ON A CLASS OF ÉTALE ANALYTIC SHEAVES

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Revised and expanded version

Contents

1. Introduction

According to N.Katz (see [Ka4], Introduction), it was B.Dwork the first person to understand that classical differential equations with *irregular* singularities had deep meaning in arithmetic algebraic geometry (against the "prevailing dogma" which held that only equations with regular singular points should have meaning). Since then, the irregular differential equations have been gradually reappropriated into the mainstream of geometry. Initially only some specific areas were affected, such as p-adic analysis and

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positive characteristic geometry, but the trend is now spreading even to the domain of complex analysis, as witnessed *e.g.* by the recent book [Mal], which reports on ideas of P.Deligne, B.Malgrange *et al.* towards establishing an irregular Riemann-Hilbert correspondence.

The aim of this paper is to explore a p-adic version of the theory developed in [Mal]. In truth, in our work the differential equations remain on the background, while the emphasis is on the "dual world" of étale local systems naturally attached to them. In this we are guided by a well known heuristic, which translates many concepts arising from the study of differential equations, into dual topological notions (see *e.g.* the table at the end of [Ka1]). In particular, it is well understood that the notion of irregular singular point should be related to the appearance of *wild ramification* on a local system. Now, in our framework, all the varieties are defined over some p-adic field k of *characteristic zero*. But for such varieties, the étale topology is very close to the classical complex analytic topology, in particular, all ramification is tame: in other words, the algebraic étale topology in characteristic zero is too coarse to describe the monodromy of irregular differential equations.

We remedy this problem by replacing the algebraic étale topology with the much finer *analytic étale topology* recently introduced by V.Berkovich. In this sense, the upgrade from algebraic to analytic étale topology is analogous to the introduction of the space $\widetilde{E}$ of Deligne, which plays a major role in chapter XI of [Mal].

In technical terms, what we need to do is to consider our algebraic varieties as special analytic spaces, and then work systematically inside the framework developed by Berkovich. We should stress here, that our main object of interest remains the category of algebraic schemes (over a fixed local field) and algebraic morphisms: the analytic spaces are always intended as auxiliary tools to define the finer topology and perform certain crucial constructions.

Once we have our candidate topology, we need to describe the class of analytic étale local systems we are interested in. In this paper, we limit ourselves to the study of local systems on smooth curves (notice that also the book [Mal] is mainly concerned with the one-dimensional case).

A priori one may see no reasons why one should not consider the category of *all* such locally constant sheaves of finite rank. However it turns out that, if the curve is not compact (and this is really the only non-trivial case), certain bounds on the ramification of the sheaf around the points at infinity must be imposed in order to obtain a reasonable theory.

In order to conveniently express this condition, we introduce first a notion of analytic local fundamental group: this is actually a pro-group $\pi_1(\eta_s, \overline{x})$ built out of the inverse system of the fundamental groups of all small punctured discs centered at a given point s on a curve. More or less tautologically, any locally constant sheaf on a small punctured disc around s yields a continuous representation of $\pi_1(\eta_s, \overline{x})$.

Next, to single out our class of sheaves, we construct a certain canonical quotient $\mu(\eta_s, \overline{x})$ of $\pi_1(\eta_s, \overline{x})$: the finite rank representations of $\pi_1(\eta_s, \overline{x})$ which factor through this quotient, classify the admissible ramification behaviours of our sheaves.

Chapter 5 is devoted to this construction. This canonical quotient should really be thought of as a topological incarnation of the local differential Galois group of [Ka1]. In particular, the upper numbering filtration defined in *loc.cit.* has a very satisfactory counterpart: that is, we have a canonical higher ramification filtration on our local fundamental group, which behaves pretty much the way it is expected of these gadgets. In terms of this filtration we define also a notion of analytic Swan conductor, which is one of the main characters in our story.

Thanks to Huber's theorem 3.2.11, the theory of *abelian* representations of the local fundamental group (*i.e.* the case of rank one sheaves) is pretty much settled. By contrast, much work remains to be done to clarify the case of higher rank: the theory proposed in this paper should be more properly regarded as a first approximation towards a better and more intrinsic understanding of the local monodromy of analytic sheaves. But lest the reader should fear of being dragged on some wild Swan chase, let us highlight few firm points already established: first, the definition of the Swan conductor itself, is given in section 5.2, together with the usual paraphernalia of representations, their slopes and so on. Second, we can prove (theorem 5.2.13) a version of the Arf-Hasse theorem: the Swan conductor of a representation of finite rank is always an integer. Third, we construct (section 4) a functor of *locally algebraic vanishing cycles* for analytic étale sheaves, for a basis of dimension one (*i.e.* essentially for a family of varieties over an open disc). This functor takes values in the category of sheaves with an action of the local fundamental group.

In view of its ties with the local differential Galois group, and since the latter group classifies connections with poles of finite order, the label "meromorphic fundamental group" which we bestow on our construction, seems appropriate enough. Hence we derive a notion of *meromorphically ramified* local system on an open curve, and the class of such sheaves is the chief object of study in this paper.

Our main tool for the investigation of the meromorphically ramified sheaves is the Fourier transform. The construction of the Fourier transform for analytic étale sheaves of Λ-modules (where Λ is some "big" torsion ring) is accomplished in chapter 7: it is really what one expects: we take the (essentially unique) rank one local system $\mathcal{L}_\psi$ on the affine line which has Swan conductor equal to one at infinity, then, for any vector bundle $\mathsf{E} \to S$ with dual $\mathsf{E}' \to S$, we have the dual pairing $\langle , \rangle : \mathsf{E} \times_S \mathsf{E}' \to S$, and the Fourier transform on E is the anti-involution

$$\mathcal{F}_\psi : \mathrm{D}^+(\mathsf{E}, \Lambda) \to \mathrm{D}^+(\mathsf{E}', \Lambda)$$

with "kernel" given by $\langle , \rangle^* \mathcal{L}_\psi$. We actually give a somewhat more general construction of the kernel, using Lubin-Tate theory: all these alternative kernels become isomorphic on the completion of the algebraic closure of our base field, but the extra generality could be useful for future arithmetic applications.

Our first application of the Fourier transform is contained in section 8.1: there we prove (see theorem 8.1.4) that the cohomology of any meromorphically ramified local system on a curve, has finite rank. We also show by a counterexample, that finiteness does not hold if the ramification is worse than meromorphic.

Wherever there is a Swan conductor, one expects also a formula of the Grothendieck-Ogg-Shafarevich type. As a second application, we prove the formula for all meromorphically ramified sheaves on any smooth open curve.

The proof makes use of Huber's theory of étale cohomology for adic spaces, and in particular exploits the possibility of working with sheaves which are (a priori) not necessarily overconvegent in the sense of [Hub]. Huber's and Berkovich's theories do not always agree, but they do in the situations which are of interest for us (*e.g.* in case of analytification of schemes over k, or more generally, of morphisms between schemes over k). There is little doubt that it would have been possible to write the entire paper in the language of adic spaces and their étale cohomology. Regrettably, the additional burden of making this translation, coming on top of an already extensive editing of the previous version, proved too much to handle for the author. Instead, we have opted for the more conservative approach of inserting a few explanatory remarks, just before Huber's theory makes its appearance in section 8.3.

On the other hand, some proofs in section 8.3 exploit in an essential way the possibility of working with sheaves which are (a priori) not necessarily overconvegent in the sense of [Hub]. For this reason, it does not seem to be easy to reproduce the arguments without leaving the framework of Berkovich's theory.

In the algebraic geometric case, the formula is established via a global argument, basically by some considerations from group cohomology and by applying Lefschetz trace formula.

By contrast, our proof is essentially a local Morse-theoretic argument, inspired by Witten's approach to Morse inequalities via the principle of stationary phase.

In section 8.4 we prove our principle of the stationary phase, and we sketch a study of the local Fourier transform by the usual global to local method. The knowledgeable reader will recognize the influence of Katz's paper [Ka3] on our presentation (except that our poor style cannot match Katz's elegant exposition). In particular our theorem 8.4.9 is formally identical to theorem 3, pag.114 in *loc.cit.*

Our last application of the Fourier transform is of arithmetic nature: the inspiration comes from the classical work [We] of Weil. In that paper, a special role is played by certain quadratic characters of a locally compact topological field F. Let $\psi : F \to \mathbb{C}^\times$ be a fixed additive character of F, V a finite dimensional F-vector space and $q : V \to F$ a non-degenerate quadratic form. Weil defines a Fourier transform $f \mapsto \widehat{f}$ from the space of distributions on V to the space of distributions on the dual V'. Next he proves the following formula (see [We],chapt.I,n.14]:

$$\widehat{\psi \circ q}(\xi) = \gamma(q) \cdot |q|^{-1/2} (\psi \circ q^t)(\xi) \qquad (\xi \in V^*)$$

where $\gamma(q)$ is a complex number of absolute value equal to one, $|q|$ is a volume factor and $q^t : V' \to k$ is the transpose of q (see *loc.cit.*).

Of the two factors, the most interesting one is, by far, $\gamma(q)$. In [We], the properties of γ as a function of the quadratic form q are studied at length. The main result is that the assignment

$$q \mapsto \gamma(q)$$

descends to a group homomorphism from the Witt group $W(F)$ of the given base field F to the group of complex roots of unity.

In case F is a finite field, a simple application of the sheaves-to-functions dictionary of [SGA4$\frac{1}{2}$] allows us to recover the value of $\gamma(q)$ by cohomological means. In fact, in this case it boils down to a finite

(Gauss) sum, and one has the formula:

$$\gamma(q) = \mathrm{Tr}(Fr, H_c^{\dim V}(V \times_F F^a, q^*\mathcal{L}_\psi)(\dim V/2)) \tag{1.0.1}$$

where $\mathcal{L}_\psi$ is the Lang torsor associated to the character ψ (which acts as a kernel for the ℓ-adic Fourier transform in the finite field case), F^a is the algebraic closure of F and $\mathrm{Tr}(Fr, M)$ denotes the trace of the action of the Frobenius generator $Fr \in Gal(F^a/F)$ on a Galois module M.

The cohomology group appearing in (1.0.1) has an obvious analogue in our theory (after all, $q^*\mathcal{L}_\psi$ is a meromorphically ramified sheaf), except that for the time being, we can only deal with torsion coefficient sheaves. But this limitation cannot stop us from considering an inverse system of kernels $\{\mathcal{L}_{\psi_n}\}$ (see chapter 9 for the notation) and then define

$$\Gamma(q) = \lim_{\overleftarrow{n}} H_c^{\dim V}(V \times_k \widehat{k}^a, q^*\mathcal{L}_{\psi_n})(\dim V/2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

In chapter 9 we show that $\Gamma(q)$ descends to a homomorphism from the Witt group of k to the group of isomorphism classes of one-dimensional ℓ-adic Galois representations of (a certain extension of) k. Furthermore, many formal properties of Weil's γ-invariant have adequate counterpart for Γ. The precise relationship between Γ and Weil's invariant is not completely clear yet; nevertheless, we hope that this example may offer a glimpse of the kind of applications which we foresee for our theory.

2. Preliminaries

Throughout this paper, k denotes a field of characteristic zero, complete with respect to a non-Archimedean metric $|\cdot|$. For any such field, we let k° be the valuation ring of k and $k^{\circ\circ}$ its maximal ideal. Also we set $\widetilde{k} = k^\circ/k^{\circ\circ}$ which is a field of characteristic $p > 0$. Furthermore, we let k^a be the algebraic closure of k, and $\widehat{k}^a$ the completion of k^a, endowed with the unique valuation which extends $|\cdot|$.

Some general notation: we denote by $\mathbb{D}(a, \rho)$ (resp. $\mathbb{E}(a, \rho)$) the closed (resp. open) disc of the affine line with radius $\rho \in \mathbb{R}$ and centered at the point $a \in \mathbb{A}^1_k$. Also, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. Lubin-Tate theory.

We recall here some well known facts from Lubin-Tate theory. The paper [LT] is the original source, but a complete account can be found in Lang's book [La].

Let k_0 be a one-dimensional local field (not necessarily of characteristic zero) with valuation $|\cdot|$; denote by π a uniformizing parameter in k_0°. Let q be the cardinality of the residue field $\widetilde{k}_0 = k_0^\circ/k_0^{\circ\circ}$. Set $p = \mathrm{char}\, \widetilde{k}_0 > 0$.

Following Lubin-Tate [LT], we let $\mathfrak{F}_\pi$ be the set of power series $f \in k^\circ[[X]]$ such that

$$\begin{aligned} f(X) &\cong \pi X \mod \text{degree } 2 \\ f(X) &\cong X^q \mod \pi. \end{aligned}$$

The simplest example is just the polynomial $f(X) = \pi X + X^q$. Recall that a formal group F is a power series $F(X, Y) = \sum_{ij} a_{ij} X^i Y^j$ with coefficients $a_{ij} \in k_0$, satisfying the identities $F(F(X, Y), Z) = F(X, F(Y, Z))$, $F(X, Y) = F(Y, X)$ and $F(X, 0) = 0$. A homomorphism of the formal group F into the formal group F' is a power series $f(X) \in k_0[[X]]$ such that $f(F(X, Y)) = F'(f(X), f(Y))$. In particular an endomorphism of F is a homomorphism of F into itself. We say that a formal group is defined over k_0° if its coefficients a_{ij} are in k_0°.

The following theorem summarizes the main features of the Lubin-Tate construction:

Theorem 2.1.1. *a) For each $f \in \mathfrak{F}_\pi$ there exists a unique formal group F_f, defined over k_0° such that f is a (formal) endomorphism of F_f. Moreover, for any two power series $f, g \in \mathfrak{F}_\pi$ and every $a \in k_0^\circ$ there is a unique $[a]_{f,g} \in k_0^\circ[[X]]$ such that $[a]_{f,g} \in \mathrm{Hom}(F_f, F_g)$ and $[a]_{f,g} \cong aX$ mod degree 2.*

b) The map $a \mapsto [a]_{f,g}$ gives a group homomorphism $k_0^\circ \to \mathrm{Hom}(F_f, F_g)$ satisfying the composition rule

$$[a]_{g,h} \circ [a]_{f,g} = [ab]_{f,h}.$$

In particular, if $f = g$, this map is a ring homomorphism $k_0^\circ \to \mathrm{End}(F_f)$.

Proof. This is theorem 1.2, chapt. 8 of [La]. □

We will write $[a]_f$ in place of $[a]_{f,f}$; in particular notice that $[\pi]_f = f$.

Given $f \in \mathfrak{F}_\pi$, the associated formal group F_f converges, as a power series, for all pairs (x, y) of elements of $\widehat{k}_0^a$ such that $|x|, |y| < 1$. It is clear that F induces an analytic group structure on $\mathbb{E}(0, 1)$. Any $a \in k_0^\circ$ induces an endomorphism $[a]_f$ of this group.

Definition 2.1.2. For any positive integer n we let $G_n \subset k_0^a$ be the kernel of the iterated power $[\pi]_f^n$. Also we define $G_\infty = \cup_{n>0} G_n$.

We collect here some well known results about G_n:

Theorem 2.1.3. *1) The action of k_0° on $\mathbb{E}(0,1)$ induces an isomorphism of k_0°-modules between G_n and the additive group $k_0^\circ/(k_0^{\circ\circ})^n$.*

2) The field $k_0(G_n)$ is a totally ramified abelian extension of k_0 with Galois group isomorphic to $(k_0^\circ/(k_0^{\circ\circ})^n)^\times$.

Proof. See theorem 2.1, chapt. 8 of [La]. □

We specialize now to characteristic zero, that is $\mathrm{char}(k_0) = 0$. In this case it is known (see [La], section 8.6) that for any formal group F over k_0, there exists a formal isomorphism

$$\boldsymbol{\lambda} : F \to \mathbb{G}_a$$

where $\mathbb{G}_a$ is the usual additive formal group over k_0, that is $\mathbb{G}_a(X,Y) = X + Y$. The isomorphism $\boldsymbol{\lambda}$ is called the logarithm of F, and it is uniquely determined by F and by the condition $d\boldsymbol{\lambda}(0)/dX = 1$.

Lemma 2.1.4. *Let F be a Lubin-Tate formal group, i.e. $F = F_f$ for some $f \in \mathfrak{F}_\pi$. Then the logarithm $\boldsymbol{\lambda} = \boldsymbol{\lambda}_F$ can be written in the form:*

$$\boldsymbol{\lambda}(X) = \sum_i g_i(X) \frac{X^{q^i}}{\pi^i}$$

with $g_i(X) \in k_0^\circ[[X]]$.

Proof. This is lemma 6.3, chapt. 8 of [La]. □

It follows easily from the lemma that $\boldsymbol{\lambda}$ converges over $\mathbb{E}(0,1)$, therefore it induces an analytic group homomorphism

$$\boldsymbol{\lambda} : \mathbb{E}(0,1) \longrightarrow (\mathbb{G}_a)^{an}.$$

Theorem 2.1.5. *Let $e_F(Z)$ be the power series (with coefficient in k_0) which is the inverse of $\boldsymbol{\lambda}_F(X)$. Then $e_F(Z)$ converges on the disc $\mathbb{E}(0, |\pi|^{1/(q-1)})$ and induces the inverse homomorphism to $\boldsymbol{\lambda}_F$ on the analytic subgroups*

$$\mathbb{E}(0, |\pi|^{1/(q-1)}) \underset{\boldsymbol{\lambda}_F}{\overset{e_F}{\rightleftarrows}} \mathbb{G}_a(0, |\pi|^{1/(q-1)}).$$

(the group on the right coincides set-theoretically with the group on the left, and we use the notation $\mathbb{G}_a$ to emphasize that it is endowed with additive group structure).

Proof. See lemma 6.4, chapt. 8 of [La]. □

Remark 2.1.6. (1) It can be shown that $\boldsymbol{\lambda}$ is a homomorphism of k_0°-modules, i.e. for all $a \in k_0^\circ$ there is an equality of power series:

$$a \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda} \circ [a]_f.$$

(2) Using theorem 2.1.5 and (a) it is not hard to show that the kernel of $\boldsymbol{\lambda}$ is the subgroup G_∞.

In what follows we will reserve the symbol ρ_1 for the constant $|\pi|^{1/(q-1)}$.

2.2. **Complements of étale cohomology.** Berkovich has defined an étale topology on his analytic varieties, and has studied the corresponding cohomology. In the work [B1], which is the reference for all the definitions which are implicit in this paper, he establishes the usual properties for his cohomology, like proper and smooth base change and Poincaré duality. In [B2] and [B3] he introduces two constructions of vanishing cycles.

We denote by $\acute{\mathrm{E}}\mathrm{t}(X)$ the category of étale analytic varieties over X and for any ring Λ, we let $\mathbf{S}(X, \Lambda)$ be the category of sheaves of Λ-modules on $\acute{\mathrm{E}}\mathrm{t}(X)$.

In his paper, Berkovich considers mainly finite rings of coefficients, of the form $\Lambda = \mathbb{Z}/n\mathbb{Z}$. For our purposes, these are not quite enough, since we have to consider characters of an infinite divisible group G_∞ into $\Lambda^\times$.

In this section we sketch briefly some arguments to extend the main results to more general torsion rings Λ: we will show that in order to compute the effect of a cohomological functor on a sheaf F of Λ-modules, it suffices to regard F as a sheaf of abelian groups and compute the cohomological functor inside the category of sheaves of abelian groups. This will allow us to quickly derive our results from the theorems of Berkovich.

To start with, let Λ be any torsion ring and let $\mathrm{D}(X,\Lambda)$ (resp.($\mathrm{D}^+(X,\Lambda)$) be the derived category of complexes (resp. of complexes vanishing in large negative degrees) of sheaves $K^\bullet$ of Λ-modules and similarly define $\mathrm{D}^-(X,\Lambda)$; denote by F_X the forgetful functor from $\mathrm{D}(X,\Lambda)$ to $\mathrm{D}(X,\mathbb{Z})$.

Let $f : X \to Y$ be a map of analytic spaces over k. First of all there is a direct image functor $Rf_* : \mathrm{D}^+(X,\Lambda) \to \mathrm{D}^+(Y,\Lambda)$.

Proposition 2.2.1. *The functor Rf_* commutes with the forgetful functor, i.e.*

$$Rf_* \circ \mathrm{F}_X = \mathrm{F}_Y \circ Rf_*.$$

Proof. For any sheaf F we will construct a resolution $I^\bullet$ by sheaves which are both injective as sheaves of Λ-modules and flabby as sheaves of abelian groups. One checks as in the algebraic case that flabby resolutions are f_*-acyclic : to do this one can look at [Mi] chapt. III sections 1,2,3 and convince oneself that all the arguments work without change in the present situation. Then $I^\bullet$ computes at the same time Rf_* in the categories $\mathrm{D}(Y,\Lambda)$ and $\mathrm{D}(X,\mathbb{Z})$, and the proposition follows.

For each $x \in X$, choose a geometric point x' localized at x, *i.e.* an imbedding of the residue field $\mathcal{H}(x)$ of x in the completion of its algebraic closure. We form the locally ringed space $X' = \cup_{x\in X} x'$ that we endow with the discrete topology. This space is an inductive limit of analytic spaces and therefore carries a natural étale site X'_{et}. Let $\pi : X'_{et} \to X_{et}$ be the obvious map.

The sheaf $\pi^* F$ is the direct product over the stalks $F_{x'} = {x'}^* F$ at the points $x' \in X'$. For every $x' \in X'$ choose an imbedding into an injective Λ-module $F_{x'} \hookrightarrow I_{x'}$: we see $I_{x'}$ as an injective sheaf of Λ-modules over the point x'. The product $I^0 = \Pi_{x'\in X'} I_{x'}$ is an injective sheaf of Λ-modules on X' and clearly F imbeds into $\pi_* I$. Since π_* preserves injective sheaves, we have constructed the first step of an injective resolution of Λ-modules; if we iterate this construction we obtain a full Godement resolution $I^\bullet$ for F. On the other hand, I is also flabby as a sheaves of abelian groups (since every sheaf on X' is flabby) and π_* preserves flabby sheaves, therefore $I^\bullet$ is also a flabby resolution, as wanted. □

Next we turn to cohomology with support. For the notation we follow section 5.1 of [B1], to which we refer the reader for all the relevant definitions.

Recall (see *loc.cit*) that a ϕ-family of supports Φ defines a left exact functor $\phi_\Phi : \mathbf{S}(Y,\Lambda) \to \mathbf{S}(X,\Lambda)$ as follows. If $F \in \mathbf{S}(Y,\Lambda)$ and $f : U \to X$ is etale, then

$$(\phi_\Phi F)(U) = \{s \in F(U_\phi) | \mathrm{Supp}(s) \in \Phi(f)\}.$$

For example, if Φ is the family of all closed subsets, then $\phi_\Phi = \phi_*$. If the map $\phi : X \to Y$ is separated then the family of all ϕ-proper subsets of X is a paracompactifying ϕ-family, and we get a left exact functor which is denoted by $\phi_!$.

We can derive the functor ϕ_Φ in the two categories $\mathrm{D}^+(X,\mathbb{Z})$ and $\mathrm{D}^+(X,\Lambda)$, and in this way we obtain two functors that we denote both by $R\phi_\Phi$. The following proposition shows that in the cases of interest no ambiguity arises from this choice of notation.

Proposition 2.2.2. *Suppose that the family Φ is paracompactifying. Then the two functors defined above coincide, i.e.*

$$R\phi_\Phi \circ \mathrm{F}_X = \mathrm{F}_Y \circ R\phi_\Phi.$$

Proof. The proof of proposition 2.2.1 produces for any sheaf of Λ-modules a resolution that is injective in the category of sheaves of Λ-modules and flabby in the category of sheaves of abelian groups.

To prove the theorem, it suffices to show that this resolution is acyclic for the functor ϕ_Φ defined on the category $\mathbf{S}(X,\mathbb{Z})$, thus the proposition follows from lemma 2.2.3 below. □

Lemma 2.2.3. *Suppose that the family Φ is paracompactifying. Let F be a flabby sheaf of abelian groups. Then $R^n\phi_\Phi(F) = 0$ for all $n > 0$.*

Proof. It is shown in [B1], proposition 5.2.1, that $R^n\phi_\Phi(F)$ is the sheaf associated with the presheaf $(U \to X) \mapsto H^n_{\Phi(f)}(U_\phi, F)$. Therefore it suffices to show that under the stated hypothesis, $H^n_{\Phi(f)}(U_\phi, F) = 0$ for all étale morphisms $U \to X$ and all $n > 0$. Since the restriction to U of a flabby sheaf of abelian groups on X, is a flabby sheaf, we have only to prove this for $U = X$.

Consider the morphism of sites $\pi : X_{et} \to |X|$, where $|X|$ is the space X with its underlying analytic topology. The morphism π induces a spectral sequence

$$H^p_\Phi(|X|, R^q\pi_* F) \Rightarrow H^{p+q}_\Phi(X, F).$$

We will prove that $R^q\pi_* F = 0$ for all $q > 0$. Assuming this for the moment, we show how to conclude. It follows from the vanishing that $H^p_\Phi(|X|, \pi_* F) = H^p_\Phi(X, F)$. Since F is flabby by hypothesis, we obtain from [B1], corollary 4.2.5, that π_*F is flabby in the analytic topology. Then π_*F is Γ_Φ-acyclic, by lemma 3.7.1 from [Gro] and the lemma is proved.

To see that $R^q\pi_*F = 0$, we can look at the stalks of this sheaf. For any point $x \in X$, let G_x be the Galois group of the algebraic closure of the residue field $\mathcal{H}(x)$. According to [B1], proposition 4.2.4, we have $(R^q\pi_*F)_x \simeq H^q(G_x, F_x)$, $q \geq 0$. Since F is flabby, it follows from [B1], corollary 4.2.5 that F_x is an acyclic G_x-module, as wanted. □

As a corollary, we get a proper base change statement for sheaves of Λ-modules.

Proposition 2.2.4. *Assume that char($\widetilde{k}$) is invertible in Λ. Let $\phi : Y \to X$ be a separated morphism of k-analytic spaces, and let $f : X' \to X$ be a morphism of analytic spaces over k, which gives rise to a cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\phi'"'] & Y \arrow[d, "\phi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

Then for any complex $K^\bullet \in \mathrm{D}^+(Y, \Lambda)$ there is a canonical isomorphism in $\mathrm{D}^+(X', \Lambda)$

$$f^*(R\phi_! K^\bullet) \simeq R\phi'_!(f'^* K^\bullet).$$

Proof. The usual devissage reduces to the case where $K^\bullet$ is concentrated in degree 0. Then the theorem follows from proposition 2.2.2 and theorem 7.7.1 of [B1]. □

Let $\mathrm{D}^b(X, \Lambda)$ be the subcategory of $\mathrm{D}^+(X, \Lambda)$ consisting of cohomologically bounded complexes. Let $\phi : Y \to X$ be as in theorem 2.2.5 and suppose that the fibres of ϕ have bounded dimension. Then, by corollary 5.3.8 of [B1] and proposition 2.2.2 we deduce that $R\phi_!$ takes $\mathrm{D}^b(X, \Lambda)$ to $\mathrm{D}^b(Y, \Lambda)$ and extends to a functor $R\phi_! : \mathrm{D}^-(X, \Lambda) \to \mathrm{D}^-(Y, \Lambda)$.

The following projection formula is proved as in [B1], theorem 5.3.9.

Proposition 2.2.5. *Suppose that $F^\bullet \in \mathrm{D}^-(X, \Lambda)$ and $G^\bullet \in \mathrm{D}^-(Y, \Lambda)$ or that $F^\bullet \in \mathrm{D}^b(X, \Lambda)$ has finite Tor-dimension and $G^\bullet \in \mathrm{D}(Y, \Lambda)$. Then there is a canonical isomorphism*

$$F^\bullet \overset{\mathbf{L}}{\otimes} R\phi_!(G^\bullet) \simeq R\phi_!(\phi^*(F^\bullet) \overset{\mathbf{L}}{\otimes} G^\bullet).$$

□

Similarly, using the propositions above we can establish the other main results of [B1], such as Poincaré duality and cohomological purity in the context of sheaves of Λ-modules. We leave the details as an exercise for the referee.

3. The analytic fundamental group of an affine curve

3.1. **The asymptotic Kummer sequence.** Let $\mathcal{X}$ be any Hausdorff analytic space over the field k. We introduce the sheaf $\mathcal{U}^1_{\mathcal{X}}$ on the étale site of $\mathcal{X}$, by setting

$$\mathcal{U}^1_{\mathcal{X}}(V) = \{f \in \mathcal{O}_V(V) \quad | \quad |1 - f|_{\sup} < 1\}$$

for any étale morphism $V \to \mathcal{X}$; the usual multiplication of functions defines an abelian sheaf structure on $\mathcal{U}^1_{\mathcal{X}}$. Moreover, the abelian sheaf $\mu_{p^n,\mathcal{X}}$ is defined as the subsheaf of p^n-torsion sections of $\mathcal{U}^1_{\mathcal{X}}$. We set $\mu_{p^\infty,\mathcal{X}} = \varinjlim_n \mu_{p^n,\mathcal{X}}$.

Lemma 3.1.1. *(Asymptotic Kummer exact sequence) There exists a short exact sequence of étale sheaves*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_{p^\infty,\mathcal{X}} & \longrightarrow & \mathcal{U}^1_{\mathcal{X}} & \xrightarrow{\lambda} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & 0. \\ & & & & f & \longmapsto & \log(f) & & \end{array} \tag{3.1.2}$$

Proof. We only have to prove the surjectivity of λ, and for this we can check on the stalks. Let $p \in \mathcal{X}$ be any point, and $f \in \mathcal{O}_{\mathcal{X},p}$. Choose some pointed étale morphism $(V, q) \to (\mathcal{X}, p)$ where f extends to an element $f \in \mathcal{O}_V(V)$. Take a compact neighborhood W of q in V so that f is bounded on W, and we can find an integer N such that $|p^N f|_{\sup,W} < p^{1/(1-p)}$. Then $g = \exp(p^N f)$ is defined and belongs to

$\mathcal{O}_W(W)$; moreover, g vanishes nowhere on W. Hence g defines an analytic map $W \to \mathbb{G}_m^{an}$. Define W' as the fibre product in the following square diagram

$$\begin{tikzcd} W' \arrow[r, "\phi"] \arrow[d] & W \arrow[d, "g"] \\ \mathbb{G}_m^{an} \arrow[r, "f \mapsto f^{p^N}"] & \mathbb{G}_m^{an}. \end{tikzcd}$$

Then W' is étale over W and $h = g^{1/p^N}$ is defined as an element of $\mathcal{O}_{W'}(W')$. One sees easily that $\lambda(h) = \phi^*(f)$ and the claim follows. □

Suppose in addition, that $\mathcal{X} = X^{an}$ where X is a connected and reduced algebraic scheme over k. Then $H^0(X^{an}, \mathcal{U}^1)$ is the group U_k^1 of elements $x \in k^\circ$ which are congruent to 1 modulo $k^{\circ\circ}$. Taking the cohomology of the exact sequence (3.1.2) we obtain

$$0 \to H^0(X^{an}, \mathcal{O}_X^{an})/\lambda(U_k^1) \to H^1(X^{an}, \mu_{p^\infty}) \to H^1(X^{an}, \mathcal{U}^1) \to H^1(X^{an}, \mathcal{O}_X^{an}). \tag{3.1.3}$$

For the rest of this chapter we make the further assumption that the field k be algebraically closed. Under this hypothesis, we have $\lambda(U_k^1) = k$.

Lemma 3.1.4. *Suppose that k is algebraically closed and let X be a reduced k-algebraic scheme. Then the natural morphism $\mu_{p^\infty} \hookrightarrow \mathcal{U}^1_{X^{an}}$ induces an imbedding in cohomology*

$$\lim_{n\to\infty} H^1(X^{an}, \mu_{p^n}) \hookrightarrow H^1(X^{an}, \mathcal{U}^1).$$

Proof. It suffices to consider the usual Kummer exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathcal{U}^1 \longrightarrow \mathcal{U}^1 \longrightarrow 0$$

and observe that the induced sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow H^0(X^{an}, \mathcal{U}^1) \longrightarrow H^0(X^{an}, \mathcal{U}^1) \longrightarrow 0$$

is exact. □

We notice that, due to the comparison theorems between algebraic and analytic étale cohomology, and the well-known compactness properties of the algebraic étale topology, the group $\lim\limits_{\overrightarrow{n}} H^1(X^{an}, \mu_{p^n})$ can be suggestively rewritten as $H^1(X, \mu_{p^\infty})$.

3.2. **Huber's theorem.** What seems to be happening is that the analytic and algebraic contributions to the (abelianized) fundamental groups are distributed onto respectively $H^0(X^{an}, \mathcal{O}_X^{an})$ and $H^1(X^{an}, \mathcal{U}^1)$. Accordingly, I do not expect any exotic coverings coming from the cohomology of $\mathcal{U}^1$, but in general I do not know how to compute it completely. However, for our purposes, the case when X is an open subscheme of the affine line is the most urgent. Luckily, this is precisely the case covered by the following theorem 3.2.11 of R.Huber. We start with some notation and three preliminary lemmas.

Let $\mathcal{X}$ be an analytic space over k and for any integer n define $\phi : H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \to H^1(\mathcal{X}, \mu_{p^n})$ as the boundary map induced by the Kummer exact sequence. Moreover, let $\psi : H^1(\mathcal{X}, \mu_{p^n}) \to H^1(\mathcal{X}, \mathcal{U}_\mathcal{X}^1)$ be the map induced by the inclusion $\mu_{p^n} \hookrightarrow \mathcal{U}_\mathcal{X}^1$.

Lemma 3.2.1. *Let $\mathcal{X}$ and n be as above. We have*

$$\mathrm{Ker}(\psi \circ \phi) = \left\{ f \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \quad | \quad \exists g \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \textit{ such that } |f/g^{p^n} - 1| < 1 \right\}.$$

Proof. Let $f \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*)$ be given. Let $q : \mathcal{Y} \to \mathcal{X}$ be the cyclic covering associated to the p^n-root of f. On $\mathcal{Y}$ we have $f^{1/p^n} \in \mathcal{O}_\mathcal{Y}(\mathcal{Y})$. Let $p_1, p_2 : \mathcal{Y} \times_\mathcal{X} \mathcal{Y} \to \mathcal{Y}$ be the projections. Then $p_1^*(f^{1/p^n}) \cdot p_2^*(f^{1/p^n})^{-1} \in \mu_{p^n}(\mathcal{Y} \times_\mathcal{X} \mathcal{Y})$ and $\phi(f)$ is given by the cocycle $c = ((\mathcal{Y} \to \mathcal{X}), p_1^*(f^{1/p^n}) \cdot p_2^*(f^{1/p^n})^{-1})$. Then $\psi(\phi(f))$ is given by the same cocycle but where we now consider $p_1^*(f^{1/p^n}) \cdot p_2^*(f^{1/p^n})^{-1}$ as an element of $\mathcal{U}_\mathcal{X}^1(\mathcal{Y} \times_\mathcal{X} \mathcal{Y})$. Since the mapping $H^1((\mathcal{Y} \to \mathcal{X}), \mathcal{U}_X^1) \to H^1(X, \mathcal{U}_\mathcal{X})$ is injective, we obtain

$$\begin{aligned} \psi(\phi(f)) = 0 &\iff c = 0 \in H^1((\mathcal{Y} \to \mathcal{X}), \mathcal{U}_\mathcal{X}^1) \\ &\iff \exists t \in \mathcal{U}_\mathcal{X}^1(\mathcal{Y}) \text{ with } p_1^*(f^{1/p^n}) \cdot p_2^*(f^{1/p^n})^{-1} = p_1^*(t) \cdot p_2^*(t)^{-1} \\ &\iff \exists t \in \mathcal{U}_\mathcal{X}^1(\mathcal{Y}) \text{ with } p_1^*(f^{1/p^n} \cdot t^{-1}) = p_2^*(f^{1/p^n} \cdot t^{-1}) \\ &\iff \exists t \in \mathcal{U}_\mathcal{X}^1(\mathcal{Y}) \text{ and } g \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \text{ with } f^{1/p^n} \cdot t^{-1} = q^*(g) \\ &\iff \exists g \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \text{ with } |f^{1/p^n}/q^*(g) - 1| < 1 \\ &\iff \exists g \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \text{ with } |q^*(f)/q^*(g)^{p^n} - 1| < 1 \text{ on } Y \\ &\iff \exists g \in H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \text{ with } |f/g^{p^n} - 1| < 1 \text{ on } X. \end{aligned}$$

□

We put $H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})(n) = \{f \in H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}}) \mid \exists g \in H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}}) \text{ with } |f/g^{p^n}-1|<1\}$.

According to lemma 3.2.1, we are interested in the group $H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})/H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})(n)$. In the following lemma we compute this group in a special situation.

Lemma 3.2.2. *Let $\mathbb{D}$ be a closed disc of $(\mathbb{A}^1_k)^{an}$ and let $\mathbb{E}_1,...,\mathbb{E}_m$ be open discs of $(\mathbb{A}^1_k)^{an}$ such that $\mathbb{E}_i \subset \mathbb{D}$ and $\mathbb{E}_i \subset \mathbb{E}_j = \emptyset$ for $i \neq j$. Let a_i be an element of $\mathbb{E}_i$. Put $\mathcal{X} = \mathbb{D} - \bigcup_i^m \mathbb{E}_i$. We assume that, for every i, the boundary of $\mathbb{E}_i$ is contained in $\mathcal{X}$. Then $H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})/H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})(n)$ is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank m with basis $T-a_1,...,T-a_m$ (where T denotes the coordinate function of $\mathbb{A}^1_k$).*

Proof. First we show the following

Claim 3.2.3. For every $f \in H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})$ there exists $\ell_1,...,\ell_m \in \mathbb{Z}$, $d \in k^*$ and $r \in H^0(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ such that

$$f = d \cdot (T-a_1)^{\ell_1} \cdot ... \cdot (T-a_m)^{\ell_m} \cdot r \text{ and } |r-1|<1.$$

Moreover $\ell_1,...,\ell_m$ are uniquely determined.

Proof of the claim: to show the existence of $\ell_1,...,\ell_m$, we fix $f \in H^0(\mathcal{X},\mathcal{O}^*_{\mathcal{X}})$. There is a $c \in \mathbb{R}$ such that $0<c<1$ and $c<|T-a_i|_{\mathcal{X}}<c^{-1}$ for $i=1,...,m$. There exists $N \in \mathbb{N}$ and $\varrho \in \mathbb{R}$, $\varrho>0$ such that, for every meromorphic function g on $\mathbb{D}$ with $g_{|\mathcal{X}}$ holomorphic and $|f-g|_{\mathcal{X}}<\varrho$, we have $\sum_{x\in\mathbb{D}} \mathrm{ord}_x(g) = N$ ([FP] lemma I.3.3). We fix a $x_0 \in \mathcal{X}$. The function f can be approximated by elements of the localization $A = k[T]_{(T-a_1)\cdot...\cdot(T-a_m)}$. We choose a $g \in A$ such that $|g(x_0)| = |f(x_0)|$ and $g(x) \neq 0$ for every $x \in X$ and $|f-g|_{\mathcal{X}} < \min(\varrho, |f(x_0)|\cdot c^{2N})$. We split $g_{|\mathbb{D}}$ into a product

$$g_{|\mathbb{D}} = (T-b_1)^{\epsilon_1} \cdot ... \cdot (T-b_s)^{\epsilon_s} \cdot g' \tag{3.2.4}$$

where $g' \in H^0(\mathbb{D},\mathcal{O}^*_{\mathbb{D}})$, $b_1,...,b_s \in \bigcup_i^m \mathbb{E}_i$ and $\epsilon_1,...,\epsilon_s \in \{+1,-1\}$ (we do not assume that $b_i \neq b_j$ for $i \neq j$). Since $g' \in H^0(\mathbb{D},\mathcal{O}^*_{\mathbb{D}})$, there exist $d \in k^*$ and $r' \in H^0(\mathbb{D},\mathcal{O}_{\mathbb{D}})$ with

$$g' = d \cdot r' \text{ and } |r'-1|_{\mathbb{D}} < 1. \tag{3.2.5}$$

(See [BGR] 5.1.3/1]). For $i=1,...,m$ put

$$\begin{aligned} L_i &= \{j \in \{1,...,s\} \mid b_j \in \mathbb{E}_i\} \\ \ell_i &= \textstyle\sum_{j\in L_i} \epsilon_j \in \mathbb{Z}. \end{aligned}$$

We show that, setting $r = f \cdot d^{-1} \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m} \in H^0(\mathcal{X},\mathcal{O}_{\mathcal{X}})$, we have

$$f = d \cdot (T-a_1)^{\ell_1} \cdot ... \cdot (T-a_m)^{\ell_m} \cdot r \qquad \text{and} \qquad |r-1|<1.$$

We have to check that $|r-1|<1$. We have

$$r = g \cdot d^{-1} \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m} + (f-g) \cdot d^{-1} \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m}.$$

We will show that

$$|g \cdot d^{-1} \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m} - 1| < 1 \tag{3.2.6}$$

and

$$|(f-g) \cdot d^{-1} \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m}|_{\mathcal{X}} < 1. \tag{3.2.7}$$

From (3.2.6) and (3.2.7) we obtain $|r-1|<1$.

Since, for every $b_j \in L_i$ we have $|\frac{T-b_j}{T-a_i}-1|_{\mathcal{X}}<1$, (3.2.6) follows from (3.2.4) and (3.2.5).

Since $|g(x_0)| = |f(x_0)|$, we obtain from (3.2.4) and (3.2.5)

$$|d^{-1}| = |f(x_0)|^{-1} \cdot |x_0-b_1|^{\epsilon_1} \cdot ... \cdot |x_0-b_s|^{\epsilon_s}.$$

Since, for $i \in \{1,...,m\}$ and $j \in L_i$, we have $|x_0-b_j| = |x_0-a_i|$, we obtain

$$|d^{-1}| = |f(x_0)|^{-1} \cdot |x_0-a_1|^{\ell_1} \cdot ... \cdot |x_0-a_m|^{\ell_m}.$$

By definition of c we have $|x_0-a_i|<c^{-1}$ for $i=1,...,m$. Since $|f-g|<\varrho$, we have by (3.2.4) and definition of N, $\ell_1+...+\ell_m = N$. Hence

$$|d^{-1}| < |f(x_0)|^{-1} \cdot c^{-N}. \tag{3.2.8}$$

By definition of c we have $|T-a_i|^{-1}<c^{-1}$. Furthermore, we have $|f-g|_{\mathcal{X}} < |f(x_0)|\cdot c^{2N}$ (by construction of g) and $\ell_1+...+\ell_m = N$. Hence

$$|(f-g) \cdot (T-a_1)^{-\ell_1} \cdot ... \cdot (T-a_m)^{-\ell_m}| < |f(x_0)| \cdot c^N. \tag{3.2.9}$$

By (3.2.8) and (3.2.9) we obtain (3.2.7). This finishes the existence part of the proof of claim 3.2.3.

Uniqueness of $\ell_1, ..., \ell_m$: for every $i \in \{1, ..., m\}$ there is a mapping $\nu_i : \mathcal{O}_{\mathcal{X}}(\mathcal{X}) - \{0\} \to \mathbb{Z}$ which satisfies the following properties: 1) $\nu_i(g \cdot h) = \nu_i(g) + \nu_i(h)$ for every $g, h \in \mathcal{O}_{\mathcal{X}}(\mathcal{X}) - \{0\}$; 2) $\nu_i(c) = 0$ for every $c \in k^*$; 3) $\nu_i(s) = 0$ for every $s \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ with $|s-1| < 1$; 4) $\nu_i(T - a_i) = 1$ and $\nu_i(T - a_j) = 0$ for $j \neq i$. (ν_i is the order function with respect to the boundary of $\mathbb{E}_i$; cf. [FP] Prop I.3.1.iii).

