A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem

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Abstract

When hit with an adverse shock, banks that do not comply with capital regulation sell risky assets to satisfy their solvency constraint. When financial markets are imperfectly competitive, this naturally gives rise to a GNEP. We consider a new framework with an arbitrary number of banks and assets, and show that Tarski's theorem can be used to prove the existence of a Nash equilibrium when markets are sufficiently competitive. We also prove the existence of ϵ-Nash equilibria.

Keywords: Generalized games, banking regulation, Cournot oligopoly, asset sales, Tarski's fixed point theorem

JEL Classification: G21, G28, D43

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1 Introduction

Compared to other financial institutions such as hedge funds, banks are regulated; their main capital ratio called the risk-based capital ratio (RBC) defined as regulatory capital or equity E divided by the risk-weighted assets RWA, that is, $\frac{E}{\text{RWA}}$, must be greater than a critical threshold around 10% at all times (see BCBS (2017)). After an adverse systemic shock, as e.g., in mid-September 2008 when Lehman Brothers failed, some banks may fail to satisfy their capital requirement. To comply with regulation, the quickest solution consists in selling a portion of their risky assets to decrease their risk-weighted assets. However, when many banks sell the same asset at the same time, something called a generalized asset shrinkage (Hanson et al. (2011)), the price of the asset will decrease through the market mechanism and will further deplete the capital of the bank, that is, this generates a kind of death spiral (e.g., Brunnermeier and Pedersen (2009), Hanson et al. (2011)). It is usual to call such a phenomenon a price-mediated contagion problem (e.g., Feinstein (2020)), as opposed to direct contagion which is related to the network of exposures (e.g., Feinstein (2017), see Glasserman and Young (2016) for a review). In Braouezec and Wagalath (2019), they consider a game theoretic price-mediated contagion model in the particular case of one risky asset and show, using Tarski's theorem, that at least one Nash equilibrium exists. They however fail to recognize that this setting is indeed a generalized game. In this paper, we consider a new setting in the spirit of Braouezec and Wagalath (2019) in the general case of $n \geq 2$ assets and prove, under some conditions on the price impacts, the existence of a Nash equilibrium using Tarski's theorem. This theorem is widely applied in financial network models (e.g., Glasserman and Young (2016)) but not in generalized games (see Dutang (2013) for a review of existence theorems, see also Arrow and Debreu (1954) and Facchinei et al. (2007)). Our contribution is intimately related to Banerjee and Feinstein (2021) and Capponi and Weber (2022) (see also Capponi and Larsson (2015)) since the authors, as we do here, consider a static strategic setting. Banerjee and Feinstein (2021) is the generalization of Braouezec and Wagalath (2019) to the case of an arbitrary number of assets and banks when each bank is subject to a risk-based capital requirement. Capponi and Weber (2022) consider a related model in which banks are subject to a non risk-based capital requirement (leverage ratio). Interestingly, as opposed to Banerjee and Feinstein (2021) and Braouezec and Wagalath (2019), in Capponi and Weber (2022) the asset allocation is not exogenous but is endogenous.

The aim of this paper is threefold. First, it is to show that due to both banking regulation and the market mechanism, a generalized game naturally occurs when one considers the asset sale problem between banks. Second, it is to show how Tarski's theorem can be used to prove the existence of a Nash equilibrium in a generalized game (see Facchinei et al. (2007), Dutang (2013)). We also characterize the optimal liquidation strategy. Third, it is to generalize our existence result to ϵ-Nash equilibria.

2 The generalized game

We consider the extension of Braouezec and Wagalath (2019) with a finite number of risky assets. Let $\mathcal{B} = \{1, ..., p\}$ be the set of banks and $\mathcal{S} = \{1, ..., n\}$ be the set of risky securities. Each bank i holds a quantity q_{ij} of security j (stocks, bonds, index...) for which the price P_j has been hit with a shock Δ_j in percentage at time $t = 0$, i.e., it is equal to $P_j\,(1 - \Delta_j)$. Right after the shock $\Delta = (\Delta_1, ..., \Delta_n)$, the total value of the assets of each bank i is equal to $A_i(\Delta) := \sum_{j=1}^{n} P_j\,(1 - \Delta_j)\,q_{ij} = A_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j$ where $\sum_{j=1}^{n} q_{ij} P_j \Delta_j$ is the *loss* incurred by the bank. The balance sheet of bank i is given below where A_i is the total assets before the shock.

