diagram $\mathscr{D}$ with the same i, j, k, l and q as in the foregoing case, and with r (resp. p) the unique left inverse of β (resp. of γ).

• If $j = n$ and $i = n - 1$, we get a diagram $\mathscr{D}$ with $i := \beta$, $j := \alpha$, $k := \gamma$ and $l := \delta$: indeed, we may then take for r (resp. for p) the unique left inverse of α (resp. of δ) that maps $n - 1$ to $n - 2$, and for q the unique left inverse of γ.

In each case, we conclude that $\mathscr{E}_{i,j}$ fulfills the conditions of (i), so the same holds for $\mathscr{D}_{i,j}$, that is obtained by applying to $\mathscr{E}_{i,j}$ the functor $\Delta^{(-)}$ of §2.3.7 □

2.3.12. For a first application, consider a finite totally ordered set $(E, \leq)$; we set

$$\boxed{\Lambda^E_I := \bigcup_{i \in [n] \setminus I} \Delta^{E \setminus \{i\}} \subset \partial\Delta^E \qquad \forall I \subset E.}$$

Hence, $\Lambda^E_\varnothing = \partial\Delta^E$, and $\Lambda^{[n]}_{\{k\}} = \Lambda^n_k$ for every $n \in \mathbb{N} \setminus \{0\}$ and every $k \in [n]$. From lemma 2.3.11(ii), we easily deduce cartesian and cocartesian diagrams :

$$\mathscr{D}^E_{I,e} \quad : \quad \begin{tikzcd} \Lambda^{E \setminus \{e\}}_I \arrow[r] \arrow[d] & \Lambda^E_{I \cup \{e\}} \arrow[d] \\ \Delta^{E \setminus \{e\}} \arrow[r] & \Lambda^E_I \end{tikzcd} \qquad \forall I \subsetneq E, \forall e \in E \setminus I.$$

We may use these diagrams to deduce :

$$\boxed{\pi_0(\Lambda^E_J) = \{\varnothing\} \qquad \forall J \in \{I \subset E \,|\, I \neq \varnothing, E\}.}$$

For the proof, we argue by induction on the cardinality c of $E \setminus J$. If $c = 1$, we have $\Lambda^E_J = \Delta^{E \setminus \{e\}}$ for some $e \in E$, and the assertion holds by example 2.1.13. Hence, suppose that $c \geq 2$ and that the sought identity is already known for all J such that the cardinality of $E \setminus J$ is $< c$. Let $e \in E \setminus J$; by applying the functor π_0 termwise to the diagram $\mathscr{D}^E_{J,e}$ we obtain a cocartesian diagram of sets $\pi_0(\mathscr{D}^E_{J,e})$ (since π_0 commutes with all representable colimits : see [13, Prop.2.49(i)]). But, by inductive assumption we have $\pi_0(\Lambda^{E \setminus \{e\}}_J) = \pi_0(\Lambda^E_{J \cup \{e\}}) = \{\varnothing\}$, and we know as well that $\pi_0(\Delta^{E \setminus \{e\}}) = \{\varnothing\}$, whence $\pi_0(\Lambda^E_J) = \{\varnothing\}$, as required. Especially :

$$(*) \qquad \boxed{\pi_0(\Lambda^n_k) = \{\varnothing\} \qquad \forall n \in \mathbb{N} \setminus \{0\}, \forall k = 0, \ldots, n.}$$

Likewise, we may check that :

$$\boxed{\pi_0(\partial\Delta^1) = \{\varnothing\} \sqcup \{\varnothing\} \qquad \text{and} \qquad \pi_0(\partial\Delta^n) = \{\varnothing\} \qquad \forall n \geq 2.}$$

Indeed, for the first identity it suffices to consider $\pi_0(\mathscr{D}^{[1]}_{\varnothing,1})$ and notice that $\Lambda^{[1]\setminus\{1\}}_{\varnothing} = \Lambda^{[0]}_{\varnothing}$ is the empty simplicial set, so $\pi_0(\Lambda^{[0]}_{\varnothing}) = \varnothing$, and moreover $\pi_0(\Delta^{[1]\setminus\{1\}}) = \pi_0(\Delta^0) = \pi_0(\Lambda^{[1]}_{\{1\}}) = \{\varnothing\}$ by $(*)$. Lastly, to compute $\pi_0(\partial\Delta^n)$ for $n \geq 2$, we consider $\pi_0(\mathscr{D}^{[n]}_{\varnothing,n})$, and argue similarly : the details are left to the reader.

2.4. **Augmented simplicial sets, joins and slices.** We denote by

$$\mathbf{\Delta}^+$$

the category with $\mathrm{Ob}(\mathbf{\Delta}^+) := \mathrm{Ob}(\mathbf{\Delta}) \cup \{\varnothing\}$, such that $\mathbf{\Delta}$ is a full subcategory of $\mathbf{\Delta}^+$ and $\varnothing$ is the unique initial object of $\mathbf{\Delta}^+$. It is convenient to set as well $[-1] := \varnothing$.

- The *categories of augmented simplicial sets* and of *augmented bisimplicial sets* are

$$\mathrm{sSet}^+ := \widehat{\mathbf{\Delta}^+} \qquad \text{and respectively} \qquad \mathrm{bSet}^+ := \widehat{\mathbf{\Delta}^+ \times \mathbf{\Delta}^+}.$$

Hence, an augmented simplicial set can be regarded as a datum $X^+ := (X, E, \phi)$, where X is a simplicial set, E is a set, and $\phi : X_0 \to E$ is a map, called *the augmentation of* X^+. We can also regard ϕ as a morphism $X \to c_E$ of simplicial sets, from X to the constant presheaf c_E with value E (notation of §1.1.8). With this notation, restriction along the inclusion functor $i : \mathbf{\Delta} \to \mathbf{\Delta}^+$ yields a well-defined functor

$$i^* : \mathrm{sSet}^+ \to \mathrm{sSet} \qquad (X, E, \phi) \mapsto X.$$

We also define for every $n \geq -1$ the *augmented n-simplicial set*

$$\Delta^n := h_{[n]}$$

where $h : \Delta^+ \to \mathrm{sSet}^+$ is the Yoneda embedding, just like in the non-augmented case; notice that the initial object of sSet^+ is $(i^*\Delta^{-1}, \varnothing, \mathbf{1}_\varnothing)$, which is *not* isomorphic to Δ^{-1}.

- The restriction functor i^* admits left and right adjoints, denoted respectively

$$i_! : \mathrm{sSet} \to \mathrm{sSet}^+ \qquad \text{and} \qquad i_* : \mathrm{sSet} \to \mathrm{sSet}^+.$$

Namely, $i_!$ assigns to every simplicial set X the augmented simplicial set $(X, \pi_0(X), p_X)$, where $p_X : X_0 \to \pi_0(X)$ is the natural projection. The functor i_* assigns to X the augmented simplicial set $(X, \{\varnothing\}, a_X)$, where $a_X : X_0 \to \{\varnothing\}$ is the unique map, which can be regarded as the unique morphism $a_X : X \to \Delta^0$ of sSet (details left to the reader). For instance, if $\varnothing$ denotes the initial object of sSet, then $i_!\varnothing$ is the initial object of sSet^+, whereas $i_*\varnothing = \Delta^{-1}$, and $i^*\Delta^{-1} = \varnothing$.

2.4.1. *Joins of augmented and non-augmented simplicial sets.* We consider the functor

$$* : \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \to \mathbf{\Delta}^+ \qquad ([m], [n]) \mapsto [m] * [n] := [m + n + 1]$$

that assigns to every pair of morphisms $f : [m] \to [m']$, $g : [n] \to [n']$ the morphism

$$f * g : [m] * [n] \to [m'] * [n'] \qquad \text{such that} \qquad i \mapsto \begin{cases} f(i) & \text{if } i \leq m \\ 1 + m' + g(i - m - 1) & \text{if } i > m. \end{cases}$$

By composing with the Yoneda embedding, we get a functor

$$F : \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \xrightarrow{*} \mathbf{\Delta}^+ \xrightarrow{h} \mathrm{sSet}^+$$

and since sSet^+ is cocomplete (remark 1.6.2(i)), F admits an extension by colimits

$$F_! : \mathsf{bSet}^+ \to \mathsf{sSet}^+$$

(theorem 1.7.5(i,ii)), which we can compose with the functor $- \boxtimes -$ of remark 1.6.4(iv), to get a *join functor for augmented simplicial sets*, denoted again by

$$* : \mathsf{sSet}^+ \times \mathsf{sSet}^+ \xrightarrow{-\boxtimes-} \mathsf{bSet}^+ \xrightarrow{F_!} \mathsf{sSet}^+ \qquad (X, Y) \mapsto X * Y$$

and fitting into the essentially commutative diagram :

$$\mathscr{D} \quad : \quad \begin{tikzcd} \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \arrow[r, "*"] \arrow[d, "h\times h"'] & \mathbf{\Delta}^+ \arrow[d, "h"] \\ \mathsf{sSet}^+ \times \mathsf{sSet}^+ \arrow[r, "*"] & \mathsf{sSet}^+. \end{tikzcd}$$

Remark 2.4.2. (i) The restriction of the join functor to the image of $h\times h$ can be described explicitly. Indeed, we have natural identifications :

$$\mathbf{\Delta}^+([n],[p]*[q]) \xrightarrow{\sim} \coprod_{i+j=n-1} \mathbf{\Delta}^+([i],[p]) \times \mathbf{\Delta}^+([j],[q]) \qquad \forall n,p,q \geq -1.$$

Namely, for every non-decreasing map $f : [n] \to [p+q+1]$ we let $i+1$ be the cardinality of $f^{-1}([p])$, and set $j := n-i-1$; the bijection assigns to f the pair $([i] \xrightarrow{g_1} [p], [j] \xrightarrow{g_2} [q])$ where g_1 is the restriction of f, and $g_2(k) := f(k+i+1) - p - 1$ for every $k = 0, \dots, j$. The inverse bijection then is given by the rule : $(g_1, g_2) \mapsto g_1 * g_2$ for every $i, j \geq -1$ with $i + j = n - 1$ and every pair of non-decreasing maps $([i] \xrightarrow{g_1} [p], [j] \xrightarrow{g_2} [q])$.

(ii) The natural identifications of (i) can be rewritten as :

$$(\Delta^p * \Delta^q)_n \xrightarrow{\sim} \coprod_{i+j=n-1} \Delta^p_i \times \Delta^q_j \qquad \forall n,p,q \geq -1.$$

Moreover, let $u : [m] \to [n]$ be any non-decreasing map, and for every $i, j \geq -1$ with $i + j = n - 1$, let $i' \geq -1$ be the unique integer such that $[i'] = u^{-1}([i])$, and set $j' := m - i' - 1$; we let $([i'] \xrightarrow{u_i} [i], [j'] \xrightarrow{u_j} [j])$ be the pair of non-decreasing maps given by the rules : $u_i(k) := u(k)$ for every $k \in [i']$, and : $u_j(k) := u(k+i'+1) - i - 1$ for every $k \in [j']$. Then it is easily seen that under the foregoing natural identifications, the map

$$(\Delta^p * \Delta^q)_u : (\Delta^p * \Delta^q)_n \to (\Delta^p * \Delta^q)_m$$

corresponds to the map

$$\coprod_{i+j=n-1} \Delta^p_{u_i} \times \Delta^q_{u_j} : \coprod_{i+j=n-1} \Delta^p_i \times \Delta^q_j \to \coprod_{i'+j'=m-1} \Delta^p_{i'} \times \Delta^q_{j'}$$

that assigns to every pair of non-decreasing maps $([i] \xrightarrow{g_1} [p], [j] \xrightarrow{g_2} [q])$ the pair $(g_1 \circ u_i, g_2 \circ u_j) \in \Delta^p_{i'} \times \Delta^q_{j'}$: the detailed verifications are left to the reader.

Proposition 2.4.3. *(i) For every $X, Y \in \mathrm{Ob}(\mathsf{sSet}^+)$ we have a natural identification :*

$$\boxed{(X * Y)_n \xrightarrow{\sim} \coprod_{i+j=n-1} X_i \times Y_j \qquad \forall n \geq -1.}$$

(ii) Let $u : [m] \to [n]$ be any morphism of $\mathbf{\Delta}^+$; then, with the notation of remark 2.4.2(ii), the bijections of (i) identify the map $(X * Y)_u : (X * Y)_n \to (X * Y)_n$ with the map :

$$\boxed{\coprod_{i+j=n-1} X_{u_i} \times Y_{u_j} : \coprod_{i+j=n-1} X_i \times Y_j \to \coprod_{i'+j'=m-1} X_{i'} \times Y_{j'}.}$$

(iii) Moreover, we have natural isomorphisms :

$$\boxed{\Delta^{-1} * X \xrightarrow{\sim} X \xleftarrow{\sim} X * \Delta^{-1} \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}^+).}$$

Proof. (iii) follows easily from (i) and (ii). To prove (i) and (ii), let us consider the functor

$$\odot : \mathrm{sSet}^+ \times \mathrm{sSet}^+ \to \mathrm{sSet}^+ \qquad (X, Y) \mapsto X \odot Y$$

where $(X \odot Y)_n := \coprod_{i+j=n-1} X_i \times Y_j$ for every $n \geq -1$, and with transition maps $(X \odot Y)_u : (X \odot Y)_n \to (X \odot Y)_m$ given by the expression of (ii), for every morphism $u : [m] \to [n]$ of $\mathbf{\Delta}^+$; then $\odot$ assigns to every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ of augmented simplicial sets, the morphism $f \odot g : X \odot Y \to X' \odot Y'$ such that

$$(f \odot g)_n := \coprod_{i+j=n-1} f_i \times g_j \qquad \forall n \geq -1.$$

By construction, both $*$ and $\odot$ commute with small colimits; hence, in light of proposition 1.7.3, it suffices to exhibit an isomorphism between the compositions of these two functors with the square of the Yoneda embedding $h \times h : \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \to \mathrm{sSet}^+ \times \mathrm{sSet}^+$. The latter is provided by remark 2.4.2. □

Remark 2.4.4. (i) Notice that the image of the Yoneda embedding $\mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \to \mathrm{bSet}^+$ is the full subcategory of bSet^+ whose set of objects is $\{\Delta^p \boxtimes \Delta^q \mid p, q \geq -1\}$. It follows that the data of $F_!$ and of the join functor $*$ are essentially equivalent : given $F_!$ we define $*$, but conversely, given $*$ we retrieve $F_!$ up to isomorphism, as the extension by colimits of the functor $\mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \to \mathrm{sSet}^+$ given by the rule : $([p], [q]) \mapsto \Delta^p * \Delta^q$. Hence, since $F_!$ is determined only up to isomorphism, the same holds for $*$. We can then simply declare that *we shall represent the join functor by the functor $\odot$ exhibited in the proof of proposition 2.4.3*; with this choice, the natural identifications of part (i) and (ii) of the proposition is just *an identity*. This very explicit choice will simplify some verifications.

(ii) For instance, we can deduce explicit natural identifications :

$$(*) \qquad (X * Y) * Z \xrightarrow{\sim} X * (Y * Z) \qquad \forall X, Y, Z \in \mathrm{Ob}(\mathrm{sSet})$$

expressing the *associativity of the join functor*. Indeed, by proposition 2.4.3(i), the evaluation at $[n]$ of both terms in $(*)$ is naturally identified with $\coprod_{i+j+k=n-2} X_i \times Y_j \times Z_k$. We need to check that these natural identifications are compatible with morphisms $u : [m] \to [n]$ of $\mathbf{\Delta}^+$. However, according to proposition 2.4.3(ii), in order to compute $((X * Y) * Z)_u$ we need to calculate first the pairs (u_{i+j+1}, u_k) for every $i, j, k \geq -1$ with $i + j + k = n - 2$; then we let $v := u_{i+j+1}$, and we need to further compute (v_i, v_j). With this notation, $((X * Y) * Z)_u$ is given by $\coprod_{i+j+k=n-2} X_{v_i} \times Y_{v_j} \times Z_{u_k}$. Likewise, to compute $(X * (Y * Z))_u$ we calculate first (u_i, u_{j+k+1}), then with $w := u_{j+k+1}$ we compute (w_j, w_k), and at last $(X * (Y * Z))_u$ is given by $\coprod_{i+j+k=n-2} X_{u_i} \times Y_{w_j} \times Z_{w_k}$. However, a direct inspection yields:

$$(v_i, v_j, u_k) = (u_i, w_j, w_k)$$

whence the sought isomorphism $(*)$.

Definition 2.4.5. The *join of simplicial sets* X and Y is the simplicial set

$$X * Y := i^*(i_* X * i_* Y).$$

Remark 2.4.6. (i) Just as in the augmented case, any two morphisms of simplicial sets $f : X \to X'$, $g : Y \to Y'$ induce a morphism

$$f * g := i^*(i_* f * i_* g) : X * Y \to X' * Y'.$$

Hence, the join operation yields a well-defined functor

$$* : \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet} \qquad (X, Y) \mapsto X * Y.$$

(ii) By virtue of proposition 2.4.3(i), we get a natural identification :

$$\boxed{(X * Y)_n \xrightarrow{\sim} X_n \sqcup Y_n \sqcup \coprod_{i+j=n-1} X_i \times Y_j \qquad \forall X, Y \in \mathrm{Ob}(\mathsf{sSet}), \forall n \in \mathbb{N}.}$$

Notice that in the foregoing expression, the indices i, j run over the values $0, \dots, n-1$, whereas in the corresponding expression of proposition 2.4.3(i) for augmented simplicial sets, the same indices run over the values $-1, \dots, n$. This difference accounts for the presence of the extra term $X_n \sqcup Y_n$ in the non-augmented case. The inclusion of this extra term then yields a natural transformation :

$$\boxed{X \sqcup Y \to X * Y \qquad \forall X, Y \in \mathrm{Ob}(\mathsf{sSet}).}$$

(iii) Likewise, from proposition 2.4.3(ii) we get natural isomorphisms of sSet :

$$\boxed{\varnothing * X \xrightarrow{\sim} X \xleftarrow{\sim} X * \varnothing \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet})}$$

where $\varnothing$ denotes the initial object of sSet. For every simplicial set X, let also $j_X : \varnothing \to X$ be the unique morphism of sSet; then, every given simplicial set T induces two functors

$$(-) \downarrow T : \mathsf{sSet} \to T/\mathsf{sSet} \qquad X \mapsto \big(T \xrightarrow{\sim} \varnothing * T \xrightarrow{j_X * T} X * T\big)$$
$$T \uparrow (-) : \mathsf{sSet} \to T/\mathsf{sSet} \qquad X \mapsto \big(T \xrightarrow{\sim} T * \varnothing \xrightarrow{T * j_X} T * X\big).$$

Lemma 2.4.7. *Both functors* $(-) \downarrow T$ *and* $T \uparrow (-)$ *preserve small colimits.*

Proof. It suffices to consider $(-) \downarrow T$. Let us first check that this functor preserves all small connected colimits. To this aim, since the target functor $\mathrm{t}_T : T/\mathsf{sSet} \to \mathsf{sSet}$ reflects all small connected colimits (corollary 1.4.6(ii)), it suffices to check that the functor

$$\mathsf{sSet} \to \mathsf{sSet} \qquad X \mapsto X * T = i^*(i_* X * i_* T)$$

preserves small connected colimits. However, since the colimits of sSet and sSet^+ are computed termwise (remark 1.6.2(i)), it is clear that i^* preserves all representable colimits, so we are reduced to checking that the functor $\mathsf{sSet} \to \mathsf{sSet}^+$ such that : $X \mapsto i_* X * i_* T$ preserves small connected colimits. However, by construction the join functor for augmented simplicial sets preserves all small colimits, so we are further reduced to checking that the functor i_* preserves small connected colimits. Again, this can be checked termwise, so we need to show that the functors $(F_n : \mathsf{sSet} \to \mathsf{Set} \,|\, n \geq -1)$ given by the rules : $X \mapsto (i_* X)_n$ preserve connected colimits. This is trivial if $n \in \mathbb{N}$. Lastly, F_{-1} is none else than the constant functor with value $\{\varnothing\}$, and it easily seen that every constant functor preserves all connected colimits : the details are left to the reader.

According to remark 1.5.10(ii), it remains only to show that $(-) \downarrow T$ preserves initial objects. The latter follows immediately from remark 2.4.6(iii). $\square$

2.4.8. By virtue of lemma 2.4.7 and theorem 1.7.5(iii), the functors $(-) \downarrow T$ and $T \uparrow (-)$ admit right adjoints, called *slice functors over and under T*, and denoted respectively

$$-/T : T/\mathsf{sSet} \to \mathsf{sSet} \qquad \text{and} \qquad T\backslash - : T/\mathsf{sSet} \to \mathsf{sSet}.$$

Hence, $-/T$ (resp. $T\backslash -$) attaches to every $(X, t : T \to X) \in \mathrm{Ob}(T/\mathsf{sSet})$ a simplicial object $(X,t)/T$ (resp. $T\backslash(X,t)$) that we may also denote more simply by X/t (resp. $t\backslash X$).

Example 2.4.9. (i) A simple inspection shows that the functors $(-) \downarrow \varnothing$ and $\varnothing \uparrow (-)$ are naturally isomorphic to the identity of sSet, under the natural isomorphism of categories $\varnothing/\mathsf{sSet} \xrightarrow{\sim} \mathsf{sSet}$, so the same holds for the functors $-/\varnothing$ and $\varnothing\backslash -$.

(ii) Let us take $T := \Delta^0$; then a morphism $t : \Delta^0 \to X$ is the same as an element $t \in X_0$; moreover, for every $n \in \mathbb{N}$ we have natural identifications :

$$(X/t)_n \xrightarrow{\sim} \mathsf{sSet}(\Delta^n, X/t) \xrightarrow{\sim} \Delta^0/\mathsf{sSet}((\Delta^0 \xrightarrow{\Delta^n \downarrow \Delta^0} \Delta^n * \Delta^0), (\Delta^0 \xrightarrow{t} X)).$$

Recall that $\Delta^n * \Delta^0 = \Delta^{n+1}$; by inspecting the definitions, we see that $\Delta^n \downarrow \Delta^0$ corresponds to the map $j_n : [0] \to [n+1]$ such that $0 \mapsto n+1$. Also, every non-decreasing map $f : [m] \to [n]$ induces the morphism $\Delta^f \downarrow \Delta^0 : \Delta^m \downarrow \Delta^0 \to \Delta^n \downarrow \Delta^0$ of Δ^0/sSet, and by tracing the natural identifications, we see that $\Delta^f \downarrow \Delta^0$ corresponds to the map $f * [0] : [m+1] \to [n+1]$ with $i \mapsto f(i)$ for $i = 0, \dots, m$, and $m+1 \mapsto n+1$. Hence, let $\pi_n := X_{j_n} : X_{n+1} \to X_0$ be the map induced by j_n; summing up, we get natural bijections :

$$(X/t)_n \xrightarrow{\sim} \pi_n^{-1}(t) \qquad \forall n \in \mathbb{N}$$

that identify the map $(X/t)_f : (X/t)_n \to (X/t)_m$ induced by $f : [m] \to [n]$ with the restriction $\pi_n^{-1}(t) \to \pi_m^{-1}(t)$ of the map $X_{n+1} \to X_{m+1}$ induced by $f * [0]$.

(iii) Likewise, for every $n \in N$ let $j'_n : [0] \to [n+1]$ be the map such that $0 \mapsto 0$, and $\pi'_n := X_{j'_n} : X_{n+1} \to X_0$ be the map induced by j'_n; we have natural bijections

$$(t/X)_n \xrightarrow{\sim} \pi'^{-1}_n(t) \qquad \forall n \in \mathbb{N}$$

that identify the map $(t/X)_f : (t/X)_n \to (t/X)_m$ induced by $f : [m] \to [n]$ with the restriction $\pi'^{-1}_n(t) \to \pi'^{-1}_n(t)$ of the map $X_{n+1} \to X_{m+1}$ induced by $[0] * [f] : [m+1] \to [n+1]$, where the latter is the map given by the rules : $0 \mapsto 0$ and $i \mapsto f(i-1)+1$ for $i = 1, \dots, m+1$: the detailed verification shall be left to the reader.

(iv) Especially, let us take $X := N(\mathscr{C})$ for some small category $\mathscr{C}$, and let $t \in N(\mathscr{C})_0$, *i.e.* t is an object of $\mathscr{C}$. Then the explicit descriptions of $N(\mathscr{C})/t$ and $t/N(\mathscr{C})$ given by (i) and (ii) boil down to natural isomorphisms of simplicial sets :

$$N(\mathscr{C})/t \xrightarrow{\sim} N(\mathscr{C}/t) \qquad \text{and} \qquad t/N(\mathscr{C}) \xrightarrow{\sim} N(t/\mathscr{C})$$

showing that the slice functors generalize the construction of the slice categories of §1.4.

Lemma 2.4.10. *(i) For every pair of monomorphisms $K \to L$, $U \to V$ of simplicial sets, the commutative diagram :*

$$\begin{tikzcd} K * U \arrow[r] \arrow[d] & K * V \arrow[d] \\ L * U \arrow[r] & L * V \end{tikzcd}$$

is cartesian, and all its arrows are monomorphisms; hence it induces a monomorphism :

$$K * V \cup L * U \to L * V.$$

(ii) We have the following identities of subobjects of $\Delta^m * \Delta^n = \Delta^{m+n+1}$ *:*

$$\begin{aligned} \partial\Delta^m * \Delta^n \cup \Delta^m * \partial\Delta^n &= \partial\Delta^{m+n+1} \\ \Lambda^m_k * \Delta^n \cup \Delta^m * \partial\Delta^n &= \Lambda^{m+n+1}_k \qquad \forall k = 0, \dots, m \\ \partial\Delta^m * \Delta^n \cup \Delta^m * \Lambda^n_k &= \Lambda^{m+n+1}_{m+k+1} \qquad \forall k = 0, \dots, n. \end{aligned}$$

Proof. (i): The assertion boils down to the trivial identities : $(K_i \times V_j) \cap (L_i \times U_j) = K_i \times U_j$ for every $i, j \in \mathbb{N}$.

(ii): The identification of $\Delta^m * \Delta^n$ with Δ^{m+n+1} is as detailed in remark 2.4.2(i) : namely, to every $i, j \geq -1$ with $k := i+j+1 \in \mathbb{N}$ and every pair of non-decreasing maps $g_1 : [i] \to [m]$, $g_2 : [j] \to [n]$, we attach $g_1 * g_2 : [k] \to [m+n+1]$ (the pairs with $i = -1$ or $j = -1$ correspond to $\Delta^m_k \sqcup \Delta^n_k$ in the explicit expression for $(\Delta^m * \Delta^n)_k$ given by remark 2.4.6(ii)). Now, recall that the k-simplices of $\partial\Delta^{m+n+1}$ are the non-surjective maps $[k] \to [m+n+1]$; however, it is easily seen that $g_1 * g_2$ is non-surjective if and only if the same holds for either g_1 or g_2, whence the first stated identity. Likewise, for every $l \in \mathbb{N}$, the l-simplices of Λ^{m+n+1}_k are the maps $[l] \to [m+n+1]$ that omit some value different from k; then the last two identities are easily verified. □

2.4.11. *Augmented front-to-back duality.* Clearly the involution $(-)^{\mathrm{op}} : \mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$ of §2.1.4 extends uniquely to an involution $(-)^{\mathrm{op}} : \mathbf{\Delta}^+ \xrightarrow{\sim} \mathbf{\Delta}^+$, which is again the identity map on objects. Then $(-)^{\mathrm{op}}$ induces a *front-to-back duality on augmented simplicial sets* :

$$(-)^{\mathrm{op}} : \mathrm{sSet}^+ \xrightarrow{\sim} \mathrm{sSet}^+ \qquad X \mapsto X^{\mathrm{op}} := X \circ (-)^{\mathrm{op}} \qquad (u : X \to Y) \mapsto u^{\mathrm{op}} := u \star (-)^{\mathrm{op}}.$$

We also consider the involution

$$(-)^{\mathrm{op}} : \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \xrightarrow{\boldsymbol{\phi}^+} \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ \xrightarrow{(-)^{\mathrm{op}} \times (-)^{\mathrm{op}}} \mathbf{\Delta}^+ \times \mathbf{\Delta}^+$$

where $\boldsymbol{\phi}^+$ is the *flip automorphism* that swaps the two factors : $([n], [m]) \mapsto ([m], [n])$. Then, we get again an induced *front-to-back duality on augmented bisimplicial sets* :

$$(-)^{\mathrm{op}} : \mathrm{bSet}^+ \xrightarrow{\sim} \mathrm{bSet}^+ \qquad X \mapsto X^{\mathrm{op}} := X \circ (-)^{\mathrm{op}} \qquad (u : X \to Y) \mapsto u^{\mathrm{op}} := u \star (-)^{\mathrm{op}}.$$

With this notation, a direct calculation yields the commutative diagram :

$$\boxed{\begin{array}{ccc} \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ & \xrightarrow{\ *\ } & \mathbf{\Delta} \\ {\scriptstyle (-)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{op}}} \\ \mathbf{\Delta}^+ \times \mathbf{\Delta}^+ & \xrightarrow{\ *\ } & \mathbf{\Delta}^+. \end{array}}$$

Lemma 2.4.12. *(i) We have a natural isomorphism in* sSet^+ *:*

$$\boxed{(X * Y)^{\mathrm{op}} \xrightarrow{\sim} Y^{\mathrm{op}} * X^{\mathrm{op}} \qquad \forall X, Y \in \mathrm{Ob}(\mathrm{sSet}^+).}$$

(ii) The natural isomorphism of (i) also exists for every $X, Y \in \mathrm{Ob}(\mathrm{sSet})$.

(iii) For every $T \in \mathrm{Ob}(\mathrm{sSet})$*, the isomorphisms of (ii) induce isomorphisms in* $T^{\mathrm{op}}/\mathrm{sSet}$ *:*

$$(X \downarrow T)^{\mathrm{op}} \xrightarrow{\sim} T^{\mathrm{op}} \uparrow X^{\mathrm{op}} \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).$$

Proof. (i): The existence of such a natural isomorphism can be shown by juggling with adjunctions, by purely categorical nonsense; however, since we have selected an explicit join functor (see remark 2.4.4), we can exhibit a natural isomorphism in a more direct manner. Namely, for every $n \geq -1$ we have :

$$(X * Y)^{\mathrm{op}}_n = (X * Y)_n = \coprod_{i+j=n-1} X_i \times Y_j \qquad (Y^{\mathrm{op}} * X^{\mathrm{op}})_n = \coprod_{i+j=n-1} Y^{\mathrm{op}}_i \times X^{\mathrm{op}}_j = \coprod_{i+j=n-1} Y_i \times X_j$$

and every non-decreasing map $u : [m] \to [n]$ induces the maps

$$(X*Y)^{\mathrm{op}}_u = (X*Y)_{u^{\mathrm{op}}} : (X*Y)_n \to (X*Y)_m \qquad (Y^{\mathrm{op}}*X^{\mathrm{op}})_u : (Y^{\mathrm{op}}*X^{\mathrm{op}})_n \to (Y^{\mathrm{op}}*X^{\mathrm{op}})_m$$

where $(X * Y)_{u^{\mathrm{op}}}$ and $(Y^{\mathrm{op}} * X^{\mathrm{op}})_u$ are given by proposition 2.4.3(ii). However, a direct inspection shows that (notation of §2.4.11) :

$$((u^{\mathrm{op}})_i, (u^{\mathrm{op}})_j) = (u_j, u_i)^{\mathrm{op}} \qquad \forall i, j \geq -1 \text{ with } i + j = n - 1.$$

It follows that the sought natural isomorphism is given by the system of maps

$$\coprod_{i+j=n-1} \omega_{X_i,Y_j} : \coprod_{i+j=n-1} X_i \times Y_j \xrightarrow{\sim} \coprod_{i+j=n-1} Y_j \times X_i \qquad \forall n \geq -1$$

where, for every pair of sets S, T, we denote by $\omega_{S,T} : S \times T \xrightarrow{\sim} T \times S$ the bijection that swaps the two factors : $(s, t) \mapsto (t, s)$.

To show (ii), it suffices to notice the obvious identities :

$$(i^*S)^{\mathrm{op}} = i^*(S^{\mathrm{op}}) \qquad (i_*T)^{\mathrm{op}} = i_*(T^{\mathrm{op}}) \qquad \forall S \in \mathrm{Ob}(\mathsf{sSet}^+), \forall T \in \mathrm{Ob}(\mathsf{sSet})$$

and apply (i). Assertion (iii) follows by direct inspection. □

Remark 2.4.13. (i) For every $T \in \mathrm{Ob}(\mathsf{sSet})$, the natural isomorphism of lemma 2.4.12(iii) induces a natural identification in sSet :

$$(X/t)^{\mathrm{op}} \xrightarrow{\sim} t^{\mathrm{op}}\backslash X^{\mathrm{op}} \qquad \forall (X, t) \in \mathrm{Ob}(T/\mathsf{sSet}).$$

(ii) Let $f : S \to T$ be any morphism of sSet; the commutative diagrams :

$$\begin{tikzcd} S \arrow[r, "\sim"] \arrow[d, "f"'] & \varnothing * S \arrow[r, "j_{X*S}"] & X * S \arrow[d, "X*f"] \\ T \arrow[r, "\sim"] & \varnothing * T \arrow[r, "j_{X*T}"] & X * T \end{tikzcd} \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet})$$

induce a system of natural morphisms $(\tau_X : T \sqcup_S (Y * S) \to X * T \mid X \in \mathrm{Ob}(\mathsf{sSet}))$ that amount to a natural transformation

$$\tau_\bullet : f^! \circ (- \downarrow S) \Rightarrow (- \downarrow T)$$

where $f^! : S/\mathsf{sSet} \to T/\mathsf{sSet}$ is the left adjoint of the functor $f_! : T/\mathsf{sSet} \to S/\mathsf{sSet}$ (see remark 1.4.2(ii)). Then, the adjoint of $\tau_\bullet$ is a natural transformation

$$\tau^\vee_\bullet : (-/T) \Rightarrow (-/S) \circ f_! \qquad (X, t : T \to X) \mapsto (\tau^\vee_{(X,t)} : X/t \to X/t \circ f).$$

Explicitly, under the natural identifications :

$$(X/t)_n \xrightarrow{\sim} \mathsf{sSet}(\Delta^n, X/t) \xrightarrow{\sim} T/\mathsf{sSet}(\Delta^n \downarrow T, (T \xrightarrow{t} X))$$

$$(X/t \circ f)_n \xrightarrow{\sim} \mathsf{sSet}(\Delta^n, X/t \circ f) \xrightarrow{\sim} S/\mathsf{sSet}(\Delta^n \downarrow S, (S \xrightarrow{t\circ f} X))$$

the map $\tau^\vee_{(X,t)} := (\tau^\vee_{(X,t),n} : (X/t)_n \to (X/t \circ f)_n \mid n \in \mathbb{N})$ corresponds to the system of maps

$$T/\mathsf{sSet}(\Delta^n \downarrow T, (T \xrightarrow{t} X)) \to S/\mathsf{sSet}(\Delta^n \downarrow S, (S \xrightarrow{t\circ f} X)) \qquad \forall n \in \mathbb{N}$$

given by the rule :

$$\begin{tikzcd}[column sep=small] & T \arrow[dl, "\Delta^n\downarrow T"'] \arrow[dr, "t"] & \\ \Delta^n * T \arrow[rr, "u"] & & X \end{tikzcd} \quad \mapsto \quad \begin{tikzcd}[column sep=small] & S \arrow[dl, "\Delta^n\downarrow S"'] \arrow[dr, "t\circ f"] & \\ \Delta^n * S \arrow[rr, "u\circ(\Delta^n * f)"] & & X. \end{tikzcd}$$

(iii) Combining (i) and (ii), we obtain as well natural transformations

$$\mu_\bullet : f^! \circ (S \uparrow -) \Rightarrow (T \uparrow -) \qquad \text{and} \qquad \mu_\bullet^\vee : (T \backslash -) \Rightarrow (S \backslash -) \circ f_!$$

which can be described explicitly as in (ii) : the details are left to the reader.

2.5. **Simplicial sets as generalized categories.** Lemma 2.3.2 allows to regard Cat as a full subcategory of the category of simplicial sets, and motivates the following definitions 2.5.1 and 2.5.4, that extend to arbitrary simplicial sets some standard terminology relating to categories :

Definition 2.5.1. (i) Let X be a simplicial set. An *object* of X is an element $x \in X_0$, or equivalently, a morphism $x : \Delta^0 \to X$ of simplicial sets.

(ii) An *arrow* of X (also called a *morphism*, or a *map*) is an element $f \in X_1$, or equivalently, a morphism $f : \Delta^1 \to X$ of simplicial sets. Such an arrow has a *source* $d_1^1(f)$ and a *target* $d_1^0(f)$ in X_0 :

$$d_1^1(f) : \Delta^0 \xrightarrow{\partial_1^1} \Delta^1 \xrightarrow{f} X \qquad d_1^0(f) : \Delta^0 \xrightarrow{\partial_0^1} \Delta^1 \xrightarrow{f} X.$$

For an arrow f of X with source x and target y, we also write $f : x \to y$; we set

$$X(x,y) := \{f \in X_1 \mid d_1^1(f) = x \text{ and } d_1^0(f) = y\}.$$

(iii) Given an object x of X, the *identity* of x is the arrow

$$\mathbf{1}_x := s_0^0(x) : x \to x \qquad \textit{i.e.} \qquad \mathbf{1}_x : \Delta^1 \xrightarrow{\sigma_0^0} \Delta^0 \xrightarrow{x} X.$$

(iv) A *functor* $f : X \to Y$ is morphism of simplicial sets. The *fibre of f over a given object* $y : \Delta^0 \to Y$ of Y is the fibre product $f^{-1}(y) := \Delta^0 \times_Y X$.

(v) Let $f, g : X \rightrightarrows Y$ be two given functors; a *natural transformation $h : f \Rightarrow g$ from f to g* is a morphism of simplicial sets

$$h : X \times \Delta^1 \to Y \quad \text{such that} \quad h \circ (X \times \partial_1^1) = f \quad \text{and} \quad h \circ (X \times \partial_0^1) = g.$$

Remark 2.5.2. (i) Let $\mathscr{C}$ and $\mathscr{C}'$ be any two small categories; by lemma 2.3.2, every functor $N(\mathscr{C}) \to N(\mathscr{C}')$ is of the form $N(F)$ for a unique functor $F : \mathscr{C} \to \mathscr{C}'$.

(ii) Let $F, G : \mathscr{C} \rightrightarrows \mathscr{C}'$ be two functors. On the one hand, notice that the nerve functor N commutes with products, since it is a right adjoint (see §2.3.3), and on the other hand, recall that Δ^1 is naturally isomorphic to $N([1])$; it follows that every natural transformation from $N(F)$ to $N(G)$ is of the form

$$N(h) : N(\mathscr{C}) \times \Delta^1 \xrightarrow{\sim} N(\mathscr{C} \times [1]) \to N(\mathscr{C}')$$

for a unique functor $h : \mathscr{C} \times [1] \to \mathscr{C}'$ such that $h \circ (\mathscr{C} \times \partial_1^1) = F$ and $h \circ (\mathscr{C} \times \partial_0^1) = G$. In turn, the datum of h is equivalent to that of a natural transformation $F \Rightarrow G$.

(iii) By remark 2.3.1(i), the front-to-back duality of §2.1.4 generalizes the functor $(-)^{\mathrm{op}}$: Cat $\to$ Cat that assigns to every small category $\mathscr{C}$ its opposite category $\mathscr{C}^{\mathrm{op}}$. Indeed, for every simplicial set X, by definition the objects of X^{op} are the same as the objects of X, and example 2.1.5(ii,iii) shows that, just as for ordinary categories, the set of morphisms $x \to y$ in X^{op} coincides with the set of morphisms $y \to x$ in X, for every pair of objects x, y; furthermore, the identity of any object x in X^{op} coincides with the identity morphism of x in X. Consequently, we shall sometimes denote by $f^{\mathrm{op}} : y^{\mathrm{op}} \to x^{\mathrm{op}}$ the arrow of X^{op} corresponding to a given arrow $f : x \to y$ of the simplicial set X, generalizing §1.1.5.

Example 2.5.3. (i) A *triangle* of X is a morphism $t : \partial\Delta^2 \to X$; this is then the datum of 3 objects x, y, z and 3 arrows f, g, h of X fitting into a diagram :

$$(f,g,h) \quad : \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

Namely, f (resp. g, resp. h) corresponds to the restriction of t to $\Delta^{\{0,1\}}$ (resp. to $\Delta^{\{1,2\}}$, resp. to $\Delta^{\{0,2\}}$). Notice that for every $i, j \in \mathbb{N}$ with $0 \le i < j \le 2$, there exists a unique isomorphism $[1] \xrightarrow{\sim} \{i, j\}$ of ordered sets, so there is no ambiguity on which object is the source and which is the target of each of these arrows. Moreover, notice that (f, g, h) is a triangle of X if and only if $(g^{\mathrm{op}}, f^{\mathrm{op}}, h^{\mathrm{op}})$ is a triangle of X^{op}.

(ii) Likewise, a morphism $Sp^2 = \Lambda^2_1 \to X$ can be regarded as a diagram :

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z. \end{tikzcd}$$

Definition 2.5.4. (i) We say that a triangle $(f, g, h) : \partial\Delta^2 \to X$ as in example 2.5.3(i) *commutes* (or *is commutative*) if it extends to a morphism $\Delta^2 \to X$:

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "(f,g,h)"] \arrow[d] & X \\ \Delta^2 \arrow[ur] & \end{tikzcd}$$

(ii) Given a pair (f, g) of morphisms of X as in example 2.5.3(ii), we say that a given arrow $h : x \to z$ of X is *a composition of f and g* if the resulting triangle (f, g, h) commutes.

(iii) We say that a morphism $f : x \to y$ of X is *left invertible* if there exists a *left inverse* of f, *i.e.* a morphism $g : y \to x$ in X that makes commute the triangle :

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathbf{1}_x"] & & x. \end{tikzcd}$$

Likewise, we say that f is *right invertible* if f^{op} is left invertible in X^{op}, and then we also say that g is a *right inverse* of f, whenever g^{op} is a left inverse of f^{op}. We say that f is *invertible* if it admits both a right and a left inverse.

(iv) A functor $F : X \to Y$ of simplicial sets is *conservative*, if for every arrow $f : x \to x'$ of X that is not invertible, the arrow $Ff : Fx \to Fy$ is not invertible (*cp.* remark 1.1.11(iii)). We say that a natural transformation $h : F \Rightarrow G$ is *invertible* if for every object x of X, the induced arrow $h_x : Fx \to Gx$ (given by restriction of h to $\{x\} \times \Delta^1 \xrightarrow{\sim} \Delta^1$) is invertible.

Remark 2.5.5. (i) If $X = N(\mathscr{C})$ for a small category $\mathscr{C}$, it is easily seen that a triangle of X commutes $\Leftrightarrow$ it commutes when regarded as a diagram of $\mathscr{C}$, and a composition of a pair of arrows of X is just their composition in $\mathscr{C}$. These assertions can be summarized by saying that the inclusion $Sp^2 \subset \Delta^2$ induces a bijection :

$$\mathrm{sSet}(\Delta^2, N(\mathscr{C})) \xrightarrow{\sim} \mathrm{sSet}(Sp^2, N(\mathscr{C})).$$

Indeed, the meaning of such bijection is that every composable pair of arrows of $N(\mathscr{C})$ has a unique composition. More generally, it is clear that, for every integer $n \ge 2$, the inclusion $Sp^n \subset \Delta^n$ induces a bijection :

$$\mathsf{sSet}(\Delta^n, N(\mathscr{C})) \xrightarrow{\sim} \mathsf{sSet}(Sp^n, N(\mathscr{C})).$$

For $n = 3$, this bijection expresses the associativity of the composition law of $\mathscr{C}$: see the proof of proposition 2.5.10.

(ii) For any simplicial set X, and any arrow $f : x \to y$ of X we get commuting triangles

$$(\mathbf{1}_x, f, f) \qquad \text{and} \qquad (f, \mathbf{1}_y, f)$$

since the (degenerate) 2-simplices $s_1^0(f)$ and $s_1^1(f)$ extend $(\mathbf{1}_x, f, f)$, respectively $(f, \mathbf{1}_x, f)$.

(iii) Moreover, example 2.1.5(ii,iii) implies that a triangle (f, g, h) of X commutes $\Leftrightarrow$ the corresponding triangle $(g^{\mathrm{op}}, f^{\mathrm{op}}, h^{\mathrm{op}})$ commutes in X^{op} (notation of remark 2.5.2(iii)).

We deduce from remark 2.5.5 the following characterization :

Lemma 2.5.6. *For every simplicial set X and every small category $\mathscr{C}$, the datum of a morphism $X \to N(\mathscr{C})$ of* sSet *is equivalent to that of a map $u : X_1 \to N(\mathscr{C})_1$ such that :*

(a) $u(\mathbf{1}_x)$ is an identity morphism of $\mathscr{C}$, for every $x \in X_0$

(b) for every commutative triangle (f, g, h) of X we have $u(h) = u(g) \circ u(f)$.

Proof. Clearly any morphism $X \to N(\mathscr{C})$ of simplicial sets yields, by restriction, a map $X_1 \to N(\mathscr{C})_1$ verifying (a) and (b).

Conversely, let u be such a map; for every $x \in X_0$ we let $u_0(x) \in N(\mathscr{C})_0 = \mathrm{Ob}(\mathscr{C})$ be the source (which is also the target) of $u(\mathbf{1}_x)$.

Claim 2.5.7. For every arrow $f : x \to y$ of X, the source and the target of $u(f)$ are respectively $u_0(x)$ and $u_0(y)$.

Proof : By remark 2.5.5(ii) we have commutative triangles $(\mathbf{1}_x, f, f)$ and $(f, \mathbf{1}_y, f)$, and by (b) it follows that $u(f) = u(f) \circ u(\mathbf{1}_x) = u(\mathbf{1}_y) \circ u(f)$, whence the claim. ◇

Next, let $n > 0$ be an integer, and $x_\bullet : \Delta^n \to X$ any n-simplex of X; the restriction of $x_\bullet$ to Sp^n amounts to a connected sequence of arrows $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$ and in light of claim 2.5.7, we may set

$$u_n(x_\bullet) := \big(u_0(x_0) \xrightarrow{u(f_1)} u_0(x_1) \xrightarrow{u(f_2)} \cdots \xrightarrow{u(f_n)} u_0(x_n)\big) \in N(\mathscr{C})_n.$$

It remains to check that the system of maps $(u_n : X_n \to N(\mathscr{C})_n \,|\, n \in \mathbb{N})$ thus defined yields a morphism of simplicial sets, and to this aim it suffices to show that

$$\begin{aligned} d_n^i \circ u_n &= u_{n-1} \circ d_n^i \qquad \forall n \geq 1, \forall i = 0, \dots, n \\ s_n^i \circ u_n &= u_{n+1} \circ s_n^i \qquad \forall n \geq 0, \forall i = 0, \dots, n. \end{aligned}$$

The identity $s_0^0 \circ u_0 = u_1 \circ s_0^0$ follows immediately from (a). Next, let $\phi : [m] \to [n]$ be any morphism of $\mathbf{\Delta}$; clearly, the composition :

$$Sp^m \hookrightarrow \Delta^m \xrightarrow{\Delta^\phi} \Delta^n$$

is the unique morphism of sSet whose restriction to $\Delta^{\{i,i+1\}}$ is induced by the unique surjection of ordered sets $\{i, i+1\} \to \{\phi(i), \phi(i+1)\}$, for every $i = 0, \dots, m-1$.

Hence, let $n > 0$ and $x_\bullet$ be as in the foregoing; it follows that the restriction of $s_n^i(x_\bullet)$ to Sp^{n+1} is the sequence of arrows :

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} x_i \xrightarrow{\mathbf{1}_{x_i}} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n$$

and likewise we may describe $s_n^i(u_n(x_\bullet))$, for every $i = 0, \dots, n$. Invoking again (a), we get the desired identity $s_n^i \circ u_n = u_{n+1} \circ s_n^i$.

Likewise, the restriction of $d_n^0(x_\bullet)$ and $d_n^n(x_\bullet)$ to Sp^{n-1} are the sequences :

$$x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n \qquad \text{and} \qquad x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1}$$

and a corresponding description applies to $d_n^0(u_n(x_\bullet))$ and respectively $d_n^n(u_n(x_\bullet))$, whence $d_n^i \circ u_n = u_{n-1} \circ d_n^i$ for $i = 0, n$. Lastly, let $n \geq 2$ and $0 < i < n$; we get a 2-simplex :

$$y_\bullet := \Delta^2 \xrightarrow{\sim} \Delta^{\{i-1,i,i+1\}} \to \Delta^n \xrightarrow{x_\bullet} X$$

whence a commutative triangle (f_i, f_{i+1}, h) with $h := d_2^1(y_\bullet) : x_{i-1} \to x_{i+1}$, and it is easily seen that the restriction to Sp^{n-1} of $d_n^i(x_\bullet)$ is given by the sequence :

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} x_{i-1} \xrightarrow{h} x_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n.$$

Likewise, $d_n^i(u_n(x_\bullet))$ is given by the sequence :

$$u_0(x_0) \xrightarrow{u(f_1)} \cdots \xrightarrow{u(f_{i-1})} u_0(x_{i-1}) \xrightarrow{u(f_{i+1})\circ u(f_i)} u_0(x_{i+1}) \xrightarrow{u(f_{i+1})} \cdots \xrightarrow{u(f_n)} u_0(x_n)$$

whence $u_{n-1} \circ d_n^i(x_\bullet) = d_n^i(u_n(x_\bullet))$, by virtue of (b). □

Proposition 2.5.8. *With the notation of §2.3.3, for every simplicial set X, the inclusion $i : Sk_2(X) \hookrightarrow X$ induces an isomorphism of categories :*

$$\boxed{\tau(Sk_2(X)) \xrightarrow{\sim} \tau(X).}$$

Proof. For every small category $\mathscr{C}$ we have a commutative diagram :

$$\begin{tikzcd}
\mathrm{Cat}(\tau(X), \mathscr{C}) \arrow[r] \arrow[d] & \mathrm{sSet}(\tau(Sk_2(X)), \mathscr{C}) \arrow[d] \\
\mathrm{sSet}(X, N(\mathscr{C})) \arrow[r] & \mathrm{sSet}(Sk_2(X), N(\mathscr{C}))
\end{tikzcd}$$

whose vertical arrows are bijections, and whose horizontal arrows are induced by i. Lemma 2.5.6 easily implies that the bottom horizontal arrow is bijective as well, so the same holds for the top horizontal arrow. The assertion then follows from Yoneda's lemma. □

Remark 2.5.5(i) motivates the following definition :

Definition 2.5.9. Let X be a simplicial set. We say that X *satisfies the Grothendieck-Segal condition*, if the inclusion $Sp^n \subset \Delta^n$ induces a bijection :

$$X_n \xrightarrow{\sim} \mathrm{sSet}(\Delta^n, X) \xrightarrow{\sim} \mathrm{sSet}(Sp^n, X) \qquad \forall n \geq 2.$$

Proposition 2.5.10. *The essential image of the nerve functor is the class of simplicial sets that satisfy the Grothendieck-Segal condition.*

Proof. Remark 2.5.5(i) shows that every simplicial set in the essential image of N satisfies the Grothendieck-Segal condition. Conversely, let X be a simplicial set verifying the Grothendieck-Segal condition. We attach to X a small category $\mathscr{C}_X$ such that :

- $\mathrm{Ob}(\mathscr{C}_X) := X_0$
- $\mathscr{C}_X(x, y)$ is the set of $f \in X_1$ with source x and target y, for every $x, y \in X_0$.

By the Grothendieck-Segal condition, every composable pair of arrows $x \xrightarrow{f} y \xrightarrow{g} z$ admits a unique composition $h : x \to z$, and we set $g \circ f := h$. From remark 2.5.5(ii) we know that $\mathbf{1}_x$ is a left and right identity for this composition law, for every $x \in X_0$. It remains to check that the composition law is associative. Hence, let $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$ be any composable sequence of 3 arrows; the datum (f, g, h) amounts to a morphism $Sp^3 \to X$ of simplicial sets, which by assumption extends to a unique morphism $t : \Delta^3 \to X$. Then

$$\partial\Delta^2 \hookrightarrow \Delta^2 \xrightarrow{d_3^3(t)} X \qquad (\text{resp. } \partial\Delta^2 \hookrightarrow \Delta^2 \xrightarrow{d_3^0(t)} X)$$

is the unique triangle whose first two arrows are f and g (resp. g and h), so that :

$$g \circ f = d_2^1 \circ d_3^3(t) \qquad h \circ g = d_2^1 \circ d_3^0(t).$$

Since $d_2^1 \circ d_1^3 = d_2^2 \circ d_3^1$ and $d_2^1 \circ d_3^0 = d_2^0 \circ d_3^2$, it follows that

$$\partial\Delta^2 \hookrightarrow \Delta^2 \xrightarrow{d_3^1(t)} X \qquad (\text{resp. } \partial\Delta^2 \hookrightarrow \Delta^2 \xrightarrow{d_3^2(t)} X)$$

is the unique triangle whose first two arrows are $g \circ f$ and h (resp. f and $h \circ g$), so that :

$$h \circ (g \circ f) = d_2^1 \circ d_3^1(t) \qquad (h \circ g) \circ f = d_2^1 \circ d_3^2(t).$$

But we have $d_2^1 \circ d_3^1 = d_2^1 \circ d_3^2$ (proposition 2.1.2(i)), whence the contention.

By construction, we have a natural identification

$$u_1 : X_1 \xrightarrow{\sim} N(\mathscr{C}_X)_1$$

that satisfies conditions (a) and (b) of lemma 2.5.6, whence a morphism of simplicial sets $X \to N(\mathscr{C}_X)$, and since both X and $N(\mathscr{C}_X)$ verify Grothendieck-Segal condition, the bijectivity of u_1 implies that u is an isomorphism, as required. □

Proposition 2.5.11. *A simplicial set X satisfies the Grothendieck-Segal condition if and only if the inclusion $\Lambda^n_k \subset \Delta^n$ induces bijections :*

$$(*) \qquad \mathrm{sSet}(\Delta^n, X) \xrightarrow{\sim} \mathrm{sSet}(\Lambda^n_k, X) \qquad \forall n \geq 2, \forall k = 1, \dots, n-1.$$

Proof. Suppose first that X satisfies the Grothendieck-Segal condition; by proposition 2.5.10, we have an isomorphism $X \xrightarrow{\sim} N(\mathscr{C})$ for some small category $\mathscr{C}$, and for every simplicial set Y we get a commutative diagram :

$$\begin{tikzcd}
\mathrm{sSet}(Y, X) \arrow[r] \arrow[d] & \mathrm{Cat}(\tau(Y), \mathscr{C}) \arrow[d] \\
\mathrm{sSet}(Sk_2(Y), X) \arrow[r] & \mathrm{Cat}(\tau(Sk_2 Y), \mathscr{C})
\end{tikzcd}$$

whose horizontal arrows are bijections, and whose vertical arrows are induced by the inclusion $Sk_2(Y) \hookrightarrow Y$. In view of proposition 2.5.8, also the right vertical arrow is bijective, so the same holds for the left vertical one. Thus, the inclusions $\Lambda^n_k \subset \Delta^n$, $Sk_2(\Delta^n) \subset \Delta^n$ and $Sk_2(\Lambda^n_k) \subset \Lambda^n_k$ induce a commutative diagram :

$$\begin{tikzcd}
\mathrm{sSet}(\Delta^n, X) \arrow[r] \arrow[d] & \mathrm{sSet}(\Lambda^n_k, X) \arrow[d] \\
\mathrm{sSet}(Sk_2(\Delta^n), X) \arrow[r] & \mathrm{sSet}(Sk_2(\Lambda^n_k), X)
\end{tikzcd}$$

whose vertical arrows are bijections. Now, suppose first that $n \geq 4$; we have

$$Sk_2(\Delta^n) = \bigcup_{0 \leq i < j < l \leq n} \Delta^{\{i,j,l\}} \qquad \Lambda^n_k := \bigcup_{k \in E \subsetneq [n]} \Delta^E$$

and notice that for every sequence $0 \leq i < j < l \leq n$, the subset $E := \{i, j, l, k\}$ is strictly contained in $[n]$, since the cardinality of the latter is $n + 1 \geq 5$. Hence, $\Delta^{\{i,j,l\}} \subset Sk_2(\Lambda^n_k)$ for every such sequence, *i.e.* :

$$Sk_2(\Delta^n) = Sk_2(\Lambda^n_k) \qquad \forall n \geq 4, \forall 0 \leq k \leq n.$$

Hence, the bottom horizontal arrow of the diagram is bijective for every $n \geq 4$ and every $0 \leq k \leq n$, so that the same holds for the top horizontal arrow.

Next, for $n = 2$, the sought bijection is clear, since $\Lambda^2_1 = Sp^2$. Lastly, we have :

$$\Lambda^3_1 = \Delta^{\{0,1,2\}} \cup \Delta^{\{0,1,3\}} \cup \Delta^{\{1,2,3\}} \qquad \Lambda^3_2 = \Delta^{\{0,1,2\}} \cup \Delta^{\{0,2,3\}} \cup \Delta^{\{1,2,3\}}.$$

Hence, the datum of a morphism $x_\bullet : \Lambda^3_1 \to N(\mathscr{C})$ of simplicial sets amounts to that of a diagram in $\mathscr{C}$ of the following shape :

$$(**) \qquad \begin{tikzcd} & & x_2 \arrow[dd, bend left] \\ x_0 \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & x_1 \arrow[ur] \arrow[dr] & \\ & & x_3 \end{tikzcd}$$

whose 3 triangles containing the vertex x_1 commute. But then, clearly the sequence of morphisms $x_0 \to x_1 \to x_2 \to x_3$ is the unique 3-simplex of $N(\mathscr{C})$ whose restriction to Λ^3_1 agrees with $x_\bullet$, as required. The same argument applies to morphisms $\Lambda^3_2 \to N(\mathscr{C})$, and concludes the proof.

Conversely, suppose that we have the bijections $(*)$. Since $\Lambda^2_1 = Sp^2$, we see in particular that every composable pair $x \xrightarrow{f} y \xrightarrow{g} z$ of arrows of X admits a unique composition, so we may define a composition law for the set of arrows of X, as in the proof of proposition 2.5.10, and remark 2.5.5(ii) shows that $\mathbf{1}_x$ is a right and left identity for this composition law, for every $x \in X_0$. In order to obtain a category $\mathscr{C}_X$ with $\mathrm{Ob}(\mathscr{C}_X) = X_0$ as in *loc.cit.*, it remains only to check the associativity of the composition law. Hence, let $x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2 \xrightarrow{h} x_3$ be any composable sequence of arrows in X; after adding $g \circ f : x_0 \to x_2$, $h \circ g : x_1 \to x_3$ and $(h \circ g) \circ f : x_0 \to x_3$, we obtain precisely a diagram of arrows of X of the shape of $(**)$, *i.e.* a morphism $\Lambda^3_1 \to X$, which by assumption extends uniquely to a morphism $\Delta^3 \to X$. The existence of such extension means that every triangle of $(**)$ commutes, which yields the sought associativity property. Arguing as in the proof of proposition 2.5.10, we deduce a morphism of simplicial sets $u : X \to N(\mathscr{C}_X)$, and we are reduced to showing that u is an isomorphism. We check, by induction on n, that $u_n : X_n \to N(\mathscr{C}_X)_n$ is a bijection for every $n \in \mathbb{N}$. The assertion is clear for $n \leq 1$; let then $n \geq 2$, and suppose that u_m is bijective for every $m < n$. We consider the commutative diagram, whose left vertical arrow is bijective by assumption :

$$\begin{tikzcd} \mathrm{sSet}(\Delta^n, X) \arrow[r] \arrow[d] & \mathrm{sSet}(\Delta^n, N(\mathscr{C}_X)) \arrow[d] \\ \mathrm{sSet}(\Lambda^n_1, X) \arrow[r] & \mathrm{sSet}(\Lambda^n_1, N(\mathscr{C}_X)) \end{tikzcd}$$

and whose right vertical arrow is also bijective, by the foregoing. Hence, we are further reduced to proving the bijectivity of the bottom horizontal arrow. However, by example

2.3.9(ii), we get a natural identification of $\mathsf{sSet}(\Lambda^n_1, X)$ (resp. of $\mathsf{sSet}(\Lambda^n_1, N(\mathscr{C}_X))$) with the set Σ_n (resp. Σ'_n) of all sequences of elements of X_{n-1} (resp. of $N(\mathscr{C}_X)_{n-1}$) :

$$(x_i \mid i \in [n] \setminus \{1\}) \qquad \text{such that} \qquad d^i_{n-1}(x_j) = d^j_{n-1}(x_i) \qquad \forall i, j \in [n] \setminus \{1\}.$$

By inductive assumption, u_{n-1} induces a bijection $\Sigma_n \xrightarrow{\sim} \Sigma_{n-1}$, QED. □

Remark 2.5.12. The proof of proposition 2.5.11 implies in particular that we have natural isomorphisms of categories :

$$\tau(\Lambda^2_1) \xrightarrow{\sim} [2] \qquad \tau(\Lambda^3_1) \xrightarrow{\sim} [3] \xleftarrow{\sim} \tau(\Lambda^3_2) \quad \text{and} \quad \tau(\Lambda^n_k) \xrightarrow{\sim} [n] \quad \forall n \geq 4, \forall 0 \leq k \leq n.$$

On the other hand, $\tau(\Lambda^2_0)$ and $\tau(\Lambda^2_2)$ are *not* isomorphic to $[2]$, and $\tau(\Lambda^3_0)$ and $\tau(\Lambda^3_3)$ are *not* isomorphic to $[3]$.

2.5.13. *Localizations of simplicial sets.* In §1.11 and §1.12 we explained how to invert given morphisms of ordinary categories; we now wish to discuss a homotopical version of these constructions, that applies to all simplicial sets. To begin with, for a given arrow $f : x \to y$ of a simplicial set X (see definition 2.5.1(ii)), let us form the push-out :

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(f,\mathbf{1}_x)"] \arrow[d] & X \arrow[d, "\eta^f"] \\ \Delta^2 \arrow[r, "L^f"] & X[gf = \mathbf{1}] \end{tikzcd}$$

where $(f, \mathbf{1}_x)$ denotes the unique morphism whose restriction to $\Delta^{\{0,1\}}$ and to $\Delta^{\{0,2\}}$ are respectively f and $\mathbf{1}_x$ (notation of §2.3.7); hence the restriction of L^f to $\Delta^{\{1,2\}}$ is an arrow $g : y \to x$ of $X[gf = \mathbf{1}]$ that is a left inverse for the image $\overline{f}$ of f in $X[gf = \mathbf{1}]$.

• To explain the universality of the resulting pair of functors $X \xrightarrow{\eta^f} X[gf = \mathbf{1}] \xleftarrow{L^f} \Delta^2$, consider any functor $u : X \to Y$ (*i.e.* any morphism of sSet : see definition 2.5.1(iv)), and suppose that we have a commuting triangle in Y :

$$(u(f), t, \mathbf{1}_{u(x)}) \quad : \quad \begin{tikzcd} & u(y) \arrow[dr, "t"] & \\ u(x) \arrow[ur, "u(f)"] \arrow[rr, "\mathbf{1}_{u(x)}"] & & u(x) \end{tikzcd}$$

that is the boundary of a given 2-simplex $H : \Delta^2 \to Y$. Then there exists a unique functor

$$v : X[gf = \mathbf{1}] \to Y \qquad \text{such that} \qquad v \circ \eta^f = u \quad \text{and} \quad v(L^f) = H$$

and clearly $v(g) = t$. So, g provides a suitably *universal left inverse* for $\overline{f}$.

• In order to add a *universal right inverse* for f, we then perform the previous operation on the arrow f^{op} of the front-to-back dual X^{op} of X (see §2.1.4); hence we let

$$X[fh = \mathbf{1}] := (X^{\mathrm{op}}[h^{\mathrm{op}} f^{\mathrm{op}} = \mathbf{1}])^{\mathrm{op}} \qquad \nu^f := (\eta^{f^{\mathrm{op}}})^{\mathrm{op}} \qquad R^f := (L^{f^{\mathrm{op}}})^{\mathrm{op}}.$$

Explicitly, this comes down to forming the push-out :

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "(\mathbf{1}_y,f)"] \arrow[d] & X \arrow[d, "\nu^f"] \\ \Delta^2 \arrow[r, "R^f"] & X[fh = \mathbf{1}] \end{tikzcd}$$

where $(\mathbf{1}_y, f)$ denotes the unique morphism whose restriction to $\Delta^{\{0,2\}}$ and to $\Delta^{\{1,2\}}$ are respectively $\mathbf{1}_y$ and f. Then the restriction of R^f to $\Delta^{\{0,1\}}$ is a *universal right inverse* $h : y \to x$ for the image of f in $X[fh = \mathbf{1}]$: the reader can spell out the corresponding universal property for the pair of functors $X \xrightarrow{\nu^f} X[fh = \mathbf{1}] \xleftarrow{R^f} \Delta^2$.

• We can combine the two foregoing constructions to add *universal right and left inverse arrows* to the given arrow f; indeed, let us set

$$X[f^{-1}] := X[gf = \mathbf{1}][\overline{f}h = \mathbf{1}]$$

and define $\mu^f : X \to X[f^{-1}]$ as the composition $X \xrightarrow{\eta^f} X[gf = \mathbf{1}] \xrightarrow{\nu^{\overline{f}}} X[f^{-1}]$. The image of f in $X[f^{-1}]$ has then a left inverse g (or rather, the image of g in $X[f^{-1}]$) and a right inverse h; the triangles $(f, g, \mathbf{1}_x)$ and $(h, f, \mathbf{1}_y)$ are the boundaries of the 2-simplices

$$\overline{L}^f : \Delta^2 \xrightarrow{L^f} X[gf = \mathbf{1}] \xrightarrow{\nu^{\overline{f}}} X[f^{-1}] \qquad \text{and} \qquad \overline{R}^f := R^{\overline{f}} : \Delta^2 \to X[f^{-1}].$$

• More generally, we wish to formally invert every element of a given subset $\Sigma \subset X_1$. To this aim, we proceed as in the proof of proposition 1.11.4 : suppose first that $\Sigma = \{f_1, \dots, f_n\}$ is a finite subset; then we set :

$$Y_0 := X \qquad \text{and} \qquad Y_i := Y_{i-1}[\overline{f}_i^{-1}] \qquad \forall i = 1, \dots, n$$

where $\overline{f}_i$ denotes the image of f_i in Y_{i-1} for every $i = 1, \dots, n$. Then we let

$$X[\Sigma^{-1}] := Y_n \qquad \text{and denote by} \qquad \mu^{X,\Sigma} : X \to X[\Sigma^{-1}]$$

the composition of the localizations $(\mu^{\overline{f}_i} : Y_{i-1} \to Y_i \mid i = 1, \dots, n)$.

• Lastly, if Σ is an arbitrary subset of X_1, we let $\mathscr{P}_0(\Sigma)$ be the set of all finite subsets of Σ, partially ordered by inclusion of subsets; by the foregoing, for every $\Delta \in \mathscr{P}_0(\Sigma)$ we get a localization $\mu^{X,\Delta} : X \to X[\Delta^{-1}]$, and we have

$$\mu^{X,\Delta'} = \mu^{X[\Delta^{-1}],\Delta'\setminus\Delta} \circ \mu^{X,\Delta} \qquad \forall \Delta, \Delta' \in \mathscr{P}_0(\Sigma) \quad \text{with} \quad \Delta \subset \Delta'.$$

Thus, we have a well-defined functor

$$X[-] : \mathscr{P}_0(\Sigma) \to \mathsf{sSet} \qquad \Delta \mapsto X[\Delta^{-1}] \qquad (\Delta \subset \Delta') \mapsto \mu^{X[\Delta^{-1}],\Delta'\setminus\Delta}$$

and we denote by $X[\Sigma^{-1}]$ the colimit of $X[-]$. If $(\delta_\Delta : X[\Delta^{-1}] \to X[\Sigma^{-1}] \mid \Delta \in \mathscr{P}_0(\Sigma))$ is a universal co-cone, we set as well $\mu^\Sigma := \delta_\varnothing : X = X[\varnothing^{-1}] \to X[\Sigma^{-1}]$.

• The simplicial set $X[\Sigma^{-1}]$ comes equipped with a distinguished system of pairs of 2-simplices :

$$\overline{L}^f \ : \ \begin{tikzcd} & y \arrow[dr, "g^f"] & \\ x \arrow[ur, "\overline{f}"] \arrow[rr, "\mathbf{1}_x"'] & & x \end{tikzcd} \qquad \overline{R}^f \ : \ \begin{tikzcd} & x \arrow[dr, "\overline{f}"] & \\ y \arrow[ur, "h^f"] \arrow[rr, "\mathbf{1}_y"'] & & y \end{tikzcd} \qquad \forall (f : x \to y) \in \Sigma$$

where $\overline{f}$ denotes the image of f in $X[\Sigma^{-1}]$, for every $f \in \Sigma$. The datum $(\mu^\Sigma, \overline{L}^\bullet, \overline{R}^\bullet)$ enjoys the following universal property : let $u : X \to Y$ be any functor such that $u(f)$ is invertible in Y for every $f \in \Sigma$, so that we have commuting triangles :

$$\begin{tikzcd} & u(y) \arrow[dr] & \\ u(x) \arrow[ur, "u(f)"] \arrow[rr, "\mathbf{1}_{u(x)}"'] & & u(x) \end{tikzcd} \qquad \begin{tikzcd} & u(x) \arrow[dr, "u(f)"] & \\ u(y) \arrow[ur] \arrow[rr, "\mathbf{1}_{u(y)}"'] & & u(y) \end{tikzcd} \qquad \forall f \in \Sigma$$

that are boundaries of 2-simplices $H^f, K^f : \Delta^2 \rightrightarrows Y$. Then there exists a unique functor

$$v : X[\Sigma^{-1}] \to Y \qquad \text{such that} \qquad v \circ \mu^\Sigma = u \quad v(\overline{L}^f) = H^f \quad v(\overline{R}^f) = K^f \qquad \forall f \in \Sigma.$$

2.5.14. Let $(\eta_\bullet, \varepsilon_\bullet)$ be the unit and counit of the adjunction for the adjoint pair (τ, N) (see §2.3.3); then $\eta_X : X \to N \circ \tau X$ consists of a system of natural maps :

$$\eta_{X,n} : X_n \to \mathsf{Cat}([n], \tau X) \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet}), \forall n \in \mathbb{N}.$$

The naturality of the maps $\eta_{X,n}$ means that we get commutative diagrams of sets :

$$\begin{tikzcd} X_m \arrow[r, "\eta_{X,m}"] \arrow[d, "f^*"'] & \mathsf{Cat}([m], \tau X) \arrow[d, "\mathsf{Cat}(f,\tau X)"] \\ X_n \arrow[r, "\eta_{X,n}"] & \mathsf{Cat}([n], \tau X) \end{tikzcd} \qquad \begin{tikzcd} Y_n \arrow[r, "\eta_{Y,n}"] \arrow[d, "u_n"'] & \mathsf{Cat}([n], \tau Y) \arrow[d, "\mathsf{Cat}([n],\tau(u))"] \\ X_n \arrow[r, "\eta_{X,n}"] & \mathsf{Cat}([n], \tau X) \end{tikzcd}$$

for every morphism $f : [n] \to [m]$ of $\mathbf{\Delta}$ and every morphism $u : Y \to X$ of sSet.

- Especially, let $x \in X_n$ for some $n \in \mathbb{N}$; if we take $Y := \Delta^n$ and let $u_x : \Delta^n \to X$ be the unique morphism of sSet such that $u_{x,n}(\mathbf{1}_{[n]}) = x$, we see that

$$\eta_{X,n}(x) = \tau(u_x) \circ \eta_{\Delta^n,n}(\mathbf{1}_{[n]}).$$

However, recall that $\varepsilon_\bullet$ is an isomorphism; then, since $\Delta^n = N([n])$, the triangular identities for $(\eta_\bullet, \varepsilon_\bullet)$ yield : $\eta_{\Delta^n,n}(\mathbf{1}_{[n]}) = N(\varepsilon_{[n]})_n^{-1}(\mathbf{1}_{[n]}) = \varepsilon_{[n]}^{-1} : [n] \xrightarrow{\sim} \tau\Delta^n$ (see [13, Prob.2.13]). In the following we will identify $[n]$ with $\tau\Delta^n$ via the (unique) isomorphism $\varepsilon_{[n]}^{-1}$, and we shall use for $\eta_{X,n}$ the notation :

$$x \mapsto \tau_n(x) := \tau(u_x) \circ \varepsilon_{[n]}^{-1} \qquad \forall n \in \mathbb{N}, \forall x \in X_n.$$

- Hence, to every $n \in \mathbb{N}$ and every subset $\Sigma \subset X_n$, we may attach the image $\tau_n\Sigma$ of Σ under the map τ_n; *e.g.* for $n = 1$, we have $\tau_1\Sigma \subset \mathrm{Mor}(\tau X)$ for every such Σ, and for every arrow $f : x \to y$ of X, the source and target of $\tau_1(f)$ are respectively $\tau_0(x)$ and $\tau_0(y)$. Also, for every $x \in X_0$, the identity $\mathbf{1}_x : x \to x$ corresponds to the 1-simplex $\Delta^1 \to \Delta^0 \xrightarrow{x} X$, so $\tau_1(\mathbf{1}_x)$ is the morphism corresponding to the functor $[1] \to [0] \xrightarrow{\tau_0(x)} \tau(X)$, *i.e.* we have $\tau_1(\mathbf{1}_x) = \mathbf{1}_{\tau_0(x)}$ for every $x \in X_0$. The maps τ_n are bijections for $n = 0$ (remark 2.3.5), but for $n > 0$ they are, in general, neither injective nor surjective.

Proposition 2.5.15. *Let $X \in \mathrm{Ob}(\mathsf{sSet})$, $\Sigma \subset X_1$ and $\mu^\Sigma : X \to X[\Sigma^{-1}]$ the localization functor of §2.5.13. With the notation of §2.5.14, the pair $(\tau(X[\Sigma^{-1}]), \tau(\mu^\Sigma))$ is a localization of the category τX along the subset $\tau_1\Sigma$;* i.e. *we have a natural identification of categories :*

$$\boxed{\tau(X[\Sigma^{-1}]) \xrightarrow{\sim} \tau X[(\tau_1\Sigma)^{-1}].}$$

Proof. In view of remark 1.11.3(iv), it suffices to check that for every small category $\mathscr{D}$, the functor μ^Σ induces a bijection :

$$\alpha : \mathsf{Cat}(\tau(X[\Sigma^{-1}]), \mathscr{D}) \xrightarrow{\sim} \{F \in \mathsf{Cat}(\tau X, \mathscr{D}) \mid F(\tau_1\Sigma) \subset \mathrm{Isom}(\mathscr{D})\} \qquad G \mapsto G \circ \tau(\mu^\Sigma)$$

where $\mathrm{Isom}(\mathscr{D})$ denotes the set of isomorphism of $\mathscr{D}$. By adjunction, we have natural bijections :

$$\omega_{X[\Sigma^{-1}]} : \mathsf{Cat}(\tau(X[\Sigma^{-1}]), \mathscr{D}) \xrightarrow{\sim} \mathsf{sSet}(X[\Sigma^{-1}], N\mathscr{D}) \quad \omega_X : \mathsf{Cat}(\tau X, \mathscr{D}) \xrightarrow{\sim} \mathsf{sSet}(X, N\mathscr{D})$$

that identify α with the corresponding map

$$\beta : \mathsf{sSet}(X[\Sigma^{-1}], N\mathscr{D}) \to \mathsf{sSet}(X, N\mathscr{D}) \qquad v \mapsto v \circ \mu^\Sigma$$

and notice that an arrow $f : a \to b$ of $N\mathscr{D}$ is invertible if and only if it is an isomorphism of $\mathscr{D}$; moreover, the sequences $a \xrightarrow{f} b \xrightarrow{f^{-1}} a$ and $b \xrightarrow{f^{-1}} a \xrightarrow{f} b$ are the unique 2-simplices of $N\mathscr{D}$ bounding the commuting triangles

$$\begin{tikzcd} & b \arrow[dr, "f^{-1}"] & \\ a \arrow[ur, "f"] \arrow[rr, "\mathbf{1}_a"'] & & a \end{tikzcd} \qquad \text{and resp.} \qquad \begin{tikzcd} & a \arrow[dr, "f"] & \\ b \arrow[ur, "f^{-1}"] \arrow[rr, "\mathbf{1}_b"'] & & b. \end{tikzcd}$$

The universal property of $X[\Sigma^{-1}]$ then says that β induces a bijection :

$$\gamma : \mathrm{sSet}(X[\Sigma^{-1}], N\mathscr{D}) \xrightarrow{\sim} \{v \in \mathrm{sSet}(X, N\mathscr{D}) \mid v(\Sigma) \subset \mathrm{Isom}(\mathscr{D})\} \qquad v \mapsto v \circ \mu^{\Sigma}.$$

Hence, we need to check that ω_X identifies the target of α with the target of γ. To this aim, let $j : \mathscr{D}_0 \to \mathscr{D}$ be the inclusion of the subcategory with $\mathrm{Ob}(\mathscr{D}_0) = \mathrm{Ob}(\mathscr{D})$, and whose morphisms are the isomorphisms of $\mathscr{D}$; also, for every $x \in \Sigma$, let $u_x : \Delta^1 \to X$ be the corresponding morphisms of sSet, as in §2.5.14. Then, a morphism $v : X \to N\mathscr{D}$ lies in the target of γ if and only if $v \circ u_x : \Delta^1 \to N\mathscr{D}$ factors through a morphism $v_x : \Delta^1 \to N\mathscr{D}_0$ for every $x \in \Sigma$, and by adjunction, v_x corresponds to a functor $F_x : \tau\Delta^1 \to \mathscr{D}_0$ such that $j \circ F_x = \varepsilon_{\mathscr{D}} \circ \tau(v \circ u_x) : \tau\Delta^1 \to \tau N\mathscr{D} \xrightarrow{\sim} \mathscr{D}$. On the other hand, a functor $F : \tau X \to \mathscr{D}$ lies in the target of α if and only if $F \circ \tau(u_x) : \tau\Delta^1 \to \mathscr{D}$ factors through a functor $\tau\Delta^1 \to \mathscr{D}_0$, for every $x \in \Sigma$. To conclude, it suffices now to notice that $\omega_X^{-1}(v) = \varepsilon_{\mathscr{D}} \circ \tau(v)$ for every morphism $v : X \to N\mathscr{D}$ of sSet. □

2.6. ∞-categories and their homotopy categories.

Definition 2.6.1. (i) An ∞-*category* (resp a *Kan complex*) is a simplicial set X such that for every integer $n \geq 2$ and every $0 < k < n$ (resp. for every integer $n \geq 1$ and every $0 \leq k \leq n$), the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ induces a surjection :

$$\mathrm{sSet}(\Delta^n, X) \to \mathrm{sSet}(\Lambda^n_k, X).$$

(ii) An ∞-*groupoid* is an ∞-category in which all morphisms are invertible.

Example 2.6.2. (i) By virtue of propositions 2.5.10 and 2.5.11, the nerve of every small category $\mathscr{C}$ is an ∞-category. Moreover, a morphism of $N(\mathscr{C})$ is invertible if and only if it is an isomorphism of $\mathscr{C}$, hence $\mathscr{C}$ is a groupoid if and only if $N(\mathscr{C})$ is an ∞-groupoid.

(ii) Since the product of any family of surjections is a surjection, we see that the product in sSet of any small family of ∞-categories is again an ∞-category. Then it also follows easily that the product of any small family of ∞-groupoids is again an ∞-groupoid.

Proposition 2.6.3. *Every Kan complex is an ∞-groupoid.*

Proof. Clearly every Kan complex X is an ∞-category. For every morphism $f : x \to y$ of X, there exists a unique morphism $u : \Lambda^2_0 \to X$ of simplicial sets whose restriction to $\Delta^{\{0,1\}}$ agrees with f, and whose restriction to $\Delta^{\{0,2\}}$ agrees with $\mathbf{1}_x$; by assumption, u extends to a morphism $\Delta^2 \to X$, whose restriction to $\partial\Delta^2$ yields a commutative diagram exhibiting a left inverse for f. Likewise one may exhibit a right inverse for f. □

Remark 2.6.4. (i) We shall see that the converse of proposition 2.6.3 holds as well.

(ii) The notion of ∞-category was introduced by Boardman and Vogt in order to understand the theory of algebraic structures up to (coherent) homotopies, under the name of *weak Kan complexes.* They were developed by Joyal under the name of *quasi-categories*, and then by Lurie under the name of ∞-categories.

(iii) With remark 2.3.8(i), it is easily seen that the front-to-back dual X^{op} of any ∞-category X is an ∞-category, which we also call simply the *opposite of* X. Also, X is an ∞-groupoid if and only if the same holds for X^{op}.

2.6.5. *The Boardman-Vogt construction.* Let X be an ∞-category; the rest of this section is dedicated to giving an explicit description of the associated category $\tau(X)$ (notation of §2.3.3). To this aim we will have to study morphisms of the form

$$x_\bullet : Sk_1(\Delta^3) \to X.$$

Such a morphism amounts to the datum of a diagram of arrows of X of the shape :

$$\begin{tikzcd} & & x_2 \arrow[dd, bend left] \\ x_0 \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & x_1 \arrow[ur] \arrow[dr] & \\ & & x_3 \end{tikzcd}$$

in which none of the triangles is required to commute. Such a diagram consists of four triangles $(d^i x_\bullet : \partial\Delta^2 \to X \,|\, i = 0, \dots, 3)$, corresponding to the restrictions of $x_\bullet$ to the subsimplicial sets $Sk_1(\Delta^{E_i})$, where $E_i := \{0, \dots, 3\} \setminus \{i\}$ for $i = 0, \dots, 3$.

Lemma 2.6.6. (Joyal's coherence lemma) *With the notation of §2.6.5, suppose that the triangles $d^0 x_\bullet$ and $d^3 x_\bullet$ commute. Then $d^1 x_\bullet$ commutes $\Leftrightarrow$ the same holds for $d^2 x_\bullet$.*

Proof. As already observed in the proof of proposition 2.5.11, the commutativity of $d^0 x_\bullet$, $d^2 x_\bullet$ and $d^3 x_\bullet$ amounts to the assertion that $x_\bullet$ is the restriction of a morphism $\Lambda^3_1 \to X$, which in turn extends to a morphism $\Delta^3 \to X$, since by assumption X is an ∞-category, whence the commutativity of $d^1 x_\bullet$. Likewise, the commutativity of $d^0 x_\bullet$, $d^1 x_\bullet$ and $d^3 x_\bullet$ means that $x_\bullet$ is the restriction of a morphism $\Lambda^3_2 \to X$, which extends to a morphism $\Delta^3 \to X$, whence the commutativity of $d^2 x_\bullet$. □

2.6.7. Given three arrows f, g, h of the ∞-category X, we shall write :

$$gf \sim h$$

if the triple (f, g, h) is a commutative triangle of X.

Lemma 2.6.8. *Let $x, y \in X_0$, and $f, g : x \rightrightarrows y$ two arrows of X. We have :*

(i) $f\mathbf{1}_x \sim f$ *and* $\mathbf{1}_y f \sim f$.

(ii) $f\mathbf{1}_x \sim g \Leftrightarrow \mathbf{1}_y f \sim g \Leftrightarrow g\mathbf{1}_x \sim f \Leftrightarrow \mathbf{1}_y g \sim f$.

(iii) Let us write $f \sim_1 g$ if $f\mathbf{1}_x \sim g$. Then $\sim_1$ is an equivalence relation on the set $X(x, y)$.

Proof. (i) follows from remark 2.5.5(ii). For (ii) we consider the diagrams :

$$\begin{tikzcd} & & y \arrow[dd, bend left, "\mathbf{1}_y"] \\ x \arrow[r, "\mathbf{1}_x"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & x \arrow[ur, "f"] \arrow[dr, "f"] & \\ & & y \end{tikzcd} \qquad \begin{tikzcd} & & y \arrow[dd, bend left, "\mathbf{1}_y"] \\ x \arrow[r, "\mathbf{1}_x"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "f"'] & x \arrow[ur, "f"] \arrow[dr, "g"] & \\ & & y \end{tikzcd}$$

In view of (i) and lemma 2.6.6, the first diagram shows that $\mathbf{1}_y f \sim g \Leftrightarrow f\mathbf{1}_x \sim g$. Likewise, the second diagram shows that $\mathbf{1}_y f \sim g \Rightarrow g\mathbf{1}_x \sim f$. After exchanging the roles of f and

g we get as well the equivalence : $\mathbf{1}_y g \sim f \Leftrightarrow g\mathbf{1}_x \sim f$ and that $\mathbf{1}_y g \sim f \Rightarrow f\mathbf{1}_x \sim g$, whence (i).

(iii): The reflexivity of $\sim_1$ holds by virtue of (i), and the symmetricity follows from (ii). For the transitivity, suppose that $f \sim_1 g$ and $g \sim_1 h$, and consider the diagram :

$$\begin{tikzcd}
 & & y \arrow[dd, bend left, "\mathbf{1}_y"] \\
x \arrow[r, "\mathbf{1}_x"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "h"'] & x \arrow[ur, "g"] \arrow[dr, "h"] & \\
 & & y.
\end{tikzcd}$$

We already know that $g \sim_1 f$ and $h \sim_1 h$, so lemma 2.6.6 yields $\mathbf{1}_y f \sim h$, which, according to (ii), is equivalent to $f \sim_1 h$, as required. ◻

2.6.9. Let X be an ∞-category; for any arrow $f : x \to y$ of X, we let $[f]$ be the equivalence class of f in the quotient

$$\mathsf{ho}(X)(x, y) := X(x, y)/{\sim_1}$$

for the equivalence relation $\sim_1$ provided by lemma 2.6.8(iii).

Lemma 2.6.10. *(i) Let $x \xrightarrow{f} y \xrightarrow{g} z$ and $h, h' : x \rightrightarrows z$ be four arrows of X such that $gf \sim h$. Then we have $gf \sim h'$ if and only if $[h] = [h']$.*

(ii) In the situation of (i), let $x \xrightarrow{f'} y \xrightarrow{g'} z$ be two more arrows of X such that $[f] = [f']$ and $[g] = [g']$. Then $g'f' \sim h$.

Proof. (i): By applying lemma 2.6.6 to the diagram :

$$\begin{tikzcd}
 & & z \arrow[dd, bend left, "\mathbf{1}_z"] \\
x \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h'"'] & y \arrow[ur, "g"] \arrow[dr, "g"] & \\
 & & z
\end{tikzcd}$$

we get $\mathbf{1}_z h \sim h' \Leftrightarrow gf \sim h'$, whence the assertion, in view of lemma 2.6.8(ii).

(ii): Let us check first that $g'f \sim h$; to this aim, we apply lemma 2.6.6 to the diagram :

$$\begin{tikzcd}
 & & z \arrow[dd, bend left, "\mathbf{1}_z"] \\
x \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & y \arrow[ur, "g"] \arrow[dr, "g'"] & \\
 & & z
\end{tikzcd}$$

and recall that $[g] = [g'] \Leftrightarrow \mathbf{1}_z g \sim g'$, again by lemma 2.6.8(ii). Next, by arguing with the opposite ∞-category X^{op}, the foregoing implies that $gf' \sim h$ (see remark 2.5.5(iii)). The assertion follows by combining these two cases. ◻

2.6.11. Lemma 2.6.10 implies that for every $x, y, z \in X_0$ we get a well-defined map

$$\mathsf{ho}(X)(x,y) \times \mathsf{ho}(X)(y,z) \to \mathsf{ho}(x,z) \qquad ([f],[g]) \mapsto [g] \circ [f] := [g \circ f]$$

where $g \circ f : x \to z$ denotes any composition of f and g in X (the existence of $g \circ f$ is assured by the assumption that X is an ∞-category).

Theorem 2.6.12. (Boardman and Vogt) *Let X be any ∞-category. We have :*

(i) There exists a well-defined small category :

$$\mathsf{ho}(X)$$

whose set of objects is X_0, and such that the set of morphism $x \to y$ in $\mathsf{ho}(X)$ is given by $\mathsf{ho}(X)(x,y)$, for every $x, y \in X_0$; the composition law of $\mathsf{ho}(X)$ is given by the system of maps as in §2.6.11. We call $\mathsf{ho}(X)$ the homotopy category *of X.*

(ii) There exists a unique morphism $u_X : X \to N(\mathsf{ho}(X))$ of simplicial sets which is the identity map on objects, and which sends every arrow f of X to its class $[f]$.

(iii) The adjoint of the morphism u_X is an isomorphism of categories :

$$\boxed{\tau(X) \xrightarrow{\sim} \mathsf{ho}(X).}$$

Proof. (i): From lemma 2.6.8(i) it is already clear that $[f] \circ [\mathbf{1}_x] = [f] = [\mathbf{1}_y] \circ [f]$ for every arrow $f : x \to y$ of X. The verification of the associativity of the composition law of $\mathsf{ho}(X)$ is analogous to that in the proof of proposition 2.5.10 : given a sequence $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$ of arrows of X, we fix a composition $g \circ f$ of f and g, and a composition $h \circ g$ of g and h, and we consider the resulting diagram :

$$\begin{tikzcd}
 & & z \arrow[dd, bend left, "h"] \\
x \arrow[r, "f"] \arrow[urr, bend left, "g\circ f"] \arrow[drr, bend right] & y \arrow[ur, "g"] \arrow[dr, "h\circ g"'] & \\
 & & t
\end{tikzcd}$$

which, by virtue of lemma 2.6.6, shows that any composition of $g \circ f$ and h is also a composition of f and $h \circ g$, as required.

(ii): The assertion follows immediately from lemma 2.5.6.

(iii): Invoking again lemma 2.5.6, we see that for every small category $\mathscr{C}$, every morphism of simplicial sets $X \to N(\mathscr{C})$ is the composition of u and a morphism of the form $N(F) : N(\mathsf{ho}(X)) \to N(\mathscr{C})$, for a unique functor $F : \mathsf{ho}(X) \to \mathscr{C}$. In other words, $(\mathsf{ho}(X), u)$ is a universal couple for the functor :

$$\mathsf{Cat} \to \mathsf{Set} \qquad \mathscr{C} \mapsto \mathsf{sSet}(X, N(\mathscr{C}))$$

whence the assertion. □

Corollary 2.6.13. *Let X be an ∞-category. We have :*

(i) X is an ∞-groupoid if and only if $\mathsf{ho}(X)$ is a groupoid.

(ii) An arrow $f : x \to y$ of X is invertible ⇔ $[f]$ is an isomorphism of $\mathsf{ho}(X)$ ⇔ there exists an arrow $g : y \to x$ of X such that $gf \sim \mathbf{1}_x$ and $fg \sim \mathbf{1}_y$.

Proof. Clearly (ii)⇒(i). To show (ii), suppose first that f is invertible in X; then there exist arrows $g, h : y \to x$ of X such that $gf \sim \mathbf{1}_x$ and $fh \sim \mathbf{1}_y$, so that $[g] \circ [f] = [\mathbf{1}_x]$ and $[f] \circ [h] = [\mathbf{1}_y]$, and therefore $[f]$ is an isomorphism of $\mathsf{ho}(X)$. Next, if the class $[f]$ of f is an isomorphism in $\mathsf{ho}(X)$, there exists an arrow $g : y \to x$ of X such that $[gf] = [\mathbf{1}_x]$ and $[fg] = [\mathbf{1}_y]$, whence $gf \sim \mathbf{1}_x$ and $fg \sim \mathbf{1}_y$, by lemma 2.6.10(i). Lastly, clearly the latter condition implies that f is an invertible arrow of X, and the proof is concluded. □

Remark 2.6.14. (i) Let $F : X \to Y$ be any functor between ∞-categories; for every pair $f, g : x \rightrightarrows x'$ of morphisms of X, it is clear that $f \sim_1 g \Rightarrow Ff \sim_1 Fg$. Hence, F induces a functor $\mathsf{ho}(F) : \mathsf{ho}(X) \to \mathsf{ho}(Y)$ that makes commute the diagram :

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "u_X"'] & Y \arrow[d, "u_Y"] \\ N\mathsf{ho}(X) \arrow[r, "N\mathsf{ho}(F)"] & N\mathsf{ho}(Y) \end{tikzcd}$$

where u_X and u_Y are the morphisms of theorem 2.6.12(ii). It follows easily that the isomorphisms $\tau(X) \xrightarrow{\sim} \mathsf{ho}(X)$ and $\tau(Y) \xrightarrow{\sim} \mathsf{ho}(Y)$ of theorem 2.6.12(iii) identify $\mathsf{ho}(F)$ with $\tau(F) : \tau(X) \to \tau(Y)$.

(ii) Let $(X_i \mid i \in I)$ be any small family of ∞-categories, and recall that $X := \prod_{i \in I} X_i$ is again an ∞-category (example 2.6.2(ii)); from the construction of the category $\mathsf{ho}(X)$, it is easily seen that the universal cone $(X \to X_i \mid i \in I)$ induces an isomorphism of categories $\mathsf{ho}(X) \xrightarrow{\sim} \prod_{i \in I} \mathsf{ho}(X_i)$. Then, by virtue of (i) and theorem 2.6.12(iii), also the functor τ commutes with arbitrary small products of ∞-categories.

(iii) From corollary 2.6.13(ii) we also see that a functor $F : X \to Y$ of ∞-categories is conservative if and only if the same holds for the induced functor $\mathsf{ho}(F) : \mathsf{ho}(X) \to \mathsf{ho}(Y)$ between homotopy categories (see definition 2.5.4(iv)).

(iv) With theorem 2.6.12(iii) and §2.3.3 we get natural isomorphisms of categories :

$$\omega_{\mathscr{C}} : \mathscr{C} \xrightarrow{\sim} \mathsf{ho}(N\mathscr{C}) \qquad \forall \mathscr{C} \in \mathsf{Ob}(\mathsf{Cat}).$$

Explicitely, $\omega_{\mathscr{C}}$ is the identity on objects, and maps every morphism $f : x \to y$ of $\mathscr{C}$ to the class $[f]$ of the corresponding 1-simplex $f : x \to y$ of $N\mathscr{C}$.

3. Basic homotopical algebra

This chapter is a review of the basic results of D.Quillen's seminal treatise [12].

The first section starts out with a study of certain *left and right lifting properties* for pairs of morphisms in an arbitrary category $\mathscr{C}$: these notions – isolated first by Quillen – abstract some general patterns that are ubiquitous in classical algebraic topology, and set the stage for the introduction of *weak factorization systems*, consisting of pairs $(\mathscr{I}, \mathscr{P})$ of classes of morphisms of $\mathscr{C}$ that are stable under retracts, such that every element of $\mathscr{I}$ has the left lifting property relative to every element of $\mathscr{P}$, and such that every morphism of $\mathscr{C}$ can be written as $p \circ i$, for some $i \in \mathscr{I}$ and $p \in \mathscr{P}$.

After these preliminaries, we come in §3.2 to the central notion of this chapter and of Quillen's homotopical algebra : that of *model category*, of which there exist in the literature several variant definitions, all essentially – though not quite completely – equivalent to the original one found in [12]; for us, it will be the datum of a finitely complete and finitely cocomplete category $\mathscr{C}$ together with three classes of morphisms : the classes $\mathscr{W}$ of *weak equivalences*, $\mathscr{F}ib$ of *fibrations*, and $\mathscr{C}of$ of *cofibrations*, fulfilling a pair of axioms, the first of which basically says that $\mathscr{W}$ resembles the class of isomorphisms of $\mathscr{C}$; the second axiom requires that both $(\mathscr{C}of, \mathscr{F}ib \cap \mathscr{W})$ and $(\mathscr{C}of \cap \mathscr{W}, \mathscr{F}ib)$ form weak factorization systems for $\mathscr{C}$. It was Quillen's insight, that such a structure captures all that is needed in order to recover, in an abstract setting, most essential constructions and several main results of the classical homotopy theory of topological spaces : notably, one has suitable *cylinder and cocylinder objects* in model categories, with which one can formulate a well-behaved notion of *homotopy equivalence* for morphisms, and then prove a generalization of *Whitehead's theorem*, stating that *a morphism of fibrant and cofibrant objects is a weak equivalence if and only if it is a homotopy equivalence* (theorem 3.3.9(i)).

An essential feature of model categories is that one can localize them by inverting weak equivalences : the resulting *homotopy categories* can be regarded as wide-reaching generalizations of the derived categories associated to abelian categories with enough injective or projective objects. Whereas derived categories provide a natural framework for the construction and investigation of derived functors of *additive* functors, homotopy categories fulfill the same role with respect to *arbitrary* functors on model categories.

This theme is developed comprehensively starting with §3.4 : while we do not strive for maximal generality, we present enough of the theory to convey Quillen's vision of *homotopical algebra as a non-abelian generalization of homological algebra*, though focusing on those aspects that will be useful in later sections of our text. So, we first explain how to derive functors defined on model categories, in rather general situations; especially, we treat in detail *Quillen adjunctions* and their induced derived adjoint pairs. Next, §3.5 studies some elementary classes of *homotopy limits and colimits*, and lastly, §3.6 adds some further complements for the special cases of *homotopy push-outs and pull-backs*.

3.1. **Lifting properties and weak factorisation systems.**

Definition 3.1.1. Let $\mathscr{C}$ be a category, and $A \xrightarrow{i} B$, $X \xrightarrow{p} Y$ two morphisms of $\mathscr{C}$.

(i) We say that *i has the left lifting property with respect to p*, or equivalently, that *p has the right lifting property with respect to i*, if every commutative square :

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

has a *diagonal filler*, *i.e.* can be completed to a commutative diagram :

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

(ii) Let $\mathscr{F}$ be a class of morphisms of $\mathscr{C}$; we say that a morphism f of $\mathscr{C}$ *has the left* (resp. *right*) *lifting property with respect to* $\mathscr{F}$, if f has the left (resp. right) lifting property with respect to every element of $\mathscr{F}$.

(iii) We denote by $l(\mathscr{F})$ (resp. $r(\mathscr{F})$) the class of morphisms of $\mathscr{C}$ that have the left (resp. right) lifting property with respect to $\mathscr{F}$.

(iv) Suppose that $\mathscr{C}$ has an initial object $\varnothing$; then we say that X is $\mathscr{F}$*-projective*, if the unique morphism $\varnothing \to X$ is in $l(\mathscr{F})$. Dually, in case $\mathscr{C}$ has a final object e, then we say that X is $\mathscr{F}$*-injective*, if the unique morphism $X \to e$ is in $r(\mathscr{F})$.

Remark 3.1.2. (i) In the situation of definition 3.1.1, the morphism $i : A \to B$ has the left lifting property with respect to $p : X \to Y \Leftrightarrow$ the morphism $i^{\mathrm{op}} : B \to A$ in $\mathscr{C}^{\mathrm{op}}$ has the right lifting property with respect to $p^{\mathrm{op}} : Y \to X$.

(ii) Let $\mathscr{E}$ be the class of epimorphisms of $\mathscr{C}$, and suppose that $\mathscr{C}$ has an initial object $\varnothing$; then X is a projective object of $\mathscr{C} \Leftrightarrow X$ is $\mathscr{E}$-projective.

(iii) Dually, let $\mathscr{M}$ be the class of monomorphisms of $\mathscr{C}$, and suppose that $\mathscr{C}$ has a final object e; then X is an injective object of $\mathscr{C} \Leftrightarrow X$ is $\mathscr{M}$-injective.

Definition 3.1.3. Let $\mathscr{C}$ be a category, $X, U \in \mathrm{Ob}(\mathscr{C})$, and $\mathscr{F} \subset \mathrm{Mor}(\mathscr{C})$.

(i) X *is a retract of* U, if there exists a commutative diagram in $\mathscr{C}$ of the form:

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathbf{1}_X"] & U \arrow[r] & X. \end{tikzcd}$$

(ii) Let $f : X \to Y$ and $g : U \to V$ be two morphisms of $\mathscr{C}$; we say that f *is a retract of* g if there exists a commutative diagram in $\mathscr{C}$ of the form :

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathbf{1}_X"] & U \arrow[r, "b"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "c"] \arrow[rr, bend right, "\mathbf{1}_Y"'] & V \arrow[r, "d"] & Y. \end{tikzcd}$$

(iii) We say that the class $\mathscr{F}$ *is stable under retracts*, if every retract of every element of $\mathscr{F}$ lies in $\mathscr{F}$.

(iv) We say that $\mathscr{F}$ *is stable under push-outs*, if for every cocartesian square :

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

in $\mathscr{C}$, such that $f \in \mathscr{F}$, we have $f' \in \mathscr{F}$. We say that $\mathscr{F}$ *is stable under pull-backs*, if $\mathscr{F}^{\mathrm{op}}$ is stable under push-outs in $\mathscr{C}^{\mathrm{op}}$.

(v) Let $\lambda > 0$ be an ordinal. An $(\mathscr{F}, \lambda)$-*sequence* is a functor $F : \lambda \to \mathscr{C}$ such that

$$L_i := \varinjlim_{j<i} F(j) \quad \forall i \in \lambda^* := \lambda \setminus \{0\} \qquad \text{and} \qquad L_\lambda := \varinjlim_{j<\lambda} F(j)$$

are representable in $\mathscr{C}$ and the induced morphism $L_i \to F(i)$ is in $\mathscr{F}$ for all $i \in \lambda^*$.

(vi) $\mathscr{F}$ *is stable under transfinite compositions*, if it contains all the isomorphisms of $\mathscr{C}$, and for every ordinal $\lambda > 0$ and every $(\mathscr{F}, \lambda)$-sequence $F : \lambda \to \mathscr{C}$, the induced morphism $F(0) \to L_\lambda$ lies in $\mathscr{F}$.

(vii) We say that a class $\mathscr{F}$ of morphisms of $\mathscr{C}$ *is weakly saturated*, if it is stable under push-outs and under transfinite compositions. We say that $\mathscr{F}$ is *saturated*, if it is weakly saturated and stable under retracts.

Remark 3.1.4. (i) In the situation of definition 3.1.3(i,ii), X (resp f) is a retract of U (resp. of g) in $\mathscr{C} \Leftrightarrow X$ (resp. f^{op}) is a retract of U (resp. of g^{op}) in $\mathscr{C}^{\mathrm{op}}$.

(ii) Let $X \xrightarrow{i} U \xrightarrow{p} X$ be two morphisms of the category $\mathscr{C}$ such that $p \circ i = \mathbf{1}_X$; then for every $Z \in \mathrm{Ob}(\mathscr{C})$, the rule $f \mapsto f \circ p$ establishes a bijection from $\mathscr{C}(X, Z)$ to the set of $g \in \mathscr{C}(U, Z)$ such that $g = g \circ i \circ p$. In other words, X represents the coequalizer of the pair $(\mathbf{1}_U, i \circ p)$, and p gives the universal co-cone for such coequalizer. Likewise, if $f : X \to Y$ is a retract of $g : U \to V$, so that we have also morphisms $Y \xrightarrow{j} V \xrightarrow{q} Y$ as in definition 3.1.3(ii), then Y represents the coequalizer of $(\mathbf{1}_V, j \circ q)$, and f is characterized as the unique morphism such that $f \circ p = q \circ g$.

(iii) Let $\mathscr{C}, \mathscr{D}$ be two categories, and $F : \mathscr{C} \to \mathscr{D}$ a functor that preserves all small representable connected colimits of $\mathscr{C}$. With (ii), we see that for every saturated (resp. weakly saturated) class $\mathscr{F} \subset \mathrm{Mor}(\mathscr{D})$, the class $F^{-1}\mathscr{F}$ is saturated (resp. weakly saturated).

Lemma 3.1.5. *For every category $\mathscr{C}$, the classes $\mathscr{I}$ of isomorphisms of $\mathscr{C}$ and $\mathscr{E}$ of epimorphisms of $\mathscr{C}$ are saturated.*

Proof. Clearly $\mathscr{I}$ is stable by push-outs, and the same holds for $\mathscr{E}$ (see [13, Exemp.2.26(v)]). Next, consider a diagram as in definition 3.1.3(ii), and suppose that $g \in \mathscr{I}$; then, it is easily seen that $bg^{-1}c$ is a left and right inverse for f, so $f \in \mathscr{I}$. In case $g \in \mathscr{E}$, notice that $d \in \mathscr{E}$ as well (see [13, Exerc.1.119(i)]), and then the same holds for dg; now, if $\alpha, \beta : Y \rightrightarrows Z$ are two morphisms of $\mathscr{C}$ such that $\alpha f = \beta f$, we get $\alpha dg = \alpha f b = \beta f b = \beta dg$, whence $\alpha = \beta$, and this shows that $f \in \mathscr{E}$.

Thus, it remains only to check that $\mathscr{I}$ and $\mathscr{E}$ are stable under trasfinite compositions. Let λ be an ordinal, and $F_\bullet : \lambda \to \mathscr{C}$ either an $(\mathscr{I}, \lambda)$-sequence or an $(\mathscr{E}, \lambda)$-sequence; we let $(\phi_j : F_j \to L \mid j \in \lambda)$ be a universal co-cone, and extend $F_\bullet$ to a functor $F_\bullet : \lambda + 1 \to \mathscr{C}$ by $F_\lambda := L$ and $F_{j\lambda} := \phi_j$ for every $j \in \lambda$. If $F_\bullet$ is an $(\mathscr{I}, \lambda)$-sequence, let us show, by transfinite induction, that $F_{0i} : F_0 \to F_i$ is an isomorphism for every $i \leq \lambda$. This is clear for $i = 0$. Next, let $i > 0$, and suppose that the assertion is already known for every $j < i$; then it is easily seen that the system $(F_{0j}^{-1} : F_j \to F_0 \mid j < i)$ is a universal co-cone $F_{\bullet|i} \Rightarrow c_{F_0}$ (where $F_{\bullet|i} : i \to \mathscr{C}$ denotes the restriction of $F_\bullet$). As $F_\bullet$ is an $(\mathscr{I}, \lambda)$-sequence, the induced morphism $F_0 \to F_i$ is then an isomorphism; but the latter is precisely F_{0i}.

Lastly, if $F_\bullet$ is an $(\mathscr{E}, \lambda)$-sequence, let us check, again by trasfinite induction, that the morphism $F_{0i} : F_0 \to F_i$ is an epimorphism for every $i \leq \lambda$. This is clear for $i = 0$. Next, let $i > 0$, suppose that the assertion is already known for every $j < i$, and let $(\tau_j : F_j \to L_i \mid j < i)$ be a universal co-cone $\tau_\bullet : F_{\bullet|i} \Rightarrow c_{L_i}$; then F_{0i} is the composition of τ_0 with the induced morphism $L_i \to F_i$, and the latter is an epimorphism, since $F_\bullet$ is an $(\mathscr{E}, \lambda)$-sequence. Thus, it suffices to check that $\tau_0 : F_0 \to L_i$ is an epimorphism.

Hence, let $\alpha, \beta : L_i \rightrightarrows X$ be two morphisms of $\mathscr{C}$ with $\alpha\tau_0 = \beta\tau_0$; it follows that $\alpha\tau_j F_{0j} = \beta\tau_j F_{0j}$ for every $j < i$, whence $\alpha\tau_j = \beta\tau_j$ for every such j, since by inductive assumption F_{0j} is an epimorphism. By the universality of $\tau_\bullet$, this implies that $\alpha = \beta$, so τ_0 is an epimorphism. □

Proposition 3.1.6. *Let $\mathscr{C}$ be a category, and $\mathscr{F} \subset \mathrm{Mor}(\mathscr{C})$ a subclass. Then there exists a smallest saturated (resp. weakly saturated) class $\mathscr{S}$ containing $\mathscr{F}$. We call $\mathscr{S}$ the* saturation *(resp. the* weak saturation*) of $\mathscr{F}$.*

Proof. We prove the existence of the saturation of $\mathscr{F}$; *mutatis mutandis*, the same argument applies to the weak saturation. Clearly, the intersection of any *finite* family of saturated classes of $\mathscr{C}$ is saturated; hence, it is tempting to define the saturation of $\mathscr{F}$ simply as the intersection of all the saturated subclasses of $\mathrm{Mor}(\mathscr{C})$ containing $\mathscr{F}$. However, such a definition runs into set-theoretic difficulties, since it is expressed by a first order formula that requires quantification over variables denoting classes, which is not admissible in many set-theoretic frameworks, and especially, not in the Bernays-Gödel axiomatisation (though it would be admissible in, *e.g.* Kelley-Morse set theory).

Instead, we use the following argument, proposed by Gabber. Let $\mathscr{A} \subset \mathscr{C}$ be any subcategory, and $i : \mathscr{A} \to \mathscr{C}$ the inclusion functor. For every cardinal κ, let us say that $\mathscr{A}$ is *relatively κ-cocomplete,* if i is full and for every κ-small category I (see definition 1.1.6(ii)) and every functor $F : I \to \mathscr{A}$, the following holds :

- the colimit of F is representable in $\mathscr{A}$ ⇔ the colimit of $i \circ F$ is representable in $\mathscr{C}$
- if the colimit of F is representable in $\mathscr{A}$, then i preserves such colimit.

Claim 3.1.7. For every infinite cardinal κ, every small subcategory $\mathscr{A} \subset \mathscr{C}$ is contained in a small relatively κ-cocomplete subcategory of $\mathscr{C}$.

Proof : Let $\mathscr{I}$ be the *set* of all κ-small categories I with $\mathrm{Ob}(I) \subset \kappa$ and $I(x, y) \subset \kappa$ for every $x, y \in \mathrm{Ob}(I)$. We construct by transfinite induction a family of small subcategories $\mathscr{A}_\lambda$, indexed by the class $\mathscr{O}$ of all ordinals, as follows. We set $\mathscr{A}_0 := \mathscr{A}$; next, let $\lambda > 0$ be any ordinal, and suppose that the family $(\mathscr{A}_\mu \,|\, \mu < \lambda)$ has already been constructed. If λ is a limit ordinal, we set $\mathscr{A}_\lambda := \bigcup_{\mu<\lambda} \mathscr{A}_\mu$. If $\lambda = \mu + 1$, denote by $i_\mu : \mathscr{A}_\mu \to \mathscr{C}$ the inclusion functor, and by $\mathscr{S}_\mu$ the *set* of all functors $F : I \to \mathscr{A}_\mu$ with $I \in \mathscr{I}$ and such that the colimit of $i_\mu \circ F$ is representable in $\mathscr{C}$; also, let $\mathscr{T}_\mu$ be the *set* of all functors $G : I \to \mathscr{A}_\mu$ with $I \in \mathscr{I}$ that admit a universal co-cone $\tau_\bullet^G : G \Rightarrow c_{L(G)}$, but such that the colimit of $i_\mu \circ G$ is *not* representable in $\mathscr{C}$. For every $F \in \mathscr{S}_\mu$ and every $G \in \mathscr{T}_\mu$ we let :

- $\mathscr{L}(F)$ be the class of all objects of $\mathscr{C}$ that represent the colimit of $i_\mu \circ F$
- $\mathscr{L}(G)$ be the class of all objects X of $\mathscr{C}$ inducing *a non-bijective map*

$$\mathscr{C}(L(G), X) \to \mathscr{C}^I(G, c_X) \qquad f \mapsto c_f \circ \tau_\bullet^G$$

Then, for every $F \in \mathscr{S}_\mu \cup \mathscr{T}_\mu$ we set

- $\rho_F := \min(r(X) \,|\, X \in \mathscr{L}(F))$ and $\mathscr{L}_0(F) := r^{-1}(\rho_F) \cap \mathscr{L}(F)$

where $r : \mathscr{U} \to \mathscr{O}$ is the rank function on the class $\mathscr{U}$ of all sets (see [13, Rem.2.9(ii)]).