Hence the equation

$$f = d \cdot (T - a_1)^{\ell_1} \cdot ... \cdot (T - a_m)^{\ell_m} \cdot r \qquad \text{with} \qquad |r - 1| < 1$$

implies $\ell_i = \nu_i(f)$. This shows that ℓ_i is uniquely determined and concludes the proof of claim 3.2.3.

We define a group homorphism

$$\begin{aligned} e : H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) &\longmapsto \mathbb{Z}^m \\ f &\longmapsto (\ell_1(f), ..., \ell_m(f)) \end{aligned}$$

where $\ell_1(f), ..., \ell_m(f)$ are the integers of claim 3.2.3. We have to show that $H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})(n) = e^{-1}(p^n\mathbb{Z}^m)$. Obviously we have $e^{-1}(p^n\mathbb{Z}^m) \subset H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})(n)$. In order to show the reverse inclusion we use the mappings ν_i introduced above. Let $f \in H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})(n)$ be given, i.e. $f \in H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$ and there is a $g \in H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$ with $|f/g^{p^n} - 1| < 1$. Then $\nu_i(f/g^{p^n}) = 0$. Hence $\ell_i(f) = \nu_i(g^{p^n}) = p^n\nu_i(g) \in p^n\mathbb{Z}^m$. This shows that $f \in e^{-1}(p^n\mathbb{Z}^m)$. □

Let $\mathbb{D}, \mathbb{D}'$ be closed discs of $(\mathbb{A}^1_k)^{an}$ and let $\mathbb{E}_1, ..., \mathbb{E}_m, \mathbb{E}'_1, ..., \mathbb{E}'_m$ be open discs of $(\mathbb{A}^1_k)^{an}$ such that $\mathbb{E}'_i \subset \mathbb{E}_i \subset \mathbb{D} \subset \mathbb{D}'$ and $\mathbb{E}_i \cap \mathbb{E}_j = \emptyset$ for $i \neq j$. Let r, r' be the radius of $\mathbb{D}, \mathbb{D}'$ and let r_i, r'_i be the radius of $\mathbb{E}_i, \mathbb{E}'_i$. Put $\mathcal{X} = \mathbb{D} - \bigcup_i^m \mathbb{E}_i$ and $\mathcal{X}' = \mathbb{D}' - \bigcup_i^m \mathbb{E}'_i$. Then $\mathcal{X} \subset \mathcal{X}'$.

Lemma 3.2.10. *With the notation above, for every $h \in \mathcal{O}^*_{\mathcal{X}'}(\mathcal{X}')$ with $h(x_0) = 0$ for some $x_0 \in \mathcal{X}$ we have*

$$|h|_{\mathcal{X}} \leq \max\left(\frac{r'_1}{r_1}, ..., \frac{r'_m}{r_m}, \frac{r}{r'}\right) \cdot |h|_{\mathcal{X}'}.$$

In particular, for every $f, g \in \mathcal{O}_{\mathcal{X}'}(\mathcal{X}')$ with $|f - 1|_{\mathcal{X}'} < 1$, $|g - 1|_{\mathcal{X}'} < 1$ and $f(x_0) = g(x_0)$ for some $x_0 \in \mathcal{X}$, we have

$$|f - g|_{\mathcal{X}} \leq \max\left(\frac{r'_1}{r_1}, ..., \frac{r'_m}{r_m}, \frac{r}{r'}\right).$$

Proof. We may assume $x_0 = 0$. For every $i \in \{1, ..., m\}$ we fix an element p_i of $\mathbb{E}_i$. We consider the automorphism $\phi : (\mathbb{P}^1_k)^{an} \to (\mathbb{P}^1_k)^{an}$, $x \mapsto \frac{1}{x}$. Then $\phi(0) = \infty$. By [FP] Prop.I.1.3 every $h \in \mathcal{O}(\phi(\mathcal{X}'))$ has a unique representation

$$h = \sum_{n \in \mathbb{N}_0} \frac{a_n}{T^n} + \sum_{n \in \mathbb{N}} \frac{a^1_n}{(T - p_1^{-1})^n} + ... + \sum_{n \in \mathbb{N}} \frac{a^m_n}{(T - p_m^{-1})^n}$$

with $a_n, a^1_n, ..., a^m_n \in k$ such that $(|a_n| \cdot r'^n)_{n \in \mathbb{N}}$ and $(|a^i_n| \cdot (\frac{|p_i|^2}{r'_i})^n)_{n \in \mathbb{N}}$ are zero sequences. Moreover,

$$|h|_{\phi(\mathcal{X}')} = \max\left(\{|a_n| \cdot r'^n\}_{n \in \mathbb{N}_0} \cup \left\{|a^i_n| \cdot \left(\frac{|p_i|^2}{r'_i}\right)^n | n \in \mathbb{N}, i = 1, ..., m\right\}\right)$$

and

$$|h|_{\phi(\mathcal{X})} = \max\left(\{|a_n| \cdot r^n\}_{n \in \mathbb{N}_0} \cup \left\{|a^i_n| \cdot \left(\frac{|p_i|^2}{r_i}\right)^n | n \in \mathbb{N}, i = 1, ..., m\right\}\right).$$

If $h(\infty) = 0$ then $a_0 = 0$ and so we obtain

$$|h|_{\phi(\mathcal{X})} \leq \max\left(\frac{r'_1}{r_1}, ..., \frac{r'_m}{r_m}, \frac{r}{r'}\right) \cdot |h|_{\phi(\mathcal{X}')}.$$

□

Theorem 3.2.11 (R.Huber). *Let X be an open subscheme of the affine line $\mathbb{A}^1_k$. Then the natural mapping*

$$\lim_{\substack{\longrightarrow \\ n \in \mathbb{N}}} H^1(X^{an}, \mu_{p^n}) \to H^1(X^{an}, \mathcal{U}^1_{X^{an}})$$

is an isomorphism.

Proof. In view of lemma 3.1.4 it suffices to show that the mapping of the theorem is surjective. Put $X = \mathbb{A}_k^1 - \{a_1, ..., a_m\}$. Let $(\mathbb{D}_s | s \in \mathbb{N})$ be an increasing sequence of closed discs of $(\mathbb{A}_k^1)^{an}$ and, for every $i \in \{1, ..., m\}$ let $(\mathbb{E}_s^i | s \in \mathbb{N})$ be a decreasing sequence of open discs of $(\mathbb{A}_k^1)^{an}$ such that $(\mathbb{A}_k^1)^{an} = \bigcup_{s\in\mathbb{N}} \mathbb{D}_s$, $\{a_i\} = \bigcap_{s\in\mathbb{N}} \mathbb{E}_s^i$, $\mathbb{E}_1^i \subset \mathbb{D}_1$ and $\mathbb{E}_1^i \cap \mathbb{E}_1^j = \emptyset$ for $i \neq j$. Put

$$\mathcal{X}_s = \mathbb{D}_s - \bigcup_{i=1}^m \mathbb{E}_s^i.$$

Then $(\mathcal{X}_s | s \in \mathbb{N})$ is an increasing admissible covering of X^{an}. We assume that, for every $i \in \{1, .., m\}$, the boundary of $\mathbb{E}_1^i$ is contained in X_1. For every $s, n \in \mathbb{N}$, let $H_{s,n} \subset H^1(\mathcal{X}_s, \mathcal{U}^1)$ be the image of the mapping $H^1(\mathcal{X}_s, \mu_{p^n}) \to H^1(\mathcal{X}_s, \mathcal{U}^1)$. Then, for every $s \in \mathbb{N}$, $(H_{s,n} | n \in \mathbb{N})$ is an increasing sequence of subgroups of $H^1(\mathcal{X}_s, \mathcal{U}^1)$. Since $\mathcal{X}_s$ is affinoid and thus $H^1(\mathcal{X}_s, \mathcal{O}) = 0$, the asymptotic Kummer sequence gives

$$H^1(\mathcal{X}_s, \mathcal{U}^1) = \bigcup_{n\in\mathbb{N}} H_{s,n}. \tag{3.2.12}$$

Since $\mathcal{O}_{\mathcal{X}_s}(\mathcal{X}_s)$ is a principal domain, we have $0 = \mathrm{Pic}(\mathcal{X}_s) = H^1(\mathcal{X}_s, \mathcal{O}^*)$. Therefore the Kummer sequence $1 \to \mu_{p^n} \to \mathcal{O}^* \to \mathcal{O}^* \to 1$ gives a surjection $H^0(\mathcal{X}_s, \mathcal{O}^*) \twoheadrightarrow H^1(\mathcal{X}_s, \mu_{p^n})$. Hence by lemma 3.2.1

$$H^0(\mathcal{X}_s, \mathcal{O}^*)/H^0(\mathcal{X}_s, \mathcal{O}^*)(n) \xrightarrow{\sim} H_{s,n}. \tag{3.2.13}$$

Then lemma 3.2.2 implies that, for every $n, s, s' \in \mathbb{N}$ with $s' > s$, the restriction homomorphism $H_{s',n} \to H_{s,n}$ is bijective. With (3.2.12) we obtain

(3.2.14) For every $s, s' \in \mathbb{N}$ with $s' > s$, the restriction map $H^1(\mathcal{X}_{s'}, \mathcal{U}^1) \to H^1(\mathcal{X}_s, \mathcal{U}^1)$ is bijective.

Let H be the image of the mapping $\varinjlim_{n\in\mathbb{N}} H^1(X^{an}, \mu_{p^n}) \to H^1(X^{an}, \mathcal{U}^1)$. We have to show $H = H^1(X^{an}, \mathcal{U}^1)$. Let $a \in H^1(X^{an}, \mathcal{U}^1)$ be given. By (3.2.12), (3.2.13) and lemma 3.2.2 there is a $b \in H$ with $a_{|\mathcal{X}_1} = b_{|\mathcal{X}_1}$. Then by (3.2.14), $a_{|\mathcal{X}_s} = b_{|\mathcal{X}_s}$ for every $s \in \mathbb{N}$. Now the following claim 3.2.15 gives $a = b$.

Claim 3.2.15. The mapping $H^1(X^{an}, \mathcal{U}^1) \to H^1(\mathcal{X}_s, \mathcal{U}^1)$ is injective.

Proof of the claim: Let $\mathcal{F}$ be a $\mathcal{U}^1$-torsor on X^{an} such that $\mathcal{F}_{|\mathcal{X}_s}$ is trivial for every $s \in \mathbb{N}$. We have to show that $\mathcal{F}$ is trivial.

For every $s \in \mathbb{N}$, we equip $H^0(\mathcal{X}_s, \mathcal{F})$ with the metric such that for one (and hence for any) trivialization $\mathcal{F}_{|\mathcal{X}_s} \xrightarrow{\sim} \mathcal{U}^1_{|\mathcal{X}_s}$, the induced mapping of global sections $H^0(\mathcal{X}_s, \mathcal{F}) \to H^0(\mathcal{X}_s, \mathcal{U}^1) (\subset H^0(\mathcal{X}_s, \mathcal{O}))$ is isometric. (If this mapping is isometric for one trivialization then it is isometric for every trivialization, since for every $x \in \mathcal{X}_s$ and $t \in H^0(\mathcal{X}_s, \mathcal{U}^1)$ we have $|t(x)| = 1$).

We fix an element $x_1 \in \mathcal{X}_1$ and an element $a \in i^*(\mathcal{F})$, where $i : \{x_1\} \to \mathcal{X}_1$ is the inclusion. For every $s \in \mathbb{N}$ we put

$$L_s = \{t \in H^0(\mathcal{X}_s, \mathcal{F}) \quad | \quad i^*(t) = a\}.$$

Then, for every $s \in \mathbb{N}$, $(L_{s'} | \mathcal{X}_s)_{s' \geq s}$ is a decreasing sequence of non-empty subsets of $H^0(\mathcal{X}_s, \mathcal{F})$. For fixed s and increasing s', the diameters of $(L_{s'|\mathcal{X}_s})$ in the metric space $H^0(\mathcal{X}_s, \mathcal{F})$ tend to zero (by lemma 3.2.10). Then, since $H^0(\mathcal{X}_s, \mathcal{F})$ is complete, the sequence $(L_{s'|\mathcal{X}_s})_{s' \geq s}$ converges to an element $t_s \in H^0(\mathcal{X}_s, \mathcal{F})$. The t_s, $s \in \mathbb{N}$ glue to a global section $t \in H^0(X^{an}, \mathcal{F})$. □

Remark 3.2.16. We take the time out to make some side remarks on the cohomology of $\mathcal{U}^1$. These will not have any bearings on the continuation, so the hurried reader is invited to skip them.

The question of the structure of $H^1(X, \mathcal{U}^1)$ is meaningful and not trivial even in the proper case. Suppose now that X is the analytification of a proper scheme. I propose the following conjectural picture. First of all, let us introduce the sheaves $\mathcal{U}_X^{\rho_1}$, $\mathcal{O}_X^{\rho_1}$ defined by

$$\begin{aligned} \mathcal{U}_X^{\rho_1}(V) &= \{f \in \mathcal{U}_X^1(V) \quad | \quad |1 - f|_{\sup} < \rho_1\} \\ \mathcal{O}_X^{\rho_1}(V) &= \{f \in \mathcal{O}_X(V) \quad | \quad |f|_{\sup} < \rho_1\} \end{aligned}$$

for any étale map $V \to X$. The restriction of λ induces an isomorphism $\mathcal{U}_X^{\rho_1} \xrightarrow{\sim} \mathcal{O}_X^{\rho_1}$. The situation is summarized by the following diagram

$$H^1(X, \mathcal{O}_X^*) \xleftarrow{j_1} H^1(X, \mathcal{U}_X^1) \xleftarrow{j_2} H^1(X, \mathcal{U}_X^{\rho_1}) \xrightarrow{H^1(\lambda)} H^1(X, \mathcal{O}_X^{\rho_1}) \xrightarrow{j_3} H^1(X, \mathcal{O}_X).$$

We recall that $H^1(X, \mathcal{O}_X)$ is canonically identified with the tangent space $T_0\mathrm{Pic}(X)$ of $\mathrm{Pic}(X)$ at the point $0 \in \mathrm{Pic}(X)$. Hence the following conjectures arise naturally:

1) the map j_3 is injective and identifies $H^1(X, \mathcal{O}_X^{\rho_1})$ with an open neighborhood (with the topology inherited from k) of the origin in $T_0\mathrm{Pic}(X)$;

2) the composition $j_1 \circ j_2 \circ H^1(\lambda)^{-1}$ corresponds, via the identification in (1) and the standard identification $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$, to the classical exponential map for (p-adic) Lie groups.

Then we expect that also the map j_1 be the imbedding of an open disc around $0 \in \mathrm{Pic}(X)$. This should be in fact the smallest open disc which contains all the p-power torsion elements in $\mathrm{Pic}(X)$.

3.3. **Flat line bundles and unitary characters.** Unitary representations of the fundamental group play an important role in complex analytic geometry. In this section we propose a (very modest) analogue of the complex analytic picture.

We consider only the analogue of unitary representations of dimension one, *i.e.* of maps of the fundamental group into the unit circle S^1. In our setting, we have a kind of "discretized" version of S^1, namely the group μ_∞ of roots of unity of arbitrary order. As an abstract group this is isomorphic to $\mathbb{Q}/\mathbb{Z}$ and we point out the split imbedding $\mu_{p^\infty} \hookrightarrow \mu_\infty$. The quotient $\mu_\infty/\mu_{p^\infty}$ is canonically identified with the group $\varinjlim_{(n,p)=1} \mu_n$.

Lemma 3.3.1. *Let X be an open subscheme of the affine line. Then*

$$H^1(X^{an}, \varinjlim_{(n,p)=1} \mu_n) \simeq H^1(X, \varinjlim_{(n,p)=1} \mu_n).$$

Proof. Write $X = \mathbb{A}^1_k - \{a_1, ..., a_m\}$ and let $(\mathbb{D}_s | s \in \mathbb{N})$ be an increasing sequence of closed discs of the affine line, with $\{a_1, ..., a_m\} \subset \mathbb{D}_1$ and for each $i \in \{1, ..., m\}$ let $(\mathbb{E}^i_s | s \in \mathbb{N})$ be a decreasing sequence of open discs such that $\bigcup_{s\in\mathbb{N}} \mathbb{D}_s = (\mathbb{A}^1_k)^{an}$, $\mathbb{E}^i_1 \subset \mathbb{D}_1$, $\{a_i\} = \bigcap_{s\in\mathbb{N}} \mathbb{E}^i_s$ and $\mathbb{E}^i_1 \cap \mathbb{E}^j_1 = \emptyset$ for $i \neq j$. Put $X_s = \mathbb{D}_s - \bigcup_i^m \mathbb{E}^i_s$. Since X_s is quasi-compact, we have

$$H^i(X_s, \varinjlim_{(n,p)=1} \mu_n) \simeq \varinjlim_{(n,p)=1} H^i(X_s, \mu_n).$$

But it is well known that $H^1(X_s, \mu_n) \simeq H^1(X, \mu_n)$ for $(n,p) = 1$. Moreover, for any $s' > s$ the restriction maps $H^0(X_{s'}, \mu_n) \to H^0(X_s, \mu_n)$ is clearly bijective, hence the claim follows from [B1] lemma 6.3.12. $\square$

Proposition 3.3.2. *Let X be an open subscheme of $\mathbb{A}^1_k$ and fix a geometric point $\overline{x}_0 \in X$. There is a canonical isomorphism*

$$\mathrm{Hom}_{cnt}(\pi_1(X^{an}, \overline{x}_0), \mu_\infty) \simeq \Gamma(X^{an}, \mathcal{O}_X^{an})/k \oplus H^1(X, \mu_\infty)$$

where $\mathrm{Hom}_{cnt}(-, \mu_\infty)$ *denotes the group of continuous homomorphisms into the discrete group* μ_∞.

Proof. The isomorphism $\mu_\infty \simeq \mu_{p^\infty} \oplus (\mu_\infty/\mu_{p^\infty})$ induces a canonical decomposition

$$\mathrm{Hom}_{cnt}(G, \mu_\infty) \simeq \mathrm{Hom}_{cnt}(G, \mu_{p^\infty}) \oplus \mathrm{Hom}_{cnt}(G, \mu_\infty/\mu_{p^\infty})$$

for any topological group G. The term $\mathrm{Hom}_{cnt}(\pi_1(X^{an}, x_0), \mu_\infty/\mu_{p^\infty})$ is computed by lemma 3.3.1. Moreover, since $H^1(X^{an}, \mathcal{O}_X^{an}) = 0$, by (3.1.3) and theorem 3.2.11 we have a short exact sequence

$$0 \to H^0(X^{an}, \mathcal{O}_X^{an})/k \to H^1(X^{an}, \mu_{p^\infty}) \to H^1(X, \mu_{p^\infty}) \to 0.$$

A splitting for this short exact sequence is provided by the sequence of imbeddings of sheaves $\mu_{p^n} \hookrightarrow \mu_{p^\infty}$ for all $n \in \mathbb{N}$. $\square$

In particular, the proposition shows that

$$\mathrm{Hom}_{cnt}(\pi_1((\mathbb{A}^1_k)^{an}, \overline{x}_0), \mu_\infty) \simeq \Gamma((\mathbb{A}^1_k)^{an}, \mathcal{O}^{an})/k. \tag{3.3.3}$$

More generally, for a smooth connected scheme X over k and a point $\overline{x}_0 \in X(k)$ N.Katz defines in [Ka1] the *differential fundamental group* $\pi_1^{diff}(X, \overline{x}_0)$ *of* X *(based at* x_0*)*. This is a pro-algebraic k-group scheme, whose algebraic representations into an algebraic group $GL(n,k)$ classify the vector bundles of rank n on X with an integrable connection. It is shown in [Ka1] that there is an isomorphism

$$\mathrm{Hom}_{k-grp.sch.}(\pi_1^{diff}(\mathbb{A}^1_k, \overline{x}_0), \mathbb{G}_{m,k}) \simeq \Gamma(\mathbb{A}^1_k, \mathcal{O})/k. \tag{3.3.4}$$

Comparing (3.3.3) and (3.3.4) we derive an imbedding

$$\phi : \mathrm{Hom}_{k-grp.sch.}(\pi_1^{diff}(\mathbb{A}^1_k, \overline{x}_0), \mathbb{G}_{m,k}) \hookrightarrow \mathrm{Hom}_{cnt}(\pi_1((\mathbb{A}^1_k)^{an}, \overline{x}_0), \mu_\infty)$$

which morally says that all line bundles on $\mathbb{A}^1_k$ with an integrable connection "come from" a unitary character of the rigid analytic fundamental group. It can be shown that in fact ϕ is a canonical map. It would be possible to define a Tannakian category of analytic flat bundles on $(\mathbb{A}^1_k)^{an}$ and hence a "rigid differential Galois group", which would induce an isomorphism in place of the imbedding ϕ above.

However, it is more interesting to make the following observation. Take a field R of characteristic $\ell \neq p$ and such that $\mu_{p^\infty} \subset R^\times$. Then to each μ_{p^∞}-torsor $T \in \mathrm{Hom}_{cnt}(\pi_1(X^{an}, \overline{x}_0), \mu_{p^\infty})$ we can associate a locally constant sheaf of free R-modules of rank one on the étale site of X^{an}, which is usually denoted $T \underset{\mu_{p^\infty}}{\times} R$ (see *e.g.* [SGA4$\frac{1}{2}$] Sommes trig. for the yoga of torsors). Let η_∞ be the generic point of henselization of $\mathbb{P}^1_k$ at the point $\infty \in \mathbb{P}^1_k$.

Claim 3.3.5. (1) The cohomology groups $H^*((\mathbb{A}^1_k)^{an}, T \underset{\mu_{p^\infty}}{\times} R)$ are finitely generated R-modules if and only if there exists a flat line bundle (L, ∇) on $\mathbb{A}^1_k$ such that $T = \phi(L, \nabla)$. (2) Suppose that $T = \phi(L, \nabla)$ for some (L, ∇) as in (1). Then we have

$$\chi((\mathbb{A}^1_k)^{an}, T \underset{\mu_{p^\infty}}{\times} R) = 1 - \mathrm{Irr}((L, \nabla)_{\eta_\infty})$$

where χ denotes the Euler-Poincaré characteristic and $\mathrm{Irr}((L, \nabla)_{\eta_\infty})$ is the *irregularity index* of the restriction of (L, ∇) to the η_∞, as defined in [Ka1].

Part (2) and the "if" direction in part (1) of the claim will be proven later in this paper, as special cases of our Grothendieck-Ogg-Shafarevich formula (theorem 8.6.2). We will also illustrate the "only if" direction with an example (see remark 8.6.7). This result suggests that there should be a class of analytic étale local systems on algebraic curves which we might call *"locally of differential origin"* which should be especially well-behaved; in particular, the étale cohomology of such sheaves should be finitely generated. The attempt to materialize this intuition will lead us to the definition of the *meromorphic local fundamental group* in chapter 5.

The case $X = \mathbb{G}_{m,k}$ is also interesting. Here we have the two formulas

$$\mathrm{Hom}_{cnt}(\pi_1((\mathbb{G}_{m,k})^{an}, \overline{x}_0), \mu_\infty) \simeq \Gamma((\mathbb{G}_{m,k})^{an}, \mathcal{O}^{an})/k \oplus (\mathbb{Q}/\mathbb{Z})$$
$$Hom_{k-grp.sch.}(\pi_1^{diff}(\mathbb{G}_{m,k}, \overline{x}_0), \mathbb{G}_{m,k}) \simeq \Gamma(\mathbb{G}_{m,k}, \mathcal{O})/k \oplus (k/\mathbb{Z}).$$

Here the terms $\mathbb{Q}/\mathbb{Z}$ and $k/\mathbb{Z}$ due their appearence to the tamely ramified (algebraic) local systems on $(\mathbb{G}_{m,k})^{an}$ and respectively to the connections with regular singularities on $\mathbb{G}_{m,k}$. More precisely, we obtain a canonical imbedding $\psi : \mathbb{Q}/\mathbb{Z} \hookrightarrow k/\mathbb{Z}$, and the image of ψ consists of the connections with regular singularities, whose residue (at the origin and at infinity) is a root of unit. Morally this means that the solutions of the remaining differential equations (*i.e.* those in the complement of $\mathrm{Im}(\psi)$) converge only on small discs, and hence do not yield any local systems on the étale site of $(\mathbb{G}_{m,k})^{an}$. Perhaps the appropriate language here would be that of Frobenius-crystals, or of unit root crystals.

4. Locally algebraic vanishing cycles

4.1. **σ-compact spaces.** In the following two definitions we introduce a class of spaces which will play a special role throughout this chapter.

Definition 4.1.1. An analytic space X is said to be *σ-compact* if it is locally compact and it is a countable union of compact analytic subdomains.

If X is also connected, it follows from [Bou] chapter I.11 exercise 14, that X can be written as an increasing union of connected compact subspaces $(X_i)_{i \geq 0}$ such that $X_i \subset Int(X_{i+1})$ (the interior of X_{i+1}) for all i. We say that the σ-compact space X is *geometrically connected* if the X_i can be chosen to be geometrically connected analytic spaces.

Definition 4.1.2. Suppose that X is a connected σ-compact analytic space and write $X = \bigcup X_i$ for a family of compact connected subspaces as above.

The category $\underline{\widetilde{\mathrm{Cov}}}_X$ of *locally algebraic coverings* consists of all the surjective connected étale morphisms $f : Y \to X$ where Y is any analytic space which can be obtained as an increasing union $Y = \bigcup_{i \geq 0} Y_i$ such that f restricts to a finite étale covering $Y_i \to X_i$ for all ≥ 0. In particular, Y is σ-compact.

Fix a geometric point $\overline{x}$ localized inside X_0. The *locally algebraic fundamental group* of X (based at $\overline{x}$) is the topological group

$$\widetilde{\pi}_1(X, \overline{x}) = \lim_{i \to \infty} \pi_1^{alg}(X_i, \overline{x})$$

endowed with the direct limit topology.

Remark 4.1.3. By a compactness argument, it is not hard to show that neither $\underline{\widetilde{\mathrm{Cov}}}_X$ nor $\widetilde{\pi}_1(X, x)$ depend on the choice of the X_i. Notice that a locally algebraic covering is an étale covering in the sense of [deJ]; the converse is not necessarily true. We also observe that a composition of locally algebraic coverings is again locally algebraic (the corresponding statement for étale coverings is false), and that if

$X' \to X$ is a morphism such that X' is also σ-compact, then for any locally algebraic covering $Y \to X$ the fibre product $Y' = Y \times_X X' \to X'$ is a disjoint union of locally algebraic coverings.

Let X be any connected k-analytic space. Recall (see [deJ] theorem 2.10) that for any geometric point $\overline{x} \in X$ the fibre functor

$$\omega_{X,\overline{x}} : \underline{\mathrm{Cov}}_X \to \pi_1(X, \overline{x}) - \underline{\mathrm{Set}}$$

is fully faithful, and induces an equivalence between the category of disjoint unions of étale coverings of X, and the category $\pi_1(X, \overline{x}) - \underline{\mathrm{Set}}$. Similarly, if X is connected and σ-compact the map $Y \mapsto Y \times_X \overline{x}$ defines a fibre functor

$$\widetilde{\omega}_{X,\overline{x}} : \widetilde{\underline{\mathrm{Cov}}}_X \to \widetilde{\pi}_1(X, \overline{x}) - \underline{\mathrm{Set}}$$

with target the category of discrete sets with a continuous action of $\widetilde{\pi}_1(X, \overline{x})$.

Proposition 4.1.4. *The fibre functor $\widetilde{\omega}_{X,\overline{x}}$ is fully faithful and every $\widetilde{\pi}_1(X, \overline{x})$-set consisting of a single orbit is contained in the essential image of $\widetilde{\omega}_{X,\overline{x}}$.*

Proof. Locally on X, this reduces to a question about finite étale coverings and their morphisms, and fully faithfulness follows easily. If O is a set consisting of a single orbit, let S be the stabilizer of some point of O. This is an open subgroup of $\widetilde{\pi}_1(X, \overline{x})$, which means that the preimage of S in $\pi_1^{alg}(X_i, \overline{x})$ is a subgroup of finite index, which by [deJ] theorem 2.10 corresponds to some finite étale covering $Y_i \to X_i$. It is easy to check that these Y_i glue to give $Y \in \widetilde{\underline{\mathrm{Cov}}}_X$ with $Y_{\overline{x}} \simeq O$. □

Definition 4.1.5. Let ℓ be a prime number different from the residue characteristic p of k. An *ℓ-coefficient ring* is a local Artinian ring Λ in which ℓ is nilpotent, such that Λ is the inductive limit of the direct system of all its finite subrings and such that $\mu_{p^\infty} \subset \Lambda^\times$.

In the following we shall be primarily interested in sheaves of Λ-modules where Λ is an ℓ-coefficient ring. Denote by $\Lambda - \underline{\mathrm{Loc}}_X$ (resp. by $\underline{\mathrm{Rep}}_{cnt}(\widetilde{\pi}_1(X, \overline{x}), \Lambda))$ the category of locally constant sheaves of finitely generated Λ-modules on X (resp. the category of continuous linear representations of $\widetilde{\pi}_1(X, \overline{x})$ into finitely generated Λ-modules).

Lemma 4.1.6. *Let X be a compact analytic variety and F a locally constant sheaf of finitely generated Λ-modules on X, where Λ is an ℓ-coefficient ring. Then we can find a finite subring $A \subset \Lambda$ and a locally constant sheaf F' of A-modules on X such that $F \simeq F' \otimes_A \Lambda$.*

Proof. Since X is compact, we can find a finite covering $\bigcup_i U_i = X$ by open subsets, and for each i a finite étale morphism $V_i \to U_i$ such that $G = F_{|V_i}$ is the constant sheaf associated to a certain finitely generated Λ-module M_i. The descent data for F from V_i to U_i is then essentially a finite set of automorphisms of M_i. These automorphisms are then defined already over some finite subring $A_i \subset \Lambda$. Hence we can find a locally constant sheaf F_i of A_i-modules on U_i such that $F_{|U_i} = F_i \otimes_{A_i} \Lambda$.

Similarly, let $U_{ij} = U_i \cap U_j$, so that F is defined by a cocycle system of morphisms $\phi_{ij} : (F_i \otimes_{A_i} \Lambda)_{|U_{ij}} \to (F_j \otimes_{A_j} \Lambda)_{|U_{ij}}$. Again, these morphisms are already defined on some big finite subring $A_{ij} \supset A_i + A_j$ and the claim follows. □

Proposition 4.1.7. *Suppose X is connected and σ-compact and Λ is some ℓ-coefficient ring. Then there is an equivalence of categories*

$$\Lambda - \underline{\mathrm{Loc}}_X \xrightarrow{\sim} \underline{\mathrm{Rep}}_{cnt}(\widetilde{\pi}_1(X, \overline{x}), \Lambda).$$

Proof. Write $X = \bigcup_{i \geq 0} X_i$ as above and pick a geometric point $\overline{x}$ localized at a point $x \in X_0$. Let $F_{\overline{x}}$ be the stalk of F at $\overline{x}$. By lemma 4.1.6 the restriction of F to X_i is already defined on some finite subring of Λ, hence $F_{|X_i}$ corresponds canonically to a representation of $\pi_1^{alg}(X_i, \overline{x})$ on $F_{\overline{x}}$. For different i, these representations are compatible, hence they define a representation of $\widetilde{\pi}_1(X, \overline{x})$ on $F_{\overline{x}}$. Viceversa, a continuous representation gives rise to a sequence of sheaves F_i on X_i which glue over all X. □

Proposition 4.1.8. *Suppose that X is σ-compact and geometrically connected. Then the sequence*

$$\widetilde{\pi}_1(X \times_k \widehat{k}^a, \overline{x}) \to \widetilde{\pi}_1(X, \overline{x}) \to \mathrm{Gal}(k^a/k) \to 1$$

is exact.

Proof. Since a direct limit of exact sequences is exact, this follows immediately from [deJ] proposition 2.13. □

Proposition 4.1.9. *Let X be a σ-compact space and G be a discrete group which is the inductive limit of its finite subgroups. Then every G-torsor over X is a disjoint union of σ-compact spaces.*

Proof. We can assume that X is connected, so that $X = \bigcup_{i\geq 0} X_i$ as usual. Let T be some G-torsor on X. The restriction of T to X_i is a G-torsor on X_i, and hence it is classified by some class in $H^1(X_i, G)$ (see [B1] corollary 4.1.9). Since X_i is compact we have $H^1(X_i, G) = \varinjlim_F H^1(X_i, F)$ where the limit ranges over all the finite subgroups $F \subset G$. The claim follows. □

Corollary 4.1.10. *Suppose that X is connected and σ-compact. Then for any group G as in proposition 4.1.9 there is a canonical isomorphism*

$$H^1(X, G) \simeq \mathrm{Hom}_{cnt}(\widetilde{\pi}_1(X, \overline{x}), G).$$

□

4.2. **pro-analytic spaces.** We recall here a few generalities about pro-analytic spaces and ind-sheaves in an equivariant setting.

Let $F : \mathcal{I} \to \mathbf{Cat}$ be a functor from the cofiltered small category $\mathcal{I}$ to the category of all small categories. We define a new category $\varinjlim_{\mathcal{I}} F$ as follows. The objects are all the pairs $(x; i)$ such that $i \in \mathcal{I}$ and $x \in F(i)$. A representative of a morphism $(x, i) \to (y, j)$ is a triple (α, β, u) where $\alpha : i \to l$ and $\beta : j \to l$ are arrows in $\mathcal{I}$ and $u : F_\alpha(x) \to F_\beta(y)$ is a morphism in $F(l)$. Two representatives (α, β, u) and (α', β', u') of a morphism $(x, i) \to (y, j)$ are said to be equivalent if there exist arrows $\gamma : l \to q$ and $\gamma' : l' \to q$ such that $\gamma \circ \alpha = \alpha' \circ \gamma'$, $\gamma \circ \beta = \gamma' \circ \beta'$ and $F_\gamma(u) = F_{\gamma'}(u')$.

Let us specialize and assume that $F : \mathcal{I} \to \mathbf{AbCat}$ is a functor from $\mathcal{I}$ to the category of all small abelian categories satifying the axioms (AB1)−(AB5) of Grothendieck. Then $\varinjlim_{\mathcal{I}} F$ is an abelian category as well. Moreover, if all the categories $F(i)$ have enough injectives and all the functors F_α take injectives to injectives, then we obtain enough injectives.in $\varinjlim_{\mathcal{I}} F$ by taking all the objects of the form (x, i), where x is injective in $F(i)$.

Next, let $\mathbb{N}$ be the set of natural numbers with its natural ordering, which we view as a cofiltered category in the standard way. Let $F : \mathbb{N} \to \mathbf{Cat}$ be a functor. We construct the category $F^{\mathbb{N}}$ as follows. The objects are all the pairs $(x; \alpha) = (\{x_i\}_{i\in\mathbb{N}} ; \{\alpha_{ij}\}_{i<j})$ where $x_i \in F(i)$ for all i and $\alpha_{ij} \in \mathrm{Hom}_{F(j)}(F_{ij}(x_i), x_j)$ for all $i < j$, and such that $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$ for all $i < j < k$. The morphisms $\mathrm{Hom}((x, \alpha), (y, \beta))$ are all the sequences $\{\gamma_i\}_{i\in\mathbb{N}}$ where $\gamma_i \in \mathrm{Hom}_{F(i)}(x_i, y_i)$ and $\beta_{ij} \circ F_{ij}(\gamma_i) = \gamma_j \circ \alpha_{ij}$ for all $i < j$.

Next, suppose that $F : \mathbb{N} \to \mathbf{AbCat}$ is given. Then $F^{\mathbb{N}}$ is also abelian. We say that an object (x, α) is eventually zero if $x_i = 0$ for all i larger than some i_0. The eventually zero objects form a thick subcategory, and we form a new category $\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F$ by localizing with respect to the eventually isomorphisms. Clearly $\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F$ is again abelian. For all i we obtain an additive functor

$$F(i) \to \underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F$$

by sending an object x of $F(i)$ to the sequence (x_j, α_{ij}) such that $x_j = 0$ if $i > j$ and $x_j = F_{ij}(x)$ if $j \geq i$, and such that α_{ij} is the identity of x_j. We say that (x_j, α_{ij}) is the *stable ind-object associated to* x. Any object of $\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F$ isomorphic to some object of this type will be said to be stable. Notice that the full subcategory of stable objects is equivalent to the limit category $\varinjlim_{i\in\mathbb{N}} F(i)$ (cf. [B2] section 2).

On the other hand, we can also form the derived category $\mathrm{D}^+(F^{\mathbb{N}})$ and then define a complex $\{K_i^\bullet\}_{i\in\mathbb{N}}$ to be eventually zero if for all integers $i > i_0$ the complexes $K_i^\bullet \in \mathrm{D}^+(F(i))$ are quasi-isomorphic to zero. The localization of $\mathrm{D}^+(F^{\mathbb{N}})$ relative to the family of eventually quasi-isomorphisms will be denoted $\mathrm{D}^+(\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F)$. It is a triangulated category with a canonical faithful functor

$$\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F \to \mathrm{D}^+(\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F).$$

We also have a notion of *stable ind-complex* defined in the obvious fashion.

Suppose now that $F, G : \mathbb{N} \to \mathbf{AbCat}$ are given, together with, for each i a left exact additive functor $\phi_i : F(i) \to G(i)$ is given such that $\phi_j F_{ij} = G_{ij} \phi_i$. This data yields a left exact functor

$$\phi^{\mathbb{N}} : F^{\mathbb{N}} \to G^{\mathbb{N}}$$

in a natural way. Suppose moreover that for all i the category $F(i)$ has enough injectives, and that all the functors F_{ij}, G_{ij} are exact. We consider the δ-functor whose component in degree q

$$T^q : F^{\mathbb{N}} \to G^{\mathbb{N}}$$

is defined as

$$(x, \alpha) \mapsto \left(\{R^q\phi_0(x_i)\}_{i\in\mathbb{N}} \ ; \ \{\alpha'_{ij}\}_{i<j}\right)$$

where $\alpha'_{ij} : G_{ij}R^q\phi_i(x_i) \to R^q\phi_j(x_j)$ is the composition of the canonical morphisms

$$G_{ij} \circ R^q\phi_i(x_i) \to R^q(G_{ij} \circ \phi_i)(x_i) = R^q(\phi_j \circ F_{ij})(x_i) \to R^q\phi_j \circ F_{ij}(x_i) \to R^q\phi_j(x_j).$$

It is clear that T^q is effaceable for all $q > 0$, hence the δ-functor $\{T^q\}_{q\in\mathbb{N}}$ is universal. In this way we obtain a derived functor

$$R(\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, \phi) : \mathrm{D}^+(\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, F) \to \mathrm{D}^+(\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, G).$$

Finally we return to analytic spaces. A pro-analytic space $\mathbf{Z} = \lim\limits_{\overleftarrow{i\in\mathcal{I}}} Z_i$ is a functor from a small cofiltered category $\mathcal{I}$ to the category of analytic spaces. For an analytic space X we denote by $X - \mathcal{A}n$ the category of X-analytic spaces, defined in the obvious way. For all $\phi \in \mathrm{Hom}_{\mathcal{I}}(j, i)$ the corresponding morphism $\phi_{\mathbf{Z}} : Z_j \to Z_i$ induces a functor

$$Z_i - \mathcal{A}n \to Z_j - \mathcal{A}n \ : \qquad X_i \mapsto X_j = X_i \times_{Z_i} Z_j.$$

Then the category of $\mathbf{Z}$-analytic spaces is by definition the direct limit category

$$\mathbf{Z} - \mathcal{A}n = \lim_{\overrightarrow{i\in\mathcal{I}^\circ}} Z_i - \mathcal{A}n.$$

Remark 4.2.1. The category of pro-analytic spaces admits fibre products and cofiltered projective limits. The category of $\mathbf{Z}$-analytic spaces admits fibre products.

Definition 4.2.2. Let $\mathbf{X}$ be a $\mathbf{Z}$-analytic space. With the notation above, let $\phi_{\mathbf{X}} : X_j \to X_i$ be the morphism induced by $\phi \in \mathrm{Hom}_{\mathcal{I}^\circ}(i, j)$. We derive a collection of functors

$$\phi^*_{\mathbf{X}} : \mathbf{S}(X_i, \Lambda) \to \mathbf{S}(X_j, \Lambda)$$

i.e. a functor $F : \mathcal{I}^\circ \to \mathbf{AbCat}$. Then we define the *category of abelian sheaves* on $\mathbf{X}$ by setting $\mathbf{S}(\mathbf{X}, \Lambda) = \lim\limits_{\overrightarrow{\mathcal{I}^\circ}} F$.

If all the morphisms $\phi_{\mathbf{Z}} : Z_j \to Z_i$ are étale, the category $\mathbf{S}(\mathbf{X}, \Lambda)$ has enough injectives. In this case, for any morphism $f : \mathbf{Y} \to \mathbf{X}$ of $\mathbf{Z}$-analytic spaces, the usual cohomological functors $f^*, Rf_*, \ldots$ extend naturally to the corresponding limit categories.

Next, a pro-group $\Gamma = \{\Gamma_i\}_{i\in\mathcal{I}}$ is a functor from the small filtered category $\mathbb{N}^\circ$ to the category of topological groups.

Definition 4.2.3. Let $\underline{\mathrm{Rep}}(\Gamma_i, \Lambda)$ (resp. $\underline{\mathrm{Rep}}_{cnt}(\Gamma_i, \Lambda)$) be the category of all Λ-modules (not necessarily finitely generated) endowed with a Γ_i-action (resp. a continuous Γ_i-action). The morphisms $\Gamma_i \to \Gamma_j$ $(i > j)$ induce restriction functors

$$\rho_{ij} : \underline{\mathrm{Rep}}(\Gamma_j, \Lambda) \to \underline{\mathrm{Rep}}(\Gamma_i, \Lambda)$$

(resp. the restriction ρ^{cnt}_{ij} to the subcategories of continuous representations) and hence a functor $\rho : \mathbb{N} \to \mathbf{AbCat}$. The *category of ind-representations* of the pro-group Γ is defined as $\underline{\mathrm{Rep}}(\Gamma, \Lambda) = \underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, \rho$. The category of *continuous ind-representations* of Γ is $\underline{\mathrm{Rep}}_{cnt}(\Gamma, \Lambda) = \underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, \rho^{cnt}$. Similarly, if X is an analytic space, we get a system of functors

$$\rho_{ij} : \mathbf{S}(X, \Lambda[\Gamma_j]) \to \mathbf{S}(X, \Lambda[\Gamma_i])$$

on the categories of sheaves of continuous $\Lambda[\Gamma_i]$-modules on X. Then we define the category $\mathbf{S}(X, \Lambda[\Gamma])$ of *ind-sheaves of continuous* Γ*-equivariant* Λ*-modules* on X as $\underset{\overrightarrow{\mathbb{N}}}{\mathrm{ind}}\, \rho$.

Remark 4.2.4. In definition 4.2.3 the continuity of the Γ_i-action on an object of $\mathbf{S}(X, \Lambda[\Gamma_i])$ is meant as in the weak sense of [B2] section 1.