Balance-sheet of bank i

Assets	Liabilities
$\mathrm{A}_{i1} - q_{i1}P_1\Delta_1$	Debt: D_i
$\vdots$	
$\mathrm{A}_{in} - q_{in}P_n\Delta_n$	Equity: $E_i(\Delta)$
$A_i - \sum_{j=1}^{n} q_{ij}P_j\Delta_j$	$E_i(\Delta) + D_i$

As long as the the bank is solvent, the total value of the assets is equal to the total value of the liabilities (i.e., $A_i(\Delta) = E_i(\Delta) + D_i$) so that the capital can absorb the loss, that is, $E_i(\Delta) = E_i - \sum_{j=1}^{n} q_{ij}P_j\Delta_j > 0$ where E_i is the capital before the shock. The positivity of the capital is however not enough. Banking regulation imposes a risk-based capital ratio for each bank i to be greater than $\theta_{i,min}$ (in practice $\theta_{i,min} \approx 10\%$). Let $\alpha_{ij} \in [0, 1]$ be the regulatory weight of asset j for bank i.

Assumption 1 *For each $i \in \mathcal{B}$*

$$\alpha_{i,1} > \alpha_{i,2} > ... > \alpha_{i,n} \tag{1}$$

The regulatory weights can be directly provided by regulators to banks (standardised approach) so that they are identical for each bank. But they can also be computed by the bank (internal model approach), which means that two different banks may have different estimates of the weight of a given asset. However, in the post-subprime crisis regulation called Basel III, banks have much less freedom than before to make use of internal models. The risk-weighted assets are equal to $\mathrm{RWA}_i(\Delta) := \sum_j \alpha_{ij}P_j(1 - \Delta_j)\, q_{ij}$. For the sake of financial interest, we assume that each bank must react after the shock.

Assumption 2 *For each $i \in \mathcal{B}$, $E_i(\Delta) > 0$ but*

$$\theta_i(\Delta) := \frac{E_i(\Delta)}{\mathrm{RWA}_i(\Delta)} < \theta_{i,min}$$

To restore their capital ratio back above $\theta_{i,min}$, banks are assumed to sell a portion $x_{ij} \in [0, 1]$ of security $j = 1, ..., n$. Let $\sum_{k\in\mathcal{B}} x_{kj}q_{kj}$ be the total quantity of security j sold by banks. When markets are imperfectly competitive, the price is impacted by such sales. For simplicity, we consider a linear price impact model (e.g., Braouezec and Wagalath (2019)) for which the price of security j after the asset sale is equal to

$$P_j^{\text{after}} = P_j^{\text{before}} \times \left(1 - \frac{\sum_{i\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}\right) \tag{2}$$

where $\Phi_j >> \sum_{k\in\mathcal{B}} q_{kj}$ is called the market depth and measures the competitiveness of market j. The greater Φ_j, the more competitive the market of security j. At the limit, when Φ_j is infinite, it is perfectly competitive. Let $x_i \in [0, 1]^n$ be the liquidation of bank i and $x \in [0, 1]^{np}$ be a vector of liquidation of the set of banks. Let

$$L_i(x_i) := \sum_{j=1}^{n} x_{ij}q_{ij}P_j(1 - \Delta_j)$$

be the total value of the assets sold by bank i. It is not difficult to show that the risk-based capital ratio of bank i is equal to

$$\theta_i(\Delta, x) := \frac{E_i(\Delta, x)}{\mathrm{RWA}_i(\Delta, x)} = \frac{E_i - \displaystyle\sum_{j=1}^{n} q_{ij}P_j \times (\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}(1 - \Delta_j))}{\displaystyle\sum_{j=1}^{n} \alpha_{ij}q_{ij}P_j\,(1 - \Delta_j)\left(1 - \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{jk}}{\Phi_j}\right)(1 - x_{ij})} \tag{3}$$

As usual in game theory, let $x = (x_i, x_{-i})$. Since the capital ratio of bank i depends upon x_i but also upon $x_{-i} \in [0,1]^{n(p-1)}$ (what banks $k \neq i$ are liquidating), the asset sale problem is *strategic*. Let

$$X_i(x_{-i}) = \{x_i \in [0,1]^n,\ \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\}$$

be the solvency constraint of a given bank i. At time $t = 0$, each bank i chooses its liquidation strategy $x_i := (x_{ij})_{j \in \mathcal{S}} \in [0,1]^n$ and prices after liquidation are disclosed, i.e., $P^{\text{after}} = (P_1^{\text{after}}, ..., P_n^{\text{after}})$ at time $t = 1$.