We define $\mathscr{A}_\lambda$ as the full subcategory of $\mathscr{C}$ with :

$$\mathrm{Ob}(\mathscr{A}_\lambda) := \mathrm{Ob}(\mathscr{A}_\mu) \cup \bigcup_{F \in \mathscr{S}_\mu \cup \mathscr{T}_\mu} \mathscr{L}_0(F).$$

It is easily seen that the sequence $(\mathscr{A}_\lambda \,|\, \lambda \in \mathscr{O})$ thus obtained is stationary : indeed we have $\mathscr{A}_\lambda = \mathscr{A}_{\kappa^+}$ for every $\lambda \geq \kappa^+$ (and even for every $\lambda \geq \kappa$, if κ is a regular cardinal : see definition 1.1.1); it follows easily that $\mathscr{A}_{\kappa^+}$ is relatively κ-cocomplete. ◇

Now, for any category $\mathscr{A}$ and any class $\mathscr{G} \subset \mathrm{Mor}(\mathscr{A})$, let us say that :

- $\mathscr{G}$ is *stable under κ-small compositions,* if for every ordinal $\lambda < \kappa$ and every $(\mathscr{G}, \lambda)$-sequence $F : \lambda \to \mathscr{A}$, the induced morphism $F0 \to \varinjlim_{i \in \lambda} Fi$ lies in $\mathscr{G}$
- $\mathscr{G}$ is *κ-saturated*, if it is stable under retracts, push-outs, and κ-small compositions.

Next, let $\mathscr{A} \subset \mathscr{C}$ be a small relatively κ-cocomplete subcategory; we define the *relative κ-saturation* $\mathscr{G}^*_{\kappa,\mathscr{A}}$ of $\mathscr{G}$ *in* $\mathscr{A}$ as the intersection of all the κ-saturated subsets of $\mathrm{Mor}(\mathscr{A})$ containing $\mathscr{G}$. Lastly, let $\mathscr{F} \subset \mathrm{Mor}(\mathscr{C})$ be any subclass, and denote by $\mathscr{E}$ the *class* of all small relatively κ-cocomplete subcategories of $\mathscr{C}$; we define the *κ-saturation* of $\mathscr{F}$ as

$$\mathscr{F}^*_\kappa := \bigcup_{\mathscr{A} \in \mathscr{E}} (\mathscr{F} \cap \mathrm{Mor}(\mathscr{A}))^*_{\kappa,\mathscr{A}}.$$

In light of claim 3.1.7 and remark 3.1.4(ii), it is easily seen that $\mathscr{F}^*_\kappa$ is indeed the smallest κ-saturated subclass of $\mathrm{Mor}(\mathscr{C})$ containing $\mathscr{F}$, for every infinite cardinal κ. To conclude, let Ω be the class of all infinite cardinals; it suffices now to observe that :

$$\bigcup_{\kappa \in \Omega} \mathscr{F}^*_\kappa$$

is the smallest saturated subclass of $\mathrm{Mor}(\mathscr{C})$ containing $\mathscr{F}$. □

Lemma 3.1.8. *Let $\mathscr{C}$ be a cocomplete (resp. finitely cocomplete) category, $\mathscr{F}$ a weakly saturated class of morphisms of $\mathscr{C}$ that contains all isomorphisms. Then $\mathscr{F}$ is stable under small (resp. finite) coproducts,* i.e. *for every small (resp. finite) family $(u_i : X_i \to Y_i \mid i \in I)$ of elements of $\mathscr{F}$, the induced morphism $\coprod_{i\in I} u_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ lies in $\mathscr{F}$.*

Proof. Let c be the cardinality of I; recall that c is the smallest ordinal number with a bijection $\omega : c \xrightarrow{\sim} I$, so we may as well replace I by c. For every ordinal $j \leq c$ let us set

$$E_j := \coprod_{i<j} Y_i \sqcup \coprod_{i \geq j} X_i.$$

In particular, $E_0 = \coprod_{i<c} X_i$ and $E_c = \coprod_{i<c} Y_i$. For every pair of ordinals $j \leq k$, we have a unique morphism

$$v_{jk} : E_j \to E_k$$

characterized as follows :

- for $i \geq k$, the restriction $X_i \hookrightarrow E_j \xrightarrow{v_{jk}} E_k$ is the natural morphism $X_i \hookrightarrow E_k$
- for $i < j$, the restriction $Y_i \hookrightarrow E_j \xrightarrow{v_{jk}} E_k$ is the natural morphism $Y_i \hookrightarrow E_k$
- for $j \leq i < k$, the restriction $X_i \hookrightarrow E_j \xrightarrow{v_{jk}} E_k$ is the composition $X_i \xrightarrow{u_i} Y_i \hookrightarrow E_k$.

It is easily seen that $v_{kl} \circ v_{jk} = v_{jl}$ for all ordinals $j \leq k \leq l$, so the system $(E_\bullet, v_{\bullet\bullet})$ defines a functor

$$E : c + 1 \to \mathscr{C}.$$

Moreover, it is easily seen that for every limit ordinal $l \leq c$ the induced morphism

$$\varinjlim_{j<l} E_j \to E_l$$

is an isomorphism, and in particular it lies in $\mathscr{F}$. Lastly, for every ordinal $j < c$ we get, by simple inspection, a cocartesian diagram :

$$\begin{tikzcd} X_j \arrow[r] \arrow[d, "u_j"'] & E_j \arrow[d, "v_{j,j+1}"] \\ Y_j \arrow[r] & E_{j+1} \end{tikzcd}$$

so $v_{j,j+1}$ lies in $\mathscr{F}$ as well. Summing up, we conclude that the natural morphism

$$E_0 \to E_c = \varinjlim_{j<c} E_j$$

is in $\mathscr{F}$; but the latter is none else than $\bigsqcup_{i \in I} u_i$. □

Proposition 3.1.9. *Let $\mathscr{C}$ be a category, and $\mathscr{F}, \mathscr{F}'$ two classes of morphisms of $\mathscr{C}$. The following holds :*

(i) $\mathscr{F} \subset r(\mathscr{F}') \Leftrightarrow \mathscr{F}' \subset l(\mathscr{F})$.
(ii) *If* $\mathscr{F} \subset \mathscr{F}'$, *then* $r(\mathscr{F}') \subset r(\mathscr{F})$ *and* $l(\mathscr{F}') \subset l(\mathscr{F})$.
(iii) $r(\mathscr{F}) = r(l(r(\mathscr{F})))$ *and* $l(\mathscr{F}) = l(r(l(\mathscr{F})))$.
(iv) $l(\mathscr{F}^{\mathrm{op}}) = r(\mathscr{F})^{\mathrm{op}}$ *and* $r(\mathscr{F}^{\mathrm{op}}) = l(\mathscr{F})^{\mathrm{op}}$ *(notation of §1.6).*
(v) $l(\mathscr{F})$ *is saturated in* $\mathscr{C}$ *and* $r(\mathscr{F})^{\mathrm{op}}$ *is saturated in* $\mathscr{C}^{\mathrm{op}}$.
(vi) *Every isomorphism of* $\mathscr{C}$ *lies in* $l(\mathscr{F}) \cap r(\mathscr{F})$.

Proof. (i) and (ii) and (vi) follow straightforwardly from the definitions, and (iv) follows from remark 3.1.2(i). Then, by virtue of (iv) the two assertions of (v) are equivalent, and likewise for the two identities of (iii). Now, we have :

$$\mathscr{G} \subset r(l(\mathscr{G})) \qquad \text{and} \qquad \mathscr{G} \subset l(r(\mathscr{G})) \qquad \text{for every class } \mathscr{G} \subset \mathrm{Mor}(\mathscr{C})$$

again by a direct inspection of the definitions. Combining with (ii) we then get :

$$r(\mathscr{F}) \subset r(l(r(\mathscr{F}))) \subset r(\mathscr{F})$$

whence (iii). In order to check that $l(\mathscr{F})$ is stable under retracts, consider a commutative diagram :

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathbf{1}_X"] & U \arrow[r] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "p"] \\ Y \arrow[r, "i"] \arrow[rr, bend right, "\mathbf{1}_Y"'] & V \arrow[r] & Y \arrow[r] & B \end{tikzcd}$$

with $p \in \mathscr{F}$ and $g \in l(\mathscr{F})$. Then we have a diagonal filler $h : V \to A$ for the square formed by U, A, V and B. It follows easily that $h \circ i : Y \to A$ is a diagonal filler for the square formed by X, A, Y and B, whence $f \in \mathscr{F}$.

Similarly, to show that $l(\mathscr{F})$ is stable under push-outs we consider a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & X' \arrow[r, "a'"] \arrow[d, "f'"] & A \arrow[d, "p"] \\ Y \arrow[r, "b"] & Y' \arrow[r, "b'"] & B \end{tikzcd}$$

with $p \in \mathscr{F}$ and $f \in l(\mathscr{F})$, and whose left square is cocartesian. Hence, we have a diagonal filler $d : Y \to A$ for the square formed by X, A, Y and B. Then, there exists a unique morphism $d' : Y' \to A$ such that $d' \circ f' = a'$ and $d' \circ b = d$; it is easily seen that d' is a diagonal filler for the diagram formed by X', A, Y' and B.

Lastly, let λ be an ordinal and $F_\bullet : \lambda \to \mathscr{C}$ an $(l(\mathscr{F}), \lambda)$-sequence; we extend $F_\bullet$ to a functor denoted again $F_\bullet : \lambda + 1 \to \mathscr{C}$, by setting

$$F_\lambda := \varinjlim_{i \in \lambda} F_i$$

where the morphisms $F_{i\lambda} : F_i \to F_\lambda$ are given by the universal co-cone. Suppose we are given an element $p : A \to B$ of $\mathscr{F}$ and a commutative diagram :

$$(*) \qquad \begin{tikzcd} F_0 \arrow[r, "f"] \arrow[d, "F_{0\lambda}"'] & A \arrow[d, "p"] \\ F_\lambda \arrow[r, "g"] & B. \end{tikzcd}$$

We construct for every ordinal $\mu \le \lambda + 1$, a co-cone $h_\bullet^{(\mu)} : F_{\bullet|\mu} \Rightarrow c_A$ such that :

- $h_0^{(\mu)} = f$ and $p \circ h_i^{(\mu)} = g \circ F_{i\lambda}$ for every $i < \mu$
- $h_i^{(\mu)} = h_i^{(\nu)}$ for every $i < \nu \le \mu$.

where $F_{\bullet|\mu}$ denotes the restriction of $F_\bullet$ to $\mu \subset \lambda+1$. Clearly the morphism $h_\lambda^{(\lambda+1)} : F_\lambda \to A$ shall be the sought diagonal filler for $(*)$.

We argue by transfinite induction : for $\mu = 0$, we have the empty co-cone, and for $\mu = 1$ we take $h_0^{(1)} := f$. Next, let $\mu > 1$, and suppose that $h_\bullet^{(\nu)}$ has already been exhibited for every ordinal $\nu < \mu$. If μ is a limit ordinal, it is easily seen that the rule :

$$i \mapsto h_i^{(\mu)} := h_i^{(i+1)} \qquad \forall i < \mu$$

yields a co-cone $h_\bullet^{(\mu)}$ with the sought properties. Lastly, if $\mu = \nu+1$, fix a universal co-cone $\tau_\bullet : F_{\bullet|\nu} \Rightarrow c_L$, so that $h_\bullet^{(\nu)}$ induces a unique morphism $b : L \to A$ with $b \circ \tau_i = h_i^{(\nu)}$ for all $i < \nu$. Then $p \circ b \circ \tau_i = g \circ F_{i\lambda}$ for every $i < \nu$. We deduce a commutative diagram :

$$\begin{tikzcd} L \arrow[r, "b"] \arrow[d, "a"'] & A \arrow[d, "p"] \\ F_\nu \arrow[r, "g \circ F_{\nu\lambda}"] & B \end{tikzcd}$$

where a is the unique morphism of $\mathscr{C}$ such that $a \circ \tau_i = F_{i\nu}$ for every $i < \nu$. By assumption, $a \in l(\mathscr{F})$, so this diagram admits a diagonal filler $d : F_\nu \to A$. We set

$$h_i^{(\mu)} := h_i^{(\nu)} \quad \forall i < \nu \qquad \text{and} \qquad h_\nu^{(\mu)} := d.$$

It is easily seen that the resulting system $(h_i^{(\mu)} \mid i \le \nu)$ yields the sought co-cone $h_\bullet^{(\mu)}$. □

Proposition 3.1.10. (Retract lemma) *Let $\mathscr{C}$ be a category, and $X \xrightarrow{i} Y \xrightarrow{p} Z$ two morphisms of $\mathscr{C}$. Suppose that $f := p \circ i$ has the right (resp. left) lifting property with respect to i (resp. to p). Then f is a retract of p (resp. of i).*

Proof. If $f \in r(\{i\})$, pick a diagonal filler $d : Y \to X$ for the commutative square :

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Y \arrow[r, "p"] & Z. \end{tikzcd}$$

We deduce a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathbf{1}_X"] & Y \arrow[r, "d"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Z \arrow[r, equal] & Z \arrow[r, equal] & Z \end{tikzcd}$$

which exhibits f as a retract of p. The case where $f \in l(\{p\})$ is reduced to the first case, by considering the opposite category, by virtue of remarks 3.1.2(i) and 3.1.4(i). □

3.1.11. For any category $\mathscr{C}$ and any class $\mathscr{F} \subset \mathrm{Mor}(\mathscr{C})$, set (notation of §1.4)

$$\mathscr{F}/X := \mathrm{s}_X^{-1}(\mathscr{F}) = \{f/X \in \mathrm{Mor}(\mathscr{C}/X) \mid f \in \mathscr{F}\} \qquad \forall X \in \mathrm{Ob}(\mathscr{C}).$$

Likewise, we shall set $X/\mathscr{F} := (\mathscr{F}^{\mathrm{op}}/X^{\mathrm{op}})^{\mathrm{op}} = \mathrm{t}_X^{-1}(\mathscr{F})$ for every $X \in \mathrm{Ob}(\mathscr{C})$.

Lemma 3.1.12. *(i) We have $r(\mathscr{F}/X) = r(\mathscr{F})/X$ and $l(\mathscr{F})/X \subset l(\mathscr{F}/X)$ for all $X \in \mathrm{Ob}(\mathscr{C})$. If moreover $\mathscr{F}$ is stable under pull-backs, then $l(\mathscr{F}/X) = l(\mathscr{F})/X$.*

(ii) $l(r(\mathscr{F}/X)) = l(r(\mathscr{F}))/X$ for every class $\mathscr{F} \subset \mathrm{Mor}(\mathscr{C})$ and all $X \in \mathrm{Ob}(\mathscr{C})$.

(iii) Dually, $l(X/\mathscr{F}) = X/l(\mathscr{F})$ and $X/r(\mathscr{F}) \subset r(X/\mathscr{F})$ for all $X \in \mathrm{Ob}(\mathscr{C})$. If moreover $\mathscr{F}$ is stable under push-outs, then $r(X/\mathscr{F}) = X/r(\mathscr{F})$.

(iv) If $\mathscr{F}$ is saturated (resp. weakly saturated), the same holds for $X/\mathscr{F}$, and if moreover $\mathscr{C}$ is cocomplete, then also $\mathscr{F}/X$ is saturated (resp. weakly saturated).

Proof. (i): The first assertion is clear from the definitions. Suppose next that $\mathscr{F}$ is stable under pull-backs, and consider an element $f/X : (K, k) \to (L, l)$ of $l(\mathscr{F}/X)$ and a commutative diagram of $\mathscr{C}$ with $p \in \mathscr{F}$:

$$\mathscr{D} \quad : \quad \begin{tikzcd} K \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ L \arrow[r, "b"] & Z. \end{tikzcd}$$

In order to check that $l(\mathscr{F}/X) \subset l(\mathscr{F})/X$, we need to exhibit a diagonal filler $L \to Y$ for $\mathscr{D}$. The latter is equivalent to giving a diagonal filler for the diagram $\mathscr{D}'$ obtained from $\mathscr{D}$ after replacing p by the base change morphism $q : L \times_Z Y \to L$; however, $q \in \mathscr{F}$ by assumption, so we may replace $\mathscr{D}$ by $\mathscr{D}'$ and assume from start that $Z = L$ and $b = \mathbf{1}_L$. Then $\mathscr{D}$ is of the form $\mathrm{s}_{/X}(\mathscr{D}/X)$ for a square diagram $\mathscr{D}/X$ whose left vertical arrow is f/X and whose right vertical arrow p/X lies in $\mathscr{F}/X$; but then $\mathscr{D}/X$ admits a diagonal filler, and hence the same holds for $\mathscr{D}$.

(ii) follows from (i), since $r(\mathscr{F})$ is stable under pull-backs, by virtue of proposition 3.1.9(v). Likewise, (iii) follows from (i), by duality.

(iv): By corollary 1.4.6(ii), the functor t_X preserves all representable connected colimits, so the assertion for $X/\mathscr{F}$ follows from remark 3.1.4(iii). Likewise, if $\mathscr{C}$ is cocomplete, the functor s_X preserves all small colimits (corollary 1.4.6(iii)), so by the same token we get the assertion for $\mathscr{F}/X$. □

Example 3.1.13. Let $\mathscr{C}$ be an abelian category, and $\mathscr{E}$ (resp. $\mathscr{M}$) the class of epimorphisms (resp. of monomorphisms) of $\mathscr{C}$; then $l(\mathscr{E})$ is the class of monomorphisms with projective cokernel. Dually, $r(\mathscr{M})$ is the class of epimorphisms with injective kernel.

• Indeed, suppose first that $i : A \to B$ is a monomorphism with projective cokernel C; for a given epimorphism $p : X \to Y$, consider a square diagram in $\mathscr{C}$:

$$(*) \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y. \end{tikzcd}$$

Since C is projective, the projection $q : B \to C$ has a section $C \to B$, so we get a decomposition $\omega : B \xrightarrow{\sim} A \oplus C$ ([13, Rem.2.99(iii)]), and we may as well assume that $B = A \oplus C$, and that i is the natural monomorphism $A \to A \oplus C$. Let then $r : B \to A$ be the natural projection, and set $h := pfr - g : B \to Y$; then hi is the zero morphism, so h factors through q and a unique morphism $k : C \to Y$. Next, since C is projective, there exists a morphism $t : C \to X$ with $pt = k$. Then $fr - tq : B \to X$ is a diagonal filler for $(*)$.

• Conversely, suppose that $i : A \to B$ lies in $l(\mathscr{E})$; since the unique morphism $A \to e$ is an epimorphism, it follows easily that i admits a left inverse $B \to A$, so it is a split monomorphism ([13, Exerc.1.119(i)]). Next, let $p : X \to Y$ be an epimorphism of $\mathscr{C}$, and $u \in \mathscr{C}(C, Y)$; then we get a square $(*)$ with f the zero morphism and $g = uq$. By assumption, such diagram admits a diagonal filler $d : B \to X$; since $di = 0$, the morphism d factors through q and a morphism $d' : C \to X$, and since $pd = uq$, we have $pd' = u$, which shows that C is projective.

• The assertion for $r(\mathscr{M})$ follows by duality from that for $l(\mathscr{E})$, in light of proposition 3.1.9(iv), since $\mathscr{M}^{\mathrm{op}}$ is the class of monomorphisms of $\mathscr{C}^{\mathrm{op}}$, and since $\mathscr{C}^{\mathrm{op}}$ is an abelian category ([13, Rem.2.81(ii)]).

Example 3.1.14. (i) For $\mathscr{C} = \mathsf{Set}$, and $i : \varnothing \to \{\varnothing\}$, the class $r(i)$ is the class of surjections, while a reformulation of the axiom of choice is the assertion that $l(r(i))$ is the class of injections. Since any small set is a small coproduct of sets with one element, lemma 3.1.8 implies that the saturation of $\{i\}$ is the class $l(r(i))$.

(ii) Let $\mathscr{A}$ be any small category; since the colimits in $\widehat{\mathscr{A}}$ are computed termwise (remark 1.6.2(i)), it follows from (i) that the class $\mathscr{M}$ of monomorphisms of $\widehat{\mathscr{A}}$ is stable under push-outs and transfinite compositions. Moreover, $\mathscr{M}^{\mathrm{op}}$ is the class of epimorphisms of $(\widehat{\mathscr{A}})^{\mathrm{op}}$, which is stable under retracts (lemma 3.1.5), so $\mathscr{M}$ is stable under retracts as well (remark 3.1.4(i)), *i.e.* $\mathscr{M}$ is saturated.

(iii) Let $\mathscr{A}$ be a small Eilenberg-Zilber category, and consider the class

$$\mathscr{F} := \{\partial h_a \to h_a \mid a \in \mathrm{Ob}(\mathscr{A})\} \subset \mathrm{Mor}(\widehat{\mathscr{A}}).$$

By lemma 3.1.8 and theorem 2.2.8, the saturation of $\mathscr{F}$ contains the class $\mathscr{M}$ of monomorphisms of $\widehat{\mathscr{A}}$; but $\mathscr{M}$ is saturated, due to (ii), and $\mathscr{F} \subset \mathscr{M}$, so $\mathscr{M}$ is the saturation of $\mathscr{F}$.

(iv) Let $\mathscr{A}$ be a small Eilenberg-Zilber category, and $\mathscr{F}$ a given class of morphisms of $\widehat{\mathscr{A}}$ such that for every $(f : X \to Y) \in \mathscr{F}$ the presheaves X and Y have only finitely many non-degenerate sections. Then *the class $r(\mathscr{F}) \subset \mathrm{Mor}(\widehat{\mathscr{A}})$ is stable under small filtered colimits* (here we regard $r(\mathscr{F})$ as a class of objects of the category $\widehat{\mathscr{A}}^{[1]}$ of morphisms of $\widehat{\mathscr{A}}$, and the assertion is that the colimit in $\widehat{\mathscr{A}}^{[1]}$ of every filtered system of elements of $r(\mathscr{F})$ lies again in $r(\mathscr{F})$). Indeed, let $u : K \to L$ and $p : X \to Y$ be any two morphisms of $\widehat{\mathscr{A}}$; then $u \in r(\{p\}) \Leftrightarrow$ the induced map

$$(u^*, p_*) : \widehat{\mathscr{A}}(L, X) \to \widehat{\mathscr{A}}(K, X) \times_{\widehat{\mathscr{A}}(K,Y)} \widehat{\mathscr{A}}(L, Y)$$

is surjective. Hence, for every fixed $(K \xrightarrow{u} L) \in \mathscr{F}$ consider the functor

$$F_u : \widehat{\mathscr{A}}^{[1]} \to \mathsf{Set}^{[1]} \qquad p \mapsto (u^*, p_*)$$

that assigns to every morphism $(X \xrightarrow{p} Y) \to (X' \xrightarrow{p'} Y')$ of $\widehat{\mathscr{A}}^{[1]}$, *i.e.* to every commutative diagram of $\widehat{\mathscr{A}}$:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

the commutative diagram of sets :

$$\begin{tikzcd} \widehat{\mathscr{A}}(L,X) \arrow[r, "(u^*,p_*)"] \arrow[d, "\widehat{\mathscr{A}}(L,i)"'] & \widehat{\mathscr{A}}(K,X) \times_{\widehat{\mathscr{A}}(K,Y)} \widehat{\mathscr{A}}(L,Y) \arrow[d, "\widehat{\mathscr{A}}(K,i)\times_{\widehat{\mathscr{A}}(K,j)}\widehat{\mathscr{A}}(L,j)"] \\ \widehat{\mathscr{A}}(L,X') \arrow[r, "(u^*,p'_*)"] & \widehat{\mathscr{A}}(K,X') \times_{\widehat{\mathscr{A}}(K,Y')} \widehat{\mathscr{A}}(L,Y'). \end{tikzcd}$$

Now, let I be a small filtered category, and consider any functor

$$p_\bullet : I \to \widehat{\mathscr{A}}^{[1]} \qquad i \mapsto (X_i \xrightarrow{p_i} Y_i) \qquad \text{with } p_i \in r(\mathscr{F})\ \forall i \in I.$$

Let $X \xrightarrow{p} Y$ be the colimit of $p_\bullet$ (recall that $\widehat{\mathscr{A}}^{[1]}$ is cocomplete, since the same holds for $\widehat{\mathscr{A}}$: see §1.3.2). Since the colimits of $\mathsf{Set}^{[1]}$ are computed termwise, and since finite limits in $\mathsf{Set}^{[1]}$ commute with all small filtered colimits (example 1.3.7(ii)), the colimit of $F_u \circ p_\bullet : I \to \mathsf{Set}^{[1]}$ is naturally identified with the induced map :

$$(*) \qquad\qquad l_{L,X_\bullet} \to l_{K,X_\bullet} \times_{l_{K,Y_\bullet}} l_{L,Y_\bullet}$$

where :

$$l_{Z,X_\bullet} := \varinjlim_{i\in I} \widehat{\mathscr{A}}(Z,X_i) \qquad l_{Z,Y_\bullet} := \varinjlim_{i\in I} \widehat{\mathscr{A}}(Z,Y_i) \qquad \forall Z \in \mathrm{Ob}(\widehat{\mathscr{A}}).$$

On the other hand, by corollary 2.2.11, we have as well natural identifications :

$$l_{Z,X_\bullet} \xrightarrow{\sim} \widehat{\mathscr{A}}(Z,X) \qquad l_{Z,Y_\bullet} \xrightarrow{\sim} \widehat{\mathscr{A}}(Z,Y) \qquad \forall Z \in \{K,L\}.$$

Under these identifications, the map $(*)$ then corresponds to (u^*, p_*); however, the colimit of any system of surjections of sets is a surjection ([13, Exerc.2.34(ii)]), so finally (u^*, p_*) is a surjection, whence the contention.

Example 3.1.15. Let R be a ring, and denote by $\mathsf{C}(R)$ the category of (unbounded) chain complexes of R-modules (see [13, §2.5]). For every $n \in \mathbb{Z}$, let $(D^n_\bullet, d^{D^n}_\bullet)$ be the chain complex with $D^n_k := R$ for $k = n-1, n$ and $D_k := 0$ for every $k \in \mathbb{Z} \setminus \{n-1, n\}$; the differential $d^{D^n}_n : D_n \to D_{n-1}$ is the identity of R. Let also $0_\bullet$ be the zero chain complex, and $R[n]_\bullet$ the chain complex concentrated in degree n, with $R[n]_0 := R$; for every $n \in \mathbb{Z}$ we have two obvious morphisms of $\mathsf{C}(R)$:

$$0_\bullet \xrightarrow{\phi^n_\bullet} D^n_\bullet \qquad R[n-1]_\bullet \xrightarrow{\psi^n_\bullet} D^n_\bullet \quad \text{with} \quad \psi^n_{n-1} := \mathbf{1}_R : R[n-1]_{n-1} \to D^n_{n-1}.$$

Let moreover $\mathscr{E}$ be the class of epimorphisms of $\mathsf{C}(R)$, and denote by $\mathscr{W}$ the class of *quasi-isomorphisms* of $\mathsf{C}(R)$, *i.e.* of morphisms $f_\bullet : X_\bullet \to Y_\bullet$ inducing an isomorphism $H_n(f_\bullet) : H_n(X_\bullet) \xrightarrow{\sim} H_n(Y_\bullet)$ for every $n \in \mathbb{Z}$ (see [13, §2.5.1]). We claim that :

$$\boxed{\mathscr{E} = r(\{\phi^n_\bullet \mid n \in \mathbb{Z}\}) \qquad \text{and} \qquad \mathscr{E} \cap \mathscr{W} = r(\{\psi^n_\bullet \mid n \in \mathbb{Z}\}).}$$

Indeed, by [13, Exerc.2.98(ii)], $\mathsf{C}(R)$ is cocomplete and its colimits are computed degree-wise; hence, a morphism $p_\bullet : (X_\bullet, d^X_\bullet) \to (Y_\bullet, d^Y_\bullet)$ of $\mathsf{C}(R)$ is an epimorphism $\Leftrightarrow$ the codiagonal morphism $\nabla_{Y_\bullet/X_\bullet} : Y_\bullet \sqcup_{X_\bullet} Y_\bullet \to Y_\bullet$ is an isomorphism in $\mathsf{C}(R)$ ([13, Exerc.2.66(iii)]), and the latter holds if and only if the codiagonal morphism $\nabla_{Y_n/X_n} : Y_n \sqcup_{X_n} Y_n \to Y_n$ is an isomorphism for every $n \in \mathbb{Z}$, *i.e.* if and only if $p_n : Y_n \to X_n$ is a surjection for every $n \in \mathbb{Z}$ ([13, Exerc.1.119(vi)]). Now, consider commutative diagrams of $\mathsf{C}(R)$:

$$(\mathrm{D}) \quad : \quad \begin{tikzcd} 0_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d, "p_\bullet"] \\ D^n_\bullet \arrow[r, "f_\bullet"] & Y_\bullet \end{tikzcd} \qquad\qquad (\mathrm{D}') \quad : \quad \begin{tikzcd} R[n-1]_\bullet \arrow[r, "g_\bullet"] \arrow[d, "\psi^n_\bullet"'] & X_\bullet \arrow[d, "p_\bullet"] \\ D^n_\bullet \arrow[r, "f_\bullet"] & Y_\bullet. \end{tikzcd}$$

It is easily seen that we have natural bijections :

$$\mathsf{C}(R)(D^n_\bullet, X_\bullet) \xrightarrow{\sim} X_n \qquad \mathsf{C}(R)(D^n_\bullet, Y_\bullet) \xrightarrow{\sim} Y_n \qquad f_\bullet \mapsto f_n(1)$$

so, for a fixed morphism $p_\bullet : X_\bullet \to Y_\bullet$, every diagram (D) admits a diagonal filler if and only if p_n is surjective, whence the first stated identity.

• Next, it is easily seen that, for a fixed $p_\bullet$, the diagrams (D') are in bijection with the set $S_n := \{(y,z) \in Y_n \oplus Z_{n-1}X_\bullet \mid d^Y_n(y) = p_{n-1}(z)\}$, where $Z_{n-1}X_\bullet \subset X_{n-1}$ denotes the submodule of cycles of $X_\bullet$ in degree $n-1$ (see [13, §2.5.1]), and if (D') corresponds to the pair (y,z), then the diagonal fillers for (D') correspond to the elements $x \in X_n$ such that $p_n(x) = y$ and $d^X_n(x) = z$. Now, *suppose that* $p_\bullet \in r(\{\psi^n_\bullet \mid n \in \mathbb{Z}\})$, and let $y \in Z_nY_\bullet$; then $(y,0) \in S_n$, so there exists $x \in X_n$ with $p_n(x) = y$ and $d^X_n(x) = 0$, *i.e.* $x \in Z_nX_\bullet$, and this shows that $H_n(p_\bullet)$ *is surjective.* Next, for every $y \in Y_n$ we have $d^Y_n(y) \in Z_{n-1}Y_\bullet$, so by the foregoing there exists $z \in Z_{n-1}X_\bullet$ with $p_{n-1}(z) = d^Y_n(y)$, and then $(y,z) \in S_n$, so again, there exists $x \in X_n$ with $p_n(x) = y$, and this show that p_n *is surjective.* Lastly, let $z \in Z_nX$ such that $p_n(z) = d^Y_{n+1}(y)$ for some $y \in Y_{n+1}$; then $(y,z) \in S_{n+1}$, so there exists $x \in X_{n+1}$ with $d^X_{n+1}(x) = z$, *i.e.* $H_n(p_\bullet)$ *is injective,* and summing up, we see that $p_\bullet \in \mathscr{W} \cap \mathscr{E}$.

• Conversely, *suppose that* $p_\bullet \in \mathscr{W} \cap \mathscr{E}$, and let $(y,z) \in S_n$; since $p_\bullet \in \mathscr{E}$, we have a short exact sequence of $\mathsf{C}(R)$:

$$0 \to K_\bullet \to X_\bullet \xrightarrow{p_\bullet} Y_\bullet \to 0$$

and since $p_\bullet \in \mathscr{W}$, by considering the associated long exact homology sequence we deduce that $H_n(K_\bullet) = 0$ for every $n \in \mathbb{Z}$ (see [13, §2.5.2]). Pick $w \in X_n$ with $p_n(w) = y$; then $d^X_n(w) - z \in Z_{n-1}(K_\bullet)$, since $p_{n-1}(d^X_n(w)) = d^Y_n(y) = p_{n-1}(z)$ and $d^X_{n-1}(d^X_n(w) - z) = d^X_{n-1} \circ d^X_n(w) = 0$. Thus, there exists $v \in K_n$ with $d^X_n(v) = d^X_n(w) - z$; with $x := w - v$ we get $p_n(x) = y$ and $d^X_n(x) = z$. This shows that $p_\bullet \in r(\{\psi^n_\bullet \mid n \in \mathbb{Z}\})$, as stated.

Definition 3.1.16. Let $\mathscr{C}$ be a category. A *weak factorization system* for $\mathscr{C}$ is a pair $(\mathscr{I}, \mathscr{P})$ of classes of morphisms of $\mathscr{C}$, such that :

(a) $\mathscr{I}$ and $\mathscr{P}$ are stable under retracts

(b) $\mathscr{I} \subset l(\mathscr{P})$, or equivalently, $\mathscr{P} \subset r(\mathscr{I})$ (proposition 3.1.9(i))

(c) Any morphism of $\mathscr{C}$ is of the form $p \circ i$, for some $i \in \mathscr{I}$ and $p \in \mathscr{P}$.

Example 3.1.17. Let $\mathscr{M}$ (resp. $\mathscr{P}$) be the class of injective (resp. surjective) maps in Set, and let $i : \varnothing \to [0]$ be the unique map. From example 3.1.14(i) we know that $\mathscr{M} = l(\mathscr{P})$ and $\mathscr{P} = r(i)$, so both $\mathscr{M}$ and $\mathscr{P}$ are stable under retracts (proposition 3.1.9(v)), and moreover every map of sets $f : X \to Y$ admits a factorization $f = p \circ i$, where $i : X \to \Gamma_f$ is the natural inclusion of X in the graph of f, and $p : \Gamma_f \to X$ is the natural projection. Hence, $(\mathscr{M}, \mathscr{P})$ is a weak factorization system for Set.

Lemma 3.1.18. *Let $(\mathscr{I}, \mathscr{P})$ be a weak factorization system for $\mathscr{C}$. We have :*

(i) $\mathscr{I} = l(\mathscr{P})$ *and* $\mathscr{P} = r(\mathscr{I})$.

(ii) $(\mathscr{P}^{\mathrm{op}}, \mathscr{I}^{\mathrm{op}})$ *is a weak factorization system for* $\mathscr{C}^{\mathrm{op}}$.

(iii) $(\mathscr{I}/X, \mathscr{P}/X)$ *is a weak factorization system for* $\mathscr{C}/X$*, for every* $X \in \mathrm{Ob}(\mathscr{C})$.

(iv) $(X/\mathscr{I}, X/\mathscr{P})$ *is a weak factorization system for* $X/\mathscr{C}$*, for every* $X \in \mathrm{Ob}(\mathscr{C})$.

Proof. (i): Let $f \in l(\mathscr{P})$ (resp. $f \in r(\mathscr{I})$), and write $f = p \circ i$ with $i \in \mathscr{I}$, $p \in \mathscr{P}$; by proposition 3.1.10, f is a retract of i (resp. of p), whence $f \in \mathscr{I}$ (resp. $f \in \mathscr{P}$). This shows that $l(\mathscr{P}) \subset \mathscr{I}$ and $r(\mathscr{I}) \subset \mathscr{P}$, whence (i).

(ii) follows directly from (i), remark 3.1.4(i) and proposition 3.1.9(iv).

(iii): Indeed, we have $\mathscr{P}/X \subset r(\mathscr{I})/X = r(\mathscr{I}/X)$ by lemma 3.1.12(i), and if $f, g \in \mathrm{Mor}(\mathscr{C}/X)$ and f is a retract of g, then $\mathrm{s}_{/X}(f)$ is a retract of $\mathrm{s}_{/X}(g)$.

(iv) follows from (ii) and (iii), by duality. □

Proposition 3.1.19. *Let $F : \mathscr{C} \rightleftarrows \mathscr{C}' : G$ be an adjoint pair of functors. Let also $(\mathscr{I}, \mathscr{P})$, $(\mathscr{I}', \mathscr{P}')$ be weak factorization systems for $\mathscr{C}$ and respectively $\mathscr{C}'$; then :*

$$F(\mathscr{I}) \subset \mathscr{I}' \Leftrightarrow G(\mathscr{P}') \subset \mathscr{P}.$$

Proof. Any adjunction $\vartheta_{\bullet\bullet}$ for (F, G) yields bijections between diagrams of the type :

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "\vartheta(f)"] \arrow[d, "i"'] & GX \arrow[d, "Gp"] \\ B \arrow[r, "\vartheta(g)"'] \arrow[ur, dashed, "\vartheta(h)"] & GY \end{tikzcd}$$

whence the assertion, in light of lemma 3.1.18(i). □

3.1.20. The following more special result, in the same vein as proposition 3.1.19, will be useful in §4.4, §5.1 and §6.1. Consider two categories $\mathscr{C}, \mathscr{C}'$, two adjoint pairs of functors

$$(F_i : \mathscr{C} \rightleftarrows \mathscr{C}' : G_i) \qquad i = 1, 2$$

with corresponding adjunctions $(\vartheta^i_{\bullet\bullet} \mid i = 1, 2)$, and a natural transformation

$$\tau_\bullet : F_1 \Rightarrow F_2.$$

Recall that the adjoint transformation $\tau^\vee_\bullet : G_2 \Rightarrow G_1$ is independent, up to natural isomorphism, of the choice of adjunctions (see §1.6.10); suppose moreover that all the fibre products of $\mathscr{C}$ and all the amalgamated sums of $\mathscr{C}'$ are representable. Then, for every morphism $f : X \to Y$ of $\mathscr{C}$, and $g : X' \to Y'$ of $\mathscr{C}'$, the commutative diagrams :

$$\begin{tikzcd} F_1X \arrow[r, "F_1f"] \arrow[d, "\tau_X"'] & F_1Y \arrow[d, "\tau_Y"] \\ F_2X \arrow[r, "F_2f"] & F_2Y \end{tikzcd} \qquad \begin{tikzcd} G_2X' \arrow[r, "G_2g"] \arrow[d, "\tau^\vee_{X'}"'] & G_2Y' \arrow[d, "\tau^\vee_{Y'}"] \\ G_1X' \arrow[r, "G_1g"] & G_1Y' \end{tikzcd}$$

induce morphisms of $\mathscr{C}'$ and respectively $\mathscr{C}$:

$$f^\diamond : F_1Y \sqcup_{F_1X} F_2X \to F_2Y \qquad g_\diamond : G_2X' \to G_1X' \times_{G_1Y'} G_2Y'.$$

For every subclass $\mathscr{S} \subset \mathrm{Mor}(\mathscr{C})$, let us also set $\mathscr{S}^\diamond := \{f^\diamond \mid f \in \mathscr{S}\}$.

Proposition 3.1.21. *(i) For every morphism $f : X \to Y$ of $\mathscr{C}$ and $g : X' \to Y'$ of $\mathscr{C}'$ we have a natural bijection between commutative diagrams of the form :*

$$(\mathsf{D}^\diamond)\quad \begin{tikzcd} F_1Y \sqcup_{F_1X} F_2X \arrow[r, "a"] \arrow[d, "f^\diamond"'] & X' \arrow[d, "g"] \\ F_2Y \arrow[r, "b"'] & Y' \end{tikzcd} \quad \textit{and} \quad (\mathsf{D}_\diamond)\quad \begin{tikzcd} X \arrow[r, "a'"] \arrow[d, "f"'] & G_2X' \arrow[d, "g_\diamond"] \\ Y \arrow[r, "b'"'] & G_1X' \times_{G_1Y'} G_2Y'. \end{tikzcd}$$

(ii) Moreover, the diagonal fillers $F_2Y \to X'$ for $(\mathsf{D}^\diamond)$ are in natural bijection with the diagonal fillers $Y \to G_2X'$ for $(\mathsf{D}_\diamond)$. Therefore :

$$\boxed{f^\diamond \in l(g) \Leftrightarrow f \in l(g_\diamond) \Leftrightarrow g \in r(f^\diamond) \Leftrightarrow g_\diamond \in r(f).}$$

(iii) With the notation of §3.1.20, for any subclass $\mathscr{S} \subset \mathrm{Mor}(\mathscr{C})$ we have:

$$\boxed{\mathscr{S}^\diamond \subset l(r(\mathscr{S}))^\diamond \subset l(r(\mathscr{S}^\diamond)).}$$

Proof. (i): Indeed, let us fix a universal co-cone and a universal cone :

$$F_1Y \xrightarrow{e'} F_1Y \sqcup_{F_1X} F_2X \xleftarrow{e} F_2X \qquad G_1X' \xleftarrow{p'} G_1X' \times_{G_1Y'} G_2Y' \xrightarrow{p} G_2Y'.$$

Then, to every given diagram $(\mathsf{D}^\diamond)$ we attach the diagram $(\mathsf{D}_\diamond)$ where :

- a' is the adjoint of $a \circ e : F_2X \to X'$ (relative to the adjunction $\vartheta^2_{\bullet\bullet}$)
- b' is the unique morphism such that $p \circ b' : Y \to G_2Y'$ is the adjoint of b (relative to $\vartheta^2_{\bullet\bullet}$) and $p' \circ b' : Y \to G_1X'$ is the adjoint of $a \circ e' : F_1Y \to X'$ (relative to $\vartheta^1_{\bullet\bullet}$).

Indeed, let us check the commutativity of $(\mathsf{D}_\diamond)$: we need to show that

$$p \circ g_\diamond \circ a' = p \circ b' \circ f \qquad \text{and} \qquad p' \circ g_\diamond \circ a' = p' \circ b' \circ f.$$

However, since $e \circ \tau_X = e' \circ F_1 f$, we may compute :

$$p' g_\diamond a' = \tau^\vee_{X'} \circ \vartheta^2_{X,X'}(ae) = \vartheta^1_{X,X'}(ae \circ \tau_X) = \vartheta^1_{X,X'}(ae' \circ F_1 f) = \vartheta^1_{Y,X'}(ae') \circ f = p'b'f$$

and on the other hand :

$$pg_\diamond a' = G_2 g \circ \vartheta^2_{X,X'}(ae) = \vartheta^2_{X,Y'}(gae) = \vartheta^2_{XY'}(bf^\diamond e) = \vartheta^2_{XY'}(b \circ F_2 f) = \vartheta^2_{Y,Y'}(b) \circ f = pb'f.$$

Conversely, to every diagram $(\mathsf{D}_\diamond)$ we attach the diagram $(\mathsf{D}^\diamond)$ where :

- a is the unique morphism such that $a \circ e$ is the adjoint of a' (relative to $\vartheta^2_{\bullet\bullet}$), and $a \circ e'$ is the adjoint of $p' \circ b'$ (relative to $\vartheta^1_{\bullet\bullet}$)
- b is the adjoint of $p \circ b'$ (relative to $\vartheta^2_{\bullet\bullet}$).

(ii): To every diagonal filler $h : F_2Y \to X'$ for $(\mathsf{D}^\diamond)$ there corresponds the diagonal filler $h' : Y \to G_2X'$ for $(\mathsf{D}_\diamond)$ given by the adjoint of h, relative to $\vartheta^2_{\bullet\bullet}$: we leave the verifications to the reader.

(iii): The first inclusion is obvious, since $\mathscr{S} \subset l(r(\mathscr{S}))$. Next, it follows from (ii) that :

$$(*) \qquad g \in r(\mathscr{S}^\diamond) \Leftrightarrow g_\diamond \in r(\mathscr{S}).$$

Applying $(*)$ with $\mathscr{S}$ replaced by $l(r(\mathscr{S}))$, we then get :

$$(**) \qquad g \in r(l(r(\mathscr{S}))^\diamond) \Leftrightarrow g_\diamond \in r(l(r(\mathscr{S}))).$$

As $r(\mathscr{S}) = r(l(r(\mathscr{S})))$ (proposition 3.1.9(iii)), we combine $(*)$ and $(**)$ to conclude :

$$r(\mathscr{S}^\diamond) = r(l(r(\mathscr{S}))^\diamond)$$

whence $l(r(\mathscr{S}))^\diamond \subset l(r(l(r(\mathscr{S}))^\diamond)) = l(r(\mathscr{S}^\diamond))$, as stated. □

3.2. **Model categories.**

Definition 3.2.1. (i) A *model category* is the datum $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ of a category $\mathscr{C}$ and three classes $\mathscr{W}, \mathscr{F}ib, \mathscr{C}of$ of morphisms of $\mathscr{C}$, such that :

(a) $\mathscr{C}$ is finitely complete and finitely cocomplete.

(b) $\mathscr{W}$ has the *2-out-of-3 property*, *i.e.* for any composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms of $\mathscr{C}$, if two among the morphisms $f, g, g \circ f$ lie in $\mathscr{W}$, then the same holds for the third one.

(c) $(\mathscr{C}of, \mathscr{F}ib \cap \mathscr{W})$ and $(\mathscr{C}of \cap \mathscr{W}, \mathscr{F}ib)$ are weak factorization systems for $\mathscr{C}$.

We call the elements of $\mathscr{W}$ (resp. of $\mathscr{F}ib$, resp. of $\mathscr{C}of$) *weak equivalences* (resp. *fibrations*, resp. *cofibrations*). A morphism that is both a weak fibration and a fibration (resp. and a cofibration) is called a *trivial fibration* (resp. a *trivial cofibration*).

(ii) Let $\varnothing$ (resp. e) be an initial (resp. final) object of $\mathscr{C}$; an object X of $\mathscr{C}$ is *fibrant* (resp. *cofibrant*) if the unique morphism $X \to e$ (resp. $\varnothing \to X$) is a fibration (resp. a cofibration).

(iii) We denote by $\mathscr{C}_c$ (resp. $\mathscr{C}_f$) the full subcategory of $\mathscr{C}$ whose objects are the cofibrant (resp. fibrant) objects of $\mathscr{C}$, and we set as well :

$$\mathscr{W}_c := \mathscr{W} \cap \mathrm{Mor}(\mathscr{C}_c) \qquad \text{and} \qquad \mathscr{W}_f := \mathscr{W} \cap \mathrm{Mor}(\mathscr{C}_f).$$

Remark 3.2.2. (i) In the situation of definition 3.2.1, notice that every isomorphism of $\mathscr{C}$ is both a trivial fibration and a trivial cofibration (proposition 3.1.9(vi)). In particular, the initial object is cofibrant, and the final object is fibrant.

(ii) By lemma 3.1.18(i), the fibrant (resp. the cofibrant) objects of $\mathscr{C}$ are precisely the $(\mathscr{C}of \cap \mathscr{W})$-injective objects (resp. the $(\mathscr{F}ib \cap \mathscr{W})$-projective objects).

Example 3.2.3. (i) Every finitely complete and finitely cocomplete category $\mathscr{C}$ admits a model category structure, for which the weak equivalences are the isomorphisms of $\mathscr{C}$, and with $\mathscr{F}ib = \mathscr{C}of = \mathrm{Mor}(\mathscr{C})$.

(ii) If $((\mathscr{C}_\lambda, \mathscr{W}_\lambda, \mathscr{F}ib_\lambda, \mathscr{C}of_\lambda) \mid \lambda \in \Lambda)$ is any small family of small model categories, then $(\prod_{\lambda\in\Lambda} \mathscr{C}_\lambda, \prod_{\lambda\in\Lambda} \mathscr{W}_\lambda, \prod_{\lambda\in\Lambda} \mathscr{F}ib_\lambda, \prod_{\lambda\in\Lambda} \mathscr{C}of_\lambda)$ is a model category (see remark 1.2.4(iv,v) for our general conventions about families of categories).

Proposition 3.2.4. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $X \in \mathrm{Ob}(\mathscr{C})$.*

(i) $(\mathscr{C}^{\mathrm{op}}, \mathscr{W}^{\mathrm{op}}, \mathscr{C}of^{\mathrm{op}}, \mathscr{F}ib^{\mathrm{op}})$ is a model category.

(ii) $(\mathscr{C}/X, \mathscr{W}/X, \mathscr{F}ib/X, \mathscr{C}of/X)$ is a model category (notation of §3.1.11).

(iii) $(X/\mathscr{C}, X/\mathscr{W}, X/\mathscr{F}ib, X/\mathscr{C}of)$ is a model category.

Proof. (i) follows from lemma 3.1.18(ii). For (ii) we invoke lemma 3.1.18(iii), and remark that $\mathscr{C}/X$ is finitely complete and finitely cocomplete, by lemma 1.4.7(i) and corollary 1.4.6(iii). Assertion (iii) follows from (i) and (ii), by duality. □

Remark 3.2.5. With the notation of proposition 3.2.4, notice that the fibrant objects of $(X/\mathscr{C}, X/\mathscr{W}, X/\mathscr{F}ib, X/\mathscr{C}of)$ are all the morphisms $X \to Y$ of $\mathscr{C}$ with $Y \in \mathrm{Ob}(\mathscr{C}_f)$, whereas the cofibrant objects are all the cofibrations $X \to Y$ of $\mathscr{C}$ with source X. Dually, the cofibrant objects of $(\mathscr{C}/X, \mathscr{W}/X, \mathscr{F}ib/X, \mathscr{C}of/X)$ are all the morphisms $Y \to X$ with $Y \in \mathrm{Ob}(\mathscr{C}_c)$, whereas the fibrant objects are all the fibrations $Y \to X$ of $\mathscr{C}$ with target X.

Proposition 3.2.6. *For every model category $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$, the classes $\mathscr{W}$, $\mathscr{F}ib$ and $\mathscr{C}of$ are stable under retracts.*

Proof. (This argument is due to André Joyal and Myles Tierney.[1], and it seems to originate in [9, Prop.A.3.1]) This is already known for $\mathscr{F}ib$ and $\mathscr{C}of$, by lemma 3.1.18(i) and proposition 3.1.9(v). Next, let $f : A \to B$ be a retract of a weak equivalence $g : X \to Y$, so that we have a commutative diagram :

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathbf{1}_A"] & X \arrow[r, "t"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "u"] \arrow[rr, bend right, "\mathbf{1}_B"'] & Y \arrow[r, "v"] & B. \end{tikzcd}$$

• Suppose first that $f \in \mathscr{F}ib$, and factor g as a composition $g : X \xrightarrow{j} Z \xrightarrow{q} Y$ with $j \in \mathscr{W} \cap \mathscr{C}of$ and $q \in \mathscr{F}if$. Then $q \in \mathscr{W}$, by the 2-out-of-3 property of $\mathscr{W}$; moreover, the commutative diagram :

$$\begin{tikzcd} X \arrow[rr, "t"] \arrow[d, "j"'] & & A \arrow[d, "f"] \\ Z \arrow[r, "q"] & Y \arrow[r, "v"] & B \end{tikzcd}$$

admits a diagonal filler $d : Z \to A$. We deduce a commutative diagram :

$$\begin{tikzcd} A \arrow[r, "js"] \arrow[d, "f"'] \arrow[rr, bend left, "\mathbf{1}_A"] & Z \arrow[r, "d"] \arrow[d, "q"] & A \arrow[d, "f"] \\ B \arrow[r, "u"] \arrow[rr, bend right, "\mathbf{1}_B"'] & Y \arrow[r, "v"] & B \end{tikzcd}$$

which exhibits f as a retract of $q \in \mathscr{F}ib \cap \mathscr{W}$; but notice that $\mathscr{F}ib \cap \mathscr{W} = l(\mathscr{F}ib)$ is stable under retracts, by proposition 3.1.9(v), so $f \in \mathscr{W}$, as stated.

• Next, let f be an arbitrary retract of g, and factor f as a composition $f : A \xrightarrow{i} E \xrightarrow{p} B$ with $i \in \mathscr{W} \cap \mathscr{C}of$ and $p \in \mathscr{F}ib$. We get a commutative diagram :

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "i"'] & X \arrow[r, "t"] \arrow[d, "e_2"] & A \arrow[d, "i"] \\ E \arrow[r, "e_1"] \arrow[d, "p"'] & E \sqcup_A X \arrow[r, "r"] \arrow[d, "k"] & E \arrow[d, "p"] \\ B \arrow[r, "u"] & Y \arrow[r, "v"] & B \end{tikzcd}$$

where (e_1, e_2) is the universal co-cone for the push-out $E \sqcup_A X$, and r (resp. k) is the unique morphism $E \sqcup_A X \to E$ such that $re_2 = it$ and $re_1 = \mathbf{1}_E$ (resp. the unique morphism $E \sqcup_A X \to Y$ such that $ke_1 = up$ and $ke_2 = g$). We then have $e_2 \in \mathscr{W} \cap \mathscr{C}of$, since $i \in \mathscr{W} \cap \mathscr{C}of$ (proposition 3.1.9(v)); since $g \in \mathscr{W}$, it follows that $k \in \mathscr{W}$, by the 2-out-of-3 property of $\mathscr{W}$. Lastly, by the foregoing case, we conclude that $p \in \mathscr{W}$, and finally $f \in \mathscr{W}$ as well. □

[1] I have borrowed this proof from `https://ncatlab.org/nlab/show/model+category`

Lemma 3.2.7. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category; we have :*

(i) Every morphism $f : X \to Y$ between cofibrant objects of $\mathscr{C}$ admits a factorization $f = p \circ g$, where g is a cofibration and p is a left inverse of a trivial cofibration.

(ii) Dually, every morphism $f : X \to Y$ between fibrant objects of $\mathscr{C}$ admits a factorization $f = h \circ g$, where h is a fibration and q is a right inverse of a trivial fibration.

Proof. (i): We consider the cocartesian diagram :

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & Y \arrow[d, "j"] \\ X \arrow[r, "i"] & X \sqcup Y \end{tikzcd}$$

Since $\mathscr{C}of$ is saturated (lemma 3.1.18(i) and proposition 3.1.9(v)), the morphisms i and j are cofibrations, so $X \sqcup Y \in \mathrm{Ob}(\mathscr{C}_c)$ (any transfinite composition of cofibrations is a cofibration, by proposition 3.1.9(v), and likewise for transfinite compositions of fibrations). The pair $(f, \mathbf{1}_Y)$ determines a unique morphism $X \sqcup Y \to Y$, that we may factor as a cofibration $k : X \sqcup Y \to T$ followed by a trivial fibration $p : T \to Y$. We thus have the commutative diagrams :

$$\begin{tikzcd} X \arrow[rr, "ki"] \arrow[dr, "f"'] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "kj"] \arrow[dr, "\mathbf{1}_Y"'] & & T \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

and notice that $T \in \mathrm{Ob}(\mathscr{C}_c)$. Since both $\mathbf{1}_Y$ and p are weak equivalences, the same holds for kj, so the latter is a trivial cofibration, and we may take $g := ki$.

(ii) follows by applying (i) to the opposite model category $(\mathscr{C}^{\mathrm{op}}, \mathscr{W}^{\mathrm{op}}, \mathscr{C}of^{\mathrm{op}}, \mathscr{F}ib^{\mathrm{op}})$ (proposition 3.2.4(i)). □

Proposition 3.2.8. (Ken Brown's lemma) *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $F : \mathscr{C}_c \to \mathscr{C}'$ a functor, $\mathscr{W}' \subset \mathrm{Mor}(\mathscr{C}')$ a class containing the isomorphisms of $\mathscr{C}'$ and enjoying the 2-out-of-3 property. Suppose that F maps every trivial cofibration between cofibrant objects to $\mathscr{W}'$; then F also maps every weak equivalence between cofibrant objects to $\mathscr{W}'$.*

Proof. Indeed, let $f : X \to Y$ be a weak equivalence between cofibrant objects of $\mathscr{C}$. By lemma 3.2.7(i), we have $f = p \circ g$ where $g : X \to T$ is a cofibration, and $p : T \to Y$ is a left inverse of a trivial cofibration $h : Y \to T$; then also T is a cofibrant object, so by assumption, $F(\mathbf{1}_Y), F(h) \in \mathscr{W}'$, whence $F(p) \in \mathscr{W}'$ as well. Moreover, since f and p are weak equivalences, the same holds for g, so the latter is another trivial cofibration between cofibrant objects, and therefore $F(p), F(g) \in \mathscr{W}'$, whence $F(f) \in \mathscr{W}'$. □

Definition 3.2.9. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $A, X \in \mathrm{Ob}(\mathscr{C})$.

(i) A *cylinder* of A is a factorization of the *codiagonal* ∇_A (see [13, Exerc.2.66(iii)]) as a cofibration (∂_0, ∂_1) followed by a weak equivalence σ :

$$\boxed{\begin{tikzcd} A \sqcup A \arrow[r, "(\partial_0, \partial_1)"] \arrow[rr, bend left, "\nabla_A"] & IA \arrow[r, "\sigma"] & A. \end{tikzcd}}$$

If $(A \xrightarrow{e_i} A \sqcup A \mid i = 0, 1)$ is the universal co-cone, then $\partial_i := (\partial_0, \partial_1) \circ e_i$ for $i = 0, 1$.

(ii) Dually, a *cocylinder* (also called a *path object*) of X is a cylinder of X in the opposite model category $\mathscr{C}^{\mathrm{op}}$, *i.e.* a factorization of the *diagonal* Δ_X as a weak equivalence s followed by a fibration (d_0, d_1) :

$$\boxed{\begin{tikzcd} X \arrow[r, "s"] \arrow[rr, bend left, "\Delta_X"] & X^I \arrow[r, "(d_0,d_1)"] & X \times X. \end{tikzcd}}$$

If $(X \times X \xrightarrow{p_i} X \mid i = 0, 1)$ is the universal cone, then $d_i := p_i \circ (d_0, d_1)$ for $i = 0, 1$.

(iii) Let $f_0, f_1 : A \rightrightarrows X$ be two morphisms of $\mathscr{C}$; a *left homotopy from f_0 to f_1* is the datum of a cylinder for A as in (i), and a morphism $h : IA \to X$ of $\mathscr{C}$ such that $h \circ \partial_i = f_i$ for $i = 0, 1$. Dually, a *right homotopy from f_0 to f_1* is a left homotopy from f_0^{op} to f_1^{op} in $\mathscr{C}^{\mathrm{op}}$, *i.e.* the datum of a cocylinder for X as in (ii), and a morphism $k : A \to X^I$ of $\mathscr{C}$ such that $d_i \circ k = f_i$ for $i = 0, 1$.

(iv) With the notation of (iii), we shall write :

$$\boxed{f_0 \overset{l}{\sim} f_1 \qquad (\text{resp. } f_0 \overset{r}{\sim} f_1)}$$

if there exists a left (resp. right) homotopy from f_0 to f_1.

Remark 3.2.10. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $A, X \in \mathrm{Ob}(\mathscr{C})$; consider a cylinder $A \sqcup A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A$ of A and a cocylinder $X \xrightarrow{s} X^I \xrightarrow{(d_0,d_1)} X \times X$ of X.

(i) For every $f \in \mathscr{C}(A, X)$, the morphism $f\sigma : IA \to X$ (resp. $sf : A \to X^I$) is a left (resp. right) homotopy from f to f. We say that $f\sigma$ (resp. sf) *is a constant left* (resp. *right*) *homotopy*.

(ii) Let $\tau : A \sqcup A \xrightarrow{\sim} A \sqcup A$ be the isomorphism that swaps the two copies of A; let also $f_0, f_1 \in \mathscr{C}(A, X)$ and $h : IA \to X$ a homotopy from f_0 to f_1. Then $(\partial_1, \partial_0) := (\partial_0, \partial_1) \circ \tau : A \sqcup A \to IA$ is still a cofibration with $\sigma \circ (\partial_1, \partial_0) = \nabla_A$, so that h is also a left homotopy from f_1 to f_0, relative to the cylinder diagram

$$A \sqcup A \xrightarrow{(\partial_1,\partial_0)} IA \xrightarrow{\sigma} A.$$

We denote this homotopy by $h^\vee$ and we call it *the inverse homotopy* of h.

(iii) *If A is cofibrant*, the morphisms $(\partial_i : A \to IA)$ are trivial cofibrations. Indeed, by virtue of the 2-out-of-3 property, the morphisms ∂_i are weak equivalences. Moreover, the components of the universal co-cone $(e_i : A \to A \sqcup A \mid i = 0, 1)$ are cofibrations, since they are obtained by push-out from the cofibration $\varnothing \to A$. But then, each $\partial_i = (\partial_0, \partial_1) \circ e_i$ is a cofibration as well, whence the assertion.

(iv) *Suppose that A is cofibrant*; consider $u, v, w \in \mathscr{C}(A, X)$, as well as cylinders:

$$A \sqcup A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A \qquad A \sqcup A \xrightarrow{(\partial'_0,\partial'_1)} I'A \xrightarrow{\sigma'} A$$

and left homotopies $h : IA \to X$, $h' : I'A \to X$ with $h\partial_0 = u$, $h\partial_1 = v = h'\partial'_0$ and $h'\partial'_1 = w$. We form the cocartesian square :

$$(*) \qquad \begin{tikzcd} A \arrow[r, "\partial_1"] \arrow[d, "\partial'_0"'] & IA \arrow[d, "e"] \\ I'A \arrow[r, "e'"] & I''A := IA \sqcup_A I'A. \end{tikzcd}$$

There exists a unique morphism $\sigma'' : I''A \to A$ such that $\sigma'' e = \sigma$ and $\sigma'' e' = \sigma'$, and we claim that we get a cylinder :

$$A \sqcup A \xrightarrow{(\partial_0'', \partial_1'')} I''A \xrightarrow{\sigma''} A \qquad \text{with } \partial_0'' := e\partial_0 \text{ and } \partial_1'' := e'\partial_1'.$$

Indeed, it is easily seen that $\sigma'' \circ (\partial_0'', \partial_1'') = \nabla_A$. Next, by (iii), ∂_0' is a trivial cofibration, so the same holds for e; since moreover σ is a weak equivalence, it follows that σ'' is a weak equivalence as well. It remains to check that $(\partial_0'', \partial_1'')$ is a cofibration. To this aim, we decompose $(*)$ as two adjacent commutative squares :

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[d, "\partial_0'"'] & A \sqcup A \arrow[r, "(\partial_0, \partial_1)"] \arrow[d, "\mathbf{1}_A \sqcup \partial_0'"'] & IA \arrow[d, "e"] \\ I'A \arrow[r] & A \sqcup I'A \arrow[r, "(\partial_0'', e')"] & I''A. \end{tikzcd}$$

Since the left square is obviously cocartesian, and since the same holds for $(*)$, it follows formally that the right square is cocartesian as well; thus, (∂_0'', e') and $\mathbf{1}_A \sqcup \partial_1'$ are cofibrations, whence the same for their composition, which is $(\partial_0'', \partial_1'')$.

Lastly, we let $h'' : I''A \to X$ be the unique morphism such that $h''e = h$ and $h''e' = h'$; it follows easily that $h''\partial_0'' = u$ and $h''\partial_1'' = w$, whence $u \overset{l}{\sim} w$. We call h'' *the composition of h and h'*, and we denote it by :

$$h * h' := h''.$$

(v) Dually, *if X is fibrant*, and if we have cocylinders

$$X \xrightarrow{s} X^I \xrightarrow{(d_0, d_1)} X \times X \qquad X \xrightarrow{s'} X^{I'} \xrightarrow{(d_0', d_1')} X \times X$$

and right homotopies $k : A \to X^I$, $k' : A \to X^{I'}$ with $d_0 k = u$, $d_1 k = v = d_0' k'$ and $d_1' k' = w$, then we may form the cartesian square :

$$\begin{tikzcd} X^{I''} \arrow[r, "p"] \arrow[d, "p'"'] & X^I \arrow[d, "d_0"] \\ X^{I'} \arrow[r, "d_1'"] & X. \end{tikzcd}$$

We set $d_0'' := d_0 p$ and $d_1'' = d_1' p'$, and we obtain the cocylinder :

$$X \xrightarrow{s''} X^{I''} \xrightarrow{(d_0'', d_1'')} X \times X \qquad \text{where } ps'' = s \text{ and } p's'' = s'.$$

Then the unique morphism $k'' : A \to X^{I''}$ of $\mathscr{C}$ with $pk'' = k$ and $p'k'' = k'$ is a right homotopy from u to w. As in (iv), we call k'' *the composition of k and k'*, and denote it

$$k * k' := k''.$$

Lemma 3.2.11. *(i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $A \in \mathrm{Ob}(\mathscr{C}_c)$, $X \in \mathrm{Ob}(\mathscr{C})$ and $f_0, f_1 \in \mathscr{C}(A, X)$. Consider the following conditions :*

(a) $f_0 \overset{l}{\sim} f_1$.

(b) For every cocylinder $X \xrightarrow{s} X^I \xrightarrow{(d_0, d_1)} X \times X$ there exists $k \in \mathscr{C}(A, X^I)$ such that $d_i \circ k = f_i$ for $i = 0, 1$.

(c) $f_0 \overset{r}{\sim} f_1$.

(d) There exists a cocylinder $X \xrightarrow{s} X^I \xrightarrow{(d_0, d_1)} X \times X$ such that s is a trivial cofibration, and $k \in \mathscr{C}(A, X^I)$ such that $d_i \circ k = f_i$ for $i = 0, 1$.

(e) $g \circ f_0 \overset{r}{\sim} g \circ f_1$ for every $Y \in \mathrm{Ob}(\mathscr{C})$ and every $g \in \mathscr{C}(X, Y)$.

Then $\overset{l}{\sim}$ *is an equivalence relation on* $\mathscr{C}(A,X)$*, and we have : (a)⇒(b)⇒(c)⇔(d)⇒(e).*

(ii) Dually, let $A \in \mathrm{Ob}(\mathscr{C})$*,* $X \in \mathrm{Ob}(\mathscr{C}_f)$*, and consider the following conditions :*

(a) $f_0 \overset{r}{\sim} f_1$.

(b) For every cylinder $A \sqcup A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A$ *there exists* $h \in \mathscr{C}(IA,X)$ *such that* $h \circ \partial_i = f_i$ *for* $i = 0,1$.

(c) $f_0 \overset{l}{\sim} f_1$.

(d) There exists a cylinder $A \sqcup A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A$ *such that* σ *is a trivial fibration, and* $h \in \mathscr{C}(IA,X)$ *such that* $h \circ \partial_i = f_i$ *for* $i = 0,1$.

(e) $f_0 \circ g \overset{l}{\sim} f_1 \circ g$ *for every* $B \in \mathrm{Ob}(\mathscr{C})$ *and every* $g \in \mathscr{C}(B,A)$.

Then $\overset{r}{\sim}$ *is an equivalence relation on* $\mathscr{C}(A,X)$*, and we have : (a)⇒(b)⇒(c)⇔(d)⇒(e).*

(iii) If $A \in \mathrm{Ob}(\mathscr{C}_c), X \in \mathrm{Ob}(\mathscr{C}_f)$*, and* $f_0, f_1 \in \mathscr{C}(A,X)$*, then* $f_0 \overset{l}{\sim} f_1 \Leftrightarrow f_0 \overset{r}{\sim} f_1$.

(iv) Suppose either that $A \in \mathrm{Ob}(\mathscr{C}_c)$ *and* $f_0 \overset{l}{\sim} f_1$*, or else that* $X \in \mathrm{Ob}(\mathscr{C}_f)$ *and* $f_0 \overset{r}{\sim} f_1$*. Then* $f_0 \in \mathscr{W} \Leftrightarrow f_1 \in \mathscr{W}$.

Proof. (i): The reflexivity, symmetricity, and transitivity of $\overset{l}{\sim}$ follow respectively from parts (i), (ii) and (iv) of remark 3.2.10. Next, to see that *(a)⇒(b)*, let $A \sqcup A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\sigma} A$ be a cylinder of A with a left homotopy $h : IA \to X$ from f_0 to f_1. Since (d_0, d_1) is a fibration and ∂_1 is a trivial cofibration (remark 3.2.10(iii)), the commutative diagram :

$$\begin{tikzcd} A \arrow[r, "sf_1"] \arrow[d, "\partial_1"'] & X^I \arrow[d, "(d_0,d_1)"] \\ IA \arrow[r, "(h,f_1\sigma)"] & X \times X \end{tikzcd}$$

has a diagonal filler $K : IA \to X^I$. To conclude, set $k := K\partial_0 : A \to X^I$, and notice that :

$$d_0k = d_0K\partial_0 = h\partial_0 = f_0 \qquad \text{and} \qquad d_1k = d_1K\partial_0 = f_1\sigma\partial_0 = f_1.$$

Clearly, *(b)⇒(c)⇐(d)*; to see that *(c)⇒(d)*, pick a cocylinder $X \xrightarrow{t} X' \xrightarrow{(d'_0,d'_1)} X \times X$ of X with a right homotopy $k' : A \to X'$ from f_0 to f_1, factor t as the composition of a trivial cofibration $s : X \to X^I$ and a trivial fibration $f : X^I \to X'$, and set $d_i := d'_i \circ f : X^I \to X$ for $i = 0,1$; then $X \xrightarrow{s} X^I \xrightarrow{(d_0,d_1)} X \times X$ is a cocylinder, and the commutative square :

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X^I \arrow[d, "f"] \\ A \arrow[r, "k'"] & X' \end{tikzcd}$$

admits a diagonal filler $k : A \to X^I$. It suffices then to notice that $d_i \circ k = f_i$ for $i = 0,1$.

(d)⇒(e) : Pick any cocylinder $Y \xrightarrow{t} Y^I \xrightarrow{(p_0,p_1)} Y \times Y$ and a diagonal filler $h : X^I \to Y^I$ for the commutative square :

$$\begin{tikzcd} X \arrow[r, "t\circ g"] \arrow[d, "s"'] & Y^I \arrow[d, "(p_0,p_1)"] \\ X^I \arrow[r, "(g\circ d_0,g\circ d_1)"] & Y \times Y. \end{tikzcd}$$

Then $h \circ k : A \to Y^I$ is a right homotopy from $g \circ f_0$ to $g \circ f_1$.

(i)⇒(ii), by considering the opposite model category $\mathscr{C}^{\mathrm{op}}$, and clearly (i,ii)⇒(iii).

(iv): If $A \in \mathrm{Ob}(\mathscr{C}_c)$ and $f_0 \overset{l}{\sim} f_1$, resume the notation of the proof of (i); since $f_i = h \circ \partial_i$ for $i = 0, 1$, the 2-out-of-3 property of $\mathscr{W}$ yields : $f_0 \in \mathscr{W} \Leftrightarrow h \in \mathscr{W} \Leftrightarrow f_1 \in \mathscr{W}$, since ∂_0 and ∂_1 are trivial cofibrations. One argues likewise if $X \in \mathrm{Ob}(\mathscr{C}_f)$ and $f_0 \overset{r}{\sim} f_1$. □

Definition 3.2.12. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $A, X \in \mathrm{Ob}(\mathscr{C})$.

(i) We denote by $\overset{l}{\approx}$ (resp. by $\overset{r}{\approx}$) the equivalence relation on $\mathscr{C}(A, X)$ generated by $\overset{l}{\sim}$ (resp. by $\overset{r}{\sim}$), and we set :

$$\boxed{[A, X]_l := \mathscr{C}(A, X)/\overset{l}{\approx} \qquad (\text{resp. } [A, X]_r := \mathscr{C}(A, X)/\overset{r}{\approx}).}$$

(ii) If A is cofibrant (resp. if X is fibrant), we have $\overset{l}{\sim}=\overset{l}{\approx}$ (resp. $\overset{r}{\sim}=\overset{r}{\approx}$) by lemma 3.2.11(i,ii). Moreover, if A is cofibrant and X is fibrant, then $[A, X]_l = [A, X]_r$ by lemma 3.2.11(iii); in this case, we denote this common quotient by $[A, X]$. It is clear that the relation of right homotopy is compatible with compositions to the right, whereas left homotopy is compatible with compositions to the left; we obtain therefore a well-defined functor

$$[-, -] : \mathscr{C}_c^{\mathrm{op}} \times \mathscr{C}_f \to \mathrm{Set}.$$

(iii) Let moreover $\mathscr{C}_{cf} := \mathscr{C}_c \cap \mathscr{C}_f$; then we obtain a category $\overline{\mathscr{C}_{cf}}$ with :

$$\boxed{\mathrm{Ob}(\overline{\mathscr{C}_{cf}}) := \mathrm{Ob}(\mathscr{C}_{cf}) \qquad \text{and} \qquad \overline{\mathscr{C}_{cf}}(A, B) := [A, B] \quad \forall A, B \in \mathrm{Ob}(\overline{\mathscr{C}_{cf}})}$$

with the composition law deduced by restriction from the foregoing functor $[-, -]$. Clearly we have a functor which is the identity on objects

$$\pi : \mathscr{C}_{cf} \to \overline{\mathscr{C}_{cf}}$$

and maps every morphism $u : A \to B$ of $\mathscr{C}_{cf}$ to its equivalence class $[u]$ in $[A, B]$.

(iv) We have furthermore two categories $\overline{\mathscr{C}_c}$ and $\overline{\mathscr{C}_f}$ with :

$$\boxed{\begin{array}{llll} \mathrm{Ob}(\overline{\mathscr{C}_c}) := \mathrm{Ob}(\mathscr{C}_c) & \text{and} & \overline{\mathscr{C}_c}(A, B) := [A, B]_r & \forall A, B \in \mathrm{Ob}(\overline{\mathscr{C}_c}) \\ \mathrm{Ob}(\overline{\mathscr{C}_f}) := \mathrm{Ob}(\mathscr{C}_f) & \text{and} & \overline{\mathscr{C}_f}(A, B) := [A, B]_l & \forall A, B \in \mathrm{Ob}(\overline{\mathscr{C}_f}) \end{array}}$$

with functors

$$\pi_c : \mathscr{C}_c \to \overline{\mathscr{C}_c} \qquad \pi_f : \mathscr{C}_f \to \overline{\mathscr{C}_f}$$

that are identities on objects, and that map every morphism $u : A \to B$ of $\mathscr{C}_c$ (resp. of $\mathscr{C}_f$) to its equivalence class $[u]_r$ in $[A, B]_r$ (resp. its equivalence class $[u]_l$ in $[A, B]_u$). Hence, the composition law of $\overline{\mathscr{C}_c}$ is given by the rule :

$$[v]_r \circ [u]_r := [v \circ u]_r \qquad \forall A, B, C \in \mathrm{Ob}(\overline{\mathscr{C}_c}), \forall u \in \mathscr{C}(A, B), \forall v \in \mathscr{C}(B, C).$$

To check that this rule yields a well-defined law, it suffices to recall that right homotopies are compatible with compositions to the right, and also to left, for morphisms of cofibrant objects, by virtue of lemma 3.2.11(i). Likewise one determines the composition law of $\overline{\mathscr{C}_f}$.

Remark 3.2.13. (i) In the situation of definition 3.2.12, endow $\mathscr{C}^{\mathrm{op}}$ with the opposite model category induced by $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$. Then we have natural identifications :

$$\overline{(\mathscr{C}^{\mathrm{op}})_c} \xrightarrow{\sim} (\overline{\mathscr{C}_f})^{\mathrm{op}} \qquad \overline{(\mathscr{C}^{\mathrm{op}})_f} \xrightarrow{\sim} (\overline{\mathscr{C}_c})^{\mathrm{op}} \qquad \overline{(\mathscr{C}^{\mathrm{op}})_{cf}} \xrightarrow{\sim} (\overline{\mathscr{C}_{cf}})^{\mathrm{op}}.$$

(ii) Also, the inclusions $\mathscr{C}_c \hookleftarrow \mathscr{C}_{cf} \hookrightarrow \mathscr{F}_f$ induce fully faithful inclusion functors :

$$\overline{\mathscr{C}_c} \hookleftarrow \overline{\mathscr{C}_{cf}} \hookrightarrow \overline{\mathscr{C}_f}.$$

Proposition 3.2.14. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $A, X \in \mathrm{Ob}(\mathscr{C})$.*

(i) If A is cofibrant, and if $f : X \to X'$ is either a trivial fibration or a weak equivalence of fibrant objects, then composition with f induces a bijection :

$$f_* : [A, X]_l \xrightarrow{\sim} [A, X']_l \qquad [h] \mapsto [fh].$$

(ii) If X is fibrant, and if $g : A' \to A$ is either a trivial cofibration or a weak equivalence between cofibrant objects, then composition with g induces a bijection :

$$g^* : [A, X]_r \xrightarrow{\sim} [A', X]_r \qquad [h] \mapsto [hg].$$

Proof. By arguing with the opposite model category $\mathscr{C}^{\mathrm{op}}$, it suffices to check (i). To this aim, we apply (the dual of) proposition 3.2.8 to the functor

$$[A, -]_l : \mathscr{C} \to \mathsf{Set} \qquad Y \mapsto [A, Y]_l$$

and to the class $\mathscr{W}' \subset \mathrm{Mor}(\mathsf{Set})$ of bijective maps : we are thus reduced to the case where f is a trivial fibration of $\mathscr{C}$. For the surjectivity of f_*, let $h' : A \to X'$ be any morphism of $\mathscr{C}$, and consider the square diagram :

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "h'"] & X'. \end{tikzcd}$$

By assumption, we have a diagonal filler $h : A \to X$, so $f_*[h] = [h']$.

For the injectivity, let $g_0, g_1 : A \rightrightarrows X$ be two morphisms of $\mathscr{C}$, and $k' : IA \to X'$ a left homotopy from fg_0 to fg_1 (for some cylinder IA of A). We get a commutative square :

$$\begin{tikzcd} A \sqcup A \arrow[r, "(g_0, g_1)"] \arrow[d, "(\partial_0, \partial_1)"'] & X \arrow[d, "f"] \\ IA \arrow[r, "k'"] & X' \end{tikzcd}$$

which admits a diagonal filler $k : IA \to X$. The latter is a left homotopy from g_0 to g_1, whence the assertion. □

3.3. **The homotopy category of a model category.** Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, with initial and final objects $\varnothing$ and e. For every $X \in \mathrm{Ob}(\mathscr{C})$ the unique morphism $\varnothing \to X$ (resp. $X \to e$) is the composition of a cofibration $\varnothing \to Y$ and a trivial fibration $Y \to X$ (resp. of a trivial cofibration $X \to Z$ and a fibration $Z \to e$), so $Y \in \mathrm{Ob}(\mathscr{C}_c)$, and $Z \in \mathrm{Ob}(\mathscr{C}_f)$. Also, if X is fibrant (resp. cofibrant), then $Y \in \mathrm{Ob}(\mathscr{C}_{cf})$ (resp. $Z \in \mathrm{Ob}(\mathscr{C}_{cf})$).

3.3.1. *Fibrant and cofibrant replacements.* We wish to define functors

$$\boxed{\overline{\mathscr{C}_c} \xleftarrow{(-)_c} \mathscr{C} \xrightarrow{(-)_f} \overline{\mathscr{C}_f} \qquad X_c \leftarrow\!\shortmid X \mapsto X_f.}$$

To this aim, we first fix pairs of morphisms (using the axiom of global choice) :

$$\boxed{X_c \xrightarrow{\beta_X} X \xrightarrow{\alpha_X} X_f \qquad \forall X \in \mathrm{Ob}(\mathscr{C})}$$

with $X_c \in \mathrm{Ob}(\mathscr{C}_c)$, $X_f \in \mathrm{Ob}(\mathscr{C}_f)$, $\alpha_X \in \mathscr{W} \cap \mathscr{C}of$, $\beta_X \in \mathscr{W} \cap \mathscr{F}ib$, and where :

$$\beta_X = \mathbf{1}_X \quad \forall X \in \mathrm{Ob}(\mathscr{C}_c) \qquad \text{and} \qquad \alpha_X = \mathbf{1}_X \quad \forall X \in \mathrm{Ob}(\mathscr{C}_f).$$

Next, for every $X, Y \in \mathrm{Ob}(\mathscr{C})$ and every $g \in \mathscr{C}(X, Y)$, the square diagrams :

$$
(*) \qquad
\begin{tikzcd}
\varnothing \arrow[rr] \arrow[d] & & Y_c \arrow[d, "\beta_Y"] \\
X_c \arrow[r, "\beta_X"] & X \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\alpha_X"'] & Y \arrow[r, "\alpha_Y"] & Y_f \arrow[d] \\
X_f \arrow[rr] & & e
\end{tikzcd}
$$

admit diagonal fillers $g_c : X_c \to Y_c$ and $g_f : X_f \to Y_f$ respectively. Hence we define :

$$[X_f, Y_f]_l = \overline{\mathscr{C}_f}(X_f, Y_f) \leftarrow \mathscr{C}(X, Y) \to \overline{\mathscr{C}_c}(X_c, Y_c) = [X_c, Y_c]_r \qquad [g_f]_l \leftarrow\!\shortmid g \mapsto [g_c]_r$$

(notation of definition 3.2.12(iv)). Notice that the class $[g_c]_r$ of g_c in $[X_c, Y_c]_r$ is independent of the choice of g_c, by proposition 3.2.14(i) and lemma 3.2.11(i). Likewise, the class $[g_f]_l$ of g_f in $[X_f, Y_f]_l$ is independent of the choice of g_f. We need to check that :

$$[(hg)_f]_l = [h_f]_l \circ [g_f]_l \qquad [(hg)_c]_r = [h_c]_r \circ [g_c]_r \qquad \forall X \xrightarrow{g} Y \xrightarrow{h} Z \text{ in } \mathscr{C}.$$

However, by a direct inspection we get commutative diagrams :

$$
\begin{tikzcd}
\varnothing \arrow[rr] \arrow[d] & & Z_c \arrow[d, "\beta_Z"] \\
X_c \arrow[r, "\beta_X"'] \arrow[urr, "h_c \circ g_c"] & X \arrow[r, "hg"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "hg"] \arrow[d, "\alpha_X"'] & Z \arrow[r, "\alpha_Z"] & Z_f \arrow[d] \\
X_f \arrow[rr] \arrow[urr, "h_f \circ g_f"] & & e
\end{tikzcd}
$$

whence the assertion. Lastly, obviously we may take $(\mathbf{1}_X)_c := \mathbf{1}_{X_c}$ and $(\mathbf{1}_X)_f := \mathbf{1}_{X_f}$, so $[(\mathbf{1}_X)_c]_r = \mathbf{1}_{X_c}$ in $\overline{\mathscr{C}_c}$, for every $X \in \mathrm{Ob}(\mathscr{C})$, and likewise for $[(\mathbf{1}_X)_f]_l$.

Lemma 3.3.2. *(i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $F : \mathscr{C}_c \to \mathscr{D}$ a functor that maps every $u \in \mathscr{W}_c$ to an isomorphism Fu of $\mathscr{D}$ (notation of definition 3.2.1(iii)). Then F factors through $\pi_c : \mathscr{C}_c \to \overline{\mathscr{C}_c}$ and a unique functor $\overline{F} : \overline{\mathscr{C}_c} \to \mathscr{D}$.*

(ii) Dually, let $F : \mathscr{C}_f \to \mathscr{D}$ be a functor that maps every $u \in \mathscr{W}_f$ to an isomorphism Fu of $\mathscr{D}$. Then F factors through $\pi_f : \mathscr{C}_f \to \overline{\mathscr{C}_f}$ and a unique functor $\overline{F} : \overline{\mathscr{C}_f} \to \mathscr{D}$.

Proof. By virtue of remark 3.2.13(i), it suffices to show (ii). The latter comes down to checking that for every $X, Y \in \mathrm{Ob}(\mathscr{C}_f)$ and every $g, g' \in \mathscr{C}_f(X, Y)$ with $g \overset{l}{\sim} g'$, we have $Fg = Fg'$. Indeed, by lemma 3.2.11(ii) there exists a cylinder of X in $\mathscr{C}$:

$$X \sqcup X \xrightarrow{(\partial_0, \partial_1)} IX \xrightarrow{\sigma} X$$

such that σ is a trivial fibration, and a morphism $h : IX \to Y$ with $h\partial_0 = g$ and $h\partial_1 = g'$; we are therefore reduced to checking that $F\partial_0 = F\partial_1$. But $F(\sigma)$ is an isomorphism, since $\sigma \in \mathscr{W}_c$, and on the other hand, $\sigma \circ \partial_0 = \mathbf{1}_X = \sigma \circ \partial_1$, so $F\partial_0 = (F\sigma)^{-1} = F\partial_1$. □

Remark 3.3.3. (i) The chosen X_c, X_f are *fibrant* and respectively *cofibrant replacements for* X. Any such choice induces a corresponding choice of fibrant and cofibrant replacements on the opposite model category $\mathscr{C}^{\mathrm{op}}$ (proposition 3.2.4(i)) : namely, we take $(X^{\mathrm{op}})_c := (X_f)^{\mathrm{op}}$ and $(X^{\mathrm{op}})_f := (X_c)^{\mathrm{op}}$; then $\alpha_{X^{\mathrm{op}}} = \beta_X^{\mathrm{op}}$ and $\beta_{X^{\mathrm{op}}} = \alpha_X^{\mathrm{op}}$ for every $X \in \mathrm{Ob}(\mathscr{C})$. We call $(-)_c$ and $(-)_f$ *the fibrant* and respectively *cofibrant replacement functors* for $\mathscr{C}$.

(ii) Notice that both $X_{cf} := (X_c)_f$ and $X_{fc} := (X_f)_c$ lie in $\mathscr{C}_{cf}$. Hence, the restriction $\mathscr{C}_c \to \overline{\mathscr{C}_f}$ of $(-)_f$ factors through the inclusion $\overline{\mathscr{C}_{cf}} \hookrightarrow \overline{\mathscr{C}_f}$ (remark 3.2.13(ii)). Moreover, notice that the resulting functor $\mathscr{C}_c \to \overline{\mathscr{C}_{cf}}$ maps every $u \in \mathscr{W}_c$ to an isomorphism $[u_f]$ of $\overline{\mathscr{C}_{cf}}$: indeed, $u_f \in \mathscr{W}$ by the 2-out-of-3 property of $\mathscr{W}$, so by proposition 3.2.14, the class $[u_f]$ admits both a left and right inverse in $\overline{\mathscr{C}_{cf}}$. Likewise, the restriction $\mathscr{C}_f \to \overline{\mathscr{C}_c}$ of $(-)_c$ induces a functor $\mathscr{C}_f \to \overline{\mathscr{C}_{cf}}$ that maps every $u \in \mathscr{W}_f$ to an isomorphism $[u_c]$. Then, by lemma 3.3.2, these latter functors in turn factor through unique functors :

$$\overline{\mathscr{C}_c} \xrightarrow{[-]_f} \overline{\mathscr{C}_{cf}} \xleftarrow{[-]_c} \overline{\mathscr{C}_f}.$$

(iii) Furthermore, we get a further pair of functors $\mathscr{C} \rightrightarrows \overline{\mathscr{C}_{cf}}$, by setting :

$$(-)_{cf} := [-]_f \circ (-)_c \qquad \text{and} \qquad (-)_{fc} := [-]_c \circ (-)_f.$$

Lastly, the weak equivalence $\alpha_X : X \to X_f$ induces a morphism $\tau_X := (\alpha_X)_{cf} : X_{cf} \to X_{fc}$. We claim that the rule $X \mapsto [\tau_X]$ yields an isomorphism of functors :

$$[\tau_\bullet] : (-)_{cf} \xrightarrow{\sim} (-)_{fc}.$$

Indeed, for every morphism $g : X \to Y$ of $\mathscr{C}$, a simple diagram chase shows that :

$$\beta_{Y_f} \circ \tau_Y \circ g_{cf} \circ \alpha_{X_c} = \alpha_Y \circ g \circ \beta_X = \beta_{Y_f} \circ g_{fc} \circ \tau_X \circ \alpha_{X_c} \qquad \text{in } \mathscr{C}(X_c, Y_f)$$

(details left to the reader). By proposition 3.2.14, we deduce that $[\tau_Y \circ g_{cf}] = [g_{fc} \circ \tau_X]$ in $[X_{cf}, Y_{fc}]$ for every such g, whence the naturality of $[\tau_\bullet]$. By construction, $\tau_X \in \mathscr{W}$, so $[\tau_X]$ is an isomorphism in $\overline{\mathscr{C}_{cf}}$, again by proposition 3.2.14, whence the assertion.

(iv) Notice as well that $[\tau_X] = [\mathbf{1}_{X_{cf}}] = [\mathbf{1}_{X_{fc}}]$ whenever $X \in \mathrm{Ob}(\mathscr{C}_c) \cup \mathrm{Ob}(\mathscr{C}_f)$ (details left to the reader).

Example 3.3.4. In the situation of §3.3.1, let $X \in \mathrm{Ob}(\mathscr{C})$, and endow the slice category $X/\mathscr{C}$ with the model structure provided by proposition 3.2.4(iii). In light of remark 3.2.5, our choice of fibrant replacements $(\alpha_Y \mid Y \in \mathrm{Ob}(\mathscr{C}))$ induces a system of fibrant replacements for $X/\mathscr{C}$: namely, for every $(Y, g : X \to Y) \in \mathrm{Ob}(X/\mathscr{C})$ we let $(Y, g)_f := (Y_f, \alpha_Y \circ g)$, and set $\alpha_{(Y,g)} := X/\alpha_Y : (Y, g) \to (Y, g)_f$. On the other hand, a cofibrant replacement for (Y, g) is obtained by fixing a factorization $g = k \circ h$ with a cofibration $h : X \to Z$ and a trivial cofibration $k : Z \to Y$; then one may set $(Y, g)_c := (Z, h)$ and $\beta_{(Y,g)} := X/k : (Y, g)_c \to (Y, g)$. Likewise one may describe fibrant and cofibrant replacements for $\mathscr{C}/X$ (with the model structure of proposition 3.2.4(ii)).

Theorem 3.3.5. *For every model category $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$, the localization $(\mathscr{C}[\mathscr{W}^{-1}], \gamma)$ of $\mathscr{C}$ along $\mathscr{W}$ exists, and the functor $(-)_{cf} : \mathscr{C} \to \overline{\mathscr{C}_{cf}}$ is the composition of $\gamma : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ and a unique equivalence of categories :*

$$\boxed{\mathscr{C}[\mathscr{W}^{-1}] \xrightarrow{\sim} \overline{\mathscr{C}_{cf}}.}$$

We call $\mathscr{C}[\mathscr{W}^{-1}]$ the homotopy category of the model category $\mathscr{C}$*, and denote it :*

$$\boxed{\mathsf{ho}(\mathscr{C}) := \mathscr{C}[\mathscr{W}^{-1}].}$$

Proof. Let $\mathsf{ho}(\mathscr{C})$ be the category with $\mathrm{Ob}(\mathsf{ho}(\mathscr{C})) := \mathrm{Ob}(\mathscr{C})$ and with

$$\mathsf{ho}(\mathscr{C})(X, Y) := [X_{cf}, Y_{cf}] \qquad \forall X, Y \in \mathrm{Ob}(\mathscr{C}).$$

The composition law for $\mathsf{ho}(\mathscr{C})$ is the same as for the category $\overline{\mathscr{C}_{cf}}$, so that $(-)_{cf}$ is the composition of an obvious functor that is the identity on objects :

$$\gamma : \mathscr{C} \to \mathsf{ho}(\mathscr{C}) \qquad X \mapsto X \qquad (X \xrightarrow{f} Y) \mapsto [X_{cf} \xrightarrow{f_{cf}} Y_{cf}]$$

and a unique equivalence of categories that is the identity on morphisms :

$$\phi : \mathsf{ho}(\mathscr{C}) \xrightarrow{\sim} \overline{\mathscr{C}_{cf}} \qquad X \mapsto X_{cf}.$$

It remains to check that $(\mathsf{ho}(\mathscr{C}), \gamma)$ is a localization of $\mathscr{C}$ along the class $\mathscr{W}$. We have already remarked that if $f \in \mathscr{W}$, then $\gamma(f)$ is an isomorphism of $\mathsf{ho}(\mathscr{C})$ (remark 3.3.3(ii)).

Lastly, let $\mathscr{D}$ be a category, and $F : \mathscr{C} \to \mathscr{D}$ a functor that maps weak equivalences of $\mathscr{C}$ to isomorphisms of $\mathscr{D}$; also, let $X, Y \in \mathrm{Ob}(\mathscr{C})$, and consider two morphisms $f, f' : X_{cf} \rightrightarrows Y_{cf}$ of $\mathscr{C}_{cf}$. Notice that if $[f] = [f']$, then $Ff = Ff'$, by lemma 3.3.2. Then, let $\overline{F}X := FX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and for every $[f] \in \mathsf{ho}(\mathscr{C})(X, Y)$ set :

$$\overline{F}[f] := F\beta_Y \circ (F\alpha_{Y_c})^{-1} \circ Ff \circ F\alpha_{X_c} \circ (F\beta_X)^{-1} : \overline{F}X \to \overline{F}Y.$$

Clearly $\overline{F}[\mathbf{1}_X] = \mathbf{1}_{\overline{F}X}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and $\overline{F}[g \circ f] = \overline{F}[g] \circ \overline{F}[f]$ for every composable pair $X \xrightarrow{[f]} Y \xrightarrow{[g]} Z$ of morphisms of $\mathsf{ho}(\mathscr{C})$. Also, for every $g \in \mathscr{C}(X, Y)$:

$$Fg = F\beta_Y \circ Fg_c \circ (F\beta_X)^{-1} = F\beta_Y \circ (F\alpha_{Y_c})^{-1} \circ Fg_{cf} \circ F\alpha_{X_c} \circ (F\beta_X)^{-1} = \overline{F} \circ \gamma(g).$$

This shows that F factors through γ and a functor $\overline{F} : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$. To prove the uniqueness of $\overline{F}$, suppose that $G : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$ is another functor with $F = G \circ \gamma$, and let again $f \in \mathscr{C}(X_{cf}, Y_{cf})$; so f induces elements of $\mathsf{ho}(\mathscr{C})(X, Y)$, $\mathsf{ho}(\mathscr{C})(X_c, Y_c)$ and $\mathsf{ho}(\mathscr{C})(X_{cf}, Y_{cf})$ that we denote respectively by $[f]$, $[f_c]$ and $[f_{cf}]$. We then have :

$$[f] \circ \gamma(\beta_X) = \gamma(\beta_Y) \circ [f_c] \qquad \text{and} \qquad \gamma(\alpha_{Y_c}) \circ [f_c] = [f_{cf}] \circ \gamma(\alpha_{X_c})$$

whence $G[f] \circ F\beta_X = F\beta_Y \circ G[f_c]$ and $F\alpha_{Y_c} \circ G[f_c] = G[f_{cf}] \circ F\alpha_{X_c}$, and corresponding identities hold for $\overline{F}[f]$, $\overline{F}[f_c]$ and $\overline{F}[f_{cf}]$. Notice that $\overline{F}X = GX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, since γ is the identity on objects. But we have as well $\overline{F}[f_{cf}] = G[f_{cf}]$, since ϕ restricts to the identity on the full subcategory $\overline{\mathscr{C}_{cf}}$ of $\mathsf{ho}(\mathscr{C})$; moreover, $F\alpha_X$ and $F\beta_X$ are isomorphisms of $\mathscr{D}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, since $\alpha_X, \beta_X \in \mathscr{W}$, so $\overline{F}[f] = G[f]$. □

Corollary 3.3.6. *For every model category $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$, the localizations $\mathscr{C}_c[\mathscr{W}_c^{-1}]$ and $\mathscr{C}_f[\mathscr{W}_f^{-1}]$ exist (notation of definition 3.2.1(iii)), and the inclusions of categories :*

$$\begin{tikzcd} \mathscr{C}_{cf} \arrow[r, hook] \arrow[d, hook] & \mathscr{C}_c \arrow[d, hook] \\ \mathscr{C}_f \arrow[r, hook] & \mathscr{C} \end{tikzcd}$$

induce equivalences of categories :

$$\boxed{\begin{tikzcd} \overline{\mathscr{C}_{cf}} \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathscr{C}_c[\mathscr{W}_c^{-1}] \arrow[d, "\wr"] \\ \mathscr{C}_f[\mathscr{W}_f^{-1}] \arrow[r, "\sim"] & \mathsf{ho}(\mathscr{C}). \end{tikzcd}}$$

Proof. The explicit construction of $\mathsf{ho}(\mathscr{C})$ implies that the inclusion $\mathscr{C}_{cf} \to \mathscr{C}$ induces an equivalence $\overline{\mathscr{C}_{cf}} \xrightarrow{\sim} \mathsf{ho}(\mathscr{C})$. Next, let $\mathsf{ho}(\mathscr{C}_c)$ be the full subcategory of $\mathsf{ho}(\mathscr{C})$ with $\mathrm{Ob}(\mathsf{ho}(\mathscr{C}_c)) := \mathrm{Ob}(\mathscr{C}_c)$, and denote by $\gamma_c : \mathscr{C}_c \to \mathsf{ho}(\mathscr{C}_c)$ the restriction of the localization $\gamma : \mathscr{C} \to \mathsf{ho}(\mathscr{C})$. Then $\gamma_c(f)$ is an isomorphism of $\mathsf{ho}(\mathscr{C}_c)$, for every $f \in \mathscr{W}_c$, and the proof of theorem 3.3.5 can be repeated *verbatim*, to show that $(\mathsf{ho}(\mathscr{C}_c), \gamma_c)$ is a localization of $\mathscr{C}_c$ along $\mathscr{W}_c$. Since, moreover, $\beta_X \in \mathscr{W}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, we deduce that the inclusion $\mathscr{C}_c \to \mathscr{C}$ induces an equivalence $\mathscr{C}_c[\mathscr{W}_c^{-1}] \xrightarrow{\sim} \mathsf{ho}(\mathscr{C})$.

In order to get the corresponding assertions for $\mathscr{C}_f[\mathscr{W}_f^{-1}]$, we argue with the opposite model category $\mathscr{C}^{\mathrm{op}}$, and invoke remark 1.11.5(iii). □

Remark 3.3.7. (i) With the notation of corollary 3.3.6, in the following we also set

$$\boxed{\mathsf{ho}(\mathscr{C}_c) := \mathscr{C}_c[\mathscr{W}_c^{-1}] \qquad \text{and} \qquad \mathsf{ho}(\mathscr{C}_f) := \mathscr{C}_f[\mathscr{W}_f^{-1}].}$$

By construction we have, for every $X, Y \in \mathrm{Ob}(\mathscr{C}_c)$ and $X', Y' \in \mathrm{Ob}(\mathscr{C}_f)$:

$$\mathsf{ho}(\mathscr{C}_c)(X,Y) = [X_f, Y_f] \qquad \mathsf{ho}(\mathscr{C}_f)(X',Y') = [X'_c, Y'_c].$$

The localization $\gamma_c : \mathscr{C}_c \to \mathsf{ho}(\mathscr{C}_c)$ (resp. $\gamma_f : \mathscr{C}_f \to \mathsf{ho}(\mathscr{C}_f)$) is the identity on objects, and maps every $g \in \mathscr{C}_c(X,Y)$ to $[g_f : X_f \to Y_f]$ (resp. every $h \in \mathscr{C}_f(X',Y')$ to $[h_c : X_c \to Y_c]$). Moreover, if $F : \mathscr{C}_c \to \mathscr{D}$ is a functor such that $F(\mathscr{W}_c) \subset \mathrm{Isom}(\mathscr{D})$, then the unique functor $\mathsf{ho}(F) : \mathsf{ho}(\mathscr{C}_c) \to \mathscr{D}$ with $\mathsf{ho}(F) \circ \gamma_c = F$ is given by the rules : $X \mapsto FX$ for every $X \in \mathrm{Ob}(\mathscr{C}_c)$ and $[g] \mapsto F(\alpha_Y)^{-1} \circ F(g) \circ F(\alpha_X)$ for every $[g] \in \mathsf{ho}(\mathscr{C}_c)(X,Y)$. Likewise, if $G : \mathscr{C}_f \to \mathscr{D}$ is a functor such that $F(\mathscr{W}_f) \subset \mathrm{Isom}(\mathscr{D})$, then the unique functor $\mathsf{ho}(G) : \mathsf{ho}(\mathscr{C}_f) \to \mathscr{D}$ with $\mathsf{ho}(G) \circ \gamma_f = G$ is given by the rules : $X' \mapsto GX'$ for every $X' \in \mathrm{Ob}(\mathscr{C}_f)$ and $[h] \mapsto F(\beta_{Y'}) \circ F(h) \circ F(\beta_{X'})^{-1}$ for every $[h] \in \mathsf{ho}(\mathscr{C}_f)(X',Y')$.

(ii) Furthermore, by lemma 3.3.2, the localization functors γ_c and γ_f factor through $\pi_c : \mathscr{C}_c \to \overline{\mathscr{C}_c}$, respectively $\pi_f : \mathscr{C}_f \to \overline{\mathscr{C}_f}$, and unique functors

$$\overline{\gamma}_c : \overline{\mathscr{C}_c} \to \mathsf{ho}(\mathscr{C}_c) \qquad \overline{\gamma}_f : \overline{\mathscr{C}_f} \to \mathsf{ho}(\mathscr{C}_f).$$

Unlike the case for $\overline{\mathscr{C}_{cf}}$, these functors are not necessarily equivalences of categories, but they induce isomorphisms of categories :

$$\boxed{\overline{\mathscr{C}_c}[\overline{\mathscr{W}_c}^{-1}] \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}_c) \qquad \overline{\mathscr{C}_f}[\overline{\mathscr{W}_f}^{-1}] \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}_f)}$$

where $\overline{\mathscr{W}_c}$ denotes the image of $\mathscr{W}_c$ in $\overline{\mathscr{C}_c}$, and likewise for $\overline{\mathscr{W}_f}$. Indeed, let $F : \overline{\mathscr{C}_c} \to \mathscr{D}$ be a functor such that $F(\overline{\mathscr{W}_c}) \subset \mathrm{Isom}(\mathscr{D})$; there exists a unique functor $\overline{F} : \mathsf{ho}(\mathscr{C}_c) \to \mathscr{D}$ with $\overline{F} \circ \gamma_c = F \circ \pi_c$, whence $\overline{F} \circ \overline{\gamma}_c \circ \pi_c = F \circ \pi_c$, and since π_c is both full and the identity on objects, we get $\overline{F} \circ \overline{\gamma}_c = F$. Clearly $\overline{F}$ is the unique functor $\mathsf{ho}(\mathscr{C}_c) \to \mathscr{D}$ enjoying the latter identity, so this shows that $(\mathsf{ho}(\mathscr{C}_c), \overline{\gamma}_c)$ is a localization of $\overline{\mathscr{C}_c}$ along $\overline{\mathscr{W}_c}$,whence the first stated isomorphism; likewise we obtain the stated isomorphism for $\overline{\mathscr{C}_f}[\overline{\mathscr{W}_f}^{-1}]$.

(iii) Also, there exist unique functors $\overline{(-)}_c$ and $\overline{(-)}_f$ that make commute the diagram :

$$\boxed{\begin{tikzcd}
\overline{\mathscr{C}_f} \arrow[d, "\overline{\gamma}_f"'] & \mathscr{C} \arrow[l, "(-)_f"'] \arrow[r, "(-)_c"] \arrow[d, "\gamma"] & \overline{\mathscr{C}_c} \arrow[d, "\overline{\gamma}_c"] \\
\mathsf{ho}(\mathscr{C}_f) & \mathsf{ho}(\mathscr{C}) \arrow[l, "\overline{(-)}_f"'] \arrow[r, "\overline{(-)}_c"] & \mathsf{ho}(\mathscr{C}_c)
\end{tikzcd}}$$

(where γ is the localization) and they are quasi-inverse functors for the equivalences

$$i_c : \mathsf{ho}(\mathscr{C}_c) \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}) \qquad i_f : \mathsf{ho}(\mathscr{C}_f) \xrightarrow{\sim} \mathsf{ho}(\mathscr{C})$$

provided by corollary 3.3.6; indeed, a direct inspection shows that $\overline{(-)}_c \circ i_c = \mathbf{1}_{\mathsf{ho}(\mathscr{C}_c)}$ and $\overline{(-)}_f \circ i_f = \mathbf{1}_{\mathsf{ho}(\mathscr{C}_f)}$. Furthermore, the rules: $X \mapsto \gamma(\beta_X)$ and $X \mapsto \gamma(\alpha_X)$ for every $X \in \mathrm{Ob}(\mathscr{C})$ define isomorphisms of functors

$$[\beta_\bullet] : i_c \circ \overline{(-)}_c \xrightarrow{\sim} \mathbf{1}_{\mathsf{ho}(\mathscr{C})} \qquad \text{and} \qquad [\alpha_\bullet] : \mathbf{1}_{\mathsf{ho}(\mathscr{C})} \xrightarrow{\sim} i_f \circ \overline{(-)}_f.$$

Here, $\gamma(\beta_X) \in \mathsf{ho}(\mathscr{C})(X_c, X) = [X_{cf}, X_{cf}]$ is $[\mathbf{1}_{X_{cf}}]$, whereas $\gamma(\alpha_X) \in \mathsf{ho}(\mathscr{C})(X, X_f) = [X_{cf}, X_{fc}]$ is the class $[\tau_X]$ of remark 3.3.3(iii). Also, for every $g \in \mathscr{C}(X,Y)$, the functor

$\overline{(-)}_c$ (resp. $\overline{(-)}_f$) sends $\gamma(g) = [g_{cf}] : X \to Y$ to $[g_{cf}] : X_c \to Y_c$ (resp. to $[g_{fc}] : X_f \to Y_f$). The detailed verifications shall be left to the reader.

(iv) Let $u : W \to X$ and $v : Y \to Z$ be two morphisms of $\mathscr{C}$, with $W \in \mathscr{C}_c$ and $Z \in \mathscr{C}_f$. Then, for every $\phi \in \mathsf{ho}(\mathscr{C})(X,Y)$ there exists $g \in \mathscr{C}(W,Z)$ such that :

$$\gamma(g) = \gamma(v) \circ \phi \circ \gamma(u) \qquad \text{in } \mathsf{ho}(\mathscr{C})(W,Z).$$

Indeed, recall that ϕ is by construction the class of some $f \in \mathscr{C}(X_{cf}, Y_{cf})$; then the same f also induces $\phi' \in \mathsf{ho}(\mathscr{C})(Y_c, X_c)$, such that the following diagram commutes in $\mathsf{ho}(\mathscr{C})$:

$$\begin{tikzcd} X \arrow[d, "\phi"'] & X_c \arrow[l, "\gamma(\beta_X)"'] \arrow[r, "\gamma(\alpha_{X_c})"] \arrow[d, "\phi'"] & X_{cf} \arrow[d, "\gamma(f)"] \\ Y & Y_c \arrow[l, "\gamma(\beta_Y)"'] \arrow[r, "\gamma(\alpha_{Y_c})"] & Y_{cf} \end{tikzcd}$$

(details left to the reader). Next, since Z is fibrant and α_{Y_c} is a trivial cofibration, there exists $t \in \mathscr{C}(Y_{cf}, Z)$ such that $t \circ \alpha_{Y_c} = v \circ \beta_Y : Y_c \to Z$. Lastly, since $W_c = W$, the morphism u induces a morphism $u_c \in \mathscr{C}(W, X_c)$ such that $\beta_X \circ u_c = u$. With this notation, we may then take $g := t \circ f \circ \alpha_{X_c} \circ u_c$.

Definition 3.3.8. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $f : X \to Y$ a morphism of $\mathscr{C}_{cf}$. We say that f *is a homotopy equivalence* if its class $[f] \in [X,Y]$ is an isomorphism of $\overline{\mathscr{C}_{cf}}$, *i.e.* if there exists $g \in \mathscr{C}_{cf}(Y,X)$ such that $\mathbf{1}_Y \overset{l}{\sim} f \circ g \overset{l}{\sim} \mathbf{1}_Y$ and $\mathbf{1}_X \overset{l}{\sim} g \circ f \overset{l}{\sim} \mathbf{1}_X$.

Theorem 3.3.9. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $f \in \mathrm{Mor}(\mathscr{C})$.*

(i) (Whitehead) *If $f \in \mathrm{Mor}(\mathscr{C}_{cf})$, then f is a homotopy equivalence $\Leftrightarrow f \in \mathscr{W}$.*

(ii) f is a weak equivalence if and only if its image in $\mathsf{ho}(\mathscr{C})$ *is an isomorphism.*

Proof. (i): Let $A, B \in \mathrm{Ob}(\mathscr{C}_{cf})$ and $f \in \mathscr{C}(A,B)$; then by construction f is a homotopy equivalence if and only if its image $[f]$ in $\mathsf{ho}(\mathscr{C})$ is an isomorphism; in particular, if f is a weak equivalence, then f is a homotopy equivalence, by virtue of proposition 3.2.14. We may then assume that f is a homotopy equivalence, and we check that $f \in \mathscr{W}$. To this aim, pick a factorization of $f : A \xrightarrow{g} C \xrightarrow{p} B$ with $g \in \mathscr{W} \cap \mathscr{C}of$ and $p \in \mathscr{F}ib$. Then $C \in \mathrm{Ob}(\mathscr{C}_{cf})$, and therefore g is a homotopy equivalence, by the foregoing.

Let f' be a homotopy inverse for f, so that there exists a left homotopy $H : IB \to B$ from ff' to $\mathbf{1}_B$. Notice that the commutative square :

$$\begin{tikzcd} B \arrow[r, "gf'"] \arrow[d, "\partial_0"'] & C \arrow[d, "p"] \\ IB \arrow[r, "H"] & B \end{tikzcd}$$

admits a diagonal filler $H' : IB \to C$, since $\partial_0 \in \mathscr{W} \cap \mathscr{C}of$ (remark 3.2.10(iii)). Set $q := H' \circ \partial_1 : B \to C$; then $pq = \mathbf{1}_B$, and H' is a left homotopy from gf' to q.