Proceeding as above we can extend the usual cohomological formalism to the categories $\underline{\mathrm{Rep}}(\Gamma, \Lambda)$ and $\mathbf{S}(X, \Lambda[\Gamma])$.

4.3. **Construction of the functor.** As customary, the functor of locally algebraic vanishing cycles will assume its values in a certain category of equivariant sheaves under the action of some fundamental group. The problem is that we do not know whether the family of all open normal subgroups of the locally algebraic fundamental group of a σ-compact space forms a fundamental system of neighborhoods of the identity. Hence we must proceed a bit more carefully than usual.

Proposition 4.3.1. *Let X be σ-compact and connected, and write $X = \bigcup_{i\geq 0} X_i$ as usual. Then for every open subgroup $S \subset \widetilde{\pi}_1(X, \overline{x})$ and for every integer $i \geq 0$ there exists another open subgroup $S' \subset S$ whose preimage in $\pi_1^{alg}(X_i, \overline{x})$ is a normal subgroup of $\pi_1^{alg}(X_i, \overline{x})$.*

Proof. Let T be the preimage of S in $\pi_1^{alg}(X_i, \overline{x})$. Since the latter group is profinite, we can find a largest subgroup $T' \subset T$ which is open and normal. Clearly $T' = \bigcap_g g^{-1}Tg$ where g ranges over all the elements of $\pi_1^{alg}(X_i, \overline{x})$. By compactness, there is a finite set P such that we have already $T' = \bigcap_{g\in P} g^{-1}Tg$. Then we can take $S' = \bigcap_{g\in P} g^{-1}Sg$. □

Let C be a smooth curve defined over k, and s a k-rational point on C. We consider the associated pro-analytic space $C(s)$. Concretely, if we fix a local coordinate z around s, $C(s)$ is isomorphic to a projective system of small discs $\mathbb{E}(s,r) = \{x \ : \ 0 \leq |z(x)| < r\}$ centered at s. Clearly for all $r > 0$ the space $\mathbb{E}^*(s,r) = \mathbb{E}(s,r) - \{s\}$ is σ-compact. The projective system of all such $\mathbb{E}^*(s,r)$ forms a pro-analytic space which we denote η_s. Similarly, we denote by $\eta_{\overline{s}}$ the pro-analytic space formed by the projective system of all $\mathbb{E}^*(s,r) \times_k \widehat{k}^a$.

To start with, we want to construct a certain pro-analytic space $\overline{\eta}_s$ with a map of pro-analytic spaces $\overline{\eta}_s \to \eta_s$. For each $r > 0$ choose a geometric point $\overline{x}_r \in \mathbb{E}^*(s,r)$ and set $\pi_1^{(r)} = \widetilde{\pi}_1(\mathbb{E}^*(s,r), \overline{x}_r)$. For all $r_1 > r_2 > 0$ pick a system of morphisms $\phi_{r_1,r_2} : \pi_1^{(r_2)} \to \pi_1^{(r_1)}$ (induced by the imbeddings $\mathbb{E}^*(s,r_2) \subset \mathbb{E}^*(s,r_1)$) such that $\phi_{r_1,r_2} \circ \phi_{r_2,r_3} = \phi_{r_1,r_3}$. These maps are unique only up to inner automorphisms, but we do not mind. We remark the exact sequence

$$\pi_1(\eta_{\overline{s}}, \overline{x}) \to \pi_1(\eta_s, \overline{x}) \to \mathrm{Gal}(k^a/k) \to 0 \tag{4.3.2}$$

which follows immediately from proposition 4.1.8.

The obvious next step would be to define the fundamental group of η_s by taking the inverse limit of the projective system defined above. Unfortunately very little is known about these groups and homomorphisms, hence for the time being I see no alternative to bringing along the whole structure.

Definition 4.3.3. The *local fundamental group* $\pi_1(\eta_s, \overline{x})$ is the pro-group indexed by the ordered set of positive real numbers and defined by the family of groups $\{\pi_1^{(r)}\}_{r>0}$ and their homomorphisms $\{\phi_{r_1,r_2} : \pi_1^{(r_2)} \to \pi_1^{(r_1)}\}_{r_1>r_2>0}$.

We denote by $\mathrm{D}^+(X, \Lambda[\pi_1(\eta_s, \overline{x})])$ the derived category (in the sense of section 4.2) of $\mathbf{S}(X, \Lambda[\pi_1(\eta_s, \overline{x})])$. The usual global section functor (and its derived functors) for Λ-sheaves extends to ind-sheaves in the obvious manner

$$H^i(X, -) : \mathbf{S}(X, \Lambda[\pi_1(\eta_s, \overline{x})]) \to \underline{\mathrm{Rep}}(\pi_1(\eta_s, \overline{x}), \Lambda) \ : \qquad F_\bullet \mapsto \{\ H^i(X, F_r)\ \}_{r>0}.$$

For any $r > 0$ choose a left inverse τ_r of the functor $\widetilde{\omega}_{\overline{x}_r} : \widetilde{\underline{\mathrm{Cov}}}_{\mathbb{E}^*(s,r)} \to \pi_1^{(r)} - \underline{\mathrm{Set}}$. Then for all open subgroups $S \subset \pi_1^{(r)}$ we obtain a locally algebraic covering $t_S : \tau_r(\pi_1^{(r)}/S) \to \mathbb{E}^*(s,r)$ and using the maps ϕ_{r_1,r_2} we also get morphisms

$$t_{S_1,S_2} : \tau_s(\pi_1^{(r_2)}/S_1) \to \tau_r(\pi_1^{(r_1)}/S_2)$$

whenever $r_1 > r_2$ and $\phi_{r_1,r_2}(S_1) \subset S_2$. By construction we have $t_{S_2} \circ t_{S_1,S_2} = t_{S_1}$ for all S_1, S_2 as above.

We define a small cofiltered category $\mathcal{I}$ whose objects are all pairs (r, S) where S is an open subgroup of $\pi_1^{(r)}$ and with morphisms $\mathrm{Hom}((r_1, S_1), (r_2, S_2))$ equal to the restriction of $\phi_{r_2,r_1} : S_1 \to S_2$ in case $r_1 \leq r_2$ and $\phi_{r_2,r_1}(S_1) \subset S_2$, the empty set otherwise. The data above defines a functor $t : \mathcal{I} \to \acute{\mathrm{E}}\mathrm{t}(\eta_s)$ *i.e.* a pro-analytic space which we denote $\overline{\eta}_s$.

With this preparation, we can now define our functor of vanishing cycles. Let $\mathbf{X}$ be a $C(s)$-analytic space so that $\mathbf{X} = X \times_C C(s)$ for some C-analytic space X and set$\mathbf{X}_{\eta_s} = \mathbf{X} \times_{C(s)} \eta_s$, $\mathbf{X}_{\overline{\eta}_s} = \mathbf{X} \times_{C(s)} \overline{\eta}_s$. The morphisms $\rho_{r_1,r_2} : X \times_C \mathbb{E}^*(s,r_1) \to X \times_C \mathbb{E}^*(s,r_2)$ induced by the imbeddings $\mathbb{E}^*(s,r_1) \hookrightarrow \mathbb{E}^*(s,r_2)$ $(r_1 \leq r_2)$ define a system of functors

$$\rho^*_{r_1,r_2} : \mathbf{S}(X \times_C \mathbb{E}^*(s,r_2), \Lambda) \to \mathbf{S}(X \times_C \mathbb{E}^*(s,r_1), \Lambda).$$

Set $\mathbf{I}(\mathbf{X}_{\eta_s}, \Lambda) = \underset{\overrightarrow{r}}{\mathrm{ind}}\, \rho$. Let also $\mathrm{D}^+(\mathbf{X}_{\eta_s}, \Lambda)$ be the derived category $\mathrm{D}^+(\underset{\overrightarrow{r}}{\mathrm{ind}}\, \rho)$ defined as in section 4.2. Moreover, for any $T \in \mathcal{I}$ set $X_T = X \times_C t(T)$ and let F_T be the restriction of the sheaf F to X_T. The image of T inside $\mathrm{Gal}(k^a/k)$ is a subgroup of finite index, corresponding to some finite extension k_T of k. If we let $X_{k_T} = X \times_k k_T$ we obtain a diagram

$$\begin{tikzcd} X_T \arrow[r, "j_T"] \arrow[d] & X_{k_T} \arrow[d] & X_{\overline{s}} \arrow[l, "i_T"'] \arrow[d] \\ t(T) \arrow[r] & C \times_k k_T & \overline{s}. \arrow[l] \end{tikzcd}$$

Let $\mathcal{I}_r$ be the full subcategory of $\mathcal{I}$ consisting of all the objects of the form (S, r) for arbitrary S and denote by $\widetilde{\Psi}^{(r)}_{\eta_s} : \mathbf{S}(X \times_C \mathbb{E}^*(s, r), \Lambda) \to \mathbf{S}(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1^{(r)}])$ the left exact functor

$$F \mapsto \varinjlim_{T \in \mathcal{I}_r} i_T^* j_{T*}(F_T). \tag{4.3.4}$$

This collection of functors determines a functor on the corresponding categories of ind-sheaves:

$$\widetilde{\Psi}_{\eta_s} = \underset{\overrightarrow{r}}{\mathrm{ind}}\, \widetilde{\Psi}^{(r)}_{\eta_s} : \mathbf{I}(\mathbf{X}_{\eta_s}, \Lambda) \to \mathbf{S}(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1(\eta_s, \overline{x})])$$

whose derived functor $R\widetilde{\Psi}_{\eta_s}$ is the *functor of locally algebraic vanishing cycles.* The only point which requires explanation is the $\pi_1(\eta_s, \overline{x})$-action on $R\widetilde{\Psi}_{\eta_s}(F_\bullet)$. Here is how it is obtained: fix r and let $F = F_r \in \mathbf{S}(X \times_C \mathbb{E}^*(s, r), \Lambda)$. Clearly it suffices to produce a compatible system of $\pi_1^{(r)}$-actions on the right-hand side of (4.3.4). To this purpose, write $\mathbb{E}^*(s, r) = \bigcup_{i \geq 0} X_i$ for an increasing sequence of compact connected subspaces. Now, for fixed r, it suffices to produce a compatible family of $\pi_1^{alg}(X_i, \overline{x}_r)$-actions for all $i \geq 0$. But for any given i, it follows from proposition 4.3.1 that the set $\mathcal{I}_{r,i}$ of all open subgroups of $\pi_1^{(r)}$ whose preimage in $\pi_1^{alg}(X_i, \overline{x}_r)$ is normal, forms a cofinal family in $\mathcal{I}_r$, hence we can replace the index category $\mathcal{I}_r$ in (4.3.4) by the smaller $\mathcal{I}_{r,i}$ and then the action of $\widetilde{\pi}_1^{alg}(X_i, \overline{x}_r)$ is apparent.

We review hereafter the standard properties of $R\widetilde{\Psi}_{\eta_s}$. For any $F_\bullet \in \mathbf{I}(\mathbf{X}_{\eta_s}, \Lambda)$ set

$$H^0(\mathbf{X}_{\overline{\eta}_s}, F_\bullet) = \{ \varinjlim_{T \in \mathcal{I}_r} H^0(X_T, F_T) \}_{r > 0}.$$

By the remarks above this defines a functor

$$H^0(\mathbf{X}_{\overline{\eta}_s}, -) : \mathbf{I}(\mathbf{X}_{\eta_s}, \Lambda) \to \underline{\mathrm{Rep}}(\pi_1(\eta_s, \overline{x}), \Lambda)$$

whose derived functor is the cohomology of the general fibre of $\mathbf{X}$.

Proposition 4.3.5. *Let $f : \mathbf{Y} \to \mathbf{X}$ be a smooth morphism of $C(s)$-analytic spaces. Then for all $F \in \mathrm{D}^+(\mathbf{X}_{\eta_s}, \Lambda)$ there is a canonical isomorphism in $\mathrm{D}^+(\mathbf{Y}_{\overline{s}}, \Lambda[\pi_1(\eta_s, \overline{x})])$*

$$f_{\overline{s}}^*(R\widetilde{\Psi}_{\eta_s} F) \simeq R\widetilde{\Psi}_{\eta_s}(f_{\eta_s}^* F).$$

Proof. Follows directly from smooth base change. □

Proposition 4.3.6. *Let $f : \mathbf{Y} \to \mathbf{X}$ be a compact morphism of $C(s)$-analytic spaces. Then for all $F \in \mathrm{D}^+(\mathbf{Y}_{\eta_s}, \Lambda)$ there is a canonical isomorphism in $\mathrm{D}^+(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1(\eta_s, \overline{x})])$*

$$R\widetilde{\Psi}_{\eta_s}(Rf_{\eta_s *} F) \simeq Rf_{\overline{s}*}(R\widetilde{\Psi}_{\eta_s} F).$$

Proof. Follows directly from compact base change. □

Corollary 4.3.7. *Let $\mathbf{X}$ be a compact $C(s)$-analytic space. Then for all $F \in \mathbf{S}(\mathbf{X}_{\eta_s}, \Lambda)$ there is a spectral sequence of $\pi_1(\eta_s, \overline{x})$-ind-representations*

$$E_2^{p,q} = H^p(\mathbf{X}_{\overline{s}}, R^q\widetilde{\Psi}_{\eta_s}(F)) \Longrightarrow H^{p+q}(\mathbf{X}_{\overline{\eta}_s}, F).$$

□

Theorem 4.3.8. *Suppose that $\mathbf{X}$ is smooth over $C(s)$. Then $\widetilde{\Psi}_{\eta_s}(\Lambda_X) \simeq \Lambda_{X_{\overline{s}}}$ and $R^i\widetilde{\Psi}_{\eta_s}(\Lambda_X) = 0$ for $i > 0$.*

Proof. By smooth base change it suffices to consider the case $X = C$. It is clear that $\widetilde{\Psi}_{\eta_s}(\Lambda_X) \simeq \Lambda_{X_{\overline{s}}}$. From Poincaré duality it follows easily that $R^i\widetilde{\Psi}_{\eta_s}(\Lambda_X) = 0$ for $i > 1$. For $i = 1$, it suffices to show that for any $T = (S, r) \in \mathcal{I}$ and any $f \in H^1(t(T), \Lambda)$ we can find T' which dominates T and such that the image of f in $H^1(t(T'), \Lambda)$ is zero. The class f defines a certain Λ-torsor over $t(T)$. But according to proposition 4.1.9 every connected component Y of this Λ-torsor is locally algebraic over $t(T)$. Hence Y is a locally algebraic covering of $\mathbb{E}^*(s, r)$ corresponding to some subgroup S'. Clearly $T' = (S', r)$ will do the job. □

Corollary 4.3.9. *Suppose that* $\mathbf{X}$ *is smooth over* $C(s)$*. Then for any locally constant sheaf* F *of finitely generated* Λ*-modules on* $\mathbf{X}$ *we have* $\widetilde{\Psi}_{\eta_s}(F) = F_{\overline{s}}$ *and* $R^i\widetilde{\Psi}_{\eta_s}(F) = 0$ *for* $i > 0$. □

An argument like in the proof of theorem 4.3.8 also shows the following

Lemma 4.3.10. *Let* F *be a locally constant sheaf of finitely generated* Λ*-modules on* η_s*. Then* $H^i(\overline{\eta}_s, F)$ *vanishes for all* $i > 0$. □

Slightly more generally, for any algebraic extension E of k we let $s' = s \times_k \widehat{E}$ and we consider the family of continuous morphisms

$$\rho_r : \widetilde{\pi}_1(\mathbb{E}^*(s,r) \times_k \widehat{E}, \overline{x}_r) \to \widetilde{\pi}_1(\mathbb{E}^*(s,r), \overline{x}_r).$$

We introduce a category $\mathcal{I}_k$ (and also $\mathcal{I}_{k,r}$) consisting of all the pairs (S', r) such that S' is a subgroup of $\widetilde{\pi}_1(\mathbb{E}^*(s,r) \times_k \widehat{E}, \overline{x}_r)$ of the form $S' = \rho_r^{-1}(S)$ for some open subgroup $S \subset \widetilde{\pi}_1(\mathbb{E}^*(s,r), \overline{x}_r)$; the morphisms are defined as for $\mathcal{I}$. For all $C(s) \times_k E$-analytic space $\mathbf{X}$ we define a functor

$$\widetilde{\Psi}_{\eta_{s'}/k} : \mathbf{I}(\mathbf{X}_{\eta_{s'}}, \Lambda) \to \mathbf{S}(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1(\eta_{s'}, \overline{x})]) \ : \qquad F_\bullet \mapsto \{ \varinjlim_{T \in \mathcal{I}_{k,r}} i_T^* j_{T*}(F_T) \}_{r>0}.$$

For the derived functor of $\widetilde{\Psi}_{\eta_{s'}/k}$ we can prove the obvious analogues of proposition 4.3.5 and 4.3.6. But I do not know whether the analogue of theorem 4.3.8 also holds. However, it follows from the following proposition that theorem 4.3.8 does hold in case $\mathbf{X}$ is obtained by base change from a $C(s)$-analytic space.

Proposition 4.3.11. *With the notation above, let* $\mathrm{F} : \mathbf{S}(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1(\eta_s, \overline{x})]) \to \mathbf{S}(\mathbf{X}_{\overline{s}}, \Lambda[\pi_1(\eta_{s'}, \overline{x})])$ *be the natural forgetful functor. Let* $F_\bullet \in \mathbf{I}(\mathbf{X}_{\eta_s}, \Lambda)$ *and denote by* $F'_\bullet$ *the inverse image of* $F_\bullet$ *on* $\mathbf{X}_{\eta_{s'}}$*. For all integers* $q \geq 0$ *there is a canonical isomorphism*

$$\mathrm{F} \circ R^q\widetilde{\Psi}_{\eta_s}(F_\bullet) \simeq R^q\widetilde{\Psi}_{\eta_{s'}/k}(F'_\bullet).$$

Proof. Clearly it suffices to treat the case $q = 0$. For this, given two pairs $T = (S, r)$ and $T' = (S', r)$ related as above, it suffices to remark the isomorphism

$$i_{T'}^* j_{T'*}(F_{T'}) \simeq i_T^* j_{T*}(F_T).$$

□

5. Local theory in dimension one

5.1. **The meromorphic quotient of the local fundamental group.** In this chapter we construct our category of sheaves which are "locally of differential origin" (see the discussion in section 3.3).

Lemma 5.1.1. *Let* $\psi : X \to Y$ *be a locally algebraic covering of* σ*-compact spaces over* k*. Fix a geometric point* $\overline{x}$ *on* X *and let* $\overline{y} = \psi(\overline{x})$*. The induced group homomorphism*

$$\psi_* : \widetilde{\pi}_1(X, \overline{x}) \to \widetilde{\pi}_1(Y, \overline{y})$$

is injective.

Proof. For any locally algebraic covering $\phi : C \to X$ we obtain a covering $\psi \circ \phi : C \to Y$. By virtue of proposition 4.1.4, the map $\phi \mapsto \psi \circ \phi$ corresponds to a functor

$$\Psi : \widetilde{\pi}_1(X, \overline{x}) - \underline{\mathrm{Set}} \to \widetilde{\pi}_1(Y, \overline{y}) - \underline{\mathrm{Set}}.$$

On the other hand we have a natural pullback functor

$$\psi^* : \widetilde{\pi}_1(Y, \overline{y}) - \underline{\mathrm{Set}} \to \widetilde{\pi}_1(X, \overline{x}) - \underline{\mathrm{Set}}$$

which is dual to the group homomorphism ψ_*. But for every $S \in \widetilde{\pi}_1(X, \overline{x}) - \underline{\mathrm{Set}}$ there is an equivariant imbedding

$$S \hookrightarrow \psi^* \circ \Psi(S)$$

and the claim follows. □

Choose a local coordinate t around 0 (so that $t(0) = 0$) on $(\mathbb{A}^1_k)^{an}$. For any $\rho > 0$, $N \in \mathbb{N}$ and any finite extension k' of k, we let $\mathbb{E}(\rho)$ be an open disc of radius ρ in $(\mathbb{A}^1_k)^{an}$ centered at $0 \in (\mathbb{A}^1_k)^{an}$, we set $\mathbb{E}^*(\rho) = \mathbb{E}(\rho) - \{0\}$ and we consider the morphism

$$\psi_{N,k'} : \mathbb{E}^*(\rho^{1/N}) \times_k k' \to \mathbb{E}^*(\rho) \qquad t \mapsto t^N. \tag{5.1.2}$$

Our point of departure is Levelt's theorem (see [Ka1] (2.2.2) and theorem 2.4.6) according to which (1) any connection on η_∞ (see section 3.3 for the notation) extends canonically to a connection on $\mathbb{G}_{m,k}$ with regular singularities at the origin; and (2) given any connection V on $\mathbb{G}_{m,k}$ with regular singularities at the origin, there exists a finite extension k' of k and an integer $N>0$ such that $(\psi^*_{N,k'}V)$ is a successive extension of flat line bundles which extend to all $\mathbb{G}_{m,k'}$ with regular singularities at the origin.

We cook up our meromorphic quotient of the local fundamental group just in such a way to ensure that a topological analogue of Levelt's theorem becomes true, basically by definition.

From the asymptotic Kummer sequence (3.1.2) we derive an imbedding

$$H^0(\mathbb{E}^*(\rho),\mathcal{O})/\log(H^0(\mathbb{E}^*(\rho),\mathcal{U}^1)) \hookrightarrow H^1(\mathbb{E}^*(\rho),\mu_{p^\infty}). \tag{5.1.3}$$

Lemma 5.1.4. *Choose a geometric point $\overline{x}_1\in\mathbb{E}(\rho)$. The map (5.1.3) induces an imbedding*

$$t^{-1}\cdot k[t^{-1}] \hookrightarrow \mathrm{Hom}_{cnt}(\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1),\mu_{p^\infty}).$$

Proof. From corollary 4.1.10 we know that $H^1(\mathbb{E}^*(\rho),\mu_{p^\infty})\simeq \mathrm{Hom}_{cnt}(\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1),\mu_{p^\infty})$. Hence it suffices to show that $k[t^{-1}]\cap\log(H^0(\mathbb{E}^*(\rho),\mathcal{U}^1))=k$.

Claim 5.1.5. the restriction map $H^0(\mathbb{E}(\rho),\mathcal{U}^1)\to H^0(\mathbb{E}^*(\rho),\mathcal{U}^1)$ is a bijection.

Proof of the claim: One proceeds as in the proof of lemma 3.2.10. The details are left to the reader.

From the claim it follows that $\log(H^0(\mathbb{E}^*(\rho),\mathcal{U}^1))\subset H^0(\mathbb{E}(\rho),\mathcal{O})$. Now the lemma follows by the stronger equality $k[t^{-1}]\cap H^0(\mathbb{E}(\rho),\mathcal{O})=k$. □

Now, for any integer $N>0$ choose a geometric point $\overline{x}_N\in\mathbb{E}^*(\rho^{1/N})$ and a coordinate $t^{1/N}$ on $\mathbb{E}(\rho^{1/N})$ such that $\psi^*_{N,k'}(t)=(t^{1/N})^N$ and $\psi_{N,k'}(\overline{x}_N)=\overline{x}_1$ for all $N\in\mathbb{N}$. We obtain a system of group homomorphisms

$$\psi_{N,k'*}:\widetilde{\pi}_1(\mathbb{E}^*(\rho^{1/N})\times_k k',\overline{x}_N)\to\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$$

and we let

$$P(\rho)=\bigcap_{k\subset k'\subset k^a}\bigcap_{N\in\mathbb{N}}\mathrm{Im}(\psi_{N,k'*})\subset\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1).$$

From lemmas 5.1.1 and 5.1.4 we obtain a homomorphism

$$\mathcal{A}=\varinjlim_{k\subset k'\subset k^a}\varinjlim_{N\in\mathbb{N}} t^{-1/N}\cdot k'[t^{-1/N}]=\varinjlim_{N\in\mathbb{N}} t^{-1/N}\cdot k^a[t^{-1/N}]\to\mathrm{Hom}_{cnt}(P(\rho),\mu_{p^\infty}) \tag{5.1.6}$$

where k' ranges over all the finite extensions of k.

Definition 5.1.7. For each $a\in\mathcal{A}$ denote by $\chi_a:P(\rho)\to\mu_{p^\infty}$ the corresponding character defined by (5.1.6). The *essential ramification subgroup* of $\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$ is defined as

$$P_{ess}(\rho)=\bigcap_{a\in\mathcal{A}}\mathrm{Ker}(\chi_a)\subset P(\rho).$$

Lemma 5.1.8. *$P_{ess}(\rho)$ is a normal subgroup of $\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$.*

Proof. Let $\gamma\in\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$ and $a\in\mathcal{A}$. After replacing k by some finite Galois extension k' (and hence $\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$ by its *open normal subgroup* $\widetilde{\pi}_1(\mathbb{E}^*(\rho)\times_k k',\overline{x}_1)$) we can assume that $a\in t^{-1/N}k'[t^{-1/N}]$. Then $\chi_a:P(\rho)\to\mu_{p^\infty}$ extends to a character $\overline{\chi}_{a(t^{-1/N})}:\mathrm{Im}(\psi_{N,k'*})\to\mu_{p^\infty}$ for some N. The conjugate $\gamma(\mathrm{Ker}(\overline{\chi}_a))\gamma^{-1}$ depends only on the class

$$\overline{\gamma}\in\widetilde{\pi}_1(\mathbb{E}^*(\rho)\times_k k',\overline{x}_1)/\mathrm{Im}(\psi_{N,k'*})\simeq\mathbb{Z}/N\mathbb{Z}.$$

Clearly $\mathbb{Z}/N\mathbb{Z}$ acts as the group of deck transformations of the covering $\psi_{N,k'}:\mathbb{E}^*(\rho^{1/N})\times_k k'\to\mathbb{E}^*(\rho)\times_k k'$, *i.e.* γ corresponds to a morphism

$$\overline{\gamma}:\mathbb{E}^*(\rho^{1/N})\times_k k'\to\mathbb{E}^*(\rho^{1/N})\times_k k' \qquad t^{1/N}\mapsto\zeta\cdot t^{1/N}$$

where ζ is some N-th root of 1. Unwinding the definitions one checks easily that $\overline{\gamma}(\mathrm{Ker}(\overline{\chi}_{a(t^{-1/N})}))\overline{\gamma}^{-1}=\mathrm{Ker}(\overline{\chi}_{a(\zeta\cdot t^{-1/N})})$ which means that $P_{ess}(\rho)$ is normal in $\bigcap_{k\subset k'}\widetilde{\pi}_1(\mathbb{E}^*(\rho)\times_k k',\overline{x}_1)$. To descend to the base field k, we need to consider the Galois action. However, let $\sigma\in\mathrm{Gal}(k^a/k)$; unwinding the definitions we see easily that

$$\sigma(Ker(\chi_a))=\mathrm{Ker}(\chi_{a^\sigma})$$

where $a\mapsto a^\sigma$ is the obvious Galois action on the group $\mathcal{A}$. The claim follows. □

Definition 5.1.9. (1) The *meromorphic quotient* of $\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)$ is the topological group $\mu(\mathbb{E}^*(\rho),\overline{x}_1)$ obtained as follows. As a group, we let

$$\mu(\mathbb{E}^*(\rho),\overline{x}_1)=\widetilde{\pi}_1(\mathbb{E}^*(\rho),\overline{x}_1)/P_{ess}(\rho).$$

A topology is specified on $\mu(\mathbb{E}^*(\rho),\overline{x}_1)$ by declaring that the intersections of finitely many subgroups

$$\mathrm{Ker}(\overline{\chi}_{a(t^{-1/N})}:\mathrm{Im}(\psi_{N,k'*})\to\mu_{p^\infty})\qquad(a\in\mathcal{A};\ k\subset k'\subset k^a)$$

introduced in the proof of lemma 5.1.8, form a cofinal system of open neighborhoods of the identity element. (By [Bou] chapter III.2 the topology is well defined and unique).

(2) Let C be a smooth curve defined over k, s a k-rational point on C and t a local coordinate on C around s. Recall from section 4.3 that the pro-analytic space η_s is isomorphic (as a pro-analytic space) to a projective system (indexed by $\rho>0$) of pointed discs $\mathbb{E}^*(s,\rho)=\{x\ :\ 0<|t(x)|<\rho\}$ centered at s. Choosing geometric points $\overline{x}_1^\rho\in\mathbb{E}^*(s,\rho)$ for all $\rho>0$ gives us, as in section 4.3, the pro-group $\pi_1(\eta_s,\overline{x}_1)$. The *meromorphic quotient* of $\pi_1(\eta_s,\overline{x}_1)$ is the topological pro-group $\mu(\eta_s,\overline{x}_1)$ indexed by the ordered set of positive real numbers and defined by

$$\rho\mapsto\mu(\mathbb{E}^*(s,\rho),\overline{x}_1^\rho)\qquad(\rho>0)$$

where, for $\rho_2>\rho_1$, the morphism $\overline{\phi}_{\rho_2,\rho_1}:\mu(\mathbb{E}^*(s,\rho_1),\overline{x}_1^{\rho_1})\to\mu(\mathbb{E}^*(s,\rho_2),\overline{x}_1^{\rho_2})$ is induced by the morphism $\phi_{\rho_2,\rho_1}:\widetilde{\pi}_1(\mathbb{E}^*(s,\rho_1),\overline{x}_1^{\rho_1})\to\widetilde{\pi}_1(\mathbb{E}^*(s,\rho_2),\overline{x}_1^{\rho_2})$ (see section 4.3 for the notation). Similarly we define the pro-groups $P(\eta_s)=\{P(\rho)\mid\rho>0\}$ and $P_{ess}(\eta_s)=\{P_{ess}(\rho)\mid\rho>0\}$ with morphisms for $\rho_2>\rho_1$ given by the restrictions of ϕ_{ρ_2,ρ_1}.

Remark 5.1.10. Notice that the definition depends on the choice of a local coordinate t around the point s. We use the same coordinate for the construction of all the quotients $\mu(\mathbb{E}^*(s,\rho),\overline{x}_1^\rho)$ (for varying ρ) which occur in part (2) of the definition. After that, it is easy to see that, given $a(t^{-1/N})\in\mathcal{A}$, the composition

$$\psi_{N,k'*}(\widetilde{\pi}_1(\mathbb{E}^*(s,\rho_1^{1/N})\times_k k',\overline{x}_N^{\rho_1}))\xrightarrow{\phi_{\rho_2,\rho_1}}\psi_{N,k'*}(\widetilde{\pi}_1(\mathbb{E}^*(s,\rho_2^{1/N})\times_k k',\overline{x}_N^{\rho_2}))\xrightarrow{\overline{\chi}_a}\mu_{p^\infty}$$

coincides (up to inner automorphisms) with $\overline{\chi}_a:\psi_{N,k'*}(\widetilde{\pi}_1(\mathbb{E}^*(s,\rho_1^{1/N}),\overline{x}_N^{\rho_1})\to\mu_{p^\infty}$. This shows that the maps ϕ_{ρ_2,ρ_1} descend to the respective meromorphic quotients.

Remark 5.1.11. (1) I tend to think that the topology of $\mu(\eta_s,\overline{x}_1)$ coincides with the quotient topology induced by the projection

$$\pi_1(\eta_s,\overline{x}_1)\to\mu(\eta_s,\overline{x}_1)\tag{5.1.12}$$

but I do not know how to prove (or disprove) this statement. In any case, the homomorphism (5.1.12) is continuous (*i.e.* all the surjections (for varying ρ) from the first projective system to the second one, which define (5.1.12), are continuous).

(2) We remark the natural imbeddings of pro-groups (by this we just mean that they are induced by imbeddings on the component groups, for varying ρ)

$$P_{ess}(\eta_s)\hookrightarrow P(\eta_s)\hookrightarrow\pi_1(\eta_s,\overline{x}_1).\tag{5.1.13}$$

We discuss the dependance on the parameter t. Given any two local coordinates t,t' as above, we can assume that $t'=t\cdot(1+h(t))$ where $h(0)=0$. For all small $\rho>0$ we obtain an automorphism

$$\tau:\mathbb{E}(s,\rho)\to\mathbb{E}(s,\rho)\qquad t\mapsto t\cdot(1+h(t))$$

such that $\tau^*(t)=t'$. In correspondence with the two parameters we obtain two meromorphic quotients $\mu(\mathbb{E}(s,\rho),\overline{x}_1^\rho)$ (resp. $\mu(\eta_s,\overline{x}_1)$) and $\mu'(\mathbb{E}(s,\rho),\overline{x}_1^\rho)$ (resp. $\mu'(\eta_s,\overline{x}_1)$). I do not know how to compare directly these two pro-groups; however we have the following result.

Proposition 5.1.14. *The automorphism τ induces an equivalence of categories*

$$\tau^*:\varinjlim_{\rho}\mu'(\mathbb{E}(s,\rho),\overline{x}_1^\rho)-\underline{\mathrm{Set}}\xrightarrow{\sim}\varinjlim_{\rho}\mu(\mathbb{E}(s,\rho),\overline{x}_1^\rho)-\underline{\mathrm{Set}}.$$

In other words, the category of stable $\mu(\eta_s,\overline{x})$-sets is an analytic invariant of η_s. The same holds for the category of stable ind-representations of $\mu(\eta_s,\overline{x})$ on finitely generated Λ-modules (where Λ is any ℓ-coefficient ring).

Proof. Let $a(t^{1/N}) \in \mathcal{A}$. Choose N-th roots $t^{1/N}$ and $t'^{1/N}$ for t and respectively t'. We have two morphisms $\psi_{N,k'}, \psi'_{N,k'} : \mathbb{E}^*(s,\rho^{1/N}) \times_k k' \to \mathbb{E}^*(s,\rho)$ defined by $\psi_{N,k'}(t^{1/N}) = t$ and $\psi'_{N,k'}(t'^{1/N}) = t'$. We can find a positive real number $\varepsilon < \rho^{1/N}$ small enough so that the function $1 + h(t)$ has a N-th root $g(t)$ on $\mathbb{E}(s,\varepsilon)$. Define an automorphism

$$\phi : \mathbb{E}^*(s,\varepsilon) \times_k k' \to \mathbb{E}^*(s,\varepsilon) \times_k k' \qquad t^{1/N} \mapsto t^{1/N} \cdot g(t).$$

We obtain a commutative diagram

$$\begin{tikzcd} \mathbb{E}^*(s,\varepsilon) \times_k k' \arrow[r, "\phi"] \arrow[d, "\psi_{N,k'}"'] & \mathbb{E}^*(s,\varepsilon) \times_k k' \arrow[d, "\psi'_{N,k'}"] \\ \mathbb{E}^*(s,\rho) \arrow[r, "\tau"] & \mathbb{E}^*(s,\rho). \end{tikzcd}$$

It follows that $\phi^*(t^{1/N}) = \zeta \cdot t'^{1/N}$ for some N-th root of unit $\zeta \in k^a$. After base change to $k' = k[\zeta]$ we can even obtain that $\phi^*(t^{1/N}) = t'^{1/N}$. Hence we can replace ρ by ε and assume that $N = 1$ and that $a = a(t)$ induces a character $\chi_a : \widetilde{\pi}_1(\mathbb{E}(s,\rho),\overline{x}) \to \mu_{p^\infty}$. Then we can find a positive real number $\varepsilon < \rho$ small enough so that

$$t^{-1} - t'^{-1} = t^{-1} - t^{-1} \cdot (1 + h(t))^{-1} \in \log(H^0(\mathbb{E}^*(s,\varepsilon),\mathcal{U}^1)).$$

By lemma 5.1.4 it follows that for any $a \in t^{-1}k[t^{-1}]$ the characters $\chi_{a(t)}, \chi_{a(t')} : \widetilde{\pi}_1(\mathbb{E}^*(s,\varepsilon),\overline{x}) \to \mu_{p^\infty}$ coincide. The proposition follows from this and from the definition of the topology on $\mu(\eta_s,\overline{x})$. $\square$

Due to this proposition, we will sometime omit to specify the choice of the parameter t.

We remark that the category of groups can be imbedded in the category of pro-groups (indexed by some fixed small category $\mathcal{I}$) by assigning to a group G the pro-group $\{G_i; \phi_{ij} \mid i,j \in \mathcal{I}\}$ such that $G_i = G$ for all $i \in \mathcal{I}$ and all the maps $\phi_{ij} : G_i \to G_j$ $(i < j)$ being the identity of G. Denote by $\pi_1^{alg}(\eta_s,\overline{\eta}_s)$ the usual algebraic local fundamental group of the generic point of the henselization C_s^h of the curve C at the point s. In other words, this is the Galois group of the algebraic closure of the fraction field of $\mathcal{O}_{C,s}^h$. With this notation we have the following proposition.

Proposition 5.1.15. *There are natural exact sequences*

$$0 \to P(\eta_s)/P_{ess}(\eta_s) \to \mu(\eta_s,\overline{x}_1) \to \pi_1^{alg}(\eta_s,\overline{\eta}_s) \to 0$$
$$\mu(\eta_s \times_k \widehat{k}^a,\overline{x}) \to \mu(\eta_s,\overline{x}) \to \mathrm{Gal}(k^a/k) \to 0.$$

Proof. The first sequence just restates the definitions. The second one follows easily from proposition 4.1.8. $\square$

Now, let us take $C = \mathbb{A}_k^1$ and $s = 0$. For the coordinate t we choose a *global algebraic section* $t \in H^0(\mathbb{A}_k^1,\mathcal{O})$ such that $t(0) = 0$ and $H^0(\mathbb{A}_k^1,\mathcal{O}) \simeq k[t]$. This is determined up to scalar multiples. All the construction of the meromorphic quotient can be repeated over $\widetilde{\pi}_1((\mathbb{G}_{m,k})^{an},\overline{x})$: we define $P(\mathbb{G}_{m,k})$ as the kernel of the canonical map $\widetilde{\pi}_1((\mathbb{G}_{m,k})^{an},\overline{x}) \to \pi_1^{alg}(\mathbb{G}_{m,k},\overline{x})$. Then, inspecting the previous arguments, we obtain also a map

$$\chi : \mathcal{A} \to Hom_{cnt}(P(\mathbb{G}_{m,k}),\mu_{p^\infty})$$

and for each $\rho > 0$, the imbedding $j_\rho : \mathbb{E}^*(0,\rho) \hookrightarrow (\mathbb{G}_{m,k})^{an}$ induces a group homomorphism $j_{\rho *} : P(\rho) \to P(\mathbb{G}_m)$ such that $\chi_a \circ j_{\rho *} = \chi_a$ for any $a \in \mathcal{A}$. Therefore we define $P_{ess}(\mathbb{G}_{m,k})$ and $\mu((\mathbb{G}_{m,k})^{an},\overline{x})$ as in the local case, and clearly there is a canonical homomorphism

$$\overline{\jmath}_{\rho *} : \mu(\mathbb{E}^*(0,\rho),\overline{x}) \to \mu((\mathbb{G}_{m,k})^{an},\overline{x}). \tag{5.1.16}$$

(Here $\mu(\mathbb{E}^*(0,\rho),\overline{x})$ is defined by the same coordinate t chosen above).

Definition 5.1.17. The category $\underline{\mathrm{Cov}}^\mu_{(\mathbb{G}_{m,k})^{an}}$ of *meromorphic coverings* of $(\mathbb{G}_{m,k})^{an}$ is the full subcategory of $\widetilde{\underline{\mathrm{Cov}}}_{(\mathbb{G}_{m,k})^{an}}$ consisting of all the locally algebraic coverings $X \to (\mathbb{G}_{m,k})^{an}$ such that the action of $\widetilde{\pi}_1((\mathbb{G}_{m,k})^{an},\overline{x})$ on the fibre $\widetilde{\omega}_{\overline{x}}(X)$ factors through a continuous action of $\mu((\mathbb{G}_{m,k})^{an},\overline{x})$.

Lemma 5.1.18. *(1) The fibre functor $\widetilde{\omega}_{\overline{x}}$ restricts to a fully faithful functor*

$$\widetilde{\omega}_{\overline{x}} : \underline{\mathrm{Cov}}^\mu_{(\mathbb{G}_{m,k})^{an}} \to \mu((\mathbb{G}_{m,k})^{an},\overline{x}) - \underline{\mathrm{Set}}$$

and every $\widetilde{\pi}_1(X,\overline{x})$-set consisting of a single orbit is contained in the essential image of $\widetilde{\omega}_{X,\overline{x}}$.

(2) For an integer $N > 0$ let $\psi_{-N} : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ be the morphism $t \mapsto t^{-N}$. Then for any covering $X \in \underline{\mathrm{Cov}}^{\mu}_{(\mathbb{G}_{m,k})^{an}}$ there exists an integer $N > 0$ and a finite extension k' of k such that $\psi^*_{-N}(X) \times_k k'$ extends to an abelian covering of $(\mathbb{A}^1_{k'})^{an}$.

Proof. Part (1) follows from proposition 4.1.4. To prove part (2) we can replace k by some finite extension k' and then assume that $\widetilde{\omega}_{\overline{x}}(X)$ is of the form $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})/(\mathrm{Ker}(\overline{\chi}_{a_1})\cap\ldots\cap\mathrm{Ker}(\overline{\chi}_{a_m}))$ for some $a_1, \ldots, a_m$. Then we can find $N > 0$ large enough so that $a_1, \ldots, a_m \in t^{-1/N}k[t^{-1/N}]$. It is clear that $\psi^*_{-N}(X) \times_k k'$ extends to an abelian covering on $(\mathbb{A}^1_{k'})^{an}$. □

Proposition 5.1.19. *The continuous map $\overline{\jmath}_{\rho *}$ in (5.1.16) induces a homeomorphism from $\mu(\mathbb{E}^*(0,\rho), \overline{x})$ onto a dense subset of $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})$.*

Proof. For the injectivity of $\overline{\jmath}_{\rho *}$ we need to show that $j^{-1}_{\rho *}(P_{ess}(\mathbb{G}_{m,k})) = P_{ess}(\rho)$. This follows directly from the equality $\chi_a \circ j_{\rho *} = \chi_a$ for any $a \in \mathcal{A}$. We show that $\overline{\jmath}_{\rho *}$ has dense image: for a given open subgroup $S \subset \mu((\mathbb{G}_{m,k})^{an}, \overline{x})$ we obtain a map of discrete sets:

$$\overline{\jmath}^S_{\rho *} : \mu(\mathbb{E}^*(0,\rho), \overline{x})/j^{-1}_*(S) \to \mu((\mathbb{G}_{m,k})^{an}, \overline{x})/S.$$

It suffices to show the following

Claim 5.1.20. For any S as above the map $\overline{\jmath}^S_{\rho *}$ is bijective.