Assumption 3 *Each bank $i \in \mathcal{B}$ solves the following constrained optimization problem.*

$$\min_{x_i} L_i(x_i) \quad s.t. \quad x_i \in X_i(x_{-i}) \tag{4}$$

The objective function is similar to Braouezec and Wagalath (2019) in that it only depends upon x_i, the decision of bank i. Interaction between banks thus takes place through the solvency constraint but not through the objective function. Let $\Phi := (\Phi_1, ..., \Phi_n)$ and let

$$K_{\Delta,\Phi} = K := \{x \in [0,1]^{np} : \forall i \in \mathcal{B}, x_i \in X_i(x_{-i})\} \subset [0,1]^{np}$$

be the set of admissible strategies of the generalized game. We now recall the definition of a Nash equilibrium for our game.

Definition 1 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the asset sale game if, for each $i \in \mathcal{B}$ and each $x_i \in [0,1]^n$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $L_i(x^*_i, x^*_{-i}) \leq L_i(x_i, x^*_{-i})$.*

3 Perfectly competitive markets

When markets are perfectly competitive, $\frac{1}{\Phi_j} = 0$ for all j. From equation (3), the risk-based capital ratio of bank i reduces to

$$\theta_i(\Delta, x_i) = \frac{E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j}{\sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij})} \tag{5}$$

so that the decision problem is not anymore strategic; the capital ratio only depends upon x_i.

Lemma 1 *The risk-based capital ratio $\theta_i(\Delta, x_i)$ is an increasing function of x_{ij} for each $i \in \mathcal{B}$ and each $j \in \mathcal{S}$.*

Proof. Since the numerator of $\theta_i(\Delta, x_i)$ is invariant with respect to x_{ij} while the denominator decreases with x_{ij}, the result follows. □

The next proposition provides a characterization of the optimal liquidation strategy. Note that the optimization problem reduces to a linear programming problem.

Proposition 1 ***(Characterization of the optimal strategy)*** *Under assumptions 1, 2, 3, when markets are perfectly competitive, there is a unique optimal liquidation vector $(x^*_{i,1}, ..., x^*_{i,2}, ..., x^*_{i,n}) \in [0,1]^n \backslash \{(1,1,...,1)\}$ of the form $(1, ..., 1, x^*_{i,h}, 0, ...0)$ where $x^*_{i,h} \in (0,1)$ for some integer $h \in \{1, ..., n\}$ is such that*

$$x^*_{i,h} = \frac{1}{\alpha_{i,h} q_{i,h} P_h (1 - \Delta_h)} \left[\sum_{j=h+1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j) \right]$$

Proof. The denominator of (5) tends to zero when x_{ij} tends to one for all j so that the capital ratio tends to infinity. Since $\theta_{i,min} < \infty$, a solution exists. For $x_i^* = (x_{i1}^*, ..., x_{in}^*)$, the constraint is clearly binding; $\theta_{i,t+1}(\Delta, x_i) = \theta_{i,min}$, that is,

$$\frac{E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j)(1-x_{ij}^*)} = \theta_{i,min} \tag{6}$$

which implies that:

$$\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) x_{ij}^* = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$$

Letting $X_{ij} = q_{ij} P_j (1-\Delta_j) x_{ij}^*$, and $C_i = \sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1-\Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j)$, equation (6) is equivalent to:

$$\sum_{j=1}^{n} \alpha_{ij} X_{ij} = C_i \tag{7}$$

Each bank i seeks to minimize $\sum_{j=1}^{n} X_{ij}$. Therefore, the unique solution consists in selling asset 1 with the highest risk weight α_{i1}, then asset 2 to asset $h \leq n$ until the capital ratio is restored. $\square$

Proposition 1 says that it is optimal for each bank i to first sell the asset with the highest regulatory weight. If this is not enough to restore the capital ratio, it is now optimal to sell 100% of asset 1 and a portion of risky asset 2. If this is not enough, it is optimal to sell 100% of asset 1 and 2 and a portion of risky asset 3 and so on and so forth. The optimal liquidation strategy follows the order of the weights.