Let $g' : C \to A$ be a homotopy inverse for g, *i.e.* $gg' \overset{l}{\sim} \mathbf{1}_C$ and $g'g \overset{l}{\sim} \mathbf{1}_A$; then :

$$p \overset{l}{\sim} pgg' \overset{l}{\sim} (pg) \circ (f'f) \circ g' \overset{l}{\sim} (pq) \circ (fg') = fg' \quad \text{and thus :} \quad qp \overset{l}{\sim} (gf') \circ (fg') \overset{l}{\sim} \mathbf{1}_C.$$

Hence, let $K : IC \to C$ be a left homotopy from $\mathbf{1}_C$ to qp; then $\mathbf{1}_C : C \xrightarrow{\partial_0} IC \xrightarrow{K} C$ lies in $\mathscr{W}$, and the same holds for $\partial_0, \partial_1 : C \rightrightarrows IC$ (remark 3.2.10(iii)), so the same holds for K, and then also for $qp = K \circ \partial_1$. Lastly, the commutative diagram :

$$\begin{tikzcd} C \arrow[r, "\mathbf{1}_C"] \arrow[d, "p"'] & C \arrow[r, "\mathbf{1}_C"] \arrow[d, "qp"] & C \arrow[d, "p"] \\ B \arrow[r, "q"] & C \arrow[r, "p"] & B \end{tikzcd}$$

exhibits p as a retract of qp, so $p \in \mathscr{W}$ (proposition 3.2.6), and finally, $f = pg \in \mathscr{W}$.

(ii): By the 2-out-of-3 property, it is clear that $f \in \mathscr{W} \Leftrightarrow f_c \in \mathscr{W} \Leftrightarrow f_{cf} \in \mathscr{W}$ (notation of §3.3.1); thus, it remains only to check that a morphism $f : X \to Y$ of $\mathscr{C}_{cf}$ is in $\mathscr{W}$ if and only if $[f]$ is an isomorphism of $\overline{\mathscr{C}_{cf}}$, and this is (i). □

Corollary 3.3.10. *(i) The functor $\overline{\gamma}_c : \overline{\mathscr{C}_c} \to \mathsf{ho}(\mathscr{C}_c)$ admits a fully faithful right adjoint, and the class $\overline{\mathscr{W}_c}$ admits a left calculus of fractions.*

(ii) Dually, The functor $\overline{\gamma}_f : \overline{\mathscr{C}_f} \to \mathsf{ho}(\mathscr{C}_f)$ admits a fully faithful left adjoint, and the class $\overline{\mathscr{W}_f}$ admits a right calculus of fractions.

Proof. (i): Recall that the restriction $\mathscr{C}_c \to \overline{\mathscr{C}_{cf}}$ of the functor $(-)_f : \mathscr{C} \to \overline{\mathscr{C}_f}$ factors uniquely through a functor $[-]_f : \overline{\mathscr{C}_c} \to \overline{\mathscr{C}_{cf}}$ (remark 3.3.3(ii)). Let also $j : \overline{\mathscr{C}_{cf}} \to \overline{\mathscr{C}_c}$ be the fully faithful inclusion of remark 3.2.13(ii); it follows easily that the rule : $X \mapsto (\alpha_X : X \to X_f)$ for every $X \in \mathrm{Ob}(\mathscr{C}_c)$ yields a natural transformation

$$\alpha_\bullet : \mathbf{1}_{\overline{\mathscr{C}_c}} \Rightarrow j \circ [-]_f.$$

On the other hand, clearly $[-]_f \circ j = \mathbf{1}_{\overline{\mathscr{C}_{cf}}}$, and it is easily seen that $\alpha_\bullet \star j = \mathbf{1}_j$. Next, a simple inspection shows that $\alpha_\bullet$ and $\mathbf{1}_{\overline{\mathscr{C}_{cf}}}$ verify the triangular identities of [13, Prob.2.13(ii)], hence they are the unit and respectively the counit of a unique adjunction for the pair of functors $([-]_f, j)$. Lastly, $\overline{\gamma}_c \circ j : \overline{\mathscr{C}_{cf}} \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}_c)$ is the equivalence provided by corollary 3.3.6, and $\overline{\gamma}_c \star \alpha_\bullet : \overline{\gamma}_c \Rightarrow \overline{\gamma}_c \circ j \circ [-]_f$ is an isomorphism of functors, whence the first assertion. The second assertion now follows from example 1.12.2 and theorem 3.3.9(ii).

(ii) follows from (i) by considering the opposite model category structure, in light of remarks 3.2.13(i) and 1.11.5(iii). □

Remark 3.3.11. By corollary 3.3.10 and example 1.12.8, we see that the morphisms of $\mathsf{ho}(\mathscr{C}_c)$ (resp. of $\mathsf{ho}(\mathscr{C}_f)$) are given by left fractions $t^{-1} \circ f$ (resp. right fractions $f \circ t^{-1}$) with denominators $t \in \overline{\mathscr{W}_c}$ (resp. $t \in \overline{\mathscr{W}_f}$). In general, the class $\mathscr{W}$ of weak equivalences of $\mathscr{C}$ does not admit a calculus of fractions, so the morphisms of $\mathsf{ho}(\mathscr{C})$ cannot always be represented as fractions with denominators in $\mathscr{W}$.

3.4. **Derived functors and Quillen adjunctions.**

Definition 3.4.1. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $\gamma : \mathscr{C} \to \mathsf{ho}(\mathscr{C})$ its homotopy localization, and $F : \mathscr{C} \to \mathscr{D}$ a functor to a category $\mathscr{D}$.

(i) A *left derived functor of F* is a *right Kan extension* $(\mathbf{L}F, a^F_\bullet)$ *of F along γ, i.e.* the datum of a functor

$$\boxed{\mathbf{L}F : \mathsf{ho}(\mathscr{C}) \to \mathscr{D} \qquad \text{and a natural transformation} \qquad a^F_\bullet : \mathbf{L}F \circ \gamma \Rightarrow F}$$

enjoying the following universal property. For every functor $G : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$ and every natural transformation $\tau_\bullet : G \circ \gamma \Rightarrow F$, there exists a unique natural transformation $f_\bullet : G \Rightarrow \mathbf{L}F$ such that $\tau_\bullet = a^F_\bullet \circ (f_\bullet \star \gamma)$.

(ii) Dually, a *right derived functor of* F is a *left Kan extension* $(\mathbf{R}F, b^F_\bullet)$ *of* F *along* γ, *i.e.* the datum of a functor

$$\boxed{\mathbf{R}F : \mathsf{ho}(\mathscr{C}) \to \mathscr{D} \qquad \text{and a natural transformation} \qquad b^F_\bullet : F \Rightarrow \mathbf{R}F \circ \gamma}$$

such that for every functor $G : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$ and every natural transformation $\eta_\bullet : F \Rightarrow G \circ \gamma$ there exists a unique natural transformation $g_\bullet : \mathbf{R}F \Rightarrow G$ with $\eta_\bullet = (g_\bullet \star \gamma) \circ b^F_\bullet$.

Remark 3.4.2. (i) Clearly $(\mathbf{L}F, a^F_\bullet)$ is a left derived functor of $F \Leftrightarrow ((\mathbf{L}F)^{\mathrm{op}}, (a^F_\bullet)^{\mathrm{op}})$ is a right derived functor of $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$, where $\mathscr{C}^{\mathrm{op}}$ is endowed with its natural opposite model category structure (proposition 3.2.4(i)).

(ii) As usual, the left derived functor of F is determined up to unique isomorphism; *i.e.* if $(\mathbf{L}'F, a'_\bullet)$ is another left derived functor of F, there exists a unique isomorphism of functors $\omega_\bullet : \mathbf{L}F \xrightarrow{\sim} \mathbf{L}'F$ such that $a^F_\bullet = a'_\bullet \circ (\omega \star \gamma)$, and likewise for right derived functors.

(iii) *If F sends every weak equivalence of $\mathscr{C}$ to an isomorphism of $\mathscr{D}$*, then it factors through γ and a unique functor $\mathsf{ho}(F) : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$, and lemma 1.11.7(i) easily implies that $(\mathsf{ho}(F), \mathbf{1}_F)$ is both a left and right derived functor of F. Especially :

$$\boxed{\mathbf{L}\gamma = \mathbf{R}\gamma = \mathbf{1}_{\mathsf{ho}(\mathscr{C})}.}$$

(iv) Let $F, F' : \mathscr{C} \rightrightarrows \mathscr{D}$ be two functors that admit left derived functors $(\mathbf{L}F, a^F_\bullet)$ and respectively $(\mathbf{L}F', a^{F'}_\bullet)$, and let $\mu_\bullet : F \Rightarrow F'$ be a natural transformation. Then there exists a unique natural transformation

$$\boxed{\mathbf{L}\mu_\bullet : \mathbf{L}F \Rightarrow \mathbf{L}F' \qquad \text{such that} \qquad \mu_\bullet \circ a^F_\bullet = a^{F'}_\bullet \circ (\mathbf{L}\mu_\bullet \star \gamma).}$$

We call $\mathbf{L}\mu_\bullet$ *the left derived transformation of* $\mu_\bullet$. Dually, if F and F' admit right derived functors $(\mathbf{R}F, b^F_\bullet)$ and respectively $(\mathbf{R}F', b^{F'}_\bullet)$, then $\mu_\bullet$ induces a unique natural transformation

$$\boxed{\mathbf{R}\mu_\bullet : \mathbf{R}F \Rightarrow \mathbf{R}F' \qquad \text{such that} \qquad b^{F'}_\bullet \circ \mu_\bullet = (\mathbf{R}\mu_\bullet \star \gamma) \circ b^F_\bullet.}$$

We call $\mathbf{R}\mu_\bullet$ *the right derived transformation of* $\mu_\bullet$.

3.4.3. Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $F : \mathscr{C} \to \mathscr{D}$ *a functor that sends trivial cofibrations between cofibrant objects of* $\mathscr{C}$ *to isomorphisms of* $\mathscr{D}$. By Ken Brown's lemma (proposition 3.2.8), F then also sends all elements of $\mathscr{W}_c$ to isomorphisms of $\mathscr{D}$, so we get unique functors $\overline{F}_c$ and $\mathsf{ho}(F_c)$ fitting into a commutative diagram :

$$\begin{tikzcd} \mathscr{C}_c \arrow[r, "\pi_c"] \arrow[dr, "F_c"'] & \overline{\mathscr{C}_c} \arrow[r, "\overline{\gamma}_c"] \arrow[d, "\overline{F}_c"] & \mathsf{ho}(\mathscr{C}_c) \arrow[dl, "\mathsf{ho}(F_c)"] \\ & \mathscr{D} & \end{tikzcd}$$

where F_c is the restriction of F, and π_c and $\overline{\gamma}_c$ are as in remark 3.3.7(ii). We set

$$\boxed{\mathbf{L}F := \mathsf{ho}(F_c) \circ \overline{(-)}_c : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}}$$

where $\overline{(-)}_c : \mathsf{ho}(\mathscr{C}) \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}_c)$ is the equivalence of categories of remark 3.3.7(iii). Explicitly, $\mathbf{L}F$ is given by the rules : $X \mapsto FX_c$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and $[g] \mapsto F(\alpha_{Y_c})^{-1} \circ F(g) \circ F(\alpha_{X_c})$ for every $[g] \in \mathsf{ho}(\mathscr{C})(X, Y) = [X_{cf}, Y_{cf}]$. Notice that :

$$\mathbf{L}F \circ \gamma = \mathsf{ho}(F_c) \circ \overline{(-)}_c \circ \gamma = \mathsf{ho}(F_c) \circ \overline{\gamma}_c \circ (-)_c = \overline{F}_c \circ (-)_c$$

so $\mathbf{L}F \circ \gamma : \mathscr{C} \to \mathscr{D}$ is given by the rules : $X \mapsto FX_c$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and $g \mapsto Fg_c$ for every $g \in \mathscr{C}(X, Y)$. We have therefore a natural transformation

$$\boxed{a^F_\bullet : \mathbf{L}F \circ \gamma \Rightarrow F \qquad X \mapsto a^F_X := (FX_c \xrightarrow{F\beta_X} FX).}$$

Proposition 3.4.4. *In the situation of §3.4.3, we have :*

(i) The pair $(\mathbf{L}F, a^F_\bullet)$ is a left derived functor of F.

(ii) More generally, for every functor $F' : \mathscr{D} \to \mathscr{D}'$, the pair $(F' \circ \mathbf{L}F, F' \star a^F_\bullet)$ is a left derived functor of $F' \circ F : \mathscr{C} \to \mathscr{D}'$.

Proof. (i): Let $G : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}$ be a functor, $\tau_\bullet : G \circ \gamma \Rightarrow F$ a natural transformation. Define as in the proof of lemma 1.11.7(i), the category $\mathscr{D}^{[1]}$ and the evaluation functors

$$\mathsf{e}_0, \mathsf{e}_1 : \mathscr{D}^{[1]} \rightrightarrows \mathscr{D}$$

and recall that the datum of $\tau_\bullet$ is equivalent to that of a functor

$$T : \mathscr{C} \to \mathscr{D}^{[1]} \qquad \text{such that} \qquad \mathsf{e}_0 \circ T = G \circ \gamma \quad \text{and} \quad \mathsf{e}_1 \circ T = F.$$

By assumption, the restriction $\mathscr{C}_c \to \mathscr{D}^{[1]}$ of T sends weak equivalences to isomorphisms of $\mathscr{D}^{[1]}$, so it factors through a functor

$$T_c : \mathsf{ho}(\mathscr{C}_c) \to \mathscr{D}^{[1]} \quad \text{with} \quad \mathsf{e}_0 \circ T_c = G \circ i_c \quad \text{and} \quad \mathsf{e}_1 \circ T_c = \mathsf{ho}(F_c)$$

(where i_c is as in remark 3.3.7(iii)). Indeed, both identities for $\mathsf{e}_j \circ T_c$ $(j = 0, 1)$ can be checked after composition with $\gamma_c : \mathscr{C}_c \to \mathsf{ho}(\mathscr{C}_c)$, and then they follow by a simple diagram chase. In turn, the composition $T_c \circ \overline{(-)_c} : \mathsf{ho}(\mathscr{C}) \to \mathscr{D}^{[1]}$ is equivalent to the datum of a natural transformation $\phi_\bullet : G \circ i_c \circ \overline{(-)_c} \Rightarrow \mathbf{L}F$, and we set

$$f_\bullet := \phi_\bullet \circ (G \star [\beta_\bullet])^{-1} : G \Rightarrow \mathbf{L}F.$$

Explicitly, for every $X \in \mathrm{Ob}(\mathscr{C})$, $f_X : GX \to \mathbf{L}FX$ is the unique morphism of $\mathscr{D}$ that makes commute the following diagram, whose left vertical arrow is an isomorphism :

$$(*) \qquad \begin{tikzcd} GX_c \arrow[r, "\tau_{X_c}"] \arrow[d, "G\gamma(\beta_X)"'] & \mathbf{L}FX = FX_c \arrow[d, "F\beta_X"] \\ GX \arrow[ur, "f_X"] \arrow[r, "\tau_X"] & FX. \end{tikzcd}$$

Since $a^F_X = F\beta_X$, we see that $a^F_\bullet \circ (f_\bullet \star \gamma) = \tau_\bullet$. It remains to check that $f_\bullet$ is the unique natural transformation $G \Rightarrow \mathbf{L}F$ verifying this identity; hence, let $g_\bullet : G \Rightarrow \mathbf{L}F$ be another natural transformation with $a^F_\bullet \circ (g_\bullet \star \gamma) = \tau_\bullet$. Then, recall that $\gamma(\beta_X) = [\mathbf{1}_{X_{cf}}]$ for the morphism $\beta_X : X_c \to X$ of $\mathscr{C}$; so $\mathbf{L}F\gamma(\beta_X) = \mathbf{1}_{FX_c}$, and we get the commutative diagram :

$$\begin{tikzcd} GX_c \arrow[r, "g_{X_c}"] \arrow[d, "G\gamma(\beta_X)"'] & \mathbf{L}FX_c = FX_c \arrow[r, "a^F_{X_c}"] \arrow[d, equal] & FX_c \arrow[d, "F\beta_X"] \\ GX \arrow[r, "g_X"] & \mathbf{L}FX = FX_c \arrow[r, "a^F_X"] & FX. \end{tikzcd}$$

Moreover, $a^F_{X_c} = F\beta_{X_c}$ and $\beta_{X_c} = \mathbf{1}_{X_c}$, so $\tau_{X_c} = a^F_{X_c} \circ g_{X_c} = g_{X_c}$, and we conclude that diagram $(*)$ commutes also after replacing f_X by g_X, whence $g_\bullet = f_\bullet$.

(ii): A direct inspection shows that $F' \circ \mathbf{L}F = \mathbf{L}(F' \circ F)$ and $F' \star a^F_\bullet = a^{F'F}_\bullet$, so the assertion follows from (i). □

Remark 3.4.5. (i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $F : \mathscr{C} \to \mathscr{D}$ *a functor that sends trivial fibrations between fibrant objects of* $\mathscr{C}$ *to isomorphisms of* $\mathscr{D}$. Then, as in §3.4.3, we have unique functors $\overline{F}_f$ and $\mathrm{ho}(F_f)$ that make commute the diagram :

$$\begin{tikzcd}
\mathscr{C}_f \arrow[r, "\pi_f"] \arrow[dr, "F_f"'] & \overline{\mathscr{C}_f} \arrow[r, "\overline{\gamma}_f"] \arrow[d, "\overline{F}_f"] & \mathrm{ho}(\mathscr{C}_f) \arrow[dl, "\mathrm{ho}(F_f)"] \\
& \mathscr{D} &
\end{tikzcd}$$

where F_f denotes the restriction of F, and π_f and $\overline{\gamma}_f$ are as in remark 3.3.7(ii). So we set

$$\boxed{\mathbf{R}F := \mathrm{ho}(F_f) \circ \overline{(-)}_f : \mathrm{ho}(\mathscr{C}) \to \mathscr{D}}$$

and a direct computation shows that $\mathbf{R}F \circ \gamma = \overline{F}_f \circ (-)_f$, *i.e.* $\mathbf{R}F \circ \gamma : \mathscr{C} \to \mathscr{D}$ is given by the rules : $X \mapsto FX_f$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and $[g] \mapsto Fg_f$ for every $[g] \in \mathrm{ho}(\mathscr{C})(X, Y)$. Then, together with the natural transformation

$$\boxed{b^F_\bullet : F \Rightarrow \mathbf{R}F \circ \gamma \qquad X \mapsto b^F_X := (FX \xrightarrow{F\alpha_X} FX_f)}$$

we get, by the dual of proposition 3.4.4(i), a right derived functor $(\mathbf{R}F, b^F_\bullet)$ of F, arguing with the opposite model category $\mathscr{C}^{\mathrm{op}}$, and in view of remarks 1.11.5(iii) and 3.4.2(i).

(ii) Let $(\mathscr{C}', \mathscr{W}', \mathscr{F}ib', \mathscr{C}of')$ be another model category, and $F : \mathscr{C} \to \mathscr{C}'$ *a functor that sends trivial cofibrations between cofibrant objects to weak equivalences.* Then the composition of F with the localization $\gamma' : \mathscr{C}' \to \mathrm{ho}(\mathscr{C}')$ verifies the condition of §3.4.3, so it admits a left derived functor, that we denote by

$$\boxed{(\mathbb{L}F : \mathrm{ho}(\mathscr{C}) \to \mathrm{ho}(\mathscr{C}'), \alpha^F_\bullet := a^{\gamma' F}_\bullet : \mathbb{L}F \circ \gamma \Rightarrow \gamma' \circ F)}$$

and we call *the total left derived functor of* F. Hence, for every functor $G : \mathrm{ho}(\mathscr{C}) \to \mathrm{ho}(\mathscr{C}')$ and every natural transformation $\tau_\bullet : G \circ \gamma \Rightarrow \gamma' \circ F$, there exists a unique natural transformation $f_\bullet : G \Rightarrow \mathbb{L}F$ with $\alpha^F_\bullet \circ (f_\bullet \star \gamma) = \tau_\bullet$.

(iii) Let $\mathscr{C}$ and $\mathscr{C}'$ be as in (ii), and $F : \mathscr{C} \to \mathscr{C}'$ *a functor that sends trivial fibrations between fibrant objects to weak equivalences.* Combining (i) and (ii) we then see that $\gamma' \circ F : \mathscr{C} \to \mathrm{ho}(\mathscr{C}')$ admits a right derived functor, denoted by

$$\boxed{(\mathbb{R}F : \mathrm{ho}(\mathscr{C}) \to \mathrm{ho}(\mathscr{C}'), \beta^F_\bullet := b^{\gamma' F}_\bullet : \gamma' \circ F \Rightarrow \mathbb{R}F \circ \gamma)}$$

(notation of definition 3.4.1(ii)) which we call *the total right derived functor of* F.

Example 3.4.6. (i) In the situation of remark 3.4.5(ii), suppose that F *sends every weak equivalence* of $\mathscr{C}$ to a weak equivalence of $\mathscr{C}'$. Then $\gamma' \circ F : \mathscr{C} \to \mathrm{ho}(\mathscr{C}')$ factors through γ and a unique functor $\mathrm{ho}(F) : \mathrm{ho}(\mathscr{C}) \to \mathrm{ho}(\mathscr{C}')$, and by remark 3.4.2(iii), the pair $(\mathrm{ho}(F), \mathbf{1}_{\gamma' F})$ is both a left and right total derived functor of F.

(ii) Especially, $\mathbb{L}\mathbf{1}_{\mathscr{C}} = \mathbb{R}\mathbf{1}_{\mathscr{C}} = \mathbf{1}_{\mathrm{ho}(\mathscr{C})}$.

(iii) Keep the situation of remark 3.4.5(ii), and suppose that *the restriction* $F_c : \mathscr{C}_c \to \mathscr{C}'$ *of* F *reflects weak equivalences*; then $\mathbb{L}F$ is conservative. Indeed, let $[g] : X \to Y$ be any morphism of $\mathrm{ho}(\mathscr{C})$ such that $\mathbb{L}F([g])$ is an isomorphism of $\mathrm{ho}(\mathscr{C}')$; by definition, $[g]$ is the homotopy class of a morphism $g : X_{cf} \to Y_{cf}$ of $\mathscr{C}_{cf}$, and it follows easily that $\gamma F(g) : FX_{cf} \to FY_{cf}$ is an isomorphism of $\mathrm{ho}(\mathscr{C}')$, *i.e.* Fg is a weak equivalence of $\mathscr{C}'$ (theorem 3.3.9(ii)), so by assumption g is a weak equivalence of $\mathscr{C}$, and hence $[g]$ is an isomorphism of $\mathrm{ho}(\mathscr{C})$, whence the claim.

(iv) Dually, in the situation of remark 3.4.5(iii), if *the restriction* $F_f : \mathscr{C}_f \to \mathscr{C}'$ *reflects weak equivalences*, then $\mathbb{R}F$ is conservative.

Remark 3.4.7. (i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $F, F' : \mathscr{C} \rightrightarrows \mathscr{D}$ two functors that send trivial cofibrations between cofibrant objects to isomorphisms of $\mathscr{D}$, and $\mu_\bullet : F \Rightarrow F'$ a natural transformation. Then, with the notation of §3.4.3, we claim that :

$$\boxed{\mathbf{L}\mu_X = \mu_{X_c} \qquad \forall X \in \mathrm{Ob}(\mathscr{C}).}$$

Indeed, the proof of proposition 3.4.4(i) shows that $\mathbf{L}F(\beta_X) = \mathbf{1}_{FX_c}$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and that $\mathbf{L}\mu_X$ is the unique morphism of $\mathscr{D}$ that makes commute the diagram :

$$\begin{tikzcd} FX_c \arrow[rr, "\mu_{X_c}"] \arrow[d, equal] & & \mathbf{L}F'X = F'X_c \arrow[d, "F'\beta_X"] \\ FX_c \arrow[r, "F\beta_X"'] \arrow[urr, "\mathbf{L}\mu_X"] & FX \arrow[r, "\mu_X"'] & F'X. \end{tikzcd}$$

This morphism is then clearly μ_{X_c}.

(ii) Dually, if both F and F' send trivial fibrations between fibrant objects to isomorphisms of $\mathscr{D}$, then for every natural transformation $\mu_\bullet : F \Rightarrow F'$ we have :

$$\boxed{\mathbf{R}\mu_X = \mu_{X_f} \qquad \forall X \in \mathrm{Ob}(\mathscr{C}).}$$

(iii) Let also $(\mathscr{C}', \mathscr{W}', \mathscr{F}ib', \mathscr{C}of')$ be another model category, $F, F' : \mathscr{C} \rightrightarrows \mathscr{C}'$ two functors that send trivial cofibrations between cofibrant objects to weak equivalences, and $\mu_\bullet : F \Rightarrow F'$ a natural transformation. Then the total derived functors $\mathbb{L}F, \mathbb{L}F' : \mathsf{ho}(\mathscr{C}) \rightrightarrows \mathsf{ho}(\mathscr{C}')$ are both well defined, and we may form the left derived transformation of $\gamma' \star \mu_\bullet$, that we denote

$$\boxed{\mathbb{L}\mu_\bullet : \mathbb{L}F \Rightarrow \mathbb{L}F' \qquad \text{such that} \qquad (\gamma' \star \mu_\bullet) \circ a^F_\bullet = a^{F'}_\bullet \circ (\mathbb{L}\mu_\bullet \star \gamma)}$$

and we call *the total left derived transformation of* $\mu_\bullet$. In light of (i), we see that :

$$\mathbb{L}\mu_X = [(\mu_{X_c})_{cf}] \qquad \forall X \in \mathrm{Ob}(\mathsf{ho}(\mathscr{C})) = \mathrm{Ob}(\mathscr{C}).$$

(iv) Dually, if both F and F' send trivial fibrations between fibrant objects to weak equivalences, then the total right derived functors $\mathbb{R}F, \mathbb{R}F' : \mathsf{ho}(\mathscr{C}) \rightrightarrows \mathsf{ho}(\mathscr{C}')$ are well defined, and we may form the right derived transformation of $\gamma' \star \mu_\bullet$ that we denote also

$$\boxed{\mathbb{R}\mu_\bullet : \mathbb{R}F \Rightarrow \mathbb{R}F' \qquad \text{such that} \qquad b^{F'}_\bullet \circ (\gamma' \star \mu_\bullet) = (\mathbb{R}\mu_\bullet \star \gamma) \circ b^F_\bullet}$$

and we call *the total right derived transformation of* $\mu_\bullet$. In light of (iv) we get :

$$\mathbb{R}\mu_X = [(\mu_{X_f})_{cf}] \qquad \forall X \in \mathrm{Ob}(\mathsf{ho}(\mathscr{C})) = \mathrm{Ob}(\mathscr{C}).$$

Remark 3.4.8. (i) Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be model categories, and $\mathscr{C} \xrightarrow{F} \mathscr{C}' \xrightarrow{F'} \mathscr{C}''$ two functors. Suppose that F *preserves trivial cofibrations between cofibrant objects*, and that F' *sends trivial cofibrations between cofibrant objects to weak equivalences.* Then the total derived functors $\mathbb{L}F$, $\mathbb{L}F'$ and $\mathbb{L}(F' \circ F)$ are well defined, and there exists *a canonical natural transformation*

$$\boxed{\phi^{F,F'}_\bullet : \mathbb{L}F' \circ \mathbb{L}F \Rightarrow \mathbb{L}(F' \circ F).}$$

Indeed, by definition we have (non-commutative!) squares :

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathscr{C}' \arrow[r, "F'"] \arrow[d, "\gamma'"'] & \mathscr{C}'' \arrow[d, "\gamma''"] \\ \mathsf{ho}(\mathscr{C}) \arrow[r, "\mathbb{L}F"'] \arrow[ur, phantom, "\alpha^F_\bullet \Uparrow"] & \mathsf{ho}(\mathscr{C}') \arrow[r, "\mathbb{L}F'"'] \arrow[ur, phantom, "\alpha^{F'}_\bullet \Uparrow"] & \mathsf{ho}(\mathscr{C}'') \end{tikzcd}$$

and we may then compose $\alpha^F_\bullet$ and $\alpha^{F'}_\bullet$ to get a natural transformation

$$\alpha^{F,F'}_\bullet := (\alpha^{F'}_\bullet \star F) \circ (\mathbb{L}F' \star \alpha^F_\bullet) : \mathbb{L}F' \circ \mathbb{L}F \circ \gamma \Rightarrow \gamma'' \circ F' \circ F.$$

Then, $\phi^{F,F'}_\bullet$ is the unique natural transformation such that

$$\alpha^{F'\circ F}_\bullet \circ (\phi^{F,F'}_\bullet \star \gamma) = \alpha^{F,F'}_\bullet.$$

(ii) Dually, if F *preserves trivial fibrations between fibrant objects*, and if F' *sends trivial fibrations between fibrant objects to weak equivalences,* then there exists a *canonical natural transformation*

$$\boxed{\psi^{F,F'}_\bullet : \mathbb{R}(F' \circ F) \Rightarrow \mathbb{R}F' \circ \mathbb{R}F}$$

characterized by a corresponding uniqueness condition.

Proposition 3.4.9. *Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be three model categories, and $\mathscr{C} \xrightarrow{F} \mathscr{C}' \xrightarrow{F'} \mathscr{C}''$ two functors. The following holds :*

(i) If F preserves cofibrant objects, and both F and F' send trivial cofibrations between cofibrant objects to weak equivalences, then the canonical natural transformation of remark 3.4.8(i) is an isomorphism

$$\boxed{\mathbb{L}F' \circ \mathbb{L}F \xrightarrow{\sim} \mathbb{L}(F' \circ F).}$$

(ii) Dually, if F preserves fibrant objects, and both F and F' send trivial fibrations between fibrant objects to weak equivalences, the canonical natural transformation is an isomorphism

$$\boxed{\mathbb{R}(F' \circ F) \xrightarrow{\sim} \mathbb{R}F' \circ \mathbb{R}F.}$$

Proof. Assertion (ii) follows from (i), by virtue of remark 3.4.2(i).

(i): Clearly the natural transformation $\phi^{F,F'}_\bullet$ depends on the choices of left derived functors for F, F' and $F'F$, but it is easily seen that if replace a given set of choices by a different one, the correponding canonical natural transformation is altered by composition with some isomorphisms; hence, its categorical properties are *intrinsic :* in particular $\phi^{F,F'}_\bullet$ is an isomorphism of functors for a given set of choices if and only if the same holds for the canonical natural transformation corresponding to any other such set of choices. Hence, we may suppose that the total derived functors $\mathbb{L}F$, $\mathbb{L}F'$ and $\mathbb{L}(F' \circ F)$ are calculated as in §3.4.3, after fixing cofibrant replacements :

$$(\beta_X : X_c \to X \mid X \in \mathrm{Ob}(\mathscr{C})) \qquad \text{and} \qquad (\beta'_Y : Y_c \to Y \mid Y \in \mathrm{Ob}(\mathscr{C}')).$$

Recall also that we take $\beta_X = \mathbf{1}_X$ if X is cofibrant, and likewise for β'_Y. Let moreover

$$\gamma : \mathscr{C} \to \mathsf{ho}(\mathscr{C}) \qquad \gamma' : \mathscr{C}' \to \mathsf{ho}(\mathscr{C}') \qquad \gamma'' : \mathscr{C}'' \to \mathsf{ho}(\mathscr{C}'')$$

be the respective localizations; then $\alpha^F_X = \gamma'(F\beta_X) : F(X_c) \to FX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and $\mathbb{L}F'(\alpha^F_X) = \gamma'' F'((F\beta_X)_c) : F'F(X_c) \to F'((FX)_c)$, where $(F\beta_X)_c : (FX_c)_c \to (FX)_c$

makes commute the diagram :

$$
\begin{tikzcd}
(FX_c)_c \arrow[r, "(F\beta_X)_c"] \arrow[d, "\beta'_{F(X_c)}"'] & (FX)_c \arrow[d, "\beta'_{FX}"] \\
F(X_c) \arrow[r, "F\beta_X"] & FX.
\end{tikzcd}
$$

Likewise, $\alpha^{F'}_{FX} = \gamma'' F' \beta'_{FX} : \mathbb{L}F'(FX) \to F'FX$, with $\mathbb{L}F'(FX) = F'((FX)_c)$, and $\alpha^{F'F}_X = \gamma'' F' F(\beta_X) : F'F(X_c) \to F'FX$. Hence :

$$\alpha^{F,F'}_X = \gamma'' F'(\beta'_{FX} \circ (F\beta_X)_c) : (FX_c)_c \to FX \qquad \forall X \in \mathrm{Ob}(\mathscr{C})$$

and finally :

$$\phi^{F,F'}_X = \gamma'' F'(\beta'_{F(X_c)}) : F'((FX_c)_c) \to F'F(X_c) \qquad \forall X \in \mathrm{Ob}(\mathscr{C}).$$

However, under our assumptions, $F(X_c)$ is a cofibrant object of $\mathscr{C}'$ for every $X \in \mathrm{Ob}(\mathscr{C})$, hence $\beta'_{F(X_C)} = \mathbf{1}_{F(X_c)}$, so $\phi^{F,F'}_\bullet$ is the identity automorphism of $\mathbb{L}F' \circ \mathbb{L}F = \mathbb{L}(F' \circ F)$. □

Definition 3.4.10. (i) Let $\mathscr{C}$ and $\mathscr{C}'$ be two model categories. A *Quillen adjunction* is an adjoint pair of functors

$$F : \mathscr{C} \rightleftarrows \mathscr{C}' : G$$

(so F is left adjoint to G) with F preserving cofibrations and G preserving fibrations.

(ii) A *left* (resp. *right*) *Quillen functor* is a functor F (resp. G) admitting a right adjoint G (resp. a left adjoint F), such that (F, G) is a Quillen adjunction.

Remark 3.4.11. (i) Let $\mathscr{C}$ and $\mathscr{C}'$ be two model categories, and $(F : \mathscr{C} \rightleftarrows \mathscr{C}' : G)$ a Quillen adjunction. Endow the opposite categories $\mathscr{C}^{\mathrm{op}}$ and $\mathscr{C}'^{\mathrm{op}}$ with the induced model category structures provided by proposition 3.2.4(i); then it is clear that the induced pair $(G^{\mathrm{op}}, F^{\mathrm{op}})$ is again a Quillen adjunction.

(ii) For every $X \in \mathrm{Ob}(\mathscr{C})$ and $Y \in \mathrm{Ob}(\mathscr{C}')$ endow $\mathscr{C}/X$, $X/\mathscr{C}$, $\mathscr{C}'/Y$ and $Y/\mathscr{C}'$ with the induced model category structures of proposition 3.2.4(ii,iii); then, by the following lemma 3.4.12(i), the Quillen adjunction (F, G) induces Quillen adjunctions :

$$
\begin{aligned}
&F_{/Y} : \mathscr{C}/GY \rightleftarrows \mathscr{C}'/Y : G_{/Y} && {}_{X/}F : X/\mathscr{C} \rightleftarrows FX/\mathscr{C}' : {}_{X/}G \\
&F_{/X} : \mathscr{C}/X \rightleftarrows \mathscr{C}'/FX : G_{/X} && {}_{Y/}F : GY/\mathscr{C} \rightleftarrows Y/\mathscr{C}' : {}_{Y/}G
\end{aligned}
$$

given by the constructions of §1.4.8 and §1.4.10.

Lemma 3.4.12. *(i) Let $\mathscr{C}$ and $\mathscr{C}'$ be two model categories, and $(F : \mathscr{C} \rightleftarrows \mathscr{C}' : G)$ an adjoint pair of functors. The following conditions are equivalent :*

(a) (F, G) is a Quillen adjunction.

(b) F preserves cofibrations and trivial cofibrations.

(c) G preserves fibrations and trivial fibrations.

(ii) In particular, for every Quillen adjunction (F, G), the functor F admits a total left derived functor, and G admits a total right derived functor. Moreover :

$$F(\mathscr{C}_c) \subset \mathscr{C}'_c \qquad \textit{and} \qquad G(\mathscr{C}'_f) \subset \mathscr{C}_f.$$

(iii) Furthermore, if (F, G) is a Quillen adjunction, we have :

(a) F preserves weak equivalences between cofibrant objects.

(b) G preserves weak equivalences between fibrant objects.

Proof. (i) follows by applying proposition 3.1.19 to the adjoint pair (F, G) and both weak factorization systems $(\mathscr{W} \cap \mathscr{C}of, \mathscr{F}ib)$ and $(\mathscr{C}of, \mathscr{W} \cap \mathscr{F}ib)$.

(ii): The first assertion is an immediate consequence of (i). For the stated inclusions, it suffices to recall that F preserves initial objects, and G preserves final objects (see [13, Rem.2.27(i,ii) and Prop.2.49]), and apply again (i).

(iii): Assertion (a) follows from (i,ii) and Ken Brown's lemma (proposition 3.2.8); then assertion (b) follows from (a), after passing to the induced model category structures on the opposite categories (proposition 3.2.4(i)) and considering the Quillen adjunction $(G^{\mathrm{op}}, F^{\mathrm{op}})$ of remark 3.4.11(i). □

Theorem 3.4.13. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two model categories; every Quillen adjunction $(F : \mathscr{C} \rightleftarrows \mathscr{C}' : G)$ induces an adjoint pair of functors :*

$$\boxed{\mathbb{L}F : \mathsf{ho}(\mathscr{C}) \rightleftarrows \mathsf{ho}(\mathscr{C}') : \mathbb{R}G.}$$

Proof. To begin with, let us observe :

Claim 3.4.14. Let $X \in \mathrm{Ob}(\mathscr{C}_c)$, $Y \in \mathrm{Ob}(\mathscr{C}'_f)$; consider a cylinder and a cocylinder:

$$X \sqcup X \xrightarrow{(\partial_0,\partial_1)} IX \xrightarrow{\sigma} X \qquad Y \xrightarrow{s} Y^I \xrightarrow{(d_0,d_1)} Y \times Y.$$

Then the induced diagrams :

$$FX \sqcup FX \xrightarrow{(F\partial_0,F\partial_1)} F(IX) \xrightarrow{F\sigma} FX \qquad GY \xrightarrow{Gs} G(Y^I) \xrightarrow{(Gd_0,Gd_1)} GY \times GY$$

are again a cylinder for FX and a cocylinder for GY, respectively.

Proof : Evidently, $(F\partial_0, F\partial_1) := F(\partial_0, \partial_1)$, and likewise for (Gd_0, Gd_1), so the morphism $(F\partial_0, F\partial_1)$ is a cofibration, and (Gd_0, Gd_1) is a fibration. Also, $IX \in \mathrm{Ob}(\mathscr{C}_c)$ and $Y^I \in \mathrm{Ob}(\mathscr{C}'_f)$ by remark 3.2.10(iii) and its dual; then, by Ken Brown's lemma (proposition 3.2.8), $F\sigma$ and Gs are weak equivalences, whence the claim. ◇

Claim 3.4.15. The adjunction ϑ for (F, G) induces natural bijections :

$$[FX, Y] \xrightarrow{\sim} [X, GY] \qquad \forall X \in \mathrm{Ob}(\mathscr{C}_c), \forall Y \in \mathrm{Ob}(\mathscr{C}'_f).$$

Proof : Let $f, g : FX \rightrightarrows Y$ be two morphisms of $\mathscr{C}'$, and suppose that $f \overset{r}{\sim} g$, so that we have a cocylinder $Y \xrightarrow{s} Y^I \xrightarrow{(d_0,d_1)} Y \times Y$ for Y and a morphism $k : FX \to Y^I$ with $d_0 k = f$ and $d_1 k = g$. By adjunction, there follows a commutative diagram :

$$\begin{tikzcd} & & GY \\ X \arrow[r, "\vartheta(k)"] \arrow[urr, bend left, "\vartheta(f)"] \arrow[drr, bend right, "\vartheta(g)"'] & GY^I \arrow[ur, "Gd_0"'] \arrow[dr, "Gd_1"] & \\ & & GY \end{tikzcd}$$

showing, in light of claim 3.4.14, that $\vartheta(k)$ is a right homotopy from $\vartheta(f)$ to $\vartheta(g)$; especially, $\vartheta(f) \overset{r}{\sim} \vartheta(g)$. Likewise we check that for every $f', g' \in \mathscr{C}'(X, GY)$ with $f' \overset{l}{\sim} g'$ we have $\vartheta^{-1}(f') \overset{l}{\sim} \vartheta^{-1}(g')$, whence the claim, with lemmata 3.2.11(iii) and 3.4.12(ii). ◇

Recall now that $\mathbb{L}F$ and $\mathbb{R}G$ are respectively the compositions :

$$\mathsf{ho}(\mathscr{C}) \xrightarrow{\overline{(-)}_c} \mathsf{ho}(\mathscr{C}_c) \xrightarrow{\mathsf{ho}(\gamma' F_c)} \mathsf{ho}(\mathscr{C}') \qquad \mathsf{ho}(\mathscr{C}') \xrightarrow{\overline{(-)}_f} \mathsf{ho}(\mathscr{C}'_f) \xrightarrow{\mathsf{ho}(\gamma G_f)} \mathsf{ho}(\mathscr{C})$$

with $\gamma : \mathscr{C} \to \mathsf{ho}(\mathscr{C})$, $\gamma' : \mathscr{C}' \to \mathsf{ho}(\mathscr{C}')$ the localizations, and $F_c : \mathscr{C}_c \to \mathscr{C}'$, $G_f : \mathscr{C}'_f \to \mathscr{C}$ the restrictions of F and G. Hence, for every $X \in \mathrm{Ob}(\mathscr{C})$ and $Y \in \mathrm{Ob}(\mathscr{C}')$ we have :

$$\mathsf{ho}(\mathscr{C}')(\mathbb{L}FX, Y) = [(FX_c)_f, Y_{cf}] \qquad \mathsf{ho}(\mathscr{C})(X, \mathbb{R}GY) = [X_{cf}, (GY_f)_c].$$

Let now $(X_c \xrightarrow{\beta_X} X \xrightarrow{\alpha_X} X_f \mid X \in \mathrm{Ob}(\mathscr{C}))$ and $(Y_c \xrightarrow{\beta'_Y} Y \xrightarrow{\alpha'_Y} Y_f \mid Y \in \mathrm{Ob}(\mathscr{C}'))$ be our chosen fibrant and cofibrant replacements (see §3.3.1); by proposition 3.2.14, the morphisms $\alpha'_{FX_c} : FX_c \to (FX_c)_f$, $\beta_{GY_f} : (GY_f)_c \to GY_f$ and $\alpha_{X_c} : X_c \to X_{cf}$ induce bijections :

$$[(FX_c)_f, Y_{cf}] \xrightarrow{\sim} [FX_c, Y_{cf}] \qquad [X_{cf}, (GY_f)_c] \xrightarrow{\sim} [X_c, (GY_f)_c] \xrightarrow{\sim} [X_c, GY_f].$$

Also, the morphism $\beta'_{Y_f} : Y_{fc} \to Y_f$ and the morphism $\tau_Y : Y_{cf} \to Y_{fc}$ of remark 3.3.3(iii) yield bijections :

$$[FX_c, Y_{cf}] \xrightarrow{\sim} [FX_c, Y_{fc}] \xrightarrow{\sim} [FX_c, Y_f].$$

By combining these bijections with claim 3.4.15, we obtain bijections :

$$\mathsf{ho}(\mathscr{C}')(\mathbb{L}FX, Y) \xrightarrow{\sim} \mathsf{ho}(\mathscr{C})(X, \mathbb{R}GY) \qquad \forall X \in \mathrm{Ob}(\mathscr{C}), \forall Y \in \mathrm{Ob}(\mathscr{C}')$$

and it is easily checked that these bijections are natural in X and Y. ◻

Remark 3.4.16. (i) The proof of theorem 3.4.13 associates with every given adjunction $\vartheta_{\bullet\bullet}$ for the pair (F, G) an explicit adjunction $\vartheta^*_{\bullet\bullet}$ for the derived pair $(\mathbb{L}F, \mathbb{R}G)$; namely, for given $\phi \in [(FX_c)_f, Y_{fc}]$ and $\psi \in [X_{cf}, (GY_f)_c]$ we have :

$$\psi = \vartheta^*_{X,Y}(\phi) \qquad \Leftrightarrow \qquad \beta_{GY_f} \circ \psi \circ \alpha_{X_c} = \vartheta_{X_c, Y_f}(\beta'_{Y_f} \circ \tau_Y \circ \phi \circ \alpha'_{FX_c}).$$

where $(X_c \xrightarrow{\beta_X} X \xrightarrow{\alpha_X} X_f \mid X \in \mathrm{Ob}(\mathscr{C}))$ and $(Y_c \xrightarrow{\beta'_Y} Y \xrightarrow{\alpha'_Y} Y_f \mid Y \in \mathrm{Ob}(\mathscr{C}'))$ are the fibrant and cofibrant replacements, and $(\tau_Y : Y_{cf} \to Y_{fc} \mid Y \in \mathrm{Ob}(\mathscr{C}'))$ is the induced system of weak equivalences, as in remark 3.3.3(iii).

(ii) The unit of $\vartheta^*_{\bullet\bullet}$ is given by the morphism $[\eta^*_X] := \vartheta^*_{X,FX_c}(\mathbf{1}_{FX_c}) \in [X_{cf}, G((FX_c)_f)_c]$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Recall also that $\beta_{(FX_c)_f} = \mathbf{1}_{(FX_c)_f}$ and $[\tau_{FX_c}] = [\mathbf{1}_{FX_c}]$ (remark 3.3.3(iv)); in light of (i), $[\eta^*_X]$ is therefore characterized as the homotopy class of any morphism η^*_X that makes commute the following diagram of $\mathscr{C}$:

$$\begin{tikzcd}
X_c \arrow[rr, "\alpha_{X_c}"] \arrow[d, "\eta_{X_c}"'] & & X_{cf} \arrow[d, "\eta^*_X"] \\
GFX_c \arrow[r, "G(\alpha'_{FX_c})"] & G((FX_c)_f) & G((FX_c)_f)_c \arrow[l, "\beta_{G((FX_c)_f)}"']
\end{tikzcd}$$

where η_{X_c} denotes the unit of $\vartheta_{\bullet\bullet}$.

Corollary 3.4.17. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two model categories, $(F : \mathscr{C} \rightleftarrows \mathscr{C}' : G)$ and $(F' : \mathscr{C} \rightleftarrows \mathscr{C}' : G')$ two Quillen adjunctions, and $\mu_\bullet : F \Rightarrow F'$ a natural transformation. Then :*

$$\boxed{(\mathbb{L}\mu_\bullet)^\vee = \mathbb{R}\mu^\vee_\bullet.}$$

Proof. Here $\mathbb{L}\mu_\bullet$ denotes the total left derived transformation of $\mu_\bullet$, and $\mathbb{R}\mu^\vee_\bullet$ denotes the total right derived transformation of the adjoint $\mu^\vee_\bullet$ of $\mu_\bullet$ (see §1.6.10 and remark 3.4.7(iii,iv)); likewise, $(\mathbb{L}\mu_\bullet)^\vee$ denotes the adjoint of $\mathbb{L}\mu_\bullet$. The construction of $\mu^\vee_\bullet$ depends on the choice of adjunctions $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$ for the pair (F, G) and respectively (F', G'), and similarly for the construction of $(\mathbb{L}\mu_\bullet)^\vee$; hence, the stated equality is meant to hold when we fix such adjunctions $\vartheta_{\bullet\bullet}$ and $\vartheta'_{\bullet\bullet}$, *and moreover we choose for the adjoint pairs* $(\mathbb{L}F, \mathbb{R}G)$ *and* $(\mathbb{L}F', \mathbb{R}G')$ *the induced adjunctions* $\vartheta^*_{\bullet\bullet}$ *and respectively* $\vartheta'^*_{\bullet\bullet}$ *explicited in remark* 3.4.16.

However, even for arbitrarily chosen adjunctions, the stated equality will at least hold up to composition with suitable isomorphisms of functors.

The sought equality comes down to the identities (see §1.6.10) :

$$\vartheta^*_{X,Y}(f \circ \mathbb{L}\mu_X) = \mathbb{R}\mu^\vee_Y \circ \vartheta'^*_{X,Y}(f) \qquad \forall(X,Y) \in \mathrm{Ob}(\mathscr{C} \times \mathscr{C}'), \forall f \in \mathsf{ho}(\mathscr{C}')(F'X,Y)$$

which in turn, by remarks 3.4.7(iii,iv) and 3.4.16, is equivalent to the identities :

$$\beta_{GY_f} \circ [(\mu^\vee_{Y_f})_c] \circ \vartheta'^*_{X,Y}(f) \circ \alpha_{X_c} = \vartheta_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ [(\mu_{X_c})_f] \circ \alpha'_{FX_c}).$$

But notice that

$$\beta_{GY_f} \circ [(\mu^\vee_{Y_f})_c] = [\mu^\vee_{Y_f}] \circ \beta_{G'Y_f} \quad \text{and} \quad \beta_{G'Y_f} \circ \vartheta'^*_{X,Y}(f) \circ \alpha_{X_c} = \vartheta'_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ \alpha'_{F'X_c}).$$

Hence, we are reduced to showing that :

$$[\mu^\vee_{Y_f}] \circ \vartheta'_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ \alpha'_{F'X_c}) = \vartheta_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ [(\mu_{X_c})_f] \circ \alpha'_{FX_c}).$$

But we have : $[\mu^\vee_{Y_f}] \circ \vartheta'_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ \alpha'_{F'X_c}) = \vartheta_{X_c,Y_f}(\beta'_{Y_f} \circ \tau_Y \circ f \circ \alpha'_{F'X_c} \circ \mu_{X_c})$, and on the other hand, $[(\mu_{X_c})_f] \circ \alpha'_{FX_c} = \alpha'_{F'X_c} \circ \mu_{X_c}$, whence the sought equality. □

Corollary 3.4.18. *Let $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ be three model categories, and*

$$F : \mathscr{C} \rightleftarrows \mathscr{C}' : G \qquad F' : \mathscr{C}' \rightleftarrows \mathscr{C}'' : G'$$

two Quillen adjunctions. Then $(F'F : \mathscr{C} \rightleftarrows \mathscr{C}'' : GG')$ is a Quillen adjunction, and we have natural isomorphisms of functors :

$$\boxed{\mathbb{L}F' \circ \mathbb{L}F \xrightarrow{\sim} \mathbb{L}(F' \circ F) \qquad \mathbb{R}(G \circ G') \xrightarrow{\sim} \mathbb{R}G \circ \mathbb{R}G'.}$$

Proof. The first assertion is obvious from the definitions (see [13, Exerc.2.17(i)]). Next, we already know both F and F' preserve cofibrant objects, and both G and G' preserve fibrant objects (lemma 3.4.12(ii)). Then the sought natural isomorphisms follow from lemma 3.4.12 and proposition 3.4.9. □

Example 3.4.19. Let $(\mathscr{C}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, $x : X \to X'$ a morphism of $\mathscr{C}$, and endow $X/\mathscr{C}$ and $X'/\mathscr{C}$ with the model structures provided by proposition 3.2.4(iii). According to remark 1.4.2(ii), the morphism x induces an adjoint pair of functors

$$x^! : X/\mathscr{C} \rightleftarrows X'/\mathscr{C} : x_!$$

and obviously $x_!$ both preserves and reflects fibrations, cofibrations and weak equivalences, so $(x^!, x_!)$ is a Quillen adjunction (lemma 3.4.12(i)), and $\mathbb{R}x_! = \mathsf{ho}(x_!)$ is conservative (example 3.4.6(i,iv)). Moreover, by example 3.3.4, any choice of fibrant replacements $\alpha_\bullet := (\alpha_Y : Y \to Y_f \mid Y \in \mathrm{Ob}(\mathscr{C}))$ for $\mathscr{C}$ induces systems of fibrant replacements $X/\alpha_\bullet$ and $X'/\alpha_\bullet$ for $X/\mathscr{C}$ and respectively $X'/\mathscr{C}$; with these choices, it is also clear that $x_!$ preserves fibrant replacements. Let $\eta_\bullet : \mathbf{1}_{X/\mathscr{C}} \Rightarrow x_! x^!$ be the unit of the adjunction for $(x^!, x_!)$; for every $(Y, g : X \to Y) \in \mathrm{Ob}(X/\mathscr{C})$, the object $x_!(x^!((Y,g)_c)_f) = (x_! x^!((Y,g)_c))_f$ is cofibrant in $X/\mathscr{C}$, so remark 3.4.16(ii) characterizes the unit $[\eta^*_{(Y,g)}] : (Y,g) \to \mathbb{R}x_! \mathbb{L}x^!(Y,g)$ of the derived pair $(\mathbb{L}x^!, \mathbb{R}x_!)$ as the homotopy class of any morphism $\eta^*_{(Y,g)} : (Y,g)_{cf} \to (x_! x^!((Y,g)_c))_f$ making commute the diagram:

$$\begin{tikzcd}
(Y,g)_c \arrow[r, "\alpha_{(Y,g)}"] \arrow[d, "\eta_{(Y,g)_c}"'] & (Y,g)_{cf} \arrow[d, "\eta^*_{(Y,g)}"] \\
x_! x^!((Y,g)_c) \arrow[r, "\alpha_{x_! x^!(Y,g)_c}"] & (x_!(x^!((Y,g)_c))_f.
\end{tikzcd}$$

In other words, we have :

$$[\eta^*_{(Y,g)}] = [(\eta_{(Y,g)_c})_f] \qquad \forall (Y,g) \in \mathrm{Ob}(X/\mathscr{C}).$$

3.5. **Homotopy limits and homotopy colimits.** Examples of useful derived functors are provided by the total derived functor of basic categorical operations, such as limits and colimits. Although we will not consider the general case here, we will need special cases which can be dealt with elementarily.

3.5.1. *Injective and projective model structures.* Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and I a small category; according to §1.3.2, since $\mathscr{C}$ is finitely complete and finitely cocomplete, the same holds for $\mathscr{C}^I$. We let $\mathscr{W}^I$ (resp. $\mathscr{F}ib^I$) be the subclass of $\mathrm{Mor}(\mathscr{C}^I)$ consisting of all natural transformations $\phi_\bullet : F_\bullet \to G_\bullet$ such that $\phi_i \in \mathscr{W}$ (resp. $\phi_i \in \mathscr{F}ib$) for every $i \in \mathrm{Ob}(I)$. Then, we also set : $\mathscr{C}of^I := l(\mathscr{W}^I \cap \mathscr{F}ib^I)$ (notation of definition 3.1.1(iii)). In general, it is not clear whether the datum

$$(\mathscr{C}^I, \mathscr{W}^I, \mathscr{F}ib^I, \mathscr{C}of^I)$$

is a model category, but when it is, we call it *the projective model structure on* $\mathscr{C}^I$.

- Dually, *the injective model structure* will be the projective model structure on

$$\mathscr{C}_I := (\mathscr{C}^{\mathrm{op}})^{I^{\mathrm{op}}}$$

when the latter is well defined (where $\mathscr{C}^{\mathrm{op}}$ is endowed with the opposite model structure of proposition 3.2.4(i)). Hence, the weak equivalences (resp. the cofibrations) of the injective model structure on $\mathscr{C}_I$ are given by $\mathscr{W}_I := (\mathscr{W}^I)^{\mathrm{op}}$ (resp. by $\mathscr{C}of_I$, defined as the class of all natural transformations $\phi_\bullet : F_\bullet \to G_\bullet$ between functors $F, G : I \to \mathscr{C}$ such that $\phi_i \in \mathscr{C}of$ for every $i \in \mathrm{Ob}(I)$).

Remark 3.5.2. (i) In the situation of §3.5.1, since $\mathscr{W}_I = (\mathscr{W}^I)^{\mathrm{op}}$, the natural isomorphism $(\mathscr{C}^I)^{\mathrm{op}} \xrightarrow{\sim} \mathscr{C}_I$ (see §1.3) induces, by remark 1.11.5(iii), an isomorphism :

$$\boxed{\mathsf{ho}(\mathscr{C}^I)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{ho}(\mathscr{C}_I).}$$

(ii) Let I, J be two small categories, and suppose that the projective model structures on $\mathscr{C}^I$ and $(\mathscr{C}^I)^J$ are well defined; notice that the natural isomorphism

$$(*) \qquad (\mathscr{C}^I)^J \xrightarrow{\sim} \mathscr{C}^{I\times J}$$

of §1.3.6 identifies $(\mathscr{W}^I)^J$ and $(\mathscr{F}ib^I)^J$ with $\mathscr{W}^{I\times J}$ and respectively $\mathscr{F}ib^{I\times J}$. Hence, the projective model structure on $\mathscr{C}^{I\times J}$ is well defined, and is identified via $(*)$ with the projective model structure on $(\mathscr{C}^I)^J$. Likewise, if the injective model structures on $\mathscr{C}_I$ and $(\mathscr{C}_I)_J$ are well defined, the same holds for that of $\mathscr{C}_{I\times J}$, and $(*)$ identifies it with the injective model structure on $(\mathscr{C}_I)_J$.

(iii) We will consider a few basic examples of small categories I such that the projective model structure exists for any model category $\mathscr{C}$. *E.g*, if I is a small discrete category, then the projective model structure exists and coincides with the product of model categories as in example 3.2.3(ii), *i.e.* the cofibrations are the natural transformations $\phi_\bullet : F_\bullet \to G_\bullet$ such that $\phi_i \in \mathscr{C}of$ for every $i \in \mathrm{Ob}(I)$; the axioms are simply verified termwise. The following three propositions exhibit some more examples.

Proposition 3.5.3. *(i) Let* $[1]$ *be the category attached to the ordered set* $\{0, 1\}$ *(see remark 1.9.3(iii)); then, for every model category* $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$*, the projective and injective model structures on* $\mathscr{C}^{[1]}$ *and respectively* $\mathscr{C}_{[1]}$ *are well defined.*

(ii) The cofibrant objects for the projective model structure on $\mathscr{C}^{[1]}$ are the cofibrations between cofibrant objects of $\mathscr{C}$.

Proof. (i): Since $[1]^{\mathrm{op}}$ is isomorphic to $[1]$, it suffices to prove the existence of the projective model structure. To this aim, for any pair $(X_0 \xrightarrow{x} X_1, Y_0 \xrightarrow{y} Y_1)$ of objects of $\mathscr{C}^{[1]}$ and any morphism $f_\bullet : x \to y$ in $\mathscr{C}^{[1]}$, *i.e.* any commutative square of $\mathscr{C}$:

$$(\dagger) \qquad \begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "x"'] & Y_0 \arrow[d, "y"] \\ X_1 \arrow[r, "f_1"] & Y_1 \end{tikzcd}$$

denote by $(f_1, y) : X_1 \sqcup_{X_0} Y_0 \to Y_1$ the induced morphism of $\mathscr{C}$. We let $\mathscr{I}$ be the class of all morphisms $f_\bullet : x \to y$ of $\mathscr{C}^{[1]}$ such that $f_0, (f_1, y) \in \mathscr{C}\!of$.

Claim 3.5.4. (i) $(\mathscr{I}, \mathscr{W}^{[1]} \cap \mathscr{F}ib^{[1]})$ is a weak factorization system for $\mathscr{C}^{[1]}$.

(ii) $\mathscr{I} = \mathscr{C}\!of^{[1]}$.

(iii) $\mathscr{W}^{[1]} \cap \mathscr{C}\!of^{[1]} = \mathscr{I}' := \{f_\bullet \in \mathscr{I} \mid f_0, (f_1, y) \in \mathscr{W}\} = l(\mathscr{F}ib^{[1]})$.

Proof : (i): Let $g_\bullet : (A_0 \xrightarrow{a} A_1) \to (B_0 \xrightarrow{b} B_1)$ be an element of $\mathscr{W}^{[1]} \cap \mathscr{F}ib^{[1]}$; a commutative square in $\mathscr{C}^{[1]}$ of the form :

$$(*) \qquad \begin{tikzcd} x \arrow[r, "h_\bullet"] \arrow[d, "f_\bullet"'] & a \arrow[d, "g_\bullet"] \\ y \arrow[r, "k_\bullet"] & b \end{tikzcd}$$

comes down to a commutative diagram of $\mathscr{C}$ of the form :

$$(\mathrm{D}) \qquad \begin{tikzcd} A_0 \arrow[rrr, "g_0"] \arrow[ddd, "a"'] & & & B_0 \arrow[ddd, "b"] \\ & X_0 \arrow[r, "f_0"] \arrow[d, "x"'] \arrow[ul, "h_0"'] & Y_0 \arrow[d, "y"] \arrow[ur, "k_0"] & \\ & X_1 \arrow[r, "f_1"] \arrow[dl, "h_1"] & Y_1 \arrow[dr, "k_1"'] & \\ A_1 \arrow[rrr, "g_1"] & & & B_1 \end{tikzcd}$$

and a diagonal filler $d_\bullet : y \to a$ for diagram $(*)$ amounts to a pair of morphisms of $\mathscr{C}$:

$$(d_i : Y_i \to A_i \mid i = 0, 1) \qquad \text{with} \qquad ad_0 = d_1 y \qquad d_i f_i = h_i \qquad g_i d_i = k_i \qquad (i = 0, 1).$$

Now, if $f_0 \in \mathscr{C}\!of$, then we may find a diagonal filler $d_0 : Y_0 \to A_0$ for the commutative square with sides f_0, k_0, g_0, h_0; since $ad_0 f_0 = ah_0 = h_1 x$, there follows a unique morphism

$$(h_1, ad_0) : X_1 \sqcup_{X_0} Y_0 \to A_1 \qquad \text{such that} \qquad k_1 \circ (f_1, y) = g_1 \circ (h_1, ad_0).$$

If $(f_1, y) \in \mathscr{C}\!of$ as well, the commutative square with sides $(h_1, ad_0), g_1, k_1, (f_1, y)$ admits a diagonal filler $d_1 : Y_1 \to A_1$, and it is easily seen that the resulting pair (d_0, d_1) is the sought diagonal filler $d_\bullet$, so $f_\bullet \in \mathscr{C}\!of^{[1]}$. Next, since $\mathscr{F}ib$ and $\mathscr{W}$ are stable under retracts (proposition 3.2.6), it is easily seen that the same holds for $\mathscr{F}ib^{[1]}$ and $\mathscr{W}^{[1]}$; to show that the same holds for $\mathscr{I}$, consider $(a \xrightarrow{g_\bullet} b) \in \mathscr{I}$ and a commutative diagram of $\mathscr{C}^{[1]}$:

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f_\bullet"'] \arrow[rr, bend left, "\mathbf{1}_x"] & a \arrow[r] \arrow[d, "g_\bullet"] & x \arrow[d, "f_\bullet"] \\ y \arrow[r] \arrow[rr, bend right, "\mathbf{1}_y"'] & b \arrow[r] & y. \end{tikzcd}$$

We deduce the commutative diagrams of $\mathscr{C}$:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] \arrow[rr, bend left, "\mathbf{1}_{X_0}"] & A_0 \arrow[r] \arrow[d, "g_0"] & X_0 \arrow[d, "f_0"] \\ Y_0 \arrow[r] \arrow[rr, bend right, "\mathbf{1}_{Y_0}"'] & B_0 \arrow[r] & Y_0 \end{tikzcd} \qquad \begin{tikzcd} X_1 \sqcup_{X_0} Y_0 \arrow[r] \arrow[d, "(f_1,y)"'] \arrow[rr, bend left, "\mathbf{1}_{X_1 \sqcup_{X_0} Y_0}"] & A_1 \sqcup_{A_0} B_0 \arrow[r] \arrow[d, "(g_1,b)"] & X_1 \sqcup_{X_0} Y_0 \arrow[d, "(f_1,y)"] \\ Y_1 \arrow[r] \arrow[rr, bend right, "\mathbf{1}_{Y_1}"'] & B_1 \arrow[r] & Y_1 \end{tikzcd}$$

where, by assumption, $g_0, (g_1, y) \in \mathscr{C}of$, so $f_0, (f_1, y) \in \mathscr{C}of$, by proposition 3.2.6.

It remains to check that every morphism (†) of $\mathscr{C}^{[1]}$ factors as $f_\bullet = p_\bullet \circ i_\bullet$ with $p_\bullet \in \mathscr{W}^{[1]} \cap \mathscr{F}ib^{[1]}$ and $i_\bullet \in \mathscr{I}$. To this aim, pick first a factorization of $f_0 : X_0 \to Y_0$ as a cofibration $i_0 : X_0 \to T_0$ followed by a trivial fibration $p_0 : T_0 \to Y_0$; then form the cocartesian square :

$$\begin{tikzcd} X_0 \arrow[r, "i_0"] \arrow[d, "x"'] & T_0 \arrow[d, "t'"] \\ X_1 \arrow[r, "i'_1"] & T'_1 \end{tikzcd}$$

and choose a factorization of the induced morphism $(f_1, yp_0) : T'_1 \to Y_1$ as a cofibration $j : T'_1 \to T_1$ followed by a trivial fibration $p_1 : T_1 \to Y_1$. Set $i_1 := ji'_1 : X_1 \to T_1$ and $t := jt' : T_0 \to T_1$; then $i_\bullet := (i_0, i_1) : x \to t$ lies in $\mathscr{C}of^{[1]}$, $p_\bullet := (p_0, p_1) : t \to y$ lies in $\mathscr{W}^{[1]} \cap \mathscr{F}ib^{[1]}$, and $f_\bullet = p_\bullet \circ i_\bullet$, as required.

(ii): This follows immediately from (i) and lemma 3.1.18(i).

(iii): *Mutatis mutandis,* the argument of (i) shows that $(\mathscr{I}', \mathscr{F}ib^{[1]})$ is a weak factorization system for $\mathscr{C}^{[1]}$, so $\mathscr{I}' = l(\mathscr{F}ib^{[1]})$, again by lemma 3.1.18(i); moreover, if $f_0 \in \mathscr{W} \cap \mathscr{C}of$, then the induced morphism $e : X_1 \to X_1 \sqcup_{X_0} Y_0$ lies in $\mathscr{W} \cap \mathscr{C}of$ as well (proposition 3.1.9(v)), and in this case, since $f_1 = (f_1, y) \circ e$, the 2-out-of-3 property implies that $f_1 \in \mathscr{W}$ if and only if $(f_1, y) \in \mathscr{W}$, whence the assertion. ◇

Claim 3.5.4 (and its proof) imply immediately (i).

(ii) follows straightforwardly from claim 3.5.4(ii). □

Proposition 3.5.5. *(i) Let* $\mathbb{V}$ *be the full subcategory of* $[1] \times [1]$ *with*

$$\mathrm{Ob}(\mathbb{V}) := \{(0,0), (0,1), (1,0)\}.$$

Then, for every model category $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$*, the projective model structure on* $\mathscr{C}^{\mathbb{V}}$ *and the injective model structure on* $\mathscr{C}_{\mathbb{V}^{\mathrm{op}}}$ *are well defined.*

(ii) The cofibrant objects of $\mathscr{C}^{\mathbb{V}}$ *are the diagrams* $x_\bullet := (X_1 \xleftarrow{x_1} X_0 \xrightarrow{x_2} X_2)$*, where* x_1 *and* x_2 *are cofibrations between cofibrant objects of* $\mathscr{C}$*.*

Proof. Again, it suffices to prove the existence of the projective model structure on $\mathscr{C}^{\mathbb{V}}$. Now, the inclusion $[0] \to [1]$ induces an obvious restriction functor

$$\rho : \mathscr{C}^{[1]} \to \mathscr{C} \qquad (X_0 \to X_1) \mapsto X_0$$

and we have a cartesian diagram of categories :

$$\begin{tikzcd} \mathscr{C}^{\mathbb{V}} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \mathscr{C}^{[1]} \arrow[d, "\rho"] \\ \mathscr{C}^{[1]} \arrow[r, "\rho"] & \mathscr{C} \end{tikzcd}$$

where $\pi_i(x_\bullet) := (X_0 \xrightarrow{x_i} X_i)$ for every $x_\bullet := (X_1 \xleftarrow{x_1} X_0 \xrightarrow{x_2} X_2) \in \mathrm{Ob}(\mathscr{C}^{\mathbb{V}})$ and $i = 1, 2$. With this notation, it is clear that :

$$\mathscr{W}^{\mathbb{V}} = \mathscr{W}^{[1]} \times_{\mathscr{W}} \mathscr{W}^{[1]} \qquad \text{and} \qquad \mathscr{F}ib^{\mathbb{V}} = \mathscr{F}ib^{[1]} \times_{\mathscr{F}ib} \mathscr{F}ib^{[1]}.$$

Claim 3.5.6. $\mathscr{C}of^{\mathbb{V}} = \mathscr{C}of^{[1]} \times_{\mathscr{C}of} \mathscr{C}of^{[1]}$ and $\mathscr{W}^{\mathbb{V}} \cap \mathscr{C}of^{\mathbb{V}} = l(\mathscr{F}ib^{\mathbb{V}})$.

Proof : Set $\mathscr{I} := \mathscr{C}of^{[1]} \times_{\mathscr{C}of} \mathscr{C}of^{[1]}$. The proof of claim 3.5.4(i) shows that, in order to construct a diagonal filler for a diagram (D) of $\mathscr{C}^{[1]}$ with $f_\bullet \in \mathscr{C}of^{[1]}$ and $g_\bullet \in \mathscr{W}^{[1]} \cap \mathscr{F}ib^{[1]}$, we may first choose a diagonal filler d_0 for the diagram $\rho(\mathrm{D})$ obtained by applying ρ termwise to (D); then, for every such choice d_0 we find a diagonal filler $d_\bullet$ for (D) with $\rho(d_\bullet) = d_0$. This easily implies that $\mathscr{I} \subset \mathscr{C}of^{\mathbb{V}}$. Likewise, the proof of proposition 3.5.3 shows that in order to factor a morphism $f_\bullet$ of $\mathscr{C}^{[1]}$ as a cofibration followed by a trivial fibration, we may first choose such a factorization for $\rho(f_\bullet)$, say $\rho(f_\bullet) = g_0 \circ h_0$; then for every such choice we find a factorization $f_\bullet = g_\bullet \circ h_\bullet$ as sought, with $\rho(g_\bullet) = g_0$ and $\rho(h_\bullet) = h_0$. We deduce easily that every morphism of $\mathscr{C}^{\mathbb{V}}$ is the composition of an element of $\mathscr{I}$ followed by an element of $\mathscr{W}^{\mathbb{V}} \cap \mathscr{F}ib^{\mathbb{V}}$. Moreover, $\mathscr{W}^{\mathbb{V}} \cap \mathscr{F}ib^{\mathbb{V}}$ and $\mathscr{C}of^{[1]}$ are stable under retracts (propositions 3.5.3(i) and 3.2.6), so the same holds for $\mathscr{I}$; summing up, $(\mathscr{I}, \mathscr{W}^{\mathbb{V}} \cap \mathscr{F}ib^{\mathbb{V}})$ is a weak factorization system for $\mathscr{C}^{\mathbb{V}}$.

In light of claim 3.5.4(ii), a similar argument proves that $(\mathscr{I} \cap \mathscr{W}^{\mathbb{V}}, \mathscr{F}ib^{\mathbb{V}})$ is also a weak factorization system for $\mathscr{C}^{\mathbb{V}}$, and then both identities of the claim follow from lemma 3.1.9(i). ◇

Assertion (i) follows straightforwardly from claim 3.5.6 and its proof, and (ii) follows immediately from claim 3.5.6 and proposition 3.5.3(ii). □

Proposition 3.5.7. *(i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $\lambda > 0$ an ordinal such that $\mathscr{C}$ is μ-cocomplete for every ordinal $\mu < \lambda$. Then the projective model structure on $\mathscr{C}^\lambda$ is well defined.*

(ii) Dually, if $\mathscr{C}$ is μ^{op}-complete for every ordinal $\mu < \lambda$, then the injective model structure on $\mathscr{C}_{\lambda^{\mathrm{op}}}$ is well defined.

(iii) A functor $X_\bullet : \lambda \to \mathscr{C}$ is a cofibrant object of $\mathscr{C}^\lambda$ $\Leftrightarrow$ the induced morphism

$$L_i^X := \varinjlim_{j<i} X_j \to X_i$$

is a cofibration of $\mathscr{C}$, for every $i \in \lambda$ (especially, every X_i is cofibrant in $\mathscr{C}$).

Proof. For every morphism $f_\bullet : X_\bullet \to Y_\bullet$ and every $i \in \lambda$, let $L^f_i : L^X_i \to L^Y_i$ be the induced morphism, so that we have a commutative diagram :

$$\begin{tikzcd} L^X_i \arrow[r, "l^X_i"] \arrow[d, "L^f_i"'] & X_i \arrow[d, "f_i"] \\ L^Y_i \arrow[r, "l^Y_i"] & Y_i \end{tikzcd}$$

and denote by $(f_i, l^Y_i) : X_i \sqcup_{L^X_i} L^Y_i \to Y_i$ the induced morphism of $\mathscr{C}$. We let $\mathscr{I}^\lambda$ be the class of all morphisms $f_\bullet : X_\bullet \to Y_\bullet$ of $\mathscr{C}^\lambda$ such that $(f_i, l^Y_i) \in \mathscr{C}of$ for every $i \in \lambda$. Also, for every two ordinals $\nu < \mu \leq \lambda$ we consider the restriction functor

$$r_{\mu\nu} : \mathscr{C}^\mu \to \mathscr{C}^\nu \qquad X_\bullet \mapsto (X_j \mid j < \nu).$$

Claim 3.5.8. $(\mathscr{I}^\lambda, \mathscr{W}^\lambda \cap \mathscr{F}ib^\lambda)$ is a weak factorization system for $\mathscr{C}^\lambda$.

Proof : Let $f_\bullet \in \mathscr{I}^\lambda$, $g_\bullet \in \mathscr{W}^\lambda \cap \mathscr{F}ib^\lambda$, and consider a commutative square of $\mathscr{C}^\lambda$:

$$(\mathrm{D}) \qquad \begin{tikzcd} X_\bullet \arrow[r, "h_\bullet"] \arrow[d, "f_\bullet"'] & A_\bullet \arrow[d, "g_\bullet"] \\ Y_\bullet \arrow[r, "k_\bullet"] & B_\bullet . \end{tikzcd}$$

We wish to exhibit a family $(d^\mu_\bullet \mid 1 \leq \mu \leq \lambda)$ such that $d^\mu_\bullet$ is a diagonal filler for $r_{\lambda\mu}(\mathrm{D})$ and $r_{\mu\nu}(d^\mu_\bullet) = d^\nu_\bullet$ for every $1 \leq \nu \leq \mu \leq \lambda$. We argue by transfinite induction on $\mu \geq 1$: for $\mu = 1$, we have $\mathscr{C}^1 = \mathscr{C}$, and we need a diagonal filler $d^1 : Y_0 \to A_0$ for the commutative square formed by h_0, g_0, k_0 and f_0; but notice that L^X_0 and L^Y_0 are both the colimits of the empty family, so $L^X_0 = L^Y_0 = \varnothing$, and therefore $(f_0, l^Y_0) = f_0$, so $f_0 \in \mathscr{C}of$, and since $g_0 \in \mathscr{W} \cap \mathscr{F}ib$, such a d^1 can indeed be found.

- Next, let $\mu \leq \lambda$ be a limit ordinal, and suppose that $d^\nu_\bullet$ has already been defined for every $\nu < \mu$; we then get the sought diagonal filler $d^\mu_\bullet$ by setting $d^\mu_i := d^{i+1}_i$ for every $i \in \mu$.

- Lastly, suppose that $\lambda > \mu \geq 1$, and that $d^\mu_\bullet$ has already been exhibited. There follows a commutative diagram :

$$(\mathrm{D}_\mu) \qquad \begin{tikzcd} L^A_\mu \arrow[rrr, "L^g_\mu"] \arrow[ddd, "l^A_\mu"'] & & & L^B_\mu \arrow[ddd, "l^B_\mu"] \\ & L^X_\mu \arrow[ul, "L^h_\mu"'] \arrow[r, "L^f_\mu"] \arrow[d, "l^X_\mu"'] & L^Y_\mu \arrow[ur, "L^k_\mu"] \arrow[d, "l^Y_\mu"] & \\ & X_\mu \arrow[r, "f_\mu"] \arrow[dl, "h_\mu"'] & Y_\mu \arrow[dr, "k_\mu"] & \\ A_\mu \arrow[rrr, "g_\mu"] & & & B_\mu \end{tikzcd}$$

and the morphism $d^\mu_\bullet : r_{\lambda\mu}(Y_\bullet) \to r_{\lambda\mu}(A_\bullet)$ induces a diagonal filler $L^d_\mu : L^Y_\mu \to L^A_\mu$ for the square with sides L^f_μ, L^k_μ, L^g_μ and L^h_μ. However, (D_μ) can be regarded as a commutative square of $\mathscr{C}^{[1]}$, and arguing as in the proof of claim 3.5.4(i), we see that the condition $(f_\mu, l^Y_\mu) \in \mathscr{C}of$ enables us to find a morphism $d : Y_\mu \to A_\mu$ such that the pair (L^d_μ, d) defines a diagonal filler for the commutative square (D_μ) of $\mathscr{C}^{[1]}$. We may then define

$d_\bullet^{\mu+1}$ by setting $d_i^{\mu+1} := d_i^\mu$ for every $i < \mu$ and $d_\mu^{\mu+1} := d$. This concludes the construction of the sequence $(d_\bullet^\mu \mid 1 \leq \mu \leq \lambda)$, and shows that $f_\bullet \in \mathscr{C}of^{\mathbb{V}}$.

• Next, clearly $\mathscr{W}^\lambda \cap \mathscr{F}ib^\lambda$ is stable under retracts; to show that the same holds for $\mathscr{I}^\lambda$, we argue as in the proof of claim 3.5.4(i) : consider a commutative diagram of $\mathscr{C}^\lambda$:

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d, "f_\bullet"'] \arrow[rr, bend left, "\mathbf{1}_{X_\bullet}"] & A_\bullet \arrow[r] \arrow[d, "g_\bullet"] & X_\bullet \arrow[d, "f_\bullet"] \\ Y_\bullet \arrow[r] \arrow[rr, bend right, "\mathbf{1}_{Y_\bullet}"'] & B_\bullet \arrow[r] & Y_\bullet \end{tikzcd}$$

with $g_\bullet \in \mathscr{I}^\lambda$. We deduce for every $i \in \lambda$ a commutative diagram :

$$\begin{tikzcd} X_i \sqcup_{L_i^X} L_i^Y \arrow[r] \arrow[d, "(f_i, l_i^Y)"'] \arrow[rr, bend left, "\mathbf{1}"] & L_i^A \sqcup_{L_i^A} L_i^B \arrow[r] \arrow[d, "(g_i, l_i^B)"] & X_i \sqcup_{L_i^X} L_i^Y \arrow[d, "(f_i, l_i^Y)"] \\ Y_i \arrow[r] \arrow[rr, bend right, "\mathbf{1}"'] & B_i \arrow[r] & Y_i \end{tikzcd}$$

where, by assumption, $(g_i, l_i^B) \in \mathscr{C}of$, so that $(f_i, l_i^Y) \in \mathscr{C}of$ as well, since $\mathscr{C}of$ is saturated.

• Lastly, let us check that every morphism $f_\bullet : X_\bullet \to Y_\bullet$ of $\mathscr{C}^\lambda$ factors as $f_\bullet = h_\bullet \circ g_\bullet$, with $g_\bullet \in \mathscr{I}$ and $h_\bullet \in \mathscr{W}^\lambda \cap \mathscr{F}ib^\lambda$. To this aim, we exhibit, by transfinite induction, a family $(g_\bullet^\mu, h_\bullet^\mu \mid 1 \leq \mu \leq \lambda)$ with :

$$(r_{\lambda\mu}(X_\bullet) \xrightarrow{g_\bullet^\mu} T_\bullet^\mu) \in \mathscr{I}^\mu \qquad (T_\bullet^\mu \xrightarrow{h_\bullet^\mu} r_{\lambda\mu}(Y_\bullet)) \in \mathscr{W}^\mu \cap \mathscr{F}ib^\mu \qquad \forall \mu \leq \lambda$$

and $h_\bullet^\mu \circ g_\bullet^\mu = r_{\lambda\mu}(f_\bullet)$, and such that moreover $r_{\mu\nu}(g_\bullet^\mu) = g_\bullet^\nu$ for every $1 \leq \nu \leq \mu$.

• For $\mu = 1$, the datum $(g_\bullet^1, h_\bullet^1)$ is just a factorization $f_0 = h_0 \circ g_0$ with $g_0 \in \mathscr{C}of$ and $h_0 \in \mathscr{W} \cap \mathscr{F}ib$. Next, if $\mu \leq \lambda$ is a limit ordinal, and if $(g_\bullet^\nu, h_\bullet^\nu)$ has already been exhibited for every $\nu < \mu$, we get $(g_\bullet^\mu, h_\bullet^\mu)$ by setting $g_i^\mu := g_i^{i+1}$ and $h_i^\mu := h_i^{i+1}$ for every $i \in \mu$.

• Lastly, let $\lambda > \mu \geq 1$, and suppose that the factorization $(g_\bullet^\mu, h_\bullet^\mu)$ has already been given; we deduce a commutative diagram :

$$\begin{tikzcd} L_\mu^X \arrow[r, "L_\mu^g"] \arrow[d, "l_\mu^X"'] & L_\mu^T \arrow[r, "L_\mu^h"] \arrow[d, "e_\mu^T"] & L_\mu^Y \arrow[d, "l_\mu^Y"] \\ X_\mu \arrow[r, "e_\mu^X"] & T'_\mu \arrow[r, "(f_\mu, l_\mu^Y L_\mu^h)"] & Y_\mu \end{tikzcd}$$

whose left square subdiagram is cocartesian. Let us then pick a factorization

$$(f_\mu, l_\mu^Y L_\mu^h) = h_\mu \circ i_\mu \quad \text{with} \quad (i_\mu : T'_\mu \to T_\mu) \in \mathscr{C}of \quad \text{and} \quad (h_\mu : T_\mu \to Y_\mu) \in \mathscr{W} \cap \mathscr{F}ib.$$

The required factorization $(g_\bullet^{\mu+1}, h_\bullet^{\mu+1})$ is given by $g_i^{\mu+1} := g_i^\mu$, $h_i^{\mu+1} := h_i^\mu$ for every $i < \mu$, and $g_\mu^{\mu+1} := i_\mu \circ e_\mu^X : X_\mu \to T_\mu$, $h_\mu^{\mu+1} := h_\mu$. ◇

Claim 3.5.9. (i) $\mathscr{W}^\lambda \cap \mathscr{I}^\lambda = \mathscr{J}^\lambda := \{(f_\bullet : X_\bullet \to Y_\bullet) \in \mathscr{I}^\lambda \mid (f_i, l_i^Y) \in \mathscr{W} \cap \mathscr{C}of \ \forall i \in \lambda\}$.

(ii) $(\mathscr{J}^\lambda, \mathscr{F}ib^\lambda)$ is a weak factorization system for $\mathscr{C}^\lambda$.

Proof : The proof of (ii) is, *mutatis mutandis,* the same as that of claim 3.5.8.

(i): Let $(f_\bullet : X_\bullet \to Y_\bullet) \in \mathscr{J}^\lambda$, and for every $\nu < \mu < \lambda$, set $M^\mu_\nu := L^X_\mu \sqcup_{X_\nu} Y_\nu$; then, for every $\nu \leq \rho < \mu$ we have an induced morphism :

$$t_{\nu\rho} := (\mathbf{1}_{L^X_\mu}, Y_{\nu\rho}) : M^\mu_\nu \to M^\mu_\rho$$

where $Y_{\nu\rho} : Y_\nu \to Y_\rho$ denotes the transition morphism for the functor $Y_\bullet$. Clearly the system $(M^\mu_\nu, t_{\nu\rho} \mid \nu \leq \rho < \mu)$ defines a functor $M^\mu_\bullet : \mu \to \mathscr{C}$. Notice also that :

$$\varinjlim_{\nu<\rho} M^\mu_\nu = L^X_\mu \sqcup_{L^X_\rho} L^Y_\rho \qquad \forall \rho < \mu \qquad \text{and} \qquad \varinjlim_{\nu<\mu} M^\mu_\nu = L^Y_\mu.$$

Moreover, it is easily seen that the natural commutative square :

$$\begin{tikzcd} X_\rho \sqcup_{L^X_\rho} L^Y_\rho \arrow[r, "(f_\rho, l^Y_\rho)"] \arrow[d] & Y_\rho \arrow[d] \\ L^X_\mu \sqcup_{L^X_\rho} L^Y_\rho \arrow[r] & L^X_\mu \sqcup_{X_\rho} Y_\rho \end{tikzcd}$$

is cocartesian for every $\rho < \mu$. So, for every $\rho < \mu$ the induced morphism :

$$\varinjlim_{\nu<\rho} M^\mu_\nu \to M^\mu_\rho$$

lies in $\mathscr{W} \cap \mathscr{C}of$ (proposition 3.1.9(v)), in other words, $M^\mu_\bullet$ is a $(\mu, \mathscr{W} \cap \mathscr{C}of)$-sequence, and since $\mathscr{W} \cap \mathscr{C}of$ is saturated, we conclude that the induced morphism :

$$(*) \qquad\qquad L^X_\mu \sqcup_{X_0} Y_0 \to L^Y_\mu$$

lies in $\mathscr{W} \cap \mathscr{C}of$ as well. Furthermore, we have a cocartesian diagram :

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d] & Y_0 \arrow[d] \\ L^X_\mu \arrow[r] & L^X_\mu \sqcup_{X_0} Y_0 \end{tikzcd}$$

and $f_0 \in \mathscr{W} \cap \mathscr{C}of$, so by the same token, the induced morphism $L^X_\mu \to L^X_\mu \sqcup_{X_0} Y_0$ lies also in $\mathscr{W} \cap \mathscr{C}of$; therefore, the same holds for its composition with $(*)$:

$$L^f_\mu : L^X_\mu \to L^Y_\mu \qquad \forall \mu < \lambda$$

and finally, also for the composition :

$$f_\mu : X_\mu \xrightarrow{X_\mu \sqcup_{L^X_\mu} L^f_\mu} X_\mu \sqcup_{L^X_\mu} L^Y_\mu \xrightarrow{(f_\mu, l^Y_\mu)} Y_\mu \qquad \forall \mu < \lambda.$$

Hence : $\mathscr{J}^\lambda \subset \mathscr{W}^\lambda \cap \mathscr{I}^\lambda$. To show the converse inclusion, let $(g_\bullet : X_\bullet \to Y_\bullet) \in \mathscr{W}^\lambda \cap \mathscr{I}^\lambda$; by (ii), we find $(i_\bullet : X_\bullet \to Z_\bullet) \in \mathscr{J}^\lambda$ and $(p_\bullet : Z_\bullet \to Y_\bullet) \in \mathscr{F}ib^\lambda$ with $g_\bullet = p_\bullet \circ i_\bullet$, and notice that $p_\bullet \in \mathscr{W}^\lambda$, by the 2-out-of-3 property of $\mathscr{W}$. Then the commutative square :

$$\begin{tikzcd} X_\bullet \arrow[r, "i_\bullet"] \arrow[d, "g_\bullet"'] & Z_\bullet \arrow[d, "p_\bullet"] \\ Y_\bullet \arrow[r, equal] & Y_\bullet \end{tikzcd}$$

admits a diagonal filler $d_\bullet : Y_\bullet \to Z_\bullet$, by virtue of claim 3.5.8. Hence, the following commutative diagram exhibits $g_\bullet$ as a retract of $i_\bullet$:

$$\begin{tikzcd} X_\bullet \arrow[r, equal] \arrow[d, "g_\bullet"'] & X_\bullet \arrow[r, equal] \arrow[d, "i_\bullet"] & X_\bullet \arrow[d, "g_\bullet"] \\ Y_\bullet \arrow[r, "d_\bullet"] & Z_\bullet \arrow[r, "p_\bullet"] & Y_\bullet \end{tikzcd}$$

so $g_\bullet \in \mathscr{J}^\lambda$ by (ii), proposition 3.1.9(v) and lemma 3.1.18(i). ◇

Assertions (i) and (ii) follow by combining claims 3.5.8 and 3.5.9.

(iii): The stated characterization of cofibrant objects of $\mathscr{C}^\lambda$ follows by direct inspection of the description of $\mathscr{C}of^\lambda$ provided in the proof of (i). Lastly, notice that this characterization amounts to saying that $X_\bullet$ is cofibrant if and only if it is a $(\lambda, \mathscr{C}of)$-sequence; then $r_{\lambda\mu}(X_\bullet)$ is a $(\mu, \mathscr{C}of)$-sequence for every $\mu \leq \lambda$, and since $\mathscr{C}of$ is saturated, the induced morphism $X_0 \to L^X_\mu$ must be a cofibration for every $1 \leq \mu \leq \lambda$, and moreover X_0 is cofibrant, since $L^X_0 = \varnothing$. Then also the composition $\varnothing \to X_0 \to L^X_\mu \to X_\mu$ is a cofibration, *i.e.* X_μ is cofibrant for every $\mu < \lambda$. □

3.5.10. *Derived limits and derived colimits.* Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, I a small category, and suppose that $\mathscr{C}$ is I-cocomplete, and that the projective model structure on $\mathscr{C}^I$ is well defined. Recall that we have an adjoint pair of functors :

$$(*) \qquad \boxed{\varinjlim_I : \mathscr{C}^I \rightleftarrows \mathscr{C} : \mathrm{c}_I}$$

where c_I denotes the functor that assigns to every $X \in \mathrm{Ob}(\mathscr{C})$ the constant functor $c_X : I \to \mathscr{C}$ of value X (see [13, Rem.2.58(i)]). Under the stated assumptions, it is clear that c_I preserves fibrations and trivial fibrations, so $(*)$ is a Quillen adjunction, by lemma 3.4.12. Moreover, c_I preserves weak equivalences, so $\mathsf{ho}(\mathrm{c}_I) : \mathsf{ho}(\mathscr{C}) \to \mathsf{ho}(\mathscr{C}^I)$ is both a left and right total derived functor of c_I, according to example 3.4.6(i), and combining with theorem 3.4.13, we get an adjunction :

$$\boxed{\mathbb{L}\varinjlim_I : \mathsf{ho}(\mathscr{C}^I) \rightleftarrows \mathsf{ho}(\mathscr{C}) : \mathsf{ho}(\mathrm{c}_I).}$$

• Dually, if $\mathscr{C}$ is I-complete, and if the injective model structure is well defined on $\mathscr{C}_I$, then, by remark 3.4.2(i), we get an adjunction :

$$\boxed{\mathsf{ho}(\mathrm{c}_I) : \mathsf{ho}(\mathscr{C}) \rightleftarrows \mathsf{ho}(\mathscr{C}_I) : \mathbb{R}\varprojlim_I .}$$

Proposition 3.5.11. *In the situation of §3.5.10, let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of cofibrant objects of $\mathscr{C}^I$ such that $f_i : X_i \to Y_i$ is a weak equivalence of $\mathscr{C}$, for every $i \in \mathrm{Ob}(I)$. Then $f_\bullet$ induces a weak equivalence of $\mathscr{C}$:*

$$\varinjlim_I f_\bullet : \varinjlim_I X_\bullet \to \varinjlim_I Y_\bullet.$$

Proof. The image of $f_\bullet$ in $\mathsf{ho}(\mathscr{C}^I)$ is an isomorphism, so

$$\mathbb{L}\varinjlim_I f_\bullet : \mathbb{L}\varinjlim_I X_\bullet \to \mathbb{L}\varinjlim_I Y_\bullet$$

is an isomorphism as well. But since $X_\bullet$ and $Y_\bullet$ are cofibrant objects of $\mathscr{C}^I$, the explicit construction of the left derived functor in §3.4.3 shows that $\mathbb{L}\varinjlim_I X_\bullet$ and $\mathbb{L}\varinjlim_I Y_\bullet$ are represented by $\varinjlim_I X_\bullet$ and $\varinjlim_I Y_\bullet$, and $\mathbb{L}\varinjlim_I f_\bullet$ is represented by $\varinjlim_I f_\bullet$. Then the assertion follows from theorem 3.3.9(ii). □

Corollary 3.5.12. *(i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and I a set, regarded as a discrete category. Suppose that $\mathscr{C}$ is I-cocomplete; then, every family $(f_i : X_i \to Y_i \mid i \in I)$ of weak equivalences between cofibrant objects of $\mathscr{C}$ induces a weak equivalence :*

$$\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i.$$

(ii) Consider also the following commutative diagram of $\mathscr{C}$, whose horizontal arrows are cofibrations between cofibrant objects :

$$\begin{tikzcd} X' \arrow[d, "f'"'] & X \arrow[l, "x'"'] \arrow[r, "x''"] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\ Y' & Y \arrow[l, "y'"'] \arrow[r, "y''"] & Y''. \end{tikzcd}$$

Then, if f, f' and f'' are weak equivalences, the same holds for the induced morphism :

$$X' \sqcup_X X'' \to Y' \sqcup_Y Y''.$$

(iii) Let λ be an ordinal, and suppose that $\mathscr{C}$ is μ-cocomplete, for every ordinal $\mu \leq \lambda$. Let furthermore $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of $\mathscr{C}^\lambda$ and suppose that $f_i : X_i \to Y_i$ is a weak equivalence, and the natural morphisms of $\mathscr{C}$:

$$\varinjlim_{j<i} X_j \to X_i \qquad \varinjlim_{j<i} Y_j \to Y_i$$

are cofibrations, for every $i \in \lambda$. Then $f_\bullet$ induces a weak equivalence :

$$\varinjlim_{i<\lambda} f_i : \varinjlim_{i<\lambda} X_i \to \varinjlim_{i<\lambda} Y_i.$$

Proof. For (i), we have observed that the projective structure on $\mathscr{C}^I$ is trivially well defined (remark 3.5.2(iii)), whence the assertion, in light of proposition 3.5.11. For (ii) and (iii) one argues likewise, with propositions 3.5.5 and 3.5.7 respectively. □

The reader can spell out the duals of proposition 3.5.11 and its corollaries, concerning limits of systems of weak equivalences between fibrant objects of an I-complete model category $\mathscr{C}$. In case $\lambda = \omega$, the smallest infinite ordinal, part (iii) of corollary 3.5.12 can be stated more explicitly as follows :

Corollary 3.5.13. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and suppose that $\mathscr{C}$ is ω-cocomplete. Consider the following commutative diagram of $\mathscr{C}$:*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & Y_{n+1} \arrow[r] & \cdots \end{tikzcd}$$

whose horizontal arrows are cofibrations between cofibrant objects of $\mathscr{C}$. Then, if all the vertical arrows are weak equivalences, the same holds for the induced morphism:

$$\varinjlim_{n\in\mathbb{N}} X_n \to \varinjlim_{n\in\mathbb{N}} Y_n.$$

Proof. It is a special case of corollary 3.5.12(iii). □

Definition 3.5.14. (i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, I a small category; suppose that $\mathscr{C}$ is I-cocomplete, and that the projective model structure on $\mathscr{C}^I$ is well defined. Let $X \in \mathrm{Ob}(\mathscr{C})$, $F \in \mathrm{Ob}(\mathscr{C}^I)$, and $\tau_\bullet : F \Rightarrow c_X$ a co-cone with vertex X and base F. We say that $\tau_\bullet$ *is homotopically universal*, if the induced morphism :

$$\mathbb{L} \varinjlim_I F \to \varinjlim_I F \to X$$

is an isomorphism of $\mathsf{ho}(\mathscr{C})$. In this case, we also call X *a homotopy colimit of* F.

(ii) Dually, suppose that $\mathscr{C}$ is I-complete, and that the injective model structure is well defined on $\mathscr{C}_I$, and let $X \in \mathrm{Ob}(\mathscr{C})$, $F \in \mathrm{Ob}(\mathscr{C}_I)$, and $\tau_\bullet : c_X \Rightarrow F$ a given cone. We say that $\tau_\bullet$ *is homotopically universal*, if the induced morphism :

$$X \to \varprojlim_I F \to \mathbb{R} \varprojlim_I F$$

is an isomorphism of $\mathsf{ho}(\mathscr{C}_I)$. In this case, we say that X *is a homotopy limit of* F.

Remark 3.5.15. (i) Keep the assumptions of definition 3.5.14(i). From the construction of derived functors, we see that the following conditions are equivalent :

(a) the co-cone $\tau_\bullet : F \Rightarrow c_X$ is homotopically universal

(b) there exist a cofibrant object F' of $\mathscr{C}^I$ with a weak equivalence $f_\bullet : F' \to F$, such that $\tau_\bullet \circ f_\bullet : F' \to X$ induces a weak equivalence $\varinjlim_I F' \to X$

(c) for every cofibrant object F' of $\mathscr{C}^I$ with a weak equivalence $f_\bullet : F' \to F$, the co-cone $\tau_\bullet \circ f_\bullet : F' \to X$ induces a weak equivalence $\varinjlim_I F' \to X$.

(ii) Dually, in the situation of definition 3.5.14(ii), the following are equivalent :

(a) the cone $\tau_\bullet : c_X \Rightarrow F$ is homotopically universal

(b) there exists a fibrant object F' of $\mathscr{C}_I$ with a weak equivalence $f_\bullet : F \to F'$, such that the cone $f_\bullet \circ \tau_\bullet : c_X \Rightarrow F'$ induces a weak equivalence $X \to \varprojlim_I F'$

(c) for every fibrant object F' of $\mathscr{C}_I$ with a weak equivalence $f_\bullet : F \to F'$, the cone $f_\bullet \circ \tau_\bullet : c_X \Rightarrow F'$ induces a weak equivalence $X \to \varprojlim_I F'$.

(iii) Especially, if a functor $F : I \to \mathscr{C}$ is cofibrant for the projective model structure on $\mathscr{C}^I$, then every universal co-cone $F \Rightarrow c_X$ is also homotopically universal, and the colimit of F in $\mathscr{C}$ also represents the homotopy colimit of F. Dually, if F is fibrant for the injective model structure on $\mathscr{C}_I$, then every universal cone $c_X \Rightarrow F$ is also homotopically universal, and the limit of F in $\mathscr{C}$ also represents the homotopy limit of F.

(iv) Let $F, F' : I \rightrightarrows \mathscr{C}$ be two functors, $\tau_\bullet : F \Rightarrow c_X$ and $\tau'_\bullet : F' \Rightarrow c_{X'}$ two co-cones, $f : X \to X'$ a weak equivalence of $\mathscr{C}$, and $\mu_\bullet : F \Rightarrow F'$ a weak equivalence of $\mathscr{C}^I$ (*i.e.* $\mu_i : Fi \to F'i$ is a weak equivalence of $\mathscr{C}$, for every $i \in \mathrm{Ob}(I)$), and suppose that

$$(*) \qquad\qquad c_f \circ \tau_\bullet = \tau'_\bullet \circ \mu_\bullet.$$

Then $\tau_\bullet$ is homotopically universal if and only if the same holds for $\tau'_\bullet$. Indeed, from $(*)$ we get a commutative diagram of $\mathsf{ho}(\mathscr{C})$:

$$\begin{tikzcd} \mathbb{L}\varinjlim_I F \arrow[r, "\mu"] \arrow[d, "\alpha"'] & \mathbb{L}\varinjlim_I F' \arrow[d, "\alpha'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \qquad \text{with } \mu := \mathbb{L} \varinjlim_I \mu_\bullet.$$

By assumption, both μ and f are isomorphisms in $\mathsf{ho}(\mathscr{C})$, so α is an isomorphism if and only if the same holds for α', which is the claim.

(v) Dually, if $\tau_\bullet : c_X \Rightarrow F$ and $\tau'_\bullet : c_{X'} \Rightarrow F'$ are two cones, $f : X \to X'$ is a weak equivalence of $\mathscr{C}$, and $\mu_\bullet : F \Rightarrow F'$ is a weak equivalence of $\mathscr{C}_I$ such that $\mu_\bullet \circ \tau_\bullet = \tau'_\bullet \circ c_f$, then $\tau_\bullet$ is homotopically universal if and only if the same holds for $\tau'_\bullet$.

3.6. **Homotopy push-outs and homotopy pull-backs.**

Definition 3.6.1. (i) Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and consider a commutative square of $\mathscr{C}$:

$$
(\mathsf{D}) \qquad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "y"] & Y'. \end{tikzcd}
$$

Then the pair (x, f) defines a functor $F : \mathbb{V} \to \mathscr{C}$, where $\mathbb{V}$ is the finite category as in proposition 3.5.5, and the pair (y, f') can be regarded as a co-cone $\tau_\bullet : F \Rightarrow c_{Y'}$. We then say that (D) *is homotopy cocartesian*, if $\tau_\bullet$ is homotopically universal, and in this case we also say that Y' *is the homotopy push-out of* (x, f).

(ii) Dually, the pair (y, f') defines a functor $G : \mathbb{V}^{\mathrm{op}} \to \mathscr{C}$, and (x, f) can be regarded as a cone $\eta_\bullet : c_X \Rightarrow G$. We then say that (D) *is homotopy cartesian*, if $\eta_\bullet$ is homotopically universal, and in this case we also say that X *is the homotopy pull-back of* (y, f').

Example 3.6.2. (i) Any cocartesian diagram (D) as in definition 3.6.1, in which all morphisms are cofibrations between cofibrant objects of $\mathscr{C}$, is homotopy cocartesian.

(ii) Dually, any cartesian diagram (D), in which all morphisms are fibrations between fibrant objects of $\mathscr{C}$, is homotopy cartesian.

(iii) Set $\square := [1] \times [1]$, where $[1] := \{0, 1\}$ endowed with its standard total order. By remark 3.5.2(ii) and proposition 3.5.3(i), the projective model structure on $\mathscr{C}^\square$ and the injective model structure on $\mathscr{C}_\square$ are both well-defined, and (D) defines both an object of $\mathscr{C}^\square$ and an object of $\mathscr{C}_\square$. By inspecting the definitions, and in light of remark 3.5.15(i), we see that (D) is homotopy cocartesian $\Leftrightarrow$ there exists a cocartesian square (E) in $\mathscr{C}$ consisting of cofibrations between cofibrant objects, and a morphism $(\mathsf{E}) \to (\mathsf{D})$ in $\mathscr{C}^\square$ that is an isomorphism in $\mathsf{ho}(\mathscr{C}^\square)$. Dually, (D) is homotopy cartesian $\Leftrightarrow$ there exists a cartesian square (E) in $\mathscr{C}$ consisting of fibrations between fibrant objects, and a morphism $(\mathsf{D}) \to (\mathsf{E})$ in $\mathscr{C}_\square$ that is an isomorphism in $\mathsf{ho}(\mathscr{C}_\square)$.

Proposition 3.6.3. *(i) Consider a commutative diagram of the model category* $\mathscr{C}$ *:*

$$
\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[r, "x'"] \arrow[d, "f'"] & X'' \arrow[d, "f''"] \\ Y \arrow[r, "y"] & Y' \arrow[r, "y'"] & Y'' \end{tikzcd}
$$

whose left and right squares (D) *and* (D') *are homotopy cocartesian (resp. homotopy cartesian); then the same holds for the composed square* (D'') *with sides* $f, y' \circ y, f''$ *and* $x' \circ x$.

(ii) If (D) *and* (D'') *are homotopy cocartesian, the same holds for* (D').

(iii) Dually, if (D') *and* (D'') *are homotopy cartesian, the same holds for* (D).

Proof. (i): It suffices to consider the case where (D) and (D') are homotopy cocartesian; then, pick a cocartesian square consisting of cofibrations between cofibrant objects of $\mathscr{C}$:

$$(\mathsf{E}) \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "g"'] & A' \arrow[d, "g'"] \\ B \arrow[r, "b"] & B' \end{tikzcd}$$

with a weak equivalence $\omega : (\mathsf{E}) \to (\mathsf{D})$ of $\mathscr{C}^{\square}$, as in example 3.6.2(iii). In particular, ω restricts to a morphism $\omega_{A'} : A' \to X'$ and $\omega_{B'} : B' \to Y'$; next, factor $x' \circ \omega_{A'} : A' \to X''$ as a cofibration $a' : A' \to A''$ followed by a trivial fibration $\omega_{A''} : A'' \to X''$, and form the cocartesian diagram of $\mathscr{C}$:

$$(\mathsf{E}') \qquad \begin{tikzcd} A' \arrow[r, "a'"] \arrow[d, "g'"'] & A'' \arrow[d, "g''"] \\ B' \arrow[r, "b'"] & B''. \end{tikzcd}$$

The pair $(f'' \circ \omega_{A''}, y' \circ \omega_{B'})$ induces a morphism $\omega_{B''} : B'' \to Y''$, and the four morphisms $\omega_{A'}, \omega_{A''}, \omega_{B'}$ and $\omega_{B''}$ yield another morphism $\omega' : (\mathsf{E}') \to (\mathsf{D}')$ of $\mathscr{C}^{\square}$. Now, b' and g'' are again cofibrations, since $\mathscr{C}of$ is stable under push-outs; according to remark 3.5.15(i), the morphism $\omega_{B''}$ is then an equivalence. Lastly, let (E'') be the composition of (E) and (E'); by combining ω and ω' we get yet another weak equivalence $(\mathsf{E}'') \to (\mathsf{D}'')$ of $\mathscr{C}^{\square}$, and (E'') is a cocartesian square of $\mathscr{C}$ consisting of cofibrations between cofibrant objects, so we may invoke example 3.6.2(iii) again, to conclude.

(ii): We construct (E), (E') and (E'') as in the proof of (i); we have again a morphism $(\mathsf{E}'') \to (\mathsf{D}'')$ of $\mathscr{C}^{\square}$, and since (E'') is a cocartesian square of $\mathscr{C}$ consisting of cofibrations between cofibrant objects, we deduce again with remark 3.5.15(i) that $\omega_{B''}$ is a weak equivalence, since by assumption (D'') is homotopy cocartesian. Hence, the morphism $\omega' : (\mathsf{E}') \to (\mathsf{D}')$ is a weak equivalence of of $\mathscr{C}^{\square}$, and we conclude by invoking again example 3.6.2(iii). Assertion (iii) follows as usual from (ii), by duality. $\square$

Lemma 3.6.4. *(i) In the situation of definition 3.6.1(i), the commutative square* (D) *is homotopy cocartesian (resp. homotopy cartesian) if and only if the same holds for the square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ X' \arrow[r, "f'"] & Y'. \end{tikzcd}$$

(ii) Suppose that the morphism x of the square (D) *is a weak equivalence. Then* (D) *is homotopy cocartesian if and only if y is a weak equivalence.*

(iii) Dually, suppose that the morphism y of the square (D) *is a weak equivalence. Then* (D) *is homotopy cartesian if and only if x is a weak equivalence.*

Proof. (i): By duality, it suffices to prove the assertion for the homotopy cocartesian case. Now, define the partially ordered set $(\mathbb{V}, \leq)$ as in proposition 3.5.5(i); we have a unique non-trivial automorphism $\phi : (\mathbb{V}, \leq) \xrightarrow{\sim} (\mathbb{V}, \leq)$, namely the map that fixes $(0,0)$ and exchanges $(0,1)$ and $(1,0)$. Then ϕ induces an automorphism :

$$\mathscr{C}^{\phi} : \mathscr{C}^{\mathbb{V}} \xrightarrow{\sim} \mathscr{C}^{\mathbb{V}} \qquad (F : \mathbb{V} \to \mathscr{C}) \mapsto F \circ \phi$$

that obviously preserves weak equivalences and fibrations for the projective model structure, so it preserves cofibrations as well. Also, $\mathscr{C}^{\phi}$ sends the object $F := (Y \xleftarrow{f} X \xrightarrow{x} X')$

of $\mathscr{C}^{\mathbb{V}}$ to $F' := (X' \xleftarrow{x} X \xrightarrow{f} Y)$, and the co-cone $\tau_\bullet : F \Rightarrow c_{Y'}$ given by (y, f'), to the co-cone $\tau'_\bullet : F' \Rightarrow c_{Y'}$ given by (f', y). Hence, $\tau_\bullet$ is homotopically universal if and only if the same holds for $\tau'_\bullet$, whence the assertion.

(ii): Pick a weak equivalence $\beta_X : X_c \to X$ with $X_c \in \mathrm{Ob}(\mathscr{C}_c)$, and factor $f \circ \beta_X : X_c \to Y$ as a cofibration $f_c : X_c \to Y_c$ followed by a trivial fibration $\beta_Y : Y_c \to Y$. We deduce a cocartesian diagram of $\mathscr{C}$:

$$(\mathsf{E}) \qquad \begin{tikzcd} X_c \arrow[r, "f_c"] \arrow[d, equal] & Y_c \arrow[d, equal] \\ X_c \arrow[r, "f_c"] & Y_c \end{tikzcd}$$

and a morphism $\gamma : (\mathsf{E}) \to (\mathsf{D})$ in $\mathscr{C}^{\square}$ (notation of example 3.6.2(iii)) given by the morphisms $\beta_X, \beta_Y, x \circ \beta_X : X_c \to X'$ and $y \circ \beta_Y : Y \to Y'$. Then, if y is a weak equivalence of $\mathscr{C}$, the morphism γ is a weak equivalence of $\mathscr{C}^{\square}$, and therefore (D) is homotopy cocartesian, by example 3.6.2(iii). Conversely, if (D) is homotopy cocartesian, then $y \circ \beta_Y$ is a weak equivalence of $\mathscr{C}$, by remark 3.5.15(i), so the same holds for y, by the 2-out-of-3 property of weak equivalences. Assertion (iii) follows from (ii) as usual, by duality. ◻

Proposition 3.6.5. *(i) In the situation of definition 3.6.1(i), suppose that* (D) *is cocartesian in* $\mathscr{C}$ *with* X *and* X' *cofibrant, and that* f *is a cofibration of* $\mathscr{C}$*. Then :*

(a) If x *is a weak equivalence, the same holds for* y*.*

(b) (D) *is a homotopy cocartesian square.*

(ii) Dually, if (D) *is cartesian in* $\mathscr{C}$ *with* Y, Y' *fibrant, and if* f' *is a fibration, then* (D) *is homotopy cartesian; if moreover* y *is a weak equivalence, the same holds for* x*.*

Proof. Clearly, it suffices to show (i). To prove (i.a), endow $X/\mathscr{C}$ and $X'/\mathscr{C}$ with the model structures provided by proposition 3.2.4(iii), and consider the adjunction associated with $x : X \to X'$ as in remark 1.4.2(ii) :

$$(*) \qquad x^! : X/\mathscr{C} \rightleftarrows X'/\mathscr{C} : x_!$$

and notice that $(Y, f) \in \mathrm{Ob}(X/\mathscr{C})$ and $x^!(Y, f) = (Y', f')$ (up to unique isomorphism of $X'/\mathscr{C}$), since (D) is cocartesian. Moreover, the discussion of remark 1.4.2 shows that the unit of the canonical adjunction for $(x^!, x_!)$ assigns to (Y, f) the morphism

$$X/y \qquad : \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "x\circ f'"] & \\ Y \arrow[rr, "y"] & & Y' \end{tikzcd}$$

of $X/\mathscr{C}$. By example 3.4.19, the right derived functor $\mathbb{R}x_! = \mathsf{ho}(x_!)$ is conservative, and $(*)$ is a Quillen adjunction which, by virtue of theorem 3.4.13, induces an adjunction :

$$\mathbb{L}x^! : \mathsf{ho}(X/\mathscr{C}) \rightleftarrows \mathsf{ho}(X'/\mathscr{C}) : \mathbb{R}x_!.$$

The cofibrant objects of $X/\mathscr{C}$ are precisely the cofibrations of $\mathscr{C}$ with source X (example 3.2.5), hence $\mathbb{L}x^!(Y, f) = (Y, f')$, and $\mathbb{R}x_! \circ \mathbb{L}x^!(Y, f) = (Y', x \circ f')$; furthermore, by example 3.4.19, the unit $\eta^*_\bullet$ of the canonical adjunction for $(\mathbb{L}x^!, \mathbb{R}x_!)$ assigns to f the homotopy class $[(X/y)_f]$ of $(X/y)_f$ in $\mathsf{ho}(X/\mathscr{C})$. The class $[(X/y)_f]$ is an isomorphism of $\mathsf{ho}(X/\mathscr{C})$ if and only if X/y is a weak equivalence of $X/\mathscr{C}$ (theorem 3.3.9(ii)), and the latter holds if and only if y is a weak equivalence of $\mathscr{C}$. Recall also that $\mathbb{L}x^!$ is fully faithful if and only if $\eta^*_\bullet$ is an isomorphism of functors ([13, Prop.2.16(iii)]). Summing up, the functor $\mathbb{L}x^!$ is fully faithful $\Leftrightarrow$ for every cofibration $f : X \to Y$ of $\mathscr{C}$ the induced morphism y in diagram (D) is a weak equivalence of $\mathscr{C}$.