Proof of the claim: Let $\widetilde{S}$ be the preimage of S in $\widetilde{\pi}_1((\mathbb{G}_{m,k})^{an}, \overline{x})$. Then $\widetilde{S}$ is an open subgroup and the quotient $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})/S = \widetilde{\pi}_1((\mathbb{G}_{m,k})^{an}, \overline{x})/\widetilde{S}$ represents a connected locally algebraic covering $X_S \to (\mathbb{G}_{m,k})^{an}$. By general nonsense, the claim amounts to saying that the restriction $X_S(\rho) = X_{S|\mathbb{E}^*(0,\rho)}$ is still connected. Take an integer N and a Galois extension k' as in lemma 5.1.18(2) such that $\psi^*_{-N}(X_S)$ extends to an étale covering over all of $(\mathbb{A}^1_{k'})^{an}$. The group $G = \mathbb{Z}/N\mathbb{Z} \rtimes \mathrm{Gal}(k'/k)$ of deck automorphisms of $\psi_{-N} : \mathbb{G}_{m,k'} \to \mathbb{G}_{m,k}$ acts on the set of connected components $\pi_0(\psi^*_{-N}(X_S))$ (resp. $\pi_0(\psi^*_{-N}(X_S(\rho)))$) of $\psi^*_{-N}(X_S)$ (resp. of $\psi^*_{-N}(X_S(\rho))$). The action is transitive on $\pi_0(\psi^*_{-N}(X_S))$ and induces a bijection between the G-orbits in $\pi_0(\psi^*_{-N}(X_S(\rho)))$ and the set $\pi_0(X_S(\rho))$. Hence we can assume from start that X is an abelian covering which extends over all $(\mathbb{P}^1_k)^{an} - \{0\}$. Then, from the definition of the topology of $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})$, we see that there exist polynomials $a_1, \ldots, a_m \in t^{-1}k[t^{-1}]$ such that $\mathrm{Ker}(\overline{\chi}_{a_1}) \cap \ldots \cap \mathrm{Ker}(\overline{\chi}_{a_m}) \subset S$. Hence it suffices to check that $\overline{\jmath}^S_{\rho *}$ is bijective when $S = \mathrm{Ker}(\overline{\chi}_{a_1}) \cap \ldots \cap \mathrm{Ker}(\overline{\chi}_{a_m})$. Moreover, using the imbedding $X_S \hookrightarrow (X_{\mathrm{Ker}(\overline{\chi}_{a_1})}) \times \ldots \times (X_{\mathrm{Ker}(\overline{\chi}_{a_m})})$ we reduce to the case $S = \mathrm{Ker}(\overline{\chi}_a)$ for an $a \in t^{-1}k[t^{-1}]$. In this case the claim follows from lemma 5.1.4. □

Combining the maps $\overline{\jmath}_{\rho *}$ of (5.1.16) for varying ρ we obtain a homomorphism of pro-groups

$$\overline{\jmath}_* : \mu(\eta_0, \overline{x}) \to \mu((\mathbb{G}_{m,k})^{an}, \overline{x})$$

which is well defined up to inner automorphisms.

Corollary 5.1.21. *The map $\overline{\jmath}_*$ induces equivalences of categories*

$$\overline{\jmath}^* : \mu((\mathbb{G}_{m,k})^{an}, \overline{x}) - \underline{\mathrm{Set}} \xrightarrow{\sim} \varinjlim_{\rho} \mu(\mathbb{E}^*(0,\rho), \overline{x}^{\rho}) - \underline{\mathrm{Set}}.$$
$$\overline{\jmath}^* : \underline{\mathrm{Rep}}_{cnt}(\mu((\mathbb{G}_{m,k})^{an}, \overline{x}), \Lambda) \xrightarrow{\sim} \varinjlim_{\rho} \underline{\mathrm{Rep}}_{cnt}(\mu((\mathbb{E}^*(0,\rho), \overline{x}^{\rho}), \Lambda)$$

for any ℓ-coefficient ring Λ. □

Finally we return to the case of a general analytic local coordinate t.

Corollary 5.1.22. *Let t be any local (analytic) coordinate centered at the point $0 \in (\mathbb{A}^1_k)^{an}$. Form the corresponding pro-group $\mu(\eta_0, \overline{x})$. For any stable object $X \in \mu(\eta_0, \overline{x}) - \underline{\mathrm{Set}}$ there exists a meromorphic covering $\mathcal{X} \in \underline{\mathrm{Cov}}^{\mu}_{(\mathbb{G}_{m,k})^{an}}$ such that $X \simeq \overline{\jmath}^* \circ \widetilde{\omega}_{\overline{x}}(\mathcal{X})$. A similar statement holds true for stable continuous ind-representations of $\mu(\eta_0, \overline{x})$.* □

We regard corollary 5.1.22 as the analogue of Levelt's theorem; of course in our situation, the result is just built into the definition.

5.2. **Swan conductor.** In this section we construct a higher ramification filtration on the local fundamental group and establish some basic facts about the linear representations of $\mu(\eta_s, \overline{x})$. By virtue of corollary 5.1.22, it is equivalent to study the representations of $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})$, *i.e.* the continuous group homomorphisms

$$\rho : \mu((\mathbb{G}_{m,k})^{an}, \overline{x}) \to GL(n, \Lambda)$$

where Λ is an ℓ-coefficient ring.

We introduce an increasing filtration on $\mathcal{A}$ by subgroups $\mathcal{A}(r)$, indexed by the ordered set of positive real numbers, as follows. For an element $a(t^{-1/N}) \in \mathcal{A}$ we define the degree $\deg(a) \in \mathbb{Q}$ which is the highest power of t^{-1} occuring in a. (So, for instance, $\deg(t^{-1/N}) = 1/N$). Then, for $r \in \mathbb{R}$, $r > 0$, we let $\mathcal{A}(r) = \{a \in \mathcal{A} \mid \deg(a) \leq r\}$.

Definition 5.2.1. The *higher ramification filtration* on $\mu(\mathbb{E}^*(s, \rho), \overline{x})$ (resp. on $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})$) consists of a sequence of subgroups $I_\rho^{(r)}$ (resp. $I^{(r)}$) indexed by the real numbers $r \geq 0$, defined as follows. We set $I_\rho^{(0)} = \mu(\mathbb{E}^*(s, \rho), \overline{x})$ (resp. $I^{(0)} = \mu((\mathbb{G}_{m,k})^{an}, \overline{x})$). For $r > 0$ we set

$$I_\rho^{(r)} = \bigcap_{a \in \mathcal{A}(r)} \mathrm{Ker}(\chi_a) \subset P(\rho)/P_{ess}(\rho)$$

(and similarly for $I^{(r)} \subset P(\mathbb{G}_{m,k})/P_{ess}(\rho)$). For varying ρ, the morphisms $\overline{\phi}_{\rho_2,\rho_1}$ (see definition 5.1.9(2)) carry $I^{(\rho_1)}$ to $I^{(\rho_2)}$ and hence the sequence $I_{\eta_s}^{(r)} = \{I_\rho^{(r)} \mid \rho > 0\}$ defines an injective morphism of pro-groups

$$I_{\eta_s}^{(r)} \hookrightarrow \mu(\eta_s, \overline{x})$$

for every $r \geq 0$. It is clear that, given $r_2 > r_1$, we have $I_{\eta_s}^{(r_2)} \subset I_{\eta_s}^{(r_1)}$ (resp. $I^{(r_2)} \subset I^{(r_1)}$) hence this defines a descending *higher ramificaion filtration* on $\mu(\eta_s, \overline{x})$ (resp. on $\mu((\mathbb{G}_{m,k})^{an}, \overline{x})$) by closed subgroups.

Using the higher ramification filtration we will define a Swan conductor for stable ind-representations of $\mu(\eta_s, \overline{x})$. Proceeding as in definition 5.2.1 we also obtain a parallel notion of Swan conductor for $\mu(\mathbb{G}_{m,k}, \overline{x})$. Everything has been set up in such a way that the corresponding representation theories become equivalent (via corollary 5.1.21). In particular, this allows to work as if the meromorphic quotient of the local fundamental group were an actual group, rather than a pro-group, which sometimes may be convenient. For this reason, in the sequel we will write simply $I^{(r)}$, P and P_{ess} to denote indifferently the local pro-objects or their global counterpart.

Lemma 5.2.2. *The morphism $\psi_{N,k}$ of* (5.1.2) *induces an isomorphism*

$$\psi_{N,k*} : I^{(r)} \xrightarrow{\sim} I^{(r/N)}.$$

Proof. It follows easily by lemma 5.1.1 and by remarking that the map $\psi^*_{N,k} : \mathcal{A} \to \mathcal{A}$ is an isomorphism (you can always take an N-th root of $t^{1/M}$). $\square$

Let G be some group and $\rho : G \to GL(V)$ a representation of G on some finite rank free Λ-module V. For any character $\chi \in \mathrm{Hom}(G, \mu_{p^\infty})$ we let V_χ be the maximal submodule of V on which G acts as χ *i.e.*

$$\rho(g)v = \chi(g)v \qquad (g \in G, \quad v \in V_\chi).$$

Notice that this definition makes sense since any ℓ-coefficient ring contains μ_{p^∞}.

Proposition 5.2.3. *Let G be a finite commutative p-group and $\rho : G \to GL(V)$ a representation of G as above. Then there is a canonical decomposition*

$$V \simeq \bigoplus_{\chi \in \mathrm{Hom}(G, \mu_{p^\infty})} V_\chi.$$

Proof. Let g be some element in G. Let p^n be the exponent of G and choose a primitive root of unity $\zeta \in \mu_{p^\infty}$ of order p^n. First of all we remark that all elements of the form $\zeta^i - \zeta^j$ ($i \not\equiv j \bmod p^n$) are invertible in Λ. This follows easily from [Wa] proposition 2.1. For $1 \leq j \leq p^n$ we define

$$C_j = \prod_{i \neq j} (\zeta^j - \zeta^i).$$

Clearly we have

$$\prod_{1 \leq i \leq p^n} (\rho(g) - \zeta^i) = 0 \tag{5.2.4}$$

as an element of $\mathrm{End}_A(V)$. Define the element $\pi_j \in \mathrm{End}_\Lambda(V)$ by setting

$$\pi_j = C_j^{-1} \prod_{j \neq i} (\rho(g) - \zeta^i).$$

From (5.2.4) it follows that the image of π_j lands into the submodule $V_{g,\zeta^j} = \mathrm{Ker}(\rho(g) - \zeta^j)$.

Claim 5.2.5. The morphism

$$\oplus_{1 \leq j \leq p^n} \pi_j : V \to \bigoplus_{1 \leq j \leq p^n} V_{g,\zeta^j}$$

is injective.

Proof of the claim: For any subset $S \subset \{1, 2, ..., p^n\}$ define more generally

$$\pi_S = \prod_{i \notin S} (\rho(g) - \zeta^i).$$

For any such S and any two distinct elements i, j in the complement of S we show that

$$\mathrm{Ker}\pi_{S \cup \{i\}} \cap \mathrm{Ker}\pi_{S \cup \{j\}} = \mathrm{Ker}\pi_S. \tag{5.2.6}$$

The claim will follow easily from (5.2.6) and a simple induction argument.

Let $v \in \mathrm{Ker}\pi_{S \cup \{i\}} \cap \mathrm{Ker}\pi_{S \cup \{j\}}$ and set $w = \pi_S(v)$. Then we have

$$(\rho(g) - \zeta^i)w = (\rho(g) - \zeta^j)w = 0$$

which implies $(\zeta^i - \zeta^j)w = 0$. Since $(\zeta^i - \zeta^j)$ is invertible, this yields $w = 0$ and proves (5.2.6).

Next we show that the composition

$$\begin{array}{ccccc} \bigoplus_j V_{g,\zeta^j} & \longrightarrow & V & \xrightarrow{\oplus_j \pi_j} & \bigoplus_j V_{g,\zeta^j} \\ (v_1, ..., v_{p^n}) & \longmapsto & \sum_j v_j & & \end{array}$$

is the identity map. This is a direct calculation:

$$\begin{aligned} \pi_j(\textstyle\sum_k v_k) &= C_j^{-1} \textstyle\prod_{j \neq i} (\rho(g) - \zeta^i)(\sum_k v_k) \\ &= \textstyle\sum_k C_j^{-1} \prod_{j \neq i} (\rho(g) - \zeta^i) v_k \\ &= C_j^{-1} \textstyle\prod_{j \neq i} (\rho(g) - \zeta^i) v_j \\ &= C_j^{-1} \textstyle\prod_{j \neq i} (\zeta^j - \zeta^i) v_j \\ &= v_j. \end{aligned}$$

Together with claim 5.2.5 this shows that V is isomorphic to the direct sum of G-stable Λ-modules $\bigoplus_j V_{g,\zeta^j}$.

Let $g_1, ..., g_m$ be a set of generators of G. To conclude the proof, it suffices to remark that, for any character $\chi \in \mathrm{Hom}(G, \mu_{p^\infty})$,

$$V_\chi = V_{g_1,\chi(g_1)} \cap ... \cap V_{g_m,\chi(g_m)}$$

and that this intersection of Λ-modules is a direct summand of V. □

Corollary 5.2.7. *Let $\rho : P \to GL(V)$ be a representation of P into a finite rank free Λ-module V. Then there is a direct sum decomposition*

$$V \simeq \bigoplus_{\chi \in \mathrm{Hom}(P, \mu_{p^\infty})} V_\chi.$$

Proof. Since V has the discrete topology, ρ factors through a discrete quotient $\widetilde{P}$ of P. Then $\widetilde{P}$ is a commutative p-power torsion group, and hence it is the direct limit of the filtered family $\mathcal{F}$ of its finite subgroups.

We argue by induction on the rank r of V. Thanks to proposition 5.2.3 we can choose for each subgroup $S \in \mathcal{F}$ a character $\chi_S : S \to \mu_{p^\infty}$ and a non-zero G-stable direct summand V_S in V such that

1) $\rho_{|V_S} = \chi_S$;

2) $V_T \subset V_S$ and χ_T restricts to χ_S on S for any $S, T \in \mathcal{F}$ such that $S \subset T$.

Then, since the rank r is finite, the submodule

$$V' = \lim_{\substack{\longleftarrow \\ S \in \mathcal{F}^\circ}} V_S$$

is non-zero and it is clearly a direct summand in V. On V' the action ρ is given by the character $\lim\limits_{\substack{\longleftarrow \\ S \in \mathcal{F}^\circ}} \chi_S$ and the complement of V' has rank strictly less than r, which shows the claim. □

Lemma 5.2.8. *Any continuous character $\chi : P/P_{ess} \to \mu_{p^\infty}$ is of the form $\chi = \chi(f)$ for some element f of the algebra $\mathcal{A}$.*

Proof. Since χ is continuous, we can find $f_1, ..., f_n \in \mathcal{A}$ such that $\mathrm{Ker}\ \chi(f_1) \cap ... \cap \mathrm{Ker}\ \chi(f_n) \subset \mathrm{Ker}\ \chi$. Define

$$\widetilde{P} = P/\bigcap_i \mathrm{Ker}\ \chi(f_i).$$

The morphism $(f_1, ..., f_n) : P \to \mu_{p^\infty}^n$ induces an imbedding $\widetilde{P} \hookrightarrow \mu_{p^\infty}^n$. Since μ_{p^∞} is injective in the category of commutative p-groups, we derive a surjection

$$\mathrm{Hom}(\mu_{p^\infty}^n, \mu_{p^\infty}) \to \mathrm{Hom}(\widetilde{P}, \mu_{p^\infty}).$$

Clearly χ factors though $\widetilde{P}$, hence it lifts to an element $\widetilde{\chi} \in \mathrm{Hom}(\mu_{p^\infty}^n, \mu_{p^\infty}) \simeq \mathbb{Z}_p^n$. Let us say $\widetilde{\chi} = (a_1, ..., a_n)$ for certain $a_i \in \mathbb{Z}_p$. Then we conclude $\chi = \sum_i a_i \cdot \chi(f_i) = \chi(\sum_i a_i f_i)$. □

Proposition 5.2.9. *For any $f \in \mathcal{A}$ we have*

$$\deg(f) = \inf\{r \in \mathbb{R} \quad | \quad I^{(r)} \subset \mathrm{Ker}(\chi(f))\}.$$

Proof. Using lemma 5.2.2 we can reduce to the case that f is a polynomial in t, hence in particular $n = \deg(f) \in \mathbb{N}$. Next, it is obvious that $I^{(n)} \subset \mathrm{Ker}(f)$, so that the infimum over the set of real numbers with this property is smaller than or equal to n. Suppose that this infimum r is strictly smaller than n. For any $g \in \mathcal{A}$ of degree less than r, set $C_g = f(\mathrm{Ker}(g)) \subset \mu_{p^\infty}$. By hypothesis:

$$\bigcap_g C_g = 0.$$

Since all the proper subgroups of μ_{p^∞} are finite and nested into each other, this means that for some g we have already $C_g = 0$. Take N an integer large enough so that both $f_N = \psi_{N,k}^*(f)$ and $g_N = \psi_{N,k}^*(g)$ extend to homomorphisms $\widetilde{\pi}_1((\mathbb{A}_k^1)^{an}, \overline{x}) \to \mu_{p^\infty}$.

By construction we have $\mathrm{Ker}(g_N) \subset \mathrm{Ker}(f_N)$ and therefore we can find an endomorphism ω of μ_{p^∞} which makes the following diagram commute:

$$\begin{tikzcd} \widetilde{\pi}_1((\mathbb{A}_k^1)^{an}, \overline{x}) \arrow[r, "f_N"] \arrow[d, "g_N"'] & \mu_{p^\infty} \\ \mu_{p^\infty} \arrow[ur, "\omega"'] & \end{tikzcd}$$

We have $\mathrm{End}(\mu_{p^\infty}) \simeq \mathbb{Z}_p$, the isomorphism being given by

$$\gamma \mapsto (\zeta \mapsto \zeta^\gamma) \qquad (\gamma \in \mathbb{Z}_p, \zeta \in \mu_{p^\infty}).$$

Suppose that $\omega = (-)^\gamma$ and consider the ladder diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mu_{p^\infty} \arrow[r] \arrow[d, "(-)^\gamma"] & \mathcal{U}^1 \arrow[r, "\lambda"] \arrow[d, "(-)^\gamma"] & \mathcal{O}_X \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \mu_{p^\infty} \arrow[r] & \mathcal{U}^1 \arrow[r, "\lambda"] & \mathcal{O}_X \arrow[r] & 0. \end{tikzcd} \tag{5.2.10}$$

From the long exact ladder for the cohomology of (5.2.10) we derive that

$$f_N = \omega_* g_N = \gamma \cdot g_N.$$

But this is a contradiction, since the degree of g_N is strictly smaller than the degree of f_N. The claim follows. □

Definition 5.2.11. The *slope* of the character $\chi(f)$ is the degree of the element f. If M is a Λ-module on which P/P_{ess} acts through its character $\chi(f)$, then the slope $\lambda(M)$ of M is defined as the slope of $\chi(f)$. In particular, the slope of a simple P/P_{ess}-module is always a rational number.

Finally, let V an arbitrary P/P_{ess}-module free of finite rank. Corollary 5.2.7 shows that V decomposes as direct sum of P-stable rank one Λ-submodules. We denote by $\lambda(V)$ the set of the slopes of the simple rank one components of V; clearly $\lambda(V)$ is a finite subset of $\mathbb{Q}$, whose elements are called the slopes of V. Gathering the simple components of V which have same slope, we obtain a canonical decomposition of V as direct sum

$$V = \bigoplus_{\lambda \in \lambda(V)} V_\lambda$$

where each V_λ is purely of slope λ.

Definition 5.2.12. The Swan conductor $sw(V)$ of a P/P_{ess}-module V, is the rational number

$$sw(V) = \Sigma_{\lambda\in\lambda(V)}\lambda\cdot \mathrm{rk}_A V_\lambda.$$

The next result is our version of the Hasse-Arf theorem.

Theorem 5.2.13. *Let V be a finite rank free Λ-module with an action of $\mu(\eta_s,\overline{x})$ (*i.e.* a stable ind-representation which is a free Λ-module). Then $sw(V)$ is a positive integer.*

Proof. For an element $f(x^{1/N})\in\mathcal{A}$ let us denote by M_f the one-dimensional Λ-module on which P/P_{ess} acts through the character $\chi(f)$. Then the P-module V has a decomposition of the kind

$$V\simeq\bigoplus_{f\in S} M_f^{n_f}$$

for some finite set S of elements of $\mathcal{A}$. Let $\gamma\in\pi_1^{alg}(\eta_s,\overline{\eta}_s)$ be any element. We can define a new action of P/P_{ess} on V, by setting

$$(p,v)\mapsto\gamma p\gamma^{-1}(v)\qquad\qquad (p\in P/P_{ess},\quad \gamma\in\pi^{alg}(\eta_s,\overline{\eta}_s)).$$

Let V^γ be the module V with the new P/P_{ess}-action. Since $\gamma(P/P_{ess})\gamma^{-1}=P/P_{ess}$ as subgroups of $\mu(\eta_s,\overline{x})$, it follows that $V^\gamma\simeq V$. On the other hand, we can write

$$M_f^\gamma = M_{f^\gamma}$$

where $f^\gamma\in\mathcal{A}$ denotes an element of the form $f(\zeta x^{1/N})$ for some $\zeta\in\mu_N$. Hence we see that the set S must be stable under the substitution $f\mapsto f^\gamma$ for any γ as above. Suppose that N has been chosen minimal among the integers such that we can write f as a polynomial in $x^{1/N}$. Then it is easy to see that the orbit $\{f^\gamma\mid\gamma\in\pi_1^{alg}(\eta_s,\overline{\eta}_s)\}$ consists of exactly N elements. On the other hand, $\lambda(M_f)=\lambda(M_{f^\gamma})$ is a rational number of the form n/N ($n\in\mathbb{N}$). The claim follows directly from these facts. ☐

Definition 5.2.14. Let (V,ρ) be a representation of $\mu(\eta_s,\overline{x})$ into a finitely generated free Λ-module V. We say that (V,ρ) is *tame* if the action of $\mu(\eta_s,\overline{x})$ factors through the quotient $\pi_1^{alg}(\eta_s,\overline{\eta}_s)$. We say that (V,ρ) is *irreducible* if it does not contain any non-trivial $\mu(\eta_s,\overline{x})$-stable free Λ-submodule (this means that for all $\rho>0$, the $\mu(\mathbb{E}^*(s,\rho),\overline{x})$-module (V,ρ) does not contain any free $\mu(\mathbb{E}^*(s,\rho),\overline{x})$-submodules). We say that (V,ρ) is *absolutely irreducible* if for all finite flat extensions $\Lambda\to\Lambda'$ of ℓ-coefficient rings, $(V,\rho)\otimes_\Lambda\Lambda'$ is an irreducile $\Lambda'[\mu(\eta_s,\overline{x})]$-module.

Remark 5.2.15. (1) We caution the reader that our terminology does not agree with the standard usage of the terms "irreducible", resp. "absolutely irreducible". (2) An easy induction argument shows that for all representations (V,ρ) as in the definition, there exists some finite flat ring extension $\Lambda\to\Lambda'$ and a $\mu(\eta_s,\overline{x})$-stable filtration on $(V,\rho)\otimes_\Lambda\Lambda'$ such that all the associated subquotients are absolutely irreducible.

Let $\phi_N:\eta_0\to\eta_0$ be the étale covering of pro-analytic spaces induced by the morphism $\mathbb{G}_m\to\mathbb{G}_m$: $x\mapsto x^N$. It induces a morphism $\phi_{N*}:\mu(\eta_0,\overline{x})\to\mu(\eta_0,\overline{x})$ and we let S_N be the image of ϕ_{N*}. Clearly S_N is an open normal subgroup of index N in $\mu(\eta_0,\overline{x})=S_1$.

Theorem 5.2.16. *Suppose that k is algebraically closed. Let (V,ρ) be an absolutely irreducible representation of $\mu(\eta_s,\overline{x})$ into a free Λ-module V of rank n. Then there exist a positive integer d dividing n, a finite flat extension Λ' of Λ, a representation (L,χ) of $S_{n/d}$ into a rank one free Λ'-module L and a rank d absolutely irreducible tame Λ'-representation (T,ρ') of $S_{n/d}$ such that*

$$(V,\rho)\otimes_\Lambda\Lambda'\simeq \mathrm{Ind}_{S_{n/d}}^{S_1}(L\otimes T,\chi\otimes\rho').$$

Proof. (Cp. [Ka1] theorem 2.6.6) By corollary 5.2.7 and lemma 5.1.18(2), there exists some N such that the restriction of ρ to S_N decomposes as direct sum of characters. Pick such an N and consider the S_N-isotypical decomposition of V

$$V=\bigoplus_{i=1}^R V_{\chi_i}$$

where χ_i ranges on a finite set of characters of S_N. Since V is S_1-irreducible and S_N is normal, S_1 acts transitively on the set of isotypical components. Consequently $d=\dim(V_\chi)$ is independent of χ and

$n = dR$. Let $S_{\chi_i} \subset S_1$ be the stabilizer of V_{χ_i}. Then $S_N \subset S_{\chi_i}$ and S_{χ_i} has index R in S_1. Since S_1/S_N is a cyclic group, we must have $S_{\chi_i} = S_R$, independent of i. Therefore, as a representation of S_1 we have

$$V = \mathrm{Ind}_{S_{n/d}}^{S_1}(V_\chi)$$

for any $\chi = \chi_i$. Clearly V_χ is an irreducible $S_{n/d}$-representation and we must show that $V_\chi \simeq L \otimes T$ for appropriate L, T. Renaming V_χ, $S_{n/d}$ as V, S_1 we are reduced to the situation: V is an irreducible representation of S_1 of rank d, but for some $N \geq 1$ the subgroup S_N acts on V by scalar matrices, *i.e.* there exists a character χ of S_N such that

$$\rho(\gamma)(v) = \chi(\gamma)(v)$$

for all $\gamma \in S_N$. By a simple computation we see that the character χ is invariant by S_1-conjugation. Since S_1/S_N is cyclic, we can then replace Λ by some finite flat extension and find an extension of χ to a character $\widetilde{\chi}$ of S_1. Twisting V by the inverse of $\widetilde{\chi}$, we reduce to the case where V is an irreducible representation of S_1 which is trivial on S_N, *i.e.* V is tame and absolutely irreducible, as stated. □

Remark 5.2.17. (1) When k is algebraically closed, it might well be that every absolutely irreducible tame representation has rank one. This is certainly the case for representations which factor through some quotient $\mathbb{Z}/m$ of the tame local fundamental group, when m is prime to ℓ. (2) It follows easily from lemma 5.2.8 that every character χ as in theorem 5.2.16 is of the form $\chi(f) \otimes \chi'$ where f is some polynomial and χ' is a tame character.

6. The Lubin-Tate torsor

In this chapter we introduce and study the sheaf that plays the role covered by the Lang torsor in positive characteristic. I believe the name "Lubin-Tate torsor" is appropriate enough for this object. We return to the setup of chapter 2: here k_0 is a one-dimensional local field of zero characteristic, i.e. a p-adic field. Let F be a fixed Lubin-Tate group. As in chapter 2, we view F as an analytic group law for the analytic space $\mathbb{E}(0,1)$, and the associated logarithm $\boldsymbol{\lambda}_F$ as a morphism of analytic groups $\boldsymbol{\lambda}_F : \mathbb{E}(0,1) \to (\mathbb{G}_{a,k_0})^{an} = (\mathbb{A}^1_{k_0})^{an}$.

6.1. **Construction of the torsor.**

Lemma 6.1.1. *The logarithm $\boldsymbol{\lambda}_F : \mathbb{E}(0,1) \to (\mathbb{A}^1_{k_0})^{an}$ is an étale covering of $(\mathbb{A}^1_{k_0})^{an}$.*

Proof. Let $(\mathbb{A}^1_{k_0})^{an} = \cup_{r>0}\mathbb{D}(r)$ be the covering of the affine line by closed discs of radius r centered at the origin. Denote by $E(r)$ the connected component of $\boldsymbol{\lambda}^{-1}(\mathbb{D}(r))$ containing 0.

From remark 2.1.6(1) we get an equality of formal power series: $\boldsymbol{\lambda} \circ [\pi]^n_f = \pi^n \cdot \boldsymbol{\lambda}$. By analytic continuation, this formal identity gives rise to a commutative diagram of analytic maps:

$$\begin{tikzcd}
\mathbb{E}(0,1) \arrow[r, "{[\pi]^n_f}"] \arrow[d, "\boldsymbol{\lambda}"'] & \mathbb{E}(0,1) \arrow[d, "\boldsymbol{\lambda}"'] & \mathbb{E}(0,\rho_1) \arrow[l] \arrow[d, "\boldsymbol{\lambda}"'] \\
(\mathbb{A}^1_{k_0})^{an} \arrow[r, "\pi^n"] & (\mathbb{A}^1_{k_0})^{an} & \mathbb{G}_a(\rho_1). \arrow[l]
\end{tikzcd}$$

We remark that, for sufficiently large n_r, $E(r)$ is the connected component of the inverse image of $e_F(\pi^n_r\mathbb{D}(r))$ by $[\pi^n_r]$. Looking at the diagram above, we see that the restriction of $\boldsymbol{\lambda}$ to $E(r)$ is a finite map, hence $E(r)$ is an affinoid domain in $\mathbb{E}(0,1)$ for all r and $\mathbb{E}(0,1) = \cup_{r>0}E(r)$. Note that for $r < s$, $E(r)$ is contained in the interior of $E(s)$. It follows easily that $\boldsymbol{\lambda}$ is étale and surjective if and only if the induced maps $E(r) \to \mathbb{D}(r)$ are étale and surjective for all r.

Given $r > 0$, choose an integer n_r large enough such that $[\pi]^{n_r}_f(E(r)) \subset \mathbb{E}(0,\rho_1)$. By theorem 2.1.5, the power series e_F converges on $\mathbb{E}(0,\rho_1)$. This means that e_F defines a morphism on the quasiaffinoid space $\mathbb{E}(0,\rho_1)$, and therefore the restriction of $\boldsymbol{\lambda}$ to $\mathbb{E}(0,\rho_1)$ is an isomorphism of quasiffinoid spaces. It follows that $\boldsymbol{\lambda} : E(r) \to \mathbb{D}(r)$ is an étale covering if and only if $[\pi]^{n_r}_f : E(r) \to \pi^{n_r} \cdot \mathbb{D}(r)$ is an étale covering. Let $g \in \mathfrak{F}_\pi$ be any other power series; the homomorphism $[1]_{f,g} : \mathbb{E}(0,1) \to \mathbb{E}(0,1)$ of quasiaffinoid spaces has an inverse $[1]_{g,f}$ and therefore it is an isomorphism. From theorem 2.1.1(b) we see that $[1]_{f,g} \circ [\pi]_f \circ [1]_{g,f} = [\pi]_g$. Therefore it suffices to prove that for some $g \in \mathfrak{F}$ the morphism $g = [\pi]_g : \mathbb{E}(0,1) \to \mathbb{E}(0,1)$ is finite and étale. Then we select $g(Z) = \pi Z + Z^q$. Now consider the map of schemes $\mathbb{A}^1_{k_0} \to \mathbb{A}^1_{k_0}$ defined by the polynomial $g(Z)$: this map ramifies over a finite set of points $x_1, ..., x_m \in \mathbb{A}^1_{k_0}(k_0^a) = k_0^a$, and using the jacobian criterion one checks easily that $|x_i| \geq 1$ for all i. On the complement of $x_1, ..., x_n$, g restricts to an étale covering $U \to \mathbb{A}^1_k - \{x_1, ..., x_n\}$. By proposition 3.3.11 of [B1], it follows that the map $g^{an} : U^{an} \to (\mathbb{A}^1_{k_0})^{an} - \{x_1, ..., x_n\}$ is also an étale covering. But clearly

$[\pi]_g$ is obtained from g^{an} by base change to $\mathbb{E}(0,1) \subset (\mathbb{A}^1_{k_0})^{an}$, and the lemma follows from corollary 3.3.8 of [B1]. $\square$

Remark 6.1.2. The proof of the lemma shows in particular that the restriction of the analytic covering $\boldsymbol{\lambda} : \mathbb{E}(0,1) \to (\mathbb{A}^1_{k_0})^{an}$ to any bounded disc $\mathbb{E}(0,\rho) \hookrightarrow (\mathbb{A}^1_{k_0})^{an}$ factors as a trivial (split) covering followed by an algebraic covering of finite degree.

For any positive integer n, let $k_n = k_0(G_n)$, $k_\infty = \cup_{n>0} k_n$ and $\widehat{k}_\infty$ the completion of k_∞. Recall (remark 2.1.6(2)) that $G_\infty = \mathrm{Ker}(\boldsymbol{\lambda} : \mathbb{E}(0,1) \to (\mathbb{A}^1_{\widehat{k}^a})^{an})$. In particular, this kernel is contained in k_∞.

As usual we obtain a sheaf of sets (in the analytic étale topology) over $(\mathbb{A}^1_{k_0})^{an}$ by taking the étale local sections of the morphism $\boldsymbol{\lambda}$; let us denote by ϕ this sheaf.

For any given complete field extension k of k_0, there is a base change map $p : (\mathbb{A}^1_k)^{an} \to (\mathbb{A}^1_{k_0})^{an}$ and we can form the pull back $\phi_k = p^*\phi$. For our purposes, the really useful sheaf is $\phi_{\widehat{k}_\infty}$; for brevity we will denote it simply by ϕ_∞.

Definition 6.1.3. The sheaf ϕ_∞ acquires a translation action of the discrete group G_∞, which as usual makes it into a G_∞-torsor. We call ϕ_∞ the *Lubin-Tate torsor* associated to F.

Let Λ be some torsion ring in which the residue characteristic of k is invertible, and $\psi : G_\infty \to \Lambda^\times$ be a character of G_∞. We can form the associated sheaf

$$\mathcal{L}_\psi = \phi_\infty \times_\psi \Lambda$$

which is a rank one local system of Λ-modules on $(\mathbb{A}^1_{\widehat{k}_\infty})^{an}$.

A note about notation: for a map $f : X \to (\mathbb{A}^1_{\widehat{k}_\infty})^{an}$ sometime we will write $\mathcal{L}(f)$ in place of $f^*\mathcal{L}$. Also, if k is a complete extension of k_∞, the base change map $\pi : (\mathbb{A}^1_k)^{an} \to (\mathbb{A}^1_{\widehat{k}_\infty})^{an}$ gives us a new sheaf $\mathcal{L}_k = \pi^*\mathcal{L}$. If it is clear from the context which base field we have in mind, we will omit the subscript k. Given a linear coordinate t on $(\mathbb{A}^1_k)^{an}$, sometime we will write $\mathbb{G}_a(\rho, t)$ for the analytic group obtained by restricting the addition law of $\mathbb{G}_a$ to the disc $\mathbb{E}(0,\rho) = \{x \in (\mathbb{A}^1_k)^{an} \mid |t(x)| < \rho\}$.

We list here some elementary properties of $\mathcal{L}_\psi$, that follow from the general yoga of torsors. Let $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ be the addition map, and $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ the projection maps on the first and second factor. Then $\mathcal{L}_\psi$ comes with:

LT1) a rigidification at the origin:

$$\mathcal{L}_{\psi,\{0\}} \simeq \Lambda_{\psi,\{0\}}$$

LT2) a trivialization:

$$m^*\mathcal{L}_\psi \otimes \mathrm{pr}_1^*\mathcal{L}_\psi^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}_\psi^{-1} \simeq \Lambda_{\mathbb{G}_a \times \mathbb{G}_a}$$

compatible with the rigidification at the origin $\{0,0\}$ induced by **LT1**.

LT3) In particular:

$$\mathcal{L}_{\psi^{-1}} \simeq \mathcal{L}_\psi^{-1}.$$

We will denote by $\rho(\psi, t)$ the supremum of all real numbers ρ such that $\mathcal{L}_\psi$ trivializes on $\mathbb{G}_a(\rho, t)$. If t happens to be the same parameter which we chose to give the power series expansion for the morphism $\boldsymbol{\lambda}$, we get $\rho(\psi, t) \geq \rho_1$ and equality holds if and only if ψ is injective. Moreover $\rho(\psi, t) = \infty$ if and only if ψ is trivial.

Before moving on, we should remark that the difference between one choice or another of the underlying Lubin-Tate group, is purely arithmetic. By this we mean the following: suppose that F, F' are two Lubin-Tate groups, and G_∞, G'_∞ the respective torsion groups. Take two characters ψ, ψ' of G_∞ and respectively G'_∞. Then over the completion of $k(G_\infty)$ (resp. of $k(G'_\infty)$) we obtain the Lubin-Tate torsor $\mathcal{L}_\psi$ (resp. $\mathcal{L}_{\psi'}$). We can pull-back both of them to the common overfield $\widehat{k}_0^a$, and there we have

Proposition 6.1.4. *For $a \in \widehat{k}_0^a$, let $\mu_a : (\mathbb{A}^1_{\widehat{k}_0^a})^{an} \to (\mathbb{A}^1_{\widehat{k}_0^a})^{an}$ be the morphism $x \mapsto ax$. Then there exists $a \in \widehat{k}_0^a$ such that, with the above notation*

$$\mathcal{L}_{\psi,\widehat{k}_0^a} \simeq \mu_a^*\mathcal{L}_{\psi',\widehat{k}_0^a}.$$

Proof. I am grateful to G. Faltings for furnishing the following explanation. It suffices to compare a general Lubin-Tate torsor F with the classical $\mathbb{G}_m$. To distinguish the two analytic groups, call $\mathbb{E}_F$ (resp. $\mathbb{E}_{\mathbb{G}_m}$) the analytic space $\mathbb{E}(0,1)$ endowed with the group law F (resp. the multiplicative group law). The torsion of $\mathbb{G}_m$ is of course μ_{p^∞}. To prove the claim it suffices to show that the group homomorphism $\psi : G_\infty \to \mu_{p^\infty}$ is induced by a morphism of analytic groups $\widetilde{\psi} : \mathbb{E}_F \to \mathbb{E}_{\mathbb{G}_m}$, because in that case we can find out the right $a \in \widehat{k}_0^a$ by noticing that $\boldsymbol{\lambda}_{\mathbb{G}_m} \circ \widetilde{\psi} \circ \boldsymbol{\lambda}_F^{-1}$ is an endomorphism of $\mathbb{G}_a$, hence of the form μ_a for a certain a. Now, the map ψ induces a map on the Tate groups $\widehat{\psi} : T(F) \to T(\mathbb{G}_m)$, or what is the same, an element of $T(G)^* \simeq T(G^t)$ (here G^t is the Cartier dual group of G). This is the same as giving a compatible system of group scheme homomorphisms

$$\widehat{\psi}_n : F[n] \to \mu_{p^n} \qquad (n > 0)$$

defined over $(k_0^a)^\circ$. In turns, this is a map of p-divisible group schemes $F[p^\infty] \to \mu_{p^\infty}$ which determines the needed morphism $\widetilde{\psi} : \mathbb{E}_F \to \mathbb{E}_{\mathbb{G}_m}$ over $(\widehat{k}^a)^\circ$. □

The proof of the following proposition is taken from [SGA4$\frac{1}{2}$], Sommes trig. We reproduce it here to stay on the safe side.

Proposition 6.1.5. *Let k be a complete extension of k_∞. Let $\psi : G_\infty \to \Lambda^\times$ be a non-trivial character. Then:*

$$H_c^*(\mathbb{G}_a(\rho, t) \times_{k_0} k, \mathcal{L}_\psi) = 0$$

for all $\rho > \rho(\psi, t)$.

Proof. Let Δ_ρ be the connected component of $\boldsymbol{\lambda}^{-1}(\mathbb{G}_a(\rho, t))$ containing 0. For a $\widehat{k}_\infty$-rational point x of Δ_ρ, let τ_x be the translation $\tau_x(g) = g[+_f]x$ on Δ_ρ, where $[+_f]$ is Lubin-Tate group law. Also, let τ'_y be the translation by $y \in \mathbb{G}_a$, with respect to usual addition law on $\mathbb{G}_a$. The formula $\boldsymbol{\lambda} \circ \tau_x = \tau'_{\boldsymbol{\lambda}(x)} \circ \boldsymbol{\lambda}$ states that the pair $(\tau_x, \tau'_{\boldsymbol{\lambda}(x)})$ is an automorphism of the diagram $\Delta_\rho \to \mathbb{G}_a(\rho, t)$.

Let $\psi(x)$ be the induced automorphism of $(\mathbb{G}_a(\rho, t), \mathcal{L}_\psi)$. For $x \in G_\infty$ this automorphism gives the identity on $\mathbb{G}_a(\rho, t)$, and multiplication by $\psi(x)^{-1}$ on $\mathcal{L}_\psi$.

Let $\psi_H(x)$ be the automorphism of $H_c^*(\mathbb{G}_a(\rho, t)_{\widehat{k}_\infty}, \mathcal{L}_\psi)$ induced by $\psi(x)$. Then $\psi_H(x)$ is multiplication by $\psi(x)^{-1}$. On the other hand, the following "homotopy" lemma (applied to $\psi : \Delta_\rho \times \mathbb{G}_a(\rho, t) \longrightarrow \Delta_\rho \times \mathbb{G}_a(\rho, t)$ defined as $\psi(x, y) = (x, y + \boldsymbol{\lambda}(x)))$ shows that $\psi_H(x) = \psi_H(0)$. Since by hypothesis $\rho > \rho(\psi, t)$, we can find $x \in \mathbb{G}(\rho, t) \cap G_\infty$ such that $1 - \psi(x)^{-1}$ is invertible; but we have seen that multiplication by $1 - \psi(x)^{-1}$ is the zero map, therefore the claim follows. □

Lemma 6.1.6 ("Homotopy" lemma)**.** *Let X and Y be two analytic spaces over a complete valued field k, with Y connected. Let $\mathcal{G}$ be a sheaf on X and (ψ, ε) a family of endomorphisms of $(X, \mathcal{G})$ parametrized by Y,* i.e.*:*

$$\begin{array}{ll} \psi : Y \times X \longrightarrow Y \times X & \textit{is a } Y\textit{-morphism and} \\ \varepsilon : \psi^* \mathrm{pr}_2^* \mathcal{G} \longrightarrow \mathrm{pr}_2^* \mathcal{G} & \textit{a morphism of sheaves.} \end{array}$$

Assume ψ is proper. For $y \in Y(k)$, let $\psi_H(y)^*$ the endomorphism of $H_c^*(X, \mathcal{G})$ induced by $\psi_y : X \longrightarrow X$ and $\varepsilon_y : \psi_y^* \mathcal{G} \longrightarrow \mathcal{G}$. Then $\psi_H(y)^*$ is independent of y.

Proof. In fact, $R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G}$ is the constant sheaf on Y with stalk $H_c^p(X, \mathcal{G})$, and $\psi_H(y)^*$ is the fiber at y of the endomorphism :

$$R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G} \xrightarrow{\psi^*} R^p \mathrm{pr}_{1!} \psi^* \mathrm{pr}_2^* \mathcal{G} \xrightarrow{\varepsilon} R^p \mathrm{pr}_{1!} \mathrm{pr}_2^* \mathcal{G}.$$

□

6.2. **The character induced by Galois action.** We conclude this chapter with some observations about the Galois action on $\mathcal{L}_\psi$. Let $\overline{\phi}$ be the pull back of ϕ_∞ to $(\mathbb{A}^1_{\widehat{k}_0^a})^{an}$; by transport of structure we get a natural action of $\mathrm{Gal}(k_0^a/k_\infty)$ on $\overline{\phi}$, covering the action on $(\mathbb{A}^1_{\widehat{k}_0^a})^{an}$. This action is inherited by $\mathcal{L}_{\psi, \widehat{k}_0^a}$. In particular, if p is a k_∞-rational point of $(\mathbb{A}^1_{\widehat{k}_0^a})^{an}$, then the stalk $\mathcal{L}_{\psi,p}$ becomes a representation of $\mathrm{Gal}(k_0^a/k_\infty)$ of rank one. For any $n \leq \infty$, let k_n^{ab} denote the maximal abelian extension of k_n. It is clear that the action on $\mathcal{L}_{\psi,p}$ factors through $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$. I do not know the complete structure of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$; in particular I don't know whether there is a canonical generator that takes the place of the Frobenius element as in the finite field case. Instead we do the following. Let k_0^{nr} be the maximal unramified extension of k_0. Clearly $k_0^{nr} \subset k_\infty^{ab}$ and $k_0^{nr} \cap k_\infty = k_0$. We say that an element $\sigma \in \mathrm{Gal}(k_\infty^{ab}/k_\infty)$ is a *Frobenius element* if the image of σ in $\mathrm{Gal}(k_0^{nr}/k_0)$ is the canonical Frobenius generator. Our aim is to give an explicit formula for the trace $\mathrm{Tr}(\sigma, \mathcal{L}_{\psi,p})$ of the endomorphism induced by the Frobenius element σ on the stalk of $\mathcal{L}_\psi$ at the point p. We start with two elementary lemmas:

Lemma 6.2.1. *The map $p \mapsto \mathrm{Tr}(\sigma, \mathcal{L}_{\psi,p})$ is a continuous group homomorphism $\mathrm{Tr}_\sigma : k_\infty \to \Lambda^\times$.*

Proof. It follows easily from **LT1** and **LT2** that the map Tr_σ is a group homomorphism. Moreover, it follows from lemma 2.1.5 that the restriction of ϕ_∞ to $\mathbb{E}(0,\rho(\psi,t))$ is the trivial G_∞-torsor; therefore the restriction of $\mathcal{L}_\psi$ to the same disc is a trivial line bundle, and we conclude that the kernel of Tr_σ contains this entire disc, i.e. the map is continuous. □

Lemma 6.2.2. $k_\infty^{ab} = \cup_{n\in\mathbb{N}} k_n^{ab}$.