4 Imperfectly competitive markets: existence result with Tarski's theorem

4.1 Preliminary results and main assumptions

We shall prove a few preliminary results.

Assumption 4

$$For\ each\ i \in \mathcal{B}\ \ and\ each\ x \in [0,1]^{np}, E_i(\Delta, x) > 0 \tag{8}$$

Lemma 2 *Under assumption 4, regardless of* $x_{-i} \in [0,1]^{n(p-1)}$, $X_i(x_{-i}) \neq \emptyset$.

Proof. From assumption 4, $E_i(\Delta, 1) > 0$ where 1 is the np-dimensional vector. Let $E_i(\Delta, 1) := \xi_i > 0$ and note that $E_i(\Delta, x_i, x_{-i}) \geq \xi_i$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$. Since for each i, $\text{RWA}_i(x_i)$ tends to zero when x_i tend to the n-dimensional vector 1, $\lim_{x_i \to 1} \frac{\xi_i}{\text{RWA}_i(x_i)} \to \infty$. Since $\theta_i(\Delta, x_i, x_{-i}) \geq \frac{\xi_i}{\text{RWA}_i(x_i)}$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$ and since $\theta_{i,min} < \infty$, there exists $x_i \in [0,1]^n \backslash \{1, ..., 1\}$ such that $\theta_i(\Delta, x_i, x_{-i}) = \theta_{i,min}$, hence $X_i(x_{-i}) \neq \emptyset$. $\square$

For notations simplicity, we may denote $\theta_i(\Delta, x_i, x_{-i})$ by $\theta_i(.)$.

Lemma 3 *Given* x_i, $\theta_i(.)$ *is a decreasing function of* x_{kj} *for all* $k \neq i$ *and all* $j = 1, ..., n$. *In particular,* $\theta_i(\Delta, x_i, x_{-i})$ *is a decreasing function of* x_{-i}.

Proof. Let $N(x) := E_i(\Delta, x)$ be the numerator of the risk-based capital ratio and $D(x)$ be its denominator (see equation (3)). Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k \neq i$:

$$\frac{\partial \theta_i(.)}{\partial x_{kj}} = \frac{q_{ij} P_j q_{kj}(1-\Delta_j)}{\Phi_j D(x)^2}[\alpha_{ij}(1-x_{ij})N(x) - D(x)]$$

which has same sign as $\alpha_{ij}(1-x_{ij})N(x) - D(x)$. We notice that $\alpha_{ij}(1-x_{ij}) \leq 1$, so if $N(x) < D(x)$, we have that $\frac{\partial \theta_i}{\partial x_{kj}} < 0$. Since $\theta_{i,min} << 1$, it is always true that $N(x) < D(x)$. Therefore, if $x_{-i} \leq y_{-i}$, we have that $\theta_i(x_i, x_{-i}) \geq \theta_i(x_i, y_{-i})$. □

Lemma 4 *When the market depths Φ are large enough, $\theta_i(.)$ is an increasing function of x_{ij} for each $j = 1, 2, ..., n$.*

Proof. Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k = i$:

$$\frac{\partial \theta_i}{\partial x_{ij}} = \frac{q_{ij} P_j (1-\Delta_j)}{D(x)^2}\left[\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1-x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x)\right] \tag{9}$$

which has same sign as $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1-x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x)$. From equation (8), $N(x) > 0$ for all x. From equation (9), it thus follows that when market depths are high enough, $\frac{\partial \theta_i(.)}{\partial x_{ij}} > 0$. □

Corollary 1 *There exists a critical smallest market depths vector $\Phi' := (\Phi'_1, ..., \Phi'_n)$ such that, regardless of $x_{-i} \in [0,1]^{n(p-1)}$, for each $i \in \mathcal{B}$ and each $j \in \mathcal{S}$, $\theta_i(.)$ is an increasing function of $x_{ij} \in [0,1]$.*

Assumption 5 *The market depths satisfy $\Phi \geq \Phi'$.*

4.2 Main result

Before proving the main result, we show that when the market depths are high enough, the optimal liquidation strategy is identical to the one found in Proposition 1.