Claim 3.6.6. $\mathbb{L}x^!$ is fully faithful if and only if it is an equivalence of categories.

Proof : We may assume that $\mathbb{L}x^!$ is fully faithful, and we check that it is an equivalence. To this aim, let $(\eta^*_\bullet, \varepsilon^*_\bullet)$ be the unit and counit of the adjunction for $(\mathbb{L}x^!, \mathbb{R}x_!)$; we know already that $\eta^*_\bullet$ is an isomorphism, and by the triangle identities (see [13, Prob.2.13(ii)]) it follows that the same holds for $\mathbb{R}x_! \star \varepsilon^*_\bullet$, and then also for $\varepsilon^*_\bullet$, since $\mathbb{R}x_!$ is conservative, whence the assertion. ◇

With claim 3.6.6, we conclude that $\mathbb{L}x^!$ is an equivalence of categories if and only if y is a weak equivalence, for every cocartesian square (D) in which f is a cofibration. Next, for every pair of morphisms $x : X \to X'$ and $x' : X' \to X''$ of $\mathscr{C}$, obviously we have $(x' \circ x)_! = x_! \circ x'_!$, whence natural isomorphisms of functors :

$$x'^! \circ x^! \xrightarrow{\sim} (x' \circ x)^! \qquad \text{and} \qquad \mathbb{L}x'^! \circ \mathbb{L}x^! \xrightarrow{\sim} \mathbb{L}(x' \circ x)^!$$

by [13, Exerc.2.14(i) and 2.17(i)] and corollary 3.4.18. Moreover, if any two of the functors $\mathbb{L}x^!, \mathbb{L}x'^!, \mathbb{L}x'^! \circ \mathbb{L}x^!$ is an equivalence of categories, the same holds for the third one. We may therefore argue as in the proof of Ken Brown's lemma (proposition 3.2.8) to reduce to the case where x is a trivial cofibration, in which case assertion (i.a) is clear, since trivial cofibrations are stable under push-outs (proposition 3.1.9(v) and lemma 3.1.18(i)).

(i.b): Choose a factorization of $x : X \to X'$ as a cofibration $x' : X \to X''$ followed by a trivial fibration $x'' : X'' \to X'$, and form the following commutative diagram with two cocartesian squares :

$$\begin{tikzcd} X \arrow[r, "x'"] \arrow[d, "f"'] & X'' \arrow[r, "x''"] \arrow[d, "f''"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "y'"] & Y'' \arrow[r, "y''"] & Y'. \end{tikzcd}$$

Then, the left square is homotopy cocartesian, by example 3.6.2(i), and the same holds for the right square, by virtue of (i.a) and lemma 3.6.4(ii), since f'' is also a cofibration; so (D) is homotopy cocartesian, by proposition 3.6.3(i). □

Corollary 3.6.7. *The conclusion of corollary 3.5.12(ii) still holds, if we assume only that x' and y' are cofibrations, and that f, f', f'' are weak equivalences between cofibrant objects.*

Proof. We apply proposition 3.6.5(i.b) to the two cocartesian diagrams formed by $X_\bullet := (X' \xleftarrow{x'} X \xrightarrow{x''} X'')$ and $Y_\bullet := (Y' \xleftarrow{y'} Y \xrightarrow{y''} Y'')$ respectively : then both such diagrams are homotopy cocartesian, and f, f', f'' induce a weak equivalence $X_\bullet \to Y_\bullet$ in the model category $\mathscr{C}^{\mathbb{V}}$ of proposition 3.5.5, whence an induced isomorphism in $\mathsf{ho}(\mathscr{C})$:

$$\mathbf{L}\varinjlim_I X_\bullet \xrightarrow{\sim} \mathbf{L}\varinjlim_I Y_\bullet$$

which is represented by the induced morphism $X' \sqcup_X X'' \to Y' \sqcup_Y Y''$. Then we conclude with theorem 3.3.9(ii). □

4. Construction of model categories

Before we attack the problem of constructing model structures on categories of interest, we need to be able to count on a reliable supply of weak factorization systems : this task is addressed in §4.1, dedicated to the so-called *small object argument*, a rather general and versatile method with a long history : its earliest modern *avatar* can be found in Grothendieck's *Tôhoku* paper [7], where it is employed to construct injective objects in abelian categories; further back, one can trace its origins to Baer's paper [2] from 1940, that deals with the special case of injective modules over associative rings. This section is complemented by §4.2, that develops some technical tools useful for checking the smallness conditions required to deploy the small object argument, especially when dealing with categories of presheaves over small categories.

It turns out that most model categories that we will encounter in our discussion shall be *cofibrantly generated*, meaning that their classes of fibrations and cofibrations are obtained by suitably *saturating certain sets of morphisms*; of this type are all the model categories that are constructed via the small object argument. In §4.3 we introduce cofibrantly generated model categories, and we study in detail two basic examples : first, the category of complexes of modules over an (associative) ring, which we endow, via the small object argument, with a model category structure whose weak equivalences are the quasi-isomorphisms of complexes, and whose fibrations are the (termwise) epimorphisms. Our second example is the category of small categories, which we endow with its *canonical model category structure*, whose weak equivalences are the equivalences of categories, and whose cofibrations are the functors that are injective on objects; though this canonical model category structure is cofibrantly generated, it is obtained by direct elementary verifications that do not involve the small object argument.

The last three sections present Cisinski's method for constructing model structures on the category $\widehat{\mathscr{A}}$ of presheaves (of sets) over any small category $\mathscr{A}$, and is borrowed from his PhD thesis [3]. Cisinski's idea is to abstract Gabriel and Zisman's work [6] on the homotopy theory of Kan complexes, whose essential aspects are shown to apply to much more general situations. Especially, Gabriel and Zisman's theory of *anodyne extensions* can be transposed to arbitrary categories of presheaves, and plays a central role in Cisinski's method and in our text, starting with §4.4. The main result here is theorem 4.5.14, that associates with every *homotopical structure* on $\mathscr{A}$ (see definition 4.4.6(ii)) a cofibrantly generated model category structure on $\widehat{\mathscr{A}}$; the whole of §4.5 is occupied with the proof of this theorem, and further useful complements are gathered in §4.6 : notably, theorem 4.6.5 shows that Cisinski's construction yields *all* the cofibrantly model category structures on $\widehat{\mathscr{A}}$ whose cofibrations are the monomorphisms.

4.1. **The small object argument.**

Definition 4.1.1. Let κ be a cardinal, $\mathscr{C}$ a cocomplete category, and $X \in \mathrm{Ob}(\mathscr{C})$.

(i) We say that a partially ordered set $(E, \leq)$ is *κ-filtered*, if for every subset $J \subset E$ of cardinality $< \kappa$ there exists $x \in E$ such that $x \geq j$ for every $j \in J$.

(ii) Let $\mathscr{F}$ a subclass of $\mathrm{Mor}(\mathscr{C})$. We say that X is *κ-small relative to $\mathscr{F}$*, if for every κ-filtered ordinal λ and every $(\mathscr{F}, \lambda)$-sequence $Y_\bullet : \lambda \to \mathscr{C}$, the natural map

$$\varinjlim_{j \in \lambda} \mathscr{C}(X, Y_j) \to \mathscr{C}(X, \varinjlim_{j \in \lambda} Y_j)$$

is a bijection. We say that a morphism $f : X \to Y$ of $\mathscr{C}$ is *κ-small relative to $\mathscr{F}$*, if the same holds for X. We say that an object or a morphism of $\mathscr{C}$ is *κ-small*, if it is κ-small relative to the class of all morphisms of $\mathscr{C}$.

(iii) We say that an object X (resp. a morphism f) of $\mathscr{C}$ is *small relative to $\mathscr{F}$*, if there exists a cardinal κ such that X (resp. f) is κ-small relative to $\mathscr{F}$. We say that X (resp. f) is *small*, if it is small relative to the class of all morphisms of $\mathscr{C}$.

Remark 4.1.2. (i) Keep the notation of definition 4.1.1(ii); then, in light of corollary 1.4.6(iii), it is easily seen that if the morphism f of $\mathscr{C}$ is κ-small relative to $\mathscr{F}$, then the object f of $\mathscr{C}/Y$ is κ-small relative to $\mathscr{F}/Y$ (notation of §3.1.11).

(ii) Likewise, if $g : X \to Y$ is a morphism of $\mathscr{C}$ such that *both X and Y are κ-small relative to $\mathscr{F}$*, then g, regarded as an object of $X/\mathscr{C}$, is κ-small relative to $X/\mathscr{F}$: indeed, $X/\mathscr{C}$ is cocomplete (lemma 1.4.7(ii)), so the assertion follows easily from corollary 1.4.6(ii) (details left to the reader).

(iii) If the partially ordered set $(E, \leq)$ is κ-filtered for some cardinal κ, then clearly it is also κ'-filtered for every cardinal $\kappa' \leq \kappa$. Moreover, if X is κ-small relative to $\mathscr{F}$, it is also κ''-small relative to $\mathscr{F}$, for every cardinal $\kappa'' \geq \kappa$.

(iv) Furthermore, if $\kappa \geq \aleph_0$ is not regular, then $(E, \leq)$ is κ-filtered if and only if it is κ^+-filtered. For the proof, by (iii) we may assume that E is κ-filtered; consider then $E' \subset E$ with $|E'| = \kappa$: as κ is not regular, we have $E' = \bigcup_{i \in I} E'_i$ for a family of subsets E_i with $|E_i| < \kappa$ for every $i \in I$, and with $|I| < \kappa$. Then, by assumption for every $i \in I$ we may find $x_i \in E$ such that $x_i \geq y$ for every $y \in E'_i$; likewise, we may find $x \in E$ such that $x \geq x_i$ for every $i \in I$, and therefore $x \geq y$ for every $y \in E'$, which shows that E is κ^+-filtered.

(v) Since the successor of every infinite cardinal is always regular (example 1.1.2(ii)), summing up the discussion of (iv) we see that if $\kappa \geq \aleph_0$, we can assume that κ is regular in definition 4.1.1, without loss of generality; if $3 \leq \kappa < \aleph_0$, then κ-filtered is the same as filtered, which is the same as $\aleph_0$-filtered.

Example 4.1.3. (i) If κ is a finite cardinal, every filtered partially ordered set is trivially κ-filtered; in particular, every ordinal is κ-filtered.

(ii) Every successor ordinal has a maximal element, so it is trivially κ-filtered for every cardinal κ.

(iii) For every $k \in \mathbb{N} \setminus \{0\}$, the ordinal ω^k (the order type of $\mathbb{N}^k$) is not $\aleph_1$-filtered.

(iv) Every regular cardinal κ is κ-filtered. Indeed, let $E \subset \kappa$ with $|E| < \kappa$, and for every $x \in E$ set $S_x := \{y \in \kappa \mid y \leq x\}$; then $|S_x| < \kappa$ for every such x, and since κ is regular we have $|\bigcup_{x \in E} S_x| < \kappa$, whence the assertion.

Example 4.1.4. (i) Every set S is $|S|^+$-small (in the category Set; notation of §1.6). Indeed, let $T_\bullet : \lambda \to \mathrm{Set}$ be any functor from an $|S|^+$-filtered ordinal λ, and denote by T the colimit of $T_\bullet$. Let $f : S \to T$ be any map; for every $s \in S$ pick $\alpha(s) \in \lambda$ and $t(s) \in T_{\alpha(s)}$ such that $f(s)$ is the image of $t(s)$ in T; by assumption there exists $\beta \in \lambda$ such that $\beta \geq \alpha(s)$ for every $s \in S$, so f factors through a map $g : S \to T_\beta$. Moreover, suppose that f also factors through another map $h : S \to T_{\beta'}$, for some $\beta' \in \lambda$; then, for every $s \in S$ there exists $\gamma(s) \in \lambda$ such that the images of $g(s)$ and $h(s)$ agree in $T_{\gamma(s)}$, and again we may find $\delta \in \lambda$ such that $\delta \geq \gamma(s)$ for every $s \in S$, so that the images of g and h coincide in $\mathrm{Set}(S, T_\delta)$.

(ii) Furthermore, let $\kappa \geq \aleph_0$ be a regular cardinal; then S is κ-small if and only if $|S| < \kappa$. Indeed, by (i) and remark 4.1.2(iii) we know already that if $|S| < \kappa$, then S is κ-small. For the converse, without of loss of generality we may assume that S is a is κ-small cardinal; suppose then, by way of contradiction, that $S \geq \kappa$ (so that $\kappa \subset S$), and let

$f : S \to \kappa$ be the map such that $f(s) := s$ for every $s \in \kappa$, and $f(s) := 0$ for every $s \in S \setminus \kappa$. Since κ is the colimit of the well-ordered system of ordinals $\lambda < \kappa$, which is indexed by κ, and since κ is κ-filtered (example 4.1.3(iv)), it follows that f factors through the inclusion $\lambda \to \kappa$, for some $\lambda < \kappa$; but this is absurd, since f is surjective.

Example 4.1.5. (i) Let $\kappa \geq \aleph_0$ be any regular cardinal, and $\mathscr{A}$ any small category; if $\mathscr{A}$ is κ-small in the sense of definition 1.1.6(ii), then it is a κ-small object of Cat, in the sense of definition 4.1.1(iii). Indeed, suppose first that $|\mathrm{Mor}(\mathscr{A})| < \kappa$; let λ be any κ-filtered ordinal, and consider any functor

$$\mathscr{C}_\bullet : \lambda \to \mathsf{Cat} \qquad j \mapsto \mathscr{C}_j \qquad \overrightarrow{ji} \mapsto \mathscr{C}_{ji}.$$

Let also $\mathscr{C}$ be the colimit of $\mathscr{C}_\bullet$, with universal co-cone $(\tau_j : \mathscr{C}_j \to \mathscr{C} \mid j \in \lambda)$ (proposition 1.10.4), and $F : \mathscr{A} \to \mathscr{C}$ any functor. According to remark 1.5.12, $\mathrm{Ob}(\mathscr{C})$ is the colimit of the induced functor $\mathrm{Ob}(\mathscr{C}_\bullet) : \lambda \to \mathsf{Set}$, and $(\mathrm{Ob}(\tau_j) : \mathrm{Ob}(\mathscr{C}_j) \to \mathrm{Ob}(\mathscr{C}) \mid j \in \lambda)$ is a universal co-cone; then, by example 4.1.4, there exists $j \in \lambda$ and a map $\Phi_j : \mathrm{Ob}(\mathscr{A}) \to \mathrm{Ob}(\mathscr{C}_j)$ such that $\mathrm{Ob}(F) = \mathrm{Ob}(\tau_j) \circ \Phi_j$. Let us set $\Phi_i := \mathrm{Ob}(\mathscr{C}_{ji}) \circ \Phi_j$ whenever $j \leq i < \lambda$, so that $\mathrm{Ob}(F) = \mathrm{Ob}(\tau_i) \circ \Phi_i$ for every such i.

For every $A, B \in \mathrm{Ob}(\mathscr{A})$, remark 1.5.12 shows moreover that $\mathscr{C}(FA, FB)$ is the colimit of the induced system of sets $(\mathscr{C}_i(\Phi_i A, \Phi_i B) \mid j \leq i < \lambda)$, with universal co-cone $(\tau_{i,AB} : \mathscr{C}_i(\Phi_i A, \Phi_i B) \to \mathscr{C}(FA, FB) \mid j \leq i < \lambda)$, and transition maps

$$\mathscr{C}_{ik,AB} : \mathscr{C}_i(\Phi_i A, \Phi_i B) \to \mathscr{C}_k(\Phi_k A, \Phi_k B) \qquad f \mapsto \mathscr{C}_{ik}(f)$$

for every $j \leq i \leq k < \lambda$. Then, by invoking again example 4.1.4, we get for every such pair (A, B) an ordinal $i_{AB} < \lambda$ with $j \leq i_{AB}$, and a map $\Psi_{AB} : \mathscr{A}(A, B) \to \mathscr{C}_{i_{AB}}(\Phi_{i_{AB}} A, \Phi_{i_{AB}} B)$ such that $F_{AB} : \mathscr{A}(A, B) \to \mathscr{C}(FA, FB)$ equals $\mu_{i,AB} \circ \Psi_{AB}$. Since $|\mathrm{Ob}(\mathscr{A})^2| < \kappa$, we may next find $l < \lambda$ such that $l \geq i_{AB}$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$, and we set

$$\Psi'_{AB} := \mathscr{C}_{i_{AB}l,AB} \circ \Psi_{AB} : \mathscr{C}(A, B) \to \mathscr{C}_l(\Phi_l A, \Phi_l B) \qquad \forall A, B \in \mathrm{Ob}(\mathscr{A}).$$

By construction, we then have $F_{AB} = \tau_{l,AB} \circ \Psi'_{AB}$ for every such A, B. Next, since we have

$$\tau_{l,AA} \circ \Psi'_{AA}(\mathbf{1}_A) = F_{AA}(\mathbf{1}_A) = \mathbf{1}_{FA} = \tau_{l,AA}(\mathbf{1}_{\Phi_l A}) \qquad \forall A \in \mathrm{Ob}(\mathscr{A})$$

it follows that for every $A \in \mathrm{Ob}(\mathscr{A})$ there exists $l \leq m(A) < \lambda$ such that

$$\mathscr{C}_{lm(A),AA} \circ \Psi'_{AA}(\mathbf{1}_A) = \mathscr{C}_{lm(A),AA}(\mathbf{1}_{\Phi_l A}) = \mathbf{1}_{\Phi_{m(A)} A}.$$

Again, we may then find $m < \lambda$ such that $m \geq m(A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and with $\Psi''_{AB} := \mathscr{C}_{lm,AB} \circ \Psi'_{AB} : \mathscr{A}(A, B) \to \mathscr{C}_m(\Phi_m A, \Phi_m B)$ for every $A, B \in \mathrm{Ob}(\mathscr{C})$, we get :

$$F_{AB} = \tau_{m,AB} \circ \Psi''_{AB} \qquad \text{and} \qquad \Psi''_{AA}(\mathbf{1}_A) = \mathbf{1}_{\Phi_m A} \qquad \forall A, B \in \mathrm{Ob}(\mathscr{A}).$$

Lastly, since $F_{AC}(g \circ f) = F_{BC}(g) \circ F_{AB}(f)$ for every $A, B, C \in \mathrm{Ob}(\mathscr{A})$ and every $(f, g) \in \mathscr{A}(A, B) \times \mathscr{A}(B, C)$, and since $|\mathrm{Ob}(\mathscr{A})^3| < \kappa$ and $|\mathscr{A}(A, B) \times \mathscr{A}(B, C)| < \kappa$ for every such A, B, C, we can argue as in the foregoing, to find $m \leq n < \lambda$ such that, with $\Psi'''_{AB} := \mathscr{C}_{mn,AB} \circ \Psi''_{AB} : \mathscr{A}(A, B) \to \mathscr{C}_n(\Phi_n A, \Phi_n B)$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$, we get :

$$\Psi'''_{BC}(g) \circ \Psi'''_{AB}(f) = \Psi'''_{AC}(g \circ f) \qquad \forall A, B, C \in \mathrm{Ob}(\mathscr{A}), \forall (f, g) \in \mathscr{A}(A, B) \times \mathscr{A}(B, C)$$

(the detailed verification shall be left to the reader). Summing up, we have obtained a well-defined functor $G : \mathscr{A} \to \mathscr{C}_n$ with $GA := \Phi_n A$ and $G_{AB} := \Psi'''_{AB}$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$, such that $F = \tau_n \circ G$; this proves the surjectivity of the natural map

$$\varinjlim_{j \in \lambda} \mathsf{Cat}(\mathscr{A}, \mathscr{C}_j) \to \mathsf{Cat}(\mathscr{A}, \mathscr{C}).$$

Lastly, let $p < \lambda$ and $G' : \mathscr{A} \to \mathscr{C}_p$ another functor such that $F = \tau_p \circ G'$; we need to show that $\mathscr{C}_{nr} \circ G = \mathscr{C}_{pr} \circ G'$ for some $\lambda > r \geq n, p$. However, from example 4.1.4 we know that there exists $q \geq n, p$ such that $\mathscr{C}_{nq} \circ GA = \mathscr{C}_{pq} \circ G'A$ for every $A \in \mathrm{Ob}(\mathscr{A})$, hence, after replacing G and G' by $\mathscr{C}_{nq} \circ G$ and $\mathscr{C}_{pq} \circ G'$, we may assume that $n = p$ and $GA = G'A$ for every $A \in \mathrm{Ob}(\mathscr{A})$. Next, for every $A, B \in \mathrm{Ob}(\mathscr{A})$ there exists $q_{AB} \geq n$ such that $\mathscr{C}_{nq_{AB}} \circ Gf = \mathscr{C}_{nq_{AB}} \circ G'f$ for every $f \in \mathscr{A}(A, B)$ (again, because $|\mathscr{A}(A, B)| < \kappa$), and since $|\mathrm{Ob}(\mathscr{A})^2| < \kappa$, we may find again $r < \lambda$ such that $r \geq q_{AB}$ for every $A, B \in \mathrm{Ob}(\mathscr{A})$; then clearly $\mathscr{C}_{nr} \circ G = \mathscr{C}_{nr} \circ G'$, as required.

(ii) Conversely, if $\mathscr{A}$ is a κ-small object of Cat, then $\lambda := |\mathrm{Ob}(\mathscr{A})| < \kappa$. Indeed, suppose by way of contradiction, that $\lambda \geq \kappa$ (so that $\kappa \subset \lambda$), and pick a bijection $\omega : \mathrm{Ob}(\mathscr{A}) \xrightarrow{\sim} \lambda$; we let $\Phi : \mathrm{Ob}(\mathscr{A}) \to \kappa$ be the map such that $\Phi(A) := \omega(A)$ for every $A \in \omega^{-1}(\kappa)$, and $\Phi(A) := 0$ for every $A \in \mathrm{Ob}(\mathscr{A}) \setminus \omega^{-1}(\kappa)$. We let $\mathscr{K}$ be the unique category that is equivalent to the final object $[0]$ of Cat, and such that $\mathrm{Ob}(\mathscr{K}) = \kappa$ (*i.e.* $\mathscr{K}(\lambda, \mu) := \{\varnothing\}$ for every $\lambda, \mu \in \kappa$); then there exists a unique functor $F : \mathscr{A} \to \mathscr{K}$ such that $FA := \Phi(A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$. Notice also that $\mathscr{K}$ is the colimit in Cat of the well-ordered system of categories $(\mathscr{K}_\lambda \mid \lambda \in \kappa)$, where $\mathscr{K}_\lambda$ is the full subcategory of $\mathscr{K}$ with $\mathrm{Ob}(\mathscr{K}_\lambda) = \lambda$, for every $\lambda \in \kappa$. Then, arguing as in example 4.1.4(ii), we deduce that F must factor through the inclusion $\mathscr{K}_\lambda \to \mathscr{K}$, for some such λ; but this is absurd, since F is surjective on objects.

Example 4.1.6. Let R be a ring, M a (left) R-module; we may find sets S_1, S_2 and an R-linear map $\phi : R^{(S_2)} \to R^{(S_1)}$ with $\mathrm{Coker}(\phi)$ isomorphic to M; set $\kappa := \max(|S_1|, |S_2|)$. Then we claim that M is κ^+-small in the category $R-\mathsf{Mod}$ of (left) R-modules. Indeed, let $N_\bullet : \lambda \to R-\mathsf{Mod}$ be a functor from a κ^+-filtered ordinal λ, denote by N the colimit of $N_\bullet$, and consider any R-linear map $f : M \to N$. Let also $(e_s \mid s \in S_1)$ and $(e'_t \mid t \in S_2)$ be the canonical bases of $R^{(S_1)}$ and $R^{(S_2)}$, and $p : R^{(S_1)} \to M$ the natural projection; we get a map $q : S_1 \to N$ with $q(s) := f \circ p(e_s)$ for every $s \in S_1$, which, arguing as in example 4.1.4, is seen to factor through a map $q_\beta : S_1 \to N_\beta$, for some $\beta \in \lambda$. The map q_β yields by adjunction an R-linear map $g : R^{(S_1)} \to N_\beta$, and by assumption, the image of $r(t) := g \circ \phi(e'_t)$ vanishes in N for every $t \in S_2$; then we may find again $\gamma \in \lambda$ such that the image of $r(t)$ vanishes already in N_γ for every $t \in S_2$, and this means that g factors through an R-linear map $g_\gamma : M \to N_\gamma$, whose composition with the natural map $N_\gamma \to N$ agrees with f. Lastly, if f factors as well through another R-linear map $g_{\gamma'} : M \to N_{\gamma'}$ for some $\gamma' \in \lambda$, arguing as in the foregoing we find $\delta \in \lambda$ such that the images of $g_\gamma \circ p(e_s)$ and $g_{\gamma'} \circ p(e_s)$ agree in N_δ for every $s \in S_1$, so that the images of g_γ and $g_{\gamma'}$ coincide in $R-\mathsf{Mod}(M, N_\delta)$.

Example 4.1.7. Let R be a ring, and denote by $\mathsf{C}(R)$ the category of (unbounded) chain complexes of (left) R-modules. Then every $(X_\bullet, d^X_\bullet) \in \mathrm{Ob}(\mathsf{C}(R))$ is small in $\mathsf{C}(R)$. Indeed, by example 4.1.6, for every $n \in \mathbb{Z}$ we may find a cardinal κ_n such that X_n is κ_n-small; let then κ be any infinite cardinal larger than every κ_n, and consider any κ-filtered ordinal λ and any functor

$$Y^\bullet_\bullet : \lambda \to \mathsf{C}(R) \qquad \alpha \mapsto (Y^\alpha_\bullet, d^\alpha_\bullet).$$

Let $(L_\bullet, d^L_\bullet)$ be the colimit of $Y^\bullet_\bullet$, and $f_\bullet : (X_\bullet, d^X_\bullet) \to (L_\bullet, d^L_\bullet)$ any morphism of $\mathsf{C}(R)$. Recall that L_n represents the colimit of $Y^\bullet_n : \lambda \to R-\mathsf{Mod} : \alpha \mapsto Y^\alpha_n$, for every $n \in \mathbb{Z}$ ([13, Exerc.2.98(ii)]); then, since $\kappa \geq \kappa_n$, the map $f_n : X_n \to L_n$ factors through an R-linear map $g_n : X_n \to Y^{\alpha_n}_n$ and the natural map $Y^{\alpha_n}_n \to L_n$, for some $\alpha_n < \lambda$. Since κ is infinite, and since λ is κ-filtered, we may then find $\alpha < \lambda$ such that $\alpha \geq \alpha_n$ for every $n \in \mathbb{Z}$, and we get a system of R-linear maps $(g_n : X_n \to Y^\alpha_n \mid n \in \mathbb{N})$ such that the composition of g_n

with the natural map $\tau_n^\alpha : Y_n^\alpha \to L_n$ agrees with f_n, for every $n \in \mathbb{Z}$. Next, set

$$h_n := d_n^\alpha \circ g_n - g_{n-1} \circ d_n^X : X_n \to Y_{n-1}^\alpha \qquad \forall n \in \mathbb{Z}.$$

Then $\tau_{n-1}^\alpha \circ h_n : X_n \to L_n$ is the zero map for every $n \in \mathbb{Z}$, and arguing as in the foregoing we find $\beta < \lambda$ with $\beta \geq \alpha$, such that the composition $X_n \to Y_n^\beta$ of h_n with the transition map $Y_n^{\alpha\beta} : Y_n^\alpha \to Y_n^\beta$ is the zero map, for every $n \in \mathbb{N}$; hence, the system $(g'_n := Y_n^{\alpha\beta} \circ g_n : X_n \to Y_n^\beta)$ is a morphism $X_\bullet \to Y_\bullet^\beta$ of $\mathsf{C}(R)$, whose composition with the natural morphism $Y_\bullet^\beta \to L_\bullet$ yields $f_\bullet$. Lastly, if $\gamma, \gamma' < \lambda$ are two ordinals, and $u_\bullet : X_\bullet \to Y_\bullet^\gamma$, $u'_\bullet : X_\bullet \to Y_\bullet^{\gamma'}$ are two morphisms of $\mathsf{C}(R)$ whose images agree in $\mathsf{C}(R)(X_\bullet, L_\bullet)$, we may find for every $n \in \mathbb{Z}$ an ordinal $\delta_n \geq \gamma, \gamma'$ such that the images of u_n and u'_n agree in $R - \mathsf{Mod}(X_n, L_n)$, and then for every ordinal δ larger than every δ_n, the images of $u_\bullet$ and u'_n agree in $\mathsf{C}(R)(X_\bullet, Y_\bullet^\delta)$.

Theorem 4.1.8. *Let $\mathscr{C}$ be a cocomplete category, $\mathscr{I}$ a subset of $\mathrm{Mor}(\mathscr{C})$ such that every element of $\mathscr{I}$ is small relative to the weak saturation of $\mathscr{I}$. Then there exist a functor*

$$\mathsf{L} : \mathscr{C} \to \mathscr{C} \quad \text{and a natural transformation} \quad \lambda_\bullet : \mathbf{1}_\mathscr{C} \Rightarrow \mathsf{L}$$

such that for every $X \in \mathrm{Ob}(\mathscr{C})$ the following holds :

(a) $\mathsf{L}X$ is an $\mathscr{I}$-injective object of $\mathscr{C}$ (see definition 3.1.1(iv))

(b) λ_X lies in the weak saturation of $\mathscr{I}$.

Proof. Let Ω be the class of all ordinals, and for every $\alpha \in \Omega$ set $\alpha^+ := \alpha + 1$, the successor of α; recall that α is the well ordered set of all ordinals $< \alpha$. We regard as usual every such well ordered set $(\alpha, \leq)$ as a category, and for every $\beta, \gamma \in \alpha$ with $\beta \leq \gamma$, we denote by $\overrightarrow{\beta\gamma}$ the unique element of $\alpha(\beta, \gamma)$ (see §1.9.2).

- We shall exhibit by transfinite induction a family of functors

$$(\mathsf{L}^\alpha : \alpha^+ \times \mathscr{C} \to \mathscr{C} \mid \alpha \in \Omega)$$

such that the restriction of L^α to $\beta^+ \times \mathscr{C}$ equals L^β, for every $\beta, \alpha \in \Omega$ with $\beta \leq \alpha$.

- To this aim, we let $\mathsf{L}^0 : 1 \times \mathscr{C} \xrightarrow{\sim} \mathscr{C}$ be the (trivial) projection. Next, set

$$\Phi(X) := \coprod_{(f:K\to L)\in\mathscr{I}} \mathscr{C}(K, X) \qquad \forall X \in \mathrm{Ob}(\mathscr{C})$$

and fix representatives in $\mathscr{C}$ for the direct sums

$$S(X) := \coprod_{(f:K\to L)\in\mathscr{I}} K^{(\mathscr{C}(K,X))} \qquad T(X) := \coprod_{(f:K\to L)\in\mathscr{I}} L^{(\mathscr{C}(K,X))}$$

as well as universal co-cones :

$$(\tau_{(f,g)}^X : K \to S(X) \mid (K \xrightarrow{f} L, g) \in \Phi(X)) \quad (\eta_{(f,g)}^X : L \to T(X) \mid (K \xrightarrow{f} L, g) \in \Phi(X)).$$

We get unique morphisms $X \xleftarrow{\mu_X} S(X) \xrightarrow{\nu_X} T(X)$ making commute the diagrams:

$$\begin{tikzcd}
X \arrow[d, equal] & K \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, "\tau_{(f,g)}^X"'] & L \arrow[d, "\eta_{(f,g)}^X"] \\
X & S(X) \arrow[l, "\mu_X"'] \arrow[r, "\nu_X"] & T(X)
\end{tikzcd} \qquad \forall (f,g) \in \Phi(X).$$

With this notation, we define FX as the push-out in the cocartesian diagram :

$$\begin{tikzcd} S(X) \arrow[r, "\mu_X"] \arrow[d, "\nu_X"'] & X \arrow[d, "\lambda^1_X"] \\ T(X) \arrow[r, "\omega_X"] & FX. \end{tikzcd}$$

Next, let $h : X \to Y$ be any morphism of $\mathscr{C}$. We attach to h the unique morphisms $S(h) : S(X) \to S(Y)$ and $T(h) : T(X) \to T(Y)$ that make commute the following diagrams, for every $(f, g) \in \Phi(X)$:

$$\begin{tikzcd} K \arrow[r, equal] \arrow[d, "\tau^X_{(f,g)}"'] & K \arrow[d, "\tau^Y_{(f,hg)}"] \\ S(X) \arrow[r, "S(h)"] & S(Y) \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, equal] \arrow[d, "\eta^X_{(f,g)}"'] & L \arrow[d, "\eta^Y_{(f,hg)}"] \\ T(X) \arrow[r, "T(h)"] & T(Y). \end{tikzcd}$$

It is easily seen that the rules : $X \mapsto S(X)$ and $h \mapsto S(h)$ yield a well-defined functor $S : \mathscr{C} \to \mathscr{C}$, and likewise we deduce a functor $T : \mathscr{C} \to \mathscr{C}$. Moreover, it is easily seen that the rules $X \mapsto \mu_X$ and $X \mapsto \nu_X$ yield natural transformations $\mu_\bullet : S \Rightarrow \mathbf{1}_\mathscr{C}$ and $\nu_\bullet : S \Rightarrow T$. It follows that every morphism $h : X \to Y$ of $\mathscr{C}$ induces a unique morphism $Fh : FX \to FY$, such that the rules : $X \mapsto FX$ and $h \mapsto Fh$ yield a functor $F : \mathscr{C} \to \mathscr{C}$, and the rules : $X \mapsto \lambda^1_X$ and $X \mapsto \omega_X$ define natural transformations $\lambda^{0,1}_\bullet : \mathbf{1}_\mathscr{C} \Rightarrow F$ and $\omega_\bullet : T \Rightarrow F$, with $\lambda^1_\bullet \circ \mu_\bullet = \omega_\bullet \circ \nu_\bullet$.

We then let $\mathsf{L}^1 : 2 \times \mathscr{C} \to \mathscr{C}$ be the unique functor whose restriction to $1 \times \mathscr{C}$ agrees with L^0, such that $\mathsf{L}^1(1, X) := FX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and

$$\mathsf{L}^1(\overrightarrow{01}, \mathbf{1}_X) := \lambda^1_X \qquad \mathsf{L}^1(\mathbf{1}_1, h) := Fh \qquad \forall X \in \mathrm{Ob}(\mathscr{C}), \forall h \in \mathrm{Mor}(\mathscr{C}).$$

• Let now $\alpha \in \Omega$ be an ordinal > 1, and suppose that L^β has already been exhibited for every $\beta < \alpha$; if $\alpha = \beta^+$ for some $\beta \in \Omega$, define $\mathsf{L}^\alpha : \alpha^+ \times \mathscr{C} \to \mathscr{C}$ as the unique functor whose restriction to $\beta^+ \times \mathscr{C}$ agrees with L^β, such that $\mathsf{L}^\alpha(\alpha, X) := F(\mathsf{L}^\beta(\beta, X))$, and

$$\mathsf{L}^\alpha(\overrightarrow{\beta\alpha}, \mathbf{1}_X) := \lambda^1_{\mathsf{L}^\beta(\beta,X)} \quad \mathsf{L}^\alpha(\mathbf{1}_\alpha, h) := F(\mathsf{L}^\beta(\mathbf{1}_\beta, h)) \quad \forall X \in \mathrm{Ob}(\mathscr{C}), \forall h \in \mathrm{Mor}(\mathscr{C}).$$

• Lastly, if α is a limit ordinal, for every $X \in \mathrm{Ob}(\mathscr{C})$ we fix a representative LX for the colimit of the functor

$$L^{<\alpha}_X : \alpha \to \mathscr{C} \qquad \beta \mapsto \mathsf{L}^\beta X := \mathsf{L}^\beta(\beta, X) \qquad \overrightarrow{\beta\gamma} \mapsto \mathsf{L}^\gamma(\overrightarrow{\beta\gamma}, \mathbf{1}_X)$$

and a universal co-cone $\tau^X_\bullet : L^{<\alpha}_X \Rightarrow c_{LX}$. Then, every morphism $h : X \to Y$ of $\mathscr{C}$ induces a unique morphism $Lh : LX \to LY$ that makes commute the diagrams :

$$\begin{tikzcd} \mathsf{L}^\beta X \arrow[r, "\mathsf{L}^\beta(\mathbf{1}_\beta, h)"] \arrow[d, "\tau^X_\beta"'] & \mathsf{L}^\beta Y \arrow[d, "\tau^Y_\beta"] \\ LX \arrow[r, "Lh"] & LY \end{tikzcd} \qquad \forall \beta < \alpha.$$

It is easily seen that the rules $X \mapsto LX$ and $h \mapsto Lh$ yield a functor $L : \mathscr{C} \to \mathscr{C}$. Then we let $\mathsf{L}^\alpha : \alpha^+ \times \mathscr{C} \to \mathscr{C}$ be the unique functor whose restriction to $\beta^+ \times \mathscr{C}$ equals L^β for every $\beta < \alpha$, such that $\mathsf{L}^\alpha(\alpha, X) := LX$ for every $X \in \mathrm{Ob}(\mathscr{C})$, and

$$\mathsf{L}^\alpha(\overrightarrow{\beta\alpha}, \mathbf{1}_X) := \tau^X_\beta \qquad \mathsf{L}^\alpha(\mathbf{1}_\alpha, h) := Lh \qquad \forall X \in \mathrm{Ob}(\mathscr{C}), \forall \beta < \alpha, \forall h \in \mathrm{Mor}(\mathscr{C}).$$

• By remark 4.1.2(iii), under the stated assumptions, we may find a cardinal κ such that every element of $\mathscr{I}$ is κ-small relative to the weak saturation of $\mathscr{I}$. Pick a κ-filtered limit ordinal α (the existence of λ is ensured by examples 4.1.3(iv) and 1.1.2), and let $\mathsf{L} : \mathscr{C} \to \mathscr{C}$ be the functor such that $\mathsf{L}(X) := \mathsf{L}^\alpha(\alpha, X)$ and $\mathsf{L}(h) := \mathsf{L}^\alpha(\mathbf{1}_\alpha, h)$ for every $X \in \mathrm{Ob}(\mathscr{C})$ and $h \in \mathrm{Mor}(\mathscr{C})$; let also $\lambda_\bullet : \mathbf{1}_\mathscr{C} \Rightarrow \mathsf{L}$ be the natural transformation such that

$$X \mapsto \mathsf{L}^\alpha(\overrightarrow{0\alpha}, \mathbf{1}_X) \qquad \forall X \in \mathrm{Ob}(\mathscr{C}).$$

We claim that the pair (L, λ) fulfills conditions (a) and (b) of the theorem. Indeed, the construction and lemma 3.1.8 make it clear that (b) holds.

Lastly, let $X \in \mathrm{Ob}(\mathscr{C})$, $(f : A \to B) \in \mathscr{I}$ and $g \in \mathscr{C}(A, \mathsf{L}X)$; we need to exhibit a morphism $h : B \to \mathsf{L}X$ such that $hf = g$. However, our choice of α implies that there exists some $\beta < \alpha$ and a morphism $g_\beta : A \to \mathsf{L}^\beta X$ with $g = g_\beta \circ \mathsf{L}^\alpha(\overrightarrow{\beta\alpha}, \mathbf{1}_X)$. By construction, we have the commutative diagram :

$$\begin{tikzcd}
A \arrow[r, "\tau^{\mathsf{L}^\beta X}_{(f,g_\beta)}"] \arrow[d, "f"'] & S(\mathsf{L}^\beta X) \arrow[r, "\mu_{\mathsf{L}^\beta X}"] \arrow[d, "\nu_{\mathsf{L}^\beta X}"] & \mathsf{L}^\beta X \arrow[d, "\mathsf{L}^\alpha(\overrightarrow{\beta\beta^+}, \mathbf{1}_X)"] \\
B \arrow[r, "\eta^{\mathsf{L}^\beta X}_{(f,g_\beta)}"] & T(\mathsf{L}^\beta X) \arrow[r, "\omega_{\mathsf{L}^\beta X}"] & \mathsf{L}^{\beta^+} X
\end{tikzcd}$$

and the composition of the top horizontal arrows is g_β. Denote $h_\beta : B \to \mathsf{L}^{\beta^+} X$ the composition of the bottom arrows; then $h := \mathsf{L}^\alpha(\overrightarrow{\beta^+\alpha}, \mathbf{1}_X) \circ h_\beta$ will do. ◻

Corollary 4.1.9. (Small object argument) *Let $\mathscr{C}$ be a cocomplete category, and $\mathscr{I}$ a set of morphisms of $\mathscr{C}$. Suppose that every element of $\mathscr{I}$ is small relative to the weak saturation $\mathscr{J}$ of $\mathscr{I}$. Then we have :*

(i) *For every morphism f of $\mathscr{C}$ there exists $i \in \mathscr{J}$ and $g \in r(\mathscr{I})$ with $f = g \circ i$.*

(ii) *The couple $(l(r(\mathscr{I})), r(\mathscr{I}))$ is a weak factorization system for $\mathscr{C}$.*

(iii) *$l(r(\mathscr{I}))$ is the saturation of $\mathscr{I}$.*

Proof. (i): Let $f : X \to Y$ be any morphism of $\mathscr{C}$, and let us regard f as an object of $\mathscr{C}/Y$. By lemma 3.1.12(i,iv), $r(\mathscr{I}/Y) = r(\mathscr{I})/Y$ and $\mathscr{J}/Y$ is weakly saturated; moreover, according to remark 4.1.2(i), every element of $\mathscr{I}/Y$ is small relative to $\mathscr{J}/Y$, so also relative the weak saturation of $\mathscr{I}/Y$. Then the assertion follows by applying theorem 4.1.8 to the cocomplete category $\mathscr{C}/Y$ (corollary 1.4.6(iii)) and the set $\mathscr{I}/Y \subset \mathrm{Mor}(\mathscr{C}/Y)$.

(ii): Since $\mathscr{J} \subset l(r(\mathscr{I}))$ (proposition 3.1.9(v)), the assertion follows from (i).

(iii): By proposition 3.1.9(v), the saturation of $\mathscr{I}$ lies in $l(r(\mathscr{I}))$. For the converse inclusion, let $f \in l(r(\mathscr{I}))$, and write $f = g \circ i$ as in (i); proposition 3.1.10 implies that f is a retract of i, so it lies in the saturation of $\mathscr{I}$. ◻

4.2. **Accessible functors.** The considerations of this §, that originate from [1, Exp.I, §9], will be used in §4.5, together with the small object argument, in order to exhibit generating sets of trivial cofibrations for suitable model categories.

Definition 4.2.1. Let $\mathscr{C}, \mathscr{D}$ be two categories, $X \in \mathrm{Ob}(\mathscr{C})$, and κ a cardinal.

(i) We say that a functor $F : \mathscr{C} \to \mathscr{D}$ *is κ-accessible*, if $\mathscr{C}$ is I-cocomplete and F preserves I-colimits, for every κ-filtered partially ordered set I.

(ii) We say that X *is κ-accessible*, if the same holds for the functor

$$h_{X^{\mathrm{op}}} : \mathscr{C} \to \mathsf{Set} \qquad Y \mapsto \mathscr{C}(X, Y).$$

The full subcategory of $\mathscr{C}$ formed by the κ-accessible objects shall be denoted :

$$\mathsf{Acc}_\kappa(\mathscr{C}).$$

(iii) We say that the functor F (resp. the object X) is *accessible,* if F (resp. X) is α-accessible for some cardinal α.

(iv) We say that a presheaf S on $\mathscr{C}$ *has size* $< \kappa$, if $|SY| < \kappa$ for every $Y \in \mathrm{Ob}(\mathscr{C})$.

Remark 4.2.2. (i) In the situation of definition 4.2.1(i), if F is κ-accessible, then it is also κ'-accessible for every cardinal $\kappa' \geq \kappa$. On the other hand, F is $\aleph_0$-accessible if and only if it preserves all small filtered colimits; hence, if F is $\aleph_0$-accessible, then it is also κ-accessible for every cardinal $\kappa \geq 3$.

(ii) Clearly, every composition of κ-accessible functors is κ-accessible.

(iii) Let J be a small category, $\mathscr{C}$ any category, $\mathscr{D}$ a J-cocomplete category, $G : J\times\mathscr{C} \to \mathscr{D}$ a functor, and for every $j \in \mathrm{Ob}(J)$ denote $G_j : \mathscr{C} \to \mathscr{D}$ the restriction of G to the full subcategory $\{j\} \times \mathscr{C} \xrightarrow{\sim} \mathscr{C}$. It follows from lemma 1.3.4 that if each G_j is κ-accessible, the same holds for the colimit $\mathrm{colim}_J\, G : \mathscr{C} \to \mathscr{D}$.

(iv) Let $\mathscr{C}$ be a category, $\mathscr{A}$ a small category, and $F : \mathscr{C} \to \widehat{\mathscr{A}}$ a functor. Since the colimits are computed termwise in $\widehat{\mathscr{A}}$, we see that F is κ-accessible $\Leftrightarrow$ the composition $\mathrm{e}_A \circ F : \mathscr{C} \to \mathsf{Set}$ with the evaluation functor $\mathrm{e}_A : \widehat{\mathscr{A}} \to \mathsf{Set}$ is κ-accessible, for every $A \in \mathrm{Ob}(\mathscr{A})$ (see §1.3).

(v) In the situation of (iv), taking into account (i), we see that F is accessible $\Leftrightarrow \mathrm{e}_A \circ F$ is accessible for every $A \in \mathrm{Ob}(\mathscr{A})$.

(vi) Clearly every κ-accessible object is κ-small; the converse holds if $3 \leq \kappa \leq \aleph_0$, by virtue of [13, Prob.2.45] and remark 4.1.2(v), but for $\kappa > \aleph_0$ it is unclear whether these two conditions are equivalent.

Lemma 4.2.3. *For every small category J, the functor* $\mathsf{Lim}_J : \mathsf{Set}^J \to \mathsf{Set}$ *is* $|\mathrm{Mor}(J)|^+$*-accessible (see definition 1.1.1(i) and §1.3).*

Proof. Set $\kappa := |\mathrm{Mor}(J)|$. Let E be the category with $\mathrm{Ob}(E) =: \{s, t\}$ and with $E(s, t) := \{d_0, d_1\}$. Also, for every $f \in \mathrm{Mor}(J)$, let $s(f)$ and $t(f)$ be the source and target of f. We have a natural functor :

$$\Phi : \mathsf{Set}^J \to \mathsf{Set}^E \qquad F \mapsto \Big(\prod_{j\in\mathrm{Ob}(J)} F_j \overset{d_0}{\underset{d_1}{\rightrightarrows}} \prod_{f\in\mathrm{Mor}(J)} F_{t(f)} \Big)$$

where d_0 (resp. d_1) is the unique map whose composition with the natural projection $\prod_{f\in\mathrm{Mor}(J)} F_{t(f)} \to F_{t(f)}$ equals the projection $p_{t(f)} : \prod_{j\in\mathrm{Ob}(J)} F_j \to F_{t(f)}$ (resp. equals $f^* \circ p_{s(f)}$) for every $f \in \mathrm{Mor}(J)$. With this notation, we have an isomorphism of functors:

$$\mathsf{Lim}_J \xrightarrow{\sim} \mathsf{Lim}_E \circ \Phi$$

(see the proof of [13, Prop.2.40]). However, finite limits commute with all filtered colimits in the category Set, so Lim_E is in particular κ-accessible, and we are reduced to checking that the same holds for the functor Φ. Moreover, since (limits and) colimits are computed termwise in the categories of functors, we are further reduced to showing that every set S of cardinality $\leq \kappa$ induces a κ^+-accessible functor

$$\mathsf{Set}^S \to \mathsf{Set} \qquad (X_s \mid s \in S) \mapsto \prod_{s\in S} X_s$$

(here we regard S as a discrete category, so the objects of Set^S are the sequences $X_\bullet := (X_s \mid s \in S)$ of sets, and the morphisms $f_\bullet : X_\bullet \to Y_\bullet$ are the systems $(f_s : X_s \to Y_s \mid s \in S)$

of maps of sets). The assertion means that for every κ^+-filtered partially ordered set I, and every functor $F : I \to \mathrm{Set}^S$, the natural map:

$$(*) \qquad \operatorname*{colim}_{i\in I} \prod_{j\in S} F(i)_j \to \prod_{j\in S} \operatorname*{colim}_{i\in I} F(i)_j$$

is bijective. Now, for every $i \in I$ and $x_\bullet := (x_j \mid j \in S) \in P_i := \prod_{j\in S} F(i)_j$, denote by $[x_\bullet]$ the class of $x_\bullet$ in $\mathrm{colim}_{i\in I} P_i$, and for every $i \in I$, $j \in S$ and $y \in F(i)_j$, denote by $[y]_j$ the class of y in $C_j := \mathrm{colim}_{i\in I} F(i)_j$. With this notation, $(*)$ is the map such that :

$$[x_\bullet] \mapsto ([x_j]_j \mid j \in J) \qquad \forall i \in I, \forall x_\bullet \in P_i.$$

To check the surjectivity of $(*)$, let $([y_j]_j \mid j \in S)$ be any element of $\prod_{j\in S} C_j$; hence, for every $j \in S$ there exists $i(j) \in I$ such that $y_j \in F(i)_j$, and since I is κ^+-filtered, we can find $l \in I$ such that $l \geq i(j)$ for every $j \in S$. For every such j, let then $z_j \in F(l)_j$ be the image of y_j; then $([y_j]_j \mid j \in S) = ([z_j]_j \mid j \in S)$, and $([z_j]_j \mid j \in S)$ is the image of $[z_j \mid j \in S]$ under $(*)$.

Lastly, let $i, i' \in I$ and $x_\bullet \in P_i, x'_\bullet \in P_{i'}$, and suppose that $(*)$ maps $[x_\bullet]$ and $[x'_\bullet]$ to the same element of $\prod_{j\in S} C_j$; this means that for every $j \in S$ there exists $i(j) \in I$ with $i(j) \geq i, i'$, such that x_j and x'_j have the same images in $P_{i(j)}$. Then we may find $l \in I$ such that $l \geq i(j)$ for every $j \in S$, whence $[x_\bullet] = [x'_\bullet]$. □

Proposition 4.2.4. *Let $\mathscr{C}$ be a cocomplete category, J a small category, α a cardinal, $F_\bullet : J \to \mathscr{C}$ a functor such that F_j is an α-accessible object of $\mathscr{C}$ for every $j \in \mathrm{Ob}(J)$. Set $\beta := \max(\alpha, |\mathrm{Mor}(J)|^+)$. Then $F := \mathrm{colim}_J F_\bullet$ is β-accessible.*

Proof. Let Λ be a β-filtered partially ordered set, $G_\bullet : \Lambda \to \mathscr{C}$ a functor, and $\tau_\bullet := (\tau_\lambda : G_\lambda \to G \mid \lambda \in \Lambda)$ a universal co-cone for $G_\bullet$; we deduce natural bijections :

$$\begin{aligned}
\varinjlim_{\lambda\in\Lambda} \mathscr{C}(F, G_\lambda) &\xrightarrow{\sim} \varinjlim_{\lambda\in\Lambda} \varprojlim_{j\in\mathrm{Ob}(J)} \mathscr{C}(F_j, G_\lambda) \\
&\xrightarrow{\sim} \varprojlim_{j\in\mathrm{Ob}(J)} \varinjlim_{\lambda\in\Lambda} \mathscr{C}(F_j, G_\lambda) \qquad \text{(by lemma 4.2.3)} \\
&\xrightarrow{\sim} \varprojlim_{j\in\mathrm{Ob}(J)} \mathscr{C}(F_j, G) \qquad \text{(since each } F_j \text{ is } \alpha\text{-accessible)} \\
&\xrightarrow{\sim} \mathscr{C}(F, G)
\end{aligned}$$

whose composition is induced by the universal co-cone $\tau_\bullet$, whence the contention. □

Corollary 4.2.5. *Let $\mathscr{C}$ be a small category, and $\mathscr{I}$ a subset of $\mathrm{Mor}(\widehat{\mathscr{C}})$. We have :*

(i) Every object F of $\widehat{\mathscr{C}}$ is $|\mathrm{Mor}(\mathscr{C}/F)|^+$-accessible.

(ii) $(l(r(\mathscr{I})), r(\mathscr{I}))$ is a weak factorization system for $\widehat{\mathscr{C}}$.

(iii) $l(r(\mathscr{I}))$ is the saturation of $\mathscr{I}$.

Proof. Since $\widehat{\mathscr{C}}$ is cocomplete (remark 1.6.2(i)), we have (i)⇒(ii,iii) by corollary 4.1.9(ii).

(i): If F is representable, say $F = h_A$ for some $A \in \mathrm{Ob}(\mathscr{C})$, then the functor $\widehat{\mathscr{C}}(F, -)$ is isomorphic to the evaluation functor : $G \mapsto G_A$, by Yoneda's lemma, and the latter preserves small colimits, so F is κ-small for every cardinal κ.

For a general F, by proposition 1.7.3 there exists a functor $F_\bullet : \mathscr{C}/F \to \widehat{\mathscr{C}}$, with F_j representable for every $j \in \mathrm{Ob}(J)$, and such that F represents the colimit of $F_\bullet$; then it suffices to invoke proposition 4.2.4 to conclude. □

Example 4.2.6. Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories, and $(G : \widehat{\mathscr{B}} \rightleftarrows \widehat{\mathscr{A}} : F)$ an adjoint pair of functors. Then we claim that F is accessible. For the proof, in light of remark 4.2.2(v), it suffices to check that the same holds for the composition $\mathrm{e}_B \circ F$, for every $B \in \mathrm{Ob}(\mathscr{B})$. However, Yoneda's lemma yields a natural identification :

$$\mathrm{e}_B \circ F(X) \xrightarrow{\sim} \widehat{\mathscr{B}}(h_B, FX) \xrightarrow{\sim} \widehat{\mathscr{A}}(G(h_B), X) \qquad \forall X \in \mathrm{Ob}(\widehat{\mathscr{A}})$$

so that we are reduced to the assertion that $G(h_B)$ is an accessible object of $\widehat{\mathscr{A}}$, for every $B \in \mathrm{Ob}(\mathscr{B})$, which holds by corollary 4.2.5(i).

Corollary 4.2.7. *Let $\mathscr{C}$ be a small category, and $\mathscr{M}$ the class of all monomorphisms in $\widehat{\mathscr{C}}$. Then $(\mathscr{M}, r(\mathscr{M}))$ is a weak factorization system for $\widehat{\mathscr{C}}$.*

Proof. For every $a \in \mathrm{Ob}(\mathscr{C})$, let $\mathscr{J}_a$ be the set of all epimorphisms $h_a \to K$ such that $K(b)$ is a quotient set of $h_a(b)$, for every $b \in \mathrm{Ob}(\mathscr{C})$ (*i.e.* $K(b) = h_a(b)/\!\sim$ for some equivalence relation $\sim$ on $h_a(b)$). Let $\mathscr{I}$ be the set of all monomorphisms of $\widehat{\mathscr{C}}$ of the form $L \hookrightarrow K$, with $K \in \bigcup_{a\in\mathrm{Ob}(\mathscr{C})} \mathscr{J}_a$. We shall show that $\mathscr{M} = l(r(\mathscr{I}))$; it will follow that $r(\mathscr{M}) = r(\mathscr{I})$ (proposition 3.1.9(iii)), whence the assertion, in view of corollary 4.2.5(ii).

Now, by corollary 4.2.5(iii), $l(r(\mathscr{I}))$ is the saturation of $\mathscr{I}$; in particular, $l(r(\mathscr{I})) \subset \mathscr{M}$, in light of example 3.1.14(ii). For the converse, consider a commutative diagram :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

where $i \in \mathscr{M}$ and $p \in r(\mathscr{I})$, and denote by $\mathscr{D}$ the set of all morphisms $h : B' \to X$ with $A \subset B' \subset B$ and such that h is a diagonal filler for the induced diagram :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ B' \arrow[r, "g_{|B'}"] & Y. \end{tikzcd}$$

Notice that $\mathscr{D} \neq \varnothing$, since $f \in \mathscr{D}$. We endow $\mathscr{D}$ with the partial order such that

$$(h_1 : B_1 \to X) \leq (h_2 : B_2 \to X) \Leftrightarrow B_1 \subset B_2 \text{ and } h_1 = h_{2|B_1}.$$

Clearly, if $(h_i : B_i \to X \,|\, i \in I)$ is any totally ordered subset of $\mathscr{D}$, then there exists a unique $(h : B' \to X) \in \mathscr{D}$ with $B' := \bigcup_{i\in I} B_i$, and such that $h \geq h_i$ for every $i \in I$. By Zorn's lemma, $\mathscr{D}$ admits therefore a maximal element $k : C \to X$, and we are reduced to checking that $C = B$. To this aim, recall that there exists a small category J and a functor $F_\bullet : J \to \widehat{\mathscr{C}}$ whose colimit is represented by B, and such that F_j is a representable presheaf for every $j \in \mathrm{Ob}(J)$ (proposition 1.7.3). Hence, if $C \neq B$, there exists some $j \in \mathrm{Ob}(J)$ such that the image K of F_j in B does not lie in C; with $L := C \cap K$ we get a commutative diagram :

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, hook] & C \arrow[r, "k"] & X \arrow[d, "p"] \\ K \arrow[r, hook] & B \arrow[r, "g"] & Y \end{tikzcd}$$

that, by assumption, admits a diagonal filler $k' : K \to X$. Set $C' := C \cup K \subset B$; then k and k' induce a morphism $k'' : C' \to X$, and it is easily seen that $k'' \in \mathscr{D}$ and $k'' > k$, a contradiction. □

Remark 4.2.8. (i) Corollary 4.2.7 also admits the following more constructive proof. Let us first notice that for every $X \in \mathrm{Ob}(\widehat{\mathscr{C}})$ there exists a monomorphism $X \to Y$ such that Y is $\mathscr{M}$-injective, *i.e.* Y is an injective object of $\widehat{\mathscr{C}}$ (remark 3.1.2(iii)) : indeed, this follows directly from lemma 1.8.6, since $\widehat{\mathscr{C}}$ is cartesian closed and has a subobject classifier (remark 1.7.8(ii) and proposition 1.8.4(i)).

Next, let $f : X' \to X$ be any morphism of $\widehat{\mathscr{C}}$; we regard f as an object of $\widehat{\mathscr{C}}/X$, which is equivalent to the category of presheaves over $\mathscr{C}/X$ (lemma 1.7.2(i)). Then, by the foregoing, there exists a monomorphism $j/X : (X', f) \to (Y, g)$ in $\widehat{\mathscr{C}}/X$ such that (Y, g) is an injective object of $\widehat{\mathscr{C}}/X$. Thus, $f = g \circ j$, and since the source functor $\widehat{\mathscr{C}}/X \to \widehat{\mathscr{C}}$ preserves and reflects monomorphisms (corollary 1.4.6(i)), it is then easily seen that j is a monomorphism of $\widehat{\mathscr{C}}$, and that $g : Y \to X$ has the right lifting property with respect to the class of monomorphisms of $\widehat{\mathscr{C}}$, as required.

(ii) For every small category $\mathscr{A}$, corollary 4.2.7 yields a model category structure on $\widehat{\mathscr{A}}$ with $\mathscr{W} = \mathrm{Mor}(\widehat{\mathscr{A}})$, whose cofibrations are the monomorphisms of $\widehat{\mathscr{A}}$.

Proposition 4.2.9. *Let α be an infinite cardinal, $\mathscr{A}$ an α-small category, and $F \in \mathrm{Ob}(\widehat{\mathscr{A}})$. Then we have :*

(i) F is the α-filtered union of its α-accessible subpresheaves.

(ii) F is α-accessible if and only if it has size $< \alpha$.

Proof. Let $\mathscr{F}$ be the set of subpresheaves of F of size $< \alpha$, endowed with the partial order induced by inclusion of presheaves. Since α is infinite, arguing as in example 1.1.2(ii) we easily see that $(\mathscr{F}, \leq)$ is α-filtered. Let $U : (\mathscr{F}, \leq) \to \widehat{\mathscr{A}}$ be the inclusion; we have an obvious co-cone $j_\bullet : U \Rightarrow c_F$, whence an induced morphism

$$j : \varinjlim_{G \in \mathscr{F}} G \to F$$

and j is a monomorphism, by example 1.3.7(iii). Notice moreover that for every $A \in \mathrm{Ob}(\mathscr{A})$, the image of every morphism of presheaves $h_A \to F$ lies in $\mathscr{F}$, since by assumption $|\mathrm{Mor}(\mathscr{A})| < \alpha$. On the other hand, F is the colimit of a system of representable presheaves (proposition 1.7.3); furthermore, for every functor $S : I \to \mathsf{Set}$ and every universal co-cone $\tau_\bullet : S \Rightarrow c_T$, we have $T = \bigcup_{i \in \mathrm{Ob}(I)} \tau_i(Si)$. Since the colimits of $\widehat{\mathscr{A}}$ are computed termwise, we conclude that j *is an isomorphism.*

Now, if F is α-accessible, it follows that there exists $G \in \mathscr{F}$ and $f \in \widehat{\mathscr{A}}(F, G)$ such that $\mathbf{1}_F$ is the composition of f with the inclusion $j_G : G \to F$; but then j_G must be an isomorphism, so F has size $< \alpha$. Conversely, if F has size $< \alpha$, then $|\mathrm{Mor}(\mathscr{A}/F)| < \alpha$, and therefore F is α-accessible, by corollary 4.2.5(i) and remark 4.2.2(i). This concludes the proof of (i) and (ii). □

Corollary 4.2.10. *Let α be an infinite cardinal, $\mathscr{A}$ an α-small category, and F an α-accessible object of $\widehat{\mathscr{A}}$. Then :*

(i) Every subobject and every quotient of F is α-accessible.

(ii) The category $\mathsf{Acc}_\alpha(\widehat{\mathscr{A}})$ is finitely complete and finitely cocomplete.

Proof. (i): By proposition 4.2.9(ii), F has size $< \alpha$, and the assertion means that the same holds for every subobject and every quotient of F, which is obvious.

(ii): Let I be a finite category and $\phi : I \to \widehat{\mathscr{A}}$ a functor such that $\phi(i)$ is α-accessible for every $i \in \mathrm{Ob}(I)$. Again by proposition 4.2.9(ii), it is clear that the limit and colimit of ϕ are α-accessible, so we conclude with [13, Lemme 2.52]. □

Proposition 4.2.11. *(i) Let $\mathscr{A}, \mathscr{B}$ be two small categories, $F : \widehat{\mathscr{A}} \to \widehat{\mathscr{B}}$ an accessible functor. Then there exists a cardinal α such that for every cardinal $\beta \geq \alpha$ we have :*

$$F(\mathsf{Acc}_\beta(\widehat{\mathscr{A}})) \subset \mathsf{Acc}_{\beta^\alpha}(\widehat{\mathscr{B}}).$$

(ii) In particular, if $\beta_0 \geq \alpha$ and $\beta = 2^{\beta_0}$, then :

$$F(\mathsf{Acc}_\beta(\widehat{\mathscr{A}})) \subset \mathsf{Acc}_\beta(\widehat{\mathscr{B}}).$$

Proof. (i): Let γ be a cardinal $> |\mathrm{Mor}(\mathscr{A})|$ such that F is γ-accessible. Let $\mathsf{Acc}^0_\gamma(\widehat{\mathscr{A}})$ be the full subcategory of $\mathsf{Acc}_\gamma(\widehat{\mathscr{A}})$ whose objects are the γ-accessible presheaves X on $\mathscr{A}$ such that $X(A) \in \gamma$ for every $A \in \mathrm{Ob}(\mathscr{A})$. Then $\mathsf{Acc}^0_\gamma(\widehat{\mathscr{A}})$ is a small category, and the inclusion functor $\mathsf{Acc}^0_\gamma(\widehat{\mathscr{A}}) \to \mathsf{Acc}_\gamma(\widehat{\mathscr{A}})$ is an equivalence of categories, by virtue of proposition 4.2.9(ii). Taking into account corollary 4.2.5(i), it follows that there exists a cardinal $\alpha \geq \gamma$ such that F sends every γ-accessible presheaf on $\mathscr{A}$ to an α-accessible presheaf on $\mathscr{B}$. Let now $\beta \geq \alpha$ be another cardinal, X a β-accessible object of $\widehat{\mathscr{A}}$, and denote by $(I, \leq)$ the partially ordered set of γ-accessible subobjects of X. Since X has size $< \beta$ and since each element of I has size $< \gamma$ (proposition 4.2.9(ii)) we easily see that $|I| < \beta^\gamma$. By proposition 4.2.9(i), $(I, \leq)$ is γ-filtered and X represents the colimit of the inclusion functor $(I, \leq) \to \widehat{\mathscr{A}}$. Since F is γ-accessible, we get an induced isomorphism :

$$\operatorname*{colim}_{Y \in I} FY \xrightarrow{\sim} FX.$$

Since $\beta^\alpha \geq \max(\beta^\gamma, \alpha)$, we then conclude that FX is β^α-accessible, by proposition 4.2.4 and remark 4.2.2(i).

(ii): Notice that $\beta_0\alpha = \beta_0$, so $\beta^\alpha = (2^{\beta_0})^\alpha = 2^{\beta_0\alpha} = 2^{\beta_0} = \beta$, and apply (i). □

4.2.12. Let us now specialize as follows the situation of theorem 4.1.8 : we take $\mathscr{C} := \widehat{\mathscr{A}}$ for a given small category $\mathscr{A}$, and $\mathscr{I}$ shall be *a set of monomorphisms of* $\widehat{\mathscr{A}}$ (corollary 4.2.5(i)); recall that Ω denotes the class of all ordinals, and for every $\alpha \in \Omega$ we let $\alpha^+ := \alpha + 1$. The proof of *loc.cit.* attaches to $\mathscr{I}$ a family of functors $(\mathsf{L}^\alpha : \alpha^+ \times \widehat{\mathscr{A}} \to \widehat{\mathscr{A}} \mid \alpha \in \Omega)$ such that the restriction of L^α to $\beta^+ \times \widehat{\mathscr{A}}$ equals L^β, for every $\beta \leq \alpha$. For every $\alpha \in \Omega$, let us then denote also

$$\mathsf{L}_\alpha : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$$

the restriction of L^α to the full subcategory $\{\alpha\} \times \widehat{\mathscr{A}} \xrightarrow{\sim} \widehat{\mathscr{A}}$.

Proposition 4.2.13. *For every ordinal α, the following holds :*

(i) L_α sends monomorphisms to monomorphisms.

(ii) Every pair of monomorphisms $X \to Z \leftarrow Y$ of $\widehat{\mathscr{A}}$ induces an isomorphism :

$$\mathsf{L}_\alpha(X \cap Y) \xrightarrow{\sim} \mathsf{L}_\alpha(X) \cap \mathsf{L}_\alpha(Y).$$

(iii) L_α is accessible.

Proof. We argue by transfinite induction on α. Since $\mathsf{L}_0 = \mathbf{1}_\mathscr{C}$, the assertions are trivial for $\alpha = 0$. Next, define the functors $S, T : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$ and the natural transformations $\mu_\bullet : S \Rightarrow \mathbf{1}_{\widehat{\mathscr{A}}}$ and $\nu_\bullet : S \Rightarrow T$ as in the proof of theorem 4.1.8, so that we have a cocartesian diagram :

$$\begin{array}{ccc} SX & \xrightarrow{\mu_X} & X \\ {\scriptstyle \nu_X}\downarrow & & \downarrow \\ TX & \longrightarrow & \mathsf{L}_1 X \end{array} \qquad \forall X \in \mathrm{Ob}(\widehat{\mathscr{A}}).$$

According to remark 4.2.2(iii), in order to check that L_1 is accessible, it then suffices to show that the same holds for the functors S and T. Let us recall the constructions of these functors : first, for every morphism $u : X \to Y$ of $\widehat{\mathscr{A}}$, let

$$S_u : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}} \qquad (\text{resp. } T_u : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}})$$

be the composition of the functor $h_{X^{\mathrm{op}}} : \widehat{\mathscr{A}} \to \mathsf{Set}$ with the functor $X^{(-)}$ (resp. with $Y^{(-)}$: notation of §1.2.14). We then have :

$$SX := \coprod_{u \in \mathscr{I}} S_u X \qquad TX := \coprod_{u \in \mathscr{I}} T_u X.$$

Now, the functor $h_{X^{\mathrm{op}}}$ is accessible for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$ (corollary 4.2.5(i)), and the same holds for $X^{(-)}$, by virtue of remark 1.6.4(i), hence S_u is accessible for every morphism u of $\widehat{\mathscr{A}}$ (remark 4.2.2(i,ii)), and then the same holds for S, by remark 4.2.2(iii). Likewise we check that T is accessible. Next, let us observe :

Claim 4.2.14. (i) Let $S_1 \xleftarrow{i_1} S_0 \xrightarrow{i_2} S_2$ be maps of sets, and suppose that i_1 is injective. Then $S := (S_1 \setminus i_1(S_0)) \sqcup S_2$ represents the amalgamated sum $S_1 \sqcup_{(i,j)} S_2$, with universal co-cone $S_1 \xrightarrow{e_1} S \xleftarrow{e_2} S_2$ given by the natural inclusion e_2, and by the map e_1 such that

$$e_1(s_1) := s_1 \quad \forall s_1 \in S_1 \setminus i_1(S_0) \qquad \text{and} \qquad e_1(i_1(s_0)) := i_2(s_0) \quad \forall s_0 \in S_0.$$

(ii) Consider a commutative diagram of $\widehat{\mathscr{A}}$:

$$(*) \qquad \begin{tikzcd} X_1 \arrow[d, "i_1"'] & X_0 \arrow[l] \arrow[r] \arrow[d, "i_0"] & X_2 \arrow[d, "i_2"] \\ S_1 & S_0 \arrow[l] \arrow[r] & S_2 \end{tikzcd}$$

and suppose that both i_2 and the induced morphism $k : X_1 \sqcup_{X_0} S_0 \to S_1$ are monomorphisms. Then the same holds for the induced morphism :

$$X_1 \sqcup_{X_0} X_2 \to S_1 \sqcup_{S_0} S_2.$$

(iii) Consider a commutative diagram of $\widehat{\mathscr{A}}$:

$$\begin{tikzcd} X_1 \arrow[d] & X_0 \arrow[l, "x_1"'] \arrow[r, "x_2"] \arrow[d] & X_2 \arrow[d] \\ S_1 & S_0 \arrow[l, "s_1"'] \arrow[r, "s_2"] & S_2 \\ Y_1 \arrow[u] & Y_0 \arrow[l, "y_1"'] \arrow[r, "y_2"] \arrow[u] & Y_2 \arrow[u] \end{tikzcd}$$

and suppose that :

(a) x_1, s_1, y_1 and all the vertical arrows are monomorphisms of $\widehat{\mathscr{A}}$

(b) the two square subdiagrams on the left are cartesian.

Then the induced morphism :

$$\omega : (X_1 \times_{S_1} Y_1) \sqcup_{X_0 \times_{S_0} Y_0} (X_2 \times_{S_2} Y_2) \to (X_1 \sqcup_{X_0} X_2) \times_{S_1 \sqcup_{S_0} S_2} (Y_1 \sqcup_{Y_0} Y_2)$$

is an isomorphism.

Proof : (i): Let $S_1 \xrightarrow{f_1} X \xleftarrow{f_2} S_2$ be two maps of sets such that $f_1 i_1 = f_2 i_2$; let $f : S \to X$ be the map such that $f(s_1) := f_1(s_1)$ for every $s_1 \in S_1 \setminus i_1(S_0)$ and $f(s_2) := f_2(s_2)$ for every $s_2 \in S_2$. It is easily seen that f is the unique map with $f e_k = f_k$ for $k = 1, 2$, whence the assertion.

(ii): We regard $(*)$ as the composition of two diagrams :

$$(**) \qquad \begin{tikzcd} X_1 \arrow[d, "i_1"'] & X_0 \arrow[l] \arrow[r] \arrow[d, "i_0"] & X_2 \arrow[d, "i_2"] \\ X_1 \sqcup_{X_0} S_0 \arrow[d, "k"'] & S_0 \arrow[l] \arrow[r] \arrow[d, equal] & S_2 \arrow[d, equal] \\ S_1 & S_0 \arrow[l] \arrow[r] & S_2 \end{tikzcd}$$

and notice that both the top left and the bottom right square subdiagrams of $(**)$ are cocartesian, and both i_2 and k are injective. Clearly, it suffices to prove the assertion separately for the top two and the bottom two square subdiagrams; hence, we are reduced to the case where the left square subdiagram of $(*)$ is cocartesian. In that case, we need to check that if i_2 is a monomorphism, then the same holds for the induced morphism

$$(\dagger) \qquad X_1 \sqcup_{X_0} X_2 \to (X_1 \sqcup_{X_0} S_0) \sqcup_{S_0} S_2 \xrightarrow{\sim} X_1 \sqcup_{X_0} S_2.$$

But $(\dagger)$ is the push-out $X_1 \sqcup_{X_0} i_2$, so the assertion follows from example 3.1.14(ii).

(iii): The morphism ω is induced by the pair of morphisms :

$$X_1 \times_{S_1} Y_1 \xrightarrow{f_1} (X_1 \sqcup_{X_0} X_2) \times_{S_1 \sqcup_{S_0} S_2} (Y_1 \sqcup_{Y_0} Y_2) \xleftarrow{f_2} X_2 \times_{S_2} Y_2$$

where $(f_i \mid i = 1, 2)$ are in turn induced by the universal co-cones

$$(e_i^X : X_i \to X_1 \sqcup_{X_0} X_2 \mid i = 1, 2) \qquad (e_i^Y : Y_i \to Y_1 \sqcup_{Y_0} Y_2 \mid i = 1, 2).$$

Since the limits and colimits of $\widehat{\mathscr{A}}$ are computed termwise (see 1.3), we are then reduced to checking the corresponding assertions, where $\widehat{\mathscr{A}}$ is replaced by the category Set. Moreover, notice that the assertion is an intrinsic property of ω, *i.e.* it is independent of the choices of representatives for the amalgamated sums and fibre products appearing in the source and target of ω. Then, since x_1, s_1 and y_1 are monomorphisms of Set (*i.e.* injections), we may take representatives as in (i) :

$$X_1 \sqcup_{X_0} X_2 \xrightarrow{\sim} (X_1 \setminus X_0) \sqcup X_2 \quad S_1 \sqcup_{S_0} S_2 \xrightarrow{\sim} (S_1 \setminus S_0) \sqcup S_2 \quad Y_1 \sqcup_{Y_0} Y_2 \xrightarrow{\sim} (Y_1 \setminus Y_0) \sqcup Y_2$$

with universal co-cone $X_1 \xrightarrow{e_1^X} (X_1 \setminus X_0) \sqcup X_2 \xleftarrow{e_2^X} X_2$ given by the obvious inclusion map e_2^X, and by the map e_1^X such that $e_1^X(a) := a$ for every $a \in X_1 \setminus X_0$, and $e_1^X(a) := x_2(a)$ for every $a \in X_0$; and likewise for the corresponding universal co-cone $(e_i^S \mid i = 1, 2)$ for $(S_1 \setminus S_0) \sqcup S_2$ (resp. $(e_i^Y \mid i = 1, 2)$ for $(Y_1 \setminus Y_0) \sqcup Y_2$).

Furthermore, since the vertical arrows are monomorphisms of Set, we may regard X_i and Y_i as subsets of S_i for $i = 0, 1, 2$, and we have natural identifications :

$$X_i \times_{S_i} Y_i \xrightarrow{\sim} X_i \cap Y_i \qquad \forall i = 0, 1, 2$$

so that ω comes down to the map :

$$(X_1 \cap Y_1) \sqcup_{X_0 \cap Y_0} (X_2 \cap Y_2) \to ((X_1 \setminus X_0) \sqcup X_2) \times_{(S_1 \setminus S_0) \sqcup S_2} ((Y_1 \setminus Y_0) \sqcup Y_2)$$

induced by the universal co-cones $e_\bullet^X, e_\bullet^S, e_\bullet^Y$. Notice as well that the maps

$$(X_1 \setminus X_0) \sqcup X_2 \to (S_1 \setminus S_0) \sqcup S_2 \leftarrow (Y_1 \setminus Y_0) \sqcup Y_2$$

restrict to injective maps

$$X_1 \setminus X_0 \to S_1 \setminus S_0 \leftarrow Y_1 \setminus Y_0 \qquad X_2 \to S_2 \leftarrow Y_2$$

whence a natural identification :

$$((X_1\setminus X_0)\sqcup X_2)\times_{(S_1\setminus S_0)\sqcup S_2}((Y_1\setminus Y_0)\sqcup Y_2)\xrightarrow{\sim}((X_1\setminus X_0)\times_{S_1\setminus S_0}(Y_1\setminus Y_0))\sqcup(X_2\times_{S_2}Y_2)$$
$$\xrightarrow{\sim}((X_1\setminus X_0)\cap(Y_1\setminus Y_0))\sqcup(X_2\cap Y_2).$$

But since the two square subdiagrams on the left are cartesian, we have as well :

$$Y_0 = Y_1\cap S_0 \qquad X_0 = X_1\cap S_0$$

so finally ω is naturally identified with a map :

$$\omega' : (X_1\cap Y_1)\sqcup_{X_1\cap Y_1\cap S_0}(X_2\cap Y_2)\to((X_1\cap Y_1)\setminus S_0)\sqcup(X_2\cap Y_2).$$

Lastly, $(X_1\cap Y_1)\sqcup_{X_1\cap Y_1\cap S_0}(X_2\cap Y_2)$ is represented as in (i) by $((X_1\cap Y_1)\setminus S_0)\sqcup(X_2\cap Y_2)$, and it is easily seen that ω' is just the identity map, under these identifications. ◇

Now, in light of claim 4.2.14(ii,iii), in order to prove assertions (i) and (ii) for L_1, it then suffices to show the following :

Claim 4.2.15. (i) Every monomorphism $X\xrightarrow{j}Y$ of $\widehat{\mathscr{A}}$ induces a cartesian diagram:

$$\mathscr{D} \quad : \quad \begin{tikzcd} SX \arrow[r, "\nu_X"] \arrow[d, "Sj"'] & TX \arrow[d, "Tj"] \\ SY \arrow[r, "\nu_Y"] & TY \end{tikzcd}$$

whose arrows are monomorphisms of $\widehat{\mathscr{A}}$, and a monomorphism $SY\sqcup_{SX}TX\to TY$.

(ii) Every pair of monomorphisms $X\to Z\leftarrow Y$ of $\widehat{\mathscr{A}}$ induces isomorphisms :

$$S(X\cap Y)\xrightarrow{\sim}SX\cap SY \qquad T(X\cap Y)\xrightarrow{\sim}TX\cap TY.$$

Proof : (i): In light of example 1.3.7(iv), in order to show that $\mathscr{D}$ is cartesian and that its arrows are monomorphisms, it suffices to check that the same holds for the diagram :

$$\mathscr{D}_u \quad : \quad \begin{tikzcd} K^{(\widehat{\mathscr{A}}(K,X))} \arrow[r, "u^{(\widehat{\mathscr{A}}(K,X))}"] \arrow[d, "K^{(j_*)}"'] & L^{(\widehat{\mathscr{A}}(K,X))} \arrow[d, "L^{(j_*)}"] \\ K^{(\widehat{\mathscr{A}}(K,Y))} \arrow[r, "u^{(\widehat{\mathscr{A}}(K,Y))}"] & L^{(\widehat{\mathscr{A}}(K,Y))} \end{tikzcd}$$

for every monomorphism $u : K \to L$ of $\widehat{\mathscr{A}}$. Since the limits and colimits of $\widehat{\mathscr{A}}$ are computed termwise, we then come down to proving that the diagram of sets :

$$\begin{tikzcd} \widehat{\mathscr{A}}(K,X)\times KA \arrow[r, "\widehat{\mathscr{A}}(K,X)\times u_A"] \arrow[d, "j_*\times KA"'] & \widehat{\mathscr{A}}(K,X)\times LA \arrow[d, "j_*\times LA"] \\ \widehat{\mathscr{A}}(K,Y)\times KA \arrow[r, "\widehat{\mathscr{A}}(K,Y)\times u_A"] & \widehat{\mathscr{A}}(K,Y)\times LA \end{tikzcd}$$

is cartesian and with injective arrows, for every $A\in\mathrm{Ob}(\mathscr{A})$. But since both j_* and u_A are injections, this follows by direct inspection. Lastly, in order to check that the induced morphism $SY\sqcup_{SX}TX\to TY$ is a monomorphism, we can again reduce to the corresponding assertion for a cartesian diagram of sets with injective arrows, which follows easily from claim 4.2.14(i).

(ii): In order to check that the natural morphism $S(X \cap Y) \to SX \cap SY$ is an isomorphism, in light of example 1.3.7(iv) it suffices to show that every monomorphism $K \to L$ of $\widehat{\mathscr{A}}$ induces a cartesian diagram :

$$\begin{array}{ccc} K^{(\widehat{\mathscr{A}}(K,X\cap Y))} & \longrightarrow & K^{(\widehat{\mathscr{A}}(K,X))} \\ \downarrow & & \downarrow \\ K^{(\widehat{\mathscr{A}}(K,X))} & \longrightarrow & K^{(\widehat{\mathscr{A}}(K,Z))} \end{array}$$

and reasoning as in the proof of (i), we are reduced to checking that for every $A \in \mathrm{Ob}(\mathscr{A})$ the induced diagram of sets :

$$\begin{array}{ccc} \widehat{\mathscr{A}}(K, X \cap Y) \times KA & \longrightarrow & \widehat{\mathscr{A}}(K, X) \times KA \\ \downarrow & & \downarrow \\ \widehat{\mathscr{A}}(K, Y) \times KA & \longrightarrow & \widehat{\mathscr{A}}(K, Z) \times KA \end{array}$$

is cartesian. But it is clear that the diagram of sets :

$$\begin{array}{ccc} \widehat{\mathscr{A}}(K, X \cap Y) & \longrightarrow & \widehat{\mathscr{A}}(K, X) \\ \downarrow & & \downarrow \\ \widehat{\mathscr{A}}(K, Y) & \longrightarrow & \widehat{\mathscr{A}}(K, Z) \end{array}$$

is cartesian, whence the contention. ◇

Next, let $\alpha > 1$ be an ordinal, and suppose that the proposition is already known for L_β, for every ordinal $\beta < \alpha$. If we have $\alpha = \beta^+$ for some such β, then $\mathsf{L}_\alpha = \mathsf{L}_1 \circ \mathsf{L}_\beta$, and since the proposition is known for L_1 and L_β, we deduce it for L_α as well (remark 4.2.2(i,ii)). Lastly, if α is a limit ordinal, then $\mathsf{L}_\alpha = \mathrm{colim}_{\beta<\alpha} \mathsf{L}_\beta$, so L_α is accessible, by virtue of remark 4.2.2(iii), and since the filtered colimits are exact in $\widehat{\mathscr{A}}$ (example 1.3.7(iii)), we also get (i) and (ii) for L_α. □

Corollary 4.2.16. *In the situation of §4.2.12, suppose moreover that, for every element $f : X \to Y$ of $\mathscr{I}$, the functor $h_{X^{\mathrm{op}}} : \widehat{\mathscr{A}} \to \mathsf{Set}$ preserves all small filtered colimits. Then the functor L_α preserves all small filtered colimits, for every ordinal α.*

Proof. It sufices to trace the proof of proposition 4.2.13, adding the condition that $h_{X^{\mathrm{op}}}$ is $\aleph_0$-accesible for every element $X \to Y$ of $\mathscr{I}$. Indeed, in this case we get that the same holds for both S_u and T_u, for every morphism u of $\widehat{\mathscr{A}}$, then also for the functors S and T, and hence also for L_1. Then the assertion follows, by transfinite induction. □

4.3. **Cofibrantly generated model categories.** As explained earlier (see corollary 4.1.9), the small object argument is one of the possible tools for constructing weak factorization systems. When, in a model category structure, the weak factorisation systems $(\mathscr{C}of, \mathscr{W} \cap \mathscr{F}ib)$ and $(\mathscr{W} \cap \mathscr{C}of, \mathscr{F}ib)$ can be constructed out of the small object argument, we say that the model category is *cofibrantly generated.* Here is a more formal definition :

Definition 4.3.1. We say that a model category $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ *is cofibrantly generated* if $\mathscr{C}$ is cocomplete and there exist *subsets* $\mathscr{I}, \mathscr{J} \subset \mathrm{Mor}(\mathscr{C})$ such that:

- $\mathscr{I}$ is small relative to $l(r(\mathscr{I}))$ and $\mathscr{J}$ is small relative to $l(r(\mathscr{J}))$
- $\mathscr{F}ib = r(\mathscr{J})$ and $\mathscr{W} \cap \mathscr{F}ib = r(\mathscr{I})$.

We then say that $\mathscr{I}$ *is a generating set of cofibrations*, and $\mathscr{J}$ *is a generating set of trivial cofibrations.*

Our first important example of cofibrantly generated model category is provided by the following theorem, which is found in [8, Th.2.3.11] :

Theorem 4.3.2. *For every ring R there exists a unique cofibrantly generated model structure on the category $\mathsf{C}(R)$ of chain complexes of R-modules, whose weak equivalences are the quasi-isomorphisms, and whose fibrations are the epimorphisms.*

Proof. Recall that a *quasi-isomorphism* is a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of $\mathsf{C}(R)$ that induces isomorphisms $H_n(f_\bullet) : H_n X_\bullet \xrightarrow{\sim} H_n Y_\bullet$ in homology, for every $n \in \mathbb{Z}$. Moreover, a complex $X_\bullet$ is *acyclic* if $H_n X_\bullet = 0$ for every $n \in \mathbb{Z}$.

For every $n \in \mathbb{Z}$, define the chain complex $D^n_\bullet$ and the morphisms $\phi^n_\bullet$ and $\psi^\bullet_n$ of $\mathsf{C}(R)$ as in example 3.1.15, and set $\mathscr{I} := \{\phi^n_\bullet \mid n \in \mathbb{Z}\}$, $\mathscr{J} := \{\psi^n_\bullet \mid n \in \mathbb{Z}\}$; let also $\mathscr{E}$ (resp. $\mathscr{W}$) be the class of epimorphisms (resp. of quasi-isomorphisms) of $\mathsf{C}(R)$, so that $\mathscr{E} = r(\mathscr{I})$ and $\mathscr{E} \cap \mathscr{W} = r(\mathscr{J})$. By example 4.1.7 and remark 4.1.2(iii), $\mathscr{I}$ and $\mathscr{J}$ are small in $\mathsf{C}(R)$, and both $(l(\mathscr{E}), \mathscr{E})$ and $(l(\mathscr{E} \cap \mathscr{W}), \mathscr{E} \cap \mathscr{W})$ are weak factorization systems, by corollary 4.1.9(ii); moreover, it is clear that $\mathscr{W}$ enjoys the 2-out-of-3 property, and it is known that $\mathsf{C}(R)$ is complete and cocomplete ([13, Exerc.2.98(ii) and Exemp.2.44(i)]). Thus, it remains only to check that $l(\mathscr{E}) = \mathscr{W} \cap l(\mathscr{E} \cap \mathscr{W})$.

• First, let $f_\bullet \in \mathscr{E}$; by applying the long exact homology sequence to the short exact sequence (see [13, §.2.5.2]) :

$$0 \to \operatorname{Ker} f_\bullet \to X_\bullet \xrightarrow{f_\bullet} Y_\bullet \to 0$$

we easily see that $f_\bullet \in \mathscr{E} \cap \mathscr{W} \Leftrightarrow f_\bullet \in \mathscr{E}$ and $\operatorname{Ker} f_\bullet$ is acyclic.

• Let us say, as usual, that a chain complex $X_\bullet$ is *cofibrant* if the unique morphism $0_\bullet \to X_\bullet$ is in $l(\mathscr{E} \cap \mathscr{W})$; moreover, let us say that $X_\bullet$ is *termwise projective*, if X_n is a projective R-module for every $n \in \mathbb{Z}$, and that a morphism $f_\bullet : X_\bullet \to Y_\bullet$ is a *termwise split injection*, if $f_n : X_n \to Y_n$ is a split monomorphism of R-modules for every $n \in \mathbb{Z}$. We notice :

Claim 4.3.3. (i) If $(X_\bullet d^X_\bullet) \in \mathrm{Ob}(\mathsf{C}(R))$ is cofibrant, then it is termwise projective.

(ii) If $(X_\bullet, d^X_\bullet)$ is termwise projective and bounded below, then it is cofibrant.

(iii) $(X_\bullet, d^X_\bullet)$ is a projective object of $\mathsf{C}(R) \Leftrightarrow (0_\bullet \to X_\bullet) \in \mathscr{W} \cap l(\mathscr{E})$.

Proof : (i): Let $q : M \to N$ be a surjective R-linear map of R-modules; then the induced morphism $q \otimes_R D^n_\bullet : M \otimes_R D^n_\bullet \to N \otimes_R D^n_\bullet$ lies in $\mathscr{E} \cap \mathscr{W}$ for every $n \in \mathbb{Z}$. Every R-linear map $f : X_{n-1} \to N$ induces a morphism $f_\bullet : X_\bullet \to N \otimes_R D^n_\bullet$ with $f_{n-1} := f$, $f_n := d^X_n \circ f$, and $f_k := 0$ for every $k \neq n, n-1$. If $X_\bullet$ is cofibrant, then $f_\bullet$ lifts to a morphism $g_\bullet : X_\bullet \to M \otimes_R D^n_\bullet$ with $(q \otimes_R D^n_\bullet) \circ g_\bullet) = f_\bullet$, and especially $g_{n-1} : X_n \to M_{n-1}$ is a lift to f, so X_{n-1} is projective.

(ii): Let $p_\bullet : (M_\bullet, d^M_\bullet) \to (N_\bullet, d^N_\bullet)$ be in $\mathscr{E} \cap \mathscr{W}$, and set $(K_\bullet, d^K_\bullet) := \operatorname{Ker}(p_\bullet)$; this means that $p_n : M_n \to N_n$ is surjective for every $n \in \mathbb{N}$, and $H_n(K_\bullet) = 0$ for every $n \in \mathbb{Z}$. Now, let $N \in \mathbb{Z}$ such that $X_k = 0$ for every $k < N$, and let $f_\bullet : X_\bullet \to N_\bullet$ be a morphism of $\mathsf{C}(R)$; we construct, by induction on $k \geq N$, a system of R-linear maps $g_\bullet := (g_k : X_k \to M_k \mid k \in \mathbb{Z})$ such that $g_\bullet$ is a morphism $X_\bullet \to M_\bullet$ of $\mathsf{C}(R)$ with $p_\bullet \circ g_\bullet = f_\bullet$. Clearly, g_k shall be the zero map for every $k < N$. Next, suppose that $n \geq N$, and that g_k has already been exhibited for every $k < n$; since p_n is surjective and X_n is projective, we may find an R-linear map $h : X_n \to M_n$ such that $p_n \circ h = f_n$. Set

$$H := d^M_n \circ h - g_{n-1} \circ d^X_n : X_n \to M_{n-1}.$$

Then $p_{n-1} \circ H = d_n^N \circ p_n \circ h - p_{n-1} \circ g_{n-1} \circ d_n^X = d_n^N \circ f_n - f_{n-1} \circ d_n^X = 0$, and moreover $d_{n-1}^M \circ H = -d_{n-1}^M \circ g_{n-1} \circ d_n^X = -g_{n-2} \circ d_{n-1}^X \circ d_n^X = 0$, so H factors through an R-linear map $H' : X_n \to Z_{n-1}K_\bullet$ and the inclusion $Z_{n-1}K_\bullet \to M_{n-1}$ (where $Z_{n-1}K_\bullet$ denotes the submodule of $(n-1)$-cycles of $K_\bullet$). Since $K_\bullet$ is acyclic, we have $Z_{n-1}K_\bullet = B_{n-1}K_\bullet$, where $B_{n-1}K_\bullet$ denotes the $(n-1)$-boundaries of $K_\bullet$ (see [13, §2.5.1]). Since X_n is projective, we may therefore find an R-linear map $G : X_n \to K_n$ such that $d_n^K \circ G = H'$. Lastly, set $g_n := h - G : X_n \to M_n$; then $d_n^M \circ g_n = d_n^M \circ h - H' = g_{n-1} \circ d_n^X$ and $p_n \circ g_n = f_n - p_n \circ G = f_n$, as required.

(iii): In light of remark 3.1.2(ii), it suffices to check that every projective object $Y_\bullet$ of $\mathsf{C}(R)$ is an acyclic complex. To this aim, let $(P_\bullet, d_\bullet^P)$ be the chain complex such that $P_n := Y_n \oplus Y_{n+1}$ for every $n \in \mathbb{Z}$, with $d_n^P(y, y') := (d_n^Y y, y - d_{n+1}^Y y')$ for every $n \in \mathbb{Z}$ and every $(y, y') \in P_n$. The system of natural projections $(p_n : P_n \to Y_n \mid n \in \mathbb{Z})$ is then an epimorphism $p_\bullet : P_\bullet \to Y_\bullet$ of $\mathsf{C}(R)$; hence, since $Y_\bullet$ is projective, there exists a morphism $s_\bullet : Y_\bullet \to P_\bullet$ of $\mathsf{C}(R)$ such that $p_\bullet \circ s_\bullet = \mathbf{1}_{Y_\bullet}$. For every $n \in \mathbb{Z}$, let also $q_n : P_n \to Y_{n+1}$ be the second natural projection, and set $D_n := q_n \circ s_n : Y_n \to Y_{n+1}$ for every such n. Then the identity $d_n^P \circ s_n = s_{n-1} \circ d_n^Y$ comes down to the system of identities :

$$d_{n+1}^Y D_n + D_{n-1} d_n^Y = \mathbf{1}_{Y_n} \qquad \forall n \in \mathbb{Z}.$$

In particular, if $y \in Z_n Y_\bullet$, then $d_{n+1}^Y D_n(y) = y$, so $y \in B_n Y_\bullet$, so $Y_\bullet$ is acyclic. ◇

Claim 4.3.4. Let $f_\bullet : (X_\bullet, d_\bullet^X) \to (Y_\bullet, d_\bullet^Y)$ be a morphism of $\mathsf{C}(R)$, with $X_\bullet$ cofibrant and $Y_\bullet$ acyclic. Then $f_\bullet$ is homotopically trivial (see [13, Def.5.4]).

Proof : Define the complex $(P_\bullet, d_\bullet^P)$ and the epimorphism $p_\bullet : P_\bullet \to Y_\bullet$ of $\mathsf{C}(R)$ as in the proof of claim 4.3.3(iii), and notice that $\mathrm{Ker}(p_\bullet)$ is isomorphic to $Y_\bullet[1]$. Especially, $\mathrm{Ker}(p_\bullet)$ is acyclic, so $p_\bullet \in \mathscr{E} \cap \mathscr{W}$. Since $X_\bullet$ is cofibrant, it follows that there exists a morphism $g_\bullet : X_\bullet \to P_\bullet$ of $\mathsf{C}(R)$ such that $p_\bullet \circ g_\bullet = f_\bullet$. The latter means that for every $n \in \mathbb{N}$ there exists an R-linear map $h_n : X_n \to Y_{n+1}$ such that $g_n(x) = (f_n(x), h_n(x))$. The condition $d_n^P \circ g_n = g_{n+1} \circ d_n^X$ then translates as the identity :

$$f_n(x) - d_{n+1}^Y h_n(x) = h_{n+1} d_n^X(x) \qquad \forall n \in \mathbb{Z}$$

i.e. $h_\bullet := (h_n \mid n \in \mathbb{Z})$ is a chain homotopy from $f_\bullet$ to the zero map. ◇

Claim 4.3.5. A morphism $i_\bullet : (X_\bullet, d_\bullet^X) \to (Y_\bullet, d_\bullet^Y)$ is in $l(\mathscr{E} \cap \mathscr{W})$ if and only if it is a termwise split injection with cofibrant cokernel.

Proof : Suppose first that $i_\bullet \in l(\mathscr{E} \cap \mathscr{W})$; consider the unique morphism

$$f_\bullet : X_\bullet \to X_n \otimes_R D_\bullet^{n+1} \qquad \text{such that} \qquad f_n = \mathbf{1}_{X_n} \quad \text{and} \quad f_{n+1} = d_{n+1}^X.$$

Since $X_n \otimes_R D_\bullet^{n+1}$ is acyclic, there is a morphism $g_\bullet : Y_\bullet \to X_n \otimes_R D_\bullet^{n+1}$ such that $g_\bullet \circ i_\bullet = f_\bullet$. In particular, i_n is a split monomorphism. On the other hand, since $l(\mathscr{E} \cap \mathscr{W})$ is saturated (proposition 3.1.9(v)), the cocartesian diagram :

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d, "i_\bullet"'] & 0_\bullet \arrow[d] \\ Y_\bullet \arrow[r] & \mathrm{Coker}\, i_\bullet \end{tikzcd}$$

shows that Coker $i_\bullet$ is cofibrant. Conversely, suppose that $i_\bullet$ is a termwise split injection with cofibrant cokernel, and consider a commutative square :

$$\begin{tikzcd} X_\bullet \arrow[r, "f_\bullet"] \arrow[d, "i_\bullet"'] & A_\bullet \arrow[d, "p_\bullet"] \\ Y_\bullet \arrow[r, "g_\bullet"] & B_\bullet \end{tikzcd}$$

with $p_\bullet \in \mathscr{E} \cap \mathscr{W}$. Set $(K_\bullet, d^K_\bullet) := \mathrm{Ker}\, p_\bullet$, and $(C_\bullet, d^C_\bullet) := \mathrm{Coker}\, i_\bullet$, so that $Y_n = X_n \oplus C_n$ and C_n is a projective R-module for every $n \in \mathbb{Z}$, by claim 4.3.3. Then $d^Y_n : X_n \oplus C_n \to X_{n-1} \oplus C_{n-1}$ and $g_n : X_n \oplus C_n \to B_n$ can be expressed as blocs of matrices :

$$d^Y_n = \begin{pmatrix} d^X_n & \tau_n \\ 0 & d^C_n \end{pmatrix} \qquad g_n = \begin{pmatrix} p_n f_n & \sigma_n \end{pmatrix} \qquad \forall n \in \mathbb{Z}$$

for certain R-linear maps $\tau_n : C_n \to X_{n-1}$ and $\sigma_n : C_n \to B_n$, and the conditions $d^Y_{n-1} \circ d^Y_n = 0$ and $d^B_n \circ g_n = g_{n-1} \circ d^Y_n$ come down to the identities :

$$d^X_{n-1} \circ \tau_n + \tau_{n-1} \circ d^C_n = 0 \qquad \text{and} \qquad d^B_n \circ \sigma_n = p_{n-1} \circ f_{n-1} \circ \tau_n + \sigma_{n-1} \circ d^C_n.$$

Likewise, a diagonal filler $h_\bullet : Y_\bullet \to A_\bullet$ for the square can be expressed as a system of R-linear maps $(h_n : X_n \oplus C_n \to A_n \mid n \in \mathbb{Z})$, given by blocs :

$$h_n = \begin{pmatrix} f_n & v_n \end{pmatrix} \qquad \forall n \in \mathbb{Z}$$

and the conditions $p_n \circ h_n = g_n$ and $d^A_n \circ h_n = h_{n-1} \circ d^Y_n$ come down to the identities:

$$p_n \circ v_n = \sigma_n \qquad \text{and} \qquad d^A_n \circ v_n = f_n \circ \tau_n + v_n \circ d^C_n.$$

Now, since C_n is projective, we find for every $n \in \mathbb{Z}$ an R-linear map $G_n : C_n \to A_n$ with $p_n G_n = \sigma_n$. Set $r_n := d^A_n G_n - G_{n-1} d^C_n - f_{n-1}\tau_n : C_n \to A_{n-1}$ for every $n \in \mathbb{Z}$. Then $p_{n-1} r_n = 0$ for every $n \in \mathbb{Z}$, so r_n is the composition of an R-linear map $s_n : C_n \to K_{n-1}$ with the inclusion $j_{n-1} : K_{n-1} \to A_{n-1}$. Moreover, $d^A_{n-1} r_n = -d^A_{n-1} G_{n-1} d^C_n + f_{n-2}\tau_{n-1} d^C_n = -r_{n-1} d^C_n$, so the system $(s_n \mid n \in \mathbb{N})$ defines a morphism $s_\bullet : C_\bullet \to \Sigma K_\bullet$ of $\mathsf{C}(R)$, where $(\Sigma K_\bullet, d^{\Sigma K}_\bullet)$ denotes the chain complex with $\Sigma K_n := K_{n-1}$ and $d^{\Sigma K}_n := -d^K_{n-1}$ for every $n \in \mathbb{Z}$. By claim 4.3.4, $s_\bullet$ is homotopically trivial, *i.e.* there exists a system of maps $(D_n : C_n \to K_n \mid n \in \mathbb{Z})$ such that $-d^K_n \circ D_n + D_{n-1} d^C_n = s_n$ for every $n \in \mathbb{Z}$. Let $v_n := G_n + j_n D_n : C_n \to A_n$ for every $n \in \mathbb{Z}$; then $p_n v_n = \sigma_n$ and $d^A_n v_n = v_{n-1} d^C_n + f_n \tau_n$, *i.e.* $(h_n := (f_n, v_n) : Y_n \to A_n \mid n \in \mathbb{Z})$ is the sought diagonal filler. ◇

We may now conclude the proof : let $(f_\bullet : X_\bullet \to Y_\bullet) \in l(\mathscr{E})$; then $f_\bullet$ is a split monomorphism and $\mathrm{Coker}(f_\bullet)$ is a projective object of $\mathsf{C}(R)$, by example 3.1.13, since $\mathsf{C}(R)$ is an abelian category ([13, Exerc.2.98(ii)]); but then $\mathrm{Coker}(f_\bullet)$ is acyclic, by claim 4.3.3(iii), so $f_\bullet \in \mathscr{W}$, by the long exact homology sequence arising from the short exact sequence $0 \to X_\bullet \xrightarrow{f_\bullet} Y_\bullet \to \mathrm{Coker}(f_\bullet) \to 0$ ([13, §2.5.2]).

Conversely, suppose that $f_\bullet \in \mathscr{W} \cap l(\mathscr{E} \cap \mathscr{W})$, and pick a factorization $f_\bullet = p_\bullet \circ i_\bullet$ with $p_\bullet \in \mathscr{E}$ and $i_\bullet \in l(\mathscr{E})$; then $i_\bullet \in \mathscr{W}$ by the foregoing, so $p_\bullet \in \mathscr{W} \cap \mathscr{E}$, by the 2-out-of-3 property of $\mathscr{W}$. Hence, $f_\bullet \in l(\{p_\bullet\})$, so $f_\bullet$ is a retract of $i_\bullet$ (proposition 3.1.10), and finally $f_\bullet \in l(\mathscr{E})$, by proposition 3.1.9(v). □

4.3.6. Let R be any ring, $\mathscr{A}$ any abelian category, and $F : R-\mathsf{Mod} \to \mathscr{A}$ any additive functor. Then F extends naturally to an additive functor

$$\mathsf{C}(F) : \mathsf{C}(R) \to \mathsf{C}(\mathscr{A}) \qquad (X_\bullet, d_\bullet^X) \mapsto FX_\bullet := (FX_n, Fd_n^X \mid n \in \mathbb{Z})$$

where $\mathsf{C}(\mathscr{A})$ denotes the category of chain complexes of objects of $\mathscr{A}$. Let also Σ be the class of quasi-isomorphisms of $\mathsf{C}(\mathscr{A})$; it is known that the localization

$$\mathsf{D}(\mathscr{A}) := \mathsf{C}(\mathscr{A})[\Sigma^{-1}]$$

exists. Likewise, set $\mathsf{D}(R) := \mathsf{ho}(\mathsf{C}(R))$, where we endow $\mathsf{C}(R)$ with the model structure provided by theorem 4.3.2, so that also $\mathsf{D}(R)$ is the localization of $\mathsf{C}(R)$ that inverts quasi-isomorphisms. Denote by $\gamma_{\mathscr{A}} : \mathsf{C}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$ the localization functor; then we claim that $\gamma_{\mathscr{A}} \circ \mathsf{C}(F) : \mathsf{C}(R) \to \mathsf{D}(\mathscr{A})$ admits a left derived functor

$$\boxed{\mathbf{L}F : \mathsf{D}(R) \to \mathsf{D}(\mathscr{A})}$$

that we call *the total left derived functor of F*. For the proof, according to proposition 3.4.4(i), it suffices to check that the functor $\mathsf{C}(F)$ sends trivial cofibrations of $\mathsf{C}(R)$ to quasi-isomorphisms of $\mathsf{C}(\mathscr{A})$. However, the proof of theorem 4.3.2 shows that the class $l(\mathscr{E})$ of trivial cofibrations of $\mathsf{C}(R)$ consists of the split monomorphisms $\boldsymbol{i}_\bullet : X_\bullet \to Y_\bullet$ such that $C_\bullet := \mathrm{Coker}(\boldsymbol{i}_\bullet)$ is a projective object of $\mathsf{C}(R)$; especially, $C_\bullet$ is acyclic and cofibrant (claim 4.3.3(iii)), and therefore $\mathbf{1}_{C_\bullet}$ is homotopically trivial (claim 4.3.4). But then we have an isomorphism $Y_\bullet \xrightarrow{\sim} X_\bullet \oplus C_\bullet$ of $\mathsf{C}(R)$ that identifies $\boldsymbol{i}_\bullet$ with the natural monomorphism $X_\bullet \to X_\bullet \oplus C_\bullet$, so $F\boldsymbol{i}_\bullet : FX_\bullet \to FX_\bullet \oplus FC_\bullet$ is again (up to isomorphism) the natural monomorphism ([13, Prob.2.76(i)]), and furthermore, $\mathbf{1}_{FC_\bullet}$ is homotopically trivial ([13, Rem.5.5(v)]); especially, $FC_\bullet$ is acyclic, and so $F\boldsymbol{i}_\bullet$ is a quasi-isomorphism, as stated.

4.3.7. *The canonical model category structure on* Cat. Our second example is a cofibrantly generated model category structure on the category Cat of small categories; this example will be relevant to our discussion of the Joyal model category structure on sSet in §6.2.

Definition 4.3.8. Let $\mathscr{A}, \mathscr{B}$ be two categories. We say that a functor $F : \mathscr{A} \to \mathscr{B}$ is an *isofibration* if for every $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$, and every isomorphism $g : FA \xrightarrow{\sim} B$ of $\mathscr{B}$, there exists $A' \in \mathrm{Ob}(\mathscr{A})$ and an isomorphism $f : A \xrightarrow{\sim} A'$ of $\mathscr{A}$ with $Ff = g$.

Theorem 4.3.9. *(i) There exists a cofibrantly generated model category structure*

$$(\mathsf{Cat}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$$

called the canonical model structure on Cat, *such that :*

- *(a) $\mathscr{W}$ is the class of equivalences of categories*
- *(b) $\mathscr{F}ib$ is the class of isofibrations*
- *(c) $\mathscr{C}of$ is the class of functors that are injective on objects.*

(ii) Every object of Cat *is both fibrant and cofibrant for the canonical model structure.*

Proof. (i): We regard as usual the ordered sets $\varnothing$, $[0]$ and $[1]$ as categories, and denote by I the category with $\mathrm{Ob}(I) := \{0,1\}$ and such that the inclusion functor $\eta : [0] \to I$ is an equivalence of categories. Let moreover S be the category with $\mathrm{Ob}(S) := \{0,1\}$ such that $S(1,0) = \varnothing$, $S(i,i) = \{\mathbf{1}_i\}$ for $i = 0,1$, and $S(0,1) := \{a, b : 0 \rightrightarrows 1\}$. We let $\partial := (\partial_0^1, \partial_1^1) : [0] \sqcup [0] \to [1]$, where $\partial_i^1 : [0] \to [1]$ are the face maps (definition 2.1.1(iii)), and denote by i (resp. j, resp. k) the unique functor $\varnothing \to [0]$ (resp. the unique functor $[0] \sqcup [0] \to [0]$, resp. the unique functor $S \to [1]$ that is the identity on objects). We observe :

Claim 4.3.10. (i) The class of essentially surjective functors of Cat is stable under retracts.

(ii) $r(\eta)$ is the class of isofibration of Cat (notation of definition 3.1.1(iii)).

(iii) $r(i)$ is the class of functors of Cat that are surjective on objects.

(iv) $r(j)$ is the class of functors of Cat that are injective on objects.

(v) $r(\partial)$ (resp. $r(k)$) is the class of full (resp. faithful) functors of Cat.

(vi) $\mathscr{W} \cap \mathscr{F}ib = \mathscr{W} \cap r(i) = r(i, k, \partial)$.