Proof. It is clear that $k_n^{ab} \subset k_\infty^{ab}$. On the other hand, let $x \in k_\infty^{ab}$ and let $x_1, ..., x_m$ be the orbit of x under the action of the full Galois group $\mathrm{Gal}(k_0^a/k_0)$; take n big enough such that $[k_n(x_1,...,x_m) : k_n] = [k_\infty(x_1,...,x_m) : k_\infty]$. We get an isomorphism $\mathrm{Gal}(k_n(x_1,...,x_m)/k_n) \simeq \mathrm{Gal}(k_\infty(x_1,...,x_m)/k_\infty)$, and this last group is abelian, being a quotient of $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$. □

It follows from the lemma that the choice of a Frobenius element σ in $\mathrm{Gal}(k_\infty^{ab}/k_\infty)$ is equivalent to the choice of a sequence $\sigma_0, \sigma_1, ...$ of liftings of Frobenius $\sigma_n \in \mathrm{Gal}(k_n^{ab}/k_n)$ such that the restriction of σ_{n+1} to k_n^{ab} acts as σ_n. Let $\beta_n \in k_n$ such that the Artin symbol $(\beta_n, k_n^{ab}/k_n)$ acts on k_n^{ab} as σ_n. Then by local class field theory, it follows $\mathrm{Nm}_{k_{n+1}/k_n}(\beta_{n+1}) = \beta_n$. Also, by Lubin-Tate theory it follows $\beta_0 = \pi$. Conversely, the choice of a compatible system of elements $\beta_n \in k_n$ as before is equivalent to the choice of a Frobenius element σ.

For the next result we need some notation. First of all we select for each positive integer n:

1) a generator v_n of G_n as an k_0°-module, such that $[\pi^{m-n}]_f(v_m) = v_n$;

2) an element $\beta_n \in k_n$ such that the sequence of these elements satisfies the compatibility condition above, and corresponds to the choice of a Frobenius element σ_β;

3) a power series $b_n(z) = z \cdot r_n(z)$, where $r(z) \in k^\circ[[z]]$ satisfies $r(0) \neq 0$ and such that $b_n(v_n) = \beta_n$.

Finally, let T_n be the trace map from k_n to k_0.

Proposition 6.2.3. *Let p be a point in $\mathbb{A}^1_{k_0}(k_\infty) = k_\infty$, and choose an integer n such that (a) $|\pi^n p| < \rho_1$ and (b) $[k_0(p) : k_0] \leq n$. Let m be any integer $\geq 2n+1$. Then, with reference to the notation above:*

$$\mathrm{Tr}(\sigma_\beta, \mathcal{L}_{\psi,p}) = \psi\left(\left[\frac{1}{\pi^{m-n}} T_m\left(\frac{p}{\boldsymbol{\lambda}'(v_m)}\frac{db_m}{dz}\Big|_{z=v_m}\right)\right]_f (v_n)\right).$$

Proof. First of all, notice that the group $\mathrm{Gal}(k_0^a/k_\infty)$ acts also on $\mathbb{E}(0,1) \times_{k_0} \widehat{k}_0^a$ in such a way that the logarithm becomes an equivariant morphism. Let $q \in \boldsymbol{\lambda}^{-1}(p)$. Let $\widetilde{\sigma}$ be any lifting of σ_β to $\mathrm{Gal}(k_0^a/k_\infty)$; then essentially by definition we have:

$$\mathrm{Tr}(\sigma_\beta, \mathcal{L}_{\psi,p}) = \psi(\widetilde{\sigma}(q)[-_f]q) \tag{6.2.4}$$

(where $[-]_f$ denotes subtraction in the formal group). Obviously this formula is independent of the choices involved. Take n such that (a) is satisfied; by inspecting the proof of lemma 6.1.1 and the remark that follows it, we obtain:

$$\boldsymbol{\lambda}^{-1}(p) = [\pi^n]_f^{-1}(e(\pi^n p))[+_f]G_\infty.$$

In particular we can take $q \in [\pi^n]_f^{-1}(e(\pi^n p))$ in (6.2.4). We recall now the definition of the generalized Kummer pairing, introduced by Fröhlich in [Fr]: let $F(k_n)$ be the subgroup of $\mathbb{E}(0,1)(k_\infty)$ consisting of the elements rational over k_n; then there is a bilinear map:

$$(\ ,\ )_n^F : F(k_n) \times k_n^\times \longrightarrow G_n$$

defined as follows. If $\beta \in k_n^\times$, let τ_β be the element of the $\mathrm{Gal}(k_n^{ab}/k_n)$ which is attached to β by the Artin symbol. If $\alpha \in F(k_n)$, choose γ in $\mathbb{E}(0,1)(k^a)$ such that $[\pi^n]_f(\gamma) = \alpha$. Then $(\alpha,\beta)_n^F = \tau_\beta(\gamma)[-_f]\gamma$. Clearly, if we take n such that both (a) and (b) are satisfied, the right-hand side in (6.2.4) translates as $\psi((e(\pi^n p), \beta_n)_n^F)$. Then the formula of the theorem follows immediately from [Wi] theorem 1. □

6.3. **Semilinear Galois action.** Since the sheaf ϕ is already defined over k_0, it is natural to expect the full Galois group $\mathrm{Gal}(k_0^a/k_0)$ to act on $\mathcal{L}_\psi$. In this section we show that this is indeed the case, at least when the Lubin-Tate formal group under consideration is the classical multiplicative group $\mathbb{G}_m$. The action thus obtained will not be linear, but rather semilinear in a precise sense. In this way, our theory acquires a "p-adic flavour" which is unusual in an ℓ-adic setting.

As announced, in this section we restrict to the Lubin-Tate group $\mathbb{G}_m$. Take a prime ℓ whose residue class generates $\mathbb{Z}/p^2$; by Dirichlet theorem on primes in arithmetic progressions, there are plenty of such ℓ. With this choice, the Galois group of $\mathbb{Q}_\ell(\mu_{p^\infty})$ over $\mathbb{Q}_\ell$ is easily seen to be isomorphic to $\mathbb{Z}_p^\times$. Let $\mathbb{O}$ be the ring of integers of $\mathbb{Q}_\ell(\mu_{p^\infty})$ and set $\Lambda_n = \mathbb{O}/\ell^n$.

The group G_∞ attached to $\mathbb{G}_m$ is just μ_{p^∞} and any character $\psi : \mu_{p^\infty} \to \Lambda_n^\times$ lifts to a character $\widetilde{\psi} : \mu_{p^\infty} \to \mathbb{O}^\times$; conversely, we can start with $\widetilde{\psi}$ and then obtain ψ by projecting onto $\Lambda_n^\times$. Clearly we can assign $\widetilde{\psi}$ by identifying the two copies of μ_{p^∞}, one in $\mathbb{Q}_\ell(\mu_{p^\infty})$ and the other in $\mathbb{Q}_p(\mu_{p^\infty})$. Such identification also induces a unique isomorphism χ between $\mathrm{Gal}(\mathbb{Q}_\ell(\mu_{p^\infty})/\mathbb{Q}_\ell)$ and $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, given explicitly by the rule

$$\sigma(\widetilde{\psi}(g)) = \widetilde{\psi}(\chi(\sigma)g)$$

for all $\sigma \in \mathrm{Gal}(\mathbb{Q}_\ell(\mu_{p^\infty})/\mathbb{Q}_\ell)$ and all $g \in \mu_{p^\infty}$. Another way of seeing this is as saying that $\widetilde{\psi}$ becomes $\mathcal{G}$-equivariant, if we endow $\mathbb{Q}_\ell(\mu_{p^\infty})$ with the $\mathcal{G}$-action

$$(\sigma, x) \mapsto \chi(\sigma)x$$

for $\sigma \in \mathcal{G}$, $x \in \mathbb{Q}_\ell(\mu_{p^\infty})$. Having equivariance for $\widetilde{\psi}$ is exactly the condition needed to transfer the $\mathcal{G}$-action from ϕ_∞ to the associated locally constant sheaf $\mathcal{L}_\psi$. The $\mathcal{G}$-action on $\mathcal{L}_\psi$ is not linear, but has the following semilinearity property:

$$\sigma(bs) = (\chi(\sigma)b) \cdot \sigma(s)$$

for any local section s of $\mathcal{L}_\psi$ and all $\sigma \in \mathcal{G}$, $b \in \mathbb{Q}_\ell(\mu_{p^\infty})$.

Next, let K be any algebraic extension of k_∞, and $\widehat{K}$ its completion. There is a natural surjection $\pi : \mathrm{Gal}(K/k_0) \to \mathcal{G}$ and the $\mathcal{G}$-action on $\mathcal{L}_{\psi,\widehat{k}_\infty}$ lifts in a natural way to an action of $\mathrm{Gal}(K/k)$ on $\mathcal{L}_{\psi,\widehat{K}}$, which satisfies again the same semilinearity condition above (after replacing χ by its composition with π). For a detailed treatment the reader is referred *e.g.* to [B2] proposition 1.4.

Remark 6.3.1. In the algebraic setting, one usually introduces the topos $\mathbf{S}_X$ of sheaves of sets on the scheme X, and then, for any given ring Λ, assigns to $\mathbf{S}_X$ a structure of Λ-ringed topos. by selecting the ring object Λ_X defined by the *constant* sheaf on X with stalks isomorphic to Λ. As the above construction illustrates, in the étale analytic setting, the choice of the constant Λ_n-sheaf is not the most natural: one should rather take the geometrically constant sheaf $\Lambda_{n,X}$, twisted by the semilinear $\mathrm{Gal}(k_0^a/k_0)$-action defined in this section.

7. Fourier Transform

We are now ready to define the Fourier transform. With the set-up of the previous chapters, we only have to mimic the construction of the Deligne-Fourier transform. The proofs of most of the main properties reduce to routine verifications, carried out by applying projection formulas, proper base change theorem and Poincaré duality, exactly as in Laumon's paper.

7.1. Definition and main properties.

We consider complexes of sheaves of Λ-modules, where Λ is an ℓ-coefficient ring. Let $\mathcal{L}_\psi$ be the locally constant Lubin-Tate Λ-sheaf of rank 1 associated to the Lubin-Tate group F defined over the field k_0, and the character $\psi : G_\infty \to \Lambda^\times$. *In this chapter and the following one, the base field is a complete extension k of k_∞.*

Let S be an analytic variety over k and $\pi : \mathsf{E} \to S$ an analytic vector bundle (defined in the obvious way) of constant rank $r \geq 1$. We denote by $\pi' : \mathsf{E}' \to S$ the vector bundle dual to $\mathsf{E} \to S$, by $\langle , \rangle : \mathsf{E} \times_S \mathsf{E}' \to (\mathbb{A}^1_k)^{an}$ the canonical dual pairing and by $\mathrm{pr} : \mathsf{E} \times_S \mathsf{E}' \to \mathsf{E}$, $\mathrm{pr}' : \mathsf{E} \times_S \mathsf{E}' \to \mathsf{E}'$ the two canonical projections.

Definition 7.1.1. The *Fourier transform* for $\mathsf{E} \to S$, associated to the character ψ, is the triangulated functor

$$\mathcal{F}_\psi : \mathrm{D}^b(\mathsf{E}, \Lambda) \longrightarrow \mathrm{D}^b(\mathsf{E}', \Lambda)$$

defined by

$$\mathcal{F}_\psi(K^\bullet) = R\mathrm{pr}'_!(\mathcal{L}_\psi(\langle , \rangle) \otimes \mathrm{pr}^* K^\bullet))[r].$$

We will usually drop the subscript ψ, unless we have to deal with more than one character at the same time. For later use we also introduce a special notation for a closely related functor: the operator $\mathcal{F}_{\psi,*}$ is given by the following formula:

$$\mathcal{F}_{\psi,*} = R\mathrm{pr}'_*(\mathcal{L}_\psi(\langle , \rangle) \otimes \mathrm{pr}^* K^\bullet))[r].$$

Next we would like to show that $\mathcal{F}$ shares some interesting properties with the Fourier transform defined over finite fields.

To start with, we state and establish involutivity: denote by $\pi'' : \mathsf{E}'' \to S$ the double dual vector bundle of E. The previous construction applies to E' and its dual E'' to give a Fourier transform $\mathcal{F}'$ (and the related functor $\mathcal{F}'_*$). We consider the composition:

$$\mathrm{D}^b(\mathsf{E}, \Lambda) \xrightarrow{\mathcal{F}} \mathrm{D}^b(\mathsf{E}', \Lambda) \xrightarrow{\mathcal{F}'} \mathrm{D}^b(\mathsf{E}'', \Lambda).$$

Denote by $a : \mathsf{E} \xrightarrow{\sim} \mathsf{E}''$ the S-isomorphism defined by $a(v) = -\langle v, \cdot \rangle$. Also, let $\sigma : S \hookrightarrow \mathsf{E}$, $\sigma' : S \hookrightarrow \mathsf{E}'$, $\sigma'' : S \hookrightarrow \mathsf{E}''$ the zero sections of π, π', π'' respectively. We denote by $s : \mathsf{E} \times_S \mathsf{E} \to \mathsf{E}$ (resp. by $s' : \mathsf{E}' \times_S \mathsf{E}' \to \mathsf{E}'$) the addition law in the vector bundle $\mathsf{E} \to S$ (resp. in $\mathsf{E}' \to S$) and by $[-1] : \mathsf{E} \to \mathsf{E}$ the inverse map for this addition law.

Theorem 7.1.2. *There is a functorial isomorphism:*

$$\mathcal{F}' \circ \mathcal{F}(K^\bullet) \simeq a_*(K^\bullet)(-r)$$

for $K^\bullet \in \mathrm{D}^b(V, \Lambda)$ (The brackets denoting Tate twist, as usual).

Proof. (Cp. [Lau], theorem (1.2.2.1)). We fix some notation: let $\alpha : \mathsf{E} \times_S \mathsf{E}' \times_S \mathsf{E}'' \longrightarrow \mathsf{E}' \times_S \mathsf{E}''$ be defined as $\alpha(e, e', e'') = (e', e'' - a(e))$ and $\beta : \mathsf{E} \times \mathsf{E}'' \longrightarrow \mathsf{E}''$ as $\beta(e, e'') = e'' - a(e)$.

Consider the commutative diagram:

$$\begin{tikzcd}
 & & \mathsf{E} \times_S \mathsf{E}'' \arrow[rrr, "\beta"] \arrow[dddll, bend right, "\mathrm{pr}"'] \arrow[dddrr, bend left, "\mathrm{pr}''"] & & & \mathsf{E}'' \\
 & & \mathsf{E} \times_S \mathsf{E}' \times \mathsf{E}'' \arrow[u, "\mathrm{pr}_{13}"] \arrow[rrr, "\alpha"] \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & & \mathsf{E}' \times \mathsf{E}'' \arrow[u, "\mathrm{pr}''"'] \\
 & \mathsf{E} \times_S \mathsf{E}' \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "\mathrm{pr}'"] & & \mathsf{E}' \times_S \mathsf{E}'' \arrow[dl, "\mathrm{pr}'"'] \arrow[dr, "\mathrm{pr}''"] & & \\
\mathsf{E} & & \mathsf{E}' & & \mathsf{E}'' &
\end{tikzcd}$$

where the two squares are fiber diagrams.

It follows easily from property **LT2** that

$$\mathrm{pr}_{12}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) = \alpha^*\mathcal{L}(\langle , \rangle). \tag{7.1.3}$$

Then we have:

$$\begin{aligned}
\mathcal{F}' \circ \mathcal{F}(K^\bullet) &\simeq \mathcal{F}'(R\mathrm{pr}'_!(\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}^*K^\bullet)[r]) \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}'^*(R\mathrm{pr}'_!(\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}^*K^\bullet)))[2r] \\
&\simeq R\mathrm{pr}''_!(\mathcal{L}(\langle , \rangle) \otimes R\mathrm{pr}_{23!}\mathrm{pr}_{12}^*(\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}^*K^\bullet))[2r] && \text{(proper base change)} \\
&\simeq R\mathrm{pr}''_! R\mathrm{pr}_{23!}(\mathrm{pr}_{23}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^*\mathrm{pr}^*K^\bullet)[2r] && \text{(proj.formula)} \\
&\simeq R\mathrm{pr}''_! R\mathrm{pr}_{23!}(\alpha^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{12}^*\mathrm{pr}^*K^\bullet)[2r] && \text{(by formula (7.1.3))} \\
&\simeq R\mathrm{pr}''_! R\mathrm{pr}_{13!}(\alpha^*\mathcal{L}(\langle , \rangle) \otimes \mathrm{pr}_{13}^*\mathrm{pr}^*K^\bullet)[2r] && \text{(functoriality)} \\
&\simeq R\mathrm{pr}''_!(\mathrm{pr}^*K^\bullet \otimes R\mathrm{pr}_{13!}\alpha^*\mathcal{L}(\langle , \rangle))[2r] && \text{(projection formula)} \\
&\simeq R\mathrm{pr}''_!(\mathrm{pr}^*K^\bullet \otimes \beta^*R\mathrm{pr}''_!\mathcal{L}(\langle , \rangle))[2r]. && \text{(proper base change)}
\end{aligned}$$

To end the proof we apply to $\pi' : \mathsf{E}' \to S$ and $L = \Lambda$ the lemma 7.1.4 below. $\square$

Lemma 7.1.4. *For any $L^\bullet \in \mathrm{D}^b(S, \Lambda)$ we have:*

$$\mathcal{F}(\pi^*L^\bullet[r]) \simeq \sigma'_*L^\bullet(-r).$$

Proof. By the projection formula:

$$\mathcal{F}(\pi^*L^\bullet[r]) = L^\bullet \otimes R\mathrm{pr}'_!\mathcal{L}(\langle , \rangle)[2r].$$

On the other hand, using proper base change, property **LT1** and proposition 6.1.5, we get:

$$\begin{aligned}
&\sigma'^*R\mathrm{pr}'_!\mathcal{L}(\langle , \rangle) = R\pi_!\Lambda = \Lambda_S(-r)[-2r] \\
&R\mathrm{pr}'_!\mathcal{L}(\langle , \rangle)_{|\mathsf{E}' - \sigma'(S)} = 0.
\end{aligned}$$

$\square$

Corollary 7.1.5. *$\mathcal{F}$ is an equivalence of triangulated categories of $\mathrm{D}^b(\mathsf{E}, \Lambda)$ onto $\mathrm{D}^b(\mathsf{E}', \Lambda)$, with inverse $a^*\mathcal{F}'(-)(r)$.* $\square$

In the case of the Fourier transform over a finite field, it is known moreover that $\mathcal{F}$ preserves the t-structure coming from middle perversity. As explained in [Lau], this boils down to the equality of functors $\mathcal{F}_\psi = \mathcal{F}_{\psi,*}$. Even in absence of a theory of perverse sheaves for analytic varieties, we can still prove the corresponding statement:

Theorem 7.1.6. *The canonical map of "forget support" induces an isomorphism of functors:*

$$\phi : \mathcal{F}_\psi(-) \xrightarrow{\sim} \mathcal{F}_{\psi,*}(-).$$

Proof. Fix as usual a coordinate t on $(\mathbb{A}^1_k)^{an}$. First of all, an argument like at the beginning of the proof of [Ka-La] Theorème 2.4.1 reduces us to the case $r = 1$. Moreover, the assertion is obviously local on S, hence we can suppose that there exists a fiberwise linear isomorphism $x : \mathsf{E} \xrightarrow{\sim} (\mathbb{A}^1_k)^{an} \times_k S = (\mathbb{A}^1)^{an}_S$. Then also E' is trivialized by a coordinate $y : \mathsf{E}' \to (\mathbb{A}^1)^{an}_S$ such that $t(\langle e, e'\rangle) = x(e)y(e')$ for all local sections e, e'. Next we can find a unique $\overline{\mathsf{E}} \supset \mathsf{E}$ such that x extends to a (unique) isomorphism $\overline{x} : \overline{\mathsf{E}} \to (\mathbb{P}^1)^{an}_S$,

Let $j : \mathsf{E} \times_S \mathsf{E}' \hookrightarrow \overline{\mathsf{E}} \times_S \mathsf{E}'$ be the natural imbedding. Clearly it suffices to show that for all points of the type $(\infty, p) \in \overline{\mathsf{E}} \times_S \mathsf{E}'$

$$Rj_*(\mathcal{L}_\psi(\langle,\rangle) \otimes \mathrm{pr}^* K^\bullet)_{(\infty,p)} = 0.$$

We consider the map $\tau : \mathsf{E}' \times_S \overline{\mathsf{E}} \times_S \mathsf{E}' \to \overline{\mathsf{E}} \times_S \mathsf{E}'$ defined as $(e'_1, e, e'_2) \mapsto (e, s'(e'_1, e'_2))$. We form the fibre product diagram

$$\begin{CD}
\mathsf{E}' \times_S \mathsf{E} \times_S \mathsf{E}' @>{j^\circ}>> \mathsf{E}' \times_S \overline{\mathsf{E}} \times_S \mathsf{E}' \\
@V{\tau^\circ}VV @VV{\tau}V \\
\mathsf{E} \times_S \mathsf{E}' @>{j}>> \overline{\mathsf{E}} \times_S \mathsf{E}'
\end{CD}$$

and by smooth base change

$$\tau^* Rj_*(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K^\bullet) \simeq Rj^\circ_* \tau^{\circ *}(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K^\bullet).$$

In particular

$$Rj_*(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K^\bullet)_{(\infty,p)} \simeq Rj^\circ_* \tau^{\circ *}(\mathcal{L}(\langle,\rangle) \otimes \mathrm{pr}^* K^\bullet)_{(p,\infty,0)}.$$

Let $\mathcal{C}_S$ be the partially ordered set of all the étale neighborhoods of $(y(p), \infty, 0)$ in $(\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S \times (\mathbb{A}^1)^{an}_S$. We introduce the family $\mathcal{C}^\delta_S$ consisting of all the varieties of the form $W \times_k B$ such that

1) B is an open disc in $(\mathbb{A}^1_k)^{an}$, centered at zero, *i.e.* $B = \{a \in (\mathbb{A}^1_k)^{an} \mid |t(a)| < r_B\}$, and $W \xrightarrow{\phi} (\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S$ is an étale neighborhood of $(p, \infty) \in (\mathbb{P}^1)^{an}_S \times_S (\mathbb{A}^1)^{an}_S$;

2) the image $\phi(W)$ is contained in an open subset of the form $N(p) \times_k C$, with C an open disc in $(\mathbb{P}^1_k)^{an}$ of radius r_C around ∞ *i.e.* $C = \{a \in (\mathbb{P}^1_k)^{an} \mid |t(a)| > r_C^{-1}\}$ and $N(p)$ some fixed open neighborhood of $y(p)$ in $(\mathbb{A}^1)^{an}_S$;

3) the ratio r_B/r_C is equal to the constant δ.

Lemma 7.1.7. *For any real number $\delta > 0$ the family $\mathcal{C}^\delta_S$ is cofinal in $\mathcal{C}_S$.*

Proof. Let $\sigma : U \to (\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S \times_S (\mathbb{A}^1)^{an}_S$ be any étale open neighborhood of $(p, \infty, 0)$ and $q \in U$ a chosen lifting of $(p, \infty, 0)$. We have an induced map of germs

$$(U, q) \to ((\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S \times_S (\mathbb{A}^1)^{an}_S, (p, \infty, 0)).$$

Notice that the residue fields of the points $(p, \infty) \in (\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S$ and $(p, \infty, 0) \in (\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S \times_S (\mathbb{A}^1)^{an}_S$ are naturally isomorphic. Therefore, it follows from theorem 3.4.1 of [B1] that the germ (U, q) is isomorphic to a product of germs $(W', q') \times_k ((\mathbb{A}^1_k)^{an}, 0)$, where $\phi : (W', q') \to ((\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S, (p, \infty))$ is a morphism of germs with an étale representative. Concretely this means that there exists an open subset $V \subset W' \times_k (\mathbb{A}^1_k)^{an}$ with an open imbedding $V \hookrightarrow U$ which make the following diagram commute

$$\begin{CD}
W' \times_k (\mathbb{A}^1_k)^{an} @<<< V @>>> U \\
@V{\phi \times 1_{\mathbb{A}^1}}VV @. @VV{\sigma}V \\
((\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S) \times_k (\mathbb{A}^1_k)^{an} @= @= (\mathbb{A}^1)^{an}_S \times_S (\mathbb{P}^1)^{an}_S \times_S (\mathbb{A}^1)^{an}_S.
\end{CD}$$

Then proposition 3.7.8 of [B1] says that we can find inside V a subset of the form $W'' \times B'$ which fulfills condition (1) above. Conditions (2) and (3) are easy to fix, by taking open subsets $B \subset B'$ and $W \subset W''$. $\square$

Fix a real number δ strictly greater than $\rho(\psi,t)$. Let $W\times_k B\in\mathcal{C}^\delta_S$ be any neighborhood as above and set $B_S=y^{-1}(B\times_k S)$, $C^\circ=C\cap(\mathbb{A}^1_k)^{an}$, $C^\circ_S=x^{-1}(C^\circ\times_k S)$, $W^\circ=W\times_{(\mathbb{A}^1\times\mathbb{P}^1)^{an}_S}(\mathsf{E}'\times_S\mathsf{E})$. Furthermore, we obtain obvious projection maps $\alpha:W^\circ\to C^\circ_S$ and $\beta:C^\circ_S\to C^\circ$.

In view of the lemma, the theorem will follow if we show that

$$H^i(W^\circ\times_S B_S,\tau^{\circ *}\mathcal{L}(\langle,\rangle)\otimes\tau^{\circ *}\mathrm{pr}^*K^\bullet)=0. \tag{7.1.8}$$

We remark the commutative diagram

$$\begin{tikzcd}
\mathsf{E}'\times_S\mathsf{E}\times_S\mathsf{E}' \arrow[r, "\mathrm{pr}_{12}"] \arrow[d, "\tau^\circ"'] & \mathsf{E}'\times_S\mathsf{E} \arrow[d, "\mathrm{pr}_2"] \\
\mathsf{E}\times_S\mathsf{E}' \arrow[r, "\mathrm{pr}"] & \mathsf{E}.
\end{tikzcd} \tag{7.1.9}$$

Moreover, let $\mu:\mathsf{E}'\times_S\mathsf{E}\to(\mathbb{A}^1_k)^{an}$ be the map $(e',e)\mapsto\langle e,e'\rangle$; an easy application of the Yoga of torsors yields

$$\tau^{\circ *}\mathcal{L}(\langle,\rangle)\simeq\mathrm{pr}^*_{23}\mathcal{L}(\langle,\rangle)\otimes\mathrm{pr}^*_{12}\mathcal{L}(\mu). \tag{7.1.10}$$

We apply the Leray spectral sequence for the morphism $\mathrm{pr}_{12}:W^\circ\times_k B\to W^\circ$.

Set $M^\bullet=\mathrm{pr}^*_2K^\bullet\otimes\mathcal{L}(\mu)$; then, in virtue of (7.1.9) and (7.1.10) it suffices to show that

$$R\mathrm{pr}_{12*}(\mathrm{pr}^*_{23}\mathcal{L}(\langle,\rangle)\otimes\mathrm{pr}^*_{12}M^\bullet)_w=0$$

for all points $w\in W^\circ$. We consider the commutative diagram

$$\begin{tikzcd}
W^\circ\times_S B_S \arrow[r, "\alpha\times 1_B"] \arrow[d] & C^\circ_S\times_S B_S \arrow[r, "\beta\times 1_B"] \arrow[d] & C^\circ\times_k B \arrow[d, "m"] \\
\mathsf{E}'\times_S\mathsf{E}\times_S\mathsf{E}' \arrow[r, "\mathrm{pr}_{23}"] & \mathsf{E}\times_S\mathsf{E}' \arrow[r, "\langle,\rangle"] & (\mathbb{A}^1_k)^{an}
\end{tikzcd}$$

where $m(a,b)=ab$. Set $u=\beta\circ\alpha(w)$, take a small open neighborhood $U\subset C^\circ$ around u, and let $\mathbb{E}(r)=m(U\times_k B)$. One checks easily that, if U has been chosen small enough, then $\mathbb{E}(r)$ is some open disc of finite radius r, centered at the origin. Denote by E the connected component of $\boldsymbol{\lambda}^{-1}(\mathbb{E}(r))\subset\mathbb{E}(0,1)$ which contains $0\in\mathbb{E}(0,1)$. We form the fibre diagram

$$\begin{tikzcd}
E \arrow[d, "\boldsymbol{\lambda}"'] & V_1 \arrow[l, "m'"'] \arrow[d, "f"'] & V_2 \arrow[l] \arrow[d, "g"'] \\
\mathbb{E}(r) & U\times_k B \arrow[l, "m"'] & ((\beta\circ\alpha)^{-1}U)\times_k B. \arrow[l, "(\beta\circ\alpha)\times 1_B"']
\end{tikzcd}$$

By construction, the sheaf $m^*\mathcal{L}$ trivializes on the étale covering of finite degree $f:V_1\to U\times_k B$. It follows that $m^*\mathcal{L}_{|U\times_k B}$ is a direct summand in $f_*\Lambda$ and hence we obtain an imbedding

$$R^q\mathrm{pr}_{12*}(\mathrm{pr}^*_{23}\mathcal{L}(\langle,\rangle)\otimes\mathrm{pr}^*_{12}M^\bullet)_{|(\beta\circ\alpha)^{-1}(U)}\hookrightarrow R^q(\mathrm{pr}_{12}\circ g)_*(\mathrm{pr}_{12}\circ g)^*M^\bullet_{|(\beta\circ\alpha)^{-1}(U)}\quad(q\geq 0). \tag{7.1.11}$$

Notice also that for all $y\in U$, the geometric fibre $(\mathrm{pr}_1\circ f)^{-1}(y)$ is a finite union of open discs. In order to apply this observation, we need the following lemma, which is a minor variation of [B1] Corollary 7.4.2, and whose proof we leave therefore as an exercise for the referee.

Lemma 7.1.12. *Let $\phi:X\to Y$ be a separated smooth morphism of pure dimension d, and suppose that the geometric fibres of ϕ are non-empty and have trivial cohomology groups H^q_c with coefficients in Λ for $q<2d$. Then for all $F\in\mathbf{S}(X,\Lambda)$ we have $R^q\phi_*\phi^*F=0$ for $q>0$.* □

Next, since we have taken $W\times_k B\in\mathcal{C}^\delta_S$ and $\delta>\rho(\psi,t)$, we see that the sheaf $m^*\mathcal{L}_{|U\times_k B}$ is never trivial on any of the geometric fibres $\{y\}\times_k B$ $(y\in U)$. From this, together with (7.1.11) and lemma 7.1.12 (applied to $\mathrm{pr}_{12}\circ g$) we derive easily that

$$R\mathrm{pr}_{12*}(\mathrm{pr}^*_{23}\mathcal{L}(\langle,\rangle)\otimes\mathrm{pr}^*_{12}M)_{|(\beta\circ\alpha)^{-1}U}=0.$$

This proves (7.1.8) and the claim of the theorem. □

Remark 7.1.13. it is well known that theorem 7.1.6 formally implies that the Fourier transform commutes with Verdier duality. A Verdier duality theory for étale analytic sheaves has been established by Berkovich in [B5].

We list hereafter a few of the other main formal properties of the Fourier transform. The proofs have the same flavour as the previous proof of involutivity, and proceed exactly as in Laumon's paper, therefore we limit ourself to give the statements and refer the reader to the corresponding results in [Lau].

Theorem 7.1.14. *(Cp.* [Lau]*, theorem (1.2.2.4)) Let* $\mathsf{E}_1 \to \mathsf{E}_2$ *a morphism of vector bundles over* S *of constant ranks* r_1 *and* r_2 *respectively, and let* $f' : \mathsf{E}'_2 \to \mathsf{E}'_1$ *be the transpose of* f*. Then there is a canonical isomorphism*

$$\mathcal{F}_2(Rf_! K_1^\bullet) \simeq f'^* \mathcal{F}_1(K_1^\bullet)[r_2 - r_1]$$

for all $K_1^\bullet \in \mathrm{D}^b(\mathsf{E}_1, \Lambda)$. □

Corollary 7.1.15. *There is a canonical isomorphism*

$$R\pi'_! \mathcal{F}(K^\bullet) \simeq \sigma^* K^\bullet(-r)[-r]$$

for all $K^\bullet \in \mathrm{D}^b(\mathsf{E}_1, \Lambda)$. □

Definition 7.1.16. The convolution product on $\mathsf{E} \to S$ is the operation

$$* : \mathrm{D}^b(\mathsf{E}, \Lambda) \times \mathrm{D}^b(\mathsf{E}, \Lambda) \to \mathrm{D}^b(\mathsf{E}, \Lambda)$$

defined as

$$K_1^\bullet * K_2^\bullet = Rs_!(K_1^\bullet \overset{\mathbf{L}}{\boxtimes} K_2^\bullet).$$

Proposition 7.1.17. *(Cp.* [Lau]*, proposition (1.2.2.7)) There is a canonical isomorphism*

$$\mathcal{F}(K_1^\bullet * K_2^\bullet) \simeq \mathcal{F}(K_1^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{F}(K_2^\bullet)[-r]$$

for all $K_1^\bullet, K_2^\bullet \in \mathrm{D}^b(\mathsf{E}, \Lambda)$. □

Proposition 7.1.18. *(Cp.* [Lau]*, proposition (1.2.2.8)). There is a canonical "Plancherel" isomorphism*

$$R\pi'_!(\mathcal{F}(K_1^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{F}(K_2^\bullet)) \simeq R\pi_!(K_1^\bullet \overset{\mathbf{L}}{\otimes} [-1]^* K_2^\bullet)(-r)$$

for all $K_1^\bullet, K_2^\bullet \in \mathrm{D}^b(\mathsf{E}, \Lambda)$. □

Proposition 7.1.19. *(Cp.* [Lau]*, proposition (1.2.3.5)). Let* $S_1 \xrightarrow{f} S$ *be a morphism of* k*-analytic varieties. Let* $\mathsf{E}_1 \xrightarrow{\pi_1} S_1$ *and* $\mathsf{E}'_1 \xrightarrow{\pi'_1} S_1$ *the vector bundles over* S_1 *obtained by base change from* $\mathsf{E} \xrightarrow{\pi} S$ *and* $\mathsf{E}' \xrightarrow{\pi'} S$*. Denote by* $f_{\mathsf{E}} : \mathsf{E}_1 \to \mathsf{E}$ *and* $f_{\mathsf{E}'} : \mathsf{E}'_1 \to \mathsf{E}'$ *the canonical projections. Then there exists a canonical isomorphism*

$$\mathcal{F}(Rf_{\mathsf{E}!} K^\bullet) \simeq Rf_{\mathsf{E}'!} \mathcal{F}_1(K^\bullet)$$

for all $K^\bullet \in \mathrm{D}^b(\mathsf{E}_1, \Lambda)$ *(we have denoted by* $\mathcal{F}_1$ *the Fourier transform for the vector bundle* $\mathsf{E}_1 \to S_1$*).* □

7.2. **Computation of some Fourier transforms.** The following examples of calculation of Fourier trasforms are taken from [Lau], with the exception of proposition 7.2.4, which has no analogue in positive characteristic.

Proposition 7.2.1. *Let* $\mathsf{F} \overset{i}{\hookrightarrow} \mathsf{E}$ *be a vector sub-bundle over* S *of constant rank* s*. Denote by* $\mathsf{F}^\perp \overset{i^\perp}{\hookrightarrow} \mathsf{E}'$ *the orthogonal of* F *in* E'*. Then there is a canonical isomorphism*

$$\mathcal{F}(i_* \Lambda_{\mathsf{F}}[s]) \simeq i^\perp_* \Lambda_{\mathsf{F}^\perp}(-s)[r-s].$$

□

Proposition 7.2.2. *Let* $e \in \mathsf{E}(S)$ *(i.e. a section of* $\mathsf{E} \xrightarrow{\pi} S$*). Denote by* $\tau_e : \mathsf{E} \to \mathsf{E}$ *the translation by* e*. Then there is a canonical isomorphism*

$$\mathcal{F}(\tau_{e*} K^\bullet) \simeq \mathcal{F}(K^\bullet) \otimes \mathcal{L}(\langle e, \ \rangle)$$

for all $K^\bullet \in \mathrm{D}^b(\mathsf{E}, \Lambda)$. □

Proposition 7.2.3. *Let* $\alpha : \mathsf{E} \xrightarrow{\sim} \mathsf{E}'$ *be a symmetric isomorphism. Denote by* $q : \mathsf{E} \to (\mathbb{A}^1_k)^{an}$ *and* $q' : \mathsf{E}' \to (\mathbb{A}^1_k)^{an}$ *the quadratic forms associated to* α *(*i.e. $q(e) = \langle e, \alpha(e) \rangle$ *and* $q'(e') = \langle \alpha^{-1}(e'), e' \rangle$*). Let* $[2] : \mathsf{E}' \to \mathsf{E}'$ *be multiplication by 2 on the vector bundle* E'*. Then there is a canonical isomorphism*

$$[2]^* \mathcal{F}(\mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes \pi'^* R\pi_! \mathcal{L}(q)[r].$$

□

For the next result, we suppose $\mathsf{E} \xrightarrow{\pi} S$ has rank one for simplicity. Let $B \xrightarrow{\beta} S$ be a sphere bundle inside E, *i.e.* a fibre bundle over S with an open imbedding $j : B \hookrightarrow \mathsf{E}$ which is a morphism of S-varieties, and such that over each point $s \in S$, the restriction $j_s : \beta^{-1}(s) \hookrightarrow \pi^{-1}(s)$ is the imbedding of an open ball of finite radius centered at $\sigma(s) \in \pi^{-1}(s)$.

We also fix some linear coordinate t on $(\mathbb{A}^1_k)^{an}$ and let $D \xrightarrow{\beta'} S$ be the dual bundle of $B \to S$, *i.e.* the fibre bundle over S with a closed S-imbedding $i : D \to \mathsf{E}'$, defined by the equation

$$|t(\langle e, e'\rangle)| < \rho(\psi, t) \qquad (e \in B, e' \in D).$$

In other words, the restriction $i_s : \beta'^*(s) \to \pi'^*(s)$ is the imbedding of a closed disc centered at $\sigma'(s)$.

Proposition 7.2.4. *With the notation above we have*
i) $\mathcal{F}(i_*\Lambda_D) = j_!\Lambda_B[1]$,
ii) $\mathcal{F}(j_!\Lambda_B) = i_*\Lambda_D(-1)[-1]$.

Proof. By theorem 7.1.2 we see that (i) and (ii) are equivalent. We will prove (ii). By proper base change we can assume that S is a point; then B is an open disc $\mathbb{G}_a(\alpha, t)$ and $D = D_\beta$ is a closed disc of radius $\beta = \rho(\psi, t)/\alpha$. Set $T = \mathbb{G}_a(\alpha, t) \times_k D_\beta$. Note that the condition $\alpha\beta = \rho(\psi, t)$ implies that $\mathcal{L}(\langle,\rangle)$ trivializes on T. It follows that the restriction of $\mathcal{F}(j_!\Lambda_{\mathbb{G}_a(\alpha,t)})$ to D_β coincides with $\Lambda[-1]$. Therefore it suffices to show that $\mathcal{F}(j_!\Lambda_{\mathbb{G}_a(\alpha,t)})$ vanishes outside D_β. To this purpose we can check on the stalks, and then the claim follows from proposition 6.1.5. □

8. Global theory in dimension one

In this chapter we study the cohomology of a local system with meromorphic ramification on an affine curve (see definition 8.1.1 below). We start by establishing the finiteness of the cohomology, then we make a detailed study of the Fourier transform of a meromorphic local system, in case C is an open subscheme of the affine line. Finally we refine our finiteness result into a formula of type Grothendieck-Ogg-Shafarevich for our class of sheaves.

8.1. **Finiteness properties.** Let C be an affine smooth geometrically connected curve over k, and let $\overline{C}$ be a compactification of C. Let $\overline{C} - C = \{s_1, ..., s_m\}$ and F a locally constant sheaf of finitely generated Λ-modules on C^{an}. For each i, choose a local parameter t_i on $\overline{C}^{an}$, centered around s_i. We obtain a family of discs $\eta_i = \{\mathbb{E}(s_i, \rho) \mid \rho \in \mathbb{R}\}$ and the restriction of F to $\mathbb{E}^*(s_i, \rho) = \mathbb{E}(s_i, \rho) - \{s_i\}$ yields a representation of $\widetilde{\pi}_1(\mathbb{E}^*(s_i, \rho), \overline{x}_\rho)$ (we choose some base point $\overline{x}_\rho$). Hence F determines a stable $\pi_1(\eta_i, \overline{x})$-module of finite rank $F(s_i) = H^0(\overline{\eta}_i, F_{\eta_i})$.

Definition 8.1.1. We say that F has *meromorphic ramification*, if the action of $\pi_1(\eta_i, \overline{x})$ on each of the Λ-modules $F(s_i)$ factors through $\mu(\eta_i, \overline{x})$.

Definition 8.1.2. Let $\chi : \pi_1^{alg}(\mathbb{G}_{m,k}, \overline{x}) \to \Lambda^\times$ be a non-trivial character. It defines a locally constant sheaf of Λ-modules $\mathcal{K}_\chi$ on $(\mathbb{G}_{m,k})^{an}$ which we call the *Kummer sheaf* associated to the character χ. Similarly, for a given tame character $\chi' : \mu(\mathbb{E}^*(0, \rho), \overline{x}) \to \Lambda^\times$ we obtain a sheaf of Λ-modules of rank one on $\mathbb{E}^*(0, \rho)$, which we call the Kummer sheaf associated to χ'.