Lemma 5 *There exists a critical market depths vector $\Phi^0 := (\Phi^0_1, ..., \Phi^0_n)$ such that, for all $\Phi \geq \Phi^0$, the optimal liquidation strategy of each bank $i \in \mathcal{B}$ is identical to the one in Proposition 1.*

Proof. See the appendix.

Assumption 6 *The market depths satisfy $\Phi \geq \Phi^0$.*

Lemma 6 *Under assumptions 1 to 6, $BR_i(x_{-i})$ is an non-decreasing function of x_{-i}.*

Proof. Let $x \in [0,1]^{np}$ and $y \in [0,1]^{np}$ such that $x \leq y$ so that $x_{-i} \leq y_{-i}$. When $\Phi \geq \Phi^0$, the best responses are given as in lemma 5. Let $BR_i(x_{-i}) = (1, ..., 1, x_{ih}, 0, ...0)$ and $BR_i(y_{-i}) = (1, ..., 1, y_{il}, 0, ..., 0)$. Since $\theta_i(BR_i(x_{-i}), x_{-i}) = \theta_i(BR_i(y_{-i}), y_{-i}) = \theta_{i,min}$, using the properties of the best responses, $\theta_i((1, ..., 1, x_{ih}, 0, ..., 0), x_{-i}) = \theta_i((1, ..., 1, y_{il}, 0, ..., 0), y_{-i}) = \theta_{i,min}$. Since $x_{-i} \leq y_{-i}$, from Lemma 3 we have that $\theta_{i,min} = \theta_i((1, ..., 1, x_{ih}, 0, ...0), x_{-i}) \geq \theta_i((1, ..., 1, x_{ih}, 0, ...0), y_{-i})$ and therefore $\theta_{i,min} = \theta_i((1, ..., 1, y_{il}, 0, ..., 0), y_{-i}) \geq \theta_i((1, ..., 1, x_{ih}, 0, ..., 0), y_{-i})$. From Lemma 4, when $\Phi \geq \Phi'$, since given z_{-i}, $\theta_i(z_i, z_{-i})$ is an increasing function of z_{ij} for all $j \in [1, .., n]$, we necessarily have that $l \geq h$ and in the case that $l = h$ we have that $y_{il} \geq x_{il}$. Therefore $BR_i(x_{-i}) = (1, ..., 1, x_{ih}, 0, ..., 0) \leq (1, ..., 1, y_{il}, 0, ..., 0) = BR_i(y_{-i})$, which concludes the proof $\square$

If the market depths are not high enough, then, there may exist $x \leq y$ such that for instance given x_{-i}, $\alpha_{i1}(1 - \delta_1(\Phi_1, x)) > \alpha_{i2}(1 - \delta_2(\Phi_2, x))$ for all x_i while given y_{-i}, $\alpha_{i2}(1 - \delta_2(\Phi'_2, y)) > \alpha_{i1}(1 - \delta_1(\Phi'_1, y))$ for all y_i. In such a case, the best responses are no more increasing.

Tarski's theorem (Tarski (1955), see also Vives (1990)). *Let $(L, \geq)$ be a complete lattice and f a non decreasing function from L to L and $\mathcal{F}$ the set of fixed points of f. Then, $\mathcal{F}$ is non-empty and $(\mathcal{F}, \geq)$ is a complete lattice. In particular, $\sup_x \mathcal{F}$ and $\inf_x \mathcal{F}$ belong to $\mathcal{F}$.*

Consider the lattice $([0, 1]^{np}, \leq)$ with $\leq$ defined by the natural order $x \leq y \Longleftrightarrow x_i \leq y_i$ for each $i = 1, ..., p$ where $x_i \leq y_i$ component-wise. Note that $[0, 1]^{np}$ is the product of compact intervals and thus is a complete lattice. Consider a function f from $([0, 1]^{np}, \leq)$ to $([0, 1]^{np}, \leq)$. We shall now show that the function f

$$f(x) = (BR_1(x_{-1}), ..., BR_p(x_{-p}))$$

is non-decreasing from $([0, 1]^{np}, \leq)$ to $([0, 1]^{np}, \leq)$ and apply Tarski's theorem to f.