Proof : We leave (ii)–(vi) to the reader. For (i), consider a commutative diagram :

$$\begin{tikzcd} \mathscr{A} \arrow[r, "H"] \arrow[d, "G"'] & \mathscr{A}' \arrow[r, "K"] \arrow[d, "F"] & \mathscr{A} \arrow[d, "G"] \\ \mathscr{B} \arrow[r, "H'"] & \mathscr{B}' \arrow[r, "K'"] & \mathscr{B} \end{tikzcd}$$

of Cat such that $K \circ H = \mathbf{1}_{\mathscr{A}}$ and $K' \circ H' = \mathbf{1}_{\mathscr{B}}$. Suppose that F is essentially surjective, let $B \in \mathrm{Ob}(\mathscr{B})$, and set $B' := H'B$, so that $B = K'B'$; by assumption there exists $A' \in \mathrm{Ob}(\mathscr{A}')$ with an isomorphism $f : FA' \xrightarrow{\sim} B'$, and then $K'f : K'FA' = GKA' \xrightarrow{\sim} B$ is an isomorphism of $\mathscr{B}$, so G is essentially surjective. ◇

• From claim 4.3.10(i,ii,iv,v) and proposition 3.1.9(v) we see that $\mathscr{W}$, $\mathscr{C}of$ and $\mathscr{F}ib$ are stable under retracts. Next, consider a commutative diagram of Cat :

$$\mathscr{E} \quad : \quad \begin{tikzcd} \mathscr{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathscr{C} \arrow[d, "G"] \\ \mathscr{B} \arrow[r, "K"] & \mathscr{D} \end{tikzcd}$$

where $F \in \mathscr{C}of$ and $G \in \mathscr{F}ib$. Suppose first that $G \in \mathscr{W}$; then G is surjective on objects (claim 4.3.10(vi)), and since F is injective on objets, we can find a diagonal filler $l : \mathrm{Ob}(\mathscr{B}) \to \mathrm{Ob}(\mathscr{C})$ for the induced diagram $\mathrm{Ob}(\mathscr{E})$ formed by the underlying maps of sets of objects. Then, for every morphism $f : B \to B'$ of $\mathscr{B}$, by assumption there exists a unique morphism $g : lB \to lB'$ of $\mathscr{C}$ such that $Gg = Kf$, and we set $Lf := g$. It is easily seen that the rules $B \mapsto lB$ for every $B \in \mathrm{Ob}(\mathscr{B})$ and $f \mapsto Lf$ for every $f \in \mathrm{Mor}(\mathscr{B})$ yield a well-defined functor $L : \mathscr{B} \to \mathscr{C}$ that is the sought diagonal filler for $\mathscr{E}$ (details left to the reader). This shows that $\mathscr{C}of \subset l(\mathscr{W} \cap \mathscr{F}ib)$.

• Suppose next that $F \in \mathscr{W}$; then for every $B \in \Sigma := \mathrm{Ob}(\mathscr{B}) \setminus F(\mathrm{Ob}(\mathscr{A}))$ we can choose (by the axiom of choice) an isomorphism $\phi_B : FA_B \xrightarrow{\sim} B$ for some $A_B \in \mathrm{Ob}(\mathscr{A})$. We deduce the isomorphism $K(\phi_B) : GH(A_B) = KF(A_B) \xrightarrow{\sim} KB$ of $\mathscr{D}$, and since G is an isofibration, we then get $C_B \in \mathrm{Ob}(\mathscr{C})$ with an isomorphism $\psi_B : HA_B \xrightarrow{\sim} C_B$ of $\mathscr{C}$ such that $GC_B = KB$ and $G(\psi_B) = K(\phi_B)$. Let $l : \mathrm{Ob}(\mathscr{B}) \to \mathrm{Ob}(\mathscr{C})$ be the map such that $l(FA) := HA$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and $lB := C_B$ for every $B \in \Sigma$. Moreover, for every $A \in \mathrm{Ob}(\mathscr{A})$ set $A_{FA} := A$, $C_{FA} := HA$, $\phi_{FA} := \mathbf{1}_{FA}$ and $\psi_{FA} := \mathbf{1}_{HA}$. With this notation, for every morphism $f : B \to B'$ of $\mathscr{B}$ there exist unique $g_f \in \mathscr{A}(A_B, A_{B'})$ and $Lf \in \mathscr{C}(C_B, C_{B'})$ that make commute the diagrams :

$$\begin{tikzcd} FA_B \arrow[r, "F(g_f)"] \arrow[d, "\phi_B"'] & FA_{B'} \arrow[d, "\phi_{B'}"] \\ B \arrow[r, "f"] & B' \end{tikzcd} \qquad \begin{tikzcd} HA_B \arrow[r, "H(g_f)"] \arrow[d, "\psi_B"'] & HA_{B'} \arrow[d, "\psi_{B'}"] \\ C_B \arrow[r, "Lf"] & C_{B'}. \end{tikzcd}$$

It is easily seen that the rules $B \mapsto lB$ and $f \mapsto Lf$ for every $B \in \mathrm{Ob}(\mathscr{B})$ and every morphism f of $\mathscr{B}$ yield a well-defined diagonal filler $L : \mathscr{B} \to \mathscr{C}$ for diagram $\mathscr{E}$ (details left to the reader); this shows that $\mathscr{W} \cap \mathscr{C}of \subset l(\mathscr{F}ib)$.

• Lastly, let $F : \mathscr{A} \to \mathscr{B}$ be any functor; we set $\Lambda := \mathrm{Ob}(\mathscr{A}) \sqcup \Sigma$ (where Σ is as in the foregoing), and let $G : \Lambda \to \mathrm{Ob}(\mathscr{B})$ be the map such that $GA := FA$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and $GB := B$ for every $B \in \Sigma$. We let $\mathscr{C}$ be the category such that $\mathrm{Ob}(\mathscr{C}) := \Lambda$ and $\mathscr{C}(X, Y) := \mathscr{B}(GX, GY)$ for every $X, Y \in \mathrm{Ob}(\mathscr{C})$, with the composition law deduced from the composition law of $\mathscr{B}$, so that G defines a functor $\mathscr{C} \to \mathscr{B}$ acting as the identity map on the sets of morphisms. Then clearly $G : \mathscr{C} \to \mathscr{B}$ is fully faithful and surjective on objects, so it lies in $\mathscr{W} \cap \mathscr{F}ib$ (claim 4.3.10(iii,vi)); moreover we have a functor $H : \mathscr{A} \to \mathscr{C}$ such that $HA := A$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and $Hf := Ff$ for every $f \in \mathrm{Mor}(\mathscr{A})$. Clearly $H \in \mathscr{C}of$ and $G \circ H = F$, so $(\mathscr{C}of, \mathscr{W} \cap \mathscr{F}ib)$ is a weak factorization system for Cat.

• It remains to check that the same holds for $(\mathscr{W} \cap \mathscr{C}of, \mathscr{F}ib)$. To this aim, we define now $\mathscr{C}$ as the full subcategory of $F\mathscr{A}/\mathscr{B}$ whose objects are the triples (A, B, f) where $f : FA \xrightarrow{\sim} B$ is an isomorphism of $\mathscr{B}$, and let $G : \mathscr{C} \to \mathscr{B}$ be the restriction of the target functor $\mathrm{t} : F\mathscr{A}/\mathscr{B} \to \mathscr{B}$ (see §1.4). Moreover, let $H : \mathscr{A} \to \mathscr{C}$ be the functor such that $HA := (A, FA, \mathbf{1}_{FA})$ for every $A \in \mathrm{Ob}(\mathscr{A})$ and $Hf := (f, Ff)$ for every $f \in \mathrm{Mor}(\mathscr{A})$. Clearly $H \in \mathscr{W} \cap \mathscr{C}of$ and $G \in \mathscr{F}of$, as required.

• By example 4.1.5(i) and claim 4.3.10(ii,vi), $\{\eta\}$ is a generating set of cofibrations and $\{i, k, \partial\}$ is a generating set of trivial cofibrations; to conclude the proof of (i) it suffices now to recall that Cat is complete and cocomplete (proposition 1.10.4), and to notice :

Claim 4.3.11. The class of equivalences $\mathscr{W}$ has the 2-out-of-3 property.
Proof : Recall that a functor $F : \mathscr{A} \to \mathscr{B}$ is an equivalence if and only if it admits a *quasi-inverse*, *i.e.* a functor $G : \mathscr{B} \to \mathscr{A}$ with isomorphisms of functors $\mathbf{1}_{\mathscr{A}} \xrightarrow{\sim} G \circ F$ and $\mathbf{1}_{\mathscr{B}} \xrightarrow{\sim} F \circ G$. Now, let $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{F'} \mathscr{C}$ be two functors, and set $F'' := F' \circ F$. If $F, F' \in \mathscr{W}$, pick quasi-inverse functors $\mathscr{C} \xrightarrow{G'} \mathscr{B} \xrightarrow{G} \mathscr{A}$ and set $G'' := G \circ G'$; then we get isomorphisms :

$$G''F'' = G(G'F')F \xrightarrow{\sim} GF \xrightarrow{\sim} \mathbf{1}_{\mathscr{A}} \qquad F''G'' = F'(FG)G' \xrightarrow{\sim} F'G' \xrightarrow{\sim} \mathbf{1}_{\mathscr{C}}$$

so that $F'' \in \mathscr{W}$. If $F, F'' \in \mathscr{W}$, pick quasi-inverse functors $\mathscr{B} \xrightarrow{G} \mathscr{A}$ and $\mathscr{C} \xrightarrow{G''} \mathscr{A}$; we then get isomorphisms :

$$F'(FG'') = F''G'' \xrightarrow{\sim} \mathbf{1}_{\mathscr{C}} \qquad (FG'')F' \xrightarrow{\sim} (FG'')F'(FG) = F(G''F'')G \xrightarrow{\sim} FG \xrightarrow{\sim} \mathbf{1}_{\mathscr{B}}$$

so that $F' \in \mathscr{W}$. Likewise, one shows that if $F', F'' \in \mathscr{W}$, then $F \in \mathscr{W}$. ◇

Assertion (ii) of the theorem is clear from the definitions. □

Corollary 4.3.12. *The canonical model structure is the unique model category structure on* Cat *whose weak equivalences are the equivalences of categories.*[2]

Proof. Let $(\mathrm{Cat}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be the canonical model category structure on Cat, and $\mathsf{M} := (\mathrm{Cat}, \mathscr{W}, \mathscr{F}ib^*, \mathscr{C}of^*)$ another model category structure on Cat having the same class $\mathscr{W}$ of weak equivalences; we need to check that $\mathscr{F}ib^* = \mathscr{F}ib$ and $\mathscr{C}of^* = \mathscr{C}of$. However, notice that it suffices to check that $\mathscr{C}of^* = \mathscr{C}of$, since then it follows that $\mathscr{F}ib^* = r(\mathscr{W} \cap \mathscr{C}of^*) = r(\mathscr{W} \cap \mathscr{C}of) = \mathscr{F}ib$ (lemma 3.1.18(i)).

We shall call M-fibrant (resp. M-cofibrant) the fibrant (resp. cofibrant) objects relative to the model structure M, and we shall call likewise M-fibrations (resp. M-cofibrations) the elements of $\mathscr{F}ib^*$ (resp. of $\mathscr{C}of^*$).

[2] I have borrowed this proof from an `nlab` webpage

Claim 4.3.13. (i) The final category $[0]$ is M-cofibrant.

(ii) $\mathscr{W} \cap \mathscr{F}ib^* \subset \mathscr{W} \cap \mathscr{F}ib$.

(iii) $\mathscr{C}of \subset \mathscr{C}of^*$ and $\mathscr{F}ib^* \subset \mathscr{F}ib$.

Proof : (i): Let $\mathscr{C}$ be any non-empty category, and choose an M-fibrant category $\mathscr{B}$ with an equivalence of categories $\mathscr{B} \xrightarrow{\sim} \mathscr{C}$ (see §3.3.1); then $\mathscr{B}$ is necessarily non-empty, so we have a functor $F : [0] \to \mathscr{B}$. The composition of F with the unique functor $\mathscr{B} \to [0]$ equals $\mathbf{1}_{[0]}$, *i.e.* $[0]$ is a retract of $\mathscr{B}$, so we conclude with proposition 3.1.9(v).

(ii): Let $G \in \mathscr{W} \cap \mathscr{F}ib^*$; then $G \in r(\mathscr{C}of^*)$, and especially, $G \in r(i : \varnothing \to [0])$, by virtue of (i), and the assertion then follows from claim 4.3.10(vi).

(iii): The first inclusion follows from (ii) and proposition 3.1.9(ii). From this first inclusion, we then deduce that $\mathscr{W} \cap \mathscr{C}of \subset \mathscr{W} \cap \mathscr{C}of^*$, whence, after invoking again proposition 3.1.9(ii), the second stated inclusion. ◇

Next, let $I := \mathsf{ch}([1])$, where $[1] := \{0, 1\}$, and $\mathsf{ch} : \mathsf{Set} \to \mathsf{Cat}$ is the functor that assigns to every set S the chaotic category structure on S; recall also that ch is right adjoint to the functor $\mathrm{Ob} : \mathsf{Cat} \to \mathsf{Set}$, and let $\eta_\bullet : \mathbf{1}_{\mathsf{Cat}} \Rightarrow \mathsf{ch} \circ \mathrm{Ob}$ be the unit of adjunction for the pair $(\mathrm{Ob}, \mathsf{ch})$ (see remark 1.2.7). Notice that, for every category $\mathscr{C}$, the datum of a functor $F : I \to \mathscr{C}$ is equivalent to that of an isomorphism $f : X \xrightarrow{\sim} Y$ of $\mathscr{C}$: namely, to every such f one attaches the unique functor F such that $F(0) := X$, $F(1) := Y$, $F(\overrightarrow{01}) := f$ and $F(\overrightarrow{10}) := f^{-1}$ for the unique morphisms $\overrightarrow{01} : 0 \xrightarrow{\sim} 1$ and $\overrightarrow{10} : 1 \xrightarrow{\sim} 0$ of I.

Claim 4.3.14. (i) The unique functor $G : I \to [0]$ *is not* an M-cofibration.

(ii) $\eta_{\mathscr{C}} : \mathscr{C} \to \mathsf{ch} \circ \mathrm{Ob}(\mathscr{C})$ is an M-cofibration for every $\mathscr{C} \in \mathrm{Ob}(\mathsf{Cat})$.

Proof : (i): Notice that G is an equivalence, so it suffices to check that G is not a trivial M-cofibration. However, let $\mathscr{C}$ be an arbitrary small category, $\mathscr{D}$ any M-fibrant category with an equivalence $\mathscr{C} \xrightarrow{\sim} \mathscr{D}$ (see §3.3.1), h any isomorphism of $\mathscr{B}$, and consider the commutative diagram of Cat :

$$\begin{tikzcd} I \arrow[r, "H"] \arrow[d, "G"'] & \mathscr{B} \arrow[d] \\ {[0]} \arrow[r, equal] & {[0]} \end{tikzcd}$$

where H is the unique functor associated with the isomorphism h. If G were a trivial M-cofibration, then the diagram would have a diagonal filler; but the existence of such a diagonal filler is equivalent to asserting that h is the identity morphism of an object of $\mathscr{B}$ (details left to the reader); since h is arbitrary, we would then conclude that every isomorphism of $\mathscr{B}$ must be an identity morphism. Especially, for every $X \in \mathrm{Ob}(\mathscr{B})$, the unique automorphism of X is $\mathbf{1}_X$; then, since $\mathscr{C}$ is equivalent to $\mathscr{B}$, the same property would apply to the objects of $\mathscr{C}$, which is absurd, since $\mathscr{C}$ is an arbitrary small category.

(ii): Indeed, $\eta_{\mathscr{C}}$ is the identity map of $\mathrm{Ob}(\mathscr{C})$ on objects, so it lies in $\mathscr{C}of$, whence the claim, by virtue of claim 4.3.13(iii). ◇

In light of claim 4.3.13(iii), we are reduced to checking that $\mathscr{C}of^* \subset \mathscr{C}of$; suppose then, by way of contradiction, that the functor $F : \mathscr{A} \to \mathscr{B}$ is an M-cofibration but does not lie in $\mathscr{C}of$, *i.e.* is not injective on objects. Hence, let $A, A' \in \mathrm{Ob}(\mathscr{A})$ with $A \neq A'$ and $FA = FA'$, pick a retraction $r : \mathrm{Ob}(\mathscr{A}) \to \{A, A'\}$ of the inclusion map $\{A, A'\} \to \mathrm{Ob}(\mathscr{A})$, and set $J := \mathsf{ch}(\{A, A'\})$; by adjunction, r corresponds to a unique functor $R : \mathscr{A} \to J$

such that $\mathrm{Ob}(R) = r$. We then consider the commutative diagram of Cat :

$$\begin{tikzcd}
{[0] \sqcup [0]} \arrow[r, "{(A,A')}"] \arrow[d] & \mathscr{A} \arrow[r, "R"] \arrow[d, "F"'] & J \arrow[d, "G"] \arrow[dr, "T"] & \\
{[0]} \arrow[r, "B"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C} \arrow[r, "\eta_{\mathscr{C}}"] & \mathsf{ch} \circ \mathrm{Ob}(\mathscr{C})
\end{tikzcd}$$

whose central square subdiagram is cocartesian, and where (A, A') is the unique functor that maps the two objects of $[0] \sqcup [0]$ to A and A'; then B is of course the unique functor that maps the unique object of $[0]$ to FA. It follows that G is an M-cofibration (proposition 3.1.9(v)), and taking into account claim 4.3.14(ii), the same then holds for T.

Lastly, we get a commutative diagram of Cat :

$$\begin{tikzcd}
J \arrow[r, equal] \arrow[d] & J \arrow[r, equal] \arrow[d, "T"] & J \arrow[d] \\
{[0]} \arrow[r, "\eta_{\mathscr{C}} \circ H \circ B"] & \mathsf{ch} \circ \mathrm{Ob}(\mathscr{C}) \arrow[r] & {[0].}
\end{tikzcd}$$

Indeed, the diagram commutes on objects by construction, and then it commutes also in Cat, since, by virtue of the adjunction $(\mathrm{Ob}, \mathsf{ch})$, any two functors $J \rightrightarrows \mathsf{ch} \circ \mathrm{Ob}(\mathscr{C})$ coincide if and only if they coincide on objects. Summing up, the unique functor $J \to [0]$ is a retract of T, so it is an M-cofibration as well (proposition 3.1.9(v)). But the bijection $\{A, A'\} \xrightarrow{\sim} [1]$ with $A \mapsto 0$ and $A' \mapsto 1$ yields an isomorphism of categories $J \xrightarrow{\sim} I$, so the unique functor $I \to [0]$ is also an M-cofibration, contradicting claim 4.3.14(i). □

4.4. **Cylinders and anodyne extensions.** In this section and the following one, we explain a general procedure for constructing cofibrantly generated model structures on categories of presheaves of sets over a fixed small category, following [3]. The idea is simple : it consists in following step by step most of the book of Gabriel and Zisman [6] on the homotopy theory of Kan complexes, and to see that a significant part of it makes sense in wide generality. The idea of promoting the book of Gabriel and Zisman into a general way to define homotopy theories ex nihilo comes from the early work of Fabien Morel on the homotopy theory of schemes [11]. We begin with the following definition :

Definition 4.4.1. Let $\mathscr{A}$ be a small category. A *cellular model* for $\widehat{\mathscr{A}}$ is a set $\mathscr{M}$ of monomorphisms of presheaves on $\mathscr{A}$ such that $l(r(\mathscr{M}))$ is the class of all monomorphisms of presheaves on $\mathscr{A}$.

Example 4.4.2. (i) The proof of corollary 4.2.7 shows that the set of all monomorphisms of $\widehat{\mathscr{A}}$ of the form $L \hookrightarrow K$, where K runs over the quotients of representable presheaves, is a cellular model for $\widehat{\mathscr{A}}$.

(ii) If $\mathscr{A}$ is an Eilenberg-Zilber category, the set $\mathscr{B} := \{\partial h_a \hookrightarrow h_a \mid \ \in \mathrm{Ob}(\mathscr{A})\}$ is a cellular model for $\widehat{\mathscr{A}}$: indeed, $l(r(\mathscr{B}))$ is the saturation of $\mathscr{B}$ (corollary 4.2.5(iii)), so the assertion follows from example 3.1.14(iii).

Definition 4.4.3. (i) Let $\mathscr{A}$ be a small category, and X an object of $\widehat{\mathscr{A}}$. A *cylinder of* X is a commutative diagram in $\widehat{\mathscr{A}}$:

$$\begin{tikzcd}
X \sqcup X \arrow[r, "{(\partial_0, \partial_1)}"] \arrow[rr, bend left, "\nabla_X"] & Y \arrow[r, "\sigma"] & X
\end{tikzcd}$$

where ∇_X is the codiagonal ([13, Exerc.2.66(iii)]) and (∂_0, ∂_1) is a monomorphism.

(ii) A *functorial cylinder on* $\widehat{\mathscr{A}}$ is the datum of an endofunctor

$$I : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$$

together with natural transformations :

$$\partial_0, \partial_1 : \mathbf{1}_{\widehat{\mathscr{A}}} \Rightarrow I \qquad \sigma : I \Rightarrow \mathbf{1}_{\widehat{\mathscr{A}}}$$

such that $X \sqcup X \xrightarrow{(\partial_0 \otimes X, \partial_1 \otimes X)} IX \xrightarrow{\sigma \otimes X} X$ is a cylinder of X, for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, where the notation $\partial_0 \otimes X$ and $\partial_1 \otimes X$ and $\sigma \otimes X$ is defined as in §1.1.5.

(iii) An *exact cylinder on* $\widehat{\mathscr{A}}$ is a functorial cylinder $(I, \partial_\bullet, \sigma)$ on $\widehat{\mathscr{A}}$ such that:

(DH1) I preserves small colimits and monomorphisms

(DH2) every monomorphism $j : K \hookrightarrow L$ of $\widehat{\mathscr{A}}$ induces cartesian squares :

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, "\partial_i \otimes K"'] & L \arrow[d, "\partial_i \otimes L"] \\ IK \arrow[r, "Ij"] & IL \end{tikzcd} \qquad \forall i = 0, 1.$$

Remark 4.4.4. (i) For any cartesian diagram of $\widehat{\mathscr{A}}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

in which every arrow is a monomorphism, the induced morphism

$$(*) \qquad\qquad Y \sqcup_X Z \to T$$

is a monomorphism. Indeed, since the limits and colimits are computed termwise in $\widehat{\mathscr{A}}$, and since monomorphisms (and epimorphisms) in $\widehat{\mathscr{A}}$ can be detected termwise (remark 1.6.2(ii)), the claim is reduced to the corresponding assertion for cartesian diagrams of sets in which all arrows are injective, and in this case we have a natural identification of $Y \sqcup_X Z$ with the union $Y \cup Z \subset T$. Then, for a diagram of $\widehat{\mathscr{A}}$ we shall denote likewise $Y \cup Z$ the image of $(*)$, so that we have an inclusion

$$Y \cup Z \subset T.$$

(ii) In the situation of definition 4.4.3(ii), notice that $\partial_0 \otimes X$ and $\partial_1 \otimes X$ have the left inverse $\sigma \otimes X$, for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, so they are monomorphisms ([13, Exerc.1.119(i)]). The subobject of IX determined by $\partial_\varepsilon \otimes X$ shall be denoted :

$$\{\varepsilon\} \otimes X \qquad \forall X \in \mathrm{Ob}(\widehat{\mathscr{A}}), \forall \varepsilon = 0, 1.$$

Hence, condition (DH2) of definition 4.4.3(iii) says that *if* $(I, \partial_\bullet, \sigma)$ *is an exact cylinder*, then every inclusion $K \subset L$ of presheaves over $\mathscr{A}$ induces inclusions :

$$\{\varepsilon\} \otimes K \subset IK \cup \{\varepsilon\} \otimes L \subset IL \qquad \forall \varepsilon = 0, 1.$$

(iii) Since $(\partial_0 \otimes X, \partial_1 \otimes X)$ is a monomorphism, we have $\{0\} \otimes X \cap \{1\} \otimes X = \varnothing$ (where $\varnothing$ denotes the empty presheaf); then we also set :

$$\partial I \otimes X := \{0\} \otimes X \sqcup \{1\} \otimes X \subset IX \qquad \forall X \in \mathrm{Ob}(\widehat{\mathscr{A}}).$$

Clearly we have :

$$(\partial I \otimes L) \times_{IL} IK = (\{0\} \otimes L) \times_{IL} IK \sqcup (\{1\} \otimes L) \times_{IL} IK.$$

Indeed, these identites can be checked termwise, by evaluating the left and right side on objects of $\mathscr{A}$, and since the evaluation functor preserves and reflects limits and colimits, we are reduced to showing the corresponding identities in the category of sets, where they are obvious. Combining with (DH2), we obtain a cartesian square:

$$\begin{tikzcd} \partial I \otimes K \arrow[r, "\partial I \otimes j"] \arrow[d] & \partial I \otimes L \arrow[d] \\ IK \arrow[r, "Ij"] & IL \end{tikzcd}$$

where all arrows are monomorphisms, whence, again by (i), an induced inclusion :

$$IK \cup \partial I \otimes L \subset IL$$

for every monomorphism $K \subset L$ in $\widehat{\mathscr{A}}$ and every exact cylinder $(I, \partial_\bullet, \sigma)$.

Example 4.4.5. (i) Let $(J, \partial^J_\bullet, \sigma^J)$ be any cylinder of the final object of $\widehat{\mathscr{A}}$ (one such final object is the presheaf E such that $E_a := \{\varnothing\}$ for every $a \in \mathrm{Ob}(\mathscr{A})$). Then it is easily seen that we get an exact cylinder on $\widehat{\mathscr{A}}$ consisting of the functor :

$$I : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}} \qquad X \mapsto J \times X \qquad (X \xrightarrow{f} Y) \mapsto (J \times X \xrightarrow{J \times f} J \times Y)$$

together with the natural transformations :

$$\partial_\varepsilon : \mathbf{1}_{\widehat{\mathscr{A}}} \Rightarrow I \qquad X \mapsto (\partial^J_\varepsilon \times X) \qquad (\varepsilon = 0, 1) \qquad \sigma : I \Rightarrow \mathbf{1}_{\widehat{\mathscr{A}}} \qquad X \mapsto (\sigma^J \times X).$$

We call $(I, \partial_\bullet, \sigma)$ *the cartesian cylinder induced by* $(J, \partial^J_\bullet, \sigma^J)$.

(ii) If E is a final object of $\widehat{\mathscr{A}}$, then $J := E \sqcup E$, with the natural monomorphisms $(\partial^J_\varepsilon : E \to J \mid \varepsilon = 0, 1)$ and the codiagonal $\sigma^J := \nabla_E$, forms a cylinder of E.

(iii) Let Ω be a subobject classifer for $\widehat{\mathscr{A}}$ (proposition 1.8.4), and E a final object of $\widehat{\mathscr{A}}$. For every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$ we have two distinguished elements $\varnothing_X, \mathbf{1}_X$ of $\mathsf{Sub}_{\widehat{\mathscr{A}}}(X)$ (where $\varnothing_X$ denotes the *empty subobject* of X, represented by the unique monomorphism $\varnothing \to X$ from the initial object $\varnothing$ of $\widehat{\mathscr{A}}$), and the rules : $X \mapsto \varnothing_X$ and $X \mapsto \mathbf{1}_X$ define morphisms of presheaves on $\widehat{\mathscr{A}}$:

$$\delta_0, \delta_1 : h_E \rightrightarrows \mathsf{Sub}_{\widehat{\mathscr{A}}}$$

(where h_E is the presheaf represented by E, which is also the final object of $\widehat{\mathscr{A}}$). By Yoneda's lemma, δ_0 and δ_1 correspond to unique morphisms $\partial_0, \partial_1 : E \rightrightarrows \Omega$ of $\widehat{\mathscr{A}}$. Clearly we have $\mathbf{1}_X = \varnothing_X$ in $\mathsf{Sub}_{\widehat{\mathscr{A}}}(X)$ if and only if $X = \varnothing$, therefore ∂_0 and ∂_1 induce a monomorphism $(\partial_0, \partial_1) : E \sqcup E \to \Omega$; we then get a well-defined cylinder of E :

$$E \sqcup E \xrightarrow{(\partial_0, \partial_1)} \Omega \xrightarrow{\sigma} E$$

whence, according to (i), an induced cartesian cylinder, that we call *the Lawvere cylinder*. This cylinder will play a special role in §4.6.

Definition 4.4.6. (i) Let $\mathscr{A}$ be a small category, $(I, \partial_\bullet, \sigma)$ an exact cylinder on $\widehat{\mathscr{A}}$, and An a class of morphisms of $\widehat{\mathscr{A}}$. We say that An *is a class of I-anodyne extensions*, if the following conditions hold :

(An0) There exists a set Λ of monomorphisms of $\widehat{\mathscr{A}}$ such that $\mathsf{An} = l(r(\Lambda))$

(An1) For every monomorphism $K \hookrightarrow L$ of $\widehat{\mathscr{A}}$, and for $\varepsilon = 0, 1$, the induced morphism $IK \cup \{\varepsilon\} \otimes L \to IL$ is An

(An2) For every $(K \to L) \in \mathsf{An}$, the induced morphism $IK \cup \partial I \otimes L \to IL$ is in An.

(ii) A *homotopical structure on* $\mathscr{A}$ is the datum of an exact cylinder $(I, \partial_\bullet, \sigma)$ on $\widehat{\mathscr{A}}$ and a class of I-anodyne extensions in $\widehat{\mathscr{A}}$.

Remark 4.4.7. (i) In case $(I, \partial_\bullet, \sigma)$ is the cartesian cylinder induced by a cylinder $(J, \partial^J_\bullet, \sigma^J)$ of the final object of $\widehat{\mathscr{A}}$ (see example 4.4.5(i)), we will also say that An is *a class of J-anodyne extensions.*

(ii) Condition (An0) implies especially that An is the saturation of Λ (corollary 4.2.5(iii)), and lies in the class of monomorphisms of $\widehat{\mathscr{A}}$ (example 3.1.14(ii)).

(iii) The class of all monomorphisms of $\widehat{\mathscr{A}}$ is a class of I-anodyne extensions, for every exact cylinder $(I, \partial_\bullet, \sigma)$ on $\widehat{\mathscr{A}}$, by virtue of example 4.4.2(i).

4.4.8. Given an exact cylinder $(I, \partial_\bullet, \sigma)$ on $\widehat{\mathscr{A}}$ and a set S of monomorphisms of $\widehat{\mathscr{A}}$, let us pick a cellular model $\mathscr{M}$ for $\widehat{\mathscr{A}}$, and let us set

$$\Lambda^0_I(S, \mathscr{M}) := S \cup \{IK \cup \{\varepsilon\} \otimes L \to IL \mid (K \to L) \in \mathscr{M}, \varepsilon = 0, 1\}.$$

Next, let us define inductively :

$$\Lambda^{n+1}_I(S, \mathscr{M}) := \{IK \cup \partial I \otimes L \to IL \mid (K \to L) \in \Lambda^n_I(S, \mathscr{M})\} \qquad \forall n \in \mathbb{N}$$

and finally :

$$\Lambda_I(S, \mathscr{M}) := \bigcup_{n\in\mathbb{N}} \Lambda^n_I(S, \mathscr{M}) \qquad \mathsf{An}_I(S) := l(r(\Lambda_I(S, \mathscr{M}))).$$

Proposition 4.4.9. *With the notation of §4.4.8, the class* $\mathsf{An}_I(S)$ *is the smallest class of I-anodyne extensions in $\widehat{\mathscr{A}}$ containing S.*

Proof. Clearly, every class $\mathscr{J}$ of I-anodyne extensions containing S, contains as well $\Lambda_I(S, \mathscr{M})$, and then must contain also $\mathsf{An}_I(S)$, by proposition 3.1.9(ii,iii), since $\mathscr{J} = l(r(\mathscr{J}))$. Thus, it suffices to check that $\mathsf{An}_I(S)$ is a class of I-anodyne extensions. Condition (An0) holds with $\Lambda := \Lambda_I(S, \mathscr{M})$. To show (An1), notice first that, by virtue of theorem 1.7.5(iii), the functor I admits a right adjoint. We then apply proposition 3.1.21(iii) to the functors $F_1 := \mathbf{1}_{\widehat{\mathscr{A}}}$ (whose adjoint is of course $\mathbf{1}_{\widehat{\mathscr{A}}}$) and $F_2 := I$, to the natural transformation $\partial_\varepsilon : \mathbf{1}_{\widehat{\mathscr{A}}} \to I$, and to the subclass $\mathscr{S} := \mathscr{M}$. Recall that $l(r(\mathscr{M}))$ is the class $\mathscr{I}$ of monomorphisms of $\widehat{\mathscr{A}}$, and notice that, with the notation of *loc.cit.* we have

$$j^\diamond = (IK \cup \{\varepsilon\} \otimes L \to IL) \qquad \forall (j : K \to L) \in \mathscr{I}.$$

Hence, by construction we have as well $\mathscr{M}^\diamond \subset \Lambda^0_I(S, \mathscr{M})$; then we get :

$$\mathscr{I}^\diamond = l(r(\mathscr{M}))^\diamond \subset l(r(\mathscr{M}^\diamond)) \subset l(r(\Lambda^0_I(S, \mathscr{M}))) \subset \mathsf{An}_I(S)$$

i.e. the inclusion $j^\diamond$ is in $\mathsf{An}_I(S)$ whenever $j \in \mathscr{I}$ and $\varepsilon = 0, 1$, which is (An1).

Next, it is easily seen that the functor $\partial I : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$ admits as well a right adjoint, so we may also apply proposition 3.1.21(iii) with $F_1 := \partial I$, $F_2 := I$, and with $\tau_\bullet$ given by the inclusion $\partial I \hookrightarrow I$. We also take $\mathscr{S} := \Lambda_I(S, \mathscr{M})$, and notice that in this case we have

$$j^\diamond = (IK \cup \partial I \otimes L \to IL) \qquad \forall (j : K \to L) \in \mathscr{I}.$$

In particular, notice that by construction we have $\mathscr{S}^\diamond \subset \mathscr{S}$. Then we get :

$$\mathsf{An}_I(S)^\diamond = l(r(\mathscr{S}))^\diamond \subset l(r(\mathscr{S}^\diamond)) \subset l(r(\mathscr{S})) = \mathsf{An}_I(S)$$

i.e. the inclusion $j^\diamond$ is in $\mathsf{An}_I(S)$ whenever $j \in \mathsf{An}_I(S)$, which is (An2). $\square$

Corollary 4.4.10. *In the situation of §4.4.8, let $(J, \partial^J_\bullet, \sigma^J)$ be a cylinder of the final object of $\widehat{\mathscr{A}}$, and suppose that :*

(a) $(I, \partial_\bullet, \sigma)$ is the cartesian cylinder induced by $(J, \partial^J_\bullet, \sigma^J)$ (see example 4.4.5(i))

(b) $(IK \cup \partial I \otimes L \to IL) \in l(r(\Lambda^0_I(S, \mathscr{M})))$ for every $(K \to L) \in S$.

Then we have :

$$\boxed{\mathsf{An}_I(S) = l(r(\Lambda^0_I(S, \mathscr{M}))).}$$

Proof. As $l(r(\Lambda^0_I(S, \mathscr{M}))) \subset \mathsf{An}_I(S)$, it suffices again to check that $l(r(\Lambda^0_I(S, \mathscr{M})))$ is a class of I-anodyne extensions. Condition (An0) holds with $\Lambda := \Lambda^0_I(S, \mathscr{M})$. To show (An1) and (An2) we argue as in the proof of proposition 4.4.9 : we apply proposition 3.1.21(iii) to the natural transformation $\partial_\varepsilon : F_1 \Rightarrow F_2$ (for ε equal to either 0 or 1), where $F_1 := \mathbf{1}_{\widehat{\mathscr{A}}}$ and $F_2 := I$, and we take $\mathscr{S} := \mathscr{M}$. Then $\mathscr{S}^\diamond \subset \Lambda^0_I(\varnothing, \mathscr{M}) \subset \Lambda^1_I(S, \mathscr{M})$ and we deduce again that $\mathscr{I}^\diamond \subset l(r(\mathscr{S}^\diamond)) \subset l(r(\Lambda^0_I(S, \mathscr{M})))$, where $\mathscr{I} := l(r(\mathscr{M}))$ is the class of monomorphisms of $\widehat{\mathscr{A}}$; this yields (An1). Next, we observe :

Claim 4.4.11. $\Lambda^1_I(S, \mathscr{M}) \subset l(r(\Lambda^0_I(S, \mathscr{M})))$.

Proof : In view of condition (b), it suffices to check that $\Lambda^1_I(\varnothing, \mathscr{M}) \subset l(r(\Lambda^0_I(S, \mathscr{M})))$. However, for every $(K \to L) \in \mathscr{M}$, the morphism :

$$J \times (J \times K \cup \{\varepsilon\} \times L) \cup \partial J \times (J \times L) \to J \times J \times L$$

is identified (by permutation of the first two factors) with the morphism :

$$J \times (J \times K \cup \partial J \times L) \cup \{\varepsilon\} \times (J \times L) \to J \times J \times L$$

which lies in $l(r(\Lambda^0_I(S, \mathscr{M})))$, by condition (An1). ◇

We then apply proposition 3.1.21(iii) to the natural transformation given by the inclusion $\partial I \to I$, and we take $\mathscr{S} := \Lambda^0_I(S, \mathscr{M})$, so that $\mathscr{S}^\diamond = \Lambda^1_I(S, \mathscr{M})$; combining with claim 4.4.11 (and with proposition 3.1.9(iii)), we then get : $l(r(\mathscr{S}))^\diamond \subset l(r(\Lambda^1_I(S, \mathscr{M}))) \subset l(r(\Lambda^0_I(S, \mathscr{M})))$, and this yields (An2). □

4.5. **Model structures on categories of presheaves.** *Henceforth, and until the end of this section, we fix a small category $\mathscr{A}$ and a homotopical structure $((I, \partial_\bullet, \sigma), \mathsf{An})$ on $\mathscr{A}$* (see definition 4.4.6(ii)). Hence, whenever we mention I-anodyne extensions, it will be understood that we refer to elements of the class An.

Definition 4.5.1. (i) Let $f_0, f_1 : X \rightrightarrows Y$ be two morphisms of $\widehat{\mathscr{A}}$. *An I-homotopy from f_0 to f_1* is a morphism of presheaves $h : IX \to Y$ such that :

$$h \circ (\partial_\varepsilon \otimes X) = f_\varepsilon \qquad \forall \varepsilon = 0, 1.$$

(ii) For every $X, Y \in \mathrm{Ob}(\widehat{\mathscr{A}})$, we set

$$[X, Y] := \widehat{\mathscr{A}}(X, Y)/\overset{I}{\sim}$$

where $\overset{I}{\sim}$ denotes the smallest equivalence relation on $\widehat{\mathscr{A}}(X, Y)$ such that $f_0 \overset{I}{\sim} f_1$ whenever there exists an I-homotopy from f_0 to f_1. Two morphisms $f, g : X \rightrightarrows Y$ are *I-homotopic*, if their images $[f], [g]$ coincide in $[X, Y]$.

Remark 4.5.2. (i) In the situation of definition 4.5.1(ii), clearly if $f_0 \overset{I}{\sim} f_1$, then $g \circ f_0 \overset{I}{\sim} g \circ f_1$ for every morphism $g : Z \to X$ of $\widehat{\mathscr{A}}$. Moreover, the functoriality of the cylinder I easily implies that $f_0 \circ g' \overset{I}{\sim} f_1 \circ g'$ for every morphism $g' : Y \to Z'$ of $\widehat{\mathscr{A}}$. Hence, there exists a well-defined category :

$$\widehat{\mathscr{A}_I}$$

of presheaves on $\mathscr{A}$ up to I-homotopy, whose objects are the presheaves on $\mathscr{A}$, and with $\widehat{\mathscr{A}_I}(X,Y) := [X,Y]$ for every $X, Y \in \mathrm{Ob}(\widehat{\mathscr{A}})$, with the unique composition law such that the system of projections $(\pi_{XY} : \widehat{\mathscr{A}}(X,Y) \to [X,Y] \mid X, Y \in \mathrm{Ob}(\widehat{\mathscr{A}}))$ yields a functor :

$$\widehat{\mathscr{A}} \to \widehat{\mathscr{A}_I} \qquad X \mapsto X \qquad (f : X \to Y) \mapsto \pi_{XY}(f).$$

(ii) We shall say that a morphism of $\widehat{\mathscr{A}}$ is an *I-homotopy equivalence*, if its image in $\widehat{\mathscr{A}_I}$ is an isomorphism.

Lemma 4.5.3. *For every $f \in \widehat{\mathscr{A}}(X,Y)$, the following conditions are equivalent :*

(a) f is an I-homotopy equivalence.

(b) There exists a morphism $g : Y \to X$ such that $fg \overset{I}{\sim} \mathbf{1}_Y$ and $gf \overset{I}{\sim} \mathbf{1}_X$.

(c) Every $K \in \mathrm{Ob}(\widehat{\mathscr{A}})$ induces a bijection $f_{*,K} : [K,X] \xrightarrow{\sim} [K,Y]$.

(d) Every $K \in \mathrm{Ob}(\widehat{\mathscr{A}})$ induces a bijection $f^*_K : [Y,K] \xrightarrow{\sim} [X,K]$.

Proof. We have (a)⇔(b) by definition. If (b) holds, then $g_{*,K} : [K,Y] \to [K,X]$ is a left and right inverse for $f_{*,K}$, whence (c). Conversely, if (c) holds, there exists a morphism $g : Y \to X$ of presheaves such that $f_{*,Y}([g]) = [\mathbf{1}_Y]$ in $[Y,Y]$, *i.e.* fg is I-homotopic to $\mathbf{1}_Y$. Then fgf is I-homotopic to $f = f\mathbf{1}_X$, and therefore $[gf] = [\mathbf{1}_X]$, again by (c), whence (b). Likewise one shows the equivalence (b)⇔(d). □

Definition 4.5.4. Let $f : X \to Y$ be a morphism of $\widehat{\mathscr{A}}$.

(i) We say that f is *a strong deformation retract* (resp. *the dual of a strong deformation retract*) if there exists $g \in \widehat{\mathscr{A}}(Y,X)$ such that :

- $gf = \mathbf{1}_X$ (resp. $fg = \mathbf{1}_Y$)
- there exists an I-homotopy h (resp. k) from $\mathbf{1}_Y$ to fg (resp. from $\mathbf{1}_X$ to gf)
- $h \circ If = \sigma_\bullet \otimes f$ (resp. $f \circ k = \sigma_\bullet \otimes f$) (notation of §1.1.5).

(ii) We say that f is *a trivial fibration*, if it has the right lifting property with respect to the class of monomorphisms of $\widehat{\mathscr{A}}$.

(iii) We say that f is *an I-fibration*, if it has the right lifting property with respect to the given class An of I-anodyne extensions. We say that a presheaf X is *I-fibrant*, if the unique morphism $X \to e$ to the final presheaf e is an I-fibration.

(iv) We call f *a weak I-equivalence*, if every I-fibrant object K induces a bijection

$$f^*_K : [Y,K] \xrightarrow{\sim} [X,K].$$

Remark 4.5.5. (i) By definition $r(\mathsf{An})$ is the class of I-fibrations, and by corollary 4.2.5(ii) and proposition 3.1.9(iii), $(\mathsf{An}, r(\mathsf{An}))$ is a weak factorization system for $\widehat{\mathscr{A}}$. More precisely, by inspecting the proof of theorem 4.1.8, we see that if Λ is any subset of $\mathrm{Mor}(\widehat{\mathscr{A}})$ such that $\mathsf{An} = l(r(\Lambda))$, then every morphism f of $\widehat{\mathscr{A}}$ admits a factorization $f = p \circ i$, where p is an I-fibration, and i lies in the weak saturation of Λ. Also, the I-fibrant objects of $\widehat{\mathscr{A}}$ are the same as the An-injective objects.

(ii) From the definitions and lemma 4.5.3, it is clear that every trivial fibration is an I-fibration, and every I-homotopy equivalence is a weak I-equivalence.

Proposition 4.5.6. *(i) Any trivial fibration of $\widehat{\mathscr{A}}$ is the dual of a strong deformation retract, and any section of a trivial fibration is a strong deformation retract. In particular, any trivial fibration has a section, and is an I-homotopy equivalence.*

(ii) Every strong deformation retract is an I-anodyne extension.

(iii) Every I-anodyne extension between I-fibrant presheaves is a strong deformation retract.

Proof. (i): Let $p : X \to Y$ be a trivial fibration; since the unique morphism $\varnothing \to Y$ from the initial presheaf on $\mathscr{A}$ (the empty presheaf) is obviously a monomorphism, the commutative diagram :

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

admits a diagonal filler $s : Y \to X$; this proves that p admits a section. Next, such a section s is a monomorphism ([13, Exerc.1.119(i)]), so it induces a monomorphism $j : IY \cup \partial I \otimes X \to IX$ (remark 4.4.4(iii)). Consider then the commutative diagram:

$$\begin{tikzcd} IY \cup \partial I \otimes X \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ IX \arrow[r, "\sigma_\bullet \otimes p"] & Y \end{tikzcd}$$

where f is the unique morphism whose restriction to IY agrees with $\sigma_\bullet \otimes s$ and whose restriction to $\{0\} \otimes X$ (resp. to $\{1\} \otimes X$) agrees with $\mathbf{1}_X$ (resp. with $s \circ p$). Again, this diagram admits a diagonal filler $k : IX \to X$. The latter is then an I-homotopy from $\mathbf{1}_X$ to $s \circ p$, such that $p \circ k = \sigma_\bullet \otimes p$ and $k \circ Is = \sigma_\bullet \otimes s$, which shows that p is the dual of a strong deformation retract, and that s is a strong deformation retract.

(ii): Let $i : K \to L$ be a strong deformation retract, so that there exists a morphism $r : L \to K$ of $\mathscr{A}$ with $ri = \mathbf{1}_K$, and an I-homotopy $h : IL \to L$ from $\mathbf{1}_L$ to ir such that $h \circ Ii = \sigma \otimes i$. Since $\mathsf{An} = l(r(\mathsf{An}))$ (proposition 3.1.9(iii)), it suffices to show that for every I-fibration $p : X \to Y$, every commutative square :

$$(*) \qquad \begin{tikzcd} K \arrow[r, "a"] \arrow[d] & X \arrow[d, "p"] \\ L \arrow[r, "b"] & Y \end{tikzcd}$$

admits a diagonal filler. To this aim, notice that the two morphisms :

$$IK \xrightarrow{\sigma_\bullet \otimes a} X \xleftarrow{ar} L \xleftarrow{\sim} \{1\} \otimes L$$

agree on $IK \cap \{1\} \otimes L = \{1\} \otimes K$, so they define a morphism $u : IK \cup \{1\} \otimes L \to X$. Then we get a commutative square :

$$\begin{tikzcd} IK \cup \{1\} \otimes L \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ IL \arrow[r, "bh"] & Y \end{tikzcd}$$

whose left vertical arrow is an I-anodyne extension, so it admits a diagonal filler $k : IL \to X$. Let us set $l := k \circ (\partial_0 \otimes L) : L \to X$. Then $li = (\sigma_\bullet \otimes a) \circ (\partial_0 \otimes K) = a$ and $pl = pbh \circ (\partial_0 \otimes L) = pb$, so l is the sought diagonal filler for $(*)$.

(iii): Let $i : K \to L$ be an I-anodyne extension, and suppose that K and L are I-fibrant; then there exists a morphism $r : L \to K$ with $ri = \mathbf{1}_K$. Notice that the morphisms :

$$IK \xrightarrow{\sigma_\bullet \otimes i} L \xleftarrow{(\mathbf{1}_L, ir)} L \sqcup L \xleftarrow{\sim} \partial I \otimes L$$

agree on $IK \cap \partial I \otimes L = \partial I \otimes K$, so they induce a morphism $u : IK \cup \partial I \otimes L \to L$, and since L is I-fibrant, the diagram (where e denotes the final presheaf) :

$$\begin{tikzcd} IK \cup \partial I \otimes L \arrow[r, "u"] \arrow[d] & L \arrow[d] \\ IL \arrow[r] & e \end{tikzcd}$$

admits a diagonal filler $h : IL \to L$. By construction, h is a homotopy from $\mathbf{1}_L$ to ir such that $h \circ Ii = \sigma_\bullet \otimes i$, whence the assertion. □

Proposition 4.5.7. *Consider a cartesian diagram of $\widehat{\mathscr{A}}$:*

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "j"] & Y \end{tikzcd}$$

where p is an I-fibration. If j is a strong deformation retract, the same holds for i.

Proof. Let us pick a retraction $r : Y \to Y'$ and an I-homotopy $h : IY \to Y$ from $\mathbf{1}_Y$ to jr such that $h \circ Ij = \sigma_\bullet \otimes j$. Notice that the morphisms :

$$IX' \xrightarrow{\sigma_\bullet \otimes i} X \xleftarrow{\sim} \{0\} \otimes X$$

agree on $\{0\} \otimes X' = IX' \cap \{0\} \otimes X$, so they induce a unique morphism $f : IX' \cup \{0\} \otimes X \to X$. Then the commutative square :

$$\begin{tikzcd} IX' \cup \{0\} \otimes X \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ IX \arrow[r, "h \circ Ip"] & Y \end{tikzcd}$$

admits a diagonal filler $k : IX \to X$. The morphisms $u := k \circ (\partial_1 \otimes X) : X \to X$ and $v := r \circ p : X \to Y'$ satisfy the relation $pu = jv$ (because $h \circ (\partial_1 \otimes Y) = jr$) and thus define a unique morphism $s : X \to X'$ such that $p's = v$ and $is = u$. We have $si = \mathbf{1}_{X'}$ because $p'si = vi = rpi = rjp' = p'$ and $isi = ui = k \circ (\partial_1 \otimes X) \circ i = k \circ Ii \circ (\partial_1 \otimes X') = (\sigma_\bullet \otimes i) \circ (\partial_1 \otimes X') = i$. Hence i is a strong deformation retract. □

Lemma 4.5.8. *Let $X, Y \in \mathrm{Ob}(\widehat{\mathscr{A}})$, $f, g \in \widehat{\mathscr{A}}(X, Y)$, and suppose that Y is I-fibrant. Then $[f] = [g]$ in $[X, Y]$ $\Leftrightarrow$ there exists an I-homotopy from f to g.*

Proof. Let us write $f \approx g$ if and only if there exists an I-homotopy from f to g; clearly, it suffices to check that $\approx$ is an equivalence relation on $\widehat{\mathscr{A}}(X, Y)$.

- For the reflexivity of $\approx$, notice that $\sigma_\bullet \otimes f$ is an I-homotopy from f to f, for every $f \in \widehat{\mathscr{A}}(X, Y)$.
- Next, let $u, v, w \in \widehat{\mathscr{A}}(X, Y)$, and suppose that we have an I-homotopy h from u to v and an I-homotopy k from u to w; we wish then to exhibit an I-homotopy from v to w. To this aim, notice first that the monomorphism $\partial I \otimes X \hookrightarrow IX$ induces an I-anodyne extension :

$$(*) \qquad I(\partial I \otimes X) \cup \{0\} \otimes IX \to I(IX).$$

On the other hand, since I preserves colimits, we get natural identifications :

$$I(\partial I \otimes X) \cup \xrightarrow{\sim} I(X \sqcup X) \xrightarrow{\sim} IX \sqcup IX \qquad \{0\} \otimes IX \xrightarrow{\sim} IX$$

and moreover :

$$I(\partial I \otimes X) \cap \{0\} \otimes IX = \{0\} \otimes (\partial I \otimes X).$$

Then, a direct inspection yields a commutative diagram :

$$\begin{tikzcd} \{0\} \otimes (\partial I \otimes X) \arrow[r] \arrow[d] & I(\partial I \otimes X) \arrow[r, "\sim"] & IX \sqcup IX \arrow[d, "(h,k)"] \\ \{0\} \otimes IX \arrow[r, "\sim"] & IX \arrow[r, "\sigma_\bullet \otimes u"] & Y. \end{tikzcd}$$

Hence, (h, k) and $\sigma_\bullet \otimes u$ induce a unique morphism

$$H_0 : I(\partial I \otimes X) \cup \{0\} \otimes IX \to Y.$$

Since Y is I-fibrant, H_0 extends via $(*)$ to a morphism

$$H : I(IX) \to Y.$$

In other words : $H \circ I(\partial_0 \otimes X) = h$, $H \circ I(\partial_1 \otimes X) = k$ and $H \circ (\partial_0 \otimes IX) = \sigma_\bullet \otimes u$. Let us then set :

$$\eta := H \circ (\partial_1 \otimes IX) : IX \to Y.$$

We compute :

$$\eta \circ (\partial_0 \otimes X) = H \circ I(\partial_0 \otimes X) \circ (\partial_1 \otimes X) = h \circ (\partial_1 \otimes X) = v$$

and likewise, $\eta \circ (\partial_1 \otimes X) = k \circ (\partial_1 \otimes X) = w$, so η is the sought I-homotopy.

- For $w = u$ and $k = \sigma_\bullet \otimes u$, the constant I-homotopy from u to itself, this shows that $\approx$ is symmetric. The general case, together with the symmetricity of $\approx$, proves the transitivity of $\approx$. □

Proposition 4.5.9. *(i) Every I-anodyne extension of $\widehat{\mathscr{A}}$ is a weak I-equivalence.*

(ii) A morphism of $\widehat{\mathscr{A}}$ between I-fibrant presheaves is a weak I-equivalence if and only if it is an I-homotopy equivalence.

(iii) An I-fibration of $\widehat{\mathscr{A}}$ is a trivial fibration if and only if it is the dual of a strong deformation retract.

(iv) An I-fibration of $\widehat{\mathscr{A}}$ between I-fibrant presheaves is a weak I-equivalence if and only if it is a trivial fibration.

Proof. (i): Let $j : X \to Y$ be an I-anodyne extension, and Z an I-fibrant presheaf on $\mathscr{A}$; we must check that j induces a bijection

$$j^* : [Y, Z] \to [X, Z].$$

Now, the surjectivity of j^* is clear, since by assumption j already induces a surjection $\widehat{\mathscr{A}}(Y, Z) \to \widehat{\mathscr{A}}(X, Z)$. Next, let $f_0, f_1 : Y \rightrightarrows Z$ be two morphisms of $\widehat{\mathscr{A}}$ such that $[f_0 j] = [f_1 j]$ in $[X, Z]$; then, by lemma 4.5.8 there exists an I-homotopy $h : IX \to Z$ from $f_0 j$ to $f_1 j$. Consider moreover the composition

$$k : \partial I \otimes Y \xrightarrow{\sim} Y \sqcup Y \xrightarrow{(h,k)} Z.$$

Clearly h and k agree on $IX \cap \partial I \otimes Y = \partial I \otimes X$, so we deduce a morphism

$$(h, k) : IX \cup \partial I \otimes Y \to Z.$$

But since j is I-anodyne, the same holds for the induced morphism $IX \cup \partial I \otimes Y \to IY$, and then, since Z is I-fibrant, (h, k) extends to a morphism $IY \to Z$. The latter is an I-homotopy from f_0 to f_1, whence $[f_0] = [f_1]$ in $[Y, Z]$, as required.

(ii): This is proved as the equivalence (b)⇔(d) of lemma 4.5.3.

(iii): The condition is necessary, due to proposition 4.5.6(i). Conversely, let $p : X \to Y$ be an I-fibration that is the dual of a strong deformation retract, so that there exist morphisms $s : Y \to X$ and $k : IX \to X$ such that :

$$ps = \mathbf{1}_Y \qquad k(\partial_0 \otimes X) = \mathbf{1}_X \qquad k(\partial_1 \otimes X) = sp \qquad pk = \sigma_\bullet \otimes p.$$

Consider then a commutative diagram :

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ L \arrow[r, "b"] & Y \end{tikzcd}$$

where j is a monomorphism of $\widehat{\mathscr{A}}$; we need to exhibit a diagonal filler $l : L \to X$.

To this aim, notice that the two morphisms :

$$IK \xrightarrow{Ia} IX \xrightarrow{k} X \xleftarrow{s} Y \xleftarrow{b} L \xleftarrow{\sim} \{1\} \otimes L$$

agree on $IK \cap \{1\} \otimes L = \{1\} \otimes K$, so they induce a unique morphism :

$$u : IK \cup \{1\} \otimes L \to X.$$

On the other hand, j induces an I-anodyne extension $IK \cap \{1\} \otimes L \to IL$, so by assumption the commutative square :

$$\begin{tikzcd} IK \cup \{1\} \otimes L \arrow[r, "u"] \arrow[d] & X \arrow[d, "p"] \\ IL \arrow[r, "\sigma_\bullet \otimes b"] & X \end{tikzcd}$$

has a diagonal filler $h : IL \to X$. We set $l := h \circ (\partial_0 \otimes L) : L \to X$. We then have :

$$\begin{aligned} pl &= ph \circ (\partial_0 \otimes L) = (\sigma_\bullet \otimes b) \circ (\partial_0 \otimes L) = b \\ lj &= h \circ (\partial_0 \otimes L) \circ j = h \circ Ij \circ (\partial_0 \otimes K) \\ &= k \circ Ia \circ (\partial_0 \otimes K) = k \circ (\partial_0 \otimes X) \circ a = a \end{aligned}$$

and this achieves the proof.

(iv): We already know that a trivial fibration is a weak I-equivalence, by proposition 4.5.6(i) and remark 4.5.5(ii). Conversely, let $p : X_\bullet \to Y_\bullet$ be an I-fibration between fibrant presheaves, and suppose that $p_\bullet$ is a weak I-equivalence; so $p_\bullet$ is an I-homotopic equivalence, by (i). Then, by lemma 4.5.8, there exists a morphism $t : Y \to X$ and an I-homotopy $k : IY \to Y$ from $\mathbf{1}_Y$ to pt. We then get a commutative square :

$$\begin{tikzcd} Y \arrow[r, "t"] \arrow[d, "\partial_1 \otimes Y"'] & X \arrow[d, "p"] \\ IY \arrow[r, "k"] & Y. \end{tikzcd}$$

But notice that $\partial_1 \otimes Y$ is an I-anodyne extension, so by assumption the square admits a diagonal filler $k' : IY \to X$; set $s := k' \circ (\partial_0 \otimes Y) : Y \to X$. Hence :

$$ps = pk' \circ (\partial_0 \otimes Y) = k \circ (\partial_0 \otimes Y) = \mathbf{1}_Y.$$

Since p is an I-homotopy equivalence, we then have $[sp] = [\mathbf{1}_X]$ as well in $[X, X]$, and by invoking again lemma 4.5.8 we get an I-homotopy h from $\mathbf{1}_X$ to sp.

Notice now that the two morphisms :

$$\{1\} \otimes IX \xrightarrow{\sim} IX \xrightarrow{\sigma_\bullet \otimes sp} X \xleftarrow{(h, sph)} IX \sqcup IX \xleftarrow{\sim} I(\partial I \otimes X)$$

agree on $\{1\} \otimes IX \cap I(\partial I \otimes X) = \{1\} \otimes (\partial I \otimes X)$, and therefore induce a morphism

$$u : \{1\} \otimes IX \cup I(\partial I \otimes X) \to X.$$

We then get another commutative square :

$$\begin{tikzcd} \{1\} \otimes IX \cup I(\partial I \otimes X) \arrow[rr, "u"] \arrow[d] & & X \arrow[d, "p"] \\ I(IX) \arrow[r, "I(\sigma_\bullet \otimes X)"] & IX \arrow[r, "ph"] & Y \end{tikzcd}$$

whose left vertical arrow is again an I-anodyne extension, hence it admits a diagonal filler $H : I(IX) \to X$. Set $K := H \circ (\partial_0 \otimes IX) : IX \to X$. We compute:

$$K \circ (\partial_0 \otimes X) = H \circ (\partial_0 \otimes IX) \circ (\partial_0 \otimes X) = h \circ (\partial_0 \otimes X) = \mathbf{1}_X$$
$$K \circ (\partial_1 \otimes X) = H \circ (\partial_0 \otimes IX) \circ (\partial_1 \otimes X) = sph \circ (\partial_0 \otimes X) = sp$$
$$pK = pH \circ (\partial_0 \otimes IX) = ph \circ (\partial_0 \otimes X) \circ (\sigma_\bullet \otimes X) = p \circ (\sigma_\bullet \otimes X).$$

So, p is the dual of a strong deformation retract, and we conclude with (iii). □

Corollary 4.5.10. *Let $i : X \to Y$ be a monomorphism of $\widehat{\mathscr{A}}$. We have :*

(i) If Y is I-fibrant, then i is an I-anodyne extension $\Leftrightarrow$ i is a weak I-equivalence.

(ii) i is a weak I-equivalence $\Leftrightarrow$ i has the left lifting property with respect to the class of I-fibrations with I-fibrant target.

Proof. (i): We already know that every I-anodyne extension is a weak I-equivalence (proposition 4.5.9(i)). Conversely, suppose that Y is I-fibrant, and that i is a weak I-equivalence. Pick a factorization $i = pj$, where j is an I-anodyne extension and p is an I-fibration (remark 4.5.5(i)); especially, j is a weak I-equivalence, so the same holds for p. Moreover, since the target of p is I-fibrant, the same holds for its source. Then p is a trivial fibration, by proposition 4.5.9(iv), *i.e.* $i \in l(\{p\})$. By the retract lemma (proposition 3.1.10), i is then a retract of j, whence the assertion.

(ii): Pick a factorization $p = qj$ for the unique morphism $p : Y \to e$, where e is a final object of $\widehat{\mathscr{A}}$, $q : Y' \to e$ is an I-fibration and $j : Y \to Y'$ is an I-anodyne extension (remark 4.5.5(i)); hence, Y' is I-fibrant. Suppose first that i is a weak I-equivalence, and consider a commutative square :

$$(*) \qquad \begin{tikzcd} X \arrow[r, "a"] \arrow[d, "i"'] & A \arrow[d, "f"] \\ Y \arrow[r, "b"] & B \end{tikzcd}$$

where f is an I-fibration with I-fibrant target. Then there exists a morphism $b' : Y' \to B$ with $b = b' \circ j$. Since ji is a monomorphism with I-fibrant target, (i) says that there exists a morphism $l' : Y' \to A$ with $fl' = b'$ and $l'ji = a$. Then $l := l'j : Y \to A$ is the sought diagonal filler for $(*)$.

Conversely, if i has the left lifting property with respect to the class of I-fibrations with I-fibrant target, choose a factorisation of ji into an I-anodyne extension k followed by an

I-fibration (with fibrant target). Then the retract lemma implies that ji is a retract of k, hence is I-anodyne. In particular, ji is a weak I-equivalence (proposition 4.5.9(i)). But the same holds for j, hence also for i. □

Corollary 4.5.11. *Let $j : K \to L$ be a monomorphism in $\widehat{\mathscr{A}}$ between I-fibrant presheaves. The following conditions are equivalent :*

(a) j is a weak I-equivalence

(b) j is an I-anodyne extension

(c) j is a strong deformation retract.

Proof. (a)⇔(b) is a special case of corollary 4.5.10, and the equivalence (b)⇔(c) follows from proposition 4.5.6(ii,iii). □

4.5.12. Let us denote by $\mathscr{W}_I$, $\mathscr{H}_I$ and $\mathscr{M}$ respectively the class of weak I-equivalences, of I-homotopy equivalences, and of monomorphisms of $\mathscr{A}$. We then have :

Proposition 4.5.13. *There exists a cardinal α_0 such that the following holds for every cardinal $\alpha \geq \alpha_0$. For every morphism $i : X \to Y$ in $\mathscr{M} \cap \mathscr{W}_I$ and every 2^α-admissible subobject $Y' \subset Y$, there exists a 2^α-accessible subobject $Y'' \subset Y$ with $Y' \subset Y''$, such that the restriction $X \cap Y'' \to Y''$ of i lies in $\mathscr{M} \cap \mathscr{W}_I$.*

Proof. By assumption, there exists a subset $\Lambda \subset \mathsf{An}$ such that $\mathsf{An} = l(r(\Lambda))$, and notice that every I-anodyne extension lies in $\mathscr{M} \cap \mathscr{W}_I$, by proposition 4.5.9(i). Let Ω be the class of ordinals, and consider the family of functors

$$(\mathsf{L}_\beta : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}} \mid \beta \in \Omega)$$

associated with the set Λ, as in §4.2.12. By theorem 4.1.8 and corollary 4.2.5(i), there exists an ordinal γ such that $\mathsf{L}_\gamma X$ is Λ-injective for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, *i.e.* $\mathsf{L}_\gamma X$ is I-fibrant for every such X, since $r(\Lambda) = r(\mathsf{An})$ (proposition 3.1.9(iii)); moreover, there exists a natural transformation $\lambda_\bullet : \mathbf{1}_{\widehat{\mathscr{A}}} \Rightarrow \mathsf{L}_\gamma$ such that $\lambda_X : X \to \mathsf{L}_\gamma X$ is an I-anodyne extension, for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$. We set $\mathsf{L} := \mathsf{L}_\gamma$; by virtue of proposition 4.2.13(i,ii), L sends monomorphisms to monomorphisms, and every pair $X \to Z \leftarrow Y$ of monomorphisms of $\widehat{\mathscr{A}}$ induces an isomorphism

$$\mathsf{L}(X \cap Y) \xrightarrow{\sim} \mathsf{L}X \cap \mathsf{L}Y.$$

Furthermore, by propositions 4.2.9(i), 4.2.11(ii) and 4.2.13(iii), there exists an infinite cardinal α_0 such that for every cardinal $\alpha \geq \alpha_0$ we have :

(a) the functors L and I are α-accessible, and send 2^α-accessible presheaves to 2^α-accessible presheaves

(b) every presheaf on $\mathscr{A}$ is the α-filtered union of its α-accessible subpresheaves.

Consider then a morphism $i : X \to Y$ in $\mathscr{M} \cap \mathscr{W}_I$; we get a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "i"'] & \mathsf{L}X \arrow[d, "\mathsf{L}i"] \\ Y \arrow[r, "\lambda_Y"] & \mathsf{L}Y. \end{tikzcd}$$

We have $\lambda_X, \lambda_Y \in \mathscr{W}_I$ by proposition 4.5.9(i), so $\mathsf{L}i \in \mathscr{M} \cap \mathscr{W}_I$ as well; since $\mathsf{L}X$ and $\mathsf{L}Y$ are I-fibrant, corollary 4.5.11 then says that $\mathsf{L}i$ is a strong deformation retract. Hence, we have morphisms $r : \mathsf{L}Y \to \mathsf{L}X$ and $h : I(\mathsf{L}Y) \to \mathsf{L}Y$ such that :

$$r \circ \mathsf{L}i = \mathbf{1}_{\mathsf{L}X} \quad h \circ (\partial_0 \otimes \mathsf{L}Y) = \mathbf{1}_{\mathsf{L}Y} \quad h \circ (\partial_1 \otimes \mathsf{L}Y) = \mathsf{L}(i) \circ r \quad h \circ I(\mathsf{L}i) = (\sigma_\bullet \otimes \mathsf{L}Y) \circ I(\mathsf{L}i).$$

Now, let $\alpha \geq \alpha_0$; recall that α^+ is regular (example 1.1.2(ii)), hence α-filtered (example 4.1.3(iv)), and $\alpha^+ \leq 2^\alpha$. Let $\mathscr{F}$ be the set of all 2^α-accessible subobjects of Y, and $Y' \in \mathscr{F}$; we shall exhibit a family $\{j_\beta : Y_\beta \to Y \mid \beta \in \alpha^+\} \subset \mathscr{F}$ such that

(c) $Y_0 = Y'$ and $Y_\delta \subset Y_\beta$ whenever $\delta \leq \beta < \alpha^+$
(d) for every $\beta \in \alpha^+$ we have a commutative diagram :

$$\begin{tikzcd} I(\mathsf{L}Y_\beta) \arrow[r, "k_\beta"] \arrow[d, "I(\mathsf{L}j_\beta)"'] & \mathsf{L}Y_{\beta+1} \arrow[d, "\mathsf{L}j_{\beta+1}"] \\ I(\mathsf{L}Y) \arrow[r, "h"] & \mathsf{L}Y. \end{tikzcd}$$

Notice that k_β is uniquely determined, since $\mathsf{L}j_{\beta+1}$ is a monomorphism. We proceed by transfinite induction : for $\beta = 0$, we set $Y_0 := Y'$; suppose then that $\alpha^+ > \beta > 0$, and that we have already exhibited Y_δ for every $\delta < \beta$. According to proposition 4.2.4, the subobject $Z := \bigcup_{\delta<\beta} Y_\delta$ is still 2^α-accessible, and then the same holds for $I(\mathsf{L}Z)$, by (a). By (b), $\mathscr{F}$ is 2^α-filtered and $Y = \bigcup_{F\in\mathscr{F}} F$; since L is 2^α-accessible and preserves monomorphisms, we deduce that $\mathsf{L}Y = \bigcup_{F\in\mathscr{F}} \mathsf{L}F$. Hence, the restriction $I(\mathsf{L}Z) \to \mathsf{L}Y$ of h to the subobject $I(\mathsf{L}Z) \to I(\mathsf{L}Y)$ factors through a morphism $I(\mathsf{L}Z) \to \mathsf{L}F$ for some $F \in \mathscr{F}$; we have $Y_\beta := Z \cup F \in \mathscr{F}$ by proposition 4.2.9(ii), and this completes the construction of the family $Y_\bullet$.

By proposition 4.2.4, the subobject $Y'' := \bigcup_{\beta\in\alpha^+} Y_\beta$ of Y is 2^α-accessible. Since L and I are α-accessible, we have natural identifications :

$$\varinjlim_{\beta\in\alpha^+} \mathsf{L}Y_\beta \xrightarrow{\sim} \mathsf{L}Y'' \qquad \varinjlim_{\beta\in\alpha^+} I(\mathsf{L}Y_\beta) \xrightarrow{\sim} I(\mathsf{L}Y'').$$

Hence, the colimit of the system of morphisms $(k_\beta \mid \beta \in \alpha^+)$ yields a morphism $k : I(\mathsf{L}Y'') \to \mathsf{L}Y''$ fitting into the commutative diagram :

$$\begin{tikzcd} I(\mathsf{L}Y'') \arrow[r, "k"] \arrow[d, "I(\mathsf{L}j)"'] & \mathsf{L}Y'' \arrow[d, "\mathsf{L}j"] \\ I(\mathsf{L}Y) \arrow[r, "h"] & \mathsf{L}Y \end{tikzcd}$$

where $j : Y'' \to Y$ is the inclusion. Moreover, for every $\beta \in \alpha^+$ we have :

$$\mathsf{L}(i) \circ r \circ \mathsf{L}(j_\beta) = h \circ (\partial_1 \otimes \mathsf{L}Y) \circ \mathsf{L}(j_\beta) = h \circ I(\mathsf{L}j_\beta) \circ (\partial_1 \otimes \mathsf{L}Y_\beta) = \mathsf{L}(j_{\beta+1}) \circ k_\beta \circ (\partial_1 \otimes \mathsf{L}Y_\beta)$$

whence a unique morphism $s_\beta : \mathsf{L}Y_\beta \to \mathsf{L}X \cap \mathsf{L}Y_{\beta+1}$ making commute the diagram:

$$\begin{tikzcd} \mathsf{L}Y_\beta \arrow[dr, "s_\beta"] \arrow[drr, bend left, "k_\beta\circ(\partial_1\otimes\mathsf{L}Y_\beta)"] \arrow[ddr, bend right, "r\circ\mathsf{L}(j_\beta)"'] & & \\ & \mathsf{L}X \cap \mathsf{L}Y_{\beta+1} \arrow[r] \arrow[d] & \mathsf{L}Y_{\beta+1} \arrow[d, "\mathsf{L}j_{\beta+1}"] \\ & \mathsf{L}X \arrow[r, "\mathsf{L}i"] & \mathsf{L}Y \end{tikzcd}$$

Since the colimits are universal in $\widehat{\mathscr{A}}$ (example 1.6.19(ii)), and since L is α-accessible, we have natural identifications :

$$\bigcup_{\beta\in\alpha^+} (X \cap Y_\beta) = X \cap Y'' \qquad \varinjlim_{\beta\in\alpha^+} \mathsf{L}X \cap \mathsf{L}Y_\beta \xrightarrow{\sim} \mathsf{L}(X \cap Y'').$$

Hence, the colimit of the system of morphisms $(s_\beta \mid \beta \in \alpha^+)$ yields a morphism $s : \mathsf{L}Y'' \to \mathsf{L}(X \cap Y'')$ fitting into the commutative diagram :

$$\begin{tikzcd}
\mathsf{L}Y'' \arrow[dr, "s"] \arrow[drr, bend left, "k\circ(\partial_1\otimes\mathsf{L}Y'')"] \arrow[ddr, bend right, "r\circ\mathsf{L}(j)"'] & & \\
& \mathsf{L}(X \cap Y'') \arrow[r, "\mathsf{L}i'"] \arrow[d, "\mathsf{L}j'"'] & \mathsf{L}Y'' \arrow[d, "\mathsf{L}j"] \\
& \mathsf{L}X \arrow[r, "\mathsf{L}i"] & \mathsf{L}Y
\end{tikzcd}$$

where $X \xleftarrow{j'} X \cap Y'' \xrightarrow{i'} Y''$ are the inclusions. We compute :

$$\begin{aligned}
\mathsf{L}(j') \circ s \circ \mathsf{L}(i') &= r \circ \mathsf{L}(j) \circ \mathsf{L}(i') = r \circ \mathsf{L}(i) \circ \mathsf{L}(j') = \mathsf{L}(j') \\
\mathsf{L}(j) \circ k \circ (\partial_0 \otimes \mathsf{L}X) &= h \circ I(\mathsf{L}j) \circ (\partial_0 \otimes \mathsf{L}X) = h \circ (\partial_0 \otimes \mathsf{L}Y) \circ \mathsf{L}(j) = \mathsf{L}(j) \\
\mathsf{L}(j) \circ k \circ I(\mathsf{L}i') &= h \circ I(\mathsf{L}j) \circ I(\mathsf{L}i') = h \circ I(\mathsf{L}i) \circ I(\mathsf{L}j') \\
&= (\sigma_\bullet \otimes \mathsf{L}Y) \circ I(\mathsf{L}i) \circ I(\mathsf{L}j') = (\sigma_\bullet \otimes \mathsf{L}Y) \circ I(\mathsf{L}j) \circ I(\mathsf{L}i') \\
&= \mathsf{L}(j) \circ (\sigma_\bullet \otimes \mathsf{L}Y'') \circ I(\mathsf{L}i')
\end{aligned}$$

and since $\mathsf{L}(j)$ and $\mathsf{L}(j')$ are monomorphisms, we then deduce :

$$s \circ \mathsf{L}(i') = \mathbf{1}_{X \cap Y''} \qquad k \circ (\partial_0 \otimes \mathsf{L}X) = \mathbf{1}_{\mathsf{L}Y''} \qquad k \circ I(\mathsf{L}i') = (\sigma_\bullet \otimes \mathsf{L}Y'') \circ I(\mathsf{L}i')$$

which shows that $\mathsf{L}i' : \mathsf{L}(X \cap Y'') \to \mathsf{L}Y''$ is a strong deformation retract, and therefore a weak I-equivalence (corollary 4.5.11). Then, the commutative square :

$$\begin{tikzcd}
X \cap Y'' \arrow[r, "\lambda_{X\cap Y''}"] \arrow[d, "i'"'] & \mathsf{L}(X \cap Y'') \arrow[d, "\mathsf{L}i'"] \\
Y'' \arrow[r, "\lambda_{Y''}"] & \mathsf{L}Y''
\end{tikzcd}$$

shows that i' lies in $\mathscr{M} \cap \mathscr{W}_I$. □

Theorem 4.5.14. *(i) There exists a unique cofibrantly generated model category*

$$(\widehat{\mathscr{A}}, \mathscr{W}_I, \mathscr{F}ib_I, \mathscr{C}of_I)$$

with $\mathscr{C}of_I := \mathscr{M}$ (notation of §4.5.12).

(ii) Every fibration for this model structure is an I-fibration.

(iii) A presheaf X is fibrant relative to this model structure $\Leftrightarrow$ X is I-fibrant.

(iv) A morphism f with fibrant target is a fibration $\Leftrightarrow$ f is an I-fibration.

(v) Every presheaf is cofibrant; every I-anodyne extension is a trivial cofibration.

Proof. (i): By definition, the class of trivial fibrations is $r(\mathscr{M})$, and we know already that $(\mathscr{M}, r(\mathscr{M}))$ is a weak factorization system for $\widehat{\mathscr{A}}$ (corollary 4.2.7). It is also clear that $\mathscr{W}_I$ enjoys the 2-out-of-3 property. Moreover, $\widehat{\mathscr{A}}$ is cocomplete (remark 1.6.2(i)).

Claim 4.5.15. (i) $r(\mathscr{M}) = \mathscr{W}_I \cap r(\mathscr{M} \cap \mathscr{W}_I)$.

(ii) $\mathscr{M} \cap \mathscr{W}_I$ is a saturated class of morphisms of $\widehat{\mathscr{A}}$.

Proof : (i): The inclusion $r(\mathscr{M}) \subset r(\mathscr{M} \cap \mathscr{W}_I)$ is trivial, and we have $r(\mathscr{M}) \subset \mathscr{H}_I$, by virtue of proposition 4.5.6(i); but obviously $\mathscr{H}_I \subset \mathscr{W}_I$. Hence, let $(p : X \to Y) \in \mathscr{W}_I \cap r(\mathscr{M} \cap \mathscr{W}_I)$; we may then factor $p = j$ with $j \in \mathscr{M}$ and $q \in r(\mathscr{M})$, and by the 2-out-of-3 property, $j \in \mathscr{M} \cap \mathscr{W}_I$, since we have just shown that $q \in \mathscr{W}_I$. By the retract lemma (proposition

3.1.10), p is then a retract of q. But $r(\mathscr{M})$ is stable under retracts (proposition 3.1.10 and remark 3.1.4(i)), so $p \in r(\mathscr{M})$.

(ii): This follows from corollary 4.5.10(ii) and proposition 3.1.9(v). ◇

The class $r(\mathscr{M} \cap \mathscr{W}_I)$ shall therefore be our designated class of fibrations; we know already that the candidate class $\mathscr{M}$ of cofibrations admits a generating set (see the proof of corollary 4.2.7); to conclude, it then suffices to exhibit a generating set of trivial cofibrations, since then corollary 4.1.9 will imply that $(\mathscr{M} \cap \mathscr{W}_I, r(\mathscr{M} \cap \mathscr{W}_I))$ is a weak factorization system for $\widehat{\mathscr{A}}$, and claim 4.5.15(i) will identify $r(\mathscr{M})$ with the class of trivial fibrations. To this aim, let α_0 be an infinite cardinal fulfilling the condition of proposition 4.5.13, and $\alpha \geq \alpha_0$ such that $\mathscr{A}$ is α-small; let also $\mathscr{F}$ be the set of all presheaves X on $\mathscr{A}$ such that $X(A) \in 2^\alpha$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and $\mathscr{G}$ the set of all elements $K \to L$ of $\mathscr{M} \cap \mathscr{W}_I$ such that $K, L \in \mathscr{F}$. According to propositions 4.2.9(ii) and 4.5.13, for every element $i : X \to Y$ of $\mathscr{M} \cap \mathscr{W}_I$ and every 2^α-accessible subpresheaf $Y' \subset Y$ there exist an element $j : K \to L$ of $\mathscr{G}$ and a 2^α-accessible subpresheaf $Y'' \subset Y$ containing Y' and with isomorphisms $\phi : K \xrightarrow{\sim} X \cap Y''$, $\psi : L \xrightarrow{\sim} Y''$ making commute the diagram :

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, "\phi"'] & L \arrow[d, "\psi"] \\ X \cap Y'' \arrow[r, "i_{|X \cap Y''}"] & Y''. \end{tikzcd}$$

We have $r(\mathscr{M} \cap \mathscr{W}_I) \subset r(\mathscr{G})$ (proposition 4.2.9(ii)); in order to show the converse inclusion, consider any commutative diagram :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

with $p \in r(\mathscr{G})$ and $i \in \mathscr{M} \cap \mathscr{W}_I$, and denote by $\mathscr{D}$ the set of all morphisms $h : B' \to X$ with $A \subset B' \subset B$ and such that h is a diagonal filler for the induced diagram :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ B' \arrow[r, "g_{|B'}"] & Y. \end{tikzcd}$$

Notice that $\mathscr{D} \neq \varnothing$, since $f \in \mathscr{D}$. We endow $\mathscr{D}$ with the partial order such that

$$(h_1 : B_1 \to X) \leq (h_2 : B_2 \to X) \qquad \Leftrightarrow \qquad B_1 \subset B_2 \quad \text{and} \quad h_1 = h_{2|B_1}.$$

Arguing as in the proof of corollary 4.2.7, we see that $\mathscr{D}$ admits a maximal element $k : C \to X$, and we are reduced to checking that $C = B$. However, B is the filtered union of its 2^α-accessible subpresheaves (propositions 4.2.9(i)); hence, if $C \neq B$ there exists a 2^α-accessible presheaf $B' \subset B$ such that $B' \not\subset C$, and then we can find a 2^α-accessible presheaf $B'' \subset B$ with $B' \subset B''$ and $(j : K \to L) \in \mathscr{G}$ fitting into a commutative diagram :

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "j"'] & C \cap B'' \arrow[rr, "k_{|C \cap B''}"] \arrow[d] & & X \arrow[d, "p"] \\ L \arrow[r, "\psi"] & B'' \arrow[r] & B \arrow[r, "g"] & Y \end{tikzcd}$$

where ϕ and ψ are isomorphisms of $\widehat{\mathscr{A}}$. Since $p \in r(\mathscr{G})$, we then get a diagonal filler $k' : B'' \to X$ for the right square subdiagram. Set $C' := C \cup B''$; then k and k' determine an element $k'' : C' \to X$ of $\mathscr{D}$ with $k'' > k$, a contradiction.

Hence, $\mathscr{G}$ is the sought generating set of trivial cofibrations.

(ii,v): Since the initial object of $\widehat{\mathscr{A}}$ is the empty presheaf, every presheaf is cofibrant. For the second assertion, it suffices to recall that $\mathsf{An} \subset \mathscr{M} \cap \mathscr{W}_I$ (proposition 4.5.9(i)), so that $r(\mathscr{M} \cap \mathscr{W}_I) \subset r(\mathsf{An})$ (proposition 3.1.9(ii)).

(iii,iv): Corollary 4.5.10(ii) shows that every I-fibration with I-fibrant target is a fibration for the model structure on $\widehat{\mathscr{A}}$ given by (i). Especially, every I-fibrant presheaf is fibrant; so, every I-fibration with fibrant target is a fibration, whence the assertions, in view of (ii). □

Set $\mathscr{F}ib_I := r(\mathscr{M} \cap \mathscr{W}_I)$; by theorem 4.5.14, we then have a well-defined model category $(\widehat{\mathscr{A}}, \mathscr{W}_I, \mathscr{F}ib_I, \mathscr{M})$, whose homotopy category we denote by :

$$\mathsf{ho}_I(\widehat{\mathscr{A}}).$$

Corollary 4.5.16. *(i) $\mathscr{W}_I$ is the smallest subclass of* $\mathrm{Mor}(\widehat{\mathscr{A}})$ *containing* An *and enjoying the 2-out-of-3 property of definition 3.2.1(b).*

(ii) The localization $\widehat{\mathscr{A}}[\mathsf{An}^{-1}]$ of $\widehat{\mathscr{A}}$ exists, and the natural functor $\widehat{\mathscr{A}} \to \mathsf{ho}_I(\widehat{\mathscr{A}})$ factors through an isomorphism of categories :

$$\boxed{\widehat{\mathscr{A}}[\mathsf{An}^{-1}] \xrightarrow{\sim} \mathsf{ho}_I(\widehat{\mathscr{A}}).}$$

(iii) For every fibrant object Y of $\widehat{\mathscr{A}}$, we have natural identifications :

$$\boxed{\mathsf{ho}_I(\widehat{\mathscr{A}})(X, Y) \xrightarrow{\sim} [X, Y] \qquad \forall X \in \mathrm{Ob}(\widehat{\mathscr{A}})}$$

where $[X, Y]$ is the set of I-homotopy classes of maps $X \to Y$ (definition 4.5.1(ii)).

Proof. (i): Let $f : X \to Y$ be any morphism of $\widehat{\mathscr{A}}$. The proof of proposition 4.5.13 exhibits an I-anodyne extension $j : Y \to Y'$ with fibrant target. Moreover, let $\Lambda \subset \mathsf{An}$ be a subset such that $\mathsf{An} = l(r(\Lambda))$; by applying corollary 4.1.9(i) with $\mathscr{I} = \Lambda$, we obtain a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

such that i and j are I-anodyne extensions, and p an I-fibration. Then p is a fibration of $\widehat{\mathscr{A}}$, by theorem 4.5.14(iv). If f is a weak equivalence of $\widehat{\mathscr{A}}$, the same then holds for p, therefore p is the dual of a strong deformation restract (proposition 4.5.9(iii)), and in particular it admits a section $s : Y' \to X'$. Hence, s lies in $\mathscr{M} \cap \mathscr{W}_I$, so it is an I-anodyne extension (corollary 4.5.11), whence the claim.

(ii): If $F : \widehat{\mathscr{A}} \to \mathscr{B}$ is any functor such that Fg is an isomorphism for every $g \in \mathsf{An}$, then (i) implies that Ff is an isomorphism for every $f \in \mathscr{W}_I$; since $\mathsf{An} \subset \mathscr{W}_I$, the assertion follows.

(iii): Since every object of $\widehat{\mathscr{A}}$ is cofibrant, the assertion follows from lemma 3.2.11(i), proposition 3.2.14(ii), and the explicit construction of the homotopy category in the proof of theorem 3.3.5. □

Proposition 4.5.17. *Endow $\widehat{\mathscr{A}}$ with the model structure given by theorem 4.5.14, and let $\mathscr{C}$ be another model category, and $F : \widehat{\mathscr{A}} \rightleftarrows \mathscr{C} : G$ an adjoint pair of functors, such that F sends monomorphisms to cofibrations. Let moreover $\Lambda \subset \mathsf{An}$ be a subset such that $\mathsf{An} = l(r(\Lambda))$. The following conditions are equivalent :*

(a) (F, G) is a Quillen adjunction.

(b) F sends every I-anodyne extension to a trivial cofibration of $\mathscr{C}$.

(c) F sends every element of Λ to a trivial cofibration of $\mathscr{C}$.

Proof. We have (a)⇒(b)⇒(c), by lemma 4.5.14 and proposition 4.5.9(i).

Next, recall that the class $\mathscr{F}$ of trivial cofibrations of $\mathscr{D}$ is saturated (proposition 3.1.9(v)), so $F^{-1}\mathscr{F}$ is a saturated class of $\mathscr{C}$ (remark 3.1.4(iii) and [13, Prop.2.49(ii)]); on the other hand, An is the saturation of Λ (remark 4.4.7(ii)), so (c)⇒(b).

Lastly, suppose that (b) holds, and let $i : X \to Y$ be a trivial cofibration; the proof of proposition 4.5.13 exhibits a commutative diagram of $\widehat{\mathscr{A}}$:

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "i"'] & \mathsf{L}X \arrow[d, "\mathsf{L}i"] \\ Y \arrow[r, "\lambda_Y"] & \mathsf{L}Y \end{tikzcd}$$

where $\mathsf{L}X$ and $\mathsf{L}Y$ are I-fibrant, λ_X and λ_Y are I-anodyne extensions, and $\mathsf{L}i$ is a monomorphism, since the same holds for i; hence $\mathsf{L}i$ is a trivial cofibration of $\widehat{\mathscr{A}}$, and so it is an I-anodyne extension (corollary 4.5.11). Then $F(\mathsf{L}i), F\lambda_X$ and $F\lambda_Y$ are isomorphisms, so the same must hold for Fi, whence (a), again by lemma 4.5.14. ◻

4.6. **Localizers and absolute weak equivalences.** This section complements theorem 4.5.14 with some further useful observations. To begin with, for any small category $\mathscr{A}$, we wish to characterize the subclasses $\mathscr{W} \subset \mathrm{Mor}(\widehat{\mathscr{A}})$ that can occur as classes of weak equivalences for some model category structure on $\widehat{\mathscr{A}}$ obtained via theorem 4.5.14. To this aim, we make the following :

Definition 4.6.1. Let $\mathscr{C}$ be a category, $\mathscr{M} \subset \mathrm{Mor}(\mathscr{C})$ the class of monomorphisms of $\mathscr{C}$, and $\mathscr{W} \subset \mathrm{Mor}(\mathscr{C})$ another class. We say that $\mathscr{W}$ is a *$\mathscr{C}$-localizer*, if the following conditions hold :

(L1) $\mathscr{W}$ satisfies the 2-out-of-3 property (see definition 3.2.1(i.b))

(L2) $r(\mathscr{M}) \subset \mathscr{W}$ (notation of definition 3.1.1(iii))

(L3) $\mathscr{W} \cap \mathscr{M}$ is weakly saturated (see definition 3.1.3(vii)).

Lemma 4.6.2. *For every category $\mathscr{C}$ and every subclass $C \subset \mathrm{Mor}(\mathscr{C})$, there exists a smallest $\mathscr{C}$-localizer $\mathscr{W}(C)$ containing C. We call $\mathscr{W}(C)$* the $\mathscr{C}$-localizer generated by C.

Proof. The only difficulties in proving this assertion are due to set-theoretic issues : we would like to just take the intersection of all the $\mathscr{C}$-localizers containing C, but it is not clear whether that is a legitimate operation, since it involves quantifications over classes. Instead, we argue as in the proof of proposition 3.1.6.

Obviously we may assume that $r(\mathscr{M}) \subset C$, where $\mathscr{M}$ denotes the class of monomorphisms of $\mathscr{C}$. Next, for every category $\mathscr{A}$, every class $B \subset \mathrm{Mor}(\mathscr{A})$, and every infinite cardinal κ, we say that B is *weakly k-saturated* if it satisfies the 2-out-of-3 property, and is stable under push-outs and κ-small compositions (where the latter are defined as in the proof of proposition 3.1.6). If $\mathscr{A}$ is small, we also define the *weak κ-saturation $B^*_{\kappa,\mathscr{A}}$ of B in $\mathscr{A}$* as the intersection of all weakly κ-saturated subclasses of $\mathrm{Mor}(\mathscr{A})$ containing B.

Let Ω be the class of all infinite cardinals, and for every $\kappa \in \Omega$ let $\mathscr{E}_\kappa$ be the class of all small relatively κ-cocomplete subcategories of $\mathscr{C}$ (defined as in the proof of proposition 3.1.6); we set

$$C^*_\kappa := \bigcup_{\mathscr{A} \in \mathscr{E}} (C \cap \mathrm{Mor}(\mathscr{A}))^*_{\kappa,\mathscr{A}} \qquad \mathscr{W}(C) := \bigcup_{\kappa \in \Omega} C^*_\kappa.$$

With claim 3.1.7, it is easily seen that $\mathscr{W}(C)$ is the smallest $\mathscr{C}$-localizer containing C. □

Definition 4.6.3. Let $\mathscr{C}$ be a category, and $\mathscr{W}$ a $\mathscr{C}$-localizer. We say that $\mathscr{W}$ is *accessible*, if it is the $\mathscr{C}$-localizer $\mathscr{W}(S)$ generated by a *subset* S of $\mathrm{Mor}(\mathscr{C})$.

Proposition 4.6.4. *(i) Let $\mathscr{A}$ be a small category, $((I, \partial_\bullet, \sigma), \mathsf{An})$ a homotopical structure on $\mathscr{A}$, and $\mathscr{W}$ the class of weak equivalences for the model category structure induced on $\widehat{\mathscr{A}}$ by $((I, \partial_\bullet, \sigma), \mathsf{An})$ via theorem 4.5.14. Then $\mathscr{W}$ is an accessible $\widehat{\mathscr{A}}$-localizer.*

(ii) More precisely, if Λ is a subset of $\mathrm{Mor}(\widehat{\mathscr{A}})$ such that $\mathsf{An} = l(r(\Lambda))$, then $\mathscr{W} = \mathscr{W}(\Lambda)$.

Proof. Since $\mathscr{M}$ and $\mathscr{W} \cap \mathscr{M}$ are the classes of cofibrations and respectively of trivial cofibrations for the model category structure on $\widehat{\mathscr{A}}$ induced by $((I, \partial_\bullet, \sigma), \mathsf{An})$, it is clear that $\mathscr{W}$ verifies conditions (L1) and (L2) of definition 4.6.1; condition (L3) also follows, by virtue of proposition 3.1.9(v). Hence $\mathscr{W}$ is a $\widehat{\mathscr{A}}$-localizer, and clearly $\mathscr{W}(\Lambda) \subset \mathscr{W}$, for any Λ as in (ii), by theorem 4.5.14(v). For the converse, let $f : X \to Y$ be any element of $\mathscr{W}$, and denote by E the final object of $\widehat{\mathscr{A}}$; by corollary 4.1.9(i), we can factor the unique morphism $h : Y \to E$ as the composition $g \circ i$, where $i : Y \to Y'$ lies in the weak saturation $\mathscr{I}$ of Λ, and $g \in r(\Lambda)$, so Y' is a fibrant object of $\widehat{\mathscr{A}}$ (theorem 4.5.14(iv)). By repeating the same argument with h replaced by $i \circ f$, we get a commutative diagram of $\widehat{\mathscr{A}}$:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "i"] & Y' \end{tikzcd}$$

with $i, j \in \mathscr{I}$, and f' is an I-fibration with fibrant target. But then $i, j \in \mathscr{W}(\Lambda)$, and on the other hand f' is a trivial fibration of $\widehat{\mathscr{A}}$, *i.e.* an element of $r(\mathscr{M})$, again by theorem 4.5.14(i,iv); hence $f' \in \mathscr{W}(\Lambda)$, by condition (L2) of definition 4.6.1. With (L1), we conclude that $f \in \mathscr{W}(\Lambda)$. □

Theorem 4.6.5. *Let $\mathscr{A}$ be a small category, and $\mathscr{W}$ a subclass of $\mathrm{Mor}(\widehat{\mathscr{A}})$. The following conditions are equivalent :*

(a) $\mathscr{W}$ is an accessible $\widehat{\mathscr{A}}$-localizer.

(b) There exists a homotopical structure $(\mathscr{L}, \mathsf{An})$, where $\mathscr{L} := (L, \partial_\bullet, \sigma)$ is the Lawvere cylinder on $\widehat{\mathscr{A}}$ (see example 4.4.5(iii)), such that $\mathscr{W}$ is the class of weak equivalences for the model category structure induced by $(\mathscr{L}, \mathsf{An})$ via theorem 4.5.14.

(c) There exists a homotopical structure $(I, \partial_\bullet, \sigma)$ on $\mathscr{A}$ such that $\mathscr{W}$ is the class of weak equivalences for the model category structure on $\widehat{\mathscr{A}}$ induced by $(I, \partial_\bullet, \sigma)$ via theorem 4.5.14.

(d) There exists a cofibrantly generated model category structure on $\widehat{\mathscr{A}}$ whose class of weak equivalences is $\mathscr{W}$, and whose cofibrations are the monomorphisms of $\widehat{\mathscr{A}}$.

Proof. (This is [3, Th.1.4.3]). Obviously, (b)⇒(c)⇒(d).

(a)⇒(b): Say that $\mathscr{W} = \mathscr{W}(T)$ for a subset $T \subset \mathrm{Mor}(\widehat{\mathscr{A}})$; by corollary 4.2.7, for every $f \in T$ we can find a monomorphism i_f and a trivial fibration p_f such that $f = p_f \circ i_f$, and we set $S := \{i_f \mid f \in T\}$. Clearly $\mathscr{W} = \mathscr{W}(S)$. Let $\mathsf{An} \subset \mathrm{Mor}(\widehat{\mathscr{A}})$ be the smallest class of L-anodyne extensions containing S, where $\mathscr{L} := (L, \partial_\bullet, \sigma)$ denotes the Lawvere cylinder on $\widehat{\mathscr{A}}$. Pick a cellular model M for $\widehat{\mathscr{A}}$, and define the subset $\Lambda_L(S, M)$ as in §4.4.8; by proposition 4.6.4(ii), the class of weak equivalences of the model category structure on $\widehat{\mathscr{A}}$ induced by $(\mathscr{L}, \mathsf{An})$ is $\mathscr{W}' := \mathscr{W}(\Lambda_L(S, M))$. Clearly, $\mathscr{W} \subset \mathscr{W}'$, so we are reduced to checking the converse inclusion. The latter follows from part (ii) of the following :

Claim 4.6.6. (i) $L(\mathscr{W}) \subset \mathscr{W}$.

(ii) $\Lambda_L(S, M) \subset \mathscr{W}$.

Proof : (i): Recall that $LX = \Omega \times X$ for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, where Ω denotes a subobject classifier for $\widehat{\mathscr{A}}$; since Ω is an injective object of $\widehat{\mathscr{A}}$ (lemma 1.8.6(iii)), it is easily seen that the projection $\sigma_X : \Omega \times X \to X$ lies in $r(\mathscr{M})$ for every $X \in \mathrm{Ob}(\widehat{\mathscr{A}})$, where $\mathscr{M}$ denotes the class of monomorphisms of $\widehat{\mathscr{A}}$. Especially $\sigma_X \in \mathscr{W}$ for every such X, and then the same holds for the morphisms $\partial_i \otimes X : X \to \Omega \times X$, for $i = 0, 1$, by the 2-out-of-3 property of $\mathscr{W}$; however, for every morphism $u : X \to Y$ of $\widehat{\mathscr{A}}$ we have a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\partial_0 \otimes X"'] & Y \arrow[d, "\partial_0 \otimes Y"] \\ LX \arrow[r, "Lu"] & LY \end{tikzcd}$$

so $u \in \mathscr{W} \Leftrightarrow Lu \in \mathscr{W}$, again by the 2-out-of-3 property.

(ii): Since $S \subset \mathscr{W}$, it suffices to check that conditions (An1) and (An2) of definition 4.4.6(i) hold with An replaced by $\mathscr{W} \cap \mathscr{M}$. Hence, let $X \to Y$ be any monomorphism of $\widehat{\mathscr{A}}$; we get a commutative diagram :

$$\begin{tikzcd} \{\varepsilon\} \otimes X \arrow[r] \arrow[d, "\partial_\varepsilon \otimes X"'] & \{\varepsilon\} \otimes Y \arrow[d, "\alpha"] \arrow[rdd, bend left, "\partial_\varepsilon \otimes Y"] & \\ LX \arrow[r] \arrow[rrd, bend right, "Lu"'] & LX \cup \{\varepsilon\} \otimes Y \arrow[rd, "\beta"] & \\ & & LY \end{tikzcd}$$

whose square subdiagram is cocartesian. We know already that $\partial_\varepsilon \otimes X \in \mathscr{W} \cap \mathscr{M}$, so the same holds for α, by condition (L3) of definition 4.6.1; then, since $\partial_\varepsilon \otimes Y \in \mathscr{W}$, we get $\beta \in \mathscr{W} \cap \mathscr{M}$, by condition (L1), and this shows that (An1) holds for $\mathscr{W} \cap \mathscr{M}$.

Lastly, let $u : X \to Y$ be an element of $\mathscr{W} \cap \mathscr{M}$; we get a commutative diagram :

$$\begin{tikzcd} \partial L \otimes X \arrow[r, "\partial L \otimes u"] \arrow[d] & \partial L \otimes Y \arrow[d, "\beta"] \arrow[rdd, bend left, "\delta"] & \\ LX \arrow[r, "\alpha"] \arrow[rrd, bend right, "Lu"'] & LX \cup \partial L \otimes Y \arrow[rd, "\gamma"] & \\ & & LY \end{tikzcd}$$

whose square subdiagram is cocartesian. Since $u \in \mathscr{W} \cap \mathscr{M}$, we have $\partial L \otimes u \in \mathscr{W} \cap \mathscr{M}$, by (L3) (lemma 3.1.8), and then $\alpha \in \mathscr{W} \cap \mathscr{M}$, again by (L3); on the other hand, $Lu \in \mathscr{W}$, by (i), so $\gamma \in \mathscr{W} \cap \mathscr{M}$, by (L1). This shows that (An2) holds for $\mathscr{W} \cap \mathscr{M}$. ◇

(d)⇒(a): Let $\mathscr{M}$ be the class of monomorphisms of $\widehat{\mathscr{A}}$; by assumption, $\mathscr{W} \cap \mathscr{M} = l(r(\Lambda))$ for some subset Λ of $\mathscr{W} \cap \mathscr{M}$, and we show more precisely that $\mathscr{W} = \mathscr{W}(\Lambda)$.

Indeed, it is clear that $\mathscr{W}$ is a $\widehat{\mathscr{A}}$-localizer, since $r(\mathscr{M})$ is the class of trivial fibrations of the given model category structure on $\widehat{\mathscr{A}}$, and since $l(r(\Lambda))$ is saturated (proposition 3.1.9(v)), so $\mathscr{W}(\Lambda) \subset \mathscr{W}$. For the converse, let $f : X \to Y$ be any element of $\mathscr{W}$; by corollary 4.1.9(i) we can write $f = g \circ i$, where i lies in the weak saturation of Λ, and $g \in r(\Lambda) = r(l(r(\Lambda))) = r(\mathscr{W} \cap \mathscr{M})$ (proposition 3.1.9(iii)). Hence, $i \in \mathscr{W}(\Lambda)$, and g is a fibration for the given model category structure; then g is a trivial fibration, *i.e.* $g \in r(\mathscr{M}) \subset \mathscr{W}(\Lambda)$, so finally $f \in \mathscr{W}(\Lambda)$. ◻

4.6.1. *Absolute weak equivalences.* Let $\mathscr{A}$ be a small category, and $\mathscr{I} := ((I, \partial_\bullet, \sigma), \mathsf{An})$ a homotopical structure on $\mathscr{A}$; in general, it is not possible to recover the selected class An of anodyne extensions solely in terms of the model category structure associated to $\mathscr{I}$; however, we wish now to explain that $\mathscr{I}$ induces on each slice category $\widehat{\mathscr{A}}/S$ a natural model category structure, and An can be described purely in terms of the classes of weak equivalences of all such model categories.

- Indeed, for any $S \in \mathrm{Ob}(\widehat{\mathscr{A}})$, recall the natural equivalence of categories :

$$(*) \qquad\qquad \widehat{\mathscr{A}/S} \xrightarrow{\sim} \widehat{\mathscr{A}}/S$$

provided by lemma 1.7.2(i). Especially, we may regard a functorial cylinder on $\widehat{\mathscr{A}/S}$ as the datum of an endofunctor $I_S : \widehat{\mathscr{A}}/S \to \widehat{\mathscr{A}}/S$ together with suitable natural transformations

$$\partial_0^S, \partial_1^S : \mathbf{1}_{\widehat{\mathscr{A}}/S} \Rightarrow I_S \qquad \text{and} \qquad \sigma^S : I_S \Rightarrow \mathbf{1}_{\widehat{\mathscr{A}}/S}.$$

- Any functorial cylinder $(I, \partial_\bullet, \sigma)$ on $\widehat{\mathscr{A}}$ induces such an endofunctor, by the rule :

$$I_S(X, f : X \to S) := (IX, \sigma_\bullet \otimes f : IX \to S) \qquad \forall (X, f) \in \mathrm{Ob}(\widehat{\mathscr{A}}/S).$$

Then clearly we get the corresponding natural transformations, by setting :

$$\partial^S_{i,(X,f)} := \partial_{i,X}/S : (X, f) \to I_S(X, f) \qquad (i = 1, 2) \qquad \sigma^S_{(X,f)} := \sigma_X/S$$

for every $(X, f) \in \mathrm{Ob}(\widehat{\mathscr{A}}/S)$ (notation of §1.4). Moreover, corollary 1.4.6(iii) shows that the coproduct of any small family $((Y_i, g_i) \mid i \in I)$ of objects of $\widehat{\mathscr{A}}/S$ is represented by (Y, g), where $Y := \bigsqcup_{i \in I} Y_i$ with universal co-cone $(e_i : Y_i \to Y \mid i \in I)$, and where $g : Y \to S$ is the unique morphism such that $g \circ e_i = g_i$ for every $i \in I$; then, taking into account corollary 1.4.6(i), we see that for every $(X, f) \in \mathrm{Ob}(\widehat{\mathscr{A}}/S)$ the pair $(\partial_i^S \mid i = 1, 2)$ induces a monomorphism of $\widehat{\mathscr{A}}/S$

$$(\partial^S_{0,(X,f)}, \partial^S_{1,(X,f)}) = (\partial_{0,X}, \partial_{1,X})/S : (X, f) \sqcup (X, f) \to I_S(X, f).$$

Hence, $(I_S, \partial^S_\bullet, \sigma^S)$ is a functorial cylinder on $\widehat{\mathscr{A}/S}$, under the equivalence $(*)$.

- Moreover, suppose that $(I, \partial_\bullet, \sigma)$ is an exact cylinder, and let

$$F : J \to \widehat{\mathscr{A}}/S \qquad j \mapsto (Fj, f_j : Fi \to S)$$

be a functor from a small category J; pick a colimit L for $\mathrm{s}_S \circ F : J \to \widehat{\mathscr{A}}$ and let $\tau_\bullet : \mathrm{s}_S \circ F \Rightarrow c_L$ be a universal co-cone; since I preserves small colimits, the induced co-cone $I \star \tau_\bullet : I \circ \mathrm{s}_S \circ F \Rightarrow c_{IL}$ is still universal. Moreover, there exists a unique morphism $f : L \to S$ such that $f \circ \tau_j = f_j$ for every $j \in \mathrm{Ob}(J)$, and corollary 1.4.6(iii) implies that the systems of morphisms

$$\tau_j/S : Fj \to (L, f) \qquad I(\tau_j)/S : (IFi, \sigma_\bullet \otimes f_i) \to (IL, \sigma_\bullet \otimes f)$$

yield universal co-cones $\tau_{/S\bullet} : F \Rightarrow c_{(L,f)}$ and $I_S \star \tau_{/S\bullet} : I_S \circ F \Rightarrow c_{I_S L}$. Hence I_S preserves small colimits; in light of corollary 1.4.6(i), we also see that I_S preserves monomorphisms, so axiom (DH1) of definition 4.4.3(iii) holds for $(I_S, \partial^S_\bullet, \sigma^S)$. Furthermore, for every monomorphism $j/S : (K, f) \to (L, g)$ of $\widehat{\mathscr{A}/S}$, the underlying morphism $j : K \to L$ is a monomorphism of $\widehat{\mathscr{A}}$ (corollary 1.4.6(i)), so the induced diagram :

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, "\partial_i \otimes K"'] & L \arrow[d, "\partial_i \otimes L"] \\ IK \arrow[r, "Ij"] & IL \end{tikzcd}$$

is cartesian in $\mathscr{A}$, for $i = 0, 1$, and since the source functor s_S reflects fibre products (corollary 1.4.6(i)), we deduce that axiom (DH2) holds for $(I_S, \partial^S_\bullet, \sigma^S)$ as well, *i.e.* $(I_S, \partial^S_\bullet, \sigma^S)$ *is an exact cylinder on* $\widehat{\mathscr{A}/S}$.

• Next, let An be a class of I-anodyne extensions, and $\Lambda \subset \mathsf{An}$ a subset such that $\mathsf{An} = l(r(\Lambda))$; by lemma 3.1.12(ii), we have $\mathsf{An}/S = l(r(\Lambda/S))$, and it is then clear that An/S is a class of I_S-anodyne extensions. We may then apply theorem 4.5.14 to the homotopical structure $((I_S, \partial^S_\bullet, \sigma^S), \mathsf{An}/S)$ to obtain a model category

$$(\widehat{\mathscr{A}/S}, \mathscr{W}_{I/S}, \mathscr{F}ib_{I/S}, \mathscr{C}of_{I/S})$$

whose cofibrations $\mathscr{C}of_{I/S}$ are the monomorphisms, and whose fibrant objects are the I-fibrations of $\mathscr{A}$ with target S. By construction, we have :

$$\mathscr{C}of_{I/S} = \mathscr{C}of_I/S \qquad \mathscr{F}ib_{I/S} \cap \mathscr{W}_{I/S} = (\mathscr{F}ib_I \cap \mathscr{W}_I)/S$$
$$\mathscr{W}_{I/S} \subset \mathscr{W}_I/S \qquad \mathscr{C}of_{I/S} \cap \mathscr{W}_{I/S} \subset (\mathscr{C}of_I \cap \mathscr{W}_I)/S \qquad \mathscr{F}ib_I/S \subset \mathscr{F}ib_{I/S}.$$

Indeed, the first equality follows from corollary 1.4.6(i), and the second one follows from the first and from lemma 3.1.12(i); the first inclusion follows from corollary 4.5.16(i), the second one follows from the first and lemma 3.1.12(i), and the third one follows from the second and from proposition 3.1.9(ii).

Definition 4.6.7. (i) We shall call S-*weak equivalences* the elements of $\mathscr{W}_{I/S}$.

(ii) Let $f : X \to Y$ be any morphism of $\widehat{\mathscr{A}}$. We say that f *is an absolute weak equivalence*, relative to the given homotopical structure $((I, \partial_\bullet, \sigma), \mathsf{An})$, if for every $S \in \mathrm{Ob}(\widehat{\mathscr{A}})$ and every morphism $g : Y \to S$, the morphism $f/S : (X, g \circ f) \to (Y, g)$ is an S-weak equivalence.

Proposition 4.6.8. *A monomorphism of* $\widehat{\mathscr{A}}$ *is in* An *if and only if it is an absolute weak equivalence. A morphism of* $\widehat{\mathscr{A}}$ *is is a trivial fibration if and only if it is both an* I*-fibration and an absolute weak equivalence.*

Proof. If $(f : X \to Y) \in \mathsf{An}$, then $f/S \in \mathsf{An}/S$ for every morphism $Y \to S$ of $\widehat{\mathscr{A}}$, so f is an absolute weak equivalence, by corollary 4.5.16(i). Conversely, if the monomorphism f is an absolute weak equivalence, then $f/Y : (X, f) \to (Y, \mathbf{1}_Y)$ is a trivial cofibration with fibrant target in $\widehat{\mathscr{A}/Y}$, so $f \in \mathsf{An}$, by corollary 4.5.10(i).

Next, we know already that every trivial fibration is both an I-fibration and an absolute weak equivalence; conversely, if $f : X \to Y$ is both an I-fibration and an absolute weak equivalence, then $f/Y : (X, f) \to (Y, \mathbf{1}_Y)$ is both an I_Y-fibration with fibrant target and a weak equivalence in $\widehat{\mathscr{A}/Y}$, so f/Y is a trivial fibration (theorem 4.5.14(iv)), and then the same holds for f. □

Proposition 4.6.9. *(i) The absolute weak equivalences of $\widehat{\mathscr{A}}$ form the smallest class $\mathscr{W}^a \subset \mathrm{Mor}(\widehat{\mathscr{A}})$ containing* An *and verifying the following condition : for every pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms of $\widehat{\mathscr{A}}$ with $f \in \mathscr{W}^a$, we have $g \in \mathscr{W}^a \Leftrightarrow gf \in \mathscr{W}^a$.*

(ii) Moreover, a morphism of $\widehat{\mathscr{A}}$ is an absolute weak equivalence if and only if it admits a factorisation into an I-anodyne extension followed by a trivial fibration.

Proof. Let us check first that $\mathscr{W}^a$ fulfills the conditions of (i) : indeed, by construction $\mathsf{An} \subset \mathscr{W}^a$; next, for f and g as in (i), and any morphism $h : Z \to S$, we get an S-weak equivalence $f/S : (X, hgf) \to (Y, hg)$, and then clearly $g/S : (Y, hg) \to (Z, h)$ is an S-weak equivalence if and only if the same holds for $gf/S : (X, hgf) \to (Z, h)$, whence assertion.