Proposition 8.1.3. *Let $\mathcal{L}_\psi$ be a rank one sheaf of Λ-module associated to some Lubin-Tate torsor and a character ψ as in section 6.1. Let $G(\chi, \psi)$ be the Λ-module with continue $\mathrm{Gal}(k^a/k)$-action defined as*

$$G(\chi, \psi) = H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{K}_\chi \otimes \mathcal{L}_\psi).$$

Then: 1) $G(\chi, \psi)$ is a free Λ-module of rank one and the $H^i_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{K}_\chi \otimes \mathcal{L}_\psi)$ vanish for $i \neq 1$;
2) if j is the imbedding of $\mathbb{G}_{m,k}$ in $\mathbb{A}^1_k$, there is a canonical isomorphism

$$\mathcal{F}_\psi(j_*\mathcal{K}_\chi) \simeq j_*\mathcal{K}_{\chi^{-1}} \otimes G(\chi, \psi).$$

Proof. The second statement can be infered, *mutatis mutandis*, from the proof of proposition 1.4.3.2 of [Lau]. It is easy to verify that the cohomology of $\mathcal{K}_\chi \otimes \mathcal{L}_\psi$ vanishes in degrees $i \neq 1$. To show that $G(\chi, \psi)$ has rank one, we can use (2) and the involutivity theorem 7.1.2 to obtain

$$(-1)^* j_*\mathcal{K}_\chi(-1) \simeq j_*\mathcal{K}_\chi \overset{\mathbf{L}}{\otimes} G(\chi, \psi) \overset{\mathbf{L}}{\otimes} G(\chi^{-1}, \psi).$$

This implies that $G(\chi, \psi) \overset{\mathbf{L}}{\otimes} G(\chi^{-1}, \psi)$ must be free of rank one, hence the claim. □

Theorem 8.1.4. *Suppose that F is a locally constant sheaf of finitely generated Λ-modules on C^{an}, with meromorphic ramification. Then the cohomology groups $H^*_c(C^{an} \times_k \widehat{k}^a, F)$ are finitely generated Λ-modules.*

Proof. Choose local parameter $t_1, ..., t_m$ as above. Let $\mathbb{E}_1(\rho), ..., \mathbb{E}_m(\rho)$ be small open discs in $\overline{C}^{an}$, with $s_i \in \mathbb{E}_i(\rho)$ for $i = 1, ...m$. Set $\mathbb{E}_i^*(\rho) = \mathbb{E}_i(\rho) - \{s_i\}$ and $V = C^{an} - \bigcup_{i=1}^m \mathbb{E}_i^*(\rho)$. By lemma 4.1.6 the restriction of F to V is already defined over some finite subring $\Lambda' \subset \Lambda$, and then, by [B3] corollary 5.6, it follows that $H^*(V, F)$ is finitely generated. Hence it suffices to show that all the groups $H_c^*(\mathbb{E}_i(\rho), F)$ are finitely generated for sufficiently small ρ.

For any $N \in \mathbb{N}$, let $\phi_N : \mathbb{E}_i^*(\rho) \to \mathbb{E}_i^*(\rho^N)$ be the morphism $t_i \mapsto t_i^N$. From the equality

$$H_c^*(\mathbb{E}_i(\rho), F) \simeq H_c^*(\mathbb{E}_i(\rho^N), \phi_{N*}F) \tag{8.1.5}$$

and from theorem 5.2.16 we derive that it suffices to consider the case when $F = L \otimes T$ where L has rank one and T is tame of finite rank. Let $\mathfrak{m}$ be the maximal ideal of Λ. Using inductively the short exact sequence

$$0 \to \mathfrak{m}F \to F \to F \to F/(\mathfrak{m}F) \to 0$$

we can reduce to the case when Λ is a field. Then, by standard modular representation theory we can assume that T has rank one, hence $T = \mathcal{K}_\chi$ is a Kummer sheaf associated to a tame character χ of $\mu(\mathbb{E}^*(\rho), \overline{x}_\rho)$. Moreover, by lemma 5.2.8 we know that L is of the form $f^*(\mathcal{L}_\psi)$ where $\mathcal{L}_\psi$ is a rank one sheaf associated to a Lubin-Tate torsor as in section 6.1 and f is a polynomial $f(t_i) \in k[t_i]$ (since we are only interested in geometric results, the choice of the Lubin-Tate torsor does not matter).

Let $n = \deg(f)$. By standard arguments, after reducing the radius ρ, we can find an automorphism $\sigma : \mathbb{E}_i^*(\rho) \to \mathbb{E}_i^*(\rho)$ such that $\sigma^*(\mathcal{L}_\psi(f)) \simeq \phi_n^*(\mathcal{L}_\psi)$. Moreover, after further reducing ρ, we can also achieve that $\sigma^*(T) \simeq T$. Hence we can assume that $f(t_i)$ is a monomial $f(t_i) = t_i^n$. Using the formula $f_*(\mathcal{L}_\psi(f) \otimes T) \simeq \mathcal{L}_\psi \otimes f_*T$ together with (8.1.5) and some standard modular representation theory, we can reduce to the case when $F = \mathcal{L}_\psi \otimes \mathcal{K}_\chi$. Then the result follows from proposition 8.1.3. $\square$

8.2. **Canonical calculation of cohomology.** The following very useful results are shamelessly adapted from [Ka2] sections 2.1 and 2.2.

Let C be a proper smooth geometrically connected curve over k and $U \subset C$ a non-empty open subscheme. We are concerned with locally constant sheaves F of Λ-modules with meromorphic ramification on U^{an}, where Λ is an ℓ-coefficient ring. For $s \in C - U$ we may speak of the slope decomposition of $F(s)$. The slopes which occur in it are called the slopes of F at s. We say that F is *totally wild* at s if $F(s)^{P_{mer}} = 0$, where $P_{mer} = P(\eta_s)/P_{ess}(\eta_s) \subset \mu(\eta_s, \overline{x})$ is the meromorphic ramification subgroup. In other words, F is totally wild at s if 0 is not a slope of F at s.

Lemma 8.2.1. *For Λ as above, and s a chosen point of $C - U$, denote by $\mathcal{W}$ the abelian category of locally constant sheaves of finitely generated Λ-modules on U^{an} which are meromorphically ramified at all points of $C - U$ and totally wild at s.*

(1) $H^0(U^{an} \times_k \widehat{k}^a, F) = 0 = H_c^2(U^{an} \times_k \widehat{k}^a, F)$ for any $F \in \mathcal{W}$.

(2) The functors $F \mapsto H_c^1(U^{an} \times_k \widehat{k}^a, F)$ and $F \mapsto H^1(U^{an} \times_k \widehat{k}^a, F)$ are exact functors from W to the category of finitely generated Λ-modules.

(3) Both functors in (2) above carry Λ-flat F in $\mathcal{W}$ to free Λ-modules of finite rank. Their formation is compatible with extensions of scalars $A \to A'$ of ℓ-coefficient rings.

Proof. (Cp. [Ka2] lemma 2.1.1) (1) We have $H^0 = 0$ because $F_{\overline{\eta}_s}$ has no non-zero P_{mer}-invariants. By corollary 5.2.7, $F_{\overline{\eta}_s}$ is semisimple as a P_{mer}-module, hence it has no coinvariants either, which shows that $H_c^2 = 0$. (2) follows immediately from (1) and from theorem 8.1.4. (3) If N is any finitely generated Λ-module, take a resolution $K_\bullet \to N$ by free finitely generated Λ-modules. For $F \in \mathcal{W}$ we obtain a complex $F \otimes_\Lambda K_\bullet$ of objects in $\mathcal{W}$ and by exactness, the functor $H_?^1$ carries its homology objects to those of the complex $H_?^1(U^{an} \times_k \widehat{k}^a, F \otimes_\Lambda K_\bullet) = H_?^1(U^{an} \times_k \widehat{k}^a, F) \otimes_\Lambda K_\bullet$. This means that

$$H_?^1(U^{an} \times_k \widehat{k}^a, \mathrm{Tor}_i^\Lambda(F, N)) = \mathrm{Tor}_i^\Lambda(H_?^1(U^{an} \times_k \widehat{k}^a, F), N).$$

Therefore, if F is Λ-flat, then so is $H_?^1(U^{an} \times_k \widehat{k}^a, F)$. Taking $i = 0$ also yields compatibility with extensions of scalars. $\square$

Lemma 8.2.2. *Suppose that $C - U = D_1 \cup D_2$ is a decomposition of $C - U$ into two disjoint non-empty finite sets of closed points. Denote by $j_1 : U \hookrightarrow C - D_2$ and $j_2 : U \hookrightarrow C - D_1$ the corresponding partial compactifications of U. Then for Λ as above and F any locally constant sheaf of finitely generated Λ-modules on U^{an} with meromorphic ramification at all points of $C - U$ we have*

(1) $H^i((C - D_2)^{an} \times_k \widehat{k}^a, j_{1!}F) = 0$ for $i \neq 1$.

(2) The H^1 is an exact functor to finitely generated Λ-modules.

(3) This functor carries Λ-flat sheaves F to free finitely generated Λ-modules. Its formation commutes with extensions $\Lambda \to \Lambda'$ of ℓ-coefficient rings.

Proof. (Cp [Ka2] lemma 2.2.1) The only point which requires attention is the proof that $H^2 = 0$. In the absence of clean-cut results on the cohomological dimension of affine varieties, we offer the following somewhat *ad hoc* argument. Let $\mathfrak{m}$ be the maximal ideal of Λ. Using inductively the short exact sequence

$$0 \to \mathfrak{m}F \to F \to F \to F/(\mathfrak{m}F) \to 0$$

we reduce to the case when Λ is a field. Since $C - D_2$ is affine, we can find a sequence of affinoid domains $(V_i)_{I\in\mathbb{N}}$ with $V_i \subset V_{i+1}$ for all i, such that $\bigcup_i V_i = (C - D_2)^{an}$. Since F is meromorphically ramified, the groups $H^i(V_i \times_k \widehat{k}^a, j_{1!}F)$ are finitely generated, and by Mittag-Leffler (see [B1] lemma 6.3.12) it follows

$$H^i((C - D_2)^{an} \times_k \widehat{k}^a, j_{1!}F) = \varprojlim_i H^i(V_i \times_k \widehat{k}^a, j_{1!}F).$$

Consequently, it suffices to show that $H^2(V \times_k \widehat{k}^a, j_{1!}F) = 0$ for all sufficiently large affinoids V. Since V is quasi-compact we have

$$H^2(V \times_k \widehat{k}^a, j_{1!}F) = H^2_c(V \times_k \widehat{k}^a, j_{1!}F) = H^2_c((V \cap U) \times_k \widehat{k}^a, F).$$

Write $V \cap U = A \cup B$ where A is an affinoid domain and B is a smooth analytic space of dimension one such that $B \cap A = \emptyset$. We consider the short exact sequence

$$0 \to F_{|B!} \to F \to F_{|A*} \to 0$$

with obvious notation. Taking cohomology with compact support we obtain an exact sequence

$$H^1(A \times_k \widehat{k}^a, F_{|A}) \xrightarrow{\alpha} H^2_c(B \times_k \widehat{k}^a, F_{|B}) \to H^2_c((V \cap U) \times_k \widehat{k}^a, F) \to H^2(A \times_k \widehat{k}^a, F_{|A}). \tag{8.2.3}$$

The rightmost term in (8.2.3) vanishes by [B1] theorem 6.4.1(i). Hence it suffices to show that α is surjective for some choice of A and B. Suppose first that $H^0(U^{an} \times_k \widehat{k}^a, F^\vee) = 0$. Given V as above, take an affinoid $W \subset V$ such that $B \subset W$. Let $A' = A \cap W$. It is clear that α factors through $\alpha' : H^1(A' \times_k \widehat{k}^a, F_{|A'}) \to H^2_c(B \times_k \widehat{k}^a, F_{|B})$. We show that the natural morphism $\beta : H^1(A \times_k \widehat{k}^a, F_{|A}) \to H^1(A' \times_k \widehat{k}^a, F_{|A'})$ is surjective if A is sufficiently large and A' is sufficiently small. Since both A and A' are quasi-compact, this can be checked in Huber's theory. The cokernel of β injects into $H^2_c((A - A') \times_k \widehat{k}^a, F) \simeq H^0((A - A') \times_k \widehat{k}^a, F^\vee(1))$. Since by hypothesis $H^0(U \times_k \widehat{k}^a, F^\vee) = 0$, it follows that for A sufficiently large, and A' sufficiently small $H^0((A - A') \times_k \widehat{k}^a, F^\vee(1)) = 0$. Hence for sufficiently large V and sufficiently small W, the map β is surjective as required. Consequently, in order to prove that α is surjective, we need only to show that α' is surjective for sufficiently small A' and B. Let $N = H^0(B \times_k \widehat{k}^a, F^\vee)$. We can find a sufficiently small affinoid $A'' \subset W$ such that $A'' \cap U^{an} = A'' \cap B$ and such that the constant sheaf $N_{A''\cap U}$ on $A'' \cap U^{an}$ injects into $F_{|A''\cap U^{an}}$. Replace V by A'', and find new $A, B \subset V$ as above, so that we have a short sequence of locally constant Λ-sheaves on $V \cap U^{an}$

$$0 \to N_{V\cap U^{an}} \to F^\vee \to Q \to 0 \tag{8.2.4}$$

and $N \subset H^0(B \times_k \widehat{k}^a, F^\vee)$. If $N' = H^0(B \times_k \widehat{k}^a, F^\vee)$ is strictly larger than N, we can repeat the above procedure and find smaller V, A, B such that the exact sequence (8.2.4) holds with N' in place of N and such that $N \subset N' \subset H^0(B \times_k \widehat{k}^a, F^\vee)$. The sequence $N \subset N' \subset ...$ obtained by iterating this procedure must stabilize, since the rank of F is an upper bound for the rank of all these modules. Hence we can assume that $N = H^0(B \times_k \widehat{k}^a, F^\vee)$. We take Huber's cohomology of the exact sequence (8.2.4) : since $H^0_c(A \times_k \widehat{k}^a, Q) = 0$ we get an imbedding $H^1_c(A \times_k \widehat{k}^a, N) \to H^1_c(A \times_k \widehat{k}^a, F^\vee)$ which dualizes to a surjection

$$H^1(A \times_k \widehat{k}^a, F) \to H^1(A \times_k \widehat{k}^a, N^\vee) \simeq H^1(A \times_k \widehat{k}^a, \Lambda) \otimes_\Lambda N^\vee.$$

Again, the same surjection holds in Berkovich's cohomology, and therefore we obtain a commutative diagram

$$\begin{tikzcd}
H^1(A \times_k \widehat{k}^a, F) \arrow[r] \arrow[d] & H^1(A \times_k \widehat{k}^a, N^\vee) \arrow[d] \\
H^2_c(B \times_k \widehat{k}^a, F) \arrow[r, "\sim"] & H^2_c(B \times_k \widehat{k}^a, N^\vee).
\end{tikzcd}$$

Thus we are reduced to the case where F is the constant sheaf Λ and we need to show that the morphism $H^1(A \times_k \widehat{k}^a, \Lambda) \to H^2_c(B \times_k \widehat{k}^a, \Lambda)$ is surjective. Going back to (8.2.3), we see that it suffices to show that $H^2_c((V \cap U^{an}) \times_k \widehat{k}^a, \Lambda) = 0$, which is easily done. This concludes the proof in case $H^0(U^{an} \times_k \widehat{k}^a, F^\vee) = 0$.

In the general case, let $M = H^0(U^{an} \times_k \widehat{k}^a, F^\vee)$; we get a short exact sequence of Λ-sheaves on U

$$0 \to K_1 \to F \to M^\vee_{U^{an}} \to 0$$

whence an exact cohomology sequence

$$H^2((C-D_2)^{an}\times_k \widehat{k}^a, j_{1!}K_1)\to H^2((C-D_2)^{an}\times_k \widehat{k}^a, j_{1!}F)\to H^2((C-D_2)^{an}\times_k \widehat{k}^a, j_{1!}\Lambda)\otimes_\Lambda M^\vee\to 0.$$

One checks easily that the rightmost term vanishes, and hence we are reduced to verify the claim for the sheaf K_1. Proceeding inductively we obtain a finite sequence of locally constant sheaves $K_1, K_2, ..., K_n$ with the property that $H^2((C-D_2)^{an}\times_k \widehat{k}^a, j_{1!}K_i)=0\Rightarrow H^2((C-D_2)^{an}\times_k \widehat{k}^a, j_{1!}F)=0$ for all i, and such that $H^0(U^{an}\times_k \widehat{k}^a, K_n^\vee)=0$, so the claim follows by the previous case. $\square$

Now let us consider the following situation: U is an affine open subset of C, $D\subset U$ is a finite set of closed points and $j: U-D\hookrightarrow U$ is the inclusion. For F locally constant on U we have a long exact cohomology sequence

$$0\to H^0(U^{an}\times_k \widehat{k}^a, F)\to H^0(D\times_k \widehat{k}^a, F_{|D})\xrightarrow{\delta} H^1(U^{an}\times_k \widehat{k}^a, j_!j^*F)\to H^1(U^{an}\times_k \widehat{k}^a, F)\to 0.$$

This means that we can calculate the cohomology groups $H^i(U^{an}\times_k \widehat{k}^a, F)$ as the cohomology of the two term complex

$$*(F): H^0(D\times_k \widehat{k}^a, F_{|D})\xrightarrow{\delta} H^1(U^{an}\times_k \widehat{k}^a, j_!j^*F).$$

Proposition 8.2.5. *(1) For Λ as above, the construction $F\mapsto *(F)$ is an exact functor from the category of meromorphically ramified locally constant sheaves of finitely generated Λ-modules on U^{an} to the category of two-term complexes of finitely generated Λ-modules.*

(2) If F is Λ-flat, then $*(F)$ is a two-term complex of free finitely generated Λ-modules.

(3) The formation of $*(F)$ commutes with extensions of ℓ-coefficient rings.

Proof. (Cp. [Ka2] Key lemma 2.2.5) The essential part is a special case of lemma 8.2.2 with $(U, D_1, D_2)=(U-D, D, C-U)$. $\square$

Next we consider how to "calculate" cohomology with compact support. For a locally constant sheaf F of finitely generated Λ-modules on U^{an}, the cohomological purity theorem [B1] 7.4.5 yields a short exact sequence on U^{an}

$$0\to F\to Rj_*j^*F\to (F(-1)_{|D})[-1]\to 0.$$

Notice now that $H^2_c(U^{an}\times_k \widehat{k}^a, Rj_*(j^*F))=0$. Indeed, in terms of the diagram of inclusions

$$\begin{tikzcd} & U \arrow[dr, hook, "k_2"] & \\ U-D \arrow[ur, hook, "j"] \arrow[dr, hook, "j_2"'] & & C \\ & C-D \arrow[ur, hook, "k_1"'] & \end{tikzcd}$$

we have

$$\begin{aligned} H^i_c(U^{an}\times_k \widehat{k}^a, Rj_*(j^*F)) &= H^i(C^{an}\times_k \widehat{k}^a, k_{2!}Rj_*(j^*F)) \\ &= H^i(C^{an}\times_k \widehat{k}^a, Rk_{1*}j_{2!}(j^*F)) \\ &= H^i((C-D)^{an}\times_k \widehat{k}^a, j_{2!}(j^*F)) \end{aligned}$$

which implies the claim by virtue of lemma 8.2.2. We derive a long exact cohomology sequence

$$0\to H^1_c(U^{an}\times_k \widehat{k}^a, F)\to H^1_c(U^{an}\times_k \widehat{k}^a, Rj_*(j^*F))\to H^0(D\times_k \widehat{k}^a, F_{|D})(-1)\to H^2_c(U^{an}\times_k \widehat{k}^a, F)\to 0.$$

Therefore the two-term complex placed in degrees 1 and 2

$$*_c(F): H^1_c(U^{an}\times_k \widehat{k}^a, Rj_*(j^*F))\to H^0(D\times_k \widehat{k}^a, F_{|D})(-1)$$

calculates the $H^i_c(U^{an}\times_k \widehat{k}^a, F)$. The analogue of proposition 8.2.5 is valid for the functor $F\mapsto *_c(F)$, with the same proof.

Definition 8.2.6. For Λ as above, let F be a locally constant sheaf of free finitely generated Λ-modules on U^{an}. Then by proposition 8.2.5, the complexes $*(F)$ and $*_c(F)$ consist of free finitely generated Λ-modules. The *Euler characteristics* $\chi(U^{an}\times_k \widehat{k}^a, F)$ and $\chi_c(U^{an}\times_k \widehat{k}^a, F)$ are the alternating sums of the Λ-ranks of the components of $*(F)$ and of $*_c(F)$ respectively.

Remark 8.2.7. (1) Of course, if each of the cohomology groups $H^i(U^{an}\times_k \widehat{k}^a, F)$, $i=0,1$ is itself a free Λ-module, then the alternating sums defined above are equal to the literal alternating sums on the cohomology groups themselves. (2) It can be shown (left to the reader) that the Euler characteristics do not depend on the choice of the finite set of points D.

8.3. **Constructibility properties.** In this and the following section we study some special features of the Fourier transform on rank one vector spaces. Hence here the base variety S of section 7.1 is reduced to a point and both E and its dual E' are affine spaces of dimension one, identified with $(\mathbb{A}^1_k)^{an}$. The main result of this section says that the Fourier transform of a meromorphically ramified sheaf K on $(\mathbb{A}^1_k)^{an}$ is locally constant outside a finite set of points S; moreover the set S can be completely determined in terms of the ramification of K at infinity. Since we are interested only in geometric results, we can and do assume that F is the formal multiplicative group $\mathbb{G}_m$. Our method consists in studying in detail certain one-parameter continuous deformations of some special local systems on the affine line (see lemma 8.3.3 below).

We fix an ℓ-coefficient ring Λ and we denote by $\mathcal{L}_\psi$ the rank one sheaf of Λ-modules associated to the Lubin-Tate torsor $\mathcal{L}$ and to a character $\psi : \mu_{p^\infty} \to \Lambda^\times$. Let $j : \mathbb{G}_{m,k} \to \mathbb{A}^1_k$ be the imbedding. We choose linear coordinates y and x on the first and second factor of $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ and a linear coordinate t on $\mathbb{A}^1_k$. Then the dual pairing $\langle , \rangle$ of section 7.1 reduces to a map $m : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by the ring homomorphism $k[t] \to k[x,y]$ which sends t to xy. For a complex $K^\bullet$ of Λ-modules on $(\mathbb{A}^1_k)^{an}$, the Fourier transform in degree i is then the functor

$$\mathcal{F}^i(K^\bullet) = R^{i+1}q_!(p^*K^\bullet \otimes m^*\mathcal{L})$$

where q, p are the two projections of $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ on the two factors.

Lemma 8.3.1. *Let $f(t) = t^n$ be a monic polynomial, seen as an algebraic map $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ and denote by $\mathcal{K}_\chi$ the locally constant sheaf of free Λ-modules on $(\mathbb{G}_{m,k})^{an}$ associated to some tame representation χ of $\widetilde{\pi}_1(\mathbb{G}_m, \overline{x})$. Then $H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(f) \otimes \mathcal{K}_\chi)$ is a free Λ-module and*

$$\mathrm{rk}\ H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(f) \otimes \mathcal{K}_\chi) = \deg(f) \cdot \mathrm{rk}(\mathcal{K}_\chi).$$

Proof. The isomorphism

$$H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(f) \otimes \mathcal{K}_\chi) \simeq H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi \otimes f_*(\mathcal{K}_\chi))$$

reduces to the case where $\deg(f) = 1$. Freenes follows from lemma 8.2.1.(3) which also shows that to compute the rank we can assume that Λ is a field of characteristic ℓ. By lemma 8.2.1.(2), after a finite extension of scalars, we can even assume that the tame representation $\mathcal{K}_\chi$ is absolutely irreducible. In that case, it follows by standard modular representation theory that χ is a character, *i.e.* $\mathrm{rk}(\mathcal{K}_\chi) = 1$. Finally we are reduced to show that $H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi \otimes \mathcal{K}_\chi)$ has Λ-rank one, which holds by proposition 8.1.3. ☐

In the remaining part of section 8.3 we will switch to Huber's theory of étale cohomology for adic spaces. Formally this means that in place of a Hausdorff strictly k-analytic space X we consider the associated rigid analytic variety, which is denoted $s(X)$ in [B1] section 1.6. By [Hub](1.1.11) the category of rigid analytic varieties over k is a full subcategory of the category of adic spaces over $\mathrm{Spa}(k, k^\circ)$ (see *loc.cit.* for the notation). Then, according to [Hub] proposition 0.7.15 one associates to the functor s a morphism of sites

$$\Theta_X : s(X)_{\acute{e}t} \to X_{\acute{e}t.s}$$

where $s(X)_{\acute{e}t}$ is the étale topology of $s(X)$ and $X_{\acute{e}t.s}$ is a certain site on X (defined in [Hub] section 8.3) with a natural morphism of sites $X_{\acute{e}t} \to X_{\acute{e}t.s}$ which induces an equivalence on the associated toposes.

Huber studies the cohomology of sheaves on the site $s(X)$. We will denote with the usual symbols $(H^i(s(X), -), R^if_! ...)$ the respective functors defined as in [Hub]. Moreover, we will actually denote the rigid analytic variety $s(X)$ again as X.

A priori this notation could lead to some conflict with our previous use of these symbols, since *in general the cohomology of* [Hub] *does not agree with Berkovich's theory.* However, Huber proves that there is agreement in a number of important cases. Notably, if $f : X \to Y$ is a *closed* morphism of analytic spaces (see [B1](1.5.3iii)), then one obtains a natural isomorphism of functors

$$\Theta^*_Y \circ Rf_! \xrightarrow{\sim} Rs(f)_! \circ \Theta^*_X.$$

Moreover, for any X as above, any abelian sheaf F on $X_{\acute{e}t.s}$ and every $n \in \mathbb{N}_0$,

$$H^n(X_{\acute{e}t.s}, F) \xrightarrow{\sim} H^n(s(X)_{\acute{e}t}, \Theta^*F).$$

Using these results, most of the results proved so far remain available after we switch to Huber's theory. We will leave to the referee the task of checking that in the following we make indeed a legal use of the comparison theorems between the two theories.

Lemma 8.3.2. *Keep the notation of lemma 8.3.1. Then for all real numbers r large enough, the natural morphism which "forgets supports"*

$$H^1_c(\mathbb{D}(0,r)\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\to H^1(\mathbb{D}(0,r)\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)$$

is an isomorphism.

Proof. For any $r>0$ we have a commutative diagram

$$\begin{tikzcd} H^1_c(\mathbb{D}(0,r)\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi) \arrow[r] \arrow[d, "a_r"'] & H^1(\mathbb{D}(0,r)\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi) \\ H^1_c((\mathbb{A}^1_k)^{an}\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi) \arrow[r, "\alpha"] & H^1((\mathbb{A}^1_k)^{an}\times_k \widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi) \arrow[u, "b_r"'] \end{tikzcd}$$

and for all r large enough, both maps a_r and b_r are isomorphisms, as one sees easily, since all these cohomology groups have finite rank. Hence it suffices to show that α is an isomorphism.

By lemma 8.2.2.(3) we know that both groups are free Λ-modules, whose formation is compatible with scalar extensions, hence to show that α is an isomorphism we can assume that Λ is a field of characteristic ℓ. By lemma 8.2.2.(2) we can also take a scalar field extension to reduce to the case when $\mathcal{K}_\chi$ is absolutely irreducible, *i.e.* it has rank one.

Suppose first that χ is not the trivial character and let $j':\mathbb{A}^1_k\hookrightarrow\mathbb{P}^1_k$ be the natural imbedding. We consider the following sequence of maps of complexes

$$j'_!(\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\xrightarrow{\sim} j'_*(\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\xrightarrow{\beta} Rj'_*(\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\xrightarrow{\sim} Rj'_*(\mathcal{L}_\psi(f)\otimes j_*\mathcal{K}_\chi)\xrightarrow{\gamma} Rj'_*(\mathcal{L}_\psi(f)\otimes Rj_*\mathcal{K}_\chi).$$

By the Leray spectral sequence, the maps in cohomology induced by β and γ are injective, consequently we obtain imbeddings

$$H^1_c((\mathbb{A}^1_k)^{an}\times_k\widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\hookrightarrow H^1((\mathbb{A}^1_k)^{an}\times_k\widehat{k}^a,\mathcal{L}_\psi(f)\otimes j_!\mathcal{K}_\chi)\hookrightarrow H^1((\mathbb{G}_{m,k})^{an}\times_k\widehat{k}^a,\mathcal{L}_\psi(f)\otimes\mathcal{K}_\chi).$$

By lemma 8.3.1 (and by Poincaré duality), the first and the third terms have the same rank, hence the claim in case χ is not trivial.

If χ is trivial, we consider the short exact sequence

$$0\to j_!j^*\mathcal{L}_\psi(f)\to\mathcal{L}_\psi(f)\to Q\to 0$$

which shows that

$$H^1_c((\mathbb{A}^1_k)^{an},j_!j^*\mathcal{L}_\psi(f))\simeq H^1((\mathbb{A}^1_k)^{an},j_!j^*\mathcal{L}_\psi(f))\iff H^1_c((\mathbb{A}^1_k)^{an},\mathcal{L}_\psi(f))\simeq H^1((\mathbb{A}^1_k)^{an},\mathcal{L}_\psi(f)).$$

Now an argument similar to the previous case concludes the proof. ☐

Now, let $f(y)=\sum_j a_jy^j$, $g(y)=\sum_j b_jy^j$ be any two polynomials with coefficients $a_j,b_j\in k$ and $R>0$ any real number. We consider the following diagram

$$\begin{tikzcd} (\mathbb{A}^1_k)^{an} & (\mathbb{A}^1_k)^{an}\times_k\mathbb{D}(0,R) \arrow[l, "q_R"'] \arrow[r, "p_R"] \arrow[d, "\mu_{f,R,g}"] & \mathbb{D}(0,R) \\ & (\mathbb{A}^1_k)^{an} & \end{tikzcd}$$

where $\mu_{f,R,N}(x,y)=f(y)+x\cdot g(y)$ and p_R,q_R are the natural projections. Moreover, for any two positive real numbers $r>\varepsilon>0$ define $U(f,\varepsilon,r)\subset(\mathbb{A}^1_k)^{an}$ as follows:

- if $\deg(f)>\deg(g)$ then $U(f,\varepsilon,r)=U(r)=\mathbb{D}(0,r)$;
- if $\deg(f)\le\deg(g)$ then $U(f,\varepsilon,r)=\mathbb{D}(0,r)-\mathbb{D}(-a_{\deg(g)},\varepsilon)$.

Notice that $\mu_{0,R,y}$ coincides with the restriction of the multiplication map $m:\mathbb{A}^1_k\times\mathbb{A}^1_k\to\mathbb{A}^1_k$, hence for $f=0$ and $g(y)=y$ we recover the Fourier transform:

$$\mathcal{F}(K)\simeq\lim_{R\to\infty}Rq_{R!}(p^*_RK\otimes\mu^*_{0,R,y}\mathcal{L}_\psi).$$

Lemma 8.3.3. *With the notation above, for any two real numbers $r>\varepsilon>0$ we can find $R_0>0$ such that the for all $R\ge R_0$ the natural morphism*

$$\theta_R:R^1q_{R!}(p^*_R\mathcal{K}_\chi\otimes\mu^*_{f,R,g}\mathcal{L}_\psi)\to R^1q_{R*}(p^*_R\mathcal{K}_\chi\otimes\mu^*_{f,R,g}\mathcal{L}_\psi)$$

restricts to an isomorphism over the open subset $U(f,\varepsilon,r)$.

Proof. For any two real numbers $R_2 > R_1 > 0$ we have imbeddings (and a projection)

$$\begin{tikzcd} (\mathbb{A}^1_k)^{an} \times \mathbb{D}(0,R_1) \arrow[r, "i"] & (\mathbb{A}^1_k)^{an} \times \mathbb{D}(0,R_2) \arrow[d, "j_{\mathbb{D}}"] & (\mathbb{A}^1_k)^{an} \times (\mathbb{D}(0,R_2) - \mathbb{D}(0,R_1)) \arrow[l, "j"'] \\ (\mathbb{A}^1_k)^{an} & (\mathbb{A}^1_k \times_k \mathbb{P}^1_k)^{an}. \arrow[l, "\overline{q}"'] & \end{tikzcd}$$

Set $G = p_R^*\mathcal{K}_\chi \otimes \mu^*_{f,R,g}\mathcal{L}_\psi$. We obtain a morphism of exact triangles in the derived category:

$$\begin{tikzcd} j_{\mathbb{D}!}j_!j^*G \arrow[r] \arrow[d] & j_{\mathbb{D}!}G \arrow[r] \arrow[d] & j_{\mathbb{D}!}i_*i^*G \arrow[d, "\alpha"] \\ Rj_{\mathbb{D}*}j_!j^*G \arrow[r] & Rj_{\mathbb{D}*}G \arrow[r] & Rj_{\mathbb{D}*}i_*i^*G \end{tikzcd} \tag{8.3.4}$$

where the map α is a quasi-isomorphism. Let $U \subset (\mathbb{A}^1_k)^{an}$ be any open subset. By applying the triangulated functor $R\overline{q}*$ to (8.3.4) we derive the following equivalence

$$R^1q_{R_2!}G_{|U} \simeq R^1q_{R_2*}G_{|U} \iff R^1q_{R_2!}(j_!j^*G)_{|U} \simeq R^1q_{R_2*}(j_!j^*G)_{|U}$$

which says that the cone of θ_{R_2} depends only on the behaviour of the sheaf G on $(\mathbb{A}^1_k)^{an} \times (\mathbb{D}(R_2) - \mathbb{D}(R_1))$.

Now, suppose that $\deg(f) = M > \deg(g)$. Then for any $r > 0$ we can find R_0 sufficiently large, so that for all $R_2 > R_1 > R_0$ there is a commutative diagram

$$\begin{tikzcd} \mathbb{D}(0,r) \times (\mathbb{D}(0,R_2) - \mathbb{D}(0,R_1)) \arrow[r, "\beta"] \arrow[d, "\mu_{f_0,R_2,0}"] & \mathbb{D}(0,r) \times (\mathbb{D}(0,R_2) - \mathbb{D}(0,R_1)) \arrow[d, "\mu_{f,R_2,g}"] \\ (\mathbb{A}^1_k)^{an} \arrow[r, equal] & (\mathbb{A}^1_k)^{an} \end{tikzcd} \tag{8.3.5}$$

where $f_0(x,y) = a_My^M$ and β is an isomorphism. Then to decide whether θ_R is an isomorphism it suffices to check on the stalks over the points $x \in \mathbb{D}(0,r)$. Finally, by quasi-compact base change we reduce to the situation of lemma 8.3.2, which shows the claim in case $\deg(f) > \deg(g)$.

For the case $\deg(f) \leq \deg(g) = N$, let $g_0(y) = b_Ny^N$ and $f_1(y) = a_Ny^N$. Pick real numbers $r > \varepsilon > 0$. Then, again for $R_2 > R_1 > R_0$ all large enough, we find a diagram like 8.3.5, except that we must take $U(f,\varepsilon,r)$ instead of $\mathbb{D}(0,r)$ and μ_{f_1,R_2,g_0} instead of $\mu_{f_0,R_2,0}$. Again we can check on the stalks, and reduce to lemma 8.3.2 as in the previous case. □

Theorem 8.3.6. *Let K be a sheaf of free Λ-modules of finite rank on $(\mathbb{A}^1_k)^{an}$, which is the extension by zero of a local system meromorphically ramified on the complement of a finite set of k-rational points. Then $\mathcal{F}^0(K)$ is locally constant on the complement of the finitely many k-rational points $x_1, ..., x_n \in (\mathbb{A}^1_k)^{an}$ such that the Swan conductor of $(p^*K \otimes m^*\mathcal{L}_\psi)_{\{x_i\}\times(\mathbb{A}^1_k)^{an}}$ is lower than its generic value.*

Proof. With the notation above, set $G = p_R^*K \otimes \mu^*_{0,R,1}\mathcal{L}_\psi$. To start with, we would like to find some large open subset $U \subset (\mathbb{A}^1_k)^{an}$ over which the following map restricts to an isomorphism:

$$\theta_R : Rq_{R!}G \to Rq_{R*}G.$$

An argument like in lemma 8.3.3 says that the cone of θ_R depends only on the behaviour of the sheaf K on some annulus $\mathbb{D}(R) - \mathbb{D}(r)$, and in particular we can assume that K is the extension by zero of a local system on $(\mathbb{G}_{m,k})^{an}$, by replacing K with the canonical extension of K_{η_∞} (provided by corollary 5.1.21). Then K will even be tamely ramified at 0. Let us introduce the map

$$\psi_N : \mathbb{A}^1_k \to \mathbb{A}^1_k \qquad y \mapsto y^N$$

Clearly θ_{R^N} is a direct summand of the map

$$\xi_R : Rq_{R!}(1 \times \psi_N)^*G \to Rq_{R*}(1 \times \psi_N)^*G.$$

We know that for some N the sheaf ψ_N^*K is unramified at zero, and since by hypothesis it is meromorphically ramified at infinity, we have (by lemma 5.2.8)

$$\psi_N^*K \simeq \bigoplus_j f_j^*\mathcal{L}_\psi$$

for some polynomials f_j. Thus $(1 \times \psi_N)^*G \simeq \bigoplus_j \mu^*_{f_j,R,\psi_N}\mathcal{L}_\psi$ and we can apply lemma 8.3.3 which shows that for all $r > \varepsilon > 0$ we can find R_0 such that for all $R > R_0$ the map ξ_R restricts to an isomorphism on the set $V(r,\varepsilon_1) = \bigcap_j U(f_j, r, \varepsilon_1)$.

This in turns means that also θ_R is an isomorphism over $V(r, \varepsilon_1)$. Now, let $y_1, ..., y_n$ be the ramification points of the sheaf K, contained inside $\mathbb{D}(0,R)$. Let $\mathbb{E}(y_i, \varepsilon_2) \subset \mathbb{D}(0,R)$ be the open disc of radius ε_2 centered at the point y_i. We can choose ε_2 small enough so that $\mu^*_{0,R,y}\mathcal{L}_\psi$ is a geometrically constant sheaf on each of the open subsets $V(r,\varepsilon_1) \times \mathbb{E}(y_i, \varepsilon_2) \subset V(r, \varepsilon_1) \times \mathbb{D}(0,R)$. Let $W(\varepsilon_2, R) = \mathbb{D}(0,R) - \bigcup_i \mathbb{E}(y_i, \varepsilon_2)$. We have the usual projections

$$V(r,\varepsilon_1) \xleftarrow{q_\varepsilon} V(r,\varepsilon_1) \times W(\varepsilon_2, R) \xrightarrow{p_\varepsilon} W(\varepsilon_2, R).$$

By [Hub] theorem 6.2.2, we know that $R^1 q_{\varepsilon !}(p^*_\varepsilon K \otimes \mu^*_{0,R,y}\mathcal{L}_\psi)$ is a constructible sheaf. Since by hypothesis K has only meromorphic ramification, this easily implies that also $R^1 q_! G$ is constructible on $V(\varepsilon_1, r)$. On the other hand, by [Hub] theorem 8.3.5 we know that $R^1 q_* G$ is an overconvergent sheaf, hence by [Hub] lemma 2.7.11, $R^1 q_! G$ is locally constant on $V(r, \varepsilon_1)$.

Finally, letting $r \to \infty$ and $\varepsilon_1 \to 0$ we obtain that $\mathcal{F}^0(K)$ is locally constant outside finitely many points, as stated. By inspection, it is clear that these points are exactly those $x \in (\mathbb{A}^1_k)^{an}$ where the Swan conductor of the sheaf $G_{|\{x\} \times (\mathbb{A}^1_k)^{an}}$ drops from its generic value. □

Remark 8.3.7. Thanks to the remarks above, we see that theorem 8.3.6 also holds true in Berkovich's theory as well. I do not how to prove this theorem without making use of Huber's theory.

8.4. **Stationary phase.** We return to Berkovich's étale cohomology of analytic spaces. In this section we establish our version of the principle of stationary phase.

We apply the constructions of section 4.3 to the pro-analytic space $C(s) = \mathbb{P}^1_k(\infty)$. Let $\mathbf{X}$ denote the pro-analytic space $(\mathbb{A}^1_k \times_k \mathbb{P}^1_k) \times_{\mathbb{P}^1_k} \mathbb{P}^1_k(\infty)$. The sheaf $m^*\mathcal{L}$ induces a sheaf on $\mathbf{X}_{\eta_\infty}$, which we will denote by the same name. Then for each $i \geq 0$ we may form $R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L})$, which is a sheaf on $\mathbf{X}_{\overline{\infty}} = (\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a$.

A bit more generally, suppose that k is the completion of an algebraic extension of a complete subfield k' which contains k_∞. All the varieties and sheaves introduced above are obtained by base change from corresponding objects defined over k', and we can consider the functor $R\widetilde{\Psi}_{\eta_\infty/k'}$.

Proposition 8.4.1. *With reference to the notation above, $R^i\widetilde{\Psi}_{\eta_\infty/k'}(m^*\mathcal{L}) = 0$ for all $i \geq 0$.*

Proof. The proof is basically a variation of the proof of theorem 7.1.6 (with the two affine axes swapped in $\mathbb{A}^1_k \times_k \mathbb{P}^1_k \times_k \mathbb{A}^1_k$). Thanks to proposition 4.3.11, it suffices to consider the case $k = k'$, and hence we need only to study $R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L})$. We will show that the stalk of $R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L})$ vanishes at all points $s \in (\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a$. By definition the stalk $(R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L}))_s$ consists of a direct system of Λ-modules $\{M^i_\rho\}$ indexed by the ordered set of positive real numbers ρ. Hence it suffices to consider a given ρ and show that $M^i_\rho = 0$.

Let $\mathbf{Y} = (\mathbb{A}^1_k \times_k \mathbb{P}^1_k \times \mathbb{A}^1_k) \times_{\mathbb{P}^1_k} \mathbb{P}^1_k(\infty)$. We define a map $\tau : \mathbb{A}^1_k \times_k \mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k \times_k \mathbb{P}^1_k$ by letting $(x, y, z) \mapsto (x + z, y)$. Then τ induces a smooth map of pro-analytic spaces $\mathbf{Y} \to \mathbf{X}$ which we denote again by τ. Proposition 4.3.5 applies and we obtain

$$\tau^*_{\overline{s}}(R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L})) \simeq R^i\widetilde{\Psi}_{\eta_\infty}(\tau^*_{\eta_\infty} m^*\mathcal{L}).$$

In particular

$$(R^i\widetilde{\Psi}_{\eta_\infty}(m^*\mathcal{L}))_s \simeq (R^i\widetilde{\Psi}_{\eta_\infty}(\tau^*_{\eta_\infty} m^*\mathcal{L}))_{(0,s)}. \tag{8.4.2}$$

By (8.4.2) (see also section 4.3 for the notation) and a standard argument we obtain

$$M^i_\rho \simeq \varinjlim_{T \in \mathcal{I}_\rho} \varinjlim_{U_T} H^i(j_T^{-1} U_T, (\tau^* m^*\mathcal{L})_T)$$

where $(\tau^* m^*\mathcal{L})_T$ denotes the restriction of $\tau^* m^*\mathcal{L}$ to $(\mathbb{A}^1_{k_T})^{an} \times_{k_T} t(T) \times_{k_T} (\mathbb{A}^1_{k_T})^{an}$ and U_T ranges on all the étale neighborhoods of $(0, \infty, s)$ inside $(\mathbb{A}^1_{k_T})^{an} \times_{k_T} \mathbb{E}(\infty, \rho)_{k_T} \times_{k_T} (\mathbb{A}^1_{k_T})^{an}$. Let $\mathcal{C}_T$ be the partially ordered set consisting of all such U_T and let $\mathcal{C} = \bigcup_{T \in \mathcal{I}_\rho} \mathcal{C}_T$. We introduce the family $\mathcal{C}^\delta_T$ consisting of all the varieties of the form $B_T \times_{k_T} W_T$ such that

1) B_T is an open disc in $(\mathbb{A}^1_{k_T})^{an}$, of radius r_B and centered at 0, and $W_T \xrightarrow{\phi} (\mathbb{P}^1_{k_T} \times_{k_T} \mathbb{A}^1_{k_T})^{an}$ is an étale neighborhood of $(\infty, s) \in (\mathbb{P}^1_{k_T} \times_{k_T} \mathbb{A}^1_{k_T})^{an}$;

2) the image $\phi(W_T)$ is contained in an open subset of the form $B' \times_{k_T} N(p)$, with B' an open disc of radius r_W around ∞ and $N(s)$ some fixed open neighborhood of s;

3) the ratio r_B/r_W is equal to the constant δ.