Proposition 2 *Under assumptions 1 to 6, there exists a smallest Nash equilibrium $x^* = (x^*_1, ..., x^*_p) \in ([0, 1]^n \backslash \{(1, 1, ..., 1)\})^p$ to the generalized game defined in (4).*

Proof of proposition 2

Let us consider $x \in [0, 1]^{np}$ and $y \in [0, 1]^{np}$ such that $x \leq y$. Therefore, $x_{-i} \leq y_{-i}$ so for all $z_i \in [0, 1]^n$ we have that $\theta_i(z_i, x_{-i}) \geq \theta_i(z_i, y_{-i})$. This implies that $X_i(y_{-i}) = \{z_i \in [0, 1]^n; \theta_i(z_i, y_{-i}) \geq \theta_{i,min}\} \subset X_i(x_{-i}) = \{z_i \in [0, 1]^n; \theta_i(z_i, x_{-i}) \geq \theta_{i,min}\}$, and by assumption these two sets are not empty. Therefore, from Lemma 6, $BR_i(x_{-i}) \leq BR_i(y_{-i})$ for all i. Hence, f is a non-decreasing function from $([0, 1]^{np}, \leq)$ to $([0, 1]^{np}, \leq)$, and therefore it satisfies the assumptions of Tarski's theorem. As a consequence, the set of fixed points of f is not empty, so that there exists at least one Nash equilibrium. If there are multiple Nash equilibria, since they are ordered, by Tarski's theorem, $\inf_x \mathcal{F}$ belongs to $\mathcal{F}$, the set of Nash equilibrium and $x^* = \inf_x \mathcal{F}$ is the smallest Nash equilibrium. At equilibrium, each bank i is solvent so that it must be the case that $x^*_i \in [0, 1]^n \backslash \{(1, 1, ..., 1)\}$ $\square$

To the best of our knowledge, this is the first application of Tarski's theorem to a generalized game.

In Banerjee and Feinstein (2021), while they do not explicitly consider a generalized game, they offer a general extension of Braouezec and Wagalath (2019) in which each bank can go bankrupt. The strategy set of each bank i thus is extended and given by $\overline{X_i}(x_{-i}) = \{x_i \in [0, 1]^n, \ \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\} \cup \{(1, .., 1)\}$. Using Berge maximum theorem, they prove in Proposition 3.7 the existence of (at least) one Nash equilibrium but they do not characterize the optimal liquidation rule.

5 Epsilon-Nash equilibria

We now consider epsilon-Nash equilibria, similar in the spirit to Marinacci (1997). Let us rename and denote by $\bar{x}$ the optimal liquidation solution in case of no price impact as studied in Section 3, that is, a vector of the form $\bar{x} = (1, ..., 1, \overline{x}_{i,h}, 0, ..., 0)$. In this section, we prove the existence of epsilon-Nash equilibria and characterize a set of epsilon-Nash equilibria as a neighborhood of $\bar{x}$. The interest of these epsilon-Nash equilibria here is twofold: 1) Given $\epsilon > 0$ there always exists some market depths such that epsilon-Nash equilibria exist. 2) We can compute and describe a set of epsilon Nash equilibria, for any $\epsilon > 0$.

Definition 2 *$x^* = (x_1^*, ..., x_p^*)$ is an ϵ-Nash Equilibrium if:*
$\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, x_{-i}^*) \geq \theta_{i,min} :\ L_i(x_i^*, x_{-i}^*) - \epsilon \leq L_i(x_i, x_{-i}^*)$

Definition 3 *Given the market depths Φ_1, ..., Φ_n, we denote the set of admissible strategies of the strategic problem $K = K_{\Phi_1,...,\Phi_n}$.*

Proposition 3 *Under assumptions 1 to 3: $\forall \epsilon > 0$, there exist a neighborhood $V(\bar{x}, \epsilon)$ of $\bar{x}$ and $\Phi_1^0, ..., \Phi_n^0 > 0$ such that $\forall \Phi_1 \geq \Phi_1^0$,..., $\forall \Phi_n \geq \Phi_n^0$, $V(\bar{x}, \epsilon) \cap K_{\Phi_1,...,\Phi_n}$ is not empty and all its elements are ϵ-Nash equilibria.*

Proof of Proposition 3

We define $\tilde{\epsilon}_{ij} = \frac{\epsilon}{n \times q_{ij} P_j (1-\Delta_j)}$ and $\tilde{\epsilon} = (\tilde{\epsilon}_1, ..., \tilde{\epsilon}_p)$.