Next, let us show (ii) : indeed, by (i) and proposition 4.6.8, any composition of an I-anodyne extension and a trivial fibration is in $\mathscr{W}^a$. Conversely, let $f \in \mathscr{W}^a$; by remark 4.5.5(i), we may write $f = p \circ i$, with $i \in \mathsf{An}$ and an I-fibration p. By (i), it follows that $p \in \mathscr{W}^a$, and then p is a trivial fibration, by proposition 4.6.8.

We can now complete the proof of (i): let $\mathscr{F} \subset \mathrm{Mor}(\widehat{\mathscr{A}})$ be a subclass fulfilling the conditions of (i); taking into account (ii), in order to see that $\mathscr{W}^a \subset \mathscr{F}$, it suffices to show that $\mathscr{F}$ contains every trivial fibration $p : X \to Y$. However, p has a section s, and s is a strong deformation retract, so it is an I-anodyne extension (proposition 4.5.6(i,ii)); then both s and $ps = \mathbf{1}_Y \in \mathsf{An}$, so by assumption $p \in \mathscr{F}$. □

Corollary 4.6.10. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be two morphisms of $\widehat{\mathscr{A}}$. We have :*

(i) If g is a monomorphism and $f \in \mathsf{An}$, then $g \in \mathsf{An} \Leftrightarrow gf \in \mathsf{An}$.

(ii) If g is an I-fibration and $f/S : (X, gf) \to (Y, g)$ is an S-weak equivalence, then f is an absolute weak equivalence.

Proof. (i) follows straightforwardly from propositions 4.6.8 and 4.6.9.

(ii): Let us factor f as a cofibration $i : X \to X'$ followed by a trivial fibration $q : X' \to Y$; then $gq : X' \to S$ is an I-fibration, *i.e.* a fibrant object of $\widehat{\mathscr{A}}/S$, so $i/S : (X, gf) \to (X', gq)$ is an S-weak equivalence if and only if $i \in \mathsf{An}$ (corollary 4.5.10(i)). On the other hand, since q is a trivial fibration, the same holds for q/S, so i/S is an S-weak equivalence if and only if the same holds for f/S. Hence, if f/S is an S-weak equivalence, f is the composition of a I-anodyne extension followed by a trivial fibration, and is thus an absolute weak equivalence (proposition 4.6.9(ii)). □

5. The homotopy theory of simplicial sets

In this chapter, we first construct the classical *Kan-Quillen model category* structure on simplicial sets, using the general method of the previous chapter. This method gives that the fibrant objects in sSet are the Kan complexes, but more work is needed to check that the fibrations are precisely the Kan fibrations : this will be achieved using *Kan's subdivision functor*. Since Kan complexes will be later reinterpreted as the ∞*-groupoids*, this precise understanding of the homotopy theory of Kan complexes will play a fundamental role throughout this work.

5.1. **Kan fibrations and the Kan-Quillen model structure.** We have an obvious cylinder in the category sSet :

$$\Delta^0 \sqcup \Delta^0 \xrightarrow{(\partial^1_1,\partial^1_0)} \Delta^1 \xrightarrow{\sigma^0_0} \Delta^0$$

where $\partial^1_1, \partial^1_0$ are the face morphisms and σ^0_0 is the degeneracy morphism, as in definition 2.1.1(v). Since Δ^0 is a final object of sSet, example 4.4.5(i) says that the cartesian product functor $\Delta^1 \times (-)$ defines an exact cylinder on sSet. Following remark 4.4.4(ii), we denote by $\{\varepsilon\} \subset \Delta^1$ the image of $\partial^1_{1-\varepsilon}$, for $\varepsilon = 0, 1$; then the image of $\partial^1_{1-\varepsilon} \otimes X = \partial^1_{1-\varepsilon} \times X$ shall be likewise denoted by $\{\varepsilon\} \times X$, for every $X \in \mathrm{Ob}(\mathsf{sSet})$.

Remark 5.1.1. Recall also that we have a cellular model for sSet, consisting of the system of boundary inclusions $\mathscr{M} := \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$ (see §2.3.7 and example 4.4.2(ii)).

Definition 5.1.2. (i) An *anodyne extension* in sSet is an element of

$$\mathsf{sAn} := \mathsf{An}_{\Delta^1}(\varnothing)$$

i.e. the smallest class of $\Delta^1 \times (-)$-anodyne extensions (see proposition 4.4.9).

(ii) A *weak homotopy equivalence* in sSet is a $\Delta^1 \times (-)$-weak equivalence (see definition 4.5.4(iv)). We say that a simplicial set X is *weakly contractible,* if the unique morphism $X \to \Delta^0$ is a weak homotopy equivalence.

(iii) A morphism $f : X \to Y$ of sSet is a *simplicial homotopy equivalence*, if there exists a morphism $g : Y \to X$ and homotopies $\Delta^1 \times X \to X$, $\Delta^1 \times Y \to Y$ respectively from $\mathbf{1}_X$ to gf, and from $\mathbf{1}_Y$ to fg.

Example 5.1.3. Notice also that for every simplicial set X and every morphism $f : x \to y$ of X, both the left and right localizations $\eta^f : X \to X[gf = \mathbf{1}]$ and $\tau^f : X \to X[fh = \mathbf{1}]$ of §2.5.13 are anodyne extensions, so the same holds for the localization $\mu^f : X \to X[f^{-1}]$.

Proposition 5.1.4. *(i) The following subsets have the same saturation in* sSet *:*

$$S := \{\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to \Delta^1 \times \Delta^n \mid n \in \mathbb{N}\} \quad \textit{and} \quad T := \{\Lambda^n_k \to \Delta^n \mid n \geq 1, k = 1, \dots, n\}.$$

(ii) Dually, the same holds for the subsets :

$$\{\Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \to \Delta^1 \times \Delta^n \mid n \in \mathbb{N}\} \quad \textit{and} \quad \{\Lambda^n_k \to \Delta^n \mid n \geq 1, k = 0, \dots, n-1\}.$$

Proof. (i): Let us check first that the saturation of S contains the inclusion

$$(*) \qquad \Delta^1 \times K \cup \{1\} \times L \to \Delta^1 \times L \qquad \text{for every monomorphism } K \to L \text{ of sSet.}$$

To this aim, in light of remark 1.7.8(ii), we may apply proposition 3.1.21(iii) to the functors $F_1, F_2 : \mathsf{sSet} \to \mathsf{sSet}$ with $F_1 := \{1\} \times (-)$ and $F_2 := \Delta^1 \times (-)$, to the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by the inclusion $\{1\} \to \Delta^1$, and to the cellular model $\mathscr{M}$ of remark

5.1.1; notice also that $F_1L \cap F_2K = F_1K$ for every monomorphism $f : K \to L$, so that $f^\diamond$ is precisely the inclusion $(*)$. We get :

$$l(r(\mathscr{M}))^\diamond \subset l(r(\mathscr{M}^\diamond))$$

and on the one hand, $l(r(\mathscr{M}))$ is the class of monomorphisms of sSet, on the other hand, $l(r(\mathscr{M}^\diamond))$ is the saturation of S (corollary 4.1.9(iii)), whence the contention.

Next, for every $n \in \mathbb{N} \setminus \{0\}$ and every $k = 1, \dots, n$ we define two maps :

$$[n] \xrightarrow{s^n_k} [1] \times [n] \xrightarrow{r^n_k} [n]$$

as follows :

$$s^n_k(i) := (0, i) \qquad r^n_k(0, i) := i \qquad r^n_k(1, i) := \begin{cases} k & \text{if } i < k \\ i & \text{if } i \geq k \end{cases} \qquad \forall i = 0, \dots, n.$$

Endow $[1] \times [n]$ with the product of the orderings of $[1]$ and $[n]$; then clearly s^n_k and r^n_k are morphisms of partially ordered sets, and since the nerve functor N preserves products (see §2.3.3), they induce morphisms of sSet that we denote again :

$$\Delta^n \xrightarrow{s^n_k} \Delta^1 \times \Delta^n \xrightarrow{r^n_k} \Delta^n.$$

Recall that for every $m \in \mathbb{N}$, the m-simplices of Λ^n_k are all the non-decreasing maps $\phi : [m] \to [n]$ whose images miss some $j \neq k$; obviously, the image of the restriction $\Lambda^n_k \to \Delta^1 \times \Delta^n$ of s^n_k lies in $\{0\} \times \Lambda^n_k$, and since $0 < k \leq n$, it is easily seen that :

$$r^n_k(\Delta^1 \times \Lambda^n_k \cup \{1\} \times \Delta^n) \subset \Lambda^n_k.$$

Moreover, it is clear that $r^n_k \circ s^n_k = \mathbf{1}_{[n]}$ for every such n and k; hence, we get a commutative diagram whose vertical arrows are the inclusions :

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathbf{1}_{\Lambda^n_k}"] & \Delta^1 \times \Lambda^n_k \cup \{1\} \times \Delta^n \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r, "s^n_k"] \arrow[rr, bend right, "\mathbf{1}_{\Delta^n}"'] & \Delta^1 \times \Delta^n \arrow[r, "r^n_k"] & \Delta^n. \end{tikzcd}$$

But we have just seen that the central vertical arrow is in the saturation of S, so the same holds for the inclusion $\Lambda^n_k \to \Delta^n$. This shows that the saturation of T lies in that of S.

Claim 5.1.5. There exists a finite filtration :

$$A_{-1} := \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \subset A_0 \subset \cdots \subset A_n := \Delta^1 \times \Delta^n$$

and a system of cocartesian diagrams of sSet whose vertical arrows are the inclusions :

$$(*) \qquad \begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & A_{i-1} \arrow[d] \\ \Delta^{n+1} \arrow[r, "c^n_i"] & A_i \end{tikzcd} \qquad \forall i = 0, \dots, n.$$

Proof : We consider, for every $n \in \mathbb{N}$ and $i = 0, \dots, n$, the unique strictly increasing map $c^n_i : [n+1] \to [1] \times [n]$ whose image contains $(0, i)$ and $(1, i)$. Again, we denote by $c^n_i : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ the corresponding morphism of sSet. We set

$$A_{-1} := \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \qquad \text{and} \qquad A_i := A_{i-1} \cup \mathrm{Im}(c^n_i) \qquad \forall i = 0, \dots, n.$$

The canonical presentation of example 2.3.10(ii) shows that $\Delta^1 \times \Delta^n = \mathrm{Im}(c_0^n) \cup \cdots \cup \mathrm{Im}(c_n^n) = A_n$. Moreover, by direct inspection we find :

$$\begin{aligned} \mathrm{Im}(c_0^n \circ \partial_0^{n+1}) &\subset \{1\} \times \Delta^n \\ \mathrm{Im}(c_i^n \circ \partial_j^{n+1}) &\subset \Delta^1 \times \partial\Delta^n \qquad \forall i = 0, \dots, n, \forall j \neq i, i+1 \\ c_i^n \circ \partial_i^{n+1} &= c_{i-1}^n \circ \partial_i^{n+1} \qquad \forall i = 1, \dots, n \end{aligned}$$

and on the other hand, $\mathrm{Im}(c_i^n \circ \partial_{i+1}^{n+1}) \not\subset A_{i-1}$ for every $i = 0, \dots, n$. Summing up, we get :

$$\mathrm{Im}(c_i^n) \cap A_{i-1} = c_i^n(\Lambda_{i+1}^{n+1}) \qquad \forall i = 0, \dots, n$$

whence a diagram $(*)$ of sSet for every $i = 0, \dots, n$. The cocartesianity of $(*)$ can be checked after evaluation on $[k]$, for every $k \in \mathbb{N}$ (remark 1.6.2(i)), and then it follows easily from claim 4.2.14(i). ◇

By claim 5.1.5, each inclusion $A_{i-1} \subset A_i$ is in the saturation of T, hence the same holds for the composition $A_{-1} \subset A_n$ of these inclusions, so the saturation of S lies in that of T.

(ii) follows from (i), by considering the front-to-back duals of the sets S and T, and by noticing that the saturation of the front-to-back dual of any class $C \subset \mathrm{Mor}(\mathrm{sSet})$ coincides with the front-to-back dual of the saturation of C (details left to the reader). □

Corollary 5.1.6. (Gabriel and Zisman) *The following subclasses of* $\mathrm{Mor}(\mathrm{sSet})$ *are equal :*

(a) the class sAn *of anodyne extensions*

(b) the saturation of the subset $\{\Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to \Delta^1 \times \Delta^n \mid n \in \mathbb{N}, \varepsilon = 0, 1\}$

(c) the saturation of the subset $\{\Lambda_k^n \to \Delta^n \mid n \geq 1, k = 0, \dots, n\}$.

Proof. Let $\mathscr{M}$ be the cellular model of sSet given by the boundary inclusions $(\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N})$; with the notation of §4.4.8, the class sAn is $l(r(\Lambda_I(\varnothing, \mathscr{M})))$, where I is the functor $\Delta^1 \times (-)$, and the class of (b) is $l(r(\Lambda_I^0(\varnothing, \mathscr{M})))$, by corollary 4.2.5(iii). Then the equality of the classes of (a) and (b) is given by corollary 4.4.10.

The equality of the classes of (b) and (c) follows directly from proposition 5.1.4. □

Remark 5.1.7. In this § only corollary 5.1.6 will be needed, but the more refined proposition 5.1.4 will be of crucial importance, starting from §6.3.

Definition 5.1.8. Set $\mathscr{F} := \{\Lambda_k^n \to \Delta^n \mid n \geq 1, k = 0, \dots, n\}$.

(i) A *Kan fibration* is a morphism of sSet that lies in $r(\mathscr{F})$.

(ii) A *Kan complex* is an $\mathscr{F}$-injective object of sSet (see definition 3.1.1(iii,iv)).

Remark 5.1.9. (i) Definition 5.1.8(ii) agrees with our previous definition 2.6.1(i).

(ii) Example 2.6.2(i) shows that if the nerve of a category $\mathscr{C}$ is a Kan complex, then $\mathscr{C}$ is a groupoid. Especially, Δ^n is *not* a Kan complex, except for $n = 0$.

(iii) In view of (ii), we *cannot* invoke proposition 4.5.6(i) in order to deduce that *any* section of the unique morphism $p : \Delta^n \to \Delta^0$ is a strong deformation retract. However, let $\phi : [0] \to [n]$ be the map such that $\phi(0) = n$; then we can at least check that the morphism $\Delta^\phi : \Delta^0 \to \Delta^n$ *is a strong deformation retract.* Indeed, consider the map

$$\psi : [1] \times [n] \to [n] \qquad \psi(a, k) := \begin{cases} k & \text{if } a = 0 \\ n & \text{if } a = 1. \end{cases}$$

Then it is easily seen that $h := \Delta^\psi : \Delta^1 \times \Delta^n \to \Delta^n$ is a homotopy from $\mathbf{1}_{\Delta^n}$ to $\Delta^\phi \circ p$ fulfilling the conditions of definition 4.5.4(i). See also lemma 5.1.17.

Theorem 5.1.10. *There exists a unique cofibrantly generated model category structure on* sSet *whose weak equivalences are the weak homotopy equivalences and whose cofibrations are the monomorphisms. Moreover, the fibrant objects are the Kan complexes, every object is cofibrant, every anodyne extension is a trivial cofibration, and the fibrations with fibrant target are precisely the Kan fibrations between Kan complexes.*

We call this model category the Kan-Quillen model category structure on sSet.

Proof. This follows directly from corollary 5.1.6 and theorem 4.5.14, applied to the exact cylinder $\Delta^1 \times (-)$ on sSet, and the class of anodyne extensions of sSet. ◻

Proposition 5.1.11. *(i) For every anodyne extension $K \to L$ and every monomorphism $U \to V$ of* sSet*, the induced inclusion $K \times V \cup L \times U \to L \times V$ is anodyne.*

(ii) For every simplicial set X, the functor $(-) \times X : \mathrm{sSet} \to \mathrm{sSet}$ preserves anodyne extensions.

(iii) The classes of Kan fibrations and of trivial fibrations of sSet *are stable under small filtered colimits.*

Proof. (i): Let $j : U \to V$ be a monomorphism; by remark 1.7.8(ii), the functors $F_1 := (-) \times U$ and $F_2 := (-) \times V$ admit right adjoints, so we may apply proposition 3.1.21(iii) to F_1 and F_2, to the natural transformation $F_1 \Rightarrow F_2$ induced by j, and to the subset $\mathscr{S} := \{\Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to \Delta^1 \times \Delta^n \mid n \in \mathbb{N}, \varepsilon = 0, 1\}$. With corollary 5.1.6, we get :

$$\mathrm{sAn}^\diamond = l(r(\mathscr{S}))^\diamond \subset l((r(\mathscr{S}^\diamond)))$$

and notice that, for every monomorphism $i : K \to L$, we have $F_1 L \cap F_2 K = F_1 K$, whence :

$$i^\diamond = (K \times V \cup L \times U \to L \times V).$$

We are then reduced to checking that $\mathscr{S}^\diamond \subset \mathrm{sAn}$, *i.e.* that for every $n \in \mathbb{N}$ and $\varepsilon = 0, 1$, the inclusion $i : \Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to \Delta^1 \times \Delta^n$ induces an anodyne extension $i^\diamond$. To this aim, set $K := \Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n$ and $L := \Delta^1 \times \Delta^n$; then :

$$\begin{aligned} K \times V \cup L \times U &= \Delta^1 \times (\partial\Delta^n \times V \cup \Delta^n \times U) \cup \{\varepsilon\} \times \Delta^n \times V \\ L \times V &= \Delta^1 \times \Delta^n \times V \end{aligned}$$

so the inclusion $i^\diamond : K \times V \cup L \times U \to L \times V$ is indeed an anodyne extension.

(ii): It suffices to apply (i) with $U = \varnothing$, the initial object of sSet, and $V := X$.

(iii): The class of trivial fibrations is $r(\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\})$ (example 4.4.2(ii)), and by definition, the class of Kan fibrations is $r(\{\Lambda^n_k \to \Delta^n \mid n \geq 1, k = 0, \dots, n\})$, so the assertion follows from example 3.1.14(iv). ◻

Corollary 5.1.12. *(i) Let $p : X \to Y$ be a morphism of* sSet*. The following conditions are equivalent :*

(a) p is a Kan fibration.

(b) Every monomorphism $i : U \to V$ of sSet *induces a Kan fibration*

$$(i^*, p_*) : \mathscr{H}om(V, X) \to \mathscr{H}om(U, X) \times_{\mathscr{H}om(U,Y)} \mathscr{H}om(V, Y).$$

(c) (i^*, p_*) is a trivial fibration of sSet *for every anodyne extension $i : U \to V$.*

(ii) $\mathscr{H}om(A, -)$ preserves Kan complexes and Kan fibrations, for every $A \in \mathrm{Ob}(\mathrm{sSet})$.

Proof. (i): According to §2.1.6, every $A \in \mathrm{Ob}(\mathsf{sSet})$ induces an adjoint pair of functors :

$$(-) \times A : \mathsf{sSet} \rightleftarrows \mathsf{sSet} : \mathscr{H}om(A, -).$$

We consider the functors $F_1 := (-) \times U$ and $F_2 := (-) \times V$, and the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by the inclusion i; hence, the adjoint transformation $\tau_\bullet^\vee$ is precisely $i^* : \mathscr{H}om(V, -) \Rightarrow \mathscr{H}om(U, -)$. Notice also that $F_1 A = F_2 A \cap F_1 B$ for every monomorphism $j : A \to B$ of sSet, so that the morphism $j^\diamond$ of §3.1.20 is the induced monomorphism $A \times V \cup B \times U \to B \times V$, and on the other hand $p_\diamond$ is precisely the morphism (i^*, p_*). We let then j be the inclusion $\Lambda^n_k \to \Delta^n$, for any $n \geq 1$ and $k = 0, \dots, n$, and invoke propositions 3.1.21(ii) and 5.1.11(i) to see that (a)⇒(b). For the converse, we take for i the unique monomorphism $\varnothing \to E$, where $\varnothing$ and E denote the initial and final objects of sSet; then $j^\diamond = j$, and the assertion follows easily again from proposition 3.1.21(ii).

Likewise, we let $j : A \to B$ be any monomorphism of sSet, and conclude again with propositions 3.1.21(ii) and 5.1.11(i) that (a)⇒(c). For the converse, we take j to be the inclusion $\varnothing \to E$, so that $j^\diamond = i$: details left to the reader.

(ii): To see that $\mathscr{H}om(A, -)$ preserves Kan fibrations, we apply (i.b) with $i : \varnothing \to A$; then we also take $p : X \to \Delta^0$, to deduce that $\mathscr{H}om(A, -)$ preserves Kan complexes. □

Corollary 5.1.13. *(i) Every simplicial set X induces a Quillen adjunction :*

$$\boxed{(-) \times X : \mathsf{sSet} \rightleftarrows \mathsf{sSet} : \mathscr{H}om(X, -)}$$

for the Kan-Quillen model category structure on sSet.

(ii) The class of weak homotopy equivalences is stable under finite products.

Proof. (i): By construction, the functor $\mathscr{H}om(X, -)$ is right adjoint to $(-) \times X$ (see §2.1.6), and clearly $(-) \times X$ preserves cofibrations (*i.e.* monomorphisms), so it suffices to check that $(-) \times X$ also sends every anodyne extension of sSet to a trivial cofibration (proposition 4.5.17), which holds by proposition 5.1.11(ii).

(ii): It suffices to check that if $f : X \to Y$ and $f' : X' \to Y'$ are two weak homotopy equivalences, the same holds for $f \times f' : X \times X' \to Y \times Y'$. However, $f \times f' = (Y \times f') \circ (f \times X')$, so we are reduced to checking that the functor $(-) \times X$ preserves weak homotopy equivalences. But by (i) and lemma 3.4.12(i), the functor $(-) \times X$ preserves trivial cofibrations; since every object of sSet is cofibrant, the assertion then follows from Ken Brown's lemma (proposition 3.2.8). □

Proposition 5.1.14. *Let $\mathscr{A}$ be a small category, $F : \mathsf{sSet} \to \widehat{\mathscr{A}}$ a functor, and for every $n \in \mathbb{N}$, let $j^n : \partial\Delta^n \to \Delta^n$ be the inclusion. We have :*

(i) If F is right exact, then Fj^n is a monomorphism for every $n \neq 1$.

(ii) If F preserves all small colimits, the following conditions are equivalent :

(a) Fj^1 is a monomorphism.

(b) F preserves all monomorphisms of sSet.

Proof. (i): For $n = 0$, notice that $\partial\Delta^0 = \varnothing$, the initial object of sSet, so $F(\partial\Delta^0)$ is the initial object of $\widehat{\mathscr{A}}$ (remark 1.1.11(ii)), *i.e.* the empty presheaf, and the assertion is clear. Suppose then that $n \geq 2$, and notice that for every pair of subsets $C \subset C' \subset [n]$, the induced inclusion $\Delta^C \subset \Delta^{C'}$ is a split monomorphism of sSet, so $F\Delta^C \to F\Delta^{C'}$ is a split monomorphism of $\widehat{\mathscr{A}}$. Hence $(F\Delta^{[n]\setminus\{i\}} \mid i \in [n])$ is a family of subpresheaves of $F\Delta^n$; on the other hand, since F is right exact, example 2.3.9(i) implies that $F(\partial\Delta^n)$ represents the coequalizer of the pair of morphisms

$$\coprod_{0\leq i<j\leq n} F\Delta^{[n]\setminus\{i,j\}} \rightrightarrows \coprod_{0\leq i\leq n} F\Delta^{[n]\setminus\{i\}}$$

and example 1.8.11(ii) reduces to checking that :

$$(*) \qquad F(\Delta^{[n]\setminus\{i,j\}}) = F(\Delta^{[n]\setminus\{i\}}) \cap F(\Delta^{[n]\setminus\{j\}}) \qquad \forall 0 \leq i < j \leq n$$

(where the intersection is taken within $F(\Delta^n) = F(\Delta^{[n]})$). However, lemma 2.3.11 easily implies that the diagram

$$F(\mathscr{D}_{i,j}) \quad : \quad \begin{array}{ccc} F(\Delta^{[n]\setminus\{i,j\}}) & \longrightarrow & F(\Delta^{[n]\setminus\{j\}}) \\ \downarrow & & \downarrow \\ F(\Delta^{[n]\setminus\{i\}}) & \longrightarrow & F(\Delta^{[n]}). \end{array}$$

is cartesian, whence $(*)$.

(ii): Obviously (b)⇒(a); conversely, if (a) holds, then by virtue of (i), Fj^n is a monomorphism for every $n \in \mathbb{N}$. However, the class of monomorphisms of $\widehat{\mathscr{A}}$ is saturated (example 3.1.14(ii)), so the class of all morphisms f of sSet such that Ff is a monomorphism, is saturated as well (remark 3.1.4(iii)), and then the assertion follows from example 3.1.14(iii). □

Proposition 5.1.15. *Let $(\mathscr{C}, \mathscr{W}, \mathscr{F}ib, \mathscr{C}of)$ be a model category, and $\mathscr{W}^a$ the class of absolute weak equivalences of $\mathscr{C}$. Let $F, G :$ sSet $\to \mathscr{C}$ be two functors that preserve all small colimits and send the monomorphisms of* sSet *to cofibrations of $\mathscr{C}$. Let moreover $\tau_\bullet : F \Rightarrow G$ be a natural transformation; the following conditions are equivalent :*

(a) $(F\Delta^n \xrightarrow{\tau_{\Delta^n}} G\Delta^n) \in \mathscr{W}$ *(resp. $\tau_{\Delta^n} \in \mathscr{W}^a$) for every $n \in \mathbb{N}$.*

(b) $(FX \xrightarrow{\tau_X} GX) \in \mathscr{W}$ *(resp. $\tau_X \in \mathscr{W}^a$) for every $X \in \mathrm{Ob}(\mathrm{sSet})$.*

Proof. Set $\mathscr{F} := \{X \in \mathrm{Ob}(\mathrm{sSet}) \mid \tau_X \in \mathscr{W}\}$ and $\mathscr{F}^a := \{X \in \mathrm{Ob}(\mathrm{sSet}) \mid \tau_X \in \mathscr{W}^a\}$.

Claim 5.1.16. (i) FX and GX are cofibrant for every $X \in \mathrm{Ob}(\mathrm{sSet})$.

(ii) $\mathscr{F}$ and $\mathscr{F}^a$ are subclasses of $\mathrm{Ob}(\mathrm{sSet})$ saturated by monomorphisms.

Proof : (i): Both F and G send the empty simplicial set $\varnothing$ to the initial object of $\mathscr{C}$, since they preserve small colimits (remark 1.1.11(ii)); since they also send every monomorphism $\varnothing \to X$ to a cofibration, the assertion follows.

(ii): We need to check that $\mathscr{F}$ and $\mathscr{F}^a$ fulfill conditions (a)–(c) of definition 2.2.9. Thus, let $(X_i \mid i \in I)$ be a small family of elements of $\mathscr{F}$, set $X := \coprod_{i\in I} X_i$; by virtue of (i) and corollary 3.5.12(i), the induced morphism of $\mathscr{C}$

$$\coprod_{i\in I} \tau_{X_i} : \coprod_{i\in I} FX_i \to \coprod_{i\in I} GX_i$$

is a weak equivalence. However, $\coprod_{i\in I} FX_i$ is naturally identified with FX, since F preserves colimits, and likewise for $\coprod_{i\in I} GX_i$; moreover, under these identifications, $\coprod_{i\in I} \tau_{X_i}$ corresponds to τ_X, whence condition (a) for $\mathscr{F}$. Next, let $g : GX \to S$ be any morphism of $\mathscr{C}$, set $f := g \circ \tau_X : FX \to S$, and for every $i \in I$ let $g_i : GX_i \to S$ be the composition of g with the natural morphism $GX_i \to GX$, and $f_i := g_i \circ \tau_{X_i} : FX_i \to S$; suppose that $\tau_{X_i}/S : (FX_i, g_i) \to (GX_i, g_i)$ is an S-weak equivalence for every $i \in I$. By corollary 1.4.6(iii), (GX, g) and (FX, f) represent in $\mathscr{C}/S$ the direct sum of the families

$((GX_i, g_i) \mid i \in I)$ and respectively $((FX_i, f_i) \mid i \in I)$, so $\tau_X/S : (FX, f) \to (GX, g)$ is again an S-weak equivalence. This shows that $\mathscr{F}^a$ fulfills condition (a).

To check condition (c) for $\mathscr{F}$, let $X_\bullet := (X_0 \xrightarrow{j_0} X_1 \xrightarrow{j_1} X_2 \xrightarrow{j_2} \cdots)$ be a countable system of monomorphisms of sSet, such that $X_i \in \mathscr{F}$ for every $i \in \mathbb{N}$, and X the colimit of $X_\bullet$. By (i), Fj_i is a cofibration between cofibrant objects of $\mathscr{C}$ for every $i \in \mathbb{N}$, so corollary 3.5.13 says that the colimit of the system $(\tau_{X_i} \mid i \in \mathbb{N})$ lies in $\mathscr{W}$; but the latter is naturally identified with τ_X, since F and G preserve colimits. This prove that $\mathscr{F}$ enjoys condition (c). Next, let $g : GX \to S$ be any morphism of $\mathscr{C}$, set $f := g \circ \tau_X : FX \to S$, and for every $i \in \mathbb{N}$ let $g_i : GX_i \to S$ (resp. $f_i : FX_i \to X$) be the composition of g (resp. of f) with the natural morphism $FX_i \to FX$ (resp. $GX_i \to GX$); suppose that $\tau_{X_i}/S : (FX_i, f_i) \to (GX_i, g_i)$ is an S-weak equivalence for every $i \in \mathbb{N}$, and notice that $Fj_i/S : (FX_i, f_i) \to (FX_{i+1}, f_{i+1})$ is a cofibration between cofibrant objects of $\mathscr{C}/S$, and likewise for Gj_i/S, for every $i \in \mathbb{N}$. Then the colimit of the system $(\tau_{X_i}/S \mid i \in \mathbb{N})$ is an S-weak equivalence; but the latter is naturally identified with τ_X/S, again by virtue of corollary 1.4.6(iii). This proves that $\mathscr{F}^a$ fulfills condition (c).

Lastly, let $X, X', Y \in \mathscr{F}$, and consider a cocartesian diagram of sSet :

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] & X' \arrow[d, "c"] \\ Y \arrow[r, "d"] & Y' \end{tikzcd}$$

whose vertical arrows are monomorphisms. There follows a commutative diagram of $\mathscr{C}$:

$$(*) \qquad \begin{tikzcd} FY \arrow[d, "\tau_Y"'] & FX \arrow[l, "Fb"'] \arrow[r, "Fa"] \arrow[d, "\tau_X"] & FX' \arrow[d, "\tau_{X'}"] \\ GY & GX \arrow[l, "Gb"'] \arrow[r, "Ga"] & GX' \end{tikzcd}$$

where Fb and Gb are cofibrations, and the vertical arrows are weak equivalences of $\mathscr{C}$. Then, by (i) and corollary 3.6.7, $(*)$ induces a weak equivalence of $\mathscr{C}$:

$$FY \sqcup_{FX} FX' \to GY \sqcup_{GX} GX'.$$

But the latter is naturally identified with $\tau_{Y \sqcup_X X'} = \tau_{Y'}$, whence condition (b) for $\mathscr{F}$. To get condition (b) for $\mathscr{F}^a$, one argues as in the verification of (c) : the details shall be left to the reader. ◇

Now, by assumption $\{\Delta^n \mid n \in \mathbb{N}\}$ lies in $\mathscr{F}$ (resp. in $\mathscr{F}^a$), and recall that $\mathbf{\Delta}$ is an Eilenberg-Zilber category (example 2.2.2(i)); then the proposition follows from corollary 2.2.10 and claim 5.1.16(ii). □

Lemma 5.1.17. *Let $\mathscr{A}$ and $\mathscr{B}$ be two small categories that have final objects. Then :*

(i) Every morphism $u : N\mathscr{A} \to N\mathscr{B}$ in sSet *is an absolute weak equivalence.*

(ii) The unique morphism $N\mathscr{A} \to \Delta^0$ is a simplicial homotopy equivalence.

Proof. (i): For $\varepsilon = 0, 1$ let $\delta_\varepsilon : \mathscr{A} \to [1] \times \mathscr{A}$ be the functor such that $A \mapsto (i, A)$ for every $A \in \mathrm{Ob}(\mathscr{A})$, and $f \mapsto (\mathbf{1}_i, f)$ for every morphism f of $\mathscr{A}$; let moreover E be any final object of $\mathscr{A}$, and for every $A \in \mathrm{Ob}(\mathscr{A})$ denote by $t_A : A \to E$ the unique morphism of $\mathscr{A}$. Consider the functor

$$h : [1] \times \mathscr{A} \to \mathscr{A} \qquad (i, A) \mapsto \begin{cases} A & \text{if } i = 0 \\ E & \text{if } i = 1 \end{cases}$$

such that for every $\phi \in \mathscr{A}(A, A')$ we have (notation of §1.9.2) :

$$h(\mathbf{1}_0, \phi) := \phi \qquad h(\mathbf{1}_1, \phi) := \mathbf{1}_E \qquad h(\overrightarrow{01}, \phi) := t_A.$$

Let furthermore $f : [0] \to \mathscr{A}$ be the functor such that $0 \mapsto E$, and $g : \mathscr{A} \to [0]$ the unique functor. We have :

$$h \circ \delta_0 = \mathbf{1}_{\mathscr{A}} \qquad \text{and} \qquad h \circ \delta_1 = f \circ g$$

and notice that $h \circ ([1] \times f) : [1] \times [0] \to \mathscr{A}$ is the composition of f with the unique functor $[1] \times [0] \to [0]$. Then, set $\omega := Nf : \Delta^0 \to N\mathscr{A}$ and $\psi := Ng : N\mathscr{A} \to \Delta^0$; it follows that $\psi \circ \omega = \mathbf{1}_{\Delta^0}$ and since N commutes with products (see §2.3.3), $\lambda := Nh : \Delta^1 \times N\mathscr{A} \to N\mathscr{A}$ is a homotopy from $\mathbf{1}_{N\mathscr{A}}$ to $\omega \circ \psi$, such that $\lambda \circ (\Delta^1 \times \omega) = \omega \circ \sigma_0^0$ (notation of §5.1); *i.e.* ω is a strong deformation retract, and hence, an anodyne extension of sSet (proposition 4.5.6(ii)).

• Let us check next, that *every morphism* $\eta : \Delta^0 \to N\mathscr{A}$ *is an anodyne extension.* Indeed, we have $\eta = N(f')$ for a unique functor $f' : [0] \to \mathscr{A}$ (lemma 2.3.2), and we consider the unique functor :

$$k : [1] \to \mathscr{A} \qquad \text{such that} \qquad 0 \mapsto f'(0) \qquad 1 \mapsto E.$$

Clearly $\mu := N(k) : \Delta^1 \to N\mathscr{A}$ is a homotopy from η to ω. On the other hand, recall that both face morphisms $d_1^\varepsilon : \Delta^0 \to \Delta^1$ are anodyne extensions (corollary 5.1.6); combining with proposition 4.6.9(i) we deduce first that μ is an absolute weak equivalence (since ω is an anodyne extension), and then that the same holds for η. Then the claim follows from proposition 4.6.8.

• Let now $u : N\mathscr{A} \to N\mathscr{B}$ be any functor; choose any morphism $\eta : \Delta^0 \to N\mathscr{A}$ and notice that both η and $u \circ \eta$ are absolute weak equivalences, by the foregoing, so the same holds for u (proposition 4.6.9(i)).

(ii): The assertion follows immediately from the proof of (i). □

5.2. **The model structure of bisimplicial sets.** Notice that $\Delta^0 \boxtimes \Delta^0$ is a final object of the category bSet of bisimplicial sets (see §2.1.8), and we have a cylinder in this category:

$$\Delta^0 \boxtimes \Delta^0 \sqcup \Delta^0 \boxtimes \Delta^0 \xrightarrow{(\Delta^0 \boxtimes \partial_1^1, \Delta^0 \boxtimes \partial_0^1)} \Delta^0 \boxtimes \Delta^1 \xrightarrow{\Delta^0 \boxtimes \sigma_0^0} \Delta^0 \boxtimes \Delta^0.$$

Then, invoking again example 4.4.5(i) we get an exact cylinder on bSet, given by the cartesian product functor $I := \Delta^0 \boxtimes \Delta^1 \times (-)$. We shall denote by bAn the smallest class of I-anodyne extensions in bSet. Also, the *weak homotopy equivalences* of bisimplicial sets shall be defined as the I-weak equivalences.

• Theorem 4.5.14 yields as well a unique model category structure on bSet whose weak equivalences are given by the weak homotopy equivalences, whose cofibrations are the monomorphisms, and such that every object of bSet is cofibrant. Moreover, the fibrant objects of bSet are the I-fibrant objects.

• For every $n \in \mathbb{N}$, let $j^n : \partial\Delta^n \to \Delta^n$ be the inclusion; to every morphism $f : X \to Y$ of bSet and every such n we attach the unique morphism of sSet

$$f_n^\dagger : \langle \Delta^n, X \rangle \to \langle \partial\Delta^n, X \rangle \times_{\langle \partial\Delta^n, Y \rangle} \langle \Delta^n, Y \rangle$$

whose composition with the projection to $\langle \partial\Delta^n, X \rangle$ (resp. with the projection to $\langle \Delta^n, Y \rangle$) equals $\langle j^n, X \rangle$ (resp. equals $\langle \Delta^n, f \rangle$).

• Lastly, we shall say that f is *a levelwise weak homotopy equivalence* (resp. *a levelwise trivial cofibration*, resp. *a levelwise anodyne extension*), if the morphism $f_{\bullet,n} : X_{\bullet,n} \to Y_{\bullet,n}$

is a weak homotopy equivalence (resp. a trivial cofibration, resp. an anodyne extension) of sSet for every $n \in \mathbb{N}$.

Lemma 5.2.1. *Let $f : X \to Y$ be a morphism of* bSet. *We have :*

(i) f is a trivial fibration $\Leftrightarrow f_n^\dagger$ is a trivial fibration of sSet *for every $n \in \mathbb{N}$.*

(ii) bAn *is the saturation of the class :*

$$\mathscr{F} := \{K \boxtimes \Delta^n \cup L \boxtimes \partial\Delta^n \to L \boxtimes \Delta^n \mid (K \to L) \in \mathsf{sAn}, n \in \mathbb{N}\}.$$

(iii) f is an I-fibration $\Leftrightarrow f_n^\dagger$ is a Kan fibration for every $n \in \mathbb{N}$.

(iv) Every anodyne extension of bSet *is a levelwise anodyne extension.*

(v) For every monomorphism $A \to B$ of sSet *and every $(K \to L) \in$* sAn*, the induced morphism $K \boxtimes B \cup L \boxtimes A \to L \boxtimes B$ is in* bAn.

(vi) Let $A \in \mathrm{Ob}(\mathsf{sSet})$; if f is an I-fibration (resp. a levelwise weak homotopy equivalence, resp. a trivial fibration), then $\langle A, f\rangle : \langle A, X\rangle \to \langle A, Y\rangle$ is a Kan fibration (resp. a weak homotopy equivalence, resp. a trivial fibration), and if X is a fibrant object of bSet*, then $\langle A, X\rangle$ is a Kan complex.*

(vii) X is a fibrant object of bSet *$\Leftrightarrow \langle j^n, X\rangle : \langle \Delta^n, X\rangle \to \langle \partial\Delta^n, X\rangle$ is a Kan fibration for every $n \in \mathbb{N}$.*

Proof. (i): By examples 2.2.7(ii) and 4.4.2(ii), the system of monomorphisms :

$$j^{m,n} : \Delta^m \boxtimes \partial\Delta^n \cup \partial\Delta^m \boxtimes \Delta^n \to \Delta^m \boxtimes \Delta^m \qquad \forall m, n \in \mathbb{N}$$

is a cellular model for bSet, so $r(\{j^{m,n} \mid m, n \in \mathbb{N}\})$ is the class of trivial fibrations of bSet (definition 4.5.4(ii)). On the other hand, by virtue of proposition 2.1.10, for every morphism $f : X \to Y$ of bSet we have a natural bijection between commutative diagrams of the form :

$$\begin{tikzcd} \Delta^m \boxtimes \partial\Delta^n \cup \partial\Delta^m \boxtimes \Delta^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^m \boxtimes \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d, "j^m"'] & \langle \Delta^n, X\rangle \arrow[d, "f_n^\dagger"] \\ \Delta^m \arrow[r] \arrow[ur] & \langle \partial\Delta^n, X\rangle \times_{\langle \partial\Delta^n, Y\rangle} \langle \Delta^n, Y\rangle. \end{tikzcd}$$

Since $\{j^m \mid m \in \mathbb{N}\}$ is a cellular model for sSet, the assertion follows.

(ii): For every $m, n \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$, the monomorphism

$$X_\varepsilon^{m,n} := I(\Delta^m \boxtimes \partial\Delta^n \cup \partial\Delta^m \boxtimes \Delta^n) \cup \{\varepsilon\} \otimes (\Delta^m \boxtimes \Delta^n) \to Y^{m,n} := I(\Delta^m \boxtimes \Delta^n)$$

lies in bAn. However, by virtue of remark 2.1.9(ii) we have :

$$X_\varepsilon^{m,n} = (\Delta^1 \times \partial\Delta^m \cup \{\varepsilon\} \times \Delta^m) \boxtimes \Delta^n \cup (\Delta^1 \times \Delta^m) \boxtimes \partial\Delta^n$$
$$Y^{m,n} = (\Delta^1 \times \Delta^m) \boxtimes \Delta^n.$$

But the saturation of $\{\Delta^1 \times \partial\Delta^m \cup \{\varepsilon\} \times \Delta^m \to \Delta^1 \times \Delta^m \mid m \in \mathbb{N}, \varepsilon = 0, 1\}$ is sAn (corollary 5.1.6), so $\mathscr{F} \subset$ bAn. For the converse inclusion, it suffices to notice that bAn is the saturation of $\{X_\varepsilon^{m,n} \to Y^{m,n} \mid m, n \in \mathbb{N}, \varepsilon = 0, 1\}$ (corollary 4.4.10).

(iii) follows immediately from (ii), arguing as in the proof of (i).

(iv): For every $n, m \in \mathbb{N}$, let $U^{n,m}$ and $V^{n,m}$ be the simplicial sets with :

$$U_i^{n,m} := \partial\Delta_m^n \qquad \text{and} \qquad V_i^{n,m} := \Delta_m^n \qquad \forall i \in \mathbb{N}$$

and such that $U^{n,m}_j \xrightarrow{\phi^*} U^{n,m}_i$ and $V^{n,m}_j \xrightarrow{\phi^*} V^{n,m}_i$ are the identities, for every morphism $\phi : i \to j$ of Δ. The inclusion $\partial\Delta^n_m \subset \Delta^n_m$ induces a monomorphism $U^{n,m} \to V^{n,m}$, and for every monomorphism $K \to L$ of sSet we have :

$$(K \boxtimes \Delta^n \cup L \boxtimes \partial\Delta^n)_{\bullet,m} = K \times V^{n,m} \cup L \times U^{n,m} \qquad (L \boxtimes \Delta^n)_{\bullet,m} = L \times V^{n,m}.$$

In light of proposition 5.1.11(i), we deduce that every element of the class $\mathscr{F}$ of (ii) is a levelwise anodyne extension. However, since the colimits of bSet are computed termwise (remark 1.6.2(i)) it is easily seen that the levelwise anodyne extensions form a saturated class, so the assertion follows from (ii).

(v): Since the set $\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$ is a cellular model for sSet (example 4.4.2(ii)), the assertion follows easily from (ii).

(vi): As a special case of (v), we see that for every $(K \to L) \in$ sAn and every $A \in$ Ob(sSet), the induced morphism $K \boxtimes A \to L \boxtimes A$ lies in bAn; by applying proposition 3.1.19 to the adjoint pair of proposition 2.1.10, we deduce that if f is an I-fibration, $\langle A, f\rangle$ is a Kan fibration. By the same token, since the functor $-\boxtimes A$ preserves monomorphisms, we see that if f is a trivial fibration, the same holds for $\langle A, f\rangle$. Next, let f be a levelwise weak homotopy equivalence; by virtue of remark 2.1.9(i), it follows that $\langle \Delta^n, f\rangle : \langle \Delta^n, X\rangle \to \langle \Delta^n, Y\rangle$ is a weak homotopy equivalence for every $n \in \mathbb{N}$. Let us then endow $\mathsf{sSet}^{\mathrm{op}}$ with the model category structure induced by the Kan-Quillen model category structure on sSet (proposition 3.2.4(i)); since the functors $\langle -, X\rangle, \langle -, Y\rangle : \mathsf{sSet} \to \mathsf{sSet}^{\mathrm{op}}$ preserve all small colimits, proposition 5.1.15 implies that $\langle A, f\rangle$ is a weak homotopy equivalence.

In order to prove the last assertion, it now suffices to check that if Y is the final object of bSet, then $\langle A, Y\rangle$ is the final object of sSet. The latter follows from proposition 2.1.10, [13, Prop.2.49(i)] and remark 1.1.11(ii).

(vii): Let Y be the final object of bSet, and $f : X \to Y$ the unique morphism; we have just observed that $\langle \Delta^n, Y\rangle$ and $\langle \partial\Delta^n, Y\rangle$ are final objects of sSet, so $f^\dagger_n = \langle j^n, X\rangle$; since the fibrant objects of bSet are the I-fibrant objects, the assertion is then a special case of (iii). □

Theorem 5.2.2. *Every levelwise weak homotopy equivalence $f : X \to Y$ of bisimplicial sets induces a weak homotopy equivalence of simplicial sets :*

$$\mathrm{diag}(\phi) : \mathrm{diag}(X) \to \mathrm{diag}(Y).$$

Proof. To begin with, we remark :

Claim 5.2.3. The functor diag preserves weak homotopy equivalences.

Proof : Since the colimits of sSet and bSet are computed termwise (remark 1.6.2(i)), it is clear that the functor diag preserves all small colimits; hence, the class $\mathscr{G}$ of all morphisms ϕ of bSet such that $\mathrm{diag}(\phi)$ is a trivial cofibration of the Kan-Quillen model category structure is saturated (remark 3.1.4(iii)). Notice as well that :

$$\mathrm{diag}(K \boxtimes \Delta^n \cup L \boxtimes \partial\Delta^n \to L \boxtimes \Delta^n) = (K \times \Delta^n \cup L \times \partial\Delta^n \to L \times \Delta^n)$$

for every monomorphism $i : K \to L$ of sSet (remark 2.1.9(ii)); especially, if $i \in$ sAn, the induced morphism $K \boxtimes \Delta^n \cup L \boxtimes \partial\Delta^n \to L \boxtimes \Delta^n$ lies in $\mathscr{G}$. With lemma 5.2.1(ii), we deduce that bAn $\subset \mathscr{G}$. Notice moreover that diag preserves all monomorphisms and admits a right adjoint, since sSet is cocomplete (theorem 1.7.5(iii)); then, proposition 4.5.17 and lemma 3.4.12 imply that diag preserves trivial cofibrations, and since all objects of bSet are cofibrant, we conclude with proposition 3.2.8. ◇

On the other hand, since every element of sAn is a weak homotopy equivalence (theorem 4.5.14(v)), lemma 5.2.1(iv) implies that every anodyne extension of bSet is a levelwise weak homotopy equivalence. Now, we may find a commutative diagram of bSet

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & U \arrow[d, "p"] \\ Y \arrow[r, "j"] & V \end{tikzcd}$$

whose horizontal arrows are in bAn, and where p is an I-fibration with fibrant target (see the proof of corollary 4.5.16(i)); since levelwise weak homotopy equivalences enjoy the 2-out-of-3 property, we deduce that f is a levelwise weak homotopy equivalence if and only if the same holds for p. Also, we already know that $\mathrm{diag}(i)$ and $\mathrm{diag}(j)$ are weak homotopy equivalences of sSet; by the 2-out-of-3 property for weak homotopy equivalences, we see that $\mathrm{diag}(f)$ is a weak homotopy equivalence if and only if the same holds for $\mathrm{diag}(p)$. Since the fibrant objects of bSet are the I-fibrant objects, we see that both X and Y are fibrant; summing up, we may replace f by p, *and assume that f is also a fibration* of bSet between fibrant objects. Hence, $\langle A, f\rangle : \langle A, X\rangle \to \langle A, Y\rangle$ is a trivial fibration between Kan complexes in sSet, for every $A \in \mathrm{Ob}(\mathrm{sSet})$ (lemma 5.2.1(vi)).

For every $n \in \mathbb{N}$ we have the commutative diagram of sSet :

$$\begin{tikzcd} \langle \Delta^n, X\rangle \arrow[r, "f_n^\dagger"] \arrow[dr, "\langle \Delta^n, f\rangle"'] & \langle \partial\Delta^n, X\rangle \times_{\langle \partial\Delta^n, Y\rangle} \langle \Delta^n, Y\rangle \arrow[r] \arrow[d, "\pi"] & \langle \partial\Delta^n, X\rangle \arrow[d, "\langle \partial\Delta^n, f\rangle"] \\ & \langle \Delta^n, Y\rangle \arrow[r, "\langle j^n, Y\rangle"] & \langle \partial\Delta^n, Y\rangle \end{tikzcd}$$

and we know that $\langle j^n, Y\rangle$ is a Kan fibration of Kan complexes (lemma 5.2.1(vi,vii)); especially, it is a fibration of sSet. Moreover, $\langle \partial\Delta^n, f\rangle$ is a trivial fibration, so the same holds for π (propositions 3.1.9(v) and 3.6.5(ii)); especially, this shows that every term in the diagram is a Kan complex, and since $\langle \Delta^n, f\rangle$ is a trivial fibration, the 2-out-of-3 property implies that $f_n^\dagger$ is a weak homotopy equivalence, and so it is a trivial fibration as well, by lemma 5.2.1(iv). We conclude that f is a trivial fibration of bSet (lemma 5.2.1(i)), and then the assertion follows from claim 5.2.3. □

5.3. **Subdivision and extension functors.** To every partially ordered set $(E, \leq)$ we attach the set $\xi(E)$ of totally ordered non-empty chains of elements of E, of arbitrary finite length; we endow $\xi(E)$ with the partial order given by inclusion of chains. Every map $f : (E, \leq) \to (F, \leq)$ of partially ordered sets induces a map

$$\xi(f) : \xi(E) \to \xi(F) \qquad C \mapsto f(C)$$

of partially ordered sets. Hence we obtain a functor

$$\xi : \mathrm{poSet} \to \mathrm{poSet} \subset \mathrm{Cat}.$$

We compose it with the nerve functor (see §2.3) to get a functor $N\xi : \mathrm{poSet} \to \mathrm{sSet}$, and we consider the restriction of $N\xi$ to the full subcategory $\mathbf{\Delta}$, that we denote

$$\sigma : \mathbf{\Delta} \to \mathrm{sSet} \qquad [n] \mapsto N\xi([n]) \qquad \forall n \in \mathbb{N}.$$

Next, the *subdivision functor* is defined as the extension by colimits of σ (see theorem 1.7.5(i,ii)), and denoted

$$\mathrm{Sd} := \sigma_! : \mathrm{sSet} \to \mathrm{sSet}.$$

Thus, Sd is left adjoint to the *extension functor*

$$\mathsf{Ex} := \sigma^* : \mathrm{sSet} \to \mathrm{sSet} \qquad X \mapsto ([n] \mapsto \mathrm{sSet}(N\xi([n]), X)).$$

- For every $(E, \leq) \in \mathrm{Ob}(\mathrm{poSet})$, we have a natural epimorphism of poSet

$$\mu_E : \xi(E) \to E \qquad C \mapsto \max(C).$$

Notice that if E is *finite and totally ordered,* then μ_E is a split epimorphism : indeed, for every such E we may define the morphism of poSet :

$$\lambda_E : E \to \xi(E) \qquad e \mapsto \{x \in E \mid x \leq e\} \qquad \text{so that} \qquad \mu_E \circ \lambda_E = \mathbf{1}_E.$$

Recalling that $N([n]) = \Delta^n$, we deduce that $\mu_\bullet$ induces a natural transformation

$$N \star \mu_\bullet : \sigma \Rightarrow h$$

where $h : \mathbf{\Delta} \to \mathrm{sSet}$ denotes as usual the Yoneda embedding; moreover, $(N \star \mu_\bullet)_{[n]} = N(\mu_{[n]})$ is a split epimorphism for every $n \in \mathbb{N}$. Since $h_!$ is isomorphic to $\mathbf{1}_{\mathrm{sSet}}$ (remark 1.7.8(i)), there follows a natural transformation

$$\alpha_\bullet := (N \star \mu_\bullet)_! : \mathsf{Sd} \Rightarrow \mathbf{1}_{\mathrm{sSet}}$$

(remark 1.7.8(iii)) whence an adjoint natural transformation (see §1.6.10)

$$\beta_\bullet : \mathbf{1}_{\mathrm{sSet}} \Rightarrow \mathsf{Ex}.$$

Explicitly, for every $X \in \mathrm{Ob}(\mathrm{sSet})$, the morphism $\beta_X : X \to \mathsf{Ex}(X)$ is the system

$$\beta_{X,n} : X_n \xrightarrow{\sim} \mathrm{sSet}(\Delta^n, X) \xrightarrow{\mathrm{sSet}(\alpha_{\Delta^n}, X)} \mathrm{sSet}(\mathsf{Sd}(\Delta^n), X) \xrightarrow{\sim} \mathsf{Ex}(X)_n \qquad \forall n \in \mathbb{N}.$$

This explicit description shows especially that $\beta_{X,n}$ is injective for every such X and n, since $\alpha_{\Delta^n} : \mathsf{Sd}(\Delta^n) \to \Delta^n$ is naturally identified with the split epimorphism $N(\mu_{[n]})$ (corollary 1.7.9(i)); hence, β_X *is a monomorphism* for every $X \in \mathrm{Ob}(\mathrm{sSet})$, and in light of corollary 1.7.9(ii), we also see that α_X *is an epimorphism* for every such X.

Example 5.3.1. (i) Clearly $\xi([0]) = [0]$ and $\xi([1]) = \{\{0\}, \{1\}, \{0,1\}\}$. There follow cocartesian diagrams of poSet and sSet :

$$\begin{tikzcd} {[0]} \arrow[r, "\partial_1^0"] \arrow[d, "\partial_1^0"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & \xi([1]) \end{tikzcd} \qquad \begin{tikzcd} \Delta^0 \arrow[r, "\partial_0^1"] \arrow[d, "\partial_0^1"'] & \Delta^1 \arrow[d] \\ \Delta^1 \arrow[r] & \mathsf{Sd}(\Delta^1). \end{tikzcd}$$

(ii) With (i), we deduce natural identifications, for every $X \in \mathrm{Ob}(\mathrm{sSet})$:

$$\mathsf{Ex}(X)_0 \xrightarrow{\sim} X_0 \qquad \mathsf{Ex}(X)_1 \xrightarrow{\sim} X_1 \times_{(d_0^1, d_0^1)} X_1 = \{(a,b) \in X_1^2 \mid d_1^0(a) = d_1^0(b)\}.$$

Under these identifications, the map $\beta_{X,0} : X_0 \to \mathsf{Ex}(X)_0$ corresponds to $\mathbf{1}_{X_0}$, and $\beta_{X,1} : X_1 \to \mathsf{Ex}(X)_1$ is the map such that $a \mapsto (a, s_0^0 d_1^0(a))$ for every $a \in X_1$. Also, the map $d_1^0 : \mathsf{Ex}(X)_1 \to \mathsf{Ex}(X)_0$ (resp. $d_1^1 : \mathsf{Ex}(X)_1 \to \mathsf{Ex}(X)_0$) is given by : $(a,b) \mapsto d_1^1(a)$ (resp. $(a,b) \mapsto d_1^1(b)$) for every $(a,b) \in \mathsf{Ex}(X)_1$.

(iii) Recall that $\pi_0(X)$ is naturally identified with the coequalizer of $d_1^0, d_1^1 : X_1 \rightrightarrows X_0$, for every $X \in \mathrm{Ob}(\mathrm{sSet})$ (see §2.1.11); in light of (ii), we easily deduce that β_X induces a natural bijection :

$$\boxed{\pi_0(\beta_X) : \pi_0(X) \xrightarrow{\sim} \pi_0(\mathsf{Ex}(X)) \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).}$$

Lemma 5.3.2. *For every* $(E, \leq) \in \mathrm{Ob}(\mathrm{poSet})$ *there exists a natural isomorphism:*

$$\boxed{\mathrm{Sd} \circ N(E) \xrightarrow{\sim} N \circ \xi(E)}$$

that makes commute the diagram :

$$\begin{tikzcd} \mathrm{Sd} \circ N(E) \arrow[rr, "\sim"] \arrow[dr, "\alpha_{NE}"'] & & N \circ \xi(E) \arrow[dl, "N\mu_E"] \\ & NE. & \end{tikzcd}$$

Proof. Recall that, by construction, $\mathrm{Sd}(N(E))$ represents the colimit of the functor:

$$\Delta/N(E) \to \mathrm{sSet} \qquad ([n], \Delta^n \to N(E)) \mapsto N\xi([n])$$

(see the proof of theorem 1.7.5(i)). However, since N is a fully faithful functor (lemma 2.3.2), and since $\Delta^n = N([n])$ for every $n \in \mathbb{N}$, the category $\Delta/N(E)$ is naturally isomorphic to the category Δ/E, defined as the full subcategory of the slice category poSet/E whose objects are the morphisms $X \to E$ of poSet with $X \in \mathrm{Ob}(\Delta)$. Then, $\mathrm{Sd}(N(E))$ represents also the colimit of the functor :

$$F_E : \Delta/E \to \mathrm{sSet} \qquad ([n] \to E) \mapsto N\xi([n]).$$

With this notation, the rule : $(f : [n] \to E) \mapsto N\xi(f)$ yields a co-cone $F_E \Rightarrow c_{N\xi(E)}$, whence a morphism $\phi_E : \mathrm{Sd} \circ N(E) \to N\xi(E)$ of sSet, and it is easily seen that the rule $(E, \leq) \mapsto \phi_E$ is natural in E. We need to check that ϕ_E is an isomorphism for every such E. Now, for every $k \in \mathbb{N}$, the k-simplices of $N\xi(E)$ are the sequences of inclusions $S_0 \subset S_1 \subset \cdots \subset S_k$ of non-empty subsets $S_i \subset E$ that are totally ordered for the ordering induced by E. Thus, for every $(f : [n] \to E) \in \mathrm{Ob}(\Delta/E)$, the morphism $N\xi(f) : N\xi([n]) \to N\xi(E)$ is given on k-simplices by the rule :

$$S_\bullet := (S_0 \subset S_1 \subset \cdots \subset S_k \subset [n]) \mapsto fS_\bullet := (fS_0 \subset fS_1 \subset \cdots \subset fS_k \subset E).$$

Hence, let $T_\bullet := (T_0 \subset T_1 \subset \cdots \subset T_k)$ be any k-simplex of $N\xi(E)$; denote by $n+1$ the cardinality of T_k, and let $j : T_k \to E$ be the inclusion map. We have a unique isomorphism $u : [n] \xrightarrow{\sim} T_k$ of poSet; then $T_\bullet = N\xi(u)_k(S_\bullet)$ for a unique $S_\bullet \in N\xi([n])$, whence $N\xi(j \circ u)_k(S_\bullet) = T_\bullet$. The surjectivity of ϕ_E easily follows.

For the injectivity, consider morphisms $f : [n] \to E$, $g : [m] \to E$ of poSet and k-simplices $S_\bullet, S'_\bullet$ of $N\xi([n])$ and respectively $N\xi([m])$ such that $fS_\bullet = gS'_\bullet$ in $N\xi(E)$; we need to check that the images of $S_\bullet$ and $S'_\bullet$ agree in $\mathrm{Sd}(N(E))$. We have a unique $n' \leq n$ such that f factors as a surjective morphism $u : [n] \to [n']$ of Δ and an injective morphism $f' : [n'] \to E$ of poSet, and $u/E : ([n] \xrightarrow{f} E) \to ([n'] \xrightarrow{f'} E)$ is a morphism of Δ/E; then the k-simplices $S_\bullet$ and $uS_\bullet$ of $N\xi([n])$ and respectively $N\xi([n'])$ have the same image in $\mathrm{Sd}(N(E))$, and obviously $fS_\bullet = f'(uS_\bullet)$ in $N\xi(E)$. Thus, after replacing f and $S_\bullet$ by f' and $uS_\bullet$, we may assume that f is an injective map, and likewise for g. Next, let $c+1$ be the cardinality of S_k; we easily find a unique morphism $v : [c] \to [n]$ and a unique k-simplex $T_\bullet$ of $N\xi([c])$ such that $vT_\bullet = S_\bullet$ and $T_k = [c]$, whence a morphism $v/E : ([c] \xrightarrow{f \circ v} E) \to ([n] \xrightarrow{f} E)$ of Δ/E; then again, $T_\bullet$ and $S_\bullet$ have the same images in $\mathrm{Sd}(N(E))$, and after replacing f and $S_\bullet$ by $f \circ v$ and $T_\bullet$, we may assume as well that $S_k = [n]$, and likewise, $S'_k = [m]$. But then clearly $n = m$, and since both f and g are injective, the condition $fS_k = gS'_k$ implies that $f = g$; we have then $fS_\bullet = fS'_\bullet$, whence $S_\bullet = S'_\bullet$, again by the injectivity of f, and the proof is concluded.

It remains to check that $N(\mu_E) \circ \phi_E = \alpha_{NE}$. To this aim, for every partially ordered set $(E, \leq)$ choose a universal co-cone $\eta^E_\bullet : F_E \Rightarrow c_{\mathsf{Sd}(NE)}$; by corollary 1.7.9(i) (and by the proof of theorem 1.7.5(i,ii)), we get a commutative diagram :

$$\begin{tikzcd} N\xi[n] \arrow[r, "\eta^{[n]}_{1_{[n]}}"] \arrow[d, "N\mu_{[n]}"'] & \mathsf{Sd}(N[n]) \arrow[r, "\mathsf{Sd}(Nf)"] \arrow[d, "\alpha_{N[n]}"] & \mathsf{Sd}(NE) \arrow[d, "\alpha_{NE}"] \\ N[n] \arrow[r, equal] & N[n] \arrow[r, "Nf"] & NE \end{tikzcd}$$

for every $(f : [n] \to E) \in \mathrm{Ob}(\mathbf{\Delta}/E)$, and moreover $\mathsf{Sd}(Nf) \circ \eta^{[n]}_{1_{[n]}} = \eta^E_f$ for every such f. On the other hand, by construction we have $\phi_E \circ \eta^E_f = N\xi(f)$; summing up, we are reduced to verify the commutativity of the diagram :

$$\begin{tikzcd} N\xi[n] \arrow[r, "N\xi(f)"] \arrow[d, "N\mu_{[n]}"'] & N\xi(E) \arrow[d, "N\mu_E"] \\ N[n] \arrow[r, "Nf"] & NE \end{tikzcd}$$

for every such f, which follows from the naturality of $\mu_\bullet$. □

Proposition 5.3.3. *(i)* Sd *preserves monomorphisms and anodyne extensions.*

(ii) The functor Ex *preserves trivial fibrations and Kan fibrations.*

(iii) $\alpha_X : \mathsf{Sd}(X) \to X$ *is an absolute weak equivalence for every* $X \in \mathrm{Ob}(\mathsf{sSet})$.

(iv) The subdivision and extension functors form a Quillen adjunction

$$\boxed{\mathsf{Sd} : \mathsf{sSet} \rightleftarrows \mathsf{sSet} : \mathsf{Ex}.}$$

Proof. (i): Clearly $\mu_{[0]} : \xi[0] \to [0]$ is an isomorphism, so the same holds for the morphism $\alpha_{\Delta^0} : \mathsf{Sd}(\Delta^0) \to \Delta^0$, since the latter is naturally identified with $N(\mu_{[0]})$ (corollary 1.7.9(i)). Moreover, Sd preserves all representable colimits ([13, Prop.2.49(ii)]), so we get a commutative diagram :

$$\begin{tikzcd} \mathsf{Sd}(\Delta^0 \sqcup \Delta^0) \arrow[r, "\mathsf{Sd}(j^1)"] \arrow[d, "\alpha_{\Delta^0 \sqcup \Delta^0}"'] & \mathsf{Sd}(\Delta^1) \arrow[d, "\alpha_{\Delta^1}"] \\ \Delta^0 \sqcup \Delta^0 \arrow[r, "j^1"] & \Delta^1 \end{tikzcd}$$

where $j^1 : \partial\Delta^1 \to \Delta^1$ is the inclusion, and $\alpha_{\Delta^0 \sqcup \Delta^0}$ is an isomorphism. It follows that $\mathsf{Sd}(j^1)$ is a monomorphism, and then proposition 5.1.14(ii) implies that Sd preserves monomorphisms.

• Let $\mathscr{F}$ be the class of all morphisms f of sSet such that $\mathsf{Sd}(f)$ is an anodyne extension; by lemma 5.1.17(i) and proposition 4.6.8, we know already that every monomorphism $j : \Delta^m \to \Delta^n$ lies in $\mathscr{F}$, for every $m, n \in \mathbb{N}$, since $\mathsf{Sd}(\Delta^n)$ is the nerve of the category $\xi[n]$, that has $[n]$ as its unique final object, for every $n \in \mathbb{N}$, and since we have just shown that $\mathsf{Sd}(j)$ is a monomorphism, for every such j.

• We also know that Sd preserves all small colimits ([13, Prop.2.49(ii)]), so $\mathscr{F}$ is saturated (remark 3.1.4(iii)). Combining with corollary 5.1.6, we are thus reduced to checking that the inclusion $\Lambda^n_k \to \Delta^n$ lies in $\mathscr{F}$ for every $n \in \mathbb{N} \setminus \{0\}$ and every $k \in [n]$. We proceed by induction on n; the case $n \leq 1$ is already known. So, let $n > 1$; for every proper non-empty subset $I \subset [n]$ we define $\Lambda^{[n]}_I$ as in §2.3.12, so that $\Lambda^{[n]}_{[n]\setminus\{k\}} = \mathrm{Im}(d^k_n)$

and $\Lambda^{[n]}_{\{k\}} = \Lambda^n_k$ for every $k \in [n]$. We check, by induction on the cardinality c of $[n] \setminus I$, that the inclusion $j^n_I : \Lambda^{[n]}_I \to \Delta^n$ lies in $\mathscr{F}$. For $c = 1$, the assertion is already known, hence suppose that $2 \leq c \leq n$, and that $j^m_K \in \mathscr{F}$ when either $m < n$ and $K \subset [m]$ is an arbitrary non-empty subset, or else $m = n$ and $K \subset [n]$ is a non-empty subset of cardinality $\leq n - c$. Pick $k \in [n] \setminus I$; according to §2.3.12 we find, for a suitable subset $K \subset [n-1]$, a cocartesian diagram in sSet :

$$\begin{tikzcd} \Lambda^{[n-1]}_K \arrow[r] \arrow[d, "j^{n-1}_K"'] & \Lambda^{[n]}_{I\cup\{k\}} \arrow[d, "i"] \\ \Delta^{n-1} \arrow[r] & \Lambda^{[n]}_I . \end{tikzcd}$$

By inductive assumption, j^{n-1}_K lies in $\mathscr{F}$, so the same holds for the inclusion i; moreover, $j^n_{I\cup\{k\}} = j^n_I \circ i$, and $j^n_{I\cup\{k\}} \in \mathscr{F}$, whence $j^n_I \in \mathscr{F}$, by proposition 4.6.9(i).

(ii) follows from (i) and proposition 3.1.19.

(iii): By proposition 5.1.15, it suffices to check that $\alpha_{\Delta^n} : \mathsf{Sd}(\Delta^n) \to \Delta^n$ is an absolute weak equivalence for every $n \in \mathbb{N}$, and this follows from lemma 5.1.17(i).

(iv): By (i), Sd preserves cofibrations, and (iii) easily implies that Sd preserves weak homotopy equivalences (the details are left to the reader), so it preserves trivial cofibrations as well, and the assertion follows from lemma 3.4.12(i). □

Corollary 5.3.4. *For every $n \in \mathbb{N}$ and every $k = 0, \dots, n$, endow*

$$\partial\Phi^n := \xi([n]) \setminus \{[n]\} \qquad \textit{and} \qquad \Phi^n_k := \partial\Phi^n \setminus \{[n] \setminus \{k\}\}$$

with the partial orderings induced by their inclusions in $\xi([n])$. Then the isomorphism $\mathsf{Sd}(\Delta^n) \xrightarrow{\sim} N\xi[n]$ of lemma 5.3.2 restricts to natural identifications :

$$\mathsf{Sd}(\partial\Delta^n) \xrightarrow{\sim} N(\partial\Phi^n) \qquad \textit{and} \qquad \mathsf{Sd}(\Lambda^n_k) \xrightarrow{\sim} N(\Phi^n_k).$$

Proof. Recall that Sd commutes with all representable colimits of sSet, since it a left adjoint ([13, Prop.2.49(ii)]), and Sd preserves monomorphisms, by proposition 5.3.3(i); then the diagram of example 2.3.9(i) and the isomorphisms of lemma 5.3.2 identify $\mathsf{Sd}(\partial\Delta^n)$ with the image of the natural morphism:

$$\coprod_{0\leq k\leq n} N\xi([n] \setminus \{k\}) \to N\xi([n]).$$

On the other hand, clearly $\partial\Phi^n = \bigcup_{k=0}^n \xi([n] \setminus \{k\})$, so it remains to only to check that the induced injective map

$$\bigcup_{k=0}^{n} N\xi([n] \setminus \{k\}) \to N(\partial\Phi^n)$$

is surjective. However, for every $p \in \mathbb{N}$, the p-simplices of $N(\partial\Phi^n)$ are the chains of inclusions $C_0 \subset C_1 \subset \cdots \subset C_p$ of proper subsets of $[n]$; hence, for every such chain $C_\bullet$ there exists $k \in [n]$ with $C_p \subset [n] \setminus \{k\}$, and then clearly $C_\bullet$ lies in the image of $N\xi([n] \setminus \{k\})$. A similar argument proves the stated identity for $\mathsf{Sd}(\Lambda^n_k)$: the details shall be left to the reader. □

Proposition 5.3.5. *(i)* Ex *preserves and reflects weak homotopy equivalences.*

(ii) (Kan) *$\beta_X : X \to \mathsf{Ex}(X)$ is a weak homotopy equivalence, for every $X \in \mathrm{Ob}(\mathrm{sSet})$.*

Proof. (i): To every simplicial set X we attach the bisimplicial set $E(X)$ such that

$$E(X)_{m,n} := \mathsf{sSet}(\Delta^m \times \mathsf{Sd}(\Delta^n), X) \qquad \forall m, n \in \mathbb{N}$$

and with $E(X)_{\phi,\psi} := \mathsf{sSet}(\phi \times \mathsf{Sd}(\psi), X)$ for every pair of morphisms ϕ, ψ of $\mathbf{\Delta}$.

The projections $\Delta^m \xleftarrow{p} \Delta^m \times \mathsf{Sd}(\Delta^n) \xrightarrow{q} \mathsf{Sd}(\Delta^n)$ induce natural morphisms of bSet :

$$X \boxtimes \Delta^0 \xrightarrow{p^*} E(X) \xleftarrow{q^*} \Delta^0 \boxtimes \mathsf{Ex}(X).$$

With the notation of §2.1.8, notice moreover that :

$$(X \boxtimes \Delta^0)_{\bullet,n} \xrightarrow{\sim} X \xrightarrow{\sim} \mathscr{H}om(\Delta^0, X) \qquad E(X)_{\bullet,n} = \mathscr{H}om(\mathsf{Sd}(\Delta^n), X) \qquad \forall n \in \mathbb{N}$$

and under these identifications, $p^*_{\bullet,n} : (X \boxtimes \Delta^0)_{\bullet,n} \to E(X)_{\bullet,n}$ corresponds to

$$u^*_n := \mathscr{H}om(u_n, X) : \mathscr{H}om(\Delta^0, X) \to \mathscr{H}om(\mathsf{Sd}(\Delta^n), X) \qquad \forall n \in \mathbb{N}$$

for the unique morphism $u_n : \mathsf{Sd}(\Delta^n) \to \Delta^0$ of $\mathbf{\Delta}$.

Claim 5.3.6. Let $f, g : K \rightrightarrows L$ be two morphisms of sSet, $h : \Delta^1 \times K \to L$ a homotopy from f to g, and $X \in \mathrm{Ob}(\mathsf{sSet})$. Let $f^*, g^* : \mathscr{H}om(L, X) \rightrightarrows \mathscr{H}om(K, X)$ be the morphisms of sSet induced by f and g. Then h induces a homotopy from f^* to g^*, and a homotopy from $\mathsf{Ex}(f)$ to $\mathsf{Ex}(g)$.

Proof : Indeed, h induces the morphism

$$h^* : \mathscr{H}om(L, X) \to \mathscr{H}om(\Delta^1 \times K, X) \xrightarrow{\sim} \mathscr{H}om(\Delta^1, \mathscr{H}om(K, X))$$

which, by adjunction, corresponds to a morphism

$$\Delta^1 \times \mathscr{H}om(L, X) \to \mathscr{H}om(K, X)$$

and it is easily seen that the latter is the sought homotopy from f^* to g^*. Next, since Ex preserves products [13, Prop.2.49(i)], h induces as well the morphism

$$\Delta^1 \times \mathsf{Ex}(K) \xrightarrow{\beta_{\Delta^1} \times \mathsf{Ex}(K)} \mathsf{Ex}(\Delta^1) \times \mathsf{Ex}(K) \xrightarrow{\sim} \mathsf{Ex}(\Delta^1 \times K) \xrightarrow{\mathsf{Ex}(h)} \mathsf{Ex}(L)$$

and it is easily seen that the latter is a homotopy from $\mathsf{Ex}(f)$ to $\mathsf{Ex}(g)$: the details shall be left to the reader. ◇

Since u_n is a simplicial homotopy equivalence (lemma 5.1.17(ii)), claim 5.3.6 implies that the same holds for u^*_n, for every $n \in \mathbb{N}$; especially, p^* is a levelwise weak homotopy equivalence, so it induces a weak homotopy equivalence :

$$(*) \qquad X \xrightarrow{\sim} \mathsf{diag}(X \boxtimes \Delta^0) \to \mathsf{diag}(E(X))$$

by theorem 5.2.2. Likewise, the unique morphism $v_m : \Delta^m \to \Delta^0$ is a simplicial homotopy equivalence for every $m \in \mathbb{N}$, hence the same holds for $v^*_m : X \xrightarrow{\sim} \mathscr{H}om(\Delta^0, X) \to \mathscr{H}om(\Delta^m, X)$. However, notice the natural identifications :

$$E(X)^\phi_{\bullet,m} \xrightarrow{\sim} \mathsf{Ex}(\mathscr{H}om(\Delta^m, X)) \qquad (\Delta^0 \boxtimes \mathsf{Ex}(X))^\phi_{\bullet,m} = (\mathsf{Ex}(X) \boxtimes \Delta^0)_{\bullet,m} \xrightarrow{\sim} \mathsf{Ex}(X)$$

where $(-)^\phi$ denotes the flip automorphism as in §2.1.8. Under these identifications, the morphism $(q^*)^\phi_{\bullet,m} : (\Delta^0 \boxtimes \mathsf{Ex}(X))^\phi_{\bullet,m} \to E(X)^\phi_{\bullet,m}$ corresponds to

$$\mathsf{Ex}(v^*_m) : \mathsf{Ex}(X) \to \mathsf{Ex}(\mathscr{H}om(\Delta^m, X))$$

and invoking again claim 5.3.6, we see that $\mathsf{Ex}(v^*_m)$ is a simplicial homotopy equivalence as well. Summing up, $(q^*)^\phi$ is a levelwise weak homotopy equivalence, so it induces a weak homotopy equivalence

$$(**) \qquad \mathsf{Ex}(K) \xrightarrow{\sim} \mathsf{diag}(\Delta^0 \boxtimes \mathsf{Ex}(K)^\phi) \to \mathsf{diag}(E(X)^\phi) = \mathsf{diag}(E(X)).$$

Now, for every morphism $f : X \to Y$ of sSet we get a commutative diagram :

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathsf{diag}(E(X)) \arrow[d, "\mathsf{diag}(E(f))"] & \mathsf{Ex}(X) \arrow[l] \arrow[d, "\mathsf{Ex}(f)"] \\ Y \arrow[r] & \mathsf{diag}(E(Y)) & \mathsf{Ex}(Y) \arrow[l] \end{tikzcd}$$

whose horizontal arows are the weak homotopy equivalences $(*)$ and $(**)$. Hence f is a weak homotopy equivalence if and only if the same holds for $\mathsf{Ex}(f)$.

(ii): Notice that $\mathbf{L}\alpha_\bullet : \mathbf{L}\mathsf{Sd} \Rightarrow \mathbf{1}_{\mathsf{ho(sSet)}}$ is an isomorphism of functors, by proposition 5.3.3(iii), example 3.4.6(ii) and remark 3.4.7(iii). Then, by proposition 5.3.3(iv), corollary 3.4.17 and the discussion of §1.6.10, the same holds for

$$\mathbf{R}\beta_\bullet = (\mathbf{L}\alpha_\bullet)^\vee : \mathbf{1}_{\mathsf{ho(sSet)}} \Rightarrow \mathbf{R}\mathsf{Ex}.$$

Since $\mathbf{R}\beta_X = [(\beta_{X_f})_c]$ for every $X \in \mathrm{Ob}(\mathrm{sSet})$ (remark 3.4.7(iv)), Whitehead's theorem 3.3.9(i) and lemma 3.4.12(ii) then imply that $(\beta_X)_c$ is a weak homotopy equivalence for every fibrant object X of sSet (*i.e.* every Kan complex), hence the same holds for β_X, for every Kan complex X.

For a general simplicial set X, we may find a trivial cofibration $i : X \to Y$ with a Kan complex Y (see §3.3.1); we deduce a commutative diagram :

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\beta_X"'] & Y \arrow[d, "\beta_Y"] \\ \mathsf{Ex}(X) \arrow[r, "\mathsf{Ex}(i)"] & \mathsf{Ex}(Y) \end{tikzcd}$$

where β_Y is a weak homotopy equivalence, by the foregoing, and the same holds for $\mathsf{Ex}(i)$, by virtue of (i); so β_X is a weak homotopy equivalence as well. □

5.3.7. For every simplicial set X we define inductively :

$$\mathsf{Ex}^0(X) := X \qquad \mathsf{Ex}^{n+1}(X) := \mathsf{Ex}(\mathsf{Ex}^n(X)) \qquad \forall n \in \mathbb{N}.$$

Hence we get a sequence of functors

$$\boxed{\mathsf{Ex}^n : \mathsf{sSet} \to \mathsf{sSet} \qquad \forall n \in \mathbb{N}}$$

which preserve trivial fibrations and Kan fibrations (proposition 5.3.3(ii)), and both preserve and reflect weak homotopy equivalences (proposition 5.3.5(i)); moreover, we have natural transformations

$$\beta^n_\bullet := \beta_\bullet \star \mathsf{Ex}^n : \mathsf{Ex}^n \Rightarrow \mathsf{Ex}^{n+1} \qquad \forall n \in \mathbb{N}$$

such that β^n_X is a trivial cofibration for every $X \in \mathrm{Ob}(\mathrm{sSet})$ (proposition 5.3.5(ii)). Thus, for every such X we get a natural system of morphisms of sSet :

$$(*) \qquad X \xrightarrow{\beta^0_X} \mathsf{Ex}^1(X) \xrightarrow{\beta^1_X} \mathsf{Ex}^2(X) \xrightarrow{\beta^2_X} \cdots$$

and we set

$$\boxed{\mathsf{Ex}^\infty(X) := \varinjlim_{n\in\mathbb{N}} \mathsf{Ex}^n(X) \qquad \forall X \in \mathrm{Ob}(\mathsf{sSet}).}$$

The colimit of $(*)$ is then a natural transformation

$$\beta^\infty_\bullet : \mathbf{1}_{\mathsf{sSet}} \Rightarrow \mathsf{Ex}^\infty.$$

Lemma 5.3.8. *(i)* $\beta^\infty_X : X \to \mathsf{Ex}^\infty(X)$ *is a trivial cofibration for all* $X \in \mathrm{Ob}(\mathrm{sSet})$.

(ii) Ex^∞ *preserves Kan fibrations and weak homotopy equivalences, and is left exact.*

(iii) β^∞_X *is a bijection on objects* $\beta^\infty_{X,0} : X_0 \xrightarrow{\sim} \mathsf{Ex}^\infty(X)_0$*, and induces a natural bijection :*

$$\boxed{\pi_0(\beta^\infty_X) : \pi_0(X) \xrightarrow{\sim} \pi_0(\mathsf{Ex}^\infty(X)) \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).}$$

Proof. (i): This is clear, since trivial cofibrations form a saturated class (see proposition 3.1.9(v)), and since β^n_X is a trivial cofibration for every $n \in \mathbb{N}$.

(ii): Since Ex^n preserves Kan fibrations for every $n \in \mathbb{N}$, proposition 5.1.11(iii) implies that the same holds for Ex^∞; moreover, with (i) and the 2-out-of-3 property of weak homotopy equivalences we see that Ex^∞ preserves weak homotopy equivalences. Next, since Ex is a right adjoint, it is left exact ([13, Prop.2.49(i)]), so the same holds for Ex^n, for every $n \in \mathbb{N}$; taking into account example 1.3.7(ii), we deduce that the same holds for Ex^∞.

(iii) This is clear from example 5.3.1(ii,iii), since the functor π_0 preserves all representable colimits ([13, Prop.2.49(ii)]). □

Theorem 5.3.9. (Kan) $\mathsf{Ex}^\infty(X)$ *is a Kan complex, for every* $X \in \mathrm{Ob}(\mathrm{sSet})$.

Proof. For every $n \in \mathbb{N}$, recall that $\xi([n])$ is the set of non-empty subsets of $[n]$, ordered by inclusion of subsets; likewise, $\xi(\xi([n]))$ is the set of strictly increasing chains $C_\bullet := (\varnothing \neq C_0 \subset \cdots \subset C_p \subset [n])$ of arbitrary length $p \geq 0$, with ordering such that $C_\bullet \leq C'_\bullet := (\varnothing \neq C'_0 \subset \cdots \subset C'_q \subset [n]) \Leftrightarrow$ for every $i = 0, \dots, p$ there exists $j \leq q$ such that $C_i = C'_j$. For every $n \in \mathbb{N}$ and $k = 0, \dots, n$ consider the map

$$\psi^n_k : \xi(\xi([n])) \to \xi([n]) \qquad (\varnothing \neq C_0 \subset \cdots \subset C_p \subset [n]) \mapsto \{c^n_k(C_i) \mid i = 0, \dots, p\}$$

where :

$$c^n_k(S) := \begin{cases} \max(S) & \text{if } S \in \Phi^n_k \\ k & \text{otherwise} \end{cases} \qquad \forall S \in \xi([n]).$$

Clearly ψ^n_k is a morphism of poSet. Now, define Φ^n_k as in corollary 5.3.4, we have :

Claim 5.3.10. $\mathrm{Im}(\psi^n_k) \subset \Phi^n_k$ for every $n \in \mathbb{N}$ and every $k = 0, \dots, n$.

Proof : We need to check that $\psi^n_k(C_\bullet) \neq [n], [n] \setminus \{k\}$ for every chain $C_\bullet \in \xi(\xi([n]))$. Thus, suppose first that $\psi^n_k(C_\bullet) = [n]$; especially, the length of the chain $C_\bullet$ must equal n (since this is also the maximal length of *any* chain of $\xi(\xi([n]))$). Moreover, $C_n = [n]$, so $c^n_k(C_n) = k$, and therefore $c^n_k(C_i) \neq k$ for every $i = 0, \dots, n-1$, so that $c^n_k(C_i) = \max(C_i) \geq i$ for every $i \leq n-1$. if $c^n_k(C_{n-1}) = n-1$, then $n \notin \{c^n_k(C_i) \mid i = 0, \dots, n-1\}$, and therefore $n = c^n_k(C_n) = k$. But then $C_{n-1} = [n] \setminus \{k\} \notin \Phi^n_k$, so that $c^n_k(C_{n-1}) = k$, a contradiction.

• Next, if $c^n_k(C_{n-1}) = n$, then $k = c^n_k(C_n) \neq n$, so $\{c^n_k(C_i) \mid i = 0, \dots, n-2\} = [n-1] \setminus \{k\}$. Hence, $C_{n-2} = [n-1] \setminus \{k\}$ (since $C_0 \subset \cdots \subset C_{n-2}$ is then a maximal chain in $[n-1] \setminus \{k\}$), and therefore $C_{n-1} = [n] \setminus \{k\}$ (since $c^n_k(C_{n-1}) = \max(C_{n-1})$); but then $c^n_k(C_{n-1}) = k$ as well, again a contradiction.

• Lastly, suppose that $\psi^n_k(C_\bullet) = [n] \setminus \{k\}$, and let $p \leq n$ be the length of $C_\bullet$. Then $c^n_k(C_i) = \max(C_i)$ for $i = 0, \dots, p$, and $p \geq n-1$; moreover, the cardinality of C_p must be $\geq n$, and $C_p \neq [n]$, so $C_p = [n] \setminus \{j\}$ for some $j \neq k$. Consequently, $c^n_k(C_i) \neq i$ for every $i = 0, \dots, p$, and then $i \notin \psi^n_k(C_\bullet)$, so $j = k$, a contradiction. ◇

By claim 5.3.10, lemma 5.3.2, and corollary 5.3.4, the morphism $N(\psi^n_k)$ induces a morphism of sSet

$$u^n_k : \mathsf{Sd}^2(\Delta^n) \to \mathsf{Sd}(\Lambda^n_k).$$

Claim 5.3.11. Let $j : \Lambda^n_k \to \Delta^n$ be the inclusion; the following diagram commutes :

$$\begin{tikzcd}
\mathsf{Sd}^2(\Lambda^n_k) \arrow[r, "\mathsf{Sd}(\alpha_{\Lambda^n_k})"] \arrow[d, "\mathsf{Sd}^2(j)"'] & \mathsf{Sd}(\Lambda^n_k) \\
\mathsf{Sd}^2(\Delta^n). \arrow[ru, "u^n_k"'] &
\end{tikzcd}$$

Proof : We come down to checking the commutativity of the diagram :

$$\begin{tikzcd}
\mathsf{Sd}^2(\Lambda^n_k) \arrow[r, "\mathsf{Sd}^2(j)"] \arrow[d, "\mathsf{Sd}(\alpha_{\Lambda^n_k})"'] & \mathsf{Sd}^2(\Delta^n) \arrow[r, "\sim"] & N\xi^2[n] \arrow[d, "N(\psi^n_k)"] \\
\mathsf{Sd}(\Lambda^n_k) \arrow[r, "\mathsf{Sd}(j)"] & \mathsf{Sd}(\Delta^n) \arrow[r, "\sim"] & N\xi[n]
\end{tikzcd}$$

where the unmarked horizontal isomorphisms are given by lemma 5.3.2. However, the composition of the top and bottom horizontal arrows are respectively :

$$\mathsf{Sd}^2(\Lambda^n_k) \xrightarrow{\sim} N\xi(\Phi^n_k) \xrightarrow{N\xi(j')} N\xi^2[n] \qquad \text{and} \qquad \mathsf{Sd}(\Lambda^n_k) \xrightarrow{\sim} N\Phi^n_k \xrightarrow{Nj'} N\xi[n]$$

where $j' : \Phi^n_k \to \xi[n]$ is the inclusion, and the unmarked isomorphisms are again the natural identifications of lemma 5.3.2, which make commute the diagram :

$$\begin{tikzcd}
\mathsf{Sd}^2(\Lambda^n_k) \arrow[r, "\sim"] \arrow[d, "\mathsf{Sd}(\alpha_{\Lambda^n_k})"'] & N\xi(\Phi^n_k) \arrow[d, "N\mu_{\Phi^n_k}"] \\
\mathsf{Sd}(\Lambda^n_k) \arrow[r, "\sim"] & N\Phi^n_k
\end{tikzcd}$$

Thus, we are reduced to checking the commutativity of the diagram :

$$\begin{tikzcd}
\xi(\Phi^n_k) \arrow[r, "\xi(j')"] \arrow[d, "\mu_{\Phi^n_k}"'] & \xi^2[n] \arrow[d, "\psi^n_k"] \\
\Phi^n_k \arrow[r, "j'"] & \xi[n]
\end{tikzcd}$$

which follows by simple inspection. ◇

Now, let $x : \Lambda^n_k \to \mathsf{Ex}^\infty(X)$ be any morphism; since Λ^n_k has finitely many non-degenerate simplices, there exists $m \in \mathbb{N} \setminus \{0\}$ such that x is the composition of a morphism $x' : \Lambda^n_k \to \mathsf{Ex}^m(X)$ and the natural morphism $\mathsf{Ex}^m(X) \to \mathsf{Ex}^\infty(X)$ (corollary 2.2.11). By adjunction, x' corresponds to a morphism $y : \mathsf{Sd}(\Lambda^n_k) \to \mathsf{Ex}^{m-1}(X)$, and we let $z : \mathsf{Sd}(\Delta^n) \to \mathsf{Ex}^m(X)$ be the adjoint of $y' := y \circ u^n_k : \mathsf{Sd}^2(\Delta^n) \to \mathsf{Ex}^{m-1}(X)$; notice that $y' \circ \mathsf{Sd}^2(j) = y \circ \mathsf{Sd}(\alpha_{\Lambda^n_k})$, by claim 5.3.11. By adjunction, we deduce first a commutative diagram :

$$\begin{tikzcd}
\mathsf{Sd}(\Lambda^n_k) \arrow[r, "\alpha_{\Lambda^n_k}"] \arrow[d, "\mathsf{Sd}(j)"'] & \Lambda^n_k \arrow[d, "x'"] \\
\mathsf{Sd}(\Delta^n) \arrow[r, "z"] & \mathsf{Ex}^m(X)
\end{tikzcd}$$

and then a second commutative diagram :

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\beta_{\Lambda^n_k}"] \arrow[d, "j"'] & \mathsf{Ex}^1(\Lambda^n_k) \arrow[d, "\mathsf{Ex}^1(x')"] \\
\Delta^n \arrow[r, "w"] & \mathsf{Ex}^{m+1}(X)
\end{tikzcd}$$

where w is again the adjoint of z (see §1.6.10). But $\mathsf{Ex}^1(x') \circ \beta_{\Lambda^n_k} = \beta_{\mathsf{Ex}^m(X)} \circ x'$, so finally we arrive at the commutative diagram :

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "x'"] \arrow[d, "j"'] & \mathsf{Ex}^m(X) \arrow[r] \arrow[d, "\beta_{\mathsf{Ex}^m(X)}"'] & \mathsf{Ex}^\infty(X) \\
\Delta^n \arrow[r, "w"] & \mathsf{Ex}^{m+1}(X) \arrow[ur]
\end{tikzcd}$$

and the proof is concluded. □

5.4. **Fibrations and weak equivalences in the Kan-Quillen model category.** In this § we exploit the functor Ex^∞ to derive useful properties of the class of weak homotopy equivalences, and of the fibrations of the Kan-Quillen model structure. In the last paragraph, we also add some complements for the model category of pointed simplicial sets.

Corollary 5.4.1. *Consider a Kan fibration $p : X \to Y$ and a cartesian square of* sSet *:*

$$\mathsf{D} \quad : \quad \begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' \arrow[r, "g"] & Y.
\end{tikzcd}$$

(i) If p (resp. g) is a weak homotopy equivalence, the same holds for p' (resp. for f).

(ii) D is homotopy cartesian for the Kan-Quillen model category.

Proof. (ii): We apply termwise the functor Ex^∞ to D, to deduce another cartesian square $\mathsf{Ex}^\infty(\mathsf{D})$, by lemma 5.3.8(ii), all of whose terms are Kan complexes, by theorem 5.3.9, and where $\mathsf{Ex}^\infty(p)$ is a Kan fibration, again by lemma 5.3.8(ii), so it is a fibration between fibrant objects for the Kan-Quillen model category (theorem 5.1.10). Also, by lemma 5.3.8(i), the natural transformation $\beta^\infty_\bullet$ of §5.3.7 yields a weak equivalence $\beta^\infty_{\mathsf{D}} : \mathsf{D} \to \mathsf{Ex}^\infty(\mathsf{D})$ of the category $\mathsf{sSet}_\square$ of example 3.6.2(iii) (endowed with the injective model structure induced by the Kan-Quillen model structure on sSet), hence it suffices to check that $\mathsf{Ex}^\infty(\mathsf{D})$ is homotopy cartesian (remark 3.5.15(v)); the latter holds by proposition 3.6.5(ii).

(i): Lemma 5.3.8(i) (and the 2-out-of-3 property for weak equivalences) easily implies that if p (resp. g) is a weak homotopy equivalence, the same holds for $\mathsf{Ex}^\infty(p)$ (resp. for $\mathsf{Ex}^\infty(g)$). Next, if $\mathsf{Ex}^\infty(g)$ is a weak homotopy equivalence, the same holds for $\mathsf{Ex}^\infty(f)$, by virtue of proposition 3.6.5(ii), since we have already remarked that $\mathsf{Ex}^\infty(\mathsf{D})$ is cartesian and that $\mathsf{Ex}^\infty(p)$ is a fibration between fibrant objects for the Kan-Quillen model category; if $\mathsf{Ex}^\infty(p)$ is a weak homotopy equivalence, then it is a trivial fibration of sSet (theorem 5.1.10), and therefore the same holds for $\mathsf{Ex}^\infty(p')$ (proposition 3.1.9(v)). Lastly, if $\mathsf{Ex}^\infty(f)$ (resp. $\mathsf{Ex}^\infty(p')$) is a weak homotopy equivalence, the same holds for f (resp. for p'), once again by lemma 5.3.8(i). □

Proposition 5.4.2. *Consider a commutative triangle of simplicial sets :*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S. & \end{tikzcd}$$

(i) If for every $n \in \mathbb{N}$, every n-simplex $\Delta^n \to S$ induces a weak homotopy equivalence $\Delta^n \times_S X \to \Delta^n \times_S Y$, then every morphism $S' \to S$ of sSet *induces a weak homotopy equivalence $S' \times_S f : S' \times_S X \to S' \times_S Y$ (especially, f is a weak homotopy equivalence).*

(ii) If moreover, p and q are Kan fibrations, the following conditions are equivalent :

(a) f is a weak homotopy equivalence.

(b) $S' \times_S f$ is a weak homotopy equivalence for every morphism $S' \to S$ of sSet.

(c) f restricts to a weak homotopy equivalence $f_s : p^{-1}(s) \to q^{-1}(s)$ for every $s \in S_0$.

Proof. (i): Recall that $\mathbf{\Delta}/S$ is an Eilenberg-Zilber category, by example 2.2.2(i,ii); moreover, we have a natural equivalence of categories $\widehat{\mathbf{\Delta}/S} \xrightarrow{\sim} \mathsf{sSet}/S$ that identifies each representable presheaf $h_{([n],u)}$ on $\mathbf{\Delta}/S$ with the object $(\Delta^n, u : \Delta^n \to S)$ of sSet/S (lemma 1.7.2). Hence, let $\Sigma \subset \mathrm{Mor}(\mathsf{sSet}/S)$ be the class of all morphism $S' \to S$ of sSet such that $S' \times_S f$ is a weak homotopy equivalence; by assumption, Σ contains all the representable presheaves of $\widehat{\mathbf{\Delta}/S}$, so we are reduced to checking that Σ is saturated by monomorphisms (corollary 2.2.10). However, condition (a) of definition 2.2.9 holds for Σ, due to corollaries 3.5.12(i) and 1.4.6(iii). Lastly, since all small colimits of sSet are universal (§2.1.6), corollaries 3.5.13 and 3.6.7 imply that also conditions (b) and (c) hold for Σ.

(ii): (In (c), $p^{-1}(s)$ and $q^{-1}(s)$ denote the fibres of p and q over s : see definition 2.5.1(iv).) Obviously (b)⇒(c). In order to check that (a)⇒(b), consider the commutative diagram :

$$\begin{tikzcd} S' \times_S X \arrow[r, "S' \times_S f"] \arrow[d] & S' \times_S Y \arrow[r, "S' \times_S q"] \arrow[d] & S' \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "q"] & S \end{tikzcd}$$

whose left and right squares D and D′ are cartesian, and we let D″ be the composition of D and D′, *i.e.* the square whose two horizontal sides are p and $S' \times_S p : S' \times_S X \to S'$. By corollary 5.4.1(ii) and the dual of lemma 3.6.4(i), both D′ and D″ are homotopy cartesian, hence the same holds for D (proposition 3.6.3(iii)); then the assertion follows from lemma 3.6.4(iii). Lastly, we check that (c)⇒(a) : to this aim, (i) reduces to showing that for every $n \in \mathbb{N}$, every n-simplex $\psi : \Delta^n \to S$ induces a weak homotopy equivalence $f_\psi := \Delta^n \times_S f : \Delta^n \times_S X \to \Delta^n \times_S Y$. However, let $\phi : [0] \to [n]$ be the map such that $0 \mapsto n$; recall that the induced morphism $\Delta^\phi : \Delta^0 \to \Delta^n$ is a strong deformation retract (remark 5.1.9(iii)), and set $s := \psi \circ \Delta^\phi : \Delta^0 \to S$. We get cartesian squares of sSet :

$$\begin{tikzcd} p^{-1}(s) \arrow[r, "\beta_X"] \arrow[d] & \Delta^n \times_S X \arrow[d, "\Delta^n \times_S p"] \\ \Delta^0 \arrow[r, "\Delta^\phi"] & \Delta^n \end{tikzcd} \qquad \begin{tikzcd} p^{-1}(s) \arrow[r, "f_s"] \arrow[d, "\beta_X"'] & q^{-1}(s) \arrow[d, "\beta_Y"] \\ \Delta^n \times_S X \arrow[r, "f_\psi"] & \Delta^n \times_S Y \end{tikzcd} \qquad \begin{tikzcd} q^{-1}(y) \arrow[r, "\beta_Y"] \arrow[d] & \Delta^n \times_S Y \arrow[d, "\Delta^n \times_Y q"] \\ \Delta^0 \arrow[r, "\Delta^\phi"] & \Delta^n \end{tikzcd}$$

where $\Delta^n \times_S p$ and $\Delta^n \times_S q$ are Kan fibrations (proposition 3.1.9(v)), so that β_X and β_Y are strong deformation retracts (proposition 4.5.7), hence they are weak homotopy equivalences (proposition 4.5.6(ii)); the same holds for f_s, by virtue of (c), so also for f_ψ. □

Corollary 5.4.3. *Consider a commutative diagram of* sSet *:*

$$\mathsf{D} \quad : \quad \begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

where f and f' are Kan fibrations. The following conditions are equivalent :

(a) D *is homotopy cartesian for the Kan-Quillen model category.*

(b) Every $y' \in Y'_0$ induces a weak homotopy equivalence $v_y : f'^{-1}(y') \to f^{-1}(u(y'))$.

Proof. From corollary 5.4.1(ii) and theorem 3.3.9(ii) it follows easily that (a) holds if and only if the induced morphism $g : X' \to Y' \times_Y X$ is a weak homotopy equivalence (details left to the reader). Notice that $Y' \times_Y f : Y' \times_Y X \to Y'$ is a Kan fibration (proposition 3.1.9(v)); then, by virtue of proposition 5.4.2(ii), g is a weak homotopy equivalence if and only if every object $y' : \Delta^0 \to Y'$ induces a weak homotopy equivalence

$$g_{y'} := \Delta^0 \times_{Y'} g : f'^{-1}(y') \to \Delta^0 \times_{Y'} (Y' \times_Y X) = f^{-1}(y).$$

But $g_{y'} = v_{y'}$, so this is precisely condition (b). ◻

Theorem 5.4.4. (Quillen) *A morphism of* sSet *is a fibration (resp. a trivial cofibration) for the Kan-Quillen model category ⇔ it is a Kan fibration (resp. an anodyne extension).*

Proof. It suffices to check the assertion concerning trivial cofibrations, and since every anodyne extension is a trivial cofibration, we need only verify the converse assertion. Now, by factoring a trivial cofibration as the composition of an anodyne extension followed by a Kan fibration, the retract lemma (proposition 3.1.10) reduces to showing that every Kan fibration $f : X \to Y$ that is a weak homotopy equivalence is a trivial fibration.

Claim 5.4.5. We may assume that $Y = \Delta^n$ for some $n \in \mathbb{N}$.

Proof : Indeed, consider a commutative diagram of sSet :

$$(*) \qquad \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y. \end{tikzcd}$$

We need to find a diagonal filler for $(*)$, and by corollary 5.4.1(i) and proposition 3.1.9(v), we may replace Y by Δ^n and X by $\Delta^n \times_Y X$, whence the claim. ◇

Thus, let $f : X \to \Delta^n$ be a Kan fibration which is a weak homotopy equivalence. Let $\phi : [0] \to [n]$ be the map such that $\phi(0) = n$, and consider the cartesian diagram :

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \Delta^0 \arrow[r, "\Delta^\phi"] & \Delta^n. \end{tikzcd}$$

Recall that Δ^ϕ is a strong deformation retract (remark 5.1.9(iii)); then, by proposition 4.5.7, the same holds for j, so the latter is an anodyne extension (proposition 4.5.6(ii)). On the other hand, f' is both a Kan fibration and a weak homotopy equivalence (corollary 5.4.1(i) and proposition 3.1.9(v)), and since Δ^0 is fibrant, it follows that f' is a trivial fibration (theorem 5.1.10); then f' is an absolute weak equivalence (proposition 4.6.8). By proposition 4.6.9, we then see that f is an absolute weak equivalence, and finally, f is therefore a trivial fibration, again by proposition 4.6.8. ◻

Proposition 5.4.6. *(i) The functor* $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$ *(see §2.1.11) sends weak homotopy equivalences to bijections, and preserves finite products.*

(ii) For every $A, X \in \mathrm{Ob}(\mathsf{sSet})$*, the set* $\pi_0(\mathscr{H}om(A, X))$ *is naturally identified with the set* $[A, X]$ *of* Δ^1*-homotopy classes of morphisms* $A \to X$.

(iii) We have a Quillen adjunction :

$$\boxed{\pi_0 : \mathsf{sSet} \rightleftarrows \mathsf{Set} : \mathsf{c}_{\Delta^{\mathrm{op}}}}$$

where $\mathsf{c}_{\Delta^{\mathrm{op}}}$ *denotes the constant simplicial set functor of §2.1.11, and where* Set *is endowed with the model category structure of example 3.2.3(i).*

Proof. (ii): Recall that $\pi_0(\mathscr{H}om(A, X))$ is naturally identified with the coequalizer of $d_1^0, d_1^1 : \mathscr{H}om(A, X)_1 \rightrightarrows \mathscr{H}om(A, X)_0$ (see §2.1.11), and by definition, $\mathscr{H}om(A, X)_n$ is the set $\mathsf{sSet}(\Delta^n \times A, X)$, for every $n \in \mathbb{N}$. Under such identifications, the map d_1^i corresponds to $\mathsf{sSet}(\partial_i^1, X) : \mathsf{sSet}(\Delta^1, X) \to \mathsf{sSet}(\Delta^0, X)$, for $i = 0, 1$, whence the assertion.

(i): Let $f : X \to Y$ be a weak homotopy equivalence; if X and Y are Kan complexes, then the induced map $[\Delta^0, X] \to [\Delta^0, Y]$ is a bijection, by corollary 4.5.16(iii). For the general case, we consider the commutative diagram :

$$\begin{tikzcd} X \arrow[r, "\beta_X^\infty"] \arrow[d, "f"'] & \mathsf{Ex}^\infty(X) \arrow[d, "\mathsf{Ex}^\infty(f)"] \\ Y \arrow[r, "\beta_Y^\infty"] & \mathsf{Ex}^\infty(Y) \end{tikzcd}$$

whose horizontal arrows are weak homotopy equivalences, by lemma 5.3.8(i), so that the same holds for $\mathsf{Ex}^\infty(f)$; by theorem 5.3.9 and the previous case, it follows that $\pi_0(\mathsf{Ex}^\infty(f))$ is a bijection, and the same holds for $\pi_0(\beta_X^\infty)$ and $\pi_0(\beta_Y^\infty)$, by lemma 5.3.8(iii), so finally also for $\pi_0(f)$.

It remains to check that π_0 preserves finite products. To this aim, let first X and Y be any two Kan complexes; we consider the map

$$\omega : (X \times Y)_0 = X_0 \times Y_0 \to \pi_0(X) \times \pi_0(Y) \qquad (a, b) \mapsto ([a], [b])$$

where $[a]$ denotes the class of a in $\pi_0(X)$, and likewise for $[b]$. If $h : \Delta^1 \to X \times Y$ is any Δ^1-homotopy from (a, b) to (a', b'), then the composition of h with the projection $p : X \times Y \to X$ (resp. $q : X \times Y \to Y$) is a Δ^1-homotopy from a to a' (resp. from b to b'), hence ω factors through the projection $(X \times Y)_0 \to \pi_0(X \times Y)$ and a unique surjection

$$\overline{\omega} : \pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$$

so it remains only to verify the injectivity of $\overline{\omega}$. Hence, let $(a, b), (a', b') \in X_0 \times Y_0$ such that $[a] = [a']$ and $[b] = [b']$; according to lemma 4.5.8 there exist Δ^1-homotopies $h : \Delta^1 \to X$ from a to a' and $k : \Delta^1 \to Y$ from b to b', and we let $H : \Delta^1 \to X \times Y$ be the unique morphism such that $p \circ H = h$ and $q \circ H = k$. Clearly H is a Δ^1-homotopy from (a, b) to (a', b'), whence the assertion.

Lastly, let X, Y be any two simplicial complexes, and recall that the functor Ex^∞ is left exact (lemma 5.3.8(ii)); we then get a commutative diagram :

$$\begin{tikzcd} \pi_0(X \times Y) \arrow[r] \arrow[d] & \pi_0(\mathsf{Ex}^\infty(X \times Y)) \arrow[r] & \pi_0(\mathsf{Ex}^\infty(X) \times \mathsf{Ex}^\infty(Y)) \arrow[d] \\ \pi_0(X) \times \pi_0(Y) \arrow[rr] & & \pi_0(\mathsf{Ex}^\infty(X)) \times \pi_0(\mathsf{Ex}^\infty(Y)) \end{tikzcd}$$

and taking into account (ii) and lemma 5.3.8(i), we see that all the arrows of this diagram are bijections, except possibly for the left vertical arrow. But then also the left vertical arrow must be bijective, and the proof is concluded.

(iii): Recall that, with the model category structure of example 3.2.3(i), the weak equivalences of Set are the bijections, and every map is both a fibration and a cofibration. Now, by virtue of proposition 4.5.17 and corollary 5.1.6, it suffices to check that the inclusion $\Lambda^n_k \to \Delta^n$ induces a bijection $\pi_0(\Lambda^n_k) \xrightarrow{\sim} \pi_0(\Delta^n)$, for every $n \in \mathbb{N} \setminus \{0\}$ and $k = 0, \dots, n$. The latter holds by example 2.1.13 and §2.3.12. □

5.4.7. *Kan-Quillen model structure on pointed simplicial sets.* Recall that we denote by $\mathrm{sSet}_\circ$ the category of pointed simplicial sets (see §2.1.14); following proposition 3.2.4(iii), the Kan-Quillen model structure on sSet induces a model structure on $\mathrm{sSet}_\circ$, that we call *the Kan-Quillen model category structure on pointed simplicial sets.* The weak equivalences (resp. the fibrations, resp. the trivial fibrations) of $\mathrm{sSet}_\circ$ are called *pointed weak homotopy equivalences* (resp. *pointed Kan fibrations*, resp. *pointed trivial fibrations*) : hence, these are the morphisms $(X, x) \to (Y, y)$ of $\mathrm{sSet}_\circ$ whose underlying morphism $X \to Y$ of simplicial sets is a weak homotopy equivalence (resp. a Kan fibration, resp. a trivial fibration of sSet). Likewise, the fibrant objects of $\mathrm{sSet}_\circ$ are called *pointed Kan complexes*; again, these are the pairs (X, x) where X is a Kan complex.

Remark 5.4.8. (i) Since the cofibrations of sSet are the monomorphisms, the same holds for the cofibrations of $\mathrm{sSet}_\circ$ (corollary 1.4.6(ii)). Moreover, clearly the initial object of $\mathrm{sSet}_\circ$ is $\underline{\Delta}^0$, so the unique morphism $\underline{\Delta}^0 \to \underline{X}$ is a monomorphism, for every $\underline{X} \in \mathrm{Ob}(\mathrm{sSet}_\circ)$; *i.e. every object of* $\mathrm{sSet}_\circ$ *is cofibrant* for the Kan-Quillen model category structure.

(ii) Let us endow the finitely complete and finitely cocomplete *category of pointed sets*

$$\mathrm{Set}_\circ := \{\varnothing\}/\mathrm{Set}$$

with the model structure provided by example 3.2.3(i), and $\mathrm{sSet}_\circ$ with its Kan-Quillen model structure; then, according to remark 3.4.11(ii), the Quillen adjunction of proposition 5.4.6(iii) induces a Quillen adjunction that we denote again :

$$\boxed{\pi_0 : \mathrm{sSet}_\circ \rightleftarrows \mathrm{Set}_\circ : c_{\Delta^{\mathrm{op}}}.}$$

Explicitly, for every $(X, x) \in \mathrm{Ob}(\mathrm{sSet}_\circ)$ we have $\pi_0(X, x) := (\pi_0(X), [x])$, where $[x]$ denotes the class in $\pi_0(X)$ of the base point x. And for every pointed set $i : \{\varnothing\} \to S$, we have $c_{\Delta^{\mathrm{op}}}(S, i) := ((\Delta^0)^{(i)} : \Delta^0 \to (\Delta^0)^{(S)})$ (notation of §1.2.14).

(iii) Moreover, since the target functor $\mathrm{t} : \mathrm{sSet}_\circ \to \mathrm{sSet}$ preserves fibrations and trivial fibrations, by lemma 3.4.12(i) we have as well the Quillen adjunction :

$$\boxed{(-)_\circ : \mathrm{sSet} \rightleftarrows \mathrm{sSet}_\circ : \mathrm{t}}$$

where the left adjoint $(-)_\circ$ is defined as in §2.1.14. Especially, $(-)_\circ$ sends weak homotopy equivalences to pointed weak homotopy equivalences (lemma 3.4.12(iii)).