Lemma 8.4.3. *For any real number $\delta > 0$ the family $\mathcal{C}^\delta = \bigcup_{T \in \mathcal{I}_\rho} \mathcal{C}^\delta_T$ is cofinal in $\mathcal{C}$.*

Proof. This is of course just a special case (up to swapping the axes) of lemma 7.1.7, with $S = \mathrm{Spec} k_T$. □

Fix a real number δ strictly greater than $\rho(\psi, t)$. Let $B_T \times_{k_T} W_T \in \mathcal{C}_T^\delta$ be any neighborhood as above, and set $\mathbb{W}_T = W_T \times_{(\mathbb{P}^1_{k_T})^{an}} t(T)$. In view of the lemma, the theorem will follow if we show that

$$H^i(B_T \times \mathbb{W}_T, (\tau^* m^* \mathcal{L})_T) = 0 \qquad (i \geq 0). \tag{8.4.4}$$

Let $\mathrm{pr}_{23} : B_T \times \mathbb{W}_T \to \mathbb{W}_T$ be the projection. Define $m' : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ by setting $(y, z) \mapsto yz$. An application of the Yoga of torsors gives us the isomorphism

$$(\tau^* m^* \mathcal{L})_T \simeq \mathrm{pr}_{12}^* \mathcal{L}(m) \otimes \mathrm{pr}_{23}^* \mathcal{L}(m').$$

Now we can proceed exactly as in the proof of theorem 7.1.6 and conclude that $R\mathrm{pr}_{23*}(\tau^* m^* \mathcal{L})_T = 0$, which, by virtue of the Leray spectral sequence for pr_{23}, implies (8.4.4). □

Next, let $\overline{\mathbf{X}}$ be the $\mathbb{P}^1_k(\infty)$-space $(\mathbb{P}^1_k \times_k \mathbb{P}^1_k) \times_{\mathbb{P}^1_k} \mathbb{P}^1_k(\infty) = \mathbb{P}^1_k \times_k \mathbb{P}^1_k(\infty)$ so that there is an embedding of $\mathbb{P}^1_k(\infty)$-spaces $\mathbf{X} \to \overline{\mathbf{X}}$. We have two natural projections

$$(\mathbb{P}^1_k)^{an} \xleftarrow{\overline{p}} \overline{\mathbf{X}} \xrightarrow{\overline{q}} \mathbb{P}^1_k(\infty).$$

Given a k-rational point $s \in \mathbb{P}^1_k$, we will also consider the pro-analytic spaces $\mathbb{P}^1_k(s)$ and η_s. For any sheaf F of Λ-modules on $(\mathbb{A}^1_k)^{an}$ we will let $F(s) = H^0(\overline{\eta}_s, F_{\eta_s})$ which carries a natural structure of $\pi_1(\eta_s, \overline{x})$-module.

For a given sheaf G on $(\mathbb{A}^1_k \times_k \mathbb{A}^1_k)^{an}$ we denote by $\overline{G}$ the extension by zero of G to $(\mathbb{P}^1_k \times_k \mathbb{A}^1_k)^{an}$; then $\overline{G}$ determines a unique sheaf on $\overline{\mathbf{X}}_{\eta_\infty}$. We are interested in studying complexes of the form

$$K^\bullet_F = R\widetilde{\Psi}_{\eta_\infty}(\overline{p^* F} \otimes \overline{m^* \mathcal{L}})$$

where F is a sheaf on $(\mathbb{A}^1_k)^{an}$.

Lemma 8.4.5. *Let F be a sheaf of finitely generated Λ-modules on $(\mathbb{A}^1_k)^{an}$ which is locally constant on the complement of a finite set $S \subset (\mathbb{A}^1_k)^{an}$. Set $U = (\mathbb{A}^1_k)^{an} - S$. Then $K^\bullet_F$ vanishes on $U \times_k \widehat{k}^a$. If, moreover, $S \subset \mathbb{A}^1_k(k^a)$ and F is the extension by zero of $F_{|U}$ then $\mathcal{F}(F)$ is a complex concentrated in degrees 0 and 1, and $\mathcal{F}^1(F)$ is supported on a finite set.*

Proof. Let $\mathbf{Y}$ be a $\mathbb{P}^1_k(\infty)$-analytic space, $j : \mathbf{Y}_{\eta_\infty} \to \mathbf{Y}$ the open imbedding and $i : \mathbf{Y}_{\overline{\infty}} \to \mathbf{Y}$ the imbedding of the special fibre. Let G be a sheaf on $\mathbf{Y}_\eta$ and H a locally constant sheaf on $\mathbf{Y}$. Then one has the standard general formula

$$R\widetilde{\Psi}_{\eta_\infty}(j^* H \otimes G) \simeq i^* H \otimes R\widetilde{\Psi}_{\eta_\infty} G. \tag{8.4.6}$$

Let $\overline{F}$ be the extension by zero of F to $(\mathbb{P}^1_k)^{an}$; clearly $\overline{p}^* \overline{F}$ is locally constant on $U \times_k \mathbf{S}$. Then from proposition 8.4.1 and (8.4.6) we derive

$$K^\bullet_{F|U} \simeq i^* F_{|U} \otimes (R\widetilde{\Psi}_{\eta_\infty}(\overline{m^* \mathcal{L}}))_{|U} = 0$$

which proves the first claim.

Assume now that F is extended by zero from U and $S \subset \mathbb{A}^1_k(k^a)$. By Poincaré duality and proper base change, it is clear that $\mathcal{F}^i(F)$ can be non-zero only for $-1 \leq i \leq 1$. Since F is extended by zero from a locally constant sheaf on U, it is also obvious that $\mathcal{F}^{-1}(F) = 0$.

We show that $\mathcal{F}^1(F)$ is supported on finitely many points. The usual argument (see *e.g.* the proof of lemma 8.2.2) shows that we can assume Λ to be a field. Let $T = \{t_1, ...t_n\}$ be any finite collection of points in $(\mathbb{A}^1_k)^{an}$, with the property that $\mathcal{F}^1(F)_{t_i} \neq 0$ for all $t_i \in T$. Let K be a complete algebraically closed extension of k large enough to contain the residue fields of all the points t_i. Let $\pi : (\mathbb{A}^1_K)^{an} \to (\mathbb{A}^1_k)^{an}$ be the base change morphism. Define $\mu_i : \mathbb{A}^1_K \to \mathbb{A}^1_K$ as $x \mapsto t_i x$. By Poincaré duality we obtain

$$H^0(U_K^{an}, \pi^* \underline{\mathcal{H}om}(F, \Lambda) \otimes \mu_i^* \pi^* \mathcal{L}_{\psi^{-1}}) \neq 0$$

for all t_i. This implies that $\pi^* F$ contains $\oplus_i \mu_i^*(\pi^* \mathcal{L}_\psi)$ as a direct summand. Since F has finitely generated stalks, it follows immediately that the cardinality of T is bounded, *i.e.* $\mathcal{F}^1(F)$ has punctual support. □

Suppose that for a certain point $s \in \mathbb{P}^1_k(k)$ the stalk $\overline{F}_s$ vanishes. The definition of $R\widetilde{\Psi}_{\eta_\infty}$ being purely local, it is clear that the stalk of $K^\bullet_F$ at s only depends on $F_{\eta_s} \in \mathbf{I}(\eta_s, \Lambda)$.

By an argument like in the proof of proposition 4.1.7 we get a fully faithful functor

$$\underline{\mathrm{Rep}}_{cnt}(\pi_1(\eta_s, \overline{x}), \Lambda) \to \mathbf{I}(\eta_s, \Lambda) \qquad V \mapsto \widetilde{V}. \tag{8.4.7}$$

Definition 8.4.8. For any point $s \in \mathbb{P}^1_k(k)$ let $\mathrm{pr}_{\eta_s} : \eta_s \times_k \eta_\infty \to \eta_s$ be the projection on the first factor. The *local Fourier transform* at the point s is the functor

$$\mathcal{F}^{(s,\infty)}_{loc,\psi} : \underline{\mathrm{Rep}}_{cnt}(\pi_1(\eta_s, \overline{x}_s), \Lambda) \to \underline{\mathrm{Rep}}(\pi_1(\eta_\infty, \overline{x}_\infty), \Lambda)$$

which sends a $\pi_1(\eta_s, \overline{x}_s)$-module V to $H^0(\overline{\infty}, R^1\widetilde{\Psi}_{\eta_\infty}(\mathrm{pr}^*_{\eta_s}\widetilde{V} \otimes (m^*\mathcal{L}_\psi)_{|\eta_s\times_k\eta_\infty})$.

Theorem 8.4.9 (Principle of Stationary Phase)**.** *Let F be a sheaf on $(\mathbb{A}^1_k)^{an}$, which is the extension by zero of a locally constant sheaf of finitely generated Λ-modules, defined on the complement of a finite subset $S \subset \mathbb{A}^1_k(k)$ and meromorphically ramified around the points of S. Then there is a canonical equivariant direct sum decomposition*

$$\mathcal{F}^0_\psi(F)(\infty) \simeq \oplus_{s\in S\cup\{\infty\}}\mathcal{F}^{(s,\infty)}_{loc,\psi}(F(s)).$$

Proof. Let $s \in S \cup \{\infty\}$ and define $\phi : \eta_s \times_k \mathbb{P}^1_k(\infty) \to \mathbf{X}$ as the map of $\mathbb{P}^1_k(\infty)$-spaces induced by the obvious imbedding. Notice that ϕ is a smooth morphism. Thus, from proposition 4.3.5 we derive

$$H^1(\phi^*_{\overline{\infty}}K^\bullet_F) \simeq R^1\widetilde{\Psi}_{\eta_\infty}(\phi^*_{\eta_\infty}(\overline{\mathrm{pr}^*F} \otimes \overline{m^*\mathcal{L}_\psi})) \simeq \mathcal{F}^{(s,\infty)}_{loc}(F(s)).$$

It follows from lemma 8.4.5 that, under the stated hypotheses, $\mathcal{F}^1_\psi(F)_{\eta_\infty} = 0$, *i.e.* $\mathcal{F}_\psi(F)_{\eta_\infty}$ reduces to a single ind-sheaf placed in degree zero. Hence the spectral sequence of corollary 4.3.7 gives

$$\mathcal{F}^0_\psi(F)(\infty) = H^0(\overline{\infty}, R^0\widetilde{\Psi}_{\eta_\infty}(\mathcal{F}_\psi(F)_{\eta_\infty})).$$

On the other hand, from proposition 4.3.6 we derive

$$R\widetilde{\Psi}_{\eta_\infty}(\mathcal{F}(F)_{\eta_\infty}) \simeq R\widetilde{\Psi}_{\eta_\infty}R\overline{q}_{\eta_\infty *}(\overline{p^*F} \otimes \overline{m^*\mathcal{L}_\psi})[1] \simeq R\overline{q}_{\overline{\infty}*}(K^\bullet_F)[1].$$

From lemma 8.4.5 we know that the complex $K^\bullet_F$ is concentrated on the set $S\cup\{\infty\}$, therefore $R^i\overline{q}_{\overline{\infty}*}(K^\bullet_F)$ vanishes for $i > 0$ and the claim of the theorem follows. ☐

Corollary 8.4.10. *For all stable ind-representations V of $\mu(\eta_s, \overline{x}_s)$ represented by a finitely generated Λ-module we have $R^i\widetilde{\Psi}_{\eta_\infty}(\mathrm{pr}^*_{\eta_s}\widetilde{V} \otimes (m^*\mathcal{L}_\psi)_{\eta_s\times_k\eta_\infty})$ vanish for $i \neq 1$.*

Proof. It follows by inspection from the proof of theorem 8.4.9, by taking for F the canonical extension of $\widetilde{V}$. ☐

Remark 8.4.11. The formula of theorem 8.4.9 holds for general locally constant sheaves of finitely generated Λ-modules. A proof valid in this generality was given in [Ra3].

To conclude this section we propose to show how our local Fourier transforms honour their name with a behaviour which, in many ways, mimicks that of their namesakes introduced by Laumon.

Proposition 8.4.12. *For any $s \in \mathbb{P}^1_k(k)$ let $\mathcal{M}(s)$ be the category of stable continuous representations of $\mu(\eta_s, \overline{x}_s)$ into free Λ-modules of finite rank. Then $\mathcal{F}^{(s,\infty)}_{\psi,loc}$ restricts to an exact functor from $\mathcal{M}(s)$ to the category of stable objects of $\underline{\mathrm{Rep}}_{cnt}(\pi_1(\eta_\infty, \overline{x}_\infty), \Lambda)$ which are represented by free Λ-modules of finite rank. Moreover $\mathcal{F}^{(s,\infty)}_{\psi,loc}$ commutes with extension of scalars $\Lambda \to \Lambda'$.*

Proof. Exactness is clear from corollary 8.4.10. Then the rest follows formally as in the proof of lemma 8.2.1. ☐

Lemma 8.4.13. *1) Let $V \in \mathcal{M}(s)$ be unramified,* i.e. *suppose that the $\mu(\eta_s, \overline{x}_s)$-action on V factors through the quotient* $\mathrm{Gal}(k^a/k)$*. Then*

$$\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(V) = 0.$$

2) If we denote by Λ the trivial representation of rank one, then

$$\mathcal{F}^{(0,\infty)}_{\psi,loc}(\Lambda) = \Lambda.$$

Proof. For (1), we observe that

$$\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(V) \simeq \mathcal{F}^{(\infty,\infty)}_{\psi,loc}(\Lambda) \otimes V$$

which allows us to reduce to the case $V = \Lambda$; from lemma 7.1.4 we derive $\mathcal{F}_\psi(\mathbb{O}_n)(\infty) = 0$ and the claim follows from theorem 8.4.9. Part (2) is dealt with in a similar way, by considering the (global) Fourier transform of the extension by zero of the trivial sheaf $\Lambda_{\mathbb{G}_m}$, and using theorem 8.4.9 to analyse the local contributions at infinity. ☐

For $s \in \mathbb{G}_{m,k}(k)$, let $\mu_s : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the map $x \mapsto sx$ and set $L(s) = (\mu_s^* \mathcal{L}_\psi)_{\eta_\infty}$. This Λ-module is a rank one object of $\mathcal{M}(s)$ of Swan conductor one. The translation map $\tau_s : \mathbb{A}^1_k \to \mathbb{A}^1_k$ defined by $x \mapsto x+s$ induces a morphism $\eta_0 \to \eta_s$ and hence a group homomorphism

$$\tau_{s*} : \mu(\eta_0, \overline{x}_0) \to \mu(\eta_s, \overline{x}_s)$$

as well as a functor

$$\tau_s^* : \mathcal{M}(s) \to \mathcal{M}(0).$$

Proposition 8.4.14. *(1) If $V \in \mathcal{M}(\infty)$ is a tame representation, then $\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(V) = 0$.*

(2) If $V \in \mathcal{M}(s)$ and $s \in \mathbb{A}^1_k(k)$ then

$$\mathcal{F}^{(s,\infty)}_{\psi,loc}(V) \simeq \mathcal{F}^{(0,\infty)}_{\psi,loc}(\tau_s^* V) \otimes L(-s).$$

Proof. For the proof of (1), thanks to proposition 4.3.11 we can base change everything to $\widehat{k}^a$, at the cost of replacing everywhere the vanishing cycle functor with its generalization $R\widetilde{\Psi}_{\eta_\infty/k}$. We leave to the reader to state the obvious variant of the principle of stationary phase for the more general functor. Basically all the statements remain formally unchanged. Therefore we can assume that $k = \widehat{k}^a$, in which case

$$\pi_1^{alg}(\eta_\infty, \overline{\eta}_\infty) \simeq \pi_1^{alg}(\mathbb{G}_{m,k}, \overline{x}) \simeq \widehat{\mathbb{Z}}(1).$$

Next, by proposition 8.4.12 we can assume that Λ is a field and V is absolutely irreducible, hence of rank one. The canonical extension of V is therefore a rank one sheaf of Kummer type $\mathcal{K}_\chi$ on $(\mathbb{G}_{m,k})^{an}$ extended by zero to $(\mathbb{A}^1_k)^{an}$. The case of a trivial character has already been taken care of in lemma 8.4.13. So we assume that χ is a non-trivial character. Now, let $\mathbb{E}$ be an open disc in $(\mathbb{A}^1_k)^{an}$, centered at 0. Denote by $\mathcal{K}'_\chi$ the extension by zero of $j_*\mathcal{K}_{\chi|\mathbb{E}}$. Then $\mathcal{K}'_\chi$ imbeds in $j_*\mathcal{K}_\chi$ and there is a short exact sequence

$$0 \to \mathcal{K}'_\chi \to j_*\mathcal{K}_\chi \to \mathcal{K}''_\chi \to 0.$$

An argument as in the proof of lemma 8.4.5 shows that $\mathcal{F}(\mathcal{K}'_\chi)$ is a complex concentrated in degree zero, and hence we obtain a short exact sequence:

$$0 \to \mathcal{F}^0(\mathcal{K}'_\chi) \to \mathcal{F}^0(j_*\mathcal{K}_\chi) \to \mathcal{F}^0(\mathcal{K}''_\chi) \to 0.$$

Let $s \in \mathbb{G}_{m,k}(k)$ be any point. It is easy to check that $\mu_s^*\mathcal{K}'_\chi$ is isomorphic to the extension by zero of $j_*\mathcal{K}_{\chi|\mu_s^{-1}(\mathbb{E})}$. It follows:

$$H^i_c((\mathbb{A}^1_k)^{an}, \mathcal{K}'_\chi \otimes \mu_s^*\mathcal{L}_\psi) \simeq H^i_c(\mu_s^{-1}(\mathbb{E}), j_*\mathcal{K}_\chi \otimes \mathcal{L}_\psi).$$

From proposition 5.2.9 of [B1] we know that

$$H^i_c((\mathbb{A}^1_k)^{an}, \mathcal{K}_\chi \otimes \mathcal{L}_\psi) \simeq \lim_{|s|\to 0} H^i_c(\mu_s^{-1}(\mathbb{E}), \mathcal{K}_\chi \otimes \mathcal{L}_\psi).$$

Proposition 8.1.3(1) says that the left-hand side of this equation has rank one, therefore the limit is already attained for some value $|s_0|$. This means that on the complement $U = (\mathbb{A}^1_k)^{an} - \mu_{s_0}^{-1}(\mathbb{E})$ we have $\mathcal{F}^0(\mathcal{K}'_\chi)_{|U} \simeq \mathcal{F}^0(\mathcal{K}_\chi)_{|U}$, and therefore $\mathcal{F}^0(\mathcal{K}''_\chi)_{|U} = 0$; in particular $\mathcal{F}^0(\mathcal{K}''_\chi)_{\eta_\infty} = 0$. Next, notice that the sheaf $\mathcal{K}''_\chi$ is locally constant on the complement of a single point p (of type (2) in the notation of [B1], paragraph 3.6) in $(\mathbb{A}^1_k)^{an}$, namely the point corresponding to the sup-norm on the disc $\mathbb{E}$ (see [B1], remark 6.3.4). Therefore lemma 8.4.5 applies, and shows that $K^\bullet_{\mathcal{K}''_\chi}$ is concentrated on $\{p, \infty\}$. It is also clear that the stalk of $K^1_{\mathcal{K}''_\chi}$ over ∞ is isomorphic to the stalk of $K^1_{\mathcal{K}_\chi}$ over the same point. Now, the same argument which was used in the proof of theorem 8.4.9 shows that $\mathcal{F}^0(\mathcal{K}''_\chi)(\infty) \simeq H^0((\mathbb{P}^1_k)^{an}, K^1_{\mathcal{K}''_\chi})$. This implies $K^1_{\mathcal{K}''_\chi} = 0$. It follows that also the stalk of $K^1_{\mathcal{K}_\chi}$ vanishes over ∞, and therefore $\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(\mathcal{K}_{\chi,\eta_\infty})$ vanishes, as stated.

For (2), let $\tau_s^*\mathcal{V}$ be the global extension of $\tau_s^* V$, as provided by lemma 5.1.21. According to part (1) and theorem 8.4.9, the only contribution to $\mathcal{F}^0_\psi(\mathcal{V})(\infty)$ (resp. $\mathcal{F}^0_\psi(\tau_s^*\mathcal{V})(\infty)$) comes from $\mathcal{F}^{(s,\infty)}_{\psi,loc}(\mathcal{V}(s))$ (resp. $\mathcal{F}^{(0,\infty)}_{\psi,loc}(\tau_s^*\mathcal{V}(0))$). Proposition 7.2.2 allows to compare the two terms and yields the claim. $\square$

Proposition 8.4.14 says that it suffices to study the functors $\mathcal{F}^{(s,\infty)}_{\psi,loc}$ for the values $s = 0$ and $s = \infty$ to know all of them.

Remark 8.4.15. If we take the formal multiplicative group $\mathbb{G}_m$ as the underlying Lubin-Tate group, then the theory above can be refined by using the constructions of section 6.3. Suppose that a sheaf F is defined over (the completion of) any algebraic extension k_0 of $\mathbb{Q}_p$. In this case the principle of stationary phase gives a canonical decomposition of the *semilinear* $\pi_1(\eta_\infty, \overline{x}_\infty)$-representation which describes the asymptotic behaviour of $\mathcal{F}_\psi(F)$, in terms of local contributions. In particular the local Fourier transforms land in the category of these semilinear representations.

8.5. **Behaviour at the origin.** The setup for this section is as in the previous one: we consider a sheaf K of free Λ-modules of finite rank on $\mathbb{A}^1$, which is the extension by zero of a meromorphic local system on the complement of finitely many k-rational points. We want to understand the behaviour of $\mathcal{F}^0(K)$ around the origin $0 \in (\mathbb{A}^1_k)^{an}$, *i.e.* we want to study the cone $R\widetilde{\Phi}_{\eta_0}(K)$ of the natural morphism

$$j_!K \to R\widetilde{\Psi}_{\eta_0}(\overline{p}^*(j_!K) \otimes m^*\mathcal{L}_\psi)$$

where $\overline{p} : \mathbb{P}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the projection and $j : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ the obvious imbedding. This is a complex concentrated on $\mathbb{P}^1_k = (\mathbb{P}^1_k \times_k \mathbb{A}^1_k) \times_{\mathbb{A}^1} \{\overline{0}\}$ and we remark that

$$R\overline{q}_{\overline{0}*}R\widetilde{\Psi}_{\eta_0}(\overline{p}^*(j_!K) \otimes m^*\mathcal{L}_\psi) \simeq R\Gamma(\overline{\eta}_0, \mathcal{F}(K)_{\eta_0}[-1]) \tag{8.5.1}$$

where as usual $\mathcal{F}(K)_{\eta_0}$ stays for the stable ind-complex associated to $\mathcal{F}(K)$.

Proposition 8.5.2. *With the notation above, $R^i\widetilde{\Phi}_{\eta_0}(K)$ is a skyscraper sheaf concentrated at the point $\overline{\infty} \in (\mathbb{P}^1_k)^{an} \times_k \widehat{k}^a$ and vanishing in degrees $i \neq 1$.*

Proof. Let $x_1, .., x_n \in \mathbb{A}^1_k$ be the finitely many points where $K_{x_i} = 0$. It follows from corollary 4.3.9 that $R\widetilde{\Phi}_{\eta_0}(K)$ vanishes outside $\{x_1, ...x_n, \infty\}$. For any $\varepsilon > 0$ let $\mathbb{E}^*(0, \varepsilon) = \mathbb{E}(0, \varepsilon) - \{0\}$. We recall the standard formula

$$R^i\widetilde{\Psi}_{\eta_0}(\overline{p}^*(j_!K) \otimes m^*\mathcal{L}_\psi)_{x_i} \simeq \lim_{\overrightarrow{U}} H^i(U \times_{\mathbb{A}^1 \times \mathbb{A}^1} (\mathbb{A}^1 \times \mathbb{G}_m), p^*K \otimes m^*\mathcal{L}_\psi)$$

where U ranges over a certain projective system of locally algebraic neighborhoods of $\mathbb{E}(x_i, \varepsilon) \times \mathbb{E}^*(0, \varepsilon)$. If U is sufficiently small, $m^*\mathcal{L}_\psi$ is constant on U, hence we reduce to studying the cohomology group

$$H^i(\mathbb{E}(x_i, \varepsilon) \times V, p^*K)$$

for V ranging over the system of locally algebraic coverings of $\mathbb{E}^*(0, \varepsilon)$. By smooth base change, we need only to understand $H^i(\mathbb{E}(x_i, \varepsilon) \times_k \widehat{k}^a, K)$. Since by hypothesis K has only meromorphic ramification, this group vanishes when $\varepsilon \to 0$. This shows that $R\widetilde{\Phi}_{\eta_0}(K)$ is concentrated at $\overline{\infty}$.

It is also easy to check that $R^0\Phi_{\eta_0}(K)_{\overline{\infty}} = 0$. For degrees > 1, we observe that $\mathcal{F}(K)_{\eta_0}$ is locally constant and concentrated in degree 0, so that we can rewrite (8.5.1) as

$$R^i\widetilde{\Psi}(K)_{\overline{\infty}} \simeq R^i\Gamma(\overline{\eta}_0, \mathcal{F}(K)_{\eta_0}[-1]) \simeq H^{i-1}(\overline{\eta}_0, \mathcal{F}^0(K)_{\eta_0}). \tag{8.5.3}$$

But according to lemma 4.3.10 the rightmost term in (8.5.3) vanishes for $i - 1 > 0$, hence $R^i\Phi_{\eta_0}(K) = 0$ for $i > 1$, as stated. $\square$

Definition 8.5.4. The functor (see (8.4.7) for the notation)

$$\mathcal{F}^{(\infty,0)}_{\psi,loc} : \underline{\mathrm{Rep}}_{cnt}(\pi_1(\eta_\infty, \overline{x}_\infty), \Lambda) \to \underline{\mathrm{Rep}}(\pi_1(\eta_0, \overline{x}_0), \Lambda)$$

sends a $\pi_1(\eta_\infty, \overline{x}_\infty)$-module V to the $\pi_1(\eta_0, \overline{x}_0)$-module $H^0(\overline{0}, R^1\widetilde{\Psi}_{\eta_0}(\overline{p}^*_{\eta_\infty}(\widetilde{V}) \otimes (m^*\mathcal{L}_\psi)_{|\eta_\infty \times \eta_0}))$.

Theorem 8.5.5. *The local Fourier transform $\mathcal{F}^{(\infty,0)}_{\psi,loc}$ is an exact functor and for every meromorphic sheaf K as in proposition 8.5.2 there is a four term exact sequence of stable $\pi_1(\eta_0, \overline{x}_0)$-ind-representations*

$$0 \to H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, K) \to \mathcal{F}^0(K)(0) \to \mathcal{F}^{(\infty,0)}_{\psi,loc}(K(\infty)) \to H^2_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, K) \to 0.$$

Proof. Since we know that $\mathcal{F}(K)_{\eta_0}$ is concentrated in degree 0, the exact sequence above follows from proposition 8.5.2 and the long exact sequence for vanishing cycles on proper varieties.

To show that $\mathcal{F}^{(\infty,0)}_{\psi,loc}$ is exact, we use the canonical extension functor to reduce to a global question, and then we apply proposition 8.5.2. $\square$

Lemma 8.5.6. *With the notation above, suppose that V is unramified. Then we have*

$$\mathcal{F}^{(\infty,0)}_{\psi,loc}(V) \simeq V(-1)$$

(as usual (-1) denotes Tate twist).

Proof. It suffices to compute the global Fourier transform of the constant sheaf on $(\mathbb{A}^1_k)^{an}$ with stalk isomorphic to V, and apply the exact sequence of theorem 8.5.5. $\square$

Proposition 8.5.7. *(1) $\mathcal{F}^{(\infty,0)}_{\psi,loc}$ restricts to a functor from $\mathcal{M}(\infty)$ to the category of stable objects of $\underline{\mathrm{Rep}}_{cnt}(\pi_1(\eta_0,\overline{x}_0),\Lambda)$ which are represented by free Λ-modules of finite rank. Moreover $\mathcal{F}^{(\infty,0)}_{\psi,loc}$ commutes with extension of scalars $\Lambda \to \Lambda'$.*

(2) If V is a $\mu(\eta_\infty,\overline{x}_\infty)$-representation with all slopes ≥ 1 then $\mathcal{F}^{(\infty,0)}_{\psi,loc}(V) = 0$.

(3) If V is a tame representation, then $\mathrm{rk}(\mathcal{F}^{(\infty,0)}_{\psi,loc}(V)) = \mathrm{rk}(V)$.

Proof. (1) follows formally from the exactness of $\mathcal{F}^{(\infty,0)}_{\psi,loc}$, as in the proof of lemma 8.2.1(3).

To show (2), let K be the canonical extension of N; it follows from theorem 8.3.6 that $\mathcal{F}^0(K)$ is locally constant around the point $0 \in (\mathbb{A}^1_k)^{an}$, so the result is immediate from the exact sequence of theorem 8.5.5.

For (3) one reduces to the case $k = \widehat{k}^a$ using proposition 4.3.11. Thanks to (1) we can also assume that Λ is a field and that V is absolutely irreducible, hence of rank one. Then the claim follows easily from proposition 8.1.3(2) and the exact sequence of theorem 8.5.5 above. □

Remark 8.5.8. It is also true that $\mathcal{F}^{(\infty,0)}_{\psi,loc}(V)$ is tame if V is, but the proof is more delicate than in the "classical" case. We leave this for later.

Lemma 8.5.9. *Let f be some polynomial in one variable and $\mathcal{K}_\chi$ some tame locally constant sheaf of free finitely generated Λ-modules on $\mathbb{G}_m$. Then $H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(f) \otimes \mathcal{K}_\chi)$ is a free Λ-module of rank equal to* $\deg(f) \cdot \mathrm{rk}(\mathcal{K}_\chi)$.

Proof. Let $n = \deg(f)$ and $f_0(y) = a_n y^n$ so that $f(y) = f_0(y) + f_1(y)$ where $f_1(y)$ has degree $< n$. We consider the morphism

$$\mu_{f_0,f_1} : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k \quad : \qquad (x,y) \mapsto f_0(y) + x \cdot f_1(y).$$

Set $K = R^1 q_!(p^*\mathcal{K}_\chi \otimes \mu^*_{f_0,f_1}\mathcal{L}_\psi)$. Clearly $H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(f) \otimes \mathcal{K}_\chi) \simeq K_{\{1\}}$. On the other hand $H^1_c((\mathbb{G}_m)^{an} \times_k \widehat{k}^a, \mathcal{K}_\chi \otimes \mathcal{L}_\psi(f_0)) \simeq K_{\{0\}}$ and we know that this cohomology group is a free Λ-module with the predicted rank, thanks to lemma 8.3.1. Hence it suffices to show that the sheaf K is locally constant on $(\mathbb{A}^1_k)^{an}$. To this purpose we apply lemma 8.3.3 to the case $f = f_0$ and $g = f_1$ and we argue using Huber's theorems, as in the proof of theorem 8.3.6. □

Theorem 8.5.10. *Let (V,ρ) be a representation of $\mu(\eta_\infty,\overline{x}_\infty)$ in a free Λ-module V of finite rank, with all slopes < 1. Then $\mathcal{F}^{(\infty,0)}_{loc,\psi}(V)$ is a free Λ-module of rank*

$$\mathrm{rk}(\mathcal{F}^{(\infty,0)}_{loc,\psi}(V)) = \mathrm{rk}(V) - sw(V).$$

Proof. We can assume that V is irreducible of rank N. and that Λ is large enough so that, by virtue of theorem 5.2.16 we can write $V = \mathrm{Ind}^{S_1}_{S_N}(M)$ where M has rank one, hence is of the form $M = \mathcal{K}_\chi \otimes \mathcal{L}_\psi(f)$ for some tame character χ and some polynomial f of degree equal to $sw(V)$. The canonical extensions $\mathcal{V}$ and $\mathcal{M}$ of V and M are locally constant sheaves on $\mathbb{G}_m$ related by $\phi_{N*}\mathcal{M} \simeq \mathcal{V}$. Let $j : \mathbb{G}_m \to \mathbb{A}^1$ be the imbedding. Then we have

$$\mathcal{F}^0_\psi(j_!\mathcal{V})_0 \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, j_!\mathcal{V}) \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, j_!\mathcal{M})$$

hence by lemma 8.5.9, $\mathcal{F}^0_\psi(j_!\mathcal{V})_0$ is free of rank equal to $sw(V)$. On the other hand, for the stalk at the point $1 \in \mathbb{A}^1$ we have

$$\mathcal{F}^0_\psi(j_!\mathcal{V})_{\{1\}} \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a j_!\mathcal{V} \otimes \mathcal{L}_\psi) \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, j_!\phi_{N*}(\mathcal{M} \otimes \phi^*_N\mathcal{L}_\psi))$$

and again lemma 8.5.9 says that $\mathcal{F}^0_\psi(j_!\mathcal{V})_{\{1\}}$ is free of rank $\mathrm{rk}(V)$. But we know that $\mathcal{F}^0_\psi(j_!\mathcal{V})$ is locally constant on $(\mathbb{G}_{m,k})^{an}$, thus the claim follows immediately from theorem 8.5.5. □

8.6. **The formula of Grothendieck-Ogg-Shafarevich.** The main result of this section is the étale analytic analogue of the formula of Grothendieck-Ogg-Shafarevich which computes the Euler-Poincaré characteristic of a meromorphically ramified sheaf on a curve in terms of Swan conductors. For the proof we reduce first to the case where the curve is the affine line, and then we apply the principle of stationary phase to analyze the situation. To this purpose we must gather some preliminary information on the local Fourier transforms of representations of $\mu(\eta_s,\overline{x})$.

Theorem 8.6.1. *For any representation (V,ρ) of $\mu(\eta_0,\overline{x})$ into a free Λ-module V of finite rank, we have*

$$\mathrm{rk}(\mathcal{F}^{(0,\infty)}_{\psi,loc}(V)) = sw(V) + \mathrm{rk}(V).$$

Proof. We can assume that V is irreducible. Let $\mathcal{V}$ be the canonical extension of V, extended by zero to $(\mathbb{A}^1_k)^{an}$, so that $\mathcal{V}_{\eta_\infty}$ is tamely ramified. By proposition 8.4.14 we have $\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(\mathcal{V}_{\eta_\infty}) = 0$. Hence by the principle of stationary phase we obtain

$$\mathcal{F}^0_\psi(\mathcal{V})(\infty) \simeq \mathcal{F}^{(0,\infty)}_\psi(V).$$

By theorem 8.5.5 we have

$$\mathrm{rk}(\mathcal{F}^0_\psi(\mathcal{V})(\infty)) = \mathrm{rk}(H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{V})) + \mathrm{rk}(\mathcal{F}^{(\infty,0)}_{\psi,loc}(\mathcal{V}(\infty)))$$

and from proposition 8.5.7 it follows that $\mathrm{rk}(\mathcal{F}^{(\infty,0)}_{\psi,loc}(\mathcal{V}(\infty))) = \mathrm{rk}(V)$. An argument like in the proof of theorem 8.5.10 shows that $\mathrm{rk}(H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{V})) = sw(V)$, which implies the claim. □

Let C be a smooth affine curve with completion $\overline{C}$. We consider locally constant sheaves F of free Λ-modules on C which are meromorphically ramified around the finitely many points $\{x_1, ..., x_n\} = \overline{C} - C$. The Euler characteristic of the constant sheaf Λ_C is also called the Euler characteristic of C and denoted simply by $\chi_c(C)$. For any point $s \in \overline{C}$ we obtain a finite rank representation of $\mu(\eta_s, \overline{x}_s)$ in the free Λ-module $F(s)$ whose Swan conductor we denote by $sw(F(s))$.

Theorem 8.6.2 (Grothendieck-Ogg-Shafarevich). *With the notation above we have the equality*

$$\chi_c(C, F) = \mathrm{rk}(F)\chi_c(C) - \sum_{i=1}^n sw(F(x_i)). \tag{8.6.3}$$

Proof. By proposition 8.2.5 we can assume that Λ is a field, hence we can compute $\chi_c(C, F)$ as the literal alternating sum of the ranks of the $H^i_c(C^{an} \times_k \widehat{k}^a, F)$. Let U be any dense open subscheme of C. One checks easily that the formula holds for $\chi_c(U, F_{|U})$ if and only if it holds for $\chi_c(C, F)$. Hence we can remove any finite number of closed points whenever we wish to. Pick a finite morphism $f : \overline{C} \to \mathbb{P}^1_k$ sufficiently general, so that f is étale around the points $x_i, ..., x_n$. Let U be the open subscheme of C obtained by removing all the fibres of f which either intersect the branch locus of f or contain one or more of the points x_i. Let $V = f(U) \subset \mathbb{P}^1_k$ and set $\{z_1, ..., z_r\} = \overline{C} - U$. Then $f : U \to V$ is a finite étale morphism and we consider $G = f_*F$. Clearly G is locally constant on V^{an} and meromorphically ramified at the points $\{y_1, ..., y_m\} = \mathbb{P}^1_k - V$. By construction, we see easily that

$$\sum_{i=1}^r sw(F(z_i)) = \sum_{i=1}^m sw(f_*F(y_i)). \tag{8.6.4}$$

We are going to show that the equality (8.6.3) for $\chi_c(V, f_*F)$ implies the same equality for $\chi_c(C, F)$. By the remarks above we can instead consider $\chi_c(U, F_{|U})$. Then we have

$$\begin{aligned} \chi_c(U) &= \chi_c(\overline{C}) - r \\ \chi_c(V) &= \chi_c(\mathbb{P}^1_k) - m. \end{aligned} \tag{8.6.5}$$

From Hurwitz formula, we derive the relation

$$\deg(f) \cdot m - r = \deg(f) \cdot \chi_c(\mathbb{P}^1_k) - \chi_c(\overline{C}). \tag{8.6.6}$$

Taking into account that $\mathrm{rk}(G) = \mathrm{rk}(F) \cdot \deg(f)$, the formula for $\chi_c(U, F_{|U})$ follows by combining (8.6.4), (8.6.5) and (8.6.6)

It remains to prove (8.6.3) for the case when Λ is a field and $j : C \hookrightarrow \mathbb{A}^1_k$ is an open imbedding. We can even assume that $F(\infty)$ is unramified. Moreover, we can assume that F is geometrically irreducible, so that either F is a geometrically constant sheaf or $H^0(C^{an} \times_k \widehat{k}^a, F) = 0$.

Formula (8.6.3) is trivial for a geometrically costant F, hence we reduce to the case $H^2_c(C^{an} \times_k \widehat{k}^a, F) = 0 = H^0_c(C^{an} \times_k \widehat{k}^a, F)$ and we have to show that

$$\mathrm{rk}(H^1_c(C^{an} \times_k \widehat{k}^a, F)) = \sum_{i=1}^n sw(F(x_i)) - \mathrm{rk}(F)\chi_c(C).$$

From theorem 8.5.5 and lemma 8.5.6 we derive

$$\mathrm{rk}(H^1_c(C^{an} \times_k \widehat{k}^a, F)) = \mathrm{rk}(\mathcal{F}^0(j_!F)(0)) - \mathrm{rk}(F).$$

On the other hand, from theorem 8.3.6 we obtain

$$\mathrm{rk}(\mathcal{F}^0(j_!F)(0)) = \mathrm{rk}(\mathcal{F}^0(j_!F)(\infty))$$

and the principle of stationary phase can be applied to compute the rank of the right-hand side. We leave it to the reader to verify that formula 8.6.3 follows by combining lemma 8.4.13 and theorem 8.6.1. □

Remark 8.6.7. One may wonder whether the condition on the ramification on F is really necessary for the finiteness of the cohomology. We will not attempt here a precise analysis, but we give an example to demonstrate the general situation.

We construct inductively a sequence of polynomials in one variable $f_i(t)$ $(i = 1, 2, ...)$ and positive real numbers $r_1 < r_2 < ...$ such that $\lim_{i\to\infty} r_i = \infty$ and $\lim_{i\to\infty} f_i = f$ is an entire power series on the affine line $(\mathbb{A}^1_k)^{an}$. Suppose f_i of degree i and r_i have already been constructed, with the property that $H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}(f_i)) = H^1_c(\mathbb{D}(0, r_i) \times_k \widehat{k}^a, \mathcal{L}(f_i))$ is a free Λ-module of rank equal to $i-1$. Choose an element $\delta \in k^\times$ of norm small enough so that $|\delta| \cdot r_i < \rho_1$. Set $f_{i+1}(t) = (1 + \delta t) f_i(t)$. Then it is clear that

$$\mathcal{L}(f_{i+1})_{|\mathbb{D}(0,r_i)} \simeq \mathcal{L}(f_i)_{|\mathbb{D}(0,r_i)}$$

and as a consequence we get an imbedding

$$H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}(f_i)) \hookrightarrow H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}(f_{i+1})).$$

On the other hand, the polynomial $f_{i+1}(t)$ has degree $i+1$, hence by the usual argument (and by lemma 8.5.9) we find $r_{i+1} > 0$ such that $H^1_c(\mathbb{D}(0, r_{i+1}) \times_k \widehat{k}^a, \mathcal{L}(f_{i+1}))$ is free of rank i. Clearly the sequence $f_1(t), f_2(t), ...$ converges to some $f(t)$ and the cohomology of $\mathcal{L}(f)$ cannot be finitely generated.

8.7. **Special calculations.** As we saw above (theorem 8.3.6) the Fourier transform $\mathcal{F}(K)$ of a sheaf K with meromorphic ramification on the affine line, is a constructible sheaf. However, we do not know at present, whether $\mathcal{F}(K)$ has again meromorphic ramification. This is clearly the first important open question in the theory, and can be translated into a problem concerning the local Fourier transforms of a meromorphic representation. I would not be overly surprised, if it turned out that the answer to this question is not always affirmative. For this reason, I will refrain from stating a precise hypothesis, and will place the whole issue under the general heading of "infomal conjectures".

However, in this section we offer some pretty calculations, adapted from unpublished notes of Katz, which could be interpreted as lending a modest support to our informal conjecture.

We start with some preparation. Let X be an affine smooth scheme over k, purely of dimension $d+1$; let C be an open algebraic smooth curve, geometrically connected and defined over k. Suppose that $f : X \to C$ is a smooth affine morphism of relative dimension d. Now, let F be an analytic étale sheaf of Λ-modules over X^{an}. Notice that X^{an} is σ-compact, hence we can find an exhaustive sequence of subspaces $\bigcup_{n\in\mathbb{N}} X_n = X^{an}$ as in definition 4.1.1.

Definition 8.7.1. With the notation above, we say that F is *locally algebraic* over X^{an} if for all integers $n \in \mathbb{N}$ there exists some algebraic constructible sheaf F_n on X such $F_{|X_n} \simeq (F_n)^{an}_{|X_n}$.

We remark that, for any k^a-rational point $s \in C$, the natural action of $\pi_1(\eta_s, \overline{x})$ on $R^d f_!(F)(s)$ restricts, via (4.3.2) to an action of the group $I_s = \pi_1(\eta_{\overline{s}}, \overline{x})$.