$\forall i,\ \theta_{i,t+1}(\bar{x}, \Phi = \infty) = \theta_{i,min}$, therefore $\theta_{i,t+1}(\bar{x} + \tilde{\epsilon}, \Phi = \infty) > \theta_{i,min}$, therefore since $\theta_{i,t+1}$ is continuous in Φ, there exist $\Phi^0 = (\Phi_1^0, ..., \Phi_n^0)$ such that $\forall \Phi \geq \Phi^0$, $\forall i,\ \theta_{i,t+1}(\bar{x} + \tilde{\epsilon}, \Phi) > \theta_{i,min}$. We consider such a Φ^0 and as from now we assume that $\Phi \geq \Phi^0$.

We notice that $L_i(\bar{x} + \tilde{\epsilon}) = \sum_{j=0}^{n} (\bar{x}_{ij} + \tilde{\epsilon}_{ij}) q_{ij} P_j (1 - \Delta_j) = \sum_{j=0}^{n} \bar{x}_{ij} q_{ij} P_j (1 - \Delta_j) + \epsilon = L_i(\bar{x}) + \epsilon$.

We also know that $\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq \theta_{i,min} :\ L_i(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq L_i(\bar{x})$. Indeed, $L_i(\bar{x})$ is the infimum cost for bank i.

Therefore, $\forall i \in \{1, ..., p\},\ \forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq \theta_{i,min} :\ L_i(x_i, \bar{x}_{-i} + \tilde{\epsilon}_{-i}) \geq L_i(\bar{x} + \tilde{\epsilon}) - \epsilon$, and therefore $\bar{x} + \tilde{\epsilon}$ is an ϵ-Nash equilibrium.

Consider $V(\bar{x}, \epsilon) = \prod]\bar{x}_{ij}, \bar{x}_{ij} + \epsilon_{ij}]$.

$\forall \Phi \geq \Phi^0$, $V(\bar{x}, \epsilon) \cap K_{\Phi_1,...,\Phi_n}$ is not empty and for $a \in V(\bar{x}, \epsilon) \cap K_{\Phi_1,...,\Phi_n}$, $f_i(a) \leq L_i(\bar{x}) + \epsilon$.

Also $\forall x_i \in [0,1]^n$ *s.t.* $\theta_{i,t+1}(x_i, a_{-i}) \geq \theta_{i,min}$: $L_i(x_i, a_{-i}) \geq L_i(\bar{x})$, which implies that $L_i(x_i, a_{-i}) \geq L_i(a) - \epsilon$ and therefore a is an ϵ-Nash equilibrium.

As a consequence, all elements of $V(\bar{x}, \epsilon) \cap K_{\Phi_1,...,\Phi_n}$ are ϵ-Nash equilibria. □

6 Technical proofs

Proof of Lemma 5 Given x_{-i}, the best response $(x_{i,1},...,x_{i,n})$ is such that the constraint is binding, that is,

$$\frac{E_i - \sum_{j=1}^n q_{ij}P_j(\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))}{\sum_{j=1}^n \alpha_{ij}q_{ij}P_j\,(1-\Delta_j)\left(1-\frac{\sum_{k\in\mathcal{B}} x_{kj}q_{jk}}{\Phi_j}\right)(1-x_{ij})} = \theta_{i,min} \tag{10}$$

We shall now show that, by suitably relabeling the quantities, equation (10) can be written, up to functions in $\frac{1}{\Phi_j}$, as (7). Define $\epsilon_j(\Phi_j,x) := \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}$. From eq (10), we have

$$\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)(1-\epsilon_j(\Phi_j,x))x_{ij} =$$
$$\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)(1-\epsilon_j(\Phi_j,x)) - \frac{E_i - \sum_