Proposition 5.4.9. *(i) For every pair $\underline{X} \to \underline{Y}$, $\underline{X}' \to \underline{Y}'$ of pointed weak homotopy equivalences, the induced morphisms $\underline{X} \vee \underline{Y} \to \underline{X}' \vee \underline{Y}'$ and $\underline{X} \wedge \underline{Y} \to \underline{X}' \wedge \underline{Y}'$ are pointed weak homotopy equivalences.*

(ii) $\underline{\mathscr{H}om}(\underline{X}, \underline{Y})$ and $\mathscr{H}om_\circ(\underline{X}, \underline{Y})$ are pointed Kan complexes, for every $\underline{X} \in \mathrm{Ob}(\mathrm{sSet}_\circ)$ and every pointed Kan complex $\underline{Y}$ (see example 2.1.15(ii)).

(iii) Let $\mathrm{ho}(\mathrm{sSet}_\circ)$ *be the homotopy category of the Kan-Quillen model category structure on* $\mathrm{sSet}_\circ$*. Then, for every pointed Kan complex* $\underline{X}$*, we have natural isomorphisms of* $\mathrm{Set}_\circ$ *:*

$$\pi_0\mathscr{H}om_\circ(\underline{A},\underline{X}) \xrightarrow{\sim} (\mathrm{ho}(\mathrm{sSet}_\circ)(\underline{A},\underline{X}),[0_{AX}]) \qquad \forall \underline{A} \in \mathrm{Ob}(\mathrm{sSet}_\circ)$$

where $[0_{AX}]$ *denotes the class of* $0_{AX} : \underline{A} \to \underline{X}$ *in* $\mathrm{ho}(\mathrm{sSet}_\circ)(\underline{A},\underline{X})$.

Proof. (i): By virtue of the natural isomorphisms $\underline{X} \vee \underline{Y} \xrightarrow{\sim} \underline{Y} \vee \underline{X}$ and $\underline{X} \wedge \underline{Y} \xrightarrow{\sim} \underline{Y} \wedge \underline{X}$ (proposition 2.1.16(ii)), it suffices to check that the functors $- \vee \underline{Y}$ and $- \wedge \underline{Y}$ preserve pointed weak homotopy equivalences. Now, say that $\underline{Y} = (Y, y)$, and let $\underline{X} = (X, x) \to \underline{X}' := (X', x')$ be a pointed weak homotopy equivalence; we get a commutative diagram :

$$\begin{tikzcd}
\{x\} \times Y \arrow[d] & \{(x,y)\} \arrow[l] \arrow[r] \arrow[d] & X \times \{y\} \arrow[d] \\
\{x'\} \times Y & \{(x',y)\} \arrow[l] \arrow[r] & X' \times \{y\}
\end{tikzcd}$$

whose vertical arrows are weak homotopy equivalences. Then the same holds for the induced morphism $X \vee Y \to X' \vee Y$, by corollary 3.5.12(ii). Likewise, we have the commutative diagram of sSet :

$$\begin{tikzcd}
\Delta^0 \arrow[d, equal] & X \vee Y \arrow[l] \arrow[r] \arrow[d] & X \times Y \arrow[d] \\
\Delta^0 & X' \vee Y \arrow[l] \arrow[r] & X' \times Y
\end{tikzcd}$$

whose central vertical arrow is a weak homotopy equivalence, by the foregoing; the same holds for the right vertical arrow, by corollary 5.1.13(ii), and then also for the induced morphism $X \wedge Y \to X' \wedge Y$, by virtue of corollary 3.6.7.

(ii): The assertion for $\underline{\mathscr{H}om}(\underline{X},\underline{Y})$ is clear, from corollary 5.1.12(ii). Next, say that $\underline{X} = (X, x)$ and $\underline{Y} = (Y, y)$; we apply corollary 5.1.12(i) to the monomorphism $x : \Delta^0 \to X$ and the Kan fibration $p : Y \to \Delta^0$ to deduce that $x^* = (x^*, p_*) : \mathscr{H}om(X, Y) \to \mathscr{H}om(\Delta^0, Y) \times_{\mathscr{H}om(\Delta^0,\Delta^0)} \mathscr{H}om(X, \Delta^0) \xrightarrow{\sim} \underline{Y}$ is a Kan fibration. Then the unique morphism $\mathscr{H}om_\circ(\underline{X},\underline{Y}) \to \underline{\Delta}^0$ is a pointed Kan fibration, by proposition 3.1.9(v), as stated.

(iii): Notice first that $(\Delta^0)_\circ = \partial\underline{\Delta}^1$ (notation of example 2.1.15(i)), and moreover, for every $\underline{A} \in \mathrm{Ob}(\mathrm{sSet}_\circ)$ we have a natural identification in $\mathrm{sSet}_\circ$:

$$\underline{A} \wedge (\partial\Delta^1)_\circ \xrightarrow{\sim} \underline{A} \sqcup \underline{A}$$

(details left to the reader). Combining with proposition 2.1.16(ii), we deduce that the inclusion $i : \partial\Delta^1 \to \Delta^1$ and the projection $p : \Delta^1 \to \Delta^0$ induce morphisms of $\mathrm{sSet}_\circ$:

$$(*) \qquad \underline{A} \sqcup \underline{A} \xrightarrow{\underline{A}\wedge i_\circ} \underline{A} \wedge (\Delta^1)_\circ \xrightarrow{\underline{A}\wedge p_\circ} \underline{A} \wedge \partial\underline{\Delta}^1 \xrightarrow{\sim} \underline{A}$$

whose composition is the codiagonal $\nabla_{\underline{A}} : \underline{A} \sqcup \underline{A} \to \underline{A}$. Recall that p is a weak homotopy equivalence (lemma 5.1.17(ii)), so $p_\circ$ is a pointed weak homotopy equivalence (remark 5.4.8(iii)), and then the same holds for $\underline{A} \wedge p_\circ$, by virtue of (i). Thus, the two morphisms of $(*)$ form a cylinder for $\underline{A}$, in the sense of definition 3.2.9(i). Since $\underline{A}$ and $\underline{X}$ are respectively cofibrant and fibrant in $\mathrm{sSet}_\circ$ (remark 5.4.8(i)), combining with lemma 3.2.11(i), proposition 3.2.14(ii), and the explicit construction of the homotopy category in the proof of theorem 3.3.5, we deduce that $\mathrm{ho}(\mathrm{sSet}_\circ)(\underline{A},\underline{X})$ is naturally identified with the set of equivalence classes $[\underline{A},\underline{X}] := \mathrm{sSet}_\circ(\underline{A},\underline{X})/\sim$, for the equivalence relation such that $f \sim g \Leftrightarrow$ there exists a morphism $h : \underline{A} \wedge (\Delta^1)_\circ \to \underline{X}$ with $h \circ (\underline{A} \wedge (\partial_1^1)_\circ) = f$ and $h \circ (\underline{A} \wedge (\partial_0^1)_\circ) = g$. Then the assertion follows, in light of corollary 2.1.17 and the discussion of §2.1.11. □

Corollary 5.4.10. *Let $\underline{X}, \underline{X}'$ be pointed Kan fibrations, and $\underline{A}, \underline{A}' \in \mathrm{Ob}(\mathsf{sSet}_\circ)$. Every pair of pointed weak homotopy equivalences $\underline{X} \to \underline{X}'$, $\underline{A}' \to \underline{A}$ induces an isomorphism of* $\mathsf{Set}_\circ$ *:*

$$\pi_0 \mathscr{H}om_\circ(\underline{A}, \underline{X}) \xrightarrow{\sim} \pi_0 \mathscr{H}om_\circ(\underline{A}', \underline{X}').$$

Proof. This follows immediately from proposition 5.4.9(iii). □

Corollary 5.4.11. *(i) For every $\underline{Y} \in \mathrm{Ob}(\mathsf{sSet}_\circ)$, the adjoint pair $(- \wedge \underline{Y}, \mathscr{H}om_\circ(\underline{Y}, -))$ of proposition 2.1.16(i) is a Quillen adjunction.*

(ii) $\mathscr{H}om_\circ(\underline{Y}, -)$ preserves pointed Kan fibrations and pointed trivial fibrations.

Proof. (i): By proposition 5.4.9(i), we know that the functor $- \wedge \underline{Y}$ preserves pointed weak homotopy equivalences, so we need only check that it also preserves cofibrations, *i.e.* monomorphisms (lemma 3.4.12(i)). However, for every monomorphism $\underline{X} \to \underline{X}'$ of $\mathsf{sSet}_\circ$, it is easily seen that $(\underline{X} \times \underline{Y}) \cap (\underline{X}' \vee \underline{Y}) = \underline{X} \vee \underline{Y}$ (details left to the reader); the assertion follows easily from this identity.

(ii) follows immediately from (i) and lemma 3.4.12(i). □

6. The homotopy theory of ∞-categories

6.1. **Inner anodyne extensions.**

Definition 6.1.1. (i) The class of *inner anodyne extensions* :

$$\mathsf{inAn} \subset \mathrm{Mor}(\mathsf{sSet})$$

is the saturation of the set of monomorphisms $\{\Lambda^n_k \to \Delta^n \mid n \geq 2, k = 1, \dots, n-1\}$.

(ii) An *inner fibration* is a morphism of simplicial sets which has the right lifting property with respect to the class of inner anodyne extensions.

Example 6.1.2. (i) A simplicial set X is an ∞-category if and only if the unique morphism $X \to \Delta^0$ is an inner fibration (see definition 2.6.1(i)). More generally, it is easily seen that if $X \to Y$ is an inner fibration and if Y is an ∞-category, then the same holds for X : the details are left to the reader.

(ii) Let X be an ∞-category, and Y a simplicial set verifying the Grothendieck-Segal condition; it follows easily from proposition 2.5.11 that *every morphism $X \to Y$ of simplicial sets is an inner fibration* (detail left to the reader).

(iii) For every simplicial set X there exists an inner anodyne extension $X \to X'$ such that X' is an ∞-category. Indeed, by corollary 4.2.5(ii,iii), the unique morphism $X \to \Delta^0$ is the composition of an inner anodyne extension $X \to X'$ and an inner fibration $X' \to \Delta^0$, so the assertion follows from (i).

(iv) In light of remark 2.3.8(i), it is easily seen that a morphism $p : X \to Y$ of sSet is an inner fibration if and only if the same holds for $p^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$.

Remark 6.1.3. (i) In the following we shall deal with simplicial sets of the form

$$\Delta^n \times \Delta^r$$

for some $n, r \in \mathbb{N}$, and with their simplicial subsets. Let us then note that $\Delta^n \times \Delta^r$ is the nerve of the partially ordered set $[n] \times [r]$ (where the product is formed in the category poSet : see §2.3.3); hence the m-simplices of $\Delta^n \times \Delta^r$ are the non-decreasing maps

$$(*) \qquad \sigma : [m] \to [n] \times [r] \qquad j \mapsto (a_j, b_j)$$

for every $m \in \mathbb{N}$. Such a map σ *is a non-degenerate simplex if and only if it is injective.* Moreover, σ induces a morphism $N(\sigma) : \Delta^m \to \Delta^n \times \Delta^r$ of simplicial sets, whose image is the smallest simplicial subset Δ_σ of $\Delta^n \times \Delta^r$ containing σ. It is easily seen that the t-simplices of Δ_σ are the non-decreasing maps $[t] \to [n] \times [r]$ *whose image lies in the image of σ*, for every $t \in \mathbb{N}$ (details left to the reader). Clearly, the image of σ is a totally ordered subset of $[n] \times [m]$, so Δ_σ is isomorphic to Δ^p for some $p \leq m$ (and with $p = m$ if and only if σ is non-degenerate). Furthermore, there exists a *unique* isomorphism $\omega : \Delta^p \xrightarrow{\sim} \Delta_\sigma$, so we shall say that a simplicial subset $\Gamma \subset \Delta_\sigma$ is a face (resp. a *horn*, resp. an *inner horn*) of Δ_σ if $\omega^{-1}(\Gamma)$ is a face (resp. a horn, resp. an inner horn of Δ^p : see §2.3.7); likewise, we define the *boundary* $\partial\Delta_\sigma$ of Δ_σ as the unique simplicial subset such that $\omega^{-1}(\partial\Delta_\sigma) = \partial\Delta^p$.

(ii) Recall that for every $m \in \mathbb{N}$ and $0 \leq k \leq n$, the m-simplices of $\partial\Delta^n$ (resp. of Λ^n_k) are the non-decreasing maps $[m] \to [n]$ whose images miss some $j \in [n]$ (resp. some $j \in [n] \setminus \{k\}$); then, the discussion of (i) implies that the m-simplices of $\Lambda^n_k \times \Delta^r \cup \Delta^n \times \partial\Delta^r \subset \Delta^n \times \Delta^r$ are the maps σ as in $(*)$, satisfying either one of the following conditions :

(C0) $j \notin \{a_0, \dots, a_m\}$ for some $j \in [n] \setminus \{k\}$

(C1) $\{b_0, \dots, b_m\} \neq [r]$.

(iii) The discussion of (i) also implies immediately that the simplicial subset Δ_σ *is determined by the (totally ordered) set* $\Delta_{\sigma,0}$ of its objects (see definition 2.5.1(i)), which is naturally identified with the image of σ. By the same token, for every pair of simplices σ, σ' of $\Delta^n \times \Delta^r$, the simplicial subset $\Delta_\sigma \cap \Delta_{\sigma'}$ is again of the form $\Delta_{\sigma''}$ for a simplex σ'' whose set of objects is $\Delta_{\sigma,0} \cap \Delta_{\sigma',0}$; especially, $\Delta_\sigma \cap \Delta_{\sigma'}$ is a face of both Δ_σ and $\Delta_{\sigma'}$.

Proposition 6.1.4. (Joyal) *The following four classes of morphisms of* sSet *are equal :*

(a) the class inAn *of inner anodyne extensions*

(b) the saturation of the subset $\{\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \to \Delta^2 \times \Delta^n \mid n \in \mathbb{N}\}$

(c) the saturation of the subset $\{\Delta^n \times \partial\Delta^r \cup \Lambda^n_k \times \Delta^r \to \Delta^n \times \Delta^r \mid r \in \mathbb{N}, n \geq 2, 0 < k < n\}$

(d) the saturation of the class $\{\Delta^2 \times K \cup \Lambda^2_1 \times L \to \Delta^2 \times L \mid (K \to L) \in \mathscr{M}\}$*, where* $\mathscr{M}$ *denotes the class of monomorphisms of* sSet.

Proof. Obviously the class (b) lies in the class (d). For the converse, recall the functors $F_1 := \Lambda^2_1 \times (-)$ and $F_2 := \Delta^2 \times (-)$ admit right adjoints (remark 1.7.8(ii)), so we may apply proposition 3.1.21(iii) to F_1 and F_2, to the natural transformation $F_1 \Rightarrow F_2$ induced by the inclusion $\Lambda^2_1 \to \Delta^2$, and to the set of monomorphisms $\mathscr{S} := \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$; we get

$$l(r(\mathscr{S}))^\diamond \subset l((r(\mathscr{S}^\diamond)))$$

and notice also that $F_1L \cap F_2K = F_1K$ for every monomorphism $K \to L$, so that

$$j^\diamond = (\Delta^2 \times K \cup \Lambda^2_1 \times L \to \Delta^2 \times L) \qquad \forall (K \to L) \in \mathscr{M}.$$

Hence, $l(r(\mathscr{S}))^\diamond$ is the class (d) (example 4.4.2(ii)) and $l((r(\mathscr{S}^\diamond)))$ is the class (b), which shows that the classes (b) and (d) coincide.

• Next, let us check that the class (a) lies in the class (d). To this aim, for every $n \geq 2$ and $k = 1, \dots, n-1$ we consider the morphisms of $\mathbf{\Delta}$:

$$[n] \xrightarrow{s^n_k} [2] \times [n] \xrightarrow{r^n_k} [n]$$

such that :

$$s^n_k(j) := \begin{cases} (0,j) & \text{if } j < k \\ (1,j) & \text{if } j = k \\ (2,j) & \text{if } j > k \end{cases} \qquad r^n_k(i,j) := \begin{cases} \min(j,k) & \text{if } i = 0 \\ k & \text{if } i = 1 \\ \max(j,k) & \text{if } i = 2. \end{cases}$$

Clearly $r^n_k \circ s^n_k = \mathbf{1}_{[n]}$ for every such n and k, and we denote by the same letters the associated morphisms of sSet :

$$\Delta^n \xrightarrow{s^n_k} \Delta^2 \times \Delta^n \xrightarrow{r^n_k} \Delta^n.$$

Claim 6.1.5. We have a commutative diagram of sSet :

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "s"] \arrow[d] & \Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n \arrow[r, "r"] \arrow[d, "i"] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r, "s^n_k"] & \Delta^2 \times \Delta^n \arrow[r, "r^n_k"] & \Delta^n \end{tikzcd}$$

whose vertical arrows are the natural inclusions.

Proof : Recall that for every $m \in \mathbb{N}$, the m-simplices of Λ^n_k are the non-decreasing maps $\phi : [m] \to [n]$ whose images miss some $j \neq k$, in which case $\mathrm{Im}(s^n_k \circ \phi) \cap ([2] \times \{j\}) = \varnothing$, so $s^n_k \circ \phi$ fulfills condition (C1) of remark 6.1.3(ii), and then the image of the restriction of s^n_k to Λ^n_k lies in $\Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n$, *i.e.* s is well defined.

Next, in order to check that r is well defined, we need to show that for every $\psi : [m] \to [2] \times [n]$ verifying either (C0) or (C1) of remark 6.1.3(ii), the map $r^n_k \circ \psi$ skips some $j \neq k$. Now, if $\mathrm{Im}(\psi) \cap ([2] \times \{j\}) = \varnothing$ for some $j \neq k$, then it is easily seen that $r^n_k \circ \psi$ skips j. Lastly, if $\mathrm{Im}(\psi) \cap (\{2\} \times [n]) = \varnothing$ (resp. if $\mathrm{Im}(\psi) \cap (\{0\} \times [n]) = \varnothing$), then the image of $r^n_k \circ \psi$ lies in $[k]$ (resp. in $[n] \setminus [k-1]$), whence the assertion. ◇

The monomorphism i of claim 6.1.5 lies in the class (c), so the inclusion $\Lambda^n_k \to \Delta^n$ is a retract of an element of the class (d), hence the class (a) lies in the class (c), as stated.

• Lastly, obviously the class (b) lies in the class (c); to conclude, we check that the class (c) lies in the class (a), following the argument of [5, Lemma A.1]; indeed, set $Y := \Delta^n \times \Delta^r$ and $Y^0 := \Lambda^n_k \times \Delta^r \cup \Delta^n \times \partial\Delta^r$. For every simplex σ of Y and every $(a, b) \in [n] \times [r]$, we shall say that σ *meets* (a, b) if (a, b) is an object of the associated simplicial subset Δ_σ (*i.e.* if (a, b) lies in the image of the map σ : see remark 6.1.3(i,iii)). For every $i = 1, \dots, r + 1$ we then define inductively Y^i as the smallest simplicial subset of Y containing Y^{i-1} and all the simplices σ of Y meeting $(k, i-1)$. Notice that for every simplex of Y, either meets (k, j) or is a face of a simplex meeting (k, j) for some $j \in [r]$, so that $Y^{r+1} = Y$.

• We are then reduced to showing that the inclusion $Y^i \to Y^{i+1}$ is an inner anodyne extension for every $i = 0, \dots, r$; to this aim we let $Y^i\langle n-1\rangle := Y^i$, and for every $t = n, \dots, n+r$ we define inductively $Y^i\langle t\rangle$ as the smallest simplicial subset of Y^{i+1} containing $Y^i\langle t-1\rangle$ and every non-degenerate t-simplex of Y meeting (k, i).

Notice that for $m \leq n-1$, every m-simplex of Y meeting (k, i) lies in Y^0, since it satisfies condition (C0) of remark 6.1.3(ii); moreover, for $m > n + r$, every m-simplex of Y is degenerate, and then it is easily seen that every m-simplex of Y^{i+1} lies already in $Y^i\langle t\rangle$ for some $t \leq n + r$; *i.e.* $Y^{i+1} = Y^i\langle n + r\rangle$. Now, for every $t = n, \dots, n + r$, let Σ_t be the set of non-degenerate t-simplices of Y^{i+1} meeting (k, i) and not contained in $Y^i\langle t-1\rangle$. For every $\sigma \in \Sigma_t$, let $\Delta_\sigma \subset Y^i\langle t\rangle$ be the smallest simplicial subset containing σ, and set $\Lambda_\sigma := \Delta_\sigma \cap Y^i\langle t-1\rangle$; by remark 6.1.3(i), Δ_σ is isomorphic to Δ^t. We are further reduced to proving that the inclusion $Y^i\langle t-1\rangle \to Y^i\langle t\rangle$ is an inner anodyne extension for every $t = n, \dots, n+r$. The latter is a consequence of lemma 3.1.8 and the following more precise:

Claim 6.1.6. For every $t = n, \dots, n + r$, the following holds :

(i) Λ_σ is an inner horn of Δ_σ, for every $\sigma \in \Sigma_t$.

(ii) The induced commutative diagram :

$$\begin{tikzcd} \coprod_{\sigma\in\Sigma_t} \Lambda_\sigma \arrow[r] \arrow[d] & Y^i\langle t-1\rangle \arrow[d] \\ \coprod_{\sigma\in\Sigma_t} \Delta_\sigma \arrow[r] & Y^i\langle t\rangle \end{tikzcd}$$

is cartesian and cocartesian in sSet.

Proof : (i): By construction σ is *not* a simplex of $Y^i\langle t-1\rangle$, so $\Lambda_\sigma \subset \partial\Delta_\sigma$ (notation of remark 6.1.3(i)). Moreover, by assumption we have $\sigma(j) = (k, i)$ for some $j \in [n]$; however, if either $j = 0$ or $j = n$, then clearly σ fulfills condition (C0) of remark 6.1.3(ii), so $\sigma \in Y^0$, a contradiction. Let then $l \in [n] \setminus \{j\}$; the l-th $(t-1)$-dimensional face of Δ_σ is the simplicial subset Δ_τ, for the non-decreasing map $\tau : [t-1] \to [n] \times [r]$ whose image is $\sigma([n] \setminus \{l\})$. Hence, τ meets (k, i), so that $\tau \in Y^i\langle t-1\rangle$; this shows that Λ_σ contains the j-th horn of Δ_σ. Lastly, let $\nu : [t-1] \to [n] \times [r]$ be the non-decreasing map whose image is $\sigma([n] \setminus \{j\})$; to conclude, it suffices to check that ν is not a simplex of Λ_σ.

To this aim, say that $\nu(j-1) = \sigma(j-1) = (a,b)$; then $a \leq k$, and in case $a = k$ we must have $b < i$, so that σ is a simplex of Y^i, a contradiction. Hence, $a < k$, and in case $a < k-1$, we see that σ fulfills condition (C0) of remark 6.1.3(ii), so σ would be a simplex of Y^0, again a contradiction. So, finally $a = k-1$; now, if we had $b < i$, then σ would be a face of the $(t+1)$-simplex σ' of Y such that

$$\sigma'(i) := \begin{cases} \sigma(i) & \text{if } i = 0, \dots, j-1 \\ (k,b) & \text{if } i = j \\ \sigma(i-1) & \text{if } i = j+1, \dots, t+1. \end{cases}$$

But σ' is a simplex of $Y^{b+1} \subset Y^i$, and therefore the same would hold for σ, a contradiction.

So, we see that $\nu(j-1) = (k-1, i)$; since σ does not verify condition (C1) of remark 6.1.3(ii), it then follows immediately that neither does ν. Moreover, since σ does not fulfill condition (C0), it is clear that neither does ν, so ν is not a simplex of Y^0.

Now, suppose that ν is a simplex of Y^i; this comes down to asserting that ν is a simplex of a simplicial subset Δ_τ, for some simplex τ of Y that meets some pair (k,l) with $l < i$. But ν cannot be a simplex of any such Δ_τ, since we have neither $(k-1,i) < (k,l)$ nor $(k,l) < (k-1,i)$. So, ν is not a simplex of Y^i. Lastly, suppose that ν is a simplex of $Y^i\langle t-1\rangle$; since ν is not a simplex of Y^i, it follows that ν is a simplex of Δ_τ, for some s-simplex τ that meets (k,i), with $s < t$. But since ν is a non-degenerate $(t-1)$-simplex of Y, it would follow that $s = t-1$ and $\tau = \nu$; but this is absurd, since ν does not meet (k,i). So ν does not lie in $Y^i\langle t-1\rangle$, and the proof is concluded.

(ii): The diagram is cartesian by construction; the cocartesianity assertion comes down to checking the identities :

$$Y^i\langle t\rangle_p = Y^i\langle t-1\rangle_p \sqcup \textstyle\bigsqcup_{\sigma\in\Sigma_t}(\Delta_{\sigma,p} \setminus \Lambda_{\sigma,p}) \qquad \forall p \in \mathbb{N}.$$

Thus, consider a p-simplex τ of $Y^i\langle t\rangle$ that does not lie in $Y^i\langle t-1\rangle$; by construction, τ must be a p-simplex of Δ_σ, for some non-degenerate t-simplex σ that meets (k,i). Moreover, σ cannot lie in $Y^i\langle t-1\rangle$, since $\tau \notin Y^i\langle t-1\rangle_p$, so $\sigma \in \Sigma_t$, and then clearly $\tau \in \Delta_{\sigma,p} \setminus \Lambda_{\sigma,p}$. We still need to check that σ is the *unique* element of Σ_t such that $\tau \in \Delta_{\sigma,p}$; so, suppose that $\tau \in \Delta_{\sigma',p}$ for some other $\sigma' \in \Sigma_t$. Then τ is a simplex of $\Delta_\sigma \cap \Delta_{\sigma'}$, and the latter is of the form $\Delta_{\sigma''}$ for some s-simplex σ'' of Y, where $s < t$ (see remark 6.1.3(iii)); moreover, since both σ and σ' meet (k,i), the same holds for σ'', and then $\Delta_{\sigma''} \subset Y^i\langle s\rangle \subset Y^i\langle t-1\rangle$, so $\tau \in Y^i\langle t-1\rangle_p$, a contradiction. □

Corollary 6.1.7. *(i) For every inner anodyne extension $K \to L$ and every monomorphism $U \to V$, the induced morphism $K \times V \cup L \times U \to L \times V$ is an inner anodyne extension.*

(ii) For every simplicial set X, the functor $(-) \times X$: sSet $\to$ sSet preserves inner anodyne extensions.

Proof. (i): Let us consider first the case where $K \to L$ is the inclusion $\Lambda^n_k \to \Delta^n$ for a given $n \in \mathbb{N}$ and $0 < k < n$. To this aim, notice that the functors $F^n_k := \Lambda^n_k \times (-)$ and $G^n := \Delta^n \times (-)$ admit right adjoints (remark 1.7.8(ii)), so we may apply proposition 3.1.21(iii) to F^n_k and G^n, to the natural transformation $F^n_k \Rightarrow G^n$ the induced by the inclusion $\Lambda^n_k \to \Delta^n$, and to the cellular model $\mathscr{S} := \{\partial\Delta^r \to \Delta^r \mid r \in \mathbb{N}\}$; we get :

$$\mathscr{M}^\diamond \subset l(r(\mathscr{S}^\diamond))$$

where $\mathscr{M}$ is the class of monomorphisms of sSet. Notice also that $F^n_k Y \cap G^n X = F^n_k X$ for every monomorphism $i : X \to Y$ of sSet, whence :

$$i^\diamond = (\Lambda^n_k \times Y \cup \Delta^n \times X \to \Delta^n \times Y).$$

It then suffices to notice that $\mathscr{S}^\diamond \subset \mathsf{inAn}$, by proposition 6.1.4.

Next, we fix a monomorphism $i : U \to V$ of sSet, and we apply again proposition 3.1.21(iii) to the functors $F := (-) \times U$ and $G := (-) \times V$, to the natural transformation $F \Rightarrow G$ induced by i, and to the set $\mathscr{I} := \{\Lambda^n_k \to \Delta^n \mid n \geq 2, k = 1, \dots, n-1\}$; we get :

$$\mathsf{inAn}^\diamond \subset l(r(\mathscr{T}^\diamond)).$$

Again, we have $FL \cap GK = FK$ for every monomorphism $K \to L$, so we are reduced to checking that $\mathscr{T}^\diamond \subset \mathsf{inAn}$. The latter is already known, by the foregoing case.

(ii): It suffices to apply (i) to $U := \varnothing$ and $V := X$. □

Corollary 6.1.8. *Let $p : X \to Y$ be a morphism of* sSet*. The following conditions are equivalent :*

(a) p is an inner fibration.

(b) Every inner anodyne extension $i : K \to L$ induces a trivial fibration of sSet *:*

$$\mathscr{H}om(L, X) \to \mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y).$$

(c) The restriction along the inclusion $\Lambda^2_1 \to \Delta^2$ induces a trivial fibration of sSet *:*

$$\mathscr{H}om(\Delta^2, X) \to \mathscr{H}om(\Lambda^2_1, X) \times_{\mathscr{H}om(\Lambda^2_1,Y)} \mathscr{H}om(\Delta^2, Y).$$

(d) Every monomorphism $i : K \to L$ induces an inner fibration :

$$(i^*, p_*) : \mathscr{H}om(L, X) \to \mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y).$$

Proof. Assertions (b) and (c) refer of course to the trivial fibrations for the Kan-Quillen model category structure. For the proof, one argues as in the proof of corollary 5.1.12(i), except that instead of invoking proposition 5.1.11, one applies proposition 6.1.4 and corollary 6.1.7 : the details shall be left to the reader. □

Corollary 6.1.9. *A simplicial set X is an ∞-category if and only if the inclusion $\Lambda^2_1 \to \Delta^2$ induces a trivial fibration :*

$$\mathscr{H}om(\Delta^2, X) \to \mathscr{H}om(\Lambda^2_1, X).$$

Proof. Again, we refer here to the trivial fibrations for the Kan-Quillen model category. In view of example 6.1.2(i), the assertion is then a special case of corollary 6.1.8. □

With the terminology of definition 2.5.1, the following corollary can be interpreted as stating, in particular, that *the functors between two ∞-categories form an ∞-category.*

Corollary 6.1.10. *For every ∞-category X and every simplicial set A, the simplicial set $\mathscr{H}om(A, X)$ is an ∞-category.*

Proof. It suffices to apply condition (d) of corollary 6.1.8 to the monomorphism $\varnothing \to A$ and the inner fibration $X \to \Delta^0$ (see example 6.1.2(i)), and notice that $\mathscr{H}om(\varnothing, X) = \mathscr{H}om(\varnothing, \Delta^0) = \mathscr{H}om(A, \Delta^0) = \Delta^0$. □

Proposition 6.1.11. *(i) The left adjoint τ :* sSet $\to$ Cat *of the nerve functor sends inner anodyne extensions to isomorphisms of categories (see §2.3.3).*

(ii) The functor τ preserves finite products.

Proof. (i): Let $\mathscr{F}$ be the class of morphisms f of sSet such that $\tau(f)$ is an isomorphism of Cat; since the functor τ commutes with all representable colimits ([13, Prop.2.49(ii)]), $\mathscr{F}$ is saturated (remark 3.1.4(iii) and lemma 3.1.5), hence it suffices to check that the inner horn inclusion $j^n_k : \Lambda^n_k \to \Delta^n$ lies in $\mathscr{F}$, for every $n \geq 2$ and every $k = 1, \dots, n$. For the latter, by Yoneda's lemma (theorem 1.6.5(iii)) it suffices to show that composition with $\tau(j^n_k)$ induces a bijection $\mathsf{Cat}(\tau\Delta^n, \mathscr{C}) \xrightarrow{\sim} \mathsf{Cat}(\tau\Lambda^n_k, \mathscr{C})$, for every small category $\mathscr{C}$. By adjunction, we are further reduced to checking that composition with j^n_k induces a bijection $\mathsf{sSet}(\Delta^n, N\mathscr{C}) \xrightarrow{\sim} \mathsf{sSet}(\Lambda^n_k, N\mathscr{C})$ for every such $\mathscr{C}$; the latter follows from propositions 2.5.10 and 2.5.11.

(ii): Let $X, Y \in \mathrm{Ob}(\mathsf{sSet})$, and pick inner anodyne extensions $X \to X'$ and $Y \to Y'$ such that X' and Y' are ∞-categories (example 6.1.2(iii)). We see from corollary 6.1.7(ii) that the induced morphism $X \times Y \to X' \times Y'$ is again an inner anodyne extension, and by (i), the induced functors $\tau X \to \tau X'$, $\tau Y \to \tau Y'$ and $\tau(X \times Y) \to \tau(X' \times Y')$ are then isomorphisms of categories. On the other hand, the canonical projections $X' \leftarrow X' \times Y' \to Y'$ induce an isomorphism $\tau(X' \times Y') \xrightarrow{\sim} \tau X' \times \tau Y'$ (remark 2.6.14(ii)), whence the assertion. □

6.2. **The Joyal model category structure.** Their name notwithstanding, inner anodyne extensions do *not* form a class of I-anodyne extensions for some exact cylinder I. In order to fit into the general framework of §4.5, our next task is to exhibit a suitable exact cylinder I on sSet, so that we may then consider the smallest class of I-anodyne extensions containing all inner anodyne extensions; this is achieved by the following :

Definition 6.2.1. (i) Let $f : 0 \to 1$ be the unique non-degenerate simplex of Δ^1 (given by the identity map of $[1]$). With the notation of §2.5.13, we define

$$J := \Delta^1[f^{-1}]$$

and let $\pi^J : J \to \Delta^0$ be the unique morphism of sSet. The localization $\mu : \Delta^1 \to J$ is a monomorphism, as it is an anodyne extension (see example 5.1.3); hence the same holds for the composition

$$(\partial^J_0, \partial^J_1) : \Delta^0 \sqcup \Delta^0 \xrightarrow{(\partial_0, \partial_1)} \Delta^1 \xrightarrow{\mu} J$$

and then the datum $(J, \partial^J_0, \partial^J_1, \pi^J)$ is a cylinder of the final object Δ^0 of sSet. By example 4.4.5(i), this datum induces an exact cartesian cylinder $(I, \partial^I_0, \partial^I_1, \pi^I)$ on sSet, where I is given by the functor $J \times (-) : \mathsf{sSet} \to \mathsf{sSet}$.

(ii) Just as in §5.1, we shall denote by $\{\varepsilon\} \subset J$ the image of $\partial^J_{1-\varepsilon}$, for $\varepsilon = 0, 1$, and $\{\varepsilon\} \times X \subset J \times X$ shall denote the image of $\partial^J_{1-\varepsilon} \times X$, for every $X \in \mathrm{Ob}(\mathsf{sSet})$.

(iii) The class of *categorical anodyne extensions*

$$\mathsf{cAn} \subset \mathrm{Mor}(\mathsf{sSet})$$

is the smallest class of J-anodyne extensions containing inAn (see definition 6.1.1(i)).

(iv) A *weak categorical equivalence* in sSet is a $J \times (-)$-weak equivalence (see definition 4.5.4(iv)). A *J-fibration* is a morphism of sSet that lies in $r(\mathsf{cAn})$. A simplicial set X is *J-fibrant* if the unique morphism $X \to \Delta^0$ is a J-fibration.

Remark 6.2.2. (i) Clearly, a 1-simplex $\Delta^1 \to X$ is an invertible arrow of X if and only if it factors through the localization $\Delta^1 \to J$ and a morphism $J \to X$.

(ii) It follows easily from (i) that a morphism of sSet is conservative (see definition 2.5.4(iv)) if and only if it has the right lifting property relative to the localization $\Delta^1 \to J$. Especially, *every trivial fibration (for the Kan-Quillen model structure) is conservative.*

(iii) By proposition 2.5.15, the category τJ is the localization of $[1]$ that inverts $\overrightarrow{01}$, *i.e.* $\mathrm{Ob}(\tau J) = \{0,1\}$ and $\mathrm{Mor}(\tau J) = \{\mathbf{1}_0, \mathbf{1}_1, \overrightarrow{01} : 0 \to 1, \overrightarrow{10} : 1 \to 0\}$ (so $\overrightarrow{10}$ is the inverse of $\overrightarrow{01}$).

Proposition 6.2.3. *(i) Let* $M := \{J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to J \times \Delta^n \mid n \in \mathbb{N}, \varepsilon = 0,1\}$ *and* $S := \{\Lambda^n_k \to \Delta^n \mid n \geq 2, 0 < k < n\}$. *Then :*

$$\boxed{\mathsf{cAn} = l(r(M \cup S)).}$$

(ii) For every categorical anodyne extension $(K \to L)$ *and every monomorphism* $U \to V$, *the induced morphism* $K \times V \cup L \times U \to L \times V$ *is a categorical anodyne extension.*

(iii) $\mathsf{inAn} \subset \mathsf{cAn} \subset \mathsf{sAn}$.

(iv) The classes of inner fibrations and of J*-fibrations are stable under small filtered colimits. Especially, the class of* ∞*-categories is stable under small filtered colimits in* sSet.

Proof. (i): Set as well $\mathscr{M} := \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$; with the notation of §4.4.8, we then have $M \cup S = \Lambda^0_I(S, \mathscr{M})$, and in view of proposition 4.4.9, we are reduced to checking that $\mathsf{An}_I(S) = l(r(\Lambda^0_I(S,\mathscr{M})))$, where $(I, \partial^I_0, \partial^I_1, \pi^I)$ is the cartesian cylinder induced by $(J, \partial^J_0, \partial^J_1, \pi^J)$. By corollary 4.4.10, it then suffices to show that $(IK \cup \partial I \otimes L \to IL) \in l(r(\Lambda^0_I(S,\mathscr{M})))$ for every $(K \to L) \in S$. But since $\mathsf{inAn} = l(r(S)) \subset l(r(\Lambda^0_I(S,\mathscr{M})))$, the latter follows from corollary 6.1.7(i).

(ii): We consider first the case where $U = \partial\Delta^r$ and $V = \Delta^r$ for some $r \in \mathbb{N}$; since the functors $(-) \times \partial\Delta^r, (-) \times \Delta^r : \mathsf{sSet} \rightrightarrows \mathsf{sSet}$ admit right adjoints (remark 1.7.8(ii)), we may apply proposition 3.1.21(iii) to the natural transformation $\partial\Delta^r \times (-) \Rightarrow \Delta^r \times (-)$ induced by the inclusion $\partial\Delta^r \to \Delta^r$, and to the set $\mathscr{S} := M \cup S$. Notice also that $(K \times \Delta^r) \cap (L \times \partial\Delta^r) = K \times \partial\Delta^r$ for every monomorphism $j : K \to L$, so that $j^\diamond$ is the induced monomorphism $K \times \Delta^r \cup L \times \partial\Delta^r \to L \times \Delta^r$, for every such j; in light of (i), we get :

$$\mathsf{cAn}^\diamond \subset l(r(M^\diamond \cup S^\diamond)).$$

However, every element of $S^\diamond$ is an inner anodyne extension, by virtue of corollary 6.1.7(i), so we are reduced to checking that $M^\diamond \subset \mathsf{cAn}$. However, if j is the inclusion $J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to J \times \Delta^n$ for some $n \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$, then $j^\diamond$ is the inclusion :

$$J \times (\Delta^r \times \partial\Delta^n \cup \partial\Delta^r \times \Delta^n) \cup \{\varepsilon\} \times (\Delta^r \times \Delta^n) \to J \times (\Delta^r \times \Delta^n)$$

which is indeed a categorical anodyne extension, as required.

Next, we fix a categorical anodyne extension $K \to L$ and we apply proposition 3.1.21(iii) to the induced natural transformation $K \times (-) \Rightarrow L \times (-)$, and with $\mathscr{S} := \mathscr{M}$; recalling that $l(r(\mathscr{M}))$ is the class $\mathscr{I}$ of monomorphisms of sSet (remark 5.1.1), we then get :

$$\mathscr{I}^\diamond \subset l(r(\mathscr{M}^\diamond)).$$

But we have just seen that $\mathscr{M}^\diamond \subset \mathsf{cAn}$, so finally $\mathscr{I}^\diamond \subset \mathsf{cAn}$, as stated.

(iii): The first inclusion holds by definition; for the second inclusion, in light of (i) it suffices to check that $M \subset \mathsf{sAn}$. By virtue of proposition 5.1.11(i), we are then reduced to checking that the inclusion $j_\varepsilon : \{\varepsilon\} \to J$ is an anodyne extension, for $\varepsilon = 0,1$; however, j_ε is the composition of the anodyne extension $d^{1-\varepsilon}_1 : \Delta^0 \to \Delta^1$ with the localization $\Delta^1 \to J$, and we have already observed that the latter is an anodyne extension as well, whence the contention.

(iv) follows directly from (i) and example 3.1.14(iv). $\square$

Corollary 6.2.4. *A morphism $p : X \to Y$ of* sSet *is a J-fibration if and only if the following two conditions hold :*

(a) *For $\varepsilon = 0, 1$, the morphism $\partial^J_\varepsilon : \Delta^0 \to J$ induces a trivial fibration of* sSet *:*

$$(\partial^{J*}_\varepsilon, p_*) : \mathscr{H}om(J, X) \to X \times_Y \mathscr{H}om(J, Y)$$

(b) *The inclusion $j : \Lambda^2_1 \to \Delta^2$ induces a trivial fibration of* sSet *:*

$$(j^*, p_*) : \mathscr{H}om(\Delta^2, X) \to \mathscr{H}om(\Lambda^2_1, X) \times_{\mathscr{H}om(\Lambda^2_1, Y)} \mathscr{H}om(\Delta^2, Y).$$

Proof. As usual, (a) and (b) refer to the trivial fibrations for the Kan-Quillen model category. Let M and S be as in proposition 6.2.3(i); since $r(\mathsf{cAn}) = r(l(r(M \cup S))) = r(M \cup S)$ (proposition 3.1.9(iii)), we see that p is a J-fibration if and only if it has the right lifting property with respect to M and to the class of inner anodyne extensions. However, arguing as in the proof of corollary 5.1.12(i) we see that $p \in r(M) \Leftrightarrow$ (a) holds, and on the other hand we know already that $p \in r(\mathsf{inAn}) \Leftrightarrow$ (b) holds (corollary 6.1.8). □

Theorem 6.2.5. *There exists a unique cofibrantly generated model category structure on* sSet *whose weak equivalences are the weak categorical equivalences and whose cofibrations are the monomorphisms. Every fibration for this model structure is a J-fibration, the fibrant objects are precisely the J-fibrant ones, and the fibrations with fibrant target are precisely the J-fibrations between J-fibrant objects. Moreover, every object is cofibrant, and every categorical anodyne extension is a trivial cofibration.*

We call this model category the Joyal model category structure on sSet.

Proof. Just as for theorem 5.1.10, this follows directly from theorem 4.5.14, applied to the exact cylinder $J \times (-)$ on sSet, and the class of categorical anodyne extensions of sSet. □

Proposition 6.2.6. *The nerve functor N and its left adjoint τ form a Quillen adjunction*

$$\boxed{\tau : \mathsf{sSet} \rightleftarrows \mathsf{Cat} : N}$$

for the Joyal model category structure on sSet *and the canonical model category structure on* Cat *(theorem 4.3.9).*

Proof. It follows easily from lemma 2.3.4 that τ preserves cofibrations. Next, define the subsets S and M of Mor(sSet) as in proposition 6.2.3(i); by virtue of proposition 4.5.17, we are reduced to checking that τ maps all the elements of $M \cup S$ to equivalences of categories. However, this is already known for the elements of S, by proposition 6.1.11(i). Hence, for $\varepsilon = 0, 1$ and every $n \in \mathbb{N}$, consider the inclusion $j_{\varepsilon,n} : J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \to J \times \Delta^n$, and notice that we have a cocartesian diagram of sSet :

$$\mathscr{D}_{\varepsilon,n} \quad : \quad \begin{tikzcd} \{\varepsilon\} \times \partial\Delta^n \arrow[r] \arrow[d] & \{\varepsilon\} \times \Delta^n \arrow[d] \\ J \times \partial\Delta^n \arrow[r] & J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n. \end{tikzcd}$$

Since τ preserves all representable colimits ([13, Prop.2.49(ii)]) and finite products (proposition 6.1.11(ii)), we deduce a cocartesian diagram of Cat :

$$\tau\mathscr{D}_{\varepsilon,n} \quad : \quad \begin{tikzcd} \tau\{\varepsilon\} \times \tau(\partial\Delta^n) \arrow[r] \arrow[d] & \tau\{\varepsilon\} \times [n] \arrow[d] \\ \tau J \times \tau(\partial\Delta^n) \arrow[r] & \tau(J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n) \end{tikzcd}$$

(recall that $\tau\Delta^n = [n]$, by §2.3.3). Now, if $n \geq 3$, the inclusion $\partial\Delta^n \to \Delta^n$ induces an isomorphism $\tau(\partial\Delta^n) \xrightarrow{\sim} [n]$ (proposition 2.5.8), so the top horizontal arrow of $\tau\mathscr{D}_{\varepsilon,n}$ is an isomorphism; then the same holds as well for the bottom horizontal arrow, and by the same token, the same also holds for the functor obtained by applying τ to the inclusion $J \times \partial\Delta^n \to J \times \Delta^n$. We conclude that $\tau(j_{\varepsilon,n})$ is an isomorphism of categories if $n \geq 3$.

Next, since $\partial\Delta^0 = \varnothing$ we see that if $n = 0$, the left vertical arrow of $\mathscr{D}_{\varepsilon,0}$ is the unique isomorphism $\varnothing \xrightarrow{\sim} \varnothing$, so the right vertical arrow is an isomorphism as well; hence, the same holds for the right vertical arrow of $\tau\mathscr{D}_{\varepsilon,0}$. Moreover, the inclusion $i : \{\varepsilon\} \times \Delta^0 \to J \times \Delta^0$ induces an equivalence of categories $\tau(i) : \tau(\{\varepsilon\}) \times [0] \xrightarrow{\sim} \tau J \times [0]$ (see remark 6.2.2(iii)), so finally $\tau(j_{\varepsilon,0})$ is an equivalence for every $\varepsilon \in \{0, 1\}$.

If $n = 1$, the diagram $\tau\mathscr{D}_{\varepsilon,1}$ becomes :

$$\begin{tikzcd}
\tau\{\varepsilon\} \times ([0] \sqcup [0]) \arrow[r] \arrow[d] & \tau\{\varepsilon\} \times [1] \arrow[d] \\
\tau J \times ([0] \sqcup [0]) \arrow[r] & \tau(J \times \partial\Delta^1 \cup \{\varepsilon\} \times \Delta^1).
\end{tikzcd}$$

Hence, let $0, 1, 0', 1'$ be the four objects of $\tau J \times [1]$; its subcategory $\tau J \times ([0] \sqcup [0])$ consists of the isomorphisms $f_0 : 0 \xrightarrow{\sim} 0'$, $f_1 : 1 \xrightarrow{\sim} 1'$ and their inverses (and of course, the identities of the four objects). Likewise, the subcategory $\tau\{0\} \times [1]$ (resp. $\tau\{1\} \times [1]$) consists of the morphism $\overrightarrow{01} : 0 \to 1$ (resp. $\overrightarrow{0'1'} : 0' \to 1'$) and the identities of the two objects 0 and 1 (resp. $0'$ and $1'$). Then, the datum of a functor $F : \tau(J \times \partial\Delta^1 \cup \{0\} \times \Delta^1) \to \mathscr{C}$ is the same as that of four objects a, b, a', b' of $\mathscr{C}$, of isomorphisms $a \xrightarrow{\sim} a'$, $b \xrightarrow{\sim} b'$, and of a morphism $a \to b$. But it is easily seen that the same datum determines a unique functor $G : \tau J \times [1] \to \mathscr{C}$, such that $F = G \circ \tau(j_{0,1})$; this shows that $\tau(j_{0,1})$ is an isomorphism of categories, and the same argument applies to $\tau(j_{1,1})$ as well.

Lastly, we consider the case where $n = 2$. To this aim, let $i : \partial\Delta^2 \to \Delta^2$ be the inclusion, and notice the cocartesian diagram of sSet :

$$\begin{tikzcd}
\partial\Delta^{\{0,2\}} \arrow[r] \arrow[d] & \Delta^{\{0,2\}} \arrow[d] \\
\Lambda^2_1 \arrow[r] & \partial\Delta^2.
\end{tikzcd}$$

After applying termwise the functor τ, we obtain a cocartesian diagram of Cat, and recall that $\tau\Lambda^2_1 = [2]$ (remark 2.5.12). Hence, the datum of a functor $F : \tau(\partial\Delta^2) \to \mathscr{C}$ amounts to that of three objects a, b, c of $\mathscr{C}$ and three morphisms $\alpha : a \to b$, $\beta : b \to c$, $\gamma : a \to c$, and F factors through $\tau(i) : \tau(\partial\Delta^2) \to [2]$ if and only if $\gamma = \beta \circ \alpha$. Likewise, the datum of a functor $G : \tau J \times \tau(\partial\Delta^2) \to \mathscr{C}$ is the same as that of six objects a, b, c, a', b', c' of $\mathscr{C}$, of isomorphisms $a \xrightarrow{\sim} a'$, $b \xrightarrow{\sim} b'$, $c \xrightarrow{\sim} c'$, and of morphisms α, β, γ as in the foregoing; then G factors through $\tau(j_{0,2})$ if and only if $\gamma = \beta \circ \alpha$. On the other hand, the cocartesianity of $\tau\mathscr{D}_{0,2}$ shows that the datum of a functor $H : \tau(J \times \partial\Delta^2 \cup \{0\} \times [2]) \to \mathscr{C}$ is precisely equivalent to that of a functor $\tau J \times \tau(\partial\Delta^2) \to \mathscr{C}$ whose restriction to $\tau\{0\} \times \tau(\partial\Delta^2)$ factors through $\tau\{0\} \times [2]$, *i.e.* with $\gamma = \beta \circ \alpha$; hence every such functor H factors uniquely through $\tau(j_{0,2})$, *i.e.* $\tau(j_{0,2})$ is an isomorphism of categories, and the same argument applies to $\tau(j_{1,2})$ as well. □

Recall that the fibrations of the canonical model category structure on Cat are the isofibrations between small categories. Proposition 6.2.6 then motivates the following :

Definition 6.2.7. A morphism $p : X \to Y$ of simplicial sets is an *isofibration*, if it is an inner fibration, and for every object x_0 in X and every invertible arrow $g : p(x_0) \to y_1$ in Y, there exists an invertible arrow $f : x_0 \to x_1$ in X such that $p(f) = g$.

Remark 6.2.8. (i) In light of example 6.1.2(ii) and proposition 2.5.10, it is easily seen that a functor $F : \mathscr{C} \to \mathscr{D}$ between small categories is an isofibration if and only if its nerve $N(F) : N\mathscr{C} \to N\mathscr{D}$ is an isofibration of sSet.

(ii) Set $S := \{\Lambda^n_k \to \Delta^n \mid n \in \mathbb{N}, 0 < k < n\}$; then the class of isofibrations of sSet is precisely $r(S \cup \{\partial^J_1\})$ (notation of definition 6.2.1(i,ii)). Taking into account propositions 6.2.3(i) and 3.1.9(iii), we deduce that *every J-fibration is an isofibration of* sSet.

(iii) In the same vein, let Y be an ∞-groupoid; then an inner fibration $X \to Y$ is an isofibration if and only if it has the right lifting property relative to the inclusion $\Lambda^1_0 \to \Delta^1$.

Proposition 6.2.9. *A morphism $p : X \to Y$ of ∞-categories is an isofibration if and only if the same holds for the morphism $p^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$.*

Proof. We know already that the opposite of an ∞-category is an ∞-category (remark 2.6.4), and that p is an inner fibration if and only if the same holds for p^{op} (example 6.1.2(iv)). Hence, we may suppose that p is an inner fibration, and that for every object x_1 in X and every invertible arrow $g : y_0 \to p(x_1)$ in Y, there exists an invertible arrow $f : x_0 \to x_1$ in X such that $p(f) = g$, and we need to deduce that p is an isofibration.

So, let x_0 be an object of X, and $g : p(x_0) \to y_1$ an invertible arrow of Y; pick a left inverse $g' : y_1 \to p(x_0)$ of g. By assumption, there exists an invertible arrow $f' : x_1 \to x_0$ of X such that $p(f') = g'$; pick a right inverse $h : x_0 \to x_1$ of f'. Then, by corollary 2.6.13(ii), the classes of g and $p(h)$ coincide in the homotopy category $\mathsf{ho}(Y)$ (see theorem 2.6.12(i)), and therefore we have a 2-simplex $c : \Delta^2 \to Y$ whose boundary is the commuting triangle :

$$\begin{tikzcd} & y_1 \arrow[dr, "\mathbf{1}_{y_1}"] & \\ p(x_0) \arrow[ur, "p(h)"] \arrow[rr, "g"] & & y_1. \end{tikzcd}$$

Moreover, the composable pair $x_0 \xrightarrow{h} x_1 \xrightarrow{\mathbf{1}_{x_1}} x_1$ defines a morphism $b : \Lambda^2_1 \to X$ fitting into a commutative square :

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "b"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "c"] & Y. \end{tikzcd}$$

Since p is an inner fibration, the square admits a diagonal filler $a : \Delta^2 \to X$; then $f := a \circ \partial^2_1 : \Delta^1 \to X$ is an arrow $x_0 \to x_1$ with $p(f) = g$. By construction we have $[f] = [h]$ in $\mathsf{ho}(X)$, and $[h]$ is an isomorphism in $\mathsf{ho}(X)$, since it is a right inverse of the isomorphism $[f']$ (corollary 2.6.13(ii)); so $[f]$ is an isomorphism of $\mathsf{ho}(X)$, *i.e.* f is invertible in X. □

6.3. **Left fibrations and right fibrations.**

Definition 6.3.1. (i) The class of *left* (resp. *right*) *anodyne extensions* is the saturation in sSet of the set $\{\Lambda^n_k \to \Delta^n \mid n \geq 1, 0 \leq k < n\}$ (resp. of $\{\Lambda^n_k \to \Delta^n \mid n \geq 1, 0 < k \leq n\}$). We denote this class by lAn (resp. rAn).

(ii) A left (resp. right) fibration is a morphism of sSet that has the right lifting property with respect to the class of left (resp. right) anodyne extensions.

Remark 6.3.2. In light of remark 2.3.8(i), it is clear that a morphism f of sSet is a left fibration if and only if its front-to-back dual $f^{\rm op}$ is a right fibration. Moreover, obviously f is a Kan fibration if and only if it is both a left and a right fibration.

Proposition 6.3.3. *Every monomorphism $K \to L$ and every left (resp. right) anodyne extension $X \to Y$ induce a left (resp. right) anodyne extension :*

$$(*) \qquad\qquad Y \times K \cup X \times L \to Y \times L.$$

Proof. Since front-to-back duality preserves monomorphisms, remark 6.3.2 reduces to checking the assertion for the case of a left anodyne extension $X \to Y$. Now, the proof of proposition 5.1.4 already shows that the assertion holds for every monomorphism $K \to L$ and for every element $X \to Y$ of $\mathscr{S} := \{\Lambda^n_k \to \Delta^n \,|\, n \geq 1, 0 \leq k < n\}$. On the other hand, in light of remark 1.7.8(ii), we may apply proposition 3.1.21(iii) to the functors $F_1, F_2 :$ sSet $\to$ sSet with $F_1 := (-) \times K$ and $F_2 := (-) \times L$, and to the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by the inclusion $K \to L$, and notice that $F_1 Y \cap F_2 X = F_1 X$ for every monomorphism $f : X \to Y$, so that $f^\diamond$ is precisely the inclusion $(*)$. We then get :

$$l(r(\mathscr{S}))^\diamond \subset l((r(\mathscr{S}^\diamond))).$$

However, $l(r(\mathscr{S}^\diamond))$ is the saturation of $\mathscr{S}^\diamond$ (corollary 4.1.9(iii)), and since $\mathscr{S}^\diamond$ consists of left anodyne extensions, we deduce that the same holds for $l(r(\mathscr{S}))^\diamond$; but $l(r(\mathscr{S}))$ is precisely the class of left anodyne extensions, whence the assertion. □

Corollary 6.3.4. *Let $p : X \to Y$ be a morphism of* sSet. *We have the equivalent conditions:*

(a) p is a left (resp. right) fibration.

(b) Every left (resp. right) anodyne extension $i : K \to L$ induces a trivial fibration :

$$\mathscr{H}om(L, X) \to \mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y).$$

(c) The restriction along $\Delta^0 \xrightarrow{\partial^1_0} \Delta^1$ (resp. $\Delta^0 \xrightarrow{\partial^1_1} \Delta^1$) induces a trivial fibration of sSet *:*

$$\mathscr{H}om(\Delta^1, X) \to X \times_Y \mathscr{H}om(\Delta^1, Y).$$

(d) Every monomorphism $i : K \to L$ induces a left (resp. right) fibration :

$$\mathscr{H}om(L, X) \to \mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y).$$

Proof. One argues as in the proof of corollary 5.1.12(i), invoking proposition 6.3.3 and proposition 5.1.4 : the details are left to the reader. □

Proposition 6.3.5. *Let $p : X \to Y$ be a morphism of ∞-categories that is either a left or right fibration. Then p is a conservative isofibration (see definitions 2.5.4(iv) and 6.2.7).*

Proof. Clearly a morphism of simplicial sets is conservative if and only if the same holds for its front-to-back dual; combining with proposition 6.2.9 and remark 6.3.2, we are then reduced to the case where p is a left fibration.

Let us check first that p is conservative. Indeed, let $f : x_0 \to x_1$ be an arrow of X such that $p(f) : p(x_0) \to p(x_1)$ is invertible in Y, and pick a 2-simplex $c : \Delta^2 \to Y$ whose boundary is the commuting triangle

$$\begin{tikzcd} & p(x_1) \arrow[dr, "g"] & \\ p(x_0) \arrow[ur, "p(f)"] \arrow[rr, "\mathbf{1}_{p(x_0)}"] & & p(x_0). \end{tikzcd}$$

Let also $b : \Lambda^2_0 \to X$ be the morphism of sSet whose restriction to $\Delta^{\{0,1\}}$ and $\Delta^{\{0,2\}}$ are respectively f and $\mathbf{1}_{x_0} : x_0 \to x_0$. By assumption, the commutative square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "b"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "c"] & Y \end{tikzcd}$$

admits a diagonal filler $\Delta^2 \to X$, whose boundary is a a commuting triangle :

$$\begin{tikzcd} & x_1 \arrow[dr, "h"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "\mathbf{1}_{x_0}"] & & x_0 \end{tikzcd}$$

Since $p(f)$ is invertible in the ∞-category Y, the same holds for $p(h)$, by virtue of corollary 2.6.13(ii). Thus, we can repeat the foregoing discussion with f replaced by h, to obtain another commuting triangle :

$$\begin{tikzcd} & x_0 \arrow[dr, "k"] & \\ x_1 \arrow[ur, "h"] \arrow[rr, "\mathbf{1}_{x_1}"] & & x_1 \end{tikzcd}$$

and we deduce that h is invertible. Since X is an ∞-category, we may then invoke again corollary 2.6.13(ii), to conclude that f is invertible, as required.

Lastly, since p is conservative and has the right lifting property with respect to the inclusion $\partial^1_1 : \Delta^0 \to \Delta^1$, it is easily seen that p is an isofibration. □

6.3.6. Recall that to every $K \in \mathrm{Ob}(\mathsf{sSet})$ we have attached in §2.4.8 an adjoint pair :

$$(- \downarrow K) : \mathsf{sSet} \rightleftarrows K/\mathsf{sSet} : (-/K).$$

Moreover, according to remark 2.4.13(ii,iii), every morphism $i : K \to L$ of sSet induces natural transformations

$$\tau_\bullet : i^! \circ (- \downarrow K) \Rightarrow (- \downarrow L) \qquad \text{and} \qquad \tau^\vee_\bullet : (-/L) \Rightarrow (-/K) \circ i_!$$

where $i^! : K/\mathsf{sSet} \to L/\mathsf{sSet}$ is the left adjoint of the functor $i_! : L/\mathsf{sSet} \to K/\mathsf{sSet}$.

- Let now $i : K \to L$ be a monomorphism of sSet; we apply the discussion of §3.1.20 to the adjoint pairs of functors :

$$i^! \circ (- \downarrow K) : \mathsf{sSet} \rightleftarrows L/\mathsf{sSet} : (-/K) \circ i_! \qquad \text{and} \qquad (- \downarrow L) : \mathsf{sSet} \rightleftarrows L/\mathsf{sSet} : (-/L)$$

and to the natural transformation $\tau_\bullet$. Hence, every morphism $j : U \to V$ of sSet and $L/p : (X,t) \to (Y, p \circ t)$ of L/sSet induces morphisms of L/sSet and respectively sSet :

$$i^!(V \downarrow K) \sqcup_{i^!(U\downarrow K)} (U \downarrow L) \xrightarrow{L/j^\diamond} (V \downarrow L) \qquad (X/t) \xrightarrow{(L/p)_\diamond} (X/t \circ i) \times_{(Y/p\circ t \circ i)} (Y/p \circ t).$$

By inspecting the constructions, we see that $L/j^\diamond$ is given by the commutative diagram

$$\begin{tikzcd} U * L \arrow[d, "e"] & & L \arrow[ll, "U\downarrow L"'] \arrow[d, "V\downarrow L"] \\ (V * K) \sqcup_{(U*K)} (U * L) \arrow[rr, "j^\diamond"] & & V * L. \end{tikzcd}$$

Moreover, by lemma 2.4.10(i), if also j is a monomorphism, $j^\diamond$ is the monomorphism $V * K \cup U * L \to V * L$. Likewise, to ease notation, in this situation we shall usually write

$X/L \to X/K \times_{Y/K} Y/L$ to denote the morphism $(L/p)_\diamond$. By proposition 3.1.21(i,ii), we have then a natural bijection between commutative diagrams of the type :

$$\mathsf{D}^\diamond \ : \ \begin{tikzcd} L \arrow[rrd, bend left, "t"] \arrow[rd, "e\circ(U\downarrow L)"] \arrow[rdd, bend right, "V\downarrow L"'] & & \\ & V*K\cup U*L \arrow[r, "a"] \arrow[d, "j^\diamond"'] & X \arrow[d, "p"] \\ & V*L \arrow[r, "b"] & Y \end{tikzcd} \qquad \text{and} \qquad \mathsf{D}_\diamond \ : \ \begin{tikzcd} U \arrow[r, "a'"] \arrow[d, "j"'] & X/L \arrow[d, "(L/p)_\diamond"] \\ V \arrow[r, "b'"] & X/K \times_{Y/K} Y/L. \end{tikzcd}$$

Furthermore, this bijection extends to a natural bijections between the sets of diagonal fillers $V * L \to X$ and $V \to X/L$ for the corresponding commutative squares.

Remark 6.3.7. The previous natural bijection can also be rephrased as follows. For every monomorphism $j : U \to V$ of sSet and every morphism $p : X \to Y$, we have a natural bijection between commutative squares of the type

$$\mathsf{E}^\diamond \ : \ \begin{tikzcd} V*K\cup U*L \arrow[r, "a"] \arrow[d, "j^\diamond"'] & X \arrow[d, "p"] \\ V*L \arrow[r, "b"] & Y \end{tikzcd} \qquad \text{and pairs} \qquad (\mathsf{E}_\diamond, t) \ : \ \begin{tikzcd} U \arrow[r, "a'"] \arrow[d, "j"'] & X/L \arrow[d, "(L/p)_\diamond"] \\ V \arrow[r, "b'"] & X/K \times_{Y/K} Y/L \end{tikzcd}$$

where $\mathsf{E}_\diamond$ is the commutative square displayed on the right, and $t : L \to X$ is a morphism of sSet; namely, a determines t, so that $\mathsf{E}^\diamond$ determines the pair $(\mathsf{E}_\diamond, t)$, and conversely, from $(\mathsf{E}_\diamond, t)$ we deduce a commutative diagram $\mathsf{D}^\diamond$, from which we can extract $\mathsf{E}^\diamond$. Again, this natural bijection extends to a bijection between the diagonal fillers for $\mathsf{E}^\diamond$ and $\mathsf{E}_\diamond$.

Proposition 6.3.8. *Let $i : K \to L$ and $j : U \to V$ be two monomorphisms of* sSet*, and let $j^\diamond : V * K \cup U * L \to V * L$ be the morphism induced by i and j, as in §6.3.6; we have :*

(i) If $j \in$ sAn *(resp. if $j \in$* rAn*), then $j^\diamond \in$* lAn *(resp. $j^\diamond \in$* inAn*).*

(ii) If $i \in$ sAn *(resp. if $i \in$* lAn*), then $j^\diamond \in$* rAn *(resp. $j^\diamond \in$* inAn*).*

Proof. Let us first remark :

Claim 6.3.9. For every $A \in \mathrm{Ob}(\mathsf{sSet})$ and every subset $\mathscr{S}$ of $\mathrm{Mor}(A/\mathsf{sSet})$, the class $l(r(\mathscr{S}))$ is the saturation of $\mathscr{S}$.

Proof : Every object of A/sSet is small, by remark 4.1.2(ii) and corollary 4.2.5(i); then also every morphism of A/sSet is small, and the claim follows from corollary 4.1.9(iii). ◇

Let us now take $i : \Lambda^m_k \to \Delta^m$ and $j : \partial\Delta^n \to \Delta^n$ to be the natural inclusions, for any $n \in \mathbb{N}$, any $m \geq 1$, and $k = 0, \dots, m$. Then $j^\diamond$ is the inclusion :

$$(*) \qquad \Delta^n * \Lambda^m_k \cup \partial\Delta^n * \Delta^m = \Lambda^{m+n+1}_{n+k+1} \to \Delta^n * \Delta^m = \Delta^{m+n+1}$$

(lemma 2.4.10(ii)). Let us then apply proposition 3.1.21 to the same adjoint pairs as in §6.3.6, and with $\mathscr{S} := \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$. The class $l(r(\mathscr{S}))$ is the class $\mathscr{M}$ of monomorphisms of sSet (remark 5.1.1); on the other hand, from $(*)$ we get $\mathscr{S}^\diamond \subset \Delta^m/\mathsf{rAn}$, and even $\mathscr{S}^\diamond \subset \Delta^m/\mathsf{inAn}$, in case $k < m$, *i.e.* in case $i \in \mathsf{lAn}$. Moreover, both Δ^m/rAn and Δ^m/inAn are saturated subclasses of $\mathrm{Mor}(\Delta^m/\mathsf{sSet})$, by lemma 3.1.12(iv); by combining with claim 6.3.9, we get therefore :

$$\mathscr{M}^\diamond \subset l(r(\mathscr{S}^\diamond)) \subset \Delta^m/\mathsf{rAn} \qquad \text{and even} \qquad \mathscr{M}^\diamond \subset \Delta^m/\mathsf{inAn} \text{ in case } i \in \mathsf{lAn}.$$

Hence, for every monomorphism $U \to V$, the inclusion $V * \Lambda^m_k \cup U * \Delta^m \to V * \Delta^m$ lies in rAn for every $m \in \mathbb{N}$ and $0 \leq k \leq m$, and even in inAn, if $k < m$.

Combining with lemma 2.4.12(ii) and remark 2.3.8(i), we see likewise that for every monomorphism $K \to L$, the inclusion $\Lambda^m_k * L \cup \Delta^m * K \to \Delta^m * L$ lies in lAn for every $m \geq 1$ and $0 \leq k \leq m$, and even in inAn, if $k > 0$.

Let us next take $i : K \to L$ to be an arbitrary monomorphism; we apply again proposition 3.1.21 to the adjoint pairs of §6.3.6, and with $\mathscr{S} := \{\Lambda^m_k \to \Delta^m \mid m \geq 1, k = 0, \dots, m\}$. Then $l(r(\mathscr{S})) = \mathsf{sAn}$ (corollary 5.1.6), and on the other hand, $\mathscr{S}^\diamond \subset L/\mathsf{lAn}$, whence :

$$\mathsf{sAn}^\diamond \subset L/\mathsf{lAn}.$$

Likewise, if we replace $\mathscr{S}$ by $\{\Lambda^m_k \to \Delta^m \mid m \geq 1, k = 1, \dots, m\}$, we get $\mathsf{rAn}^\diamond \subset L/\mathsf{inAn}$, and this completes the proof of (i). Assertion (ii) follows from (i), in light of lemma 2.4.12(ii). □

Corollary 6.3.10. *Let $i : K \to L$ be a monomorphism of* sSet, *(X, t) an object of $L/$*sSet, *$p : X \to Y$ another morphisms of* sSet, *and $(L/p)_\diamond : X/L \to X/K \times_{Y/K} Y/L$ the morphism induced by p and i, as in §6.3.6. We have :*

(i) If p is a left (resp. inner) fibration, then $(L/p)_\diamond$ is a Kan (resp. a right) fibration.

(ii) $(L/p)_\diamond$ is a trivial fibration if either one of the following two conditions holds :

(a) i is an anodyne extension and p is a right fibration

(b) i is a left anodyne extension and p is an inner fibration.

Proof. All the assertions follow directly from proposition 6.3.8 and remark 6.3.7 : the details shall be left to the reader. □

Corollary 6.3.11. *Let $i : K \to L$ be a monomorphism of* sSet, *(X, t) an object of $L/$*sSet, *and $p : X \to Y$ an inner fibration; the following holds :*

(i) If X is an ∞-category, the morphism $\tau^\vee_\bullet : X/L \to X/K$ is a right fibration.

(ii) If Y is an ∞-category, the same holds for both X/L and $X/K \times_{Y/K} Y/L$, and the projections $X/K \times_{Y/K} Y/L \to X/K$ and $X/K \to X$ are right fibrations.

Proof. (i): The functor $-/K$ preserves all representable limits ([13, Prop.2.49(i)]), hence it preserves also the final object Δ^0, for every $K \in \mathrm{Ob}(\mathsf{sSet})$; moreover, X is an ∞-category if and only if the unique morphism $X \to \Delta^0$ is an inner fibration (example 6.1.2(i)). Then the assertion is equivalent to the special case of corollary 6.3.10(i) where $Y = \Delta^0$.

(ii): If Y is an ∞-category, then the same holds for X, since p is an inner fibration (example 6.1.2(i)). Morever, taking $K := \varnothing$, we deduce from corollary 6.3.10(i) that the morphism $\tau^\vee_\bullet : X/L \to X/\varnothing \xrightarrow{\sim} X$ is a right fibration (see example 2.4.9(i)), so X/L is an ∞-category as well (example 6.1.2(i)). Furthermore, the projection $X/K \times_{Y/K} Y/L \to X/K$ is a pull-back of the right fibration $Y/L \to Y/K$ (by (i)), so it is a right fibration as well (proposition 3.1.9(v)), and since we know already that X/K is an ∞-category, it follows that the same holds for $X/K \times_{Y/K} Y/L$, again by example 6.1.2(i). □

Theorem 6.3.12. (Joyal) *Let Y be an ∞-category, $p : X \to Y$ an inner fibration, $i : K \to L$ a monomorphism of* sSet, *and consider a commutative square :*

$$\begin{tikzcd} \{0\} * L \cup \Delta^1 * K \arrow[r, "a"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 * L \arrow[r] & Y. \end{tikzcd}$$

Suppose that the arrow $a_0 \to a_1$ given by the restriction of a to $\Delta^1 = \Delta^1 * \varnothing \subset \Delta^1 * K$ is invertible in X. Then the square admits a diagonal filler $\Delta^1 * L \to X$.

Proof. By remark 6.3.7, the commutative square of the theorem corresponds to the pair $(\mathsf{E}_\diamond, t)$, where $t : L \to X$ is the restriction of a to $L = \{0\} * L$, and $\mathsf{E}_\diamond$ is the induced commutative square :

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X/L \arrow[d, "(L/p)_\diamond"] \\ \Delta^1 \arrow[r, "\alpha"] & X/K \times_{Y/K} Y/L \end{tikzcd}$$

and it suffices to check that $\mathsf{E}_\diamond$ has a diagonal filler. However, by corollaries 6.3.10(i) and 6.3.11(ii), $(L/p)_\diamond$ is a right fibration between ∞-categories, so it is an isofibration (proposition 6.3.5). Hence, we are reduced to showing that α is an invertible arrow of $X/K \times_{Y/K} Y/L$. But, invoking again corollary 6.3.11(ii), we see that also the composed projection $\pi : X/K \times_{Y/K} Y/L \to X/K \to X$ is a right fibration, so it is conservative, again by proposition 6.3.5; thus, it suffices to check that $\pi \circ \alpha$ is an invertible arrow of X. But the latter is precisely the arrow $a_0 \to a_1$, and the proof is concluded. □

Corollary 6.3.13. (Joyal) *Let Y be an ∞-category, $p : X \to Y$ an inner fibration, $n \geq 2$ an integer, and consider a commutative square :*

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "a"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y. \end{tikzcd}$$

Suppose that the arrow $a_0 \to a_1$ given by the restriction of a to $\Delta^{\{0,1\}}$ is invertible in X. Then the square admits a diagonal filler $\Delta^n \to X$.

Proof. We apply theorem 6.3.12 to the monomorphism $i : \partial\Delta^{n-2} \to \Delta^{n-2}$, and use the natural isomorphism $\Lambda^1_0 * \Delta^{n-2} \cup \Delta^1 * \partial\Delta^{n-2} \xrightarrow{\sim} \Lambda^n_0$ (lemma 2.4.10(ii)) to identify the commutative square of the theorem with that of the corollary. Under this identification, the arrow $a_0 \to a_1$ of the theorem corresponds to the arrow $a_0 \to a_1$ of the corollary, whence the assertion. □

Corollary 6.3.14. (Joyal) *A simplicial set is an ∞-groupoid ⇔ it is a Kan complex.*

Proof. We know already that every Kan complex is an ∞-groupoid (proposition 2.6.3). For the converse, it suffices to check that every ∞-groupoid X lies in $r(\mathscr{S})$, with $\mathscr{S} := \{\Lambda^n_k \to \Delta^n \mid n \geq 1, k = 0, n\}$. However, X^{op} is also an ∞-groupoid (remark 2.6.4(iii)), so the assertion follows from corollary 6.3.13 and its front-to-back dual. □

Proposition 6.3.15. *Let $p : X \to Y$ be either a left or right fibration. If Y is a Kan complex, then the same holds for X, and p is a Kan fibration.*

Proof. Indeed, p is conservative (proposition 6.3.5) and Y is an ∞-groupoid (corollary 6.3.14), so the same holds for X, *i.e.* X is a Kan complex. Next, p is also an isofibration, again by proposition 6.3.5, so it is an inner fibration, and then, by virtue of corollary 6.3.13 and its front-to-back dual, p has the right lifting property relative to all inclusions $\Lambda^n_k \to \Delta^n$, with $n \geq 2$ and $0 \leq k \leq n$. Moreover, by remark 6.2.8(iii) and proposition 6.2.9, p has also the right lifting property relative to the inclusions $\Lambda^1_k \to \Delta^1$, for $k = 0, 1$. Hence, p is a Kan fibration. □

Corollary 6.3.16. *For every left or right fibration $p : X \to Y$ and every object y of Y, the fibre $f^{-1}(y)$ is a Kan complex (see definition 2.5.1(iv)).*

Proof. Indeed, the unique morphism $f^{-1}(y) \to \Delta^0$ is a pull-back of p, hence it is a left or right fibration (proposition 3.1.9(v)), and clearly Δ^0 is a Kan complex, so the assertion follows from proposition 6.3.15. □

6.4. **Invertible natural transformations.** In §1.11.12 we exhibited two adjunctions :

$$\pi_1 : \mathsf{Cat} \rightleftarrows \mathsf{Gpd} : i : \mathsf{Gpd} \rightleftarrows \mathsf{Cat} : k$$

where $i : \mathsf{Gpd} \to \mathsf{Cat}$ is the inclusion of the category of small groupoids into the category of all small categories. Consider now the full subcategories of sSet denoted

$$\mathsf{Gpd}^\infty \qquad \text{and} \qquad \mathsf{Cat}^\infty$$

whose objects are respectively the ∞-groupoids and the ∞-categories; we shall similarly construct a right adjoint $k : \mathsf{Cat}^\infty \to \mathsf{Gpd}^\infty$ for the inclusion functor $i : \mathsf{Gpd}^\infty \to \mathsf{Cat}^\infty$.

To this aim, for every ∞-category X we form the cartesian square of sSet :

$$\begin{tikzcd} k(X) \arrow[r] \arrow[d] & X \arrow[d, "\eta_X"] \\ N(k(\tau X)) \arrow[r] & N(\tau X) \end{tikzcd}$$

where N and τ are the nerve functor and its left adjoint (see §2.3.3), and η_X denotes the unit of adjunction. Recall that $k(\tau X)$ is the largest groupoid contained in τX; hence, the inclusion $k(\tau X) \to \tau X$ is trivially an isofibration, so the same holds for the induced monomorphism $N(k(\tau X)) \to N(\tau X)$ of sSet (remark 6.2.8(i)). It follows that the morphism $k(X) \to X$ is a monomorphism and an inner fibration (proposition 3.1.9(v)), so $k(X)$ is an ∞-category (example 6.1.2(i)). Moreover, $k(X)$ and X have the same objects, and by theorem 2.6.12(ii,iii) and corollary 2.6.13(ii) it is easily seen that the arrows of $k(X)$ are precisely the invertible arrows of X, so $k(X)$ is an ∞-groupoid. The construction of $k(X)$ is natural with respect to morphisms $X \to Y$ of Cat^∞, so we get a well-defined functor $k : \mathsf{Cat}^\infty \to \mathsf{Gpd}^\infty$. Moreover, if X is a groupoid, then $\tau X = k(\tau X)$, again by corollary 2.6.13(ii), so that $k(X) = X$ in this case. Especially, for every ∞-groupoid $Y \subset X$, the inclusion $Y \to X$ factors through the inclusion $Y \to k(X)$, *i.e.* $k(X)$ *is the largest* ∞*-groupoid contained in* X, and it follows easily that k is the sought right adjoint for i.

Lemma 6.4.1. *(i) An inner fibration $p : X \to Y$ of ∞-categories is an isofibration $\Leftrightarrow$ $k(p) : k(X) \to k(Y)$ has the right lifting property relative to the inclusion $\partial^1_1 : \{0\} \hookrightarrow \Delta^1$.*

(ii) The inclusion $k(X) \to X$ is a conservative isofibration, for every $X \in \mathrm{Ob}(\mathsf{Cat}^\infty)$.

(iii) For every $n > 0$, the n-simplices of $k(X)$ are the morphisms $\Delta^n \to X$ whose restriction to $\Delta^{\{i,i+1\}}$ is an invertible arrow of X, for every $i = 0, \dots, n-1$.

(iv) Every conservative morphism $f : X \to Y$ of ∞-categories induces a cartesian square :

$$\begin{tikzcd} k(X) \arrow[r] \arrow[d, "k(f)"'] & X \arrow[d, "f"] \\ k(Y) \arrow[r] & Y. \end{tikzcd}$$

(v) $k(\tau X) = \tau(k(X))$ for every $X \in \mathrm{Ob}(\mathsf{Cat}^\infty)$.

Proof. Assertions (i), (ii) and (iv) are clear from the definitions, and (v) follows from (iii), theorem 2.6.12(iii), and the explicit construction of the homotopy category $\mathrm{ho}(X)$.

(iii): Indeed, set $Y_0 := X_0$, and for every $n > 0$, let $Y_n \subset X_n$ be the subset of n-simplices verifying the stated condition; it is easily seen that the system $(Y_n \mid n \in \mathbb{N})$ yields a subsimplex Y of X, and then Y is obviously an ∞-groupoid. On the other hand, it is clear that every groupoid contained in X lies already in Y, so $Y = k(X)$. □

6.4.2. Next, to every simplicial set A we attach the simplicial set

$$\mathrm{Ob}(A) := (\Delta^0)^{(A_0)}$$

(the disjoint union of copies of Δ^0 indexed by A_0 : see §1.2.14). Hence, the set of n-simplices of $\mathrm{Ob}(A)$ is naturally identified with A_0 for every $n \in \mathbb{N}$, and all face and degeneracy maps are identities, under this identification. We have a unique monomorphism

$$i_A : \mathrm{Ob}(A) \to A \qquad \text{such that} \qquad i_{A,0} = \mathbf{1}_{A_0} : \mathrm{Ob}(A)_0 \to A_0.$$

Remark 6.4.3. Every morphism $f : A \to B$ of sSet induces a map $f_0 : A_0 \to B_0$, whence a morphism of sSet

$$\mathrm{Ob}(f) := (\Delta^0)^{(f)} : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$$

(see §1.2.14). Clearly, the rules : $A \mapsto \mathrm{Ob}(A)$ and $f \mapsto \mathrm{Ob}(f)$ yield a well-defined functor

$$\mathrm{Ob} : \mathrm{sSet} \to \mathrm{sSet}$$

and it is easily seen that the rule : $A \mapsto i_A$ defines a natural transformation $\boldsymbol{i_\bullet} : \mathrm{Ob} \Rightarrow \mathbf{1}_{\mathrm{sSet}}$.

By virtue of corollaries 6.1.8 and 6.1.10, every ∞-category X induces an inner fibration between ∞-categories

$$i^* : \mathscr{H}\!om(A, X) \to \mathscr{H}\!om(\mathrm{Ob}(A), X) \xrightarrow{\sim} X^{A_0} \qquad F \mapsto (Fa \mid a \in A_0).$$

We then define $k(A, X)$ as the pull-back in the cartesian square :

$$\begin{tikzcd} k(A,X) \arrow[r] \arrow[d] & \mathscr{H}\!om(A,X) \arrow[d, "i^*"] \\ k(\mathscr{H}\!om(\mathrm{Ob}(A),X)) \arrow[r] & \mathscr{H}\!om(\mathrm{Ob}(A),X) \end{tikzcd}$$

whose bottom horizontal arrow is the natural inclusion. The left vertical arrow is also an inner fibration (proposition 3.1.9(v)), and we know already that $k(\mathscr{H}\!om(\mathrm{Ob}(A), X))$ is an ∞-category, so the same holds for $k(A, X)$ (example 6.1.2(i)). Recall that $\mathscr{H}\!om(A, X)$ is the ∞-category whose n-simplices are the morphisms $\Delta^n \times A \to X$ of sSet, for every $n \in \mathbb{N}$; especially, in the language of definition 2.5.1, the arrows of $\mathscr{H}\!om(A, X)$ are the natural transformations $\eta : F \Rightarrow G$ between functors $F, G : A \rightrightarrows X$. Then such a natural transformation lies in the ∞-subcategory $k(A, X)$ if and only if its evaluation $\eta_a : Fa \to Ga$ is an invertible arrow of X, for every $a \in A_0$, *i.e.* if and only if η is *invertible* in the sense of definition 2.5.4(iv). Notice also that

$$k(\mathscr{H}\!om(\mathrm{Ob}(A), X)) \xrightarrow{\sim} k(X^{A_0}) = k(X)^{A_0}$$

is the largest ∞-groupoid in X^{A_0}. Hence, more generally, the n-simplices of $k(A, X)$ are the morphisms $x : \Delta^n \times A \to X$ such that for every $a \in A_0$, *the evaluation of x at a* :

$$x(a) : \Delta^n \xrightarrow{\sim} \Delta^n \times \Delta^0 \xrightarrow{\mathbf{1}_{\Delta^n} \times a} \Delta^n \times A \xrightarrow{x} X$$

is an n-simplex of $k(X)$. Moreover, by virtue of remark 6.4.3, the construction of $k(A, X)$ is natural in both A and X (details left to the reader), so we get a well-defined functor

$$\boxed{k(-,-) : \mathrm{sSet}^{\mathrm{op}} \times \mathrm{Cat}^\infty \to \mathrm{Cat}^\infty \qquad (A, X) \mapsto k(A, X).}$$

Remark 6.4.4. (i) From the construction and from remark 2.6.4(iii), it is clear that :

$$k(X^{\mathrm{op}}) = k(X)^{\mathrm{op}} \qquad \forall X \in \mathrm{Ob}(\mathrm{Cat}^\infty).$$

(ii) From (i), we easily deduce a commutative diagram of ∞-categories :

$$\begin{tikzcd} k(A,X)^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d] & k(A^{\mathrm{op}}, X^{\mathrm{op}}) \arrow[d] \\ \mathscr{H}om(A,X)^{\mathrm{op}} \arrow[r, "\sim"] & \mathscr{H}om(A^{\mathrm{op}}, X^{\mathrm{op}}) \end{tikzcd}$$

whose vertical arrows are the inclusions, and whose bottom horizontal arrow is the natural identification of remark 2.1.7 : the details are left to the reader.

Example 6.4.5. Since $\mathscr{H}om(\varnothing, X) = \Delta^0$ and $\mathscr{H}om(\Delta^0, X) = X$, we get :

$$k(\varnothing, X) = \Delta^0 \qquad \text{and} \qquad k(\Delta^0, X) = k(X) \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).$$

Lemma 6.4.6. *(i)* $k(\mathscr{H}om(A,X)) \subset k(A,X) \subset \mathscr{H}om(A,X)$.

(ii) Every ∞-category X, and every morphism $f : A \to B$ of sSet *such that $f_0 : A_0 \to B_0$ is a bijection induce a cartesian diagram of* sSet *:*

$$\begin{tikzcd} k(B,X) \arrow[r] \arrow[d] & \mathscr{H}om(B,X) \arrow[d] \\ k(A,X) \arrow[r] & \mathscr{H}om(A,X). \end{tikzcd}$$

(iii) $k(A,X) = \mathscr{H}om(A,X)$ *for every* $A \in \mathrm{Ob}(\mathrm{sSet})$ *and every* $X \in \mathrm{Ob}(\mathrm{Gpd}^\infty)$.

Proof. (i): We have already observed that the morphism $k(A,X) \to \mathscr{H}om(A,X)$ is a monomorphism, since it is a pull-back of a monomorphism. Next, we have a commutative diagram of sSet whose horizontal arrows are the natural inclusions :

$$\begin{tikzcd} k(\mathscr{H}om(A,X)) \arrow[r] \arrow[d, "k(i^*)"'] & \mathscr{H}om(A,X) \arrow[d, "i^*"] \\ k(\mathscr{H}om(\mathrm{Ob}(A),X)) \arrow[r] & \mathscr{H}om(\mathrm{Ob}(A),X). \end{tikzcd}$$

Hence the inclusion $k(\mathscr{H}om(A,X)) \subset \mathscr{H}om(A,X)$ factors through $k(A,X)$.

(ii) and (iii) follow by a direct inspection of the constructions. □

6.4.7. The main goal of this § is to prove that the first inclusion of lemma 6.4.6(i) is in fact an equality; *i.e.* we want to show that *any invertible natural transformation is indeed an invertible arrow of the ∞-category of functors.* To this aim, we shall need an adjoint of the functor $k(-,X)$: for every $B \in \mathrm{Ob}(\mathrm{sSet})$, $X \in \mathrm{Ob}(\mathrm{Cat}^\infty)$, and $n \in \mathbb{N}$, let

$$h(B,X)_n \subset \mathscr{H}om(B,X)_n$$

be the set of n-simplices $f : \Delta^n \to \mathscr{H}om(B,X)$ of $\mathscr{H}om(B,X)$ such that that the associated morphism $\phi_f : B \to \mathscr{H}om(\Delta^n, X)$ factors though $k(\Delta^n, X)$. For every morphism $u : [m] \to [n]$ of sSet, the map $u^* : \mathscr{H}om(B,X)_n \to \mathscr{H}om(B,X)_m$ is given by the rule : $f \mapsto f \circ \Delta^u$, and notice that the morphism $\phi_{u^*(f)} : B \to \mathscr{H}om(\Delta^m, X)$ associated with

$u^*(f)$ is $\mathscr{H}om(\Delta^u, \mathbf{1}_X) \circ \phi_f$, where $\mathscr{H}om(\Delta^u, \mathbf{1}_X) : \mathscr{H}om(\Delta^n, X) \to \mathscr{H}om(\Delta^m, X)$ is defined as in §2.1.6. However, the commutative diagram :

$$\begin{tikzcd} k(\Delta^n, X) \arrow[r, "k(\Delta^u, \mathbf{1}_X)"] \arrow[d] & k(\Delta^m, X) \arrow[d] \\ \mathscr{H}om(\Delta^n, X) \arrow[r, "\mathscr{H}om(\Delta^u, \mathbf{1}_X)"] & \mathscr{H}om(\Delta^m, X) \end{tikzcd}$$

(whose vertical arrows are the inclusions) shows that if ϕ_f factors through $k(\Delta^n, X)$, then $\mathscr{H}om(\Delta^u, \mathbf{1}_X) \circ \phi_f$ factors through $k(\Delta^m, X)$; *i.e.* the system $(h(B, X)_n \mid n \in \mathbb{N})$ yields a subsimplicial set :

$$h(B, X) \subset \mathscr{H}om(B, X)$$

natural in both B and X, whence a well-defined functor

$$\boxed{h(-,-) : \mathrm{sSet}^{\mathrm{op}} \times \mathrm{Cat}^\infty \to \mathrm{sSet} \qquad (B, X) \mapsto h(B, X).}$$

One may think of $h(B, X)$ as the full subcategory of $\mathscr{H}om(B, X)$ consisting of those functors $B \to X$ that map every arrow of B to an invertible arrow of X.

Example 6.4.8. Since $\mathscr{H}om(\varnothing, X) = \Delta^0$, it is easily seen that :

$$h(\varnothing, X) = \Delta^0 \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).$$

Moreover, for every $n \in \mathbb{N}$, the set $h(\Delta^0, X)_n$ is the set of n-simplices

$$f : \Delta^n \to \mathscr{H}om(\Delta^0, X) \xrightarrow{\sim} X$$

such that the associated morphism $\phi_f : \Delta^0 \to \mathscr{H}om(\Delta^n, X)$ factors through $k(\Delta^n, X)$, *i.e.* with $f \circ u \in k(X)_0$ for every $u \in \mathrm{sSet}(\Delta^0, \Delta^n)$. But since $k(X)_0 = X_0$, we conclude that :

$$h(\Delta^0, X) = X \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}).$$

Lemma 6.4.9. *Every* $X \in \mathrm{Ob}(\mathrm{Cat}^\infty)$ *induces an adjoint pair of functors :*

$$\boxed{k(-, X) : \mathrm{sSet} \rightleftarrows \mathrm{sSet}^{\mathrm{op}} : h(-, X).}$$

Proof. By adjunction, $\mathrm{sSet}(A, \mathscr{H}om(B, X))$ and $\mathrm{sSet}(B, \mathscr{H}om(A, X))$ are identified with $\mathrm{sSet}(A \times B, X)$ and respectively $\mathrm{sSet}(B \times A, X)$, for every $A, B \in \mathrm{Ob}(\mathrm{sSet})$. Hence, the isomorphism $\omega : A \times B \xrightarrow{\sim} B \times A$ that swaps the factors induces a natural bijection :

$$(*) \qquad \mathrm{sSet}(A, \mathscr{H}om(B, X)) \xrightarrow{\sim} \mathrm{sSet}(B, \mathscr{H}om(A, X)).$$

Now, we have $h(B, X) \subset \mathscr{H}om(B, X)$ and $k(A, X) \subset \mathscr{H}om(A, X)$, and we come down to checking more precisely that the bijection $(*)$ induces by restriction a bijection :

$$\mathrm{sSet}(A, h(B, X)) \xrightarrow{\sim} \mathrm{sSet}(B, k(A, X)).$$

However, unwinding the definitions, we see that the morphisms $A \to h(B, X)$ correspond, under the foregoing natural identification, to the morphisms $u_\bullet : A \times B \to X$ such that $u_n(\pi_n^*(a), b) \in k(X)_n$ for every $n \in \mathbb{N}$, every $b \in B_n$ and every $a \in A_0$, where $\pi_n : [n] \to [0]$ is the unique map. Likewise, the morphisms $B \to k(A, X)$ correspond to the morphisms $v : B \times A \to X$ whose composition with ω fulfills the previous condition, whence the assertion. □

Remark 6.4.10. (i) For every $A, B, C \in \mathrm{Ob}(\mathsf{sSet})$ and every $X \in \mathrm{Ob}(\mathsf{Cat}^\infty)$, the isomorphism $C \times B \xrightarrow{\sim} B \times C$ that swaps the factors induces a commutative diagram :

$$\begin{tikzcd} \mathsf{sSet}(A \times C, k(B,X)) \arrow[r] \arrow[d] & \mathsf{sSet}(A \times C, \mathscr{H}om(B,X)) \arrow[r, "\sim"] & \mathsf{sSet}(A \times C \times B, X) \arrow[d, "\mathsf{sSet}(A\times\omega,X)"] \\ \mathsf{sSet}(A \times B, h(C,X)) \arrow[r] & \mathsf{sSet}(A \times B, \mathscr{H}om(C,X)) \arrow[r, "\sim"] & \mathsf{sSet}(A \times B \times C, X) \end{tikzcd}$$

whose three unmarked arrows are injections. Indeed, on the one hand, the composition of the top horizontal arrows identifies $\mathsf{sSet}(A \times C, k(B,X))$ with the set of morphisms $u_\bullet : A \times C \times B \to X$ of sSet such that $u_n(a, c, \pi_n^*(b)) \in k(X)_n$ for every $n \in \mathbb{N}$ and every $(a,b,c) \in A_n \times B_0 \times C_n$, where $\pi_n : [n] \to [0]$ is the unique map. On the other hand, the composition of the bottom horizontal arrows identifies $\mathsf{sSet}(A \times B, h(C,X))$ with the set of morphisms $v_\bullet : A \times B \times C \to X$ of sSet such that $v_n(\pi_n^*(a), \pi_n^*(b), c) \in k(X)_n$ for every $n \in \mathbb{N}$ and every $(a,b,c) \in A_0 \times B_0 \times C_n$.

(ii) Let $p : X \to Y$ be any morphism of Cat^∞; in light of lemma 6.4.9, we may apply proposition 3.1.21 to the functors $F_1, F_2 : \mathsf{sSet} \rightrightarrows \mathsf{sSet}^{\mathrm{op}}$ with $F_1 := k(-,Y)$ and $F_2 := k(-,X)$, and to the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by p. We then get, for every pair $f : A \to B$ and $g : C \to D$ of morphisms of sSet, a natural bijection between commutative squares of the type :

$$\mathsf{D}^\diamond \ : \ \begin{tikzcd} C \arrow[r] \arrow[d, "g"] & k(B,X) \arrow[d, "f^\diamond"] \\ D \arrow[r] & k(B,Y) \times_{k(A,Y)} k(A,X) \end{tikzcd} \qquad \text{and} \qquad \mathsf{D}_\diamond \ : \ \begin{tikzcd} A \arrow[r] \arrow[d, "f"] & h(D,X) \arrow[d, "g_\diamond"] \\ B \arrow[r] & h(C,X) \times_{h(C,Y)} h(D,Y). \end{tikzcd}$$

as well as a natural bijection between diagonal fillers for $\mathsf{D}^\diamond$ and $\mathsf{D}_\diamond$.

(iii) For instance take $f : \varnothing \to \Delta^0$ be the unique morphism, and $g := \partial_1^1 : \{0\} \hookrightarrow \Delta^0$. Taking into account examples 6.4.5 and 6.4.8, we see that in this case, (ii) yields a bijection between commutative squares of the type :

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, "\partial_1^1"'] & k(X) \arrow[d, "k(p)"] \\ \Delta^1 \arrow[r] & k(Y) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d] & h(\Delta^1, X) \arrow[d, "(\partial_1^1)_\diamond"] \\ \Delta^0 \arrow[r] & X \times_Y h(\Delta^1, Y) \end{tikzcd}$$

as well as a bijection between the diagonal fillers for such squares. Combining with lemma 6.4.1(i), we conclude that *p is an isofibration $\Leftrightarrow$ p is an inner fibration and the morphism $(\partial_1^1)_\diamond : h(\Delta^1, X) \to X \times_Y h(\Delta^1, Y)$ is surjective on objects.*

Theorem 6.4.11. *For every inner fibration $p : X \to Y$ between ∞-categories, the morphism $(\partial_1^1)_\diamond : h(\Delta^1, X) \to X \times_Y h(\Delta^1, Y)$ of remark 6.4.10(iii) lies in $r(\{\partial\Delta^n \to \Delta^n \mid n > 0\})$.*

Proof. Remark 6.4.10(ii) yields a natural bijection between the commutative squares :

$$\mathsf{D}^\diamond \ : \ \begin{tikzcd} \{0\} \arrow[r] \arrow[d, "\partial_1^1"] & k(\Delta^n, X) \arrow[d, "j_n^\diamond"] \\ \Delta^1 \arrow[r] & k(\Delta^n, Y) \times_{k(\partial\Delta^n, Y)} k(\partial\Delta^n, X) \end{tikzcd} \qquad \text{and} \qquad \mathsf{D}_\diamond \ : \ \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, "j_n"] & h(\Delta^1, X) \arrow[d, "(\partial_1^1)_\diamond"] \\ \Delta^n \arrow[r] & X \times_Y h(\Delta^1, Y) \end{tikzcd}$$

and a bijection between the respective diagonal fillers. Thus, we are reduced to checking that $\partial_1^1 \in l(\{j_n^\diamond \mid n > 0\})$. To this aim, let us fix an integer $n > 0$; we apply proposition 3.1.21 to the functors $F_1, F_2 : \mathsf{sSet} \rightrightarrows \mathsf{sSet}$ with $F_1 := (-) \times \partial\Delta^n$ and $F_2 := (-) \times \Delta^n$ and to the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by the inclusion j_n. Notice also that $F_2K \cap F_1L = F_1K$ for every monomorphism $K \to L$ of sSet; we then get yet another natural bijection between commutative squares of the type :

$$\mathsf{E}^\diamond \ : \quad \begin{tikzcd} \Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \arrow[r, "a"] \arrow[d, "(\partial_1^1)^\diamond"'] & X \arrow[d, "p"] \\ \Delta^1 \times \Delta^n \arrow[r, "b"] & Y \end{tikzcd} \qquad \text{and} \qquad \mathsf{E}_\diamond \ : \quad \begin{tikzcd} \{0\} \arrow[r, "a'"] \arrow[d, "\partial_1^1"'] & \mathscr{H}om(\Delta^n, X) \arrow[d, "p_\diamond"] \\ \Delta^1 \arrow[r, "b'"] & Z \end{tikzcd}$$

where $Z := \mathscr{H}om(\Delta^n, Y) \times_{\mathscr{H}om(\partial\Delta^n, Y)} \mathscr{H}om(\partial\Delta^n, X)$, and as usual, we have also a bijection between the respective diagonal fillers. Now, a direct inspection shows that the simplicial set $k(\Delta^n, Y) \times_{k(\partial\Delta^n, Y)} k(\partial\Delta^n, X)$ is a subobject of Z, and under this identification, $j_n^\diamond$ is the restriction of $p_\diamond$ to the subsimplicial set $k(\Delta^n, X) \subset \mathscr{H}om(\Delta^n, X)$. So, we are further reduced to showing that if the image of a' lies in $k(\Delta^n, X)$ and if the image of b' lies in $k(\Delta^n, Y) \times_{k(\partial\Delta^n, Y)} k(\partial\Delta^n, X)$, then $\mathsf{E}^\diamond$ has a diagonal filler : indeed, in this case $\mathsf{E}_\diamond$ has a diagonal filler $d : \Delta^1 \to \mathscr{H}om(\Delta^n, X)$, and by construction the image of the composition $\Delta^1 \to \mathscr{H}om(\partial\Delta^n, X)$ of d with the morphism $\mathscr{H}om(j_n, X) : \mathscr{H}om(\Delta^n, X) \to \mathscr{H}om(\partial\Delta^n, X)$ lies in $k(\partial\Delta^n, X)$; but since j_n induces a bijection $(\partial\Delta^n)_0 \xrightarrow{\sim} (\Delta^n)_0$ (recall that $n > 0$), lemma 6.4.6(ii) then implies that the image of d lies in $k(\Delta^n, X)$, so d will be also a diagonal filler for $\mathsf{D}^\diamond$.

Next, since the inclusion $k(\Delta^n, X) \subset \mathscr{H}om(\Delta^n, X)$ is a bijection on objects, the stated condition on a' is empty; on the other hand, the stated condition on b' amounts to asking that b is an arrow of $k(\Delta^n, Y)$ and the restriction of a to $\Delta^1 \times \partial\Delta^n$ is an arrow of $k(\partial\Delta^n, X)$. But since $\partial\Delta^n$ and Δ^n have the same objects, the latter condition on a actually *implies* that b is an arrow of $k(\Delta^n, Y)$.

Summing up, we have a diagram $\mathsf{E}^\diamond$ such that the restriction of a to $\Delta^1 \times \partial\Delta^n$ is an arrow of $k(\partial\Delta^n, X)$, and under this condition, we have to exhibit a diagonal filler for $\mathsf{E}^\diamond$. To this aim, we consider the front-to-back dual of the finite filtration of claim 5.1.5 :

$$B_{-1} := \Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \subset B_0 \subset \cdots \subset B_n := \Delta^1 \times \Delta^n.$$

By remark 2.3.8(i) we get cocartesian diagrams whose vertical arrows are the inclusions :

$$(*) \qquad \begin{tikzcd} \Lambda^{n+1}_{n-i} \arrow[r, "\alpha_i"] \arrow[d] & B_{i-1} \arrow[d] \\ \Delta^{n+1} \arrow[r, "\beta_i"] & B_i \end{tikzcd} \qquad \forall i = 0, \dots, n.$$

Hence, the inclusion $B_{i-1} \to B_i$ is inner anodyne for every $i = 0, \dots, n-1$, so the same holds for the inclusion $j : B_{-1} \to B_{n-1}$. Let $c : B_{n-1} \to Y$ be the restriction of b; since p is an inner fibration, we deduce that the commutative square :

$$\begin{tikzcd} \Delta^1 \times \partial\Delta^n \cup \{0\} \times \Delta^n \arrow[r, "a"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B_{n-1} \arrow[r, "c"] & Y \end{tikzcd}$$

admits a diagonal filler $c' : B_{n-1} \to X$. So, it remains only to exhibit a diagonal filler for :

$$\begin{tikzcd} \Lambda^{n+1}_0 \arrow[r, "c' \circ \alpha_n"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "b \circ \beta_n"] & Y. \end{tikzcd}$$

However, β_n is the front-to-back dual of the morphism $c^n_n : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ of claim 5.1.5, *i.e.* it is the morphism corresponding to the unique strictly increasing map $[n+1] \to [1] \times [n]$ whose image contains $(0,0)$ and $(1,0)$. Hence, the restriction of $c' \circ \alpha_n$ to $\Delta^{\{0,1\}}$ is the arrow of X given by the restriction of c' to $\Delta^1 \times \{0\}$, which is the same as the restriction of a to $\Delta^1 \times \{0\}$, and the latter is an invertible arrow of X, by assumption. Then, the sought diagonal filler is provided by corollary 6.3.13. □

Corollary 6.4.12. *An inner fibration $p : X \to Y$ between ∞-categories is an isofibration if and only if the associated morphism $(\partial^1_1)_\diamond : h(\Delta^1, X) \to X \times_Y h(\Delta^1, Y)$ of remark 6.4.10(iii) is a trivial fibration (for the Kan-Quillen model structure).*

Proof. Since $\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$ is a cellular model for sSet (remark 5.1.1), the assertion follows from remark 6.4.10(iii) and theorem 6.4.11. □

Theorem 6.4.13. *(i) Every isofibration $p : X \to Y$ between ∞-categories, and every monomorphism $i : K \to L$ of* sSet *induce a Kan fibration :*

$$(i^*, p_*) : k(L, X) \to k(L, Y) \times_{k(K,Y)} k(K, X).$$

(ii) $k(A, X)$ is an ∞-groupoid, for every $A \in \mathrm{Ob}(\mathrm{sSet})$ and every $X \in \mathrm{Ob}(\mathrm{Cat}^\infty)$.

Proof. (i): Notice first that the natural isomorphisms of remark 6.4.4(ii) identify $(i^*, p_*)^{\mathrm{op}}$ with $((i^{\mathrm{op}})^*, (p^{\mathrm{op}})^*)$, and p^{op} is an isofibration (proposition 6.2.9); in view of remark 6.3.2, it then suffices to check that (i^*, p_*) is a left fibration.

To this aim, in light of proposition 5.1.4(ii), we are reduced to checking that for every $n \in \mathbb{N}$, every commutative square of the following type admits a diagonal filler :

$$(*) \qquad \begin{tikzcd} \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \arrow[r, "a"] \arrow[d] & k(L, X) \arrow[d, "(i^*, p_*)"] \\ \Delta^n \times \Delta^1 \arrow[r, "b"] & k(L, Y) \times_{k(K,Y)} k(K, X). \end{tikzcd}$$

Now, if $n = 0$, $(*)$ is the same as the diagram $\mathrm{D}^\diamond$ of remark 6.4.10(ii), with $f := i$ and $g := \partial^1_1$; in this case, we are then reduced to exhibiting a diagonal filler for the associated diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "i"'] & h(\Delta^1, X) \arrow[d, "(\partial^1_1)_\diamond"] \\ L \arrow[r] & X \times_Y h(\Delta^1, Y). \end{tikzcd}$$

The latter is provided by corollary 6.4.12. Next, suppose that $n > 0$; by remark 6.4.10(i), the restriction of a to $\partial\Delta^n \times \Delta^1$ can be regarded as a morphism $a'_1 : \partial\Delta^n \times L \to h(K, X)$, and the composition of b with the projection to $k(K, X)$ can be regarded as a morphism $a'_2 : \Delta^n \times K \to h(\Delta^1, X)$. Moreover, the commutativity of $(*)$ implies that a'_1 and a'_2 agree on $\partial\Delta^n \times K$, whence a well-defined morphism $a' : \partial\Delta^n \times L \cup \Delta^n \times K \to h(\Delta^1, X)$ which restricts to a'_1 on $\partial\Delta^n \times L$ and to a'_2 on $\Delta^n \times K$. Likewise, the restriction of a to $\Delta^n \times \{0\}$ can be regarded as a morphism $b'_1 : \Delta^n \times L \to h(\Delta^0, X) \xrightarrow{\sim} X$, and the composition of b with the projection to $k(L, Y)$ can be regarded as a morphism $b'_2 : \Delta^n \times L \to h(\Delta^1, Y)$; moreover, the commutativity of $(*)$ implies that $p \circ b'_1$ equals the composition of b'_2 with $h(\partial^1_1, Y) : h(\Delta^1, Y) \to h(\Delta^0, Y) \xrightarrow{\sim} Y$, whence a well-defined morphism $b' : \Delta^n \times L \to X \times_Y h(\Delta^1, Y)$ whose composition with the natural projections agrees with b'_1 and respectively b'_2. Lastly, the commutativity of $(*)$ is equivalent to the commutativity of the square :

$$\begin{tikzcd} \partial\Delta^n \times L \cup \Delta^n \times K \arrow[r, "a'"] \arrow[d] & h(\Delta^1, X) \arrow[d, "(\partial_1^1)_\diamond"] \\ \Delta^n \times L \arrow[r, "b'"] & X \times_Y h(\Delta^1, Y) \end{tikzcd}$$

and the latter admits a diagonal filler $d' : \Delta^n \times L \to h(\Delta^1, X)$, by corollary 6.4.12. To conclude, it remains to check that d' corresponds to a morphism $d : \Delta^n \times \Delta^1 \to k(L, X)$, under the natural identifications of remark 6.4.10(i), since in that case, d will be the sought diagonal filler for $(*)$. To this aim, since $n > 0$, lemma 6.4.1(iii) further reduces to checking that the restriction $\delta' : \partial\Delta^n \times L \to h(\Delta^1, X)$ of d' corresponds, under the same natural identifications, to a morphism $\delta : \partial\Delta^n \times \Delta^1 \to k(L, X)$; however, δ is none else than the restriction of a to $\partial\Delta^n \times \Delta^1$, and the proof is concluded.

(ii): We apply (i) with $i : \varnothing \to A$ and with $Y := \Delta^0$; then (i, p_*) is the unique morphism $k(A, X) \to \Delta^0$ (example 6.4.5), and we conclude with proposition 2.6.3. □

We can now show the promised :

Corollary 6.4.14. *For every $A \in \mathrm{Ob}(\mathrm{sSet})$ and every $X \in \mathrm{Ob}(\mathrm{Cat}^\infty)$ we have :*

$$\boxed{k(A, X) = k(\mathscr{H}om(A, X)).}$$

Proof. Since $k(\mathscr{H}om(A, X))$ is the largest ∞-groupoid contained in $\mathscr{H}om(A, X)$, the assertion follows straightforwardly from theorem 6.4.13(ii) and lemma 6.4.6(i). □

Corollary 6.4.15. *Every isofibration $p : X \to Y$ between ∞-categories, and every anodyne extension $g : C \to D$ induce a trivial fibration of* sSet

$$g_\diamond : h(D, X) \to h(C, X) \times_{h(C,X)} h(D, Y).$$

Proof. (We refer here to the trivial fibrations of the Kan-Quillen model structure.) According to theorem 6.4.13(i), for every monomorphism $f : A \to B$ of sSet, the induced morphism $f^\diamond = (f^*, p_*) : k(B, X) \to k(B, Y) \times_{k(A,Y)} k(A, X)$ is a Kan fibration, so the assertion follows from remark 6.4.10(ii). □

Definition 6.4.16. Let X and Y be two ∞-categories; we say that a functor $f : X \to Y$ is *an equivalence of ∞-categories* if there exist functors $g, g' : Y \rightrightarrows X$ and invertible natural transformations $fg \Rightarrow \mathbf{1}_Y$ and $\mathbf{1}_X \Rightarrow g'f$ (see definitions 2.5.1(iv) and 2.5.4(iv)).

Remark 6.4.17. (i) Let $\mathscr{C}$ and $\mathscr{C}'$ be two small categories, and $f, g : N(\mathscr{C}) \rightrightarrows N(\mathscr{C}')$ two functors; we know already that $f = N(F)$ and $g = N(G)$ for a unique pair of functors $F, G : \mathscr{C} \rightrightarrows \mathscr{C}'$, and that all natural transformations $h : N(\mathscr{C}) \times \Delta^1 \to N(\mathscr{C}')$ from f to g are of the form $N(H)$ for a unique natural trasformation $H : \mathscr{C} \times [1] \to \mathscr{C}'$ from F to G (see remark 2.5.2). Moreover, taking into account example 2.6.2(i), we see that h is invertible if and only if H is an isomorphism of functors. Likewise, f is an equivalence of ∞-categories if and only if F is an equivalence of categories.

(ii) In the situation of corollary 6.4.14, let $\eta_\bullet : \Delta^1 \to \mathscr{H}om(A, X)$ be a natural transformation between two functors $f, g : A \rightrightarrows X$. With corollary 6.4.14 we now know that $\eta_\bullet$ is invertible $\Leftrightarrow$ $\eta_\bullet$ extends to a morphism $J \to \mathscr{H}om(A, X)$, or equivalently, a *$J$-homotopy* $\eta' : J \times A \to X$ from f to g (see definition 4.5.1(i)).

6.5. **∞-categories as fibrant objects.** The following result of Joyal justifies the introduction of the model category structure that bears his name :

Theorem 6.5.1. *(i) The fibrant objects of the Joyal model category are the ∞-categories.*

(ii) A functor between ∞-categories is a J-fibration if and only if it is an isofibration.