Lemma 8.7.2. *Suppose that F is a locally algebraic and locally constant sheaf of Λ-modules on X^{an}. Suppose also that $G = R^d f_!(F)$ is a constructible sheaf on C^{an} (*i.e., *it is locally constant of finite rank outside finitely many k^a-rational points). Then we have an injection*

$$G_{\overline{s}} \hookrightarrow G(s)^{I_s}$$

for any k^a-rational point $s \in C$.

Proof. It suffices to show that $H^0_c(C^{an}, G) = 0$.

Claim 8.7.3. $R^i f_!(F) = 0$ for all $i < d$.

Proof of the claim: For this it suffices to check on the stalks. By Poincaré duality, we are reduced to show that $H^i(f^{-1}(x), F) = 0$ for all $i > d$ and any geometric point x of C^{an}. For all integers $n \in \mathbb{N}$ we let $Y_n = X_n \cap f^{-1}(x)$. Then, by [B3] corollary 5.5, all the groups $H^i(Y_n, F)$ are finite Λ-modules, therefore by Mittag-Leffler, we are further reduced to prove that $H^i(Y_n, F) = 0$ for all $n \in \mathbb{N}$ and $i > d$. But since by hypothesis, F is locally algebraic, this follows from [B6] theorem 6.1.

From the claim and the Leray spectral sequence for f it follows easily that $H^0_c(C^{an}, R^d f_! F) = H^d_c(X^{an}, F)$. But again, dualizing and applying [B6] theorem 6.1 we obtain that $H^d_c(X^{an}, F)$ vanishes and the claim follows. $\square$

For our applications, we will need a slight twist of lemma 8.7.2, *i.e.* we want to add a group action to the picture. So, let $\overline{T}$ a smooth affine curve with an action $\sigma : G \to \mathrm{Aut}(X)$ of a finite group G, and suppose that there is a point $\infty \in \overline{T}$ which is fixed by G and such that the G-action is free on the complement $T = \overline{T} - \{\infty\}$. Set $C = T/G$ and let $f : X \to C$ be a morphism which satisfies the hypothesis of lemma 8.7.2. Let $f_T : X \times_C T \to T$ be the base change of f and let $g \in G$ act on $X \times_C T$ by the morphism $1_X \times_C \sigma(g)$. This action makes f_T a G-equivariant morphism. Finally, let $j : X \times_C T \to \overline{Y}$ be an equivariant morphism into a G-variety Y, such that there exists a smooth morphism $\overline{f}_T$ fitting into a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times_C T \arrow[l, "\pi"'] \arrow[r, hook, "j"] \arrow[d, "f_T"'] & \overline{Y} \arrow[d, "\overline{f}_T"] \\ C & T \arrow[l, "\pi_T"'] \arrow[r, hook] & \overline{T}. \end{tikzcd} \tag{8.7.4}$$

Now, suppose that F is an étale sheaf on X^{an} which satisfies the conditions of lemma 8.7.2. Then $\pi^*(F)$ is a G-sheaf on $X \times_C T$. Suppose that we can find a G-sheaf $\overline{F}$ on $\overline{Y}$ such that $j^*(\overline{F}) \simeq \pi^*(F)$. In this situation, lemma 8.7.2 yields a G-equivariant imbedding

$$(R^d \overline{f}_{T!} \overline{F})_\infty \hookrightarrow (R^d f_{T!}(\pi^* F))(\infty)^{I_\infty}. \tag{8.7.5}$$

Let x be a coordinate on $\mathbb{A}^1_k$ and let ζ be the dual coordinate on $(\mathbb{A}^1_k)' \simeq \mathbb{A}^1_k$. Let $K = k((1/\zeta))$ be the completion of the field of fractions of the local ring $\mathcal{O}_{\mathbb{P}^1_k,\infty}$. Let $\mathbb{E}(\infty)$ a disc of small radius ρ in $(\mathbb{P}^1_k)^{an}$ centered at ∞, and let $\mathbb{E}^*(\infty) = \mathbb{E}(\infty) - \{\infty\}$. Let also K_N be some finite extension of K of degree N. The imbedding $K \subset K_N$ corresponds to a finite analytic morphism $\psi_N : \mathbb{E}^*(\infty) \to \mathbb{E}^*(\infty)$ of degree N (for all radiuses ρ sufficiently small). If k is algebraically closed, any such K_N is isomorphic to a field $k((1/\tau))$ where $\tau^N = \zeta$. By lemma 5.1.1 the map ψ_N induces an injection $\psi_{N*} : \pi_1(\eta_\infty, \overline{x}) \hookrightarrow \pi_1(\eta_\infty, \overline{x})$. Hence, for any representation V of $\pi_1(\eta_\infty, \overline{x})$ we can consider the induced representation from the subgroup $\mathrm{Im}(\psi_{N*})$ to the whole group. We denote by

$$\mathrm{Ind}_K^{K_N}(V)$$

this induced representation. Clearly, if V is meromorphic (*i.e.* factors through the meromorphic quotient $\mu(\eta_\infty, \overline{x})$) then $\mathrm{Ind}_K^{K_N}(V)$ will again be meromorphic.

Lemma 8.7.6. *With the notation above, let χ be a rank one meromorphic character of $\pi_1(\eta_\infty, \overline{x})$ whose Swan conductor is an integer $a \geq 1$ prime to N. Then $\mathrm{Ind}_K^{K_N}(\chi)$ is an absolutely meromorphic representation.*

Proof. The argument is well known: the induced representation has all slopes equal to a/N. But since $(a, N) = 1$, the existence of non-trivial subrepresentations would contradict the Hasse-Arf theorem 5.2.13. □

Corollary 8.7.7. *Suppose that k is algebraically closed. Let ρ be a meromorphic representation of $\pi_1(\eta_\infty, \overline{x})$ for which there exists some $N \geq 1$ such that the restriction of ρ to the subgroup $\mathrm{Im}(\psi_{N*})$ contains as a subrepresentation a character χ whose Swan conductor is an integer a prime to N. Then ρ contains the representation $\mathrm{Ind}_K^{K_N}(\chi)$.*

Proof. By Frobenius reciprocity we obtain

$$\mathrm{Hom}_{\pi_1(\eta_\infty, \overline{x})}(\mathrm{Ind}_K^{K_N}(\chi), \rho) \simeq \mathrm{Hom}_{\mathrm{Im}(\psi_{N*})}(\chi, \rho_{|\mathrm{Im}(\psi_{N*})}).$$

Since, by lemma 8.7.6, the representation $\mathrm{Ind}_K^{K_N}(\chi)$ is irreducible, the claim follows. □

We denote by $\mathcal{K}_2$ the unique Kummer character (see definition 8.1.2) of order 2 (*i.e.* $\mathcal{K}_2^{\otimes 2}$ is trivial). Recall that for any morphism $\phi : X \to \mathbb{A}^1_k$ we denote by $\mathcal{L}_\psi(\phi)$ the sheaf $\phi^* \mathcal{L}_\psi$. Similarly we may write $\mathcal{K}_2(\psi)$.

Proposition 8.7.8. *Suppose that k is algebraically closed. Let $f \in k[x]$ be some polynomial of degree N. Let $K_N = k((1/\tau))$ be the finite extension of $K = k((1/\zeta))$ such that $f'(\tau) + \zeta = 0$. Then we have*

$$\mathcal{F}^{(\infty,\infty)}_{\psi,loc}(\mathcal{L}_\psi(f(x))) = \mathrm{Ind}_K^{K_N}(\mathcal{K}_2(f''(\zeta)) \otimes \mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta))). \tag{8.7.9}$$

Proof. In view of corollary 8.7.7 it suffices to show that there is a $\pi_1(\eta_\infty, \overline{x})$-equivariant imbedding

$$\mathcal{K}_2(f''(\zeta)) \otimes \mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta)) \hookrightarrow (-f'(\zeta))^* \mathcal{F}_\psi(\mathcal{L}_\psi(f(x))(\infty)$$

because then, the principle of stationary phase and theorem 8.6.2 imply that the two sides of (8.7.9) are both free modules of the same rank. Let us introduce the morphism $g : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1$ defined by $g(x, \zeta) = f(x) - xf'(\zeta)$. We obtain

$$(-f'(\zeta))^* \mathcal{F}_\psi(\mathcal{L}_\psi(f(x)) \simeq Rp_{2!}(p_1^* \mathcal{L}_\psi(f(x)) \otimes \mathcal{L}_\psi(-x \cdot f'(x))) \simeq Rp_{2!}(g^* \mathcal{L}_\psi).$$

Let $\tau : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the translation map $(x, \zeta) \mapsto (x + \zeta, \zeta)$. It follows

$$Rp_{2!}(g^*\mathcal{L}_\psi) \simeq Rp_{2!}(\tau^* g^* \mathcal{L}_\psi) \simeq \mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta)) \otimes Rp_{2!}(\mathcal{L}_\psi(f(x+\zeta) - f(\zeta) - x \cdot f'(\zeta))).$$

Write $f^{(n)}(\zeta) = \frac{1}{n!}\partial^n_\zeta f(\zeta)$. Then we derive

$$\mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta)) \otimes (-f'(\zeta))^* \mathcal{F}_\psi(\mathcal{L}_\psi(f(x)) \simeq Rp_{2!}(\mathcal{L}_\psi(\sum_{n \geq 2} x^n \cdot f^{(n)}(\zeta))).$$

Let $\widetilde{T}$ be the ramified double covering of $\mathbb{A}^1_k$ defined by $T = \mathrm{Spec} k[\zeta, t]/(f^{(2)}(\zeta) - t^2)$. The restriction of $\widetilde{T} \to \mathbb{A}^1_k$ to some open subscheme $C \subset \mathbb{A}^1_k$ is a $\mathbb{Z}/2\mathbb{Z}$-Galois covering $T \to C$; if ∞ denotes the unique point "at infinity" of T, we set $\overline{T} = T \cup \{\infty\}$ which is a smooth open curve with a $\mathbb{Z}/2\mathbb{Z}$-action fixing the point ∞.

Now, set $X = p_2^{-1}(C)$ and let $f : X \to C$ be the restriction of p_2. We define f_T as in (8.7.4) and we take for $\overline{Y}$ the variety $\mathbb{A}^1_k \times_k \overline{T}$. A $\mathbb{Z}/2\mathbb{Z}$-equivariant imbedding $j : X \times_C T \hookrightarrow \overline{Y}$ is obtained by

$$(x, t) \mapsto (x \cdot t, t).$$

Then we let $F = \mathcal{L}_\psi(\sum_{n \geq 2} x^n \cdot f^{(n)}(\zeta))$. To define the sheaf $\overline{F}$ on $\overline{Y}$ we introduce the morphism $\varphi : \overline{Y} \to \mathbb{A}^1_k$ given by

$$(x, t) \mapsto \left(x^2 + \sum_{n \geq 3} x^n \cdot \frac{f^{(n)}(\zeta)}{t^n}, t \right).$$

We let $\overline{F} = \varphi^* \mathcal{L}_\psi$, and it is easily seen that $\pi^*(F) \simeq j^*(\overline{F})$. Moreover, since $\mathcal{L}_\psi$ is locally algebraic, the same holds for $\overline{F}$; *i.e.* we are in the situation of (8.7.4). The restriction of φ to $\mathbb{A}^1_k \times_k \{\infty\}$ is the morphism $x \mapsto x^2$, hence $V = H^1_c(\overline{f}_T^{-1}(\infty), \overline{F})$ has Λ-rank one and by (8.7.5) there is an imbedding

$$V \hookrightarrow \pi^*_T(\mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta)) \otimes (-f'(\zeta))^* \mathcal{F}_\psi(\mathcal{L}_\psi(f(x)))(\infty).$$

Hence for either $i = 0$ or $i = 1$ we have

$$\mathcal{K}_2(f^{(2)}(\zeta))^{\otimes i} \otimes \mathcal{L}_\psi(f(\zeta) - \zeta \cdot f'(\zeta)) \hookrightarrow (-f'(x))^* \mathcal{F}_\psi(\mathcal{L}_\psi(f(x)))(\infty).$$

We have to show that indeed $i = 1$, *i.e.* that the $\mathbb{Z}/2\mathbb{Z}$-action on V is non-trivial. This follows easily from the following lemma 8.7.10. □

Lemma 8.7.10. *The action of $\mathbb{Z}/2\mathbb{Z}$ on $H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x^2))$ induced by the automorphism $x \mapsto -x$, is non-trivial.*

Proof. The imbedding $j : \mathbb{G}_{m,k} \hookrightarrow \mathbb{A}^1_k$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant, therefore, from the exact sequence

$$0 \to j_!(\mathcal{L}_\psi(x^2)_{|\mathbb{G}_{m,k}}) \to \mathcal{L}_\psi(x^2) \to \Lambda_{\{0\}} \to 0$$

we derive an equivariant short exact sequence

$$0 \to \Lambda \to H^1_c((\mathbb{G}_{m,k})^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x^2)) \to H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x^2)) \to 0.$$

In particular, it suffices to show that the $\mathbb{Z}/2\mathbb{Z}$-action on $H^1_c((\mathbb{G}_{m,k})^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x^2))$ is non-trivial. But by the Leray spectral sequence for the Galois $\mathbb{Z}/2\mathbb{Z}$-covering $\mathbb{G}_{m,k} \to \mathbb{G}_{m,k} : x \mapsto x^2$ we see that

$$H^1_c((\mathbb{G}_{m,k})^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x^2))^{\mathbb{Z}/2\mathbb{Z}} \simeq H^1_c(\mathbb{G}_{m,k})^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi(x)) \tag{8.7.11}$$

and the right-hand side of (8.7.11) has rank one, which implies the claim. □

9. The homomorphism Γ

9.1. **Definition and basic properties.** From now on we restrict for simplicity to the Lubin-Tate torsor arising from the multiplicative group $\mathbb{G}_m$; moreover, the base field k is taken to be equal to the field k_0 of section 2.1. Accordingly, the value ρ_1 equals $p^{-1/(p-1)}$. Also, G_n equals the group μ_{p^n} of p^n-th roots of unity. We pick a non-trivial character ψ of the group $G_\infty = \mu_{p^\infty}$ with values in the ring of integers $\mathbb{O}$ of the ℓ-adic completion of $\mathbb{Q}_\ell(\mu_{p^\infty})$. Then, by composing with the natural projections we obtain a compatible sequence of characters $\psi_n : \mu_{p^\infty} \to \mathbb{O}/\ell^n$. If q is the cardinality of the residue field of k, we denote by $\mathbb{E}_\lambda$ the ℓ-adic completion of the field $\mathbb{Q}_\ell(\mu_{p^\infty}, q^{1/2})$, and by $\mathbb{O}_\lambda$ the ring of integers in $\mathbb{E}_\lambda$. We need the extension $\mathbb{E}_\lambda$ to make sense of the "half Tate twist": the Tate module $\mathbb{E}_\lambda(1/2)$ is the unramified Galois representation on which Frobenius acts as multiplication by $q^{-1/2}$.

Let V be a k-vector space, $\sigma : V \to V'$ a symmetric k-linear isomorphism and $f : V \to k$ the associated non-degenerate quadratic form. We take inspiration from formula (1.0.1) of the introduction to make the following definition:

$$\Gamma(f) = \lim_{\substack{\longleftarrow \\ n}} H_c^{\dim V}(V^{an} \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n}) \otimes_{\mathbb{O}} \mathbb{E}_\lambda(\dim V/2).$$

In this chapter we will be concerned with the study of the $\mathrm{Gal}(k^a/k_\infty)$-module $\Gamma(f)$, seen as a function of f. With the present setup, this cohomology group carries also a semilinear action of $\mathrm{Gal}(k^a/k)$, as explained in section 6.3. Even though it may be interesting and worth exploring, we will not deal here with this extra structure.

The next two results establish the elementary properties of Γ.

Lemma 9.1.1. *For any f as above, $\Gamma(f)$ is a $\mathrm{Gal}(k^a/k_\infty)$-module of rank one, which depends only on the isomorphism class of f.*

Proof. It suffices to prove the corresponding result for the torsion modules $\Gamma_n(f) = H_c^{\dim V}(V^{an} \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n})$. Let g be another non-degenerate quadratic form, in the same isomorphism class as f. Then we have $g = f \circ h$ for some automorphism $h : V \to V$. We get

$$H_c^{\dim V}(V^{an} \times_k \widehat{k}^a, g^*\mathcal{L}_{\psi_n}) \simeq H_c^{\dim V}(V^{an} \times_k \widehat{k}^a, h^*f^*\mathcal{L}_{\psi_n}) \simeq H_c^{\dim V}(V^{an} \times_k \widehat{k}^a, f^*\mathcal{L}_{\psi_n})$$

which proves the second assertion. Since the characteristic of k is different from 2, we can always find a basis $\{\mathbf{e}_1, ..., \mathbf{e}_m\}$ of V, such that the quadratic form f diagonalizes in this basis. Let V_i for $i = 1, ..., m$ be the span of $\mathbf{e}_i$, and let $p_i : V \to V_i$ be the projection such that $p_i(\mathbf{e}_j) = \delta_{ij}\mathbf{e}_i$. Denote also by f_i the restriction of f to V_i. The yoga of torsors (for which we refer to [SGA4$\frac{1}{2}$]) implies the formula

$$f^*\mathcal{L} \simeq p_1^*f_1^*\mathcal{L} \otimes ... \otimes p_m^*f_m^*\mathcal{L}.$$

Since $H_c^0(V_i^{an} \times_k \widehat{k}^a, f_i^*\mathcal{L}) = H^0(V_i^{an} \times_k \widehat{k}^a, f_i^*\mathcal{L}) = 0$, it follows that $H_c^j(V_i^{an} \times_k \widehat{k}^a, f_i^*\mathcal{L}) \neq 0$ if and only if $j = 1$. Then, by Kunneth formula we have:

$$\Gamma_n(f) \simeq \Gamma(f_1) \overset{\mathbf{L}}{\otimes} ... \overset{\mathbf{L}}{\otimes} \Gamma_n(f_m).$$

Hence, to prove the first assertion it suffices to assume $\dim V = 1$. Let f' be the inverse transpose of f, defined as in proposition 7.2.3. Combining proposition 7.2.3 and the involutivity theorem 7.1.2 we obtain

$$\mathcal{L}(f) \simeq \mathcal{L}(f) \overset{\mathbf{L}}{\otimes} \Gamma_n(f) \overset{\mathbf{L}}{\otimes} \Gamma_n(f')$$

which implies that $\Gamma_n(f)$ is free of rank one. □

Remark 9.1.2. The proof also shows that the groups $H_c^i(V^{an} \times_k \widehat{k}^a, f^*\mathcal{L})$ vanish for $i \neq \dim V$.

Proposition 9.1.3. *The map $f \mapsto \Gamma(f)$ descends to a group homomorphism from the Witt group $W(k)$ of k to the group of isomorphism classes of rank one $\mathrm{Gal}(k^a/k_\infty)$-modules (with multiplication given by tensor product).*

Proof. Again, we reduce easily to the corresponding statement for torsion coefficients. Let $f : V \to k$, $g : W \to k$ be two nondegenerate quadratic forms, and let $f \oplus g : V \oplus W \to k$ be their sum. Denote also by p_V (resp. p_W) the projection of $V \oplus W$ onto V (resp. onto W). From another application of the yoga of torsors, one obtains

$$(f \oplus g)^*\mathcal{L} \simeq p_V^*f^*\mathcal{L} \otimes p_W^*g^*\mathcal{L}. \tag{9.1.4}$$

Using (9.1.4) and the Kunneth formula it follows

$$\Gamma_n(f) \otimes \Gamma_n(g) \simeq H_c^{\dim V + \dim W}((V \oplus W) \times_k \widehat{k}^a, p_V^*f^*\mathcal{L} \otimes p_W^*g^*\mathcal{L}) \simeq \Gamma_n(f \oplus g)$$

which says that Γ_n induces a homomorphism from the monoid of isomorphism classes of quadratic forms, to the group of isomorphism classes of $\mathrm{Gal}(k^a/k_\infty)$-modules of rank one. Let $f_V : V \oplus V' \to k$ be the standard quadratic form induced by the dual pairing: $f_V(x, \xi) = \langle x, \xi \rangle$ for all $x \in V, \xi \in V'$. We want to show that $\Gamma_n(f_V)$ is the trivial $\mathrm{Gal}(k^a/k_\infty)$-representation. But this is nothing else than a special case of lemma 7.1.4. Since the relations in the Witt group are generated by all the isotropic quadratic forms of the form f_V, the claim follows. □

9.2. **Computation of $\Gamma(f)$.** In this section we obtain some information on the Galois structure of $\Gamma(f)$.

For $a \in k^\times$, let M_a denote the ℓ-adic representation of $\mathrm{Gal}(k^a/k_\infty)$ corresponding to the character $\sigma \mapsto \sigma(\sqrt{a})/\sqrt{a} = \pm 1$ and let $f_a : k \to k$ be the quadratic form $x \mapsto ax^2$.

Lemma 9.2.1. *With the notation above*

$$\Gamma(f_a) \simeq \Gamma(f_1) \otimes M_a.$$

Proof. Define a projective system of sheaves $\mathcal{M}_a = \{\mathcal{M}_{a,n}\}_{n\in\mathbb{N}}$ on $(\mathbb{A}^1_k)^{an}$, by requiring $f_{a*}(\mathbb{O}_\lambda/\ell^n) = (\mathbb{O}_\lambda/\ell^n) \oplus \mathcal{M}_{a,n}$. Then we have

$$H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, f_a^*\mathcal{L}_{\psi_n}) \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}_{\psi_n} \otimes f_{a*}(\mathbb{O}_\lambda/\ell^n)) \simeq H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}_{\psi_n} \otimes \mathcal{M}_{a,n}).$$

By $\mathcal{M}_a \simeq \mathcal{M}_1 \otimes M_a$, the assertion follows. □

Given a general non-degenerate quadratic form $f : V \to k$ on a vector space of dimension n, denote by $D(f)$ the discriminant of f. Set $H^n_c(V^{an}, f^*\mathcal{L}_\psi) = \varprojlim_n H^n_c(V, f^*\mathcal{L}_{\psi_n}) \otimes_{\mathbb{O}} \mathbb{E}_\lambda$.

Proposition 9.2.2. *With the notation above, let $n = 2m$ (resp. $= 2m+1$) and $d = (-1)^m D(f)$. Then we have*

$$H^n_c(V^{an}, f^*\mathcal{L}_\psi) \simeq \begin{cases} M_d(-m) & n \text{ even} \\ H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, \mathcal{L}_\psi \otimes \mathcal{M}_d)(-m) & n \text{ odd.} \end{cases}$$

Proof. Let $U = f^{-1}(\mathbb{G}_{m,k} \times \widehat{k}^a)$ and $W = f^{-1}(0)$. Then from Theoreme 3.3 and Table 3.7 of [SGA7] Exp. XII Quadriques, we derive

$$R^q f_{|U!}\mathbb{E}_\lambda \simeq \begin{cases} M_d(-(m-1)) & q = n-1, n \text{ even} \\ \mathcal{M}_d(-m) & q = n-1, n \text{ odd} \\ \mathbb{E}_\lambda(-(n-1)) & q = 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

From this and the projection formula we obtain

$$H^q_c(U^{an}, f^*\mathcal{L}_\psi) \simeq \varprojlim_n H^1_c((\mathbb{G}_{m,k} \times \widehat{k}^a, \mathcal{L}_{\psi_n} \otimes R^{q-1} f_{|U!}(\mathbb{O}/\ell^n)) \otimes_{\mathbb{O}} \mathbb{E}_\lambda.$$

Since W is the affine cone over the non-singular quadric $Q \subset \mathbb{P}(V)$ defined by f, we can compute $H^q_c(W^{an}, f^*\mathcal{L}_\psi) = H^q_c(W, \mathbb{E}_\lambda)$ by using [SGA7] Exp. XV Formule de Picard-Lefschetz. We have $H^q_c(W, \mathbb{E}_\lambda) = H^q_{\{0\}}(W, \mathbb{E}_\lambda)$ by Prop. 2.1.2(ii) *loc.cit.* In the long exact sequence

$$\ldots \to H^q_{\{0\}}(W, \mathbb{E}_\lambda) \to H^q(W, \mathbb{E}_\lambda) \to H^q(W - \{0\}, \mathbb{E}_\lambda) \to \ldots$$

we have $H^q(W, \mathbb{E}_\lambda) = \mathbb{E}_\lambda$ for $q = 0$ and $= 0$ otherwise by Prop. 2.1.2(i) *loc.cit.* Finally, since $W - \{0\}$ is a $\mathbb{G}_m$-bundle over Q, we obtain

$$H^q_c(W, \mathbb{E}_\lambda) \simeq \begin{cases} M_d(-(m-1)) & q = n-1, n \text{ even} \\ M_d(-m) & q = n, n\text{even} \\ \mathbb{E}_\lambda(-(n-1)) & q = 2n-2 \\ 0 & \text{otherwise.} \end{cases}$$

From these computation we can easily deduce the claim. (Warning: in this proof we have used somewhat freely an ℓ-adic language: this is only a harmless abbreviation for some more cumbersome notation, and does not imply that we rely on a formalism of analytic ℓ-adic sheaves). □

Corollary 9.2.3. *With the notation above*

$$\Gamma(f_a)^{\otimes 2} \simeq M_{-1}$$

and the $\mathrm{Gal}(k^a/k_\infty)$*-action on* $\Gamma(f)$ *factors through* μ_4.

Proof. It follows immediately from proposition 9.2.2 and proposition 9.1.3. □

As an example we consider the classical case of the norm of the quaternion algebras. Recall that for any pair of elements $a, b \in k$, one obtains an associative k-algebra $\left(\frac{a,b}{k}\right)$ of dimension 4, with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and multiplication fixed by the rules:

$$\mathbf{i}^2 = a \quad \mathbf{j}^2 = b \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

Let π be a uniformizing parameter for k. If $a \in (k^\circ)^\times$ is not a quadratic residue modulo π, then the algebra $\left(\frac{a,\pi}{k}\right)$ is a division algebra and any two division algebras arising in this way are isomorphic. We denote by $\mathbb{H}$ this division algebra: it is the quaternion algebra over k. The algebra $\mathbb{H}$ is endowed with a *norm map* $N : \mathbb{H} \to k$. The norm map induces a homomorphism from the multiplicative group $\mathbb{H}^\times$ to $k^\times$. In terms of the basis given above, one has

$$N(x \cdot 1 + y \cdot \mathbf{i} + z \cdot \mathbf{j} + w \cdot \mathbf{k}) = x^2 - ay^2 - \pi z^2 + a\pi w^2.$$

The following result is now a straightforward consequence of proposition 9.2.2 and corollary 9.2.3.

Theorem 9.2.4. *The action of* $\mathrm{Gal}(k^a/k_\infty)$ *on* $\Gamma(N)$ *is trivial.* □

In [We] it is proved that, with the notation of the introduction, the constant $\gamma(N)$ equals -1. This shows that Weil's invariant is not a homomorphic image of our Γ.

9.3. **The deformation from Kummer to Artin-Schreier.** The aim of this section is to obtain an explicit formula for the action of a Frobenius element on the stalks of a Kummer sheaf $\mathcal{K}_\psi$. This formula will be applied in the next section, to determine the Galois action on $\Gamma(x^2)$, thus completing the computation started in section 9.2. The method followed here exploits the group scheme $\mathcal{G}^{(\lambda)}$ of Oort-Sekiguchi-Suwa, originally introduced in [O-S-S] for other purposes. Alternatively, the main result of this section could be seen as a special case of the general formula of proposition 6.2.3, and could have also been obtained just by quoting some of the classical Iwasawa's explicit reciprocity laws, which would have made the treatment somewhat shorter. Our choice is based mainly on a matter of personal taste.

We start with some notation. Let $[p] : \mathbb{D}(1, \rho_1) \to \mathbb{D}(1, \rho_1^p)$ the étale covering $x \mapsto (x+1)^p$. The étale local sections of the map $[p]$ gives us the usual Kummer torsor $\mathcal{K}$, and therefore, given any homomorphism $\psi : \mu_p \to \Lambda^\times$, the rank one sheaf of Λ-modules $\mathcal{K}_\psi$. To study $\mathcal{K}_\psi$ we introduce the group scheme $\mathcal{G}^{(t)}$ of [O-S-S]. We recall here the main features of this theory. First, for any $t \in k_1^{\circ\circ} = k(\mu_p)^{\circ\circ}$ we define

$$\mathcal{G}^{(t)} = \mathrm{Spec} k_1^\circ[x, 1/(1+tx)].$$

It is shown in [O-S-S] that $\mathcal{G}^{(\lambda)}$ is a group scheme over $\mathrm{Spec} k_1^\circ$, with addition law given by

$$(x, y) \mapsto txy + x + y$$

. The ring homomorphism

$$k_1^\circ[y, y^{-1}] \to k_1^\circ[x, 1/(1+tx)] \qquad y \mapsto tx + 1$$

defines a morphism of group schemes $\alpha_t : \mathcal{G}^{(t)} \to \mathbb{G}_{m,k_1^\circ}$ which restricts to an isomorphism over the generic fibre of $\mathrm{Spec} k_1^\circ$. On the other hand, the special fibre of $\mathcal{G}^{(t)}$ is the additive group $\mathbb{G}_{a,\widetilde{k}}$.

In particular, let ζ_p be a generator of the cyclic group μ_p; we define $\lambda = 1 - \zeta_p$. The deformation from Kummer to Artin-Schreier is the étale morphism $\omega : \mathcal{G}^{(\lambda^p)} \to \mathcal{G}^{(\lambda)}$ of group schemes over k_1° induced by the ring homomorphism

$$k_1^\circ[x, 1/(1+\lambda^p x)] \to k_1^\circ[y, 1/(1+\lambda y)] \qquad x \mapsto \lambda^{-p} \cdot ((\lambda y + 1)^p - 1).$$

One checks easily that there is a commutative diagram

$$\begin{tikzcd} \mathcal{G}^{(\lambda)} \arrow[r, "\omega"] \arrow[d, "\alpha_\lambda"'] & \mathcal{G}^{(\lambda^p)} \arrow[d, "\alpha_{\lambda^p}"] \\ \mathbb{G}_{m,k_1^\circ} \arrow[r, "{[p]}"] & \mathbb{G}_{m,k_1^\circ} \end{tikzcd}$$

Let $\underline{0}$ be the canonical zero section of the group scheme $\mathcal{G}^{(\lambda^p)}$, *i.e.* $\underline{0} : \mathrm{Spec} k_1^\circ \to \mathcal{G}^{(\lambda^p)}$ is induced by the ring homomorphism

$$k_1^\circ[x, 1/(1+\lambda^p x)] \to k_1^\circ \qquad x \mapsto 0.$$

Then clearly $\omega^{-1}(\underline{0})$ is a reduced finite group scheme isomorphic to $(\mathbb{Z}/p\mathbb{Z})_{k_1^\circ}$, with a map of group schemes $\omega^{-1}(\underline{0}) \hookrightarrow \mathcal{G}^{(\lambda)}$ given explicitly by the ring homomorphism

$$k_1^\circ[x, 1/(1+\lambda x)] \to k_1^\circ \oplus \ldots \oplus k_1^\circ \qquad x \mapsto \bigoplus_{i=0}^{p-1} (\zeta_p^i - 1)/\lambda.$$

Let $\nu : k_1^\circ \to k_1^\circ / k_1^{\circ\circ} = \widetilde{k}_1 = \widetilde{k}$ be the map "reduction modulo $k_1^{\circ\circ}$". The assignment $\zeta \mapsto \nu((\zeta - 1)/\lambda)$ defines a group isomorphism $\sigma : \mu_p \to \mathbb{F}_p$, and we set $\overline{\psi} = \psi \circ \sigma^{-1} : \mathbb{F}_p \to \Lambda^\times$. Again, the étale sections of ω define a $\omega^{-1}(0)$-torsor, and via the character $\overline{\psi}$ this gives rise to a locally constant sheaf of Λ-modules of rank one on the étale site of $\mathcal{G}^{(\lambda^p)}$, which we can denote by $\mathcal{G}_{\overline{\psi}}$. By the remarks above it is clear that the restriction of $\mathcal{G}_{\overline{\psi}}$ to $\mathbb{G}_{a,\widetilde{k}} = \mathcal{G}^{(\lambda^p)} \times_{k_1^\circ} \widetilde{k}$ is nothing else than the Lang torsor usually denoted $\mathcal{L}_{\overline{\psi}}$. On the other hand, we have

$$(\alpha_\lambda \times_{k_1^\circ} k_1)^* \mathcal{K}_\psi \simeq (\mathcal{G}_{\overline{\psi}})_{|\mathcal{G}^{(\lambda^p)} \times_{k_1^\circ} k_1}.$$

Let $x \in \mathbb{D}(1,\rho_1^p)(k_1)$ be any k_1-rational point. We can see x as a morphism $x : \mathrm{Spec} k_1 \to \mathbb{G}_{m,k_1}$ and clearly we can find a unique morphism $\widetilde{x} : \mathrm{Spec} k_1^\circ \to \mathcal{G}^{(\lambda^p)}$ which fits into a commutative diagram

$$\begin{tikzcd} \mathrm{Spec} k_1 \arrow[r, hook] \arrow[d, "x"] & \mathrm{Spec} k_1^\circ \arrow[r, "\widetilde{x}"] & \mathcal{G}^{(\lambda^p)} \arrow[d, "\alpha_{\lambda^p}"] \\ \mathbb{G}_{m,k_1} \arrow[rr, hook] & & \mathbb{G}_{m,k_1^\circ}. \end{tikzcd}$$

Let Fr denote any lifting of the canonical Frobenius generator on $\mathrm{Gal}(\widetilde{k}^a/\widetilde{k})$, $tr_{\widetilde{k}/\mathbb{F}_p}$ the trace for the residue field extension $\mathbb{F}_p \subset \widetilde{k}$ and denote by $\mathrm{Tr}(Fr, M)$ the trace of Fr on the Galois module M.

Proposition 9.3.1. *Let $x \in \mathbb{D}(0,\rho_1^p)(k_1)$ be any k_1-rational point. Then the stalk $(\mathcal{K}_\psi)_x$ is the unramified Λ-representation of $\mathrm{Gal}(k_1^a/k_1)$ such that*

$$\mathrm{Tr}(Fr, (\mathcal{K}_\psi)_x) = \overline{\psi}(tr_{\widetilde{k}/\mathbb{F}_p}\nu((1-x)/\lambda^p)).$$

Proof. The restriction $\widetilde{x}^*(\mathcal{G}_{\overline{\psi}})$ is a locally constant sheaf on $\mathrm{Spec} k_1^\circ$, corresponding to some unramified representation of $\mathrm{Gal}(k_1^a/k_1)$. For any x as above we derive

$$\overline{\psi}(tr_{\widetilde{k}/\mathbb{F}_p}\nu((x-1)/\lambda^p))^{-1} = \mathrm{Tr}(Fr, (\mathcal{L}_{\overline{\psi}})_{\nu((x-1)/\lambda)}) = \mathrm{Tr}(Fr, \widetilde{x}^*\mathcal{G}_{\overline{\psi}}) = \mathrm{Tr}(Fr, (\mathcal{K}_\psi)_x).$$

□

9.4. **Quadratic Gauss sums.** In this final section we obtain an explicit description of the Galois action on $\Gamma(x^2)$, thus complementing proposition 9.2.2. Unfortunately our method works only when the residue characteristic is different from 2. Therefore in this chapter we assume throughout that p is odd.

Let $f_1 : \mathbb{A}^1 \to \mathbb{A}^1$ be the quadratic form in one variable $x \mapsto x^2$. Let $\mathbb{D}(r)$ be the closed disc of radius r in $(\mathbb{A}^1_k)^{an}$, centered at the origin and $j : (\mathbb{A}^1_k)^{an} - \mathbb{D}(r) \to (\mathbb{A}^1_k)^{an}$ the imbedding of the complement of $\mathbb{D}(r)$. Suppose that the restriction of $f_1^*\mathcal{L}_\psi$ to $\mathbb{D}(r)$ is not the constant sheaf. We derive an exact sequence in cohomology

$$H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \to H^1(\mathbb{D}(r) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \to H^2_c(((\mathbb{A}^1_k)^{an} - \mathbb{D}(r)) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi).$$

By Poincaré duality $H^2_c(((\mathbb{A}^1_k)^{an} - \mathbb{D}(r)) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \simeq \mathrm{Hom}(H^0(((\mathbb{A}^1_k)^{an} - \mathbb{D}(r)) \times_k \widehat{k}^{,} f_1^*\mathcal{L}_\psi^{-1}), \Lambda) = 0$. An argument like in the proof of lemma 8.2.1 shows that all these groups are free Λ-modules, and therefore $H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \simeq H^1(\mathbb{D}(r) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi)$ if and only if $H^1(\mathbb{D}(r) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \neq 0$. Hence, let us assume that r is large enough, so that $H^1(\mathbb{D}(r) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \neq 0$. In this case,a little juggling (see the remarks in section 8.3) shows that the group $H^1_c((\mathbb{A}^1_k)^{an} \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi)$ computed in Berkovich's theory coincides with the group $H^1_c(\mathbb{D}(r) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi)$ computed in Huber's theory.

Hence *in the following we will switch again to Huber's theory*. This change is not strictly necessary, but in our view it simplifies the exposition (and it also shows once more, how much more desirable would have been to use Huber's theory consistently throughout the paper).

Set $r_1 = \rho_1^{1/2}$. We will show that indeed $H^1_c(\mathbb{D}(r_1) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi)$ does not vanish.

Let $\tau : \mathbb{D}(0,\rho_1) \to \mathbb{D}(1,\rho_1)$ be the translation map $x \mapsto x+1$ and $\varepsilon : \mathbb{D}(0,\rho_1) \to \mathbb{D}(1,\rho_1^p)$ the analytic isomorphism $x \mapsto e^{px}$. Looking back at section 2.1 we obtain easily the equality

$$[p] \circ \tau = \varepsilon \circ \boldsymbol{\lambda}.$$

We derive an isomorphism of rank one sheaves of Λ-modules:

$$\varepsilon^*\mathcal{K}_\psi \simeq \mathcal{L}_{\psi|\mathbb{D}(0,\rho_1)}. \tag{9.4.1}$$

Let $\alpha : \mathbb{D}(0,\rho_1^{1/2}) \to \mathbb{D}(1,\rho_1^p)$ be the morphism $x \mapsto 1+px^2$. The morphism f_1 restricts to a map $f_1 : \mathbb{D}(0,\rho_1^{1/2}) \to \mathbb{D}(0,\rho_1)$ and by a standard calculation we can find an analytic isomorphism (at least if the residue characteristic is odd) $\beta : \mathbb{D}(0,\rho_1^{1/2}) \to \mathbb{D}(0,\rho_1^{1/2})$ such that

$$\alpha \circ \beta = \varepsilon \circ f_1.$$

Then from (9.4.1) it follows

$$f_1^*\mathcal{L}_\psi \simeq \beta^* \circ \alpha^*\mathcal{K}_\psi.$$

In particular we have, for all integers $i \geq 0$

$$H^i_c(\mathbb{D}(0,r_1) \times_k \widehat{k}^a, f_1^*\mathcal{L}_\psi) \simeq H^i_c(\mathbb{D}(0,r_1) \times_k \widehat{k}^a, \alpha^*\mathcal{K}_\psi)$$

After base change to the overfield $K = k_1(\lambda^{1/2})$ we can find a formal model $\mathfrak{D}$ for the analytic variety $\mathbb{D}(0, \rho_1^{1/2})$; the simplest such $\mathfrak{D}$ is given by $\mathrm{Spf} K^\circ\{T\}$, whose special fibre $\mathfrak{D}_s$ is the affine line over the residue field $\widetilde{k}$. Denote by X the scheme $\mathbb{A}^1_{K^\circ}$. We can realise $\mathfrak{D}$ as the completion $\widehat{X}$ of X along its closed fibre $X_s = \mathfrak{D}_s$. We observe that the sheaf $\alpha^*\mathcal{K}_\psi$ is the restriction to $\mathbb{D}(0, r_1)$ of an algebraic constructible étale sheaf defined over the generic fibre X_η of X. Notice also that the set of K-rational points $\mathbb{D} = \mathbb{D}(0, r_1)(K)$ is a compact Hausdorff topological group, and denote by $d\mu$ the invariant measure on $\mathbb{D}$, normalized so that the total mass is equal to one. We introduce a Λ-valued function

$$\varphi : \mathbb{D}(0, r_1)(K) \to \Lambda \qquad x \mapsto \overline{\psi}(tr_{\widetilde{k}/\mathbb{F}_p}(\nu(p \cdot x^2/\lambda^2)).$$

Theorem 9.4.2. *Let $\mathcal{H}$ denote the open subgroup $\mathrm{Gal}(k^a/K(\mu_{p^\infty}))$ of the Galois group $\mathcal{G} = \mathrm{Gal}(k^a/k_\infty)$. The $\mathcal{G}$-action on the Λ-module $\Gamma(x^2)$ restricts to an unramified action of $\mathcal{H}$, and we have the following trace formula for the action of any lifting of the Frobenius generator to an element $Fr \in \mathcal{H}$*

$$\mathrm{Tr}(Fr, \Gamma(x^2)) = q^{-1/2} \int_{\mathbb{D}} \varphi \cdot d\mu$$

Proof. Let $R\Psi_\eta : \mathrm{D}^+(X_\eta, \Lambda) \to \mathrm{D}^+(X_{\overline{s}}, \Lambda)$ be the usual nearby cycle functor; by [Hub] theorem 5.7.6 we have an isomorphism

$$H^i_c(\mathbb{D}(0, r_1) \times_k \widehat{k}^a, \alpha^*\mathcal{K}_\psi) \simeq H^i_c(\mathfrak{D}_{\overline{s}}, R\Psi_\eta(\alpha^*\mathcal{K}_\psi)).$$

Let $\widetilde{x} \in \mathfrak{D}_{\overline{s}}(\widetilde{k})$ be any $\widetilde{k}$-rational point. The stalk $(R\Psi_\eta(\alpha^*\mathcal{K}_\psi))_{\widetilde{x}}$ is a complex of Λ-modules with an action of $\mathrm{Gal}(K^a/K)$. Let $\mathbb{E}(x, r_1) \subset \mathbb{D}(0, r_1)$ be the open disc, centered at some k-rational point $x \in \mathbb{D}(0, r_1)$, and consisting of all the points which specialize to $\widetilde{x}$ (in Huber's notation, this is the analytic variety $\lambda^{-1}(\{\widetilde{x}\})$). According to [Hub] theorem 5.7.9 we have

$$(R^i\Psi_\eta(\alpha^*\mathcal{K}_\psi))_{\widetilde{x}} \simeq H^i_c(\mathbb{E}(x, r_1), \alpha^*\mathcal{K}_\psi).$$

The restriction of $\alpha^*\mathcal{K}_\psi$ to $\mathbb{E} = \mathbb{E}(x, r_1)$ is geometrically constant, hence, if $\Lambda_{\mathbb{E}}$ denotes the constant sheaf of Λ-modules on $\mathbb{E}(x, r_1)$, we have

$$\alpha^*\mathcal{K}_{\psi|\mathbb{E}} \simeq \Lambda_{\mathbb{E}} \otimes (\mathcal{K}_\psi)_{\alpha(x)}.$$

Then the trace formula of the claim follows easily from the remarks at the beginning of this section, and from proposition 9.3.1. $\square$

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