(iii) A functor between ∞-categories is an equivalence if and only if it is a weak categorical equivalence, if and only if it is a J-homotopy equivalence (see remark 4.5.2(ii)).

(iv) A functor between ∞-groupoids is an equivalence if and only if it is a weak homotopy equivalence, if and only if it is a simplicial homotopy equivalence.

Proof. (ii): In light of remark 6.2.8(ii), it suffices to check that every isofibration $p : X \to Y$ between ∞-categories is a J-fibration. We apply the criterion of corollary 6.2.4 : condition (b) of the corollary holds, since p is an inner fibration (corollary 6.1.8). Next, notice that, for every ∞-category Z, the image of every morphism $f : J \to Z$ lies in $k(Z)$: indeed, the restriction of f to $\Delta^1 \subset J$ is an invertible arrow of Z (remark 6.2.2(i)), *i.e.* a 1-simplex of $k(Z)$; then the unique extension of $f_{|\Delta^1} : \Delta^1 \to k(Z)$ to J is a morphism $J \to k(Z)$ that must coincide with f. Letting $Z := \mathscr{H}om(\Delta^n, X)$ for any $n \in \mathbb{N}$ and any $X \in \mathrm{Ob}(\mathrm{Cat}^\infty)$ (corollary 6.1.10), we conclude that :

$$h(J, X) = \mathscr{H}om(J, X) \qquad \forall X \in \mathrm{Ob}(\mathrm{Cat}^\infty).$$

Recall also that $\partial^J_\varepsilon : \Delta^0 \to J$ is an anodyne extension for $\varepsilon = 0, 1$ (see definition 6.2.1(i)); combining with corollary 6.4.15, we get that $(\partial^{J*}_\varepsilon, p_*) : \mathscr{H}om(J, X) \to X \times_Y \mathscr{H}om(J, Y)$ is a trivial fibration, and this is precisely condition (a) of corollary 6.2.4.

(i): By example 6.1.2(i), every J-fibrant object of sSet is an ∞-category. Conversely, if X is an ∞-category, obviously the unique morphism $X \to \Delta^0$ is an isofibration, so it is a J-fibration, by (ii), and therefore X is J-fibrant.

(iii): In light of (i), proposition 4.5.9(ii) yields the equivalence between the two last conditions of (iii). It is also clear that every J-homotopy equivalence between ∞-categories is an equivalence of ∞-categories (lemma 4.5.3). Lastly, with remark 6.4.17(ii) and lemma 4.5.8, we easily see that the equivalences of ∞-categories are precisely the J-homotopy equivalences.

(iv): In light of corollary 6.3.14, proposition 4.5.9(ii) yields the equivalence between the two last conditions of (iv). Next, notice that if X is an ∞-groupoid and $A \in \mathrm{Ob}(\mathrm{sSet})$, then every natural transformation $\Delta^1 \to \mathscr{H}om(A, X)$ between any two functors $f, g : A \rightrightarrows X$ is invertible; *i.e.* every simplicial homotopy from f to g extends (uniquely) to a J-homotopy. This means that the equivalences of ∞-groupoids are precisely the simplicial homotopy equivalences. □

Proposition 6.5.2. *(i) The class of weak categorical equivalences is the smallest of the classes $\mathscr{W} \subset \mathrm{Mor}(\mathrm{sSet})$ satisfying the following three conditions :*

- *(a) $\mathscr{W}$ enjoys the 2-out-of-3 property*
- *(b) Every inner anodyne extension lies in $\mathscr{W}$*
- *(c) Every trivial fibration lies in $\mathscr{W}$.*

(ii) Every weak categorical equivalence is a weak homotopy equivalence.

Proof. (i): Clearly the class $\mathscr{C}$ of weak categorical equivalences satisfies these conditions. Conversely, let us show that if $\mathscr{W}$ satisfies (a)-(c), then $\mathscr{C} \subset \mathscr{W}$. Indeed, let $f : X \to Y$

be any weak categorical equivalence; arguing as in the proof of proposition 4.6.4, we can find ∞-categories X' and Y', and a commutative square :

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "j"] & Y' \end{tikzcd}$$

in which i and j are inner anodyne extensions. Then also f' is a weak categorical equivalence, and it suffices to check that $f' \in \mathscr{W}$; hence, we may assume from start that X and Y are ∞-categories. In this case, by lemma 3.2.7(ii), we can write $f = h \circ q$, where h is an isofibration and q is a right inverse of a trivial fibration, and therefore $q \in \mathscr{W}$, by (a) and (c). Also, h is a weak categorical equivalence, so $h \in \mathscr{W}$ by (c), and finally $f \in \mathscr{W}$.

(ii) is an immediate consequence of (i). □

Corollary 6.5.3. *The class of weak categorical equivalences is stable under finite products.*

Proof. We are easily reduced to checking that for every $X \in \mathrm{Ob}(\mathsf{sSet})$, the functor $(-)\times X : \mathsf{sSet} \to \mathsf{sSet}$ preserves weak categorical equivalences. Hence, let us consider the class $\mathscr{W}$ of morphisms f of sSet such that $f \times X$ is a weak categorical equivalence. Clearly $\mathscr{W}$ enjoys the 2-out-of-3 property; also $\mathsf{inAn} \subset \mathscr{W}$, by corollary 6.1.7(ii). Moreover, every trivial fibration lies in $\mathscr{W}$ (proposition 3.1.9(v)), hence every weak categorical equivalence lies in $\mathscr{W}$, by proposition 6.5.2(i). □

Corollary 6.5.4. *(i) Let $i : K \to L$ and $j : U \to V$ be two monomorphisms of* sSet*, and suppose that either i or j is a weak categorical equivalence. Then the induced morphism $K \times V \cup L \times U \to L \times V$ is a weak categorical equivalence.*

(ii) Every trivial cofibration (resp. cofibration) $i : K \to L$, and every fibration $p : X \to Y$ of the Joyal model category structure induce a trivial fibration (resp. a fibration) :

$$\mathscr{H}om(L, X) \to \mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y).$$

(iii) For every isofibration $p : X \to Y$ between ∞-categories, and every monomorphism $i : K \to L$ of sSet*, $\mathscr{H}om(K, X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L, Y)$ is an ∞-category, , and we have :*

$$\boxed{k(L, Y) \times_{k(K,Y)} k(K, X) = k(\mathscr{H}om(L, Y) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(K, X)).}$$

(iv) Every monomorphism $i : K \to L$ of sSet *induces a Kan fibration*

$$k(i, X) : k(L, X) \to k(K, X) \qquad \forall X \in \mathrm{Ob}(\mathsf{Cat}^\infty).$$

Proof. (i): We consider the commutative diagram of sSet :

$$\begin{tikzcd} K \times U \arrow[r, "K\times j"] \arrow[d, "i\times U"'] & K \times V \arrow[d, "f"'] \arrow[dr, "i\times V"] & \\ L \times U \arrow[r] & K \times V \cup L \times U \arrow[r] & L \times V \end{tikzcd}$$

whose square subdiagram is cocartesian. If i is a weak categorical equivalence, the same holds for $i \times U$, by corollary 6.5.3, hence the latter is a trivial cofibration for the Joyal model category structure, and then the same holds for f (proposition 3.1.9(v)). By the same token, $i \times V$ is a weak categorical equivalence, whence the assertion, by the 2-out-of-3 property of weak categorical equivalences.

(ii): We apply the discussion of §3.1.20 to the functors $F_1, F_2 : \mathsf{sSet} \rightrightarrows \mathsf{sSet}$ with $F_1 := K \times (-)$ and $F_2 := L \times (-)$ and to the natural transformation $\tau_\bullet : F_1 \Rightarrow F_2$ induced by the inclusion $i : K \to L$; by §2.1.6 and proposition 3.1.21, we get, for every monomorphism $j : U \to V$ a bijection between commutative squares of the type :

$$
\begin{tikzcd}
K \times V \cup L \times U \arrow[r] \arrow[d, "j^\diamond"] & X \arrow[d, "p"] \\
L \times V \arrow[r] & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
U \arrow[r] \arrow[d, "j"'] & \mathscr{H}om(L,X) \arrow[d, "p_\diamond"] \\
V \arrow[r] & \mathscr{H}om(K,X) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(L,Y)
\end{tikzcd}
$$

and between diagonal fillers for such pairs of corresponding squares. Now, if i is a trivial cofibration, then for every such j the morphism $j^\diamond$ is a trivial cofibration, by (i), so the left square admits a diagonal filler, and therefore the same holds for the right square, which shows that $p_\diamond$ is a trivial cofibration. If i is only a cofibration, then $j^\diamond$ is still a trivial cofibration, provided that j is a trivial cofibration, again by (i), and the same argument shows that in this case $p_\diamond$ is a fibration of the Joyal model category.

(iii): The first assertion follows immediately from (ii), corollary 6.1.10, and theorem 6.5.1(ii). Next, under the stated conditions on i and p, the associated morphism (i^*, p_*) of theorem 6.4.13(i) is a Kan fibration, and then $Z := k(L,Y) \times_{k(K,Y)} k(K,X)$ is an ∞-groupoid, by combining with part (ii) of the same theorem. It follows easily that Z also represents the fibre product of $k(L,Y)$ and $k(K,X)$ over $k(K,Y)$ *in the category* Gpd^∞; likewise, $\mathscr{H}om(L,Y) \times_{\mathscr{H}om(K,Y)} \mathscr{H}om(K,X)$ represents the fibre product of $\mathscr{H}om(L,Y)$ and $\mathscr{H}om(K,X)$ over $\mathscr{H}om(K,Y)$ *in the category* Cat^∞. On the other hand, since the functor $k : \mathsf{Cat}^\infty \to \mathsf{Gpd}^\infty$ is a right adjoint, it preserves such fibre products ([13, Prop.2.49(i)]), whence the stated identity, by combining with corollary 6.4.14.

(iv) is the special case of theorem 6.4.13(i) with $Y := \Delta^0$; indeed, the unique morphism $p : X \to \Delta^0$ is an isofibration, since X is an ∞-category (theorem 6.5.1(i,ii)). □

Corollary 6.5.5. *Let $X \to Y$ be an isofibration between ∞-categories. We have :*

(i) Every trivial cofibration $A \to B$ of the Joyal model structure induces a trivial fibration:

$$\alpha : k(B,X) \to k(B,Y) \times_{k(A,Y)} k(A,X).$$

(ii) Every monomorphism $C \to D$ of sSet *induces a fibration of the Joyal model structure:*

$$\beta : h(D,X) \to h(C,X) \times_{h(C,Y)} h(D,Y).$$

(iii) $h(A,X)$ is an ∞-category, for every $A \in \mathrm{Ob}(\mathsf{sSet})$ and every $X \in \mathrm{Ob}(\mathsf{Cat}^\infty)$.

Proof. (i): We consider the commutative square :

$$
\begin{tikzcd}
k(B,X) \arrow[r] \arrow[d, "\alpha"'] & \mathscr{H}om(B,Y) \arrow[d, "\phi"] \\
k(B,Y) \times_{k(A,Y)} k(A,X) \arrow[r] & \mathscr{H}om(B,Y) \times_{\mathscr{H}om(A,Y)} \mathscr{H}om(A,X)
\end{tikzcd}
$$

whose bottom (resp. top) horizontal arrow is identified, via corollary 6.5.4(iii), with the inclusion of $k(\mathscr{H}om(B,Y) \times_{\mathscr{H}om(A,Y)} \mathscr{H}om(A,X))$ into $\mathscr{H}om(B,Y) \times_{\mathscr{H}om(A,Y)} \mathscr{H}om(A,X))$ (resp. via corollary 6.4.14with the inclusion of $k(\mathscr{H}om(B,X))$ into $\mathscr{H}om(B,X)$). Then, under these natural identifications, α corresponds to $k(\phi)$. Now, ϕ is a trivial fibration, by 6.5.4(ii), hence it is conservative (remark 6.2.2(ii)), and therefore the square is cartesian (lemma 6.4.1(iv)); but then, also α is a trivial fibration (proposition 3.1.9(v)).

(ii) follows immediately from (i) and remark 6.4.10(ii).

(iii): We apply (ii) to the monomorphism $\varnothing \to A$ and with $Y := \Delta^0$; then β is the unique morphism $h(A,X) \to \Delta^0$ (example 6.4.8), whence the assertion, by theorem 6.5.1(i). □

Lemma 6.5.6. *(i) The functor* $k : \mathrm{Cat}^\infty \to \mathrm{Gpd}^\infty$ *preserves equivalences of* ∞*-categories.*

(ii) $\tau : \mathrm{sSet} \to \mathrm{Cat}$ *sends weak categorical equivalences to equivalences of categories.*

Proof. (i): In light of theorem 6.5.1(i), we may regard $(\mathrm{Cat}^\infty)^{\mathrm{op}}$ as the subcategory $(\mathrm{sSet}^{\mathrm{op}})_c$ of cofibrant objects of the model category $\mathrm{sSet}^{\mathrm{op}}$, where the latter is endowed with the opposite of the Joyal model category structure (proposition 3.2.4(i)). Then Ken Brown's lemma (proposition 3.2.8) applies to k, and reduces to showing that k sends every trivial fibration $p : X \to Y$ between ∞-categories to a trivial fibration $k(p) : k(X) \to k(Y)$. However, every such p is conservative (remark 6.2.2(ii)), hence $k(p)$ is a pull-back of p (lemma 6.4.1(iv)), so the assertion follows as usual from proposition 3.1.9(v).

(ii) follows from proposition 6.2.6 and lemma 3.4.12(iii.a). □

Theorem 6.5.7. *Let* $f : X \to Y$ *be a morphism of simplicial sets. The following conditions are equivalent :*

(a) f *is a weak categorical equivalence.*

(b) For every ∞*-category* W*, the induced morphism* $f_W^* : \mathscr{H}om(Y, W) \to \mathscr{H}om(X, W)$ *is an equivalence of* ∞*-categories.*

(c) For every ∞*-category* W*, the induced functor* $\tau f_W^* : \tau\mathscr{H}om(Y, W) \to \tau\mathscr{H}om(X, W)$ *is an equivalence of categories.*

(d) For every ∞*-category* W*, the induced morphism* $k(f, W) : k(Y, W) \to k(X, W)$ *is an equivalence of* ∞*-groupoids.*

Proof. (a)⇒(b): By proposition 3.2.8 and theorem 6.5.1(iii), it suffices to show that the functor $\mathscr{H}om(-, W) : \mathrm{sSet} \to \mathrm{sSet}^{\mathrm{op}}$ sends trivial cofibrations of the Joyal model category to weak categorical equivalences, and the latter follows from corollary 6.5.4(ii).

(b)⇒(c): This follows from theorem 6.5.1(iii) and lemma 6.5.6(ii).

(c)⇒(a): By definition of the Joyal model category structure, and, by virtue of theorem 6.5.1(i), the morphism f is a weak categorical equivalence if and only if every ∞-category W induces a bijection $[Y, W] \xrightarrow{\sim} [X, W]$, where $[X, W]$ denotes the set of morphisms from X to W up to J-homotopy equivalence (see definition 4.5.4(iv)). However, by lemma 4.5.8, for given morphisms $g_1, g_2 : X \rightrightarrows W$ we have $[g_1] = [g_2]$ in $[X, W]$ if and only if there exists a J-homotopy from g_1 to g_2, if and only if there exists an invertible arrow $g_1 \xrightarrow{\sim} g_2$ in $\mathscr{H}om(X, W)$ (remark 6.2.2(i)), if and only if there exists an isomorphism $g_1 \xrightarrow{\sim} g_2$ in $\tau\mathscr{H}om(X, W)$ (corollary 2.6.13(ii)), so if (c) holds, (a) holds as well.

(b)⇒(d): In light of corollary 6.4.14, the assertion follows from lemma 6.5.6(i).

(d)⇒(a): The objects of $k(X, W)$ are the morphisms $X \to W$ of sSet, and we have already remarked that the arrows $\Delta^1 \to k(X, W)$ are the invertible natural transformations between such morphisms, which are the same as the J-homotopies $J \times X \to W$. Combining with proposition 5.4.6(ii), it follows that :

$$\pi_0(k(X, W)) = [X, W] \qquad \forall X \in \mathrm{Ob}(\mathrm{sSet}), \forall W \in \mathrm{Ob}(\mathrm{Cat}^\infty).$$

On the other hand, the equivalence of ∞-groupoids $k(f, W)$ is a J-homotopy equivalence, and every J-homotopy equivalence yields, by restriction, a Δ^1-homotopy equivalence, hence a weak homotopy equivalence, which in turns induces a bijection $\pi_0(k(f, W)) : \pi_0(k(Y, W)) \xrightarrow{\sim} \pi_0(k(X, W))$ (proposition 5.4.6(i)), whence (a). □

Theorem 6.5.8. *Let* $f : X \to Y$ *be a functor between* ∞*-categories. The following conditions are equivalent :*

(a) f *is an equivalence of* ∞*-categories.*
(b) *For every simplicial set* A*, the induced morphism* $f_{A*} : \mathscr{H}om(A, X) \to \mathscr{H}om(A, Y)$ *is an equivalence of* ∞*-categories.*
(c) *For every simplicial set* A*, the induced functor* $\tau f_{A*} : \tau\mathscr{H}om(A, X) \to \tau\mathscr{H}om(A, Y)$ *is an equivalence of categories.*
(d) *For every simplicial set* A*, the induced morphism* $k(A, f) : k(A, X) \to k(A, Y)$ *is an equivalence of* ∞*-groupoids.*
(e) f *induces an equivalence of* ∞*-groupoids* $k(\Delta^n, X) \to k(\Delta^n, Y)$ *for every* $n \in \mathbb{N}$*.*

Proof. (a)⇒(b): Let us endow $\mathrm{sSet}^{\mathrm{op}}$ with the opposite of the Joyal model category structure (proposition 3.2.4(i)); by proposition 3.2.8 and theorem 6.5.1(i,iii), it suffices to show that the functor $\mathscr{H}om(A, -) : \mathrm{sSet}^{\mathrm{op}} \to \mathrm{sSet}^{\mathrm{op}}$ sends trivial cofibrations of $\mathrm{sSet}^{\mathrm{op}}$ (that is, trivial fibrations of the Joyal model category) to weak categorical equivalences, and the latter follows from corollary 6.5.4(ii).

(b)⇒(c),(d): These are proved as the corresponding implications of theorem 6.5.7.

(c)⇒(a): Arguing as in the proof of the corresponding implication of theorem 6.5.7, we see that (c) implies that every $A \in \mathrm{Ob}(\mathrm{sSet})$ yields a bijection $f_{A*} : [A, X] \xrightarrow{\sim} [A, Y]$, where $[A, X]$ denotes the set of morphisms from A to X up to J-homotopy equivalence, and likewise for $[A, Y]$; then f is a J-homotopy equivalence (lemma 4.5.3), whence (a).

(d)⇒(a): Arguing as in the proof of the same implication of theorem 6.5.7, we see that (d) yields a bijection $f_{A*} : [A, X] \xrightarrow{\sim} [A, Y]$ for every $A \in \mathrm{Ob}(\mathrm{sSet})$, whence (a).

Obviously (d)⇒(e); for the converse, let $\mathscr{F}$ be the class of all simplicial sets A such that $k(A, f)$ is an equivalence of ∞-groupoids; by corollary 2.2.10, it suffices to check :

Claim 6.5.9. $\mathscr{F}$ is saturated by monomorphisms.

Proof : The functors $k(-, X)$ and $k(-, Y)$ preserve all small colimits, since they are left adjoints (lemma 6.4.9 and [13, Prop.2.49(ii)]), so every small family $(A_i \mid i \in I)$ of elements of $\mathscr{F}$ induces a commutative diagram whose vertical arrows are isomorphisms of sSet :

$$\begin{tikzcd} k(\coprod_{i\in I} A_i, X) \arrow[r, "\alpha"] \arrow[d] & k(\coprod_{i\in I} A_i, Y) \arrow[d] \\ \prod_{i\in I} k(A_i, X) \arrow[r, "\beta"] & \prod_{i\in I} k(A_i, Y) \end{tikzcd}$$

and in order to check that $\coprod_{i\in I} A_i \in \mathscr{F}$, we are reduced to showing that β is a weak homotopy equivalence (theorem 6.5.1(iv)). To this aim, we endow $\mathrm{sSet}^{\mathrm{op}}$ with the model category structure induced by the Kan-Quillen model structure of sSet (proposition 3.2.4(i)); then each morphism $k(A_i, Y) \to k(A_i, X)$ is a weak equivalence between cofibrant objects of $\mathrm{sSet}^{\mathrm{op}}$, so the assertion follows from corollary 3.5.12(i).

Next, let us consider morphisms $L \xleftarrow{i} K \to K'$ of sSet, where i is a monomorphism and $K, L, K' \in \mathscr{F}$; we get an induced commutative diagram of sSet :

$$\mathscr{D} \quad : \quad \begin{tikzcd} k(L, X) \arrow[r, "k(i,X)"] \arrow[d] & k(K, X) \arrow[d] & k(K', X) \arrow[l] \arrow[d] \\ k(L, Y) \arrow[r, "k(i,Y)"] & k(K, Y) & k(K', Y) \arrow[l] \end{tikzcd}$$

where $k(i, X)$ and $k(i, Y)$ are Kan fibrations, *i.e.* cofibrations of $\mathrm{sSet}^{\mathrm{op}}$ (corollary 6.5.4(iv)), and by assumption the vertical arrows are weak homotopy equivalences. By applying corollary 3.6.7 to the diagram $\mathscr{D}^{\mathrm{op}}$ in the opposite model category $\mathrm{sSet}^{\mathrm{op}}$, we deduce a weak homotopy equivalence of sSet :

$$k(L,X) \times_{k(K,X)} k(K',X) \to k(L,Y) \times_{k(K,Y)} k(K',Y).$$

However, since $k(-,X)$ and $k(-,Y)$ preserve small colimits, this morphism is naturally identified with $k(L \sqcup_K K', f)$, so $L \sqcup_K K' \in \mathscr{F}$.

Lastly, consider a countable sequence $K_0 \xrightarrow{i_0} K_1 \xrightarrow{i_1} K_2 \to \cdots$ of monomorphisms of sSet, with $K_n \in \mathscr{F}$ for every $n \in \mathbb{N}$; we get an induced commutative diagram of sSet :

$$\begin{tikzcd} \cdots \arrow[r] & k(K_2,X) \arrow[r, "k(i_1,X)"] \arrow[d] & k(K_1,X) \arrow[r, "k(i_0,X)"] \arrow[d] & k(K_0,X) \arrow[d] \\ \cdots \arrow[r] & k(K_2,Y) \arrow[r, "k(i_1,Y)"] & k(K_1,Y) \arrow[r, "k(i_0,Y)"] & k(K_0,Y) \end{tikzcd}$$

whose vertical arrows are weak homotopy equivalences, and whose horizontal arrows are Kan fibrations, *i.e.* cofibrations of $\mathrm{sSet}^{\mathrm{op}}$, again by corollary 6.5.4(iv). By corollary 3.5.13 (applied to the opposite model category $\mathrm{sSet}^{\mathrm{op}}$), the induced morphism of sSet :

$$\varprojlim_{n\in\mathbb{N}} k(K_n,X) \to \varprojlim_{n\in\mathbb{N}} k(K_n,Y)$$

is a weak homotopy equivalence; but since the functors $k(-,X)$ and $k(-,Y)$ preserve small colimits, this morphism is naturally identified with the morphism

$$k(\varinjlim_{n\in\mathbb{N}} K_n, X) \to k(\varinjlim_{n\in\mathbb{N}} K_n, Y)$$

so the colimit of the sequence $K_\bullet$ lies in $\mathscr{F}$, and the proof is concluded. □

Remark 6.5.10. Condition (e) of theorem 6.5.8 shall be sharpened later : see theorem 6.8.1.

6.6. **Path spaces and loop spaces.** Let X be an ∞-category, and $x_\bullet := (x_0, \dots, x_n)$ a sequence of objects of X, for some $n \geq 0$. Recall that $(\Delta^n)_0$ is naturally identified with $[n] = \{0, \dots, n\}$, hence the monomorphism $i : \mathrm{Ob}(\Delta^n) = (\Delta^0)^{([n])} \to \Delta^n$ as in §6.4.2 yields an *evaluation morphism* $i^* : \mathscr{H}om(\Delta^n, X) \to X^{n+1}$. We form the cartesian square :

$$\mathscr{D}_{x_\bullet} \quad : \quad \begin{tikzcd} X(x_\bullet) \arrow[r] \arrow[d] & \mathscr{H}om(\Delta^n,X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x_\bullet"] & X^{n+1}. \end{tikzcd}$$

By construction, we have then $X(x) = \Delta^0$ for every $x \in X_0$, and if $n = 1$, the objects of $X(x_\bullet)$ are precisely the arrows of X of the form $x_0 \to x_1$.

Lemma 6.6.1. *$X(x_\bullet)$ is an ∞-groupoid, for every $n \in \mathbb{N}$ and every $x_\bullet \in (X_0)^{n+1}$.*

Proof. By construction, we have a commutative diagram :

$$\begin{tikzcd} X(x_\bullet) \arrow[r] \arrow[d, "q"'] & k(\Delta^n,X) \arrow[r] \arrow[d, "k(i^*)"] & \mathscr{H}om(\Delta^n,X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x_\bullet"] & k(X)^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

whose two square subdiagrams are cartesian. Now, since X is an ∞-category, the unique morphism $p : X \to \Delta^0$ is an isofibration; by theorem 6.4.13(i), the induced morphism $k(i^*) = (i^*, p_*) : k(\Delta^n, X) \to k(\Delta^n, \Delta^0) \times_{k(\mathrm{Ob}(\Delta^n),\Delta^0)} k(\mathrm{Ob}(\Delta^n), X) = k(X)^{n+1}$ is a Kan fibration, so the same holds for q (proposition 3.1.9(v)), *i.e.* $X(x_\bullet)$ is a Kan complex, and we conclude with corollary 6.3.14. □

Example 6.6.2. For every category $\mathscr{C}$ and every $x_\bullet := (x_0, \dots, x_n) \in \mathrm{Ob}(\mathscr{C})^{n+1}$, let us denote by $\mathscr{C}(x_\bullet)$ the set of all sequences of morphisms of $\mathscr{C}$ of the form $x_0 \to x_1 \to \cdots \to x_n$. We have natural identifications :

$$\boxed{N(\mathscr{C})(x_\bullet) \xrightarrow{\sim} N(\mathscr{C}(x_\bullet)) = (\Delta^0)^{(\mathscr{C}(x_\bullet))} \qquad \forall n \geq 0, \forall x_\bullet \in \mathrm{Ob}(\mathscr{C})^{n+1}}$$

where the set $\mathscr{C}(x_\bullet)$ is regarded as a discrete category, as usual. Indeed, example 2.3.6 yields a natural identification : $\mathscr{H}om(\Delta^n, N\mathscr{C}) \xrightarrow{\sim} N(\mathscr{C}^{[n]})$, under which, the morphism i^* corresponds to the nerve of the functor $\mathscr{C}^j : \mathscr{C}^{[n]} \to \mathscr{C}^{n+1}$ induced by the functor $j : \mathrm{Ob}([n], \leq) = \{0, \dots, n\} \to ([n], \leq)$ that is the identity on objects. Then, since the nerve functor is left exact ([13, Prop.2.49(i)]), the diagram $\mathscr{D}_{x_\bullet}$ is obtained by applying termwise the nerve functor to the cartesian diagram of Cat :

$$\begin{tikzcd} \mathscr{C}(x_\bullet) \arrow[r] \arrow[d] & \mathscr{C}^{[n]} \arrow[d, "\mathscr{C}^j"] \\ {[0]} \arrow[r, "x_\bullet"] & \mathscr{C}^{n+1} \end{tikzcd}$$

whence the sought isomorphism. In particular, notice that for $n = 1$, the set $\mathscr{C}(x, y)$ associated to any given sequence (x, y) of objects of X is just the usual Hom-set of morphisms $x \to y$ in $\mathscr{C}$. More generally, for any ∞-category X, the special case where $n = 1$ is singled out in the following definition :

Definition 6.6.3. Let X be an ∞-category, and (x, y) a given pair of objects of X.

(i) The ∞-groupoid $X(x, y)$ is called *the path space from x to y in X.*

(ii) Notice that $X(x, x)$ has a distinguished object $\mathbf{1}_x : x \to x$. The pair :

$$\boxed{\Omega^1(X, x) := (X(x, x), \mathbf{1}_x)}$$

is called *the loop space of X at the point x.*

Remark 6.6.4. (i) Let X be an ∞-category, $n, m \in \mathbb{N}$, $x_\bullet := (x_0, \dots, x_n) \in (X_0)^{n+1}$, and $\phi : [m] \to [n]$ any non-decreasing map, inducing a morphism $\Delta^\phi : \Delta^m \to \Delta^n$ of sSet; we set $x_\bullet^\phi := (x_{\phi(0)}, \dots, x_{\phi(m)})$. Then there exists a unique morphism of sSet

$$X(\phi) : X(x_\bullet) \to X(x_\bullet^\phi)$$

fitting into the commutative diagram :

$$\begin{tikzcd} X(x_\bullet) \arrow[rrr] \arrow[dr, "X(\phi)"] \arrow[ddd] & & & \mathscr{H}om(\Delta^n, X) \arrow[dl, "\mathscr{H}om(\Delta^\phi, X)"'] \arrow[ddd] \\ & X(x_\bullet^\phi) \arrow[r] \arrow[d] & \mathscr{H}om(\Delta^m, X) \arrow[d] & \\ & \Delta^0 \arrow[r, "x_\bullet^\phi"] & X^{m+1} & \\ \Delta^0 \arrow[ur, equal] \arrow[rrr, "x_\bullet"] & & & X^{n+1} \arrow[ul, "X^\phi"'] \end{tikzcd}$$

where X^ϕ is defined as in §1.2.14.

(ii) Moreover, the datum of $(X, x_\bullet)$ can be regarded as an object of the slice category $(\Delta^0)^{([n])}/\mathrm{Cat}^\infty$, and it is easily seen that the rule $(X, x_\bullet) \mapsto X(x_\bullet)$ extends to a functor

$$\Pi_n : (\Delta^0)^{([n])}/\mathrm{Cat}^\infty \to \mathrm{Gpd}^\infty.$$

Namely, every morphism $f : (X, x_\bullet) \to (Y, y_\bullet)$ of $(\Delta^0)^{([n])}/\mathsf{Cat}^\infty$ yields a commutative diagram of sSet :

$$\begin{tikzcd} \mathscr{H}om(\Delta^n, X) \arrow[r] \arrow[d] & \mathscr{H}om(\Delta^n, Y) \arrow[d] \\ X^{n+1} \arrow[r, "f^{n+1}"] & Y^{n+1} \end{tikzcd}$$

and $f^{n+1}(x_\bullet) = y_\bullet$, whence an induced morphism $\Pi_n(f) : X(x_\bullet) \to Y(y_\bullet)$ as required.

(iii) By the same token we get a well-defined functor

$$\Omega^1 : \Delta^0/\mathsf{Cat}^\infty \to \Delta^0/\mathsf{Gpd}^\infty \qquad (X, x) \mapsto \Omega^1(X, x).$$

(iv) Furthermore, since the functors $\mathscr{H}om(\Delta^n, -)$ and $\mathscr{H}om(\mathrm{Ob}(\Delta^n), -)$ preserve all representable limits (lemma 1.6.14), combining with corollary 1.4.6(iii), it is easily seen that the same holds for both Π_n (for every $n \in \mathbb{N}$) and Ω^1 : details left to the reader.

(v) Likewise, since the functors $\mathscr{H}om(\Delta^n, -)$ and $\mathscr{H}om(\mathrm{Ob}(\Delta^n), -)$ also preserve all small filtered colimits (example 2.2.12(i)), and since the finite limits of sSet commute with all small filtered colimits (see §2.1.6), combining with corollary 1.4.6(ii,iii) it follows easily that both Π_n (for every $n \in \mathbb{N}$) and Ω^1 preserve all small filtered colimits.

Example 6.6.5. (i) Let X be an ∞-category, $n \in \mathbb{N}$, and $x_\bullet \in X_0^{n+1}$. Recall that the inclusion $j : k(X) \to X$ is an isofibration (lemma 6.4.1(ii)); arguing as in the proof of lemma 6.6.1 we get a commutative diagram :

$$\begin{tikzcd} k(X)(x_\bullet) \arrow[r, "\Pi_n(j)"] \arrow[d] & X(x_\bullet) \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x_\bullet"] \\ k(\Delta^n, k(X)) \arrow[r, "k(\Delta^n, j)"] & k(\Delta^n, X) \arrow[r, "k(i^*)"] & k(X)^{n+1} \end{tikzcd}$$

whose two square subdiagrams are cartesian, and $k(\Delta^n, j)$ is a Kan fibration, by theorem 6.4.13(i). Hence $\Pi_n(j) : k(X)(x_\bullet) \to X(x_\bullet)$ *is a Kan fibration* (proposition 3.1.9(v)).

(ii) Especially, for every $x \in X_0$, the inclusion $j : k(X) \to X$ *induces a Kan fibration* $\Omega^1(j) : \Omega^1(k(X), x) \to \Omega^1(X, x)$.

Proposition 6.6.6. *Let X be an ∞-category; we have a natural bijection :*

$$\boxed{\pi_0(X(x, y)) \xrightarrow{\sim} \mathsf{ho}(X)(x, y) \qquad \forall x, y \in X_0}$$

where $\mathsf{ho}(X)$ *is the Boardman-Vogt homotopy category of X (see theorem 2.6.12).*

Proof. The morphism $X \to N(\mathsf{ho}(X))$ of theorem 2.6.12(ii) induces a morphism of sSet :

$$X(x, y) \to N(\mathsf{ho}(X))(x, y) = N(\mathsf{ho}(x, y))$$

(example 6.6.2), that is given on objects by the rule : $(f : x \to y) \mapsto [f]$, where $[f]$ denotes the class of f in $\mathsf{ho}(x, y)$. There follows a surjective map of sets :

$$\pi_0(X(x, y)) \to \pi_0(N(\mathsf{ho}(x, y))) = \mathsf{ho}(x, y)$$

that is clearly natural with respect to morphisms of $(\Delta^0)^{([1])}/\mathsf{Cat}^\infty$, and it remains to check the injectivity of this map. Hence, let $f, g : x \rightrightarrows y$ be two arrows of X with $[f] = [g]$; this means that there exists a 2-simplex $t : \Delta^2 \to X$ whose restriction to $\partial\Delta^2$ is the commuting triangle :

$$\begin{tikzcd} & y \arrow[dr, "1_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"] & & y. \end{tikzcd}$$

Let $p : \Delta^1 \times \Delta^1 \to \Delta^2$ be the nerve of the surjective morphism $[1] \times [1] \to [2]$ of poSet that maps both $(0, 0)$ and $(0, 1)$ to 0; then $tp : \Delta^1 \times \Delta^1 \to X$ corresponds to an arrow $\tau : f \to g$ of $\mathscr{H}om(\Delta^1, X)$, whose compositions with s and t are respectively $\mathbf{1}_x$ and $\mathbf{1}_y$. So, τ lies in $X(x, y)$, and shows that the classes of f and g agree in $\pi_0(X(x, y))$, as required. □

The aim of this section is to lift the composition law of the homotopy category of any ∞-category X, to operations on the path spaces and loop spaces of X. To this purpose, we shall need the following :

Proposition 6.6.7. *For every $n \geq 1$ we have :*

(i) The inclusion $i_n : \Delta^{\{n\}} \to Sp^n$ is a right anodyne extension (see §2.3.7).

(ii) The inclusion $j_n : Sp^n \to \Delta^n$ is an inner anodyne extension.

Proof. (i): We argue by induction on $n \geq 1$: the case $n = 1$ is trivial. Next, suppose that (i) holds for some $n \geq 1$; we consider the commutative diagram :

$$(*) \qquad \begin{tikzcd} \Delta^{\{n\}} \arrow[r, "l_n"] \arrow[d, "i_n"'] & \Delta^{\{n,n+1\}} \arrow[d, "i'_n"] & \Delta^{\{n+1\}} \arrow[l, "l_{n+1}"'] \arrow[dl, "i_{n+1}"] \\ Sp^n \arrow[r] & Sp^{n+1} & \end{tikzcd}$$

all whose arrows are the natural inclusions. Then the square subdiagram of $(*)$ is cocartesian, and since i_n is a right anodyne extension, the same holds for i'_n; but l_{n+1} is right anodyne as well, so the same follows for i_{n+1}.

(ii): We argue again by induction on $n \geq 1$; the cases $n = 1, 2$ are trivial; hence, let $n \geq 2$, and suppose that j_n is an inner anodyne extension. We apply proposition 6.3.8(i) with $i : \varnothing \to \Delta^{\{n+1\}}$, and $j := i_n$; in light of (i), it follows that the inclusion :

$$Sp^n * \varnothing \cup \Delta^{\{n\}} * \Delta^{\{n+1\}} \to Sp^n * \Delta^{\{n+1\}}$$

is an inner anodyne extension. Recall as well the natural identification (remark 2.4.6(ii))

$$(**) \qquad \Delta^n * \Delta^{\{n+1\}} \xrightarrow{\sim} \Delta^{n+1}$$

under which, $\Delta^n * \varnothing$ and $\Delta^{\{n\}} * \Delta^{\{n+1\}}$ correspond respectively to $\Delta^{[n]} \subset \Delta^{n+1}$ (the image of $\partial^{n+1}_{n+1} : \Delta^n \to \Delta^{n+1}$) and to $\Delta^{\{n,n+1\}} \subset \Delta^{n+1}$. Hence, $Sp^n * \varnothing \cup \Delta^{\{n\}} * \Delta^{\{n+1\}}$ corresponds to $Sp^{n+1} \subset \Delta^{n+1}$, under the same identification.

Next, we apply again proposition 6.3.8(i) with $i : \varnothing \to \Delta^{\{n+1\}}$ and with $j := j_n$; since by inductive assumption $j_n \in \mathsf{inAn}$, the same then follows for the inclusion :

$$\Delta^n * \varnothing \cup Sp^n * \Delta^{\{n+1\}} \to \Delta^n * \Delta^{\{n+1\}}$$

which, under the identification $(**)$, corresponds to $\Delta^{[n]} \cup Sp^n * \Delta^{\{n+1\}} \subset \Delta^{n+1}$.

We are thus reduced to checking that the inclusion $Sp^n * \Delta^{\{n+1\}} \to \Delta^{[n]} \cup Sp^n * \Delta^{\{n+1\}}$ is an inner anodyne extension. However, lemma 2.4.10(i) implies that the diagram :

$$\begin{tikzcd} Sp^n \arrow[r] \arrow[d, "j_n"'] & Sp^n * \Delta^{\{n+1\}} \arrow[d] \\ \Delta^{[n]} \arrow[r] & \Delta^{[n]} \cup Sp^n * \Delta^{\{n+1\}} \end{tikzcd}$$

is cocartesian, and since $j_n \in \mathsf{inAn}$, the assertion follows. □

Corollary 6.6.8. *(i) Every isofibration $X \to Y$ of ∞-categories induces trivial fibrations :*

$$\mathscr{H}om(\Delta^n, X) \to \mathscr{H}om(Sp^n, X) \times_{\mathscr{H}om(Sp^n,Y)} \mathscr{H}om(\Delta^n, Y) \qquad \forall n \geq 1.$$

(ii) In particular, every ∞-category X induces trivial fibrations :

$$\mathscr{H}om(\Delta^n, X) \to \mathscr{H}om(Sp^n, X) \qquad \forall n \geq 1.$$

Proof. (i) follows from corollary 6.5.4(ii) and proposition 6.6.7; assertion (ii) is the special case of (i) where $Y = \Delta^0$. □

Remark 6.6.9. (i) Let X be an ∞-category. Notice that the objects of $\mathscr{H}om(Sp^2, X)$ are precisely the sequences $x_\bullet := (x_0 \to x_1 \to x_2)$ of arrows of X; since, by corollary 6.6.8(ii), the morphism $\mathscr{H}om(\Delta^2, X) \to \mathscr{H}om(Sp^n, X)$ is a trivial fibration (especially, it admits a section), any such $x_\bullet$ can be lifted to an object of $\mathscr{H}om(\Delta^2, X)$, and any such lifting amounts to the choice of a composition for the two arrows $x_0 \to x_1$ and $x_1 \to x_2$ composing $x_\bullet$.

(ii) With the following proposition, we shall extend the observation of (i) to sequences $x_\bullet$ of any length $n \geq 1$. To this aim, notice first that the inclusion $Sp^n \to \Delta^n$ induces an isomorphism $\mathrm{Ob}(Sp^n) \xrightarrow{\sim} \mathrm{Ob}(\Delta^n) = (\Delta^0)^{([n])}$; thus, the inclusion $i : \mathrm{Ob}(Sp^n) \to Sp^n$ induces again an evaluation morphism $i^* : \mathscr{H}om(Sp^n, X) \to X^{n+1}$. We may then state :

Proposition 6.6.10. *Let X be an ∞-category, and $x_\bullet := (x_0, \dots, x_n)$ a sequence of objects of X, for some $n \geq 1$. The following holds :*

(i) We have a cartesian diagram of sSet *:*

$$\begin{tikzcd} \prod_{0 \leq i < n} X(x_i, x_{i+1}) \arrow[r] \arrow[d] & \mathscr{H}om(Sp^n, X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "x_\bullet"] & X^{n+1}. \end{tikzcd}$$

(ii) The trivial fibration of corollary 6.6.8(ii) restricts to a trivial fibration :

$$X(x_\bullet) \to \prod_{0 \leq i < n} X(x_i, x_{i+1})$$

whose composition with the projection onto $X(x_i, x_{i+1})$ is the morphism $X(\phi_i) : X(x_\bullet) \to X(x_i, x_{i+1})$ induced by the inclusion $\phi_i : \Delta^{\{i,i+1\}} \to \Delta^n$, for $i = 0, \dots, n-1$ (remark 6.6.4(i)).

Proof. (i): Notice first that $Sp^n = \Delta^{\{0,1\}} \sqcup_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \sqcup_{\Delta^{\{2\}}} \cdots \sqcup_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}$. By virtue of lemma 1.6.14, there follows a natural identification :

$$\mathscr{H}om(Sp^n, X) \xrightarrow{\sim} \mathscr{H}om(\Delta^{\{0,1\}}, X) \times_{\mathscr{H}om(\Delta^{\{1\}},X)} \cdots \times_{\mathscr{H}om(\Delta^{\{n-1\}},X)} \mathscr{H}om(\Delta^{\{n-1,n\}}, X).$$

Next, let us remark :

Claim 6.6.11. Let $\mathscr{C}$ be a category whose fibre products are representable, and let

$$(f_i : A'_i \to A_i \mid i = 0, \dots, n) \qquad (g_i : B_{i,i+1} \to A_{i,i+1} := A_i \times A_{i+1} \mid i = 0, \dots, n-1)$$

be two sequences of morphisms of $\mathscr{C}$. Set $A := \prod_{i=0}^n A_i$, $A' := \prod_{i=0}^n A'_i$, $A'_{i,i+1} := A'_i \times A'_{i+1}$ and $B'_{i,i+1} := A'_{i,i+1} \times_{A_{i,i+1}} B_{i,i+1}$ for $i = 0, \dots, n-1$. Then we have a natural isomorphism :

$$A' \times_A (B_{0,1} \times_{A_1} B_{1,2} \times_{A_2} \cdots \times_{A_{n-1}} B_{n-1,n}) \xrightarrow{\sim} B'_{0,1} \times_{A'_1} \cdots \times_{A'_{n-1}} B'_{n-1,n}.$$

Proof : Left to the reader. ◇

We apply claim 6.6.11 with $A'_i := \Delta_0$ and $A_i := X^{\{i\}} = X = \mathscr{H}om(\Delta^{\{i\}}, X)$ for $i = 0, \dots n$ (notation of §1.2.14) and with $B_{i,i+1} := \mathscr{H}om(\Delta^{\{i,i+1\}}, X)$; then $A_{i,i+1} = X^{\{i,i+1\}}$, and we take $f_i : \Delta^0 \to X$ to be the object x_i of X for $i = 0, \dots n$, and $g_i : \mathscr{H}om(\Delta^{\{i,i+1\}}, X) \to X^{\{i,i+1\}}$ the evaluation morphism induced by the inclusion $\mathrm{Ob}(\Delta^{\{i,i+1\}}) \to \Delta^{\{i,i+1\}}$. With this notation, $A = X^{\{0,\dots,n\}} = X^{n+1}$ and $A' = \Delta^0$, and the induced morphism $A' \to A$ is $x_\bullet : \Delta^0 \to X^{n+1}$. On the other hand, $A'_{i,i+1} = X(x_i, x_{i+1})$ for $i = 0, \dots, n-1$, whence (i).

(ii): Indeed, by (i) and by construction of $X(x_\bullet)$ we have a cartesian diagram of sSet :

$$\begin{tikzcd} X(x_\bullet) \arrow[r] \arrow[d] & \mathscr{H}om(\Delta^n, X) \arrow[d] \\ \prod_{0 \le i < n} X(x_i, x_i + 1) \arrow[r] & \mathscr{H}om(Sp^n, X) \end{tikzcd}$$

(since $\mathrm{Ob}(\Delta^n) = \mathrm{Ob}(Sp^n)$) and we know already that the right vertical morphism is a trivial fibration, so the same holds for the left vertical one (proposition 3.1.9(v)). ☐

6.6.12. Now, let X be an ∞-category, and $x_\bullet := (x_0, \dots, x_n)$ a sequence of objects of X, for some $n \ge 2$. Let $\psi_n : [1] \to [n]$ be the map such that $0 \mapsto 0$ and $1 \mapsto n$; *a composition law for path spaces* is defined as any morphism of the form :

$$(*) \qquad \prod_{0 \le i < n} X(x_i, x_{i+1}) \xrightarrow{j} X(x_\bullet) \xrightarrow{X(\psi_n)} X(x_0, x_n)$$

where j is any section of the trivial fibration provided by proposition 6.6.10(ii) (the existence of such a section is ensured by proposition 4.5.6(i)). Applying the functor π_0 to such a law $(*)$ yields, by virtue of propositions 6.6.6 and 5.4.6(i), a map :

$$\prod_{0 \le i < n} \mathsf{ho}(X)(x_i, x_{i+1}) \to \mathsf{ho}(X)(x_0, x_n)$$

and a direct inspection of the construction easily shows that the latter is nothing else than the composition law of the homotopy category $\mathsf{ho}(X)$.

6.6.13. We proceed similarly for loop spaces : for a given object x of X and for $n \ge 1$, set $x_\bullet^{(n)} := (x, \dots, x) \in X^{n+1}$. We notice that the ∞-groupoid $X(x_\bullet^{(n)})$ has a distiguished object : namely, the unique morphism $\pi_n : \Delta^n \to \Delta^0$ induces a morphism $\pi_n^* : \mathscr{H}om(\Delta^0, X) \to \mathscr{H}om(\Delta^n, X)$; there follows a commutative diagram :

$$\begin{tikzcd} \Delta^0 \arrow[r, "x"] \arrow[d, equal] & X = \mathscr{H}om(\Delta^0, X) \arrow[r, "\pi_n^*"] & \mathscr{H}om(\Delta^n, X) \arrow[d, "i^*"] \\ \Delta^0 \arrow[rr, "x_\bullet"] & & X^{n+1} \end{tikzcd}$$

whence an object called *the base point of* $X(x_\bullet^{(n)})$ and denoted also $x : \Delta^0 \to X(x_\bullet^{(n)})$, by a slight abuse of notation. Since $\pi_n \circ \Delta^\phi = \pi_m$ for every non-decreasing map $\phi : [m] \to [n]$, it is easily seen that $X(\phi) : X(x_\bullet^{(n)}) \to X(x_\bullet^{(m)})$ *preserves the respective base points*, for every such ϕ. Clearly, the base point of $X(x_\bullet^{(1)}) = X(x, x)$ is precisely the distinguished object $\mathbf{1}_x : x \to x$ as in definition 6.6.3(ii); let $\{\mathbf{1}_x\} \subset X(x, x)$ be the image of this base point $x : \Delta^0 \to X(x,x)$. We consider the commutative diagram (notation of §2.1.14) :

$$(**)\qquad \begin{tikzcd} X(x,x) \vee X(x,x) \arrow[r, "a"] \arrow[d] & X(x_\bullet^{(2)}) \arrow[d] \\ X(x,x)^2 \arrow[r, equal] & X(x,x)^2 \end{tikzcd}$$

whose left vertical arrow is the natural inclusion, whose right vertical arrow is the trivial fibration of proposition 6.6.10(ii), and where a is the morphism whose restrictions to $X(x,x) \times \{\mathbf{1}_x\}$ and to $\{\mathbf{1}_x\} \times X(x,x)$ are respectively the compositions :

$$X(x,x) \times \{\mathbf{1}_x\} \xrightarrow{\sim} X(x,x) \xrightarrow{X(\sigma_1^2)} X(x_\bullet^{(2)}) \qquad \{\mathbf{1}_x\} \times X(x,x) \xrightarrow{\sim} X(x,x) \xrightarrow{X(\sigma_0^2)} X(x_\bullet^{(2)})$$

associated with the degeneracy maps $\sigma_0^2, \sigma_1^2 : [2] \to [1]$; notice that these morphisms coincide on $X(x,x) \times \{\mathbf{1}_x\} \cap \{\mathbf{1}_x\} \times X(x,x) = \{\mathbf{1}_x\} \times \{\mathbf{1}_x\} \xrightarrow{\sim} \Delta^0$, and on this intersection they agree with the base point $x : \Delta^0 \to X(x_\bullet^{(2)}) = X(x,x,x)$.

We pick a diagonal filler $j : X(x,x)^2 \to X(x_\bullet^{(2)})$ for $(**)$; by construction, j maps $\{\mathbf{1}_x\} \times \{\mathbf{1}_x\}$ to the base point of $X(x_\bullet^{(2)})$, so we may regard it as a morphism of $\Delta^0/\mathsf{Gpd}^\infty$:

$$\Omega^1(X,x)^2 \to (X(x_\bullet^{(2)}), x).$$

We compose the latter with the morphism $X(\psi_2) : (X(x_\bullet^{(2)}), x) \to \Omega^1(X,x)$ defined as in §6.6.12, to get *a composition law for loop spaces* :

$$\boxed{\gamma_{X,x} : \Omega^1(X,x)^2 \to \Omega^1(X,x).}$$

Remark 6.6.14. The (unique) isomorphism $\omega : \Delta^0 \xrightarrow{\sim} \{\mathbf{1}_x\}$ can be regarded as an object of $\Delta^0/\mathsf{Gpd}^\infty$, and then the inclusion $\{\mathbf{1}_x\} \to X(x,x)$ is a morphism $(\{\mathbf{1}_x\}, \omega) \to \Omega^1(X,x)$ of $\Delta^0/\mathsf{Gpd}^\infty$. Since moreover $\sigma_0^2 \circ \psi_2 = \mathbf{1}_{[1]} = \sigma_1^2 \circ \psi_2$, the composition law $\gamma_{X,x}$ fits into the commutative diagram of $\Delta^0/\mathsf{Gpd}^\infty$:

$$\begin{tikzcd} \Omega^1(X,x) \times \{\mathbf{1}_x\} \arrow[r, hook] \arrow[dr, "p_1"'] & \Omega^1(X,x)^2 \arrow[d, "\gamma_{X,x}"] & \{\mathbf{1}_x\} \times \Omega^1(X,x) \arrow[l, hook'] \arrow[dl, "p_2"] \\ & \Omega^1(X,x) & \end{tikzcd}$$

where p_1 and p_2 are the natural identifications.

6.7. **The Serre long exact sequence.** Recall that for every ∞-groupoid X, the category $\mathsf{ho}(X)$ is a groupoid (corollary 2.6.13(i)), hence $\pi_0(\Omega^1(X,x)) := \pi_0(X,x)$ is a group, for every $x \in X_0$ (notation of definition 6.6.3(ii)), with the composition law inherited from $\mathsf{ho}(x,x)$ via the natural bijection $\pi_0(\Omega^1(X,x)) \xrightarrow{\sim} \mathsf{ho}(x,x)$ of proposition 6.6.6.

6.7.1. Let us introduce the slice category $\mathsf{Gpd}_\circ^\infty := \Delta^0/\mathsf{Gpd}^\infty$; then we define inductively

$$\Omega^{n+1}(X,x) := \Omega^1(\Omega^n(X,x)) \qquad \forall n \geq 1, \forall (X,x) \in \mathrm{Ob}(\mathsf{Gpd}_\circ^\infty).$$

For every $n \geq 1$, we then get *the n-th homotopy group of X at the point x*, defined as

$$\boxed{\pi_n(X,x) := \pi_0(\Omega^n(X,x)).}$$

Likewise, we shall sometimes use the notation :

$$\boxed{\pi_0(X,x) := (\pi_0(X), [x])}$$

where $[x] \in \pi_0(X)$ denotes the class of $x : \Delta^0 \to X$. Let Grp be the category of groups; clearly, we get well-defined functors :

$$\Omega^n : \mathsf{Gpd}^\infty_\circ \to \mathsf{Gpd}^\infty_\circ \qquad \pi_0 : \mathsf{Gpd}^\infty_\circ \to \mathsf{Set}_\circ \qquad \pi_n : \mathsf{Gpd}^\infty_\circ \to \mathsf{Grp} \quad \forall n \geq 1.$$

Lemma 6.7.1. *(i) The functor $\pi_0 : \mathsf{Gpd}^\infty_\circ \to \mathsf{sSet}_\circ$ preserves all small filtered colimits.*

(ii) The functor $\pi_n : \mathsf{Gpd}^\infty_\circ \to \mathsf{Grp}$ preserves all small filtered colimits, for every $n \geq 1$.

Proof. (i): This functor π_0 is the restriction of the functor ${}_{\Delta^0/}\pi_0 : \mathsf{sSet}_\circ \to \mathsf{Set}_\circ$ induced by $\pi_0 : \mathsf{sSet} \to \mathsf{Set}$ according to §1.4.8, which shows that ${}_{\Delta^0/}\pi_0$ is still a left adjoint, and therefore preserves all representable colimits of $\mathsf{sSet}_\circ$ ([13, Prop.2.49(ii)]).

Next, by proposition 5.1.11(iii), the class $\mathrm{Ob}(\mathsf{Gpd}^\infty)$ of Kan complexes is stable under filtered colimits in sSet; in light of [13, Lemma 2.52(i)] and corollary 1.4.6(ii), we deduce that $\mathsf{Gpd}^\infty_\circ$ is I-cocomplete for every small filtered category I, and the inclusion functor $\mathsf{Gpd}^\infty_\circ \to \mathsf{sSet}_\circ$ preserves and reflects I-colimits, for every such I, whence the assertion.

(ii): Likewise, Ω^n preserves all small filtered colimits, for every $n \geq 1$ (remark 6.6.4(v)), and moreover the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ preserves and reflects all small filtered colimits; combining with (i), we deduce that π_n preserves all small filtered colimits. □

Proposition 6.7.2. *The group $\pi_n(X, x)$ is abelian for every $n \geq 2$.*

Proof. Clearly, it suffices to show that $\pi_2(X, x)$ is abelian. To this aim, we consider the composition law of §6.6.13

$$\gamma_{X,x} : \Omega^1(X, x) \times \Omega^1(X, x) \to \Omega^1(X, x).$$

Since Ω^1 preserves products (remark 6.6.4(iv)), we deduce two morphisms of $\Delta^0/\mathsf{Gpd}^\infty$:

$$\Omega^2(X, x) \xleftarrow{\gamma_{\Omega^1(X,x)}} \Omega^2(X, x) \times \Omega^2(X, x) \xrightarrow{\Omega^1(\gamma_{X,x})} \Omega^2(X, , x)$$

and recall that also π_0 preserves finite products (proposition 5.4.6(i)). Then, let us set :

$$a \bullet b := \pi_0(\Omega^1(c_X))(a, b) \qquad a \circ b := \pi_0(c_{\Omega^1(X,x)})(a, b) \qquad \forall a, b \in \pi_2(X, x).$$

Hence, $- \circ -$ is just the composition law of the group $\pi_2(X, x)$, whose neutral element is the class $[x]$ of the base point $x : \Delta^0 \to \Omega^2(X, x)$; whereas $-\bullet-$ is a group homomorphism $\pi_2(X, x) \times \pi_2(X, x) \to \pi_2(X, x)$, so that :

$$(a \bullet b) \circ (c \bullet d) = (a \circ c) \bullet (b \circ d) \qquad \forall a, b, c, d \in \pi_2(X, x).$$

Claim 6.7.3. We have $a \bullet [x] = a = [x] \bullet a$ for every $a \in \pi_2(X, x)$.

Proof : With the notation of §6.6.13, let $\omega : \Delta^0 \xrightarrow{\sim} \{\mathbf{1}_x\}$ be the unique isomorphism; it is easily seen that there exists a unique isomorphism $\Omega^1(\{\mathbf{1}_x\}, \omega) \xrightarrow{\sim} (\Delta^0, \mathbf{1}_{\Delta^0})$, and since the inclusion of $\{\mathbf{1}_x\}$ into $X(x, x)$ is a morphism $i : (\{\mathbf{1}_x\}, \omega) \to \Omega^1(X, x)$ of $\Delta^0/\mathsf{Gpd}^\infty$, we get a morphism $\Omega^1(i) : (\Delta^0, \mathbf{1}_{\Delta^0}) \to \Omega^2(X, x)$ of $\Delta^0/\mathsf{Gpd}^\infty$. This means that $\Omega^1(i)$ preserves the base points, *i.e.* $\Omega^1(i) : \Delta^0 \to \Omega^2(X, x)$ *is* the base point x of $\Omega^2(X, x)$. Therefore, by applying the functor Ω^1 to the commutative diagram of remark 6.6.14, we get a commutative diagram :

$$\begin{tikzcd}
\Omega^2(X,x) \times \Delta^0 \arrow[r, "\Omega^2(X,x)\times x"] \arrow[dr, "p_1"'] & \Omega^2(X,x) \times \Omega^2(X,x) \arrow[d, "\Omega^1(\gamma_{X,x})"] & \Delta^0 \times \Omega^1(X,x) \arrow[l, "x\times\Omega^2(X,x)"'] \arrow[dl, "p_2"] \\
& \Omega^2(X,x) &
\end{tikzcd}$$

where p_1 and p_2 are the natural identifications. The assertion then follows, after applying the functor π_0 to this diagram. ◇

We may now compute :

$$a \circ b = (a \bullet [x]) \circ ([x] \bullet b) = (a \circ [x]) \bullet ([x] \circ b) = a \bullet b$$
$$b \circ a = ([x] \bullet b) \circ (a \bullet [x]) = ([x] \circ a) \bullet (b \circ [x]) = a \bullet b$$

whence $a \circ b = b \circ a$, as required. □

Proposition 6.7.4. *Let $f : X \to Y$ be an equivalence of ∞-groupoids, and $x \in X_0$. Then :*

(i) f induces an equivalence of ∞-groupoids $\Omega^1(f,x) : \Omega^1(X,x) \to \Omega^1(Y,f(x))$.

(ii) The induced map $\pi_n(X,x) \to \pi_n(Y,f(x))$ is a group isomorphism, for every $n \geq 1$.

Proof. (i): We have the commutative square :

$$\mathsf{D} \quad : \quad \begin{tikzcd} \mathscr{H}om(\Delta^1, X) \arrow[r, "f_*"] \arrow[d, "p"'] & \mathscr{H}om(\Delta^1, Y) \arrow[d, "q"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd}$$

whose top (resp. bottom) horizontal arrow is an equivalence of ∞-groupoids, by virtue of theorem 6.5.8 and lemma 6.4.6(iii) (resp. by virtue of corollary 6.5.3, theorem 6.5.1(iv) and example 2.6.2(ii)), and where p and q are the evaluation morphisms as in §6.6. Hence, f_* and $f \times f$ are weak homotopy equivalences (theorem 6.5.1(iv)), so D is homotopy cartesian for the Kan-Quillen model category (lemma 3.6.4(iii)). Moreover, p and q are Kan fibrations, by corollary 5.1.12(i); hence, f_* restricts to a weak homotopy equivalence $p^{-1}(x,x) \to q^{-1}(f(x),f(x))$, by corollary 5.4.3. The latter is precisely the morphism $\Omega^1(f,x)$, and we conclude with theorem 6.5.1(iv).

(ii): In light of (i) (and again, with theorem 6.5.1(iv)), an easy induction shows that f induces a weak homotopy equivalence $\Omega^n(X,x) \to \Omega^n(Y,f(x))$ for every $n \geq 1$; then we conclude with proposition 5.4.6(i). □

6.7.5. Let X be a simplicial set and $x \in X_0$; recall that $\mathsf{Ex}^\infty(X)$ is an ∞-groupoid (theorem 5.3.9), and the natural transformation $\beta_X^\infty : X \to \mathsf{Ex}^\infty(X)$ is a weak homotopy equivalence, that yields a natural identification on objects : $X_0 \xrightarrow{\sim} \mathsf{Ex}^\infty(X)_0$ (lemma 5.3.8(i,iii)), so by a small abuse of notation we may define :

$$\boxed{\pi_n(X,x) := \pi_n(\mathsf{Ex}^\infty(X), x) \qquad \forall n \in \mathbb{N}.}$$

By virtue of propositions 5.4.6(i) and 6.7.4(ii), and theorem 6.5.1(iv), the resulting functors

$$\pi_0 : \mathsf{sSet}_\circ \to \mathsf{Set}_\circ \qquad \text{and} \qquad \pi_n : \mathsf{sSet}_\circ \to \mathsf{Grp} \qquad \forall n \geq 1$$

extend, up to natural isomorphism, the functors π_n defined in §6.7.1 on the full subcategory $\mathsf{Gpd}^\infty_\circ$ (notation of §2.1.14).

Corollary 6.7.6. *Every weak homotopy equivalence $f : X \to Y$ of* sSet *induces group isomorphisms :*

$$\boxed{\pi_n(f,x) : \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y,f(x)) \qquad \forall n \geq 1, \forall x \in X_0.}$$

Proof. This follows immediately from proposition 6.7.4(ii), lemma 5.3.8(i) and theorem 6.5.1(iv). □

6.7.7. For every $n \in \mathbb{N}$ we define *the simplicial n-sphere* $\underline{S}^n$, as follows. For $n = 0$, we set $\underline{S}^0 := \partial\underline{\Delta}^1$, and for $n = 1$ we let $\underline{S}^1$ as the push-out in the cocartesian diagram :

$$\begin{tikzcd} \underline{S}^0 \arrow[r] \arrow[d] & \underline{\Delta}^1 \arrow[d] \\ \underline{\Delta}^0 \arrow[r] & \underline{S}^1 \end{tikzcd}$$

(notation of example 2.1.15(i)). Lastly, we define inductively

$$\underline{S}^n := \underline{S}^1 \wedge \underline{S}^{n-1} \qquad \forall n \geq 2.$$

Proposition 6.7.8. *Let* $\underline{X} \in \mathrm{Ob}(\mathrm{Gpd}^\infty_\circ)$*, and* $n \geq 1$ *an integer. Then :*

(i) With the notation of §6.7.7, we have a natural isomorphism of $\mathrm{sSet}_\circ$ *:*

$$\boxed{\mathscr{H}om_\circ(\underline{S}^n, \underline{X}) \xrightarrow{\sim} \Omega^n(\underline{X}).}$$

(ii) Also, with the notation of example 2.1.15(i), we have natural isomorphisms of $\mathrm{Set}_\circ$ *:*

$$\boxed{\pi_0(\mathscr{H}om_\circ((\partial\Delta^{n+1}, a), \underline{X})) \xrightarrow{\sim} \pi_n(\underline{X}) \qquad \forall a \in (\partial\Delta^{n+1})_0.}$$

(iii) The functor Ω^n *preserves pointed Kan fibrations between pointed Kan complexes.*

Proof. (i): We argue by induction on $n \geq 1$. The case where $n = 1$ follows directly from lemma 1.6.14. Now, suppose that we have already exhibited such a natural isomorphism for some $n \geq 1$; with proposition 2.1.16(ii), we get induced isomorphisms :

$$\begin{aligned} \mathscr{H}om_\circ(\underline{S}^{n+1}, \underline{X}) = \mathscr{H}om_\circ(\underline{S}^1 \wedge \underline{S}^n, \underline{X}) &\xrightarrow{\sim} \mathscr{H}om_\circ(\underline{S}^1, \mathscr{H}om_\circ(\underline{S}^n, \underline{X})) \\ &\xrightarrow{\sim} \mathscr{H}om_\circ(\underline{S}^1, \Omega^n(\underline{X})) \\ &\xrightarrow{\sim} \Omega^1(\Omega^n(\underline{X})) = \Omega^{n+1}(\underline{X}). \end{aligned}$$

(ii): Here, the group $\pi_n(\underline{X})$ is naturally a pointed set, with base point given by the neutral element; also, the stated naturality is meant with respect to morphisms $\underline{X} \to \underline{X}'$ of $\mathrm{Gpd}^\infty_\circ$. Now, a simple induction shows that, for every $n \geq 1$, the simplicial set S^n underlying $\underline{S}^n$ has a unique object $0_n : \Delta^0 \to S^n$, so that $\underline{S}^n = (S^n, 0_n)$. We prove first :

Claim 6.7.9. For every $n \in \mathbb{N}$, there exists an isomorphism $\partial\Delta^{n+1} \xrightarrow{\sim} S^n$ in $\mathrm{ho}(\mathrm{sSet})$.

Proof : We argue by induction on n : for $n = 0$, the sought isomorphism is just the identity of $\partial\Delta^1$. Next, suppose that the assertion is known for some $n \in \mathbb{N}$; recall that the functor $- \wedge \underline{S}^{n-1} : \mathrm{sSet}_\circ \to \mathrm{sSet}_\circ$ admits a right adjoint (proposition 2.1.16(i)), so it preserves all representable colimits ([13, Prop.2.49(ii)]); combining with corollary 1.4.6(ii), we deduce that both commutative squares :

$$(*) \qquad \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \Lambda^{n+1}_{n+1} \arrow[d] \\ \Delta^n \arrow[r] & \partial\Delta^{n+1} \end{tikzcd} \qquad \begin{tikzcd} S^0 \wedge S^{n-1} \arrow[r] \arrow[d] & \Delta^1 \wedge S^{n-1} \arrow[d] \\ \Delta^0 \wedge S^{n-1} \arrow[r] & S^n \end{tikzcd}$$

are cocartesian in sSet. Moreover, we have natural identifications : $S^0 \wedge S^{n-1} \xrightarrow{\sim} S^{n-1}$ and $\Delta^0 \wedge S^{n-1} \xrightarrow{\sim} \Delta^0$ (proposition 2.1.16(ii)). Futhermore, since the projection $\Delta^n \to \Delta^0$ is a weak homotopy equivalence for every $n \in \mathbb{N}$ (lemma 5.1.17(ii)), the same holds for the induced morphism $\Delta^1 \wedge S^{n-1} \to \Delta^0 \wedge S^{n-1} \xrightarrow{\sim} \Delta^0$. Likewise, the inclusion $\Lambda^{n+1}_{n+1} \to \Delta^{n+1}$ is an anodyne extension (corollary 5.1.6), so it is a weak homotopy equivalence, and therefore the same holds for the projection $\Lambda^{n+1}_{n+1} \to \Delta^0$, by the 2-out-of-3 property of the class of weak homotopy equivalences. Summing up, we get commutative diagrams, whose vertical arrows are weak homotopy equivalences :

$$(**)\qquad \begin{tikzcd} \Delta^n \arrow[d] & \partial\Delta^n \arrow[l] \arrow[r] \arrow[d, equal] & \Lambda^{n+1}_{n+1} \arrow[d] \\ \Delta^0 & \partial\Delta^n \arrow[l] \arrow[r] & \Delta^0 \end{tikzcd} \qquad \begin{tikzcd} \Delta^0 \arrow[d, equal] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & \Delta^1 \wedge S^{n-1} \arrow[d] \\ \Delta^0 & S^{n-1} \arrow[l] \arrow[r] & \Delta^0. \end{tikzcd}$$

Recall also that the functor $-\wedge\underline{S}^{n-1}$ preserves monomorphisms (corollary 5.4.11(i)); hence all the horizontal arrows of both squares $(*)$ are monomorphisms, and therefore *these two squares are also homotopy cocartesian* (proposition 3.6.5(i.b)).

Let $\beta^\infty_{S^{n-1}} : S^{n-1} \to \mathsf{Ex}^\infty(S^{n-1})$ be the trivial cofibration provided by lemma 5.3.8(i); by inductive assumption we have an isomorphism $\phi : \partial\Delta^n \xrightarrow{\sim} S^{n-1}$ in $\mathsf{ho}(\mathsf{sSet})$, and since $\mathsf{Ex}^\infty(S^{n-1})$ is a Kan complex (theorem 5.3.9), there exists a morphism $f : \partial\Delta^n \to \mathsf{Ex}^\infty(S^{n-1})$ in sSet such that $\gamma(f) = \gamma(\beta^\infty_{S^{n-1}}) \circ \phi$, where $\gamma : \mathsf{sSet} \to \mathsf{ho}(\mathsf{sSet})$ is the localization (remark 3.3.7(iv)). Clearly $\gamma(f)$ is an isomorphism of $\mathsf{ho}(\mathsf{sSet})$, so f is a weak homotopy equivalence (theorem 3.3.9(ii)). Hence, we get a commutative diagram of sSet:

$$\begin{tikzcd} \Delta^0 \arrow[d, equal] & \partial\Delta^n \arrow[l] \arrow[r] \arrow[d, "f"'] & \Delta^0 \arrow[d, equal] \\ \Delta^0 \arrow[d, equal] & \mathsf{Ex}^\infty(S^{n-1}) \arrow[l] \arrow[r] & \Delta^0 \arrow[d, equal] \\ \Delta^0 & S^{n-1} \arrow[l] \arrow[r] \arrow[u, "\beta^\infty_{S^{n-1}}"] & \Delta^0 \end{tikzcd}$$

whose vertical arrows are weak homotopy equivalences. Combining with $(**)$, we conclude that, both with $X = \partial\Delta^{n+1}$ and with $X = S^n$, we get a homotopy cocartesian square:

$$\begin{tikzcd} \mathsf{Ex}^\infty(S^{n-1}) \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\ \Delta^0 \arrow[r] & X \end{tikzcd}$$

whence an isomorphism $\partial\Delta^{n+1} \xrightarrow{\sim} S^n$ of $\mathsf{ho}(\mathsf{sSet})$. ◇

Now, we have already remarked that claim 6.7.9 implies the existence of a weak homotopy equivalence $f_n : \partial\Delta^{n+1} \to \mathsf{Ex}^\infty(S^n)$ for every $n \in \mathbb{N}$. Moreover, recall that $\mathsf{Ex}^\infty(S^n)$ has a unique object $0_n : \Delta^0 \to \mathsf{Ex}^\infty(S^n)$ (lemma 5.3.8(iii)), hence any object a of $\partial\Delta^{n+1}$ yields a pointed weak homotopy equivalence $f_n : (\partial\Delta^{n+1}, a) \to (\mathsf{Ex}^\infty(S^n), 0_n)$, and we have as well the pointed weak homotopy equivalence $\beta^\infty_{S^n} : (S^n, 0_n) \to (\mathsf{Ex}^\infty(S^n), 0_n)$. Hence, assertion (ii) follows, by combining with (i) and corollary 5.4.10.

(iii) follows from (i) and corollary 5.1.12(ii). □

Lemma 6.7.10. *Denote by* $\Delta^1/\mathsf{Gpd}^\infty$ *the full subcategory of the slice category* Δ^1/sSet *whose objects are the pairs* $\underline{X} := (X, h : \Delta^1 \to X)$ *with* $X \in \mathrm{Ob}(\mathsf{Gpd}^\infty)$*; also, for* $i = 0, 1$ *let* $d_i : \Delta^1/\mathsf{Gpd}^\infty \to \mathsf{Gpd}^\infty_\circ$ *be the restriction of the functor* $\partial^1_{i!} : \Delta^1/\mathsf{sSet} \to \mathsf{sSet}_\circ$ *induced by* $\partial^1_i : \Delta^0 \to \Delta^1$ *(see §1.4.1). For every* $n \geq 1$ *and* $i = 0, 1$ *there exist a functor*

$$L^n : \Delta^1/\mathsf{Gpd}^\infty \to \Delta^1/\mathsf{Gpd}^\infty \quad \textit{and natural transformations} \quad \omega^{n,i}_\bullet : \Omega^n \circ d_i \Rightarrow d_i \circ L^n$$

such that $\omega^{n,i}_{\underline{X}}$ *is a pointed weak homotopy equivalence for every* $\underline{X} \in \mathrm{Ob}(\Delta^1/\mathsf{Gpd}^\infty)$.

Proof. Let $\underline{X} := (X,h) \in \mathrm{Ob}(\Delta^1/\mathsf{Gpd}^\infty)$, and set $x_i := h \circ \partial_i^1 : \Delta^0 \to X$ for $i = 0, 1$; we consider the cartesian squares of sSet :

$$\begin{tikzcd} \Omega^1(X,x_i) \arrow[r, "j_i"] \arrow[d] & F \arrow[r] \arrow[d, "q"] & \mathscr{H}om(\Delta^1,X) \arrow[d, "(\partial_1^{1*},\partial_0^{1*})"] \\ \Delta^0 \arrow[r, "\partial_i^1"] & \Delta^1 \arrow[r, "(h,h)"] & X \times X. \end{tikzcd}$$

By applying corollary 5.1.12(i) to the projection $p : X \to \Delta^0$ and the monomorphism $i := (\partial_0^1, \partial_1^1) : \Delta^0 \sqcup \Delta^0 \to \Delta^1$ we see that $(\partial_1^{1*}, \partial_0^{1*})$ is a Kan fibration, hence the same holds for q (proposition 3.1.9(v)). On the other hand, ∂_i^1 is a weak homotopy equivalence for $i = 0, 1$ (lemma 5.1.17(i)), hence the same holds for j_i (corollary 5.4.1).

- Next, let $p_1 : \Delta^1 \times \Delta^1 \to \Delta^1$ be the projection onto the first factor; the morphism $h \circ p_1 : \Delta^1 \times \Delta^1 \to X$ defines a 1-simplex $H : \Delta^1 \to \mathscr{H}om(\Delta^1, X)$ of $\mathscr{H}om(\Delta^1, X)$ (see §2.1.6) such that $H \circ \partial_i^1 : \Delta^0 \to \mathscr{H}om(\Delta^1, X)$ corresponds to the unique morphism $\Delta^1 \to X$ that factors through x_i, *i.e.* $H \circ \partial_i^1 = \mathbf{1}_{x_i}$ for $i = 0, 1$. On the other hand, $(\partial_1^{1*}, \partial_0^{1*}) \circ H = (h, h)$, so H yields a 1-simplex of F that we denote again $H : \Delta^1 \to F$.
- By theorem 5.3.9 and lemma 5.3.8(i), we may then set

$$L^1(\underline{X}) := (\mathsf{Ex}^\infty(F), \beta_F^\infty \circ H) \in \mathrm{Ob}(\Delta^1/\mathsf{Gpd}^\infty) \qquad \text{and} \qquad \omega_{\underline{X}}^{1,i} := \beta_F^\infty \circ j_i \quad \text{for } i = 0, 1.$$

Recall that the base point of $\Omega^1(X, x_i)$ is $\mathbf{1}_{x_i}$, so $\omega_{\underline{X}}^{1,i}$ is indeed a morphism of pointed simplicial sets $\Omega^1(X, x_i) \to d_i \circ L^1(\underline{X}) = (\mathsf{Ex}^\infty(F), \beta_F^\infty \circ \mathbf{1}_{x_i})$, for $i = 0, 1$. The functoriality of L^1 and the naturality of $\omega_\bullet^{1,i}$ follow by a direct inspection of the constructions : details left to the reader; this completes the proof for $n = 1$.

- Lastly, for every $n \geq 1$ and $i = 0, 1$ we define inductively $L^{n+1} := L^n \circ L$, and we let $\omega_\bullet^{n+1,i}$ be the composition :

$$\Omega^{n+1} \circ d_i = \Omega^n \circ \Omega^1 \circ d_i \xRightarrow{\Omega^n \star \omega^{1,i}} \Omega^n \circ d_i \circ L \xRightarrow{\omega^{n,i} \star L} d_i \circ L^n.$$

By theorem 6.5.1(iv), proposition 6.7.4(i), and the foregoing case $n = 1$, we see inductively that $\omega_{\underline{X}}^{n,i}$ is a pointed weak homotopy equivalence for every $n \geq 1$, for $i = 0, 1$, and for every $\underline{X} \in \mathrm{Ob}(\Delta^1/\mathsf{Gpd}^\infty)$, so the proof is concluded. □

Proposition 6.7.11. *Let X be a simplicial set. Then X is weakly contractible (see definition 5.1.2(ii)) if and only if the following two conditions hold :*

(a) *X is connected (see §2.1.11).*

(b) *For every $n \geq 1$ there exists $x_n \in X_0$ such that $\pi_n(X, x_n)$ is a trivial group.*

Proof. The conditions are necessary, in view of corollary 6.7.6, proposition 5.4.6(i) and example 2.1.13. For the converse, after replacing X by $\mathsf{Ex}^\infty(X)$, we may assume that X is a Kan complex. We first remark :

Claim 6.7.12. If (a) and (b) hold, $\pi_n(X, x)$ is a trivial group for all $n \geq 1$ and all $x \in X_0$.

Proof : (i): Let $x_n \in X$ such that $\pi_n(X, x_n)$ is a trivial group, and x any other object of X. By assumption, the morphisms $x, x_n : \Delta^0 \to X$ are Δ^1-homotopy equivalent (proposition 5.4.6(ii)); since X is a Kan complex, we have then a Δ^1-homotopy $h : \Delta^1 \to X$ from x to x_n (lemma 3.2.11(ii)). We apply lemma 6.7.10 to $\underline{X} := (X, h)$, to get $L(\underline{X}) = (LX, H : \Delta^1 \to LX) \in \mathrm{Ob}(\Delta^1/\mathsf{Gpd}^\infty)$ and pointed weak homotopy equivalences :

$$\Omega^n(X, x) \to (LX, H \circ \partial_1^1) \qquad \Omega^n(X, x_n) \to (LX, H \circ \partial_0^1).$$

Since $\Omega^n(X, x_n)$ is connected, it follows that the same holds for LX (proposition 5.4.6(i)), and then also for $\Omega^n(X, x)$, whence the claim. ◇

By virtue of proposition 6.7.8(ii), claim 6.7.12 means that for every such n and every morphism $f : \partial\Delta^n \to X$, there exists $x \in X_0$ and a morphism $h : \Delta^1 \times \partial\Delta^n \to X$ that is a homotopy from f to the unique morphism $g : \partial\Delta^n \to X$ that factors through $x : \Delta^0 \to X$. Let also $p : \Delta^n \to X$ be the projection; we then get a morphism $(h, x \circ p) : \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \to X$, which extends to a morphism $H : \Delta^1 \times \Delta^n \to X$, since X is a Kan complex (corollary 5.1.6). The restriction of H to $\{0\} \times \Delta^n$ extends f; summing up, we have shown that for every $n \geq 1$, every morphism $\partial\Delta^n \to X$ extends to a morphism $\Delta^n \to X$. The same holds also for $n = 0$, since X is connected (hence, non-empty); since $\{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$ is a cellular model for sSet, we conclude that the unique morphism $X \to \Delta^0$ is a trivial fibration, and especially, a weak homotopy equivalence. □

Theorem 6.7.13. *Let $f : X \to Y$ be a Kan fibration of* sSet, *and $x \in X_0$. Set $y := f(x)$, $F := f^{-1}(y)$ (notation of definition 2.5.1(iv)), $\underline{X} := (X, x)$, $\underline{Y} := (Y, y)$, $\underline{F} := (F, x)$, and let $j : F \to X$ be the inclusion. Then f induces a natural long exact sequence of pointed sets :*

$$\boxed{\cdots \to \pi_n(\underline{F}) \xrightarrow{\pi_n(j)} \pi_n(\underline{X}) \xrightarrow{\pi_n(f)} \pi_n(\underline{Y}) \xrightarrow{\partial_{f,n}} \pi_{n-1}(\underline{F}) \to \cdots \to \pi_0(\underline{F}) \to \pi_0(\underline{X}) \to \pi_0(\underline{Y})}$$

that we call the Serre long exact sequence attached to f and x. *Its restriction to the terms $\pi_n(-)$ with $n > 0$ (resp. $n > 1$) is a long exact sequence of groups (resp. of abelian groups).*

Proof. We say that a sequence $(f_n : (S_n, s_n) \to (S_{n-1}, s_{n-1}) \mid n \in \mathbb{Z})$ of morphisms of $\mathrm{Set}_\circ$ is *exact* if $f_{n+1}(S_{n+1}) = f_n^{-1}(s_{n-1})$ for every $n \in \mathbb{Z}$. To begin with, we remark :

Claim 6.7.14. Consider a homotopy cartesian diagram of sSet :

$$(\mathsf{D}) \qquad : \qquad \begin{tikzcd} A' \arrow[r, "h"] \arrow[d, "p'"'] & A \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

with B' weakly contractible. Then every $a' \in A'_0$ induces an exact sequence of $\mathrm{Set}_\circ$:

$$(*) \qquad \pi_0(A', a') \xrightarrow{\pi_0(h)} \pi_0(A, a) \xrightarrow{\pi_0(p)} \pi_0(B, b) \qquad \text{with } a := h(a') \text{ and } b := p(a).$$

Proof : According to example 3.6.2(iii), there exists a cartesian square (E) of sSet consisting of Kan fibrations between Kan complexes, and a weak equivalence $\omega : (\mathsf{D}) \to (\mathsf{E})$ of $\mathrm{sSet}_\square$. By proposition 5.4.6(i), ω identifies $(*)$ with the corresponding sequence induced by (E) and the image of a' in (E). Thus, after replacing (D) by (E) we may assume that (D) is cartesian and that both p and p' are Kan fibrations between Kan complexes.

Next, let $b' := p'(a')$, and denote by $A'' \subset A'$ the fibre of p' over b'; we get another cartesian and homotopy cartesian square of sSet :

$$(\mathsf{D}') \qquad : \qquad \begin{tikzcd} A'' \arrow[r, "j"] \arrow[d] & A' \arrow[d, "p'"] \\ \Delta^0 \arrow[r, "b'"] & B' \end{tikzcd}$$

and moreover, since the unique morphism $B' \to \Delta^0$ is a weak homotopy equivalence, the same holds for b', so also for the inclusion j (corollary 5.4.1). The composition of (D) and (D') is yet another cartesian and homotopy cartesian square (D'') whose top horizontal arrow is $h \circ j : A'' \to A'$ (proposition 3.6.3(i)), and since $\pi_0(j) : \pi_0(A'') \to \pi_0(A')$ is a bijection (proposition 5.4.6(i)), $(*)$ is again naturally identified with the corresponding sequence induced by (D''). Thus, we may replace (D) by (D''), and assume that $B' = \Delta^0$.

In this situation, since $g(b') = b$, it is clear that the composition of the maps in $(*)$ is the constant map with value the class $[b]$ of b. Lastly, let $x : \Delta^0 \to A$ be any object of A such that the class of $p \circ x : \Delta^0 \to B$ in $\pi_0(B)$ equals the class of b; since B is a Kan complex, this means that there exists a homotopy $h : \Delta^1 \to B$ from $p \circ x$ to b (lemma 3.2.11(ii)), whence a commutative square in sSet :

$$\begin{tikzcd} \Delta^0 \arrow[r, "x"] \arrow[d, "\partial^1_1"'] & A \arrow[d, "p"] \\ \Delta^1 \arrow[r, "h"] & B \end{tikzcd}$$

and since p is a Kan fibration, this square admits a diagonal filler $H : \Delta^1 \to A$. Set $y := H \circ \partial^1_0 : \Delta^0 \to A$, and notice that $p \circ y = h \circ \partial^1_0 = b$, so $y \in A'_0$; then the class $[y]$ of y in $\pi_0(A')$ maps to the class of $H \circ \partial^1_1 = x$ in $\pi_0(A)$, and the claim is proved. ◇

- Now, let us first assume that Y and X are Kan complexes; then the cartesian square:

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

is also homotopy cartesian (proposition 3.6.5(ii)), and we get the exact sequence :

$$(\dagger) \qquad \pi_0(\underline{F}) \xrightarrow{\pi_0(j)} \pi_0(\underline{X}) \xrightarrow{\pi_0(f)} \pi_0(\underline{Y}).$$

with claim 6.7.14. Next, we consider the commutative diagram of $\mathsf{sSet}_\circ$:

$$\begin{tikzcd} \Omega^1(\underline{X}) \arrow[r, "i_{\underline{X}}"] \arrow[d] & Q(\underline{X}) \arrow[r] \arrow[d, "q"'] & P(\underline{X}) \arrow[r] \arrow[d, "p_{\underline{X}}"'] & \underline{\mathscr{H}om}(\underline{\Delta}^1, \underline{X}) \arrow[d, "(\partial^{1*}_1, \partial^{1*}_0)"] \\ \underline{\Delta}^0 \arrow[r, "x"] & \underline{F} \arrow[r, "j"] & \underline{X} \arrow[r, "(\mathbf{1}_X, x)"] \arrow[d] & \underline{X} \times \underline{X} \arrow[d, "p_2"] \\ & & \underline{\Delta}^0 \arrow[r, "x"] & \underline{X} \end{tikzcd}$$

whose square subdiagrams are all cartesian (and where p_2 is the projection on the second factor). Recall that $\mathscr{H}om(\Delta^1, X)$ is a Kan complex, $(\partial^{1*}_1, \partial^{1*}_0)$ is a Kan fibration and $p_2 \circ (\partial^{1*}_1, \partial^{1*}_0) = \partial^{1*}_0$ is a trivial fibration (corollary 5.1.12); hence, all the terms in the diagram are Kan complexes, and all the squares are homotopy cartesian, again by proposition 3.6.5(ii). Moreover, the unique morphism $P(\underline{X}) \to \Delta^0$ is a trivial fibration, *i.e.* $P(\underline{X})$ is weakly contractible. We may then apply claim 6.7.14 to the two squares whose bottom sides are given respectively by x and j ; we get a single exact sequence of $\mathsf{Set}_\circ$:

$$(\dagger\dagger) \qquad \pi_1(\underline{X}) \xrightarrow{\pi_0(i_{\underline{X}})} \pi_0 Q(\underline{X}) \xrightarrow{\pi_0(q)} \pi_0(\underline{F}) \xrightarrow{\pi_0(j)} \pi_0(\underline{X}).$$

We have a unique morphism $P(f) : P(\underline{X}) \to P(\underline{Y})$ that makes commute the diagram :

$$\begin{tikzcd}
P(\underline{Y}) \arrow[rrr] \arrow[ddd, "p_{\underline{Y}}"'] & & & \mathscr{H}om(\underline{\Delta}^1, \underline{Y}) \arrow[ddd, "(\partial_1^{1*}, \partial_0^{1*})"] \\
& P(\underline{X}) \arrow[ul, "P(f)"'] \arrow[r] \arrow[d, "p_{\underline{X}}"'] & \mathscr{H}om(\underline{\Delta}^1, \underline{X}) \arrow[ur, "f_*"] \arrow[d, "(\partial_1^{1*}, \partial_0^{1*})"] & \\
& \underline{X} \arrow[r, "(\mathbf{1}_X, x)"] \arrow[dl, "f"'] & \underline{X} \times \underline{X} \arrow[dr, "f \times f"] & \\
\underline{Y} \arrow[rrr, "(\mathbf{1}_Y, y)"] & & & \underline{Y} \times \underline{Y}
\end{tikzcd}$$

whose inner and outer squares are cartesian. Since both $P(\underline{X})$ and $P(\underline{Y})$ are weakly contractible, $P(f)$ is clearly a weak homotopy equivalence. Lastly, there exists a unique morphism $Q(f) : Q(\underline{X}) \to \Omega^1(\underline{Y})$ that makes commute the diagram :

$$\begin{tikzcd}
Q(\underline{X}) \arrow[rrr] \arrow[ddd] \arrow[dr, "Q(f)"] & & & P(\underline{X}) \arrow[ddd, "f \circ p_{\underline{X}}"] \arrow[dl, "P(f)"'] \\
& \Omega^1(\underline{Y}) \arrow[r] \arrow[d] & P(\underline{Y}) \arrow[d, "p_{\underline{Y}}"] & \\
& \underline{\Delta}^0 \arrow[r, "y"] \arrow[dl, equal] & \underline{Y} \arrow[dr, equal] & \\
\underline{\Delta}^0 \arrow[rrr, "y"] & & & \underline{Y}
\end{tikzcd}$$

whose inner and outer squares are again cartesian. Since $p_{\underline{Y}}$ and $f \circ p_{\underline{X}}$ are Kan fibrations, it follows that these two squares are also homotopy cartesian (proposition 3.6.5(ii)), and since $P(f)$ is a weak homotopy equivalence, we conclude that the same holds for $Q(f)$ (remark 3.5.15(ii)). A direct inspection of the construction shows that :

$$Q(f) \circ i_{\underline{X}} = \Omega^1(f) : \Omega^1(\underline{X}) \to \Omega^1(\underline{Y})$$

(details left to the reader); since $\pi_0 Q(f)$ is a bijection (proposition 5.4.6(i)), we let

$$\partial_{f,1} := \pi_0(q) \circ \pi_0(Q(f))^{-1} : \pi_1(\underline{Y}) = \pi_0 \Omega^1(\underline{Y}) \to \pi_0(\underline{F})$$

and we combine with (†) and (††) to get the first five terms of the Serre exact sequence.

• For the following terms, we notice that the functor Ω^n preserves all representable limits, for every $n \geq 1$ (remark 6.6.4(iv)), hence $\Omega^n(\underline{\Delta}^0) = \underline{\Delta}^0$ for every such n (remark 1.1.11(ii)), and therefore $\Omega^n(\underline{F})$ represents the fibre of $\Omega^n(f) : \Omega^n(\underline{X}) \to \Omega^n(\underline{Y})$; moreover, $\Omega^n(f)$ is a pointed Kan fibration (proposition 6.7.8(iii)), so we may apply the foregoing discussion to $\Omega^n(f)$, and this yields inductively the initial segment of the Serre long exact sequence, up to the term $\pi_{n+1}(\underline{X})$.

• Lastly, if f is a Kan fibration between given $X, Y \in \mathrm{Ob}(\mathsf{sSet})$, we have defined $\pi_n(f) := \pi_n(\mathsf{Ex}^\infty(f))$, and likewise for $\pi_n(j)$; however, $\mathsf{Ex}^\infty(f)$ is still a Kan fibration, and Ex^∞ is left exact (lemma 5.3.8(ii)), so we have $\mathsf{Ex}^\infty(\Delta^0) = \Delta^0$ (remark 1.1.11(ii)), and we get a cartesian diagram of $\mathsf{sSet}_\circ$:

$$\begin{tikzcd}
(\mathsf{Ex}^\infty(F), x) \arrow[r] \arrow[d] & (\mathsf{Ex}^\infty(X), x) \arrow[d, "\mathsf{Ex}^\infty(f)"] \\
\underline{\Delta}^0 \arrow[r, "y"] & (\mathsf{Ex}^\infty(Y), y)
\end{tikzcd}$$

for the natural identification of $x \in X_0$ and $y \in Y_0$ with their images in $\mathsf{Ex}^\infty(X)$ and $\mathsf{Ex}^\infty(Y)$, under the natural weak homotopy equivalences $\beta_X^\infty : X \to \mathsf{Ex}^\infty(X)$ and respectively $\beta_Y^\infty : Y \to \mathsf{Ex}^\infty(Y)$. Hence, $\mathsf{Ex}^\infty(F)$ is the fibre of $\mathsf{Ex}^\infty(f)$ over y, and the previous case applies to $\mathsf{Ex}^\infty(f)$, to conclude the construction of the long Serre exact sequence. The naturality of the sequence with respect to morphisms $(X \xrightarrow{f} Y) \to (X' \xrightarrow{f'} Y')$ of $\mathsf{sSet}^{[1]}$ (for any two Kan fibrations f, f') is clear from the construction. □

Corollary 6.7.15. *A morphism $f : X \to Y$ of simplicial sets is a weak homotopy equivalence if and only if the following two conditions hold :*

(a) f induces a bijection $\pi_0(f) : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$.

(b) f induces isomorphisms of groups

$$\pi_n(f, x) : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x)) \qquad \forall x \in X_0, \forall n \geq 1.$$

Proof. The conditions are necessary, by virtue of corollary 6.7.6 and proposition 5.4.6(i). Conversely, suppose that (a) and (b) hold for f; since $\beta_X^\infty : X \to \mathsf{Ex}^\infty(X)$ and $\beta_Y^\infty : Y \to \mathsf{Ex}^\infty(Y)$ are weak homotopy equivalences (lemma 5.3.8(i)), it suffices to check that the same holds for $\mathsf{Ex}^\infty(f)$, and since β_X^∞ and β_Y^∞ induce bijections on objects (lemma 5.3.8(iii)), proposition 5.4.6(i) and corollary 6.7.6 easily imply that $\mathsf{Ex}^\infty(f)$ stilll fulfills conditions (a) and (b). Thus, after replacing f by $\mathsf{Ex}^\infty(f)$ we may assume that X and Y are Kan complexes (theorem 5.3.9).

Now, let us write $f = g \circ h$ for a weak homotopy equivalence $h : X \to X'$ and a Kan fibration $g : X' \to Y$. Since h fulfills condition (a), the same clearly holds for g, and notice that X' is a Kan complex, since the same holds for Y. Next, let $x' \in X'_0$; condition (a) for h says that there exists $x \in X_0$ such that x' is Δ^1-homotopy equivalent to $h(x)$, so we have a Δ^1-homotopy $u : \Delta^1 \to X'$ from x' to $h(x)$ (lemma 3.2.11(ii)). According to lemma 6.7.10, we then get for every $n \geq 1$ a commutative diagram of Gpd^∞ :

$$\begin{tikzcd}
\Omega^n(X', x') \arrow[r] \arrow[d, "\Omega^n(g,x')"'] & L(X, u) \arrow[d] & \Omega^n(X', h(x)) \arrow[l] \arrow[d, "\Omega^n(g,h(x))"] \\
\Omega^n(Y, g(x')) \arrow[r] & L(Y, g \circ u) & \Omega^n(Y, f(x)) \arrow[l]
\end{tikzcd}$$

whose horizontal arrows are weak homotopy equivalences. In light of proposition 5.4.6(i), we then see that $\pi_n(g, x') : \pi_n(X', x') \to \pi_n(Y, g(x'))$ is a group isomorphism if and only if the same holds for $\pi_n(g, h(x)) : \pi_n(X', h(x)) \to \pi_n(f(x))$. On the other hand, $\pi_n(f, x) = \pi_n(g, h(x)) \circ \pi_n(h, x)$, and $\pi_n(h, x)$ is an isomorphism, since h is a weak homotopy equivalence; the same holds by assumption for $\pi_n(f, x)$, so also for $\pi_n(g, h(x))$. Summing up, we find that g fulfills again conditions (a) and (b), and clearly it suffices to check that g is a weak homotopy equivalence; thus, we may replace f by g, and assume that f is a Kan fibration between Kan complexes.

In this situation, theorem 6.7.13 implies that for every $y \in Y_0$, the fibre $F_y := f^{-1}(y)$ is a connected Kan complex with trivial homotopy groups in every degree $n \geq 1$; by virtue of proposition 6.7.11, F_y is therefore weakly contractible, for every $y \in Y_0$. Let us then apply proposition 5.4.2(ii) with $S := Y$, $p := f$ and $q := \mathbf{1}_Y$; we have just shown that f induces a weak homotopy equivalence $f_y : p^{-1}(y) \to q^{-1}(y) = \Delta^0$ for every $y \in Y$, so we get that f is a weak homotopy equivalence. □

6.8. **Fully faithful functors and essentially surjective functors.** In this section we extend to ∞-categories the usual categorical classes of fully faithful functors and essentially surjective functors, and we use them to give another characterization of equivalences of ∞-categories. We begin with the following refinement of theorem 6.5.8 :

Theorem 6.8.1. *Let $f : X \to Y$ be a morphism between ∞-categories. The following conditions are equivalent :*

(a) f is an equivalence of ∞-categories.

(b) f induces an equivalence of ∞-groupoids $k(\Delta^n, f) : k(\Delta^n, Y) \to k(\Delta^n, X)$ for $n = 0, 1$.

Proof. We have (a)⇒(b) by theorem 6.5.8; for the converse, in light of the same theorem, it suffices to show that if (b) holds, then Δ^n lies in the class $\mathscr{F}$ of simplicial sets A such that $k(A, f)$ is an equivalence of ∞-groupoids, for every $n \in \mathbb{N}$. However, recall that the inclusion $Sp^n \to \Delta^n$ is an inner anodyne extension for every $n \geq 1$ (proposition 6.6.7(ii)), so the vertical arrows in the induced commutative diagram :

$$\begin{tikzcd} k(\Delta^n, X) \arrow[r, "k(\Delta^n, f)"] \arrow[d] & k(\Delta^n, Y) \arrow[d] \\ k(Sp^n, X) \arrow[r, "k(Sp^n, f)"] & k(Sp^n, Y) \end{tikzcd}$$

are equivalences of ∞-groupoids, by theorem 6.5.7. Thus, we are reduced to showing that $Sp^n \in \mathscr{F}$ for every $n \geq 1$. We argue by induction on $n \geq 1$: since $Sp^1 = \Delta^1$, the assertion holds for $n = 1$ by assumption. Next, suppose that the assertion holds for some $n \geq 1$; we have a cocartesian square of sSet :

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\ Sp^n \arrow[r] & Sp^{n+1} \end{tikzcd}$$

and by assumption Δ^0, Δ^1 and Sp^n lie in $\mathscr{F}$; then claim 6.5.9 implies that the same holds for Sp^{n+1}, as required. □

Definition 6.8.2. Let $f : X \to Y$ be a functor between ∞-categories.

(i) We say that f is *fully faithful,* if it induces equivalences of ∞-groupoids :

$$X(x, x') \to Y(f(x), f(x')) \qquad \forall x, x' \in X_0.$$

(ii) We say that f is *essentially surjective*, if for every $y \in Y_0$ there exists $x \in X_0$ with an invertible morphism $f(x) \to y$ of Y.

Remark 6.8.3. (i) By corollary 2.6.13(ii), a functor $f : X \to Y$ of ∞-categories is essentially surjective ⇔ the same holds for the induced functor of categories $\tau(f) : \tau(X) \to \tau(Y)$.

(ii) Moreover, propositions 6.6.6 and 5.4.6 and theorem 6.5.1(iv) imply that if f is fully faithful, the same holds for the functor $\tau(f)$.

(iii) In light of example 6.6.2, it is also clear that a functor $F : \mathscr{A} \to \mathscr{B}$ between small categories is fully faithful (resp. essentially surjective) if and only if the same holds for the induced functor $N(f) : N(\mathscr{A}) \to N(\mathscr{B})$ of ∞-categories (see also example 2.6.2(i)).

Example 6.8.4. Let X be an ∞-category, and A any subset of X_0; the *full subcategory generated by* A is the simplicial subset $X_A \subset X$ whose n-simplices are all the morphisms

$f : \Delta^n \to X$ such that $f(0), \dots, f(n) \in A$, for every $n \in \mathbb{N}$. We then get a cartesian diagram of sSet :

$$\begin{tikzcd} X_A \arrow[r] \arrow[d, "v"'] & X \arrow[d, "u"] \\ N(\mathsf{ho}(X)_A) \arrow[r, "N(j)"] & N(\mathsf{ho}(X)) \end{tikzcd}$$

where $j : \mathsf{ho}(X)_A \to \mathsf{ho}(X)$ denotes the inclusion of the full subcategory $\mathsf{ho}(X)_A$ of the homotopy category $\mathsf{ho}(X)$ whose set of objects is A, and where u is the functor of theorem 2.6.12(ii). In view of example 6.1.2(ii) and corollary 2.6.13(ii), it is easily seen that u is a conservative isofibration, and then the same holds for v : details left to the reader. Hence, X_A is an ∞-category (example 6.1.2(i)); moreover, the induced morphism $X_A(a,b) \to X(a,b)$ is the identity, for every $a, b \in A$, so *the inclusion $X_A \to X$ is a fully faithful functor.*

Proposition 6.8.5. *(i) A functor $f : X \to Y$ between ∞-categories is fully faithful ⇔ it induces a homotopy cartesian square of ∞-groupoids for the Kan-Quillen model structure :*

$$\begin{tikzcd} k(\Delta^1, X) \arrow[r, "k(\Delta^1, f)"] \arrow[d, "(\partial_1^{1*}, \partial_0^{1*})"'] & k(\Delta^1, Y) \arrow[d, "(\partial_1^{1*}, \partial_0^{1*})"] \\ k(X) \times k(X) \arrow[r, "k(f)\times k(f)"] & k(Y) \times k(Y). \end{tikzcd}$$

(ii) Moreover, f is an equivalence of ∞-categories ⇔ f is fully faithful and $k(f) : k(X) \to k(Y)$ is an equivalence of ∞-groupoids.

Proof. (i): The proof of lemma 6.6.1 shows that both vertical arrows of this commutative square are Kan fibrations; then the assertion follows from corollary 5.4.3.

(ii): If f is an equivalence of ∞-categories, then both $k(f) = k(\Delta^0, f)$ and $k(\Delta^1, f)$ are equivalences of ∞-groupoids, by theorem 6.5.8 (see example 6.4.5); then, by corollary 5.1.13(ii), both horizontal arrows of the commutative square of (i) are weak homotopy equivalences (theorem 6.5.1(iv)), so this square is homotopy cartesian (lemma 3.6.4(iii)), and therefore f is fully faithful. Conversely, if $k(f)$ is an equivalence of ∞-groupoids and f is fully faithful, then $k(\Delta^1, f)$ is an equivalence of ∞-groupoids as well, by (i) and lemma 3.6.4(iii), and then f is an equivalence of ∞-categories, by theorem 6.8.1. □

Theorem 6.8.6. *A functor between ∞-categories is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective.*

Proof. Let $f : X \to Y$ be such a functor. If f is an equivalence of ∞-categories, then $\tau(f) : \tau(X) \to \tau(Y)$ is an equivalence of categories, by theorem 6.5.8, so f is essentially surjective, by remark 6.8.3(i). Moreover, f is fully faithful by proposition 6.8.5(ii).

Conversely, suppose that f is fully faithful and essentially surjective; by proposition 6.8.5(ii), we are reduced to checking that $k(f) : k(X) \to k(Y)$ is an equivalence of ∞-groupoids. Now, the induced functor $\tau(f)$ is an equivalence of categories, by remark 6.8.3(i,ii), so it induces an equivalence of categories $k \circ \tau(f) : k \circ \tau(X) \to k \circ \tau(Y)$ (notation of §1.11.12), whence a bijection $\pi_0(k\tau f) : \pi_0(k\tau X) \xrightarrow{\sim} \pi_0(k\tau Y)$; but by lemmata 2.3.4 and 6.4.1(v), $\pi_0(k\tau f)$ is naturally identified with $\pi_0 \circ k(f) : \pi_0(kX) \xrightarrow{\sim} \pi_0(kY)$.

By corollary 6.7.15, we are thus further reduced to showing that f induces pointed weak homotopy equivalences $\Omega^1(kX, x) \to \Omega^1(kY, f(x))$ for every $x \in k(X)_0 = X_0$. To this aim, notice that $\tau(f)$ is conservative (since it is an equivalence) so the same holds for f (remark 2.6.14(iii)), so we get the cartesian square of sSet :

$$\begin{tikzcd} k(X) \arrow[r, "k(f)"] \arrow[d] & k(Y) \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

by virtue of lemma 6.4.1(iv). Since the functor Ω^1 preserves all representable limits (remark 6.6.4(iv)), we deduce a cartesian square of $\mathrm{sSet}_\circ$:

$$\begin{tikzcd} \Omega^1(kX, x) \arrow[r, "\Omega^1(k(f),x)"] \arrow[d] & \Omega^1(kY, f(x)) \arrow[d] \\ \Omega^1(X, x) \arrow[r, "\Omega^1(f,x)"] & \Omega^1(Y, f(x)) \end{tikzcd}$$

whose vertical arrows are pointed Kan fibrations, by example 6.6.5. Since by assumption, f is fully faithful, we know that $\Omega^1(f, x)$ is a pointed weak homotopy equivalence, and then the same holds for $\Omega^1(k(f), x)$, by virtue of corollary 5.4.1(i). □

Corollary 6.8.7. *The classes of fully faithful functors, of essentially surjective functors, and of weak categorical equivalences are stable under small filtered colimits.*

Proof. Let I be a small filtered category, $X_\bullet, Y_\bullet : I \to \mathrm{sSet}$ two functors, and $f_\bullet : X_\bullet \to Y_\bullet$ a natural transformation; let also X (resp. Y) be the colimit of $X_\bullet$ (resp. of $Y_\bullet$), and suppose first that $f_i : X_i \to Y_i$ is an essentially surjective functor of sSet for every $i \in \mathrm{Ob}(I)$. We need to check that the colimit $f : X \to Y$ of $f_\bullet$ is still essentially surjective. Hence, let $y \in Y_0$; since I is filtered, and since the colimits of sSet are computed termwise, there exists $i \in \mathrm{Ob}(I)$ such that y is the image of some object y_i of Y_i. Then by assumption there exists an object x_i of X_i with an invertible arrow $u : f_i(x_i) \to y_i$, and we let x be the image of x_i in X; then the image of u in X is an invertible arrow $f(x) \to y$ in Y, whence the assertion.

• Next, suppose that X_i and Y_i are Kan complexes and f_i is a weak homotopy equivalence for every $i \in \mathrm{Ob}(I)$; then X and Y are Kan complexes (proposition 5.1.11(iii)), and we claim that f is a weak homotopy equivalence. Indeed, notice that $\pi_0(f_i)$ is bijective for every $i \in \mathrm{Ob}(I)$ (corollary 6.7.15); since π_0 preserves all small filtered colimits (lemma 6.7.1(i)), we get the bijectivity of $\pi_0(f)$. Next, given an object x of X, we find $i \in \mathrm{Ob}(I)$ such that x is the image of some object x_i of X_i, and we let $y_i := f_i(x_i)$ and $y := f(x)$; moreover, for every morphism $\phi : i \to j$ of I we set $x_\phi := \phi(x_i)$, where $\phi(x_i)$ denotes the image of x_i under the morphism $X_i \to X_j$ induced by ϕ, and likewise we define y_ϕ. We then get well-defined functors

$$\underline{X}_\bullet, \underline{Y}_\bullet : i/I \to \mathrm{sSet}_\circ \qquad (i \xrightarrow{\phi} j) \mapsto (X_j, x_\phi) \qquad (i \xrightarrow{\phi} j) \mapsto (Y_j, y_\phi)$$

and $f_\bullet$ induces a natural transformation $f_\bullet : \underline{X}_\bullet \to \underline{Y}_\bullet$ with $f_\phi := f_j : (X_j, x_\phi) \to (Y_j, y_\phi)$ for every $\phi \in \mathrm{Ob}(i/I)$. Since the target functor $i/I \to I$ is cofinal (example 1.5.9(i)), the colimits of $\underline{X}_\bullet$ and $\underline{Y}_\bullet$ are respectively (X, x) and (Y, y) (corollaries 1.5.4(i) and 1.4.6(ii)); also, i/I is filtered, so $\pi_n(f, x)$ is the colimit of the induced system of group isomorphisms

$$\pi_n(f_\phi, x_\phi) : \pi_n(X_j, x_\phi) \xrightarrow{\sim} \pi_n(Y_j, y_\phi) \qquad \forall (i \xrightarrow{\phi} j) \in \mathrm{Ob}(i/I), \forall n \geq 1$$

by virtue of lemma 6.7.1(ii). Hence $\pi_n(f, x)$ is an isomorphism for every object x of X and every $n \geq 1$, and therefore f is a weak homotopy equivalence (corollary 6.7.15).

• Next, suppose that X_i and Y_i are ∞-categories and f_i is a fully faithful functor for every $i \in \mathrm{Ob}(I)$; then we already know that X and Y are also two ∞-categories (proposition 6.2.3(iv)) and we need to check that f is fully faithful, in this case. Hence, let x, x' be two objects of X; arguing as in the foregoing, we may assume that I has an initial object i_0 and that x and x' are images of objects x_0, x'_0 of X_{i_0}. Then, for every $i \in \mathrm{Ob}(I)$ we let x_i, x'_i be the images of x_0 and x'_0 under the morphism $X_{i_0} \to X_i$ induced by the unique morphism $i_0 \to i$ of I; by assumption, each $f_i : X_i \to Y_i$ induces an equivalence of ∞-groupoids $X_i(x_i, x'_i) \to Y_i(f_i(x_i), f_i(x'_i))$, and by the foregoing and remark 6.6.4(v) it follows that f induces an equivalence of ∞-groupoids $X(x, x') \to Y(f(x), f(x'))$, as required.

• Lastly, suppose that f_i is a weak categorical equivalence for every $i \in \mathrm{Ob}(I)$; from theorem 6.8.6 and the foregoing cases, we already see that if moreover X_i and Y_i are ∞-categories for every such i, then $f : X \to Y$ is a weak categorical equivalence.

For the general case we observe that, by corollary 2.2.11, we may apply theorem 4.1.8 and corollary 4.2.16 to the subset $\mathscr{I} := \{\Lambda^n_k \to \Delta^n \mid n \geq 2, k = 1, \dots, n-1\}$ of $\mathrm{Mor}(\mathsf{sSet})$. We then get a functor $\mathsf{L} : \mathsf{sSet} \to \mathsf{sSet}$ that preserves all small filtered colimits and such that $\mathsf{L}(X) \in \mathrm{Ob}(\mathsf{Cat}^\infty)$ for every $X \in \mathrm{Ob}(\mathsf{sSet})$; moreover, we get a natural transformation $\lambda_\bullet : \mathbf{1}_{\mathsf{sSet}} \Rightarrow \mathsf{L}$ such that $\lambda_X : X \to \mathsf{L}(X)$ is an inner anodyne extension, for every $X \in \mathrm{Ob}(\mathsf{sSet})$. In view of theorem 6.5.1(iii) and the 2-out-of-3 property of weak categorical equivalences, we deduce that $f : X \to Y$ is a weak categorical equivalence if and only if $\mathsf{L}f : \mathsf{L}X \to \mathsf{L}Y$ is an equivalence of ∞-categories; moreover, $\mathsf{L}f$ is the colimit of the system of morphisms $\mathsf{L}f_i$, each of which is an equivalence of ∞-categories, by the same token. Then the foregoing case allows us to conclude that $\mathsf{L}f$ is an equivalence of ∞-categories, so f is a weak categorical equivalence, as stated. □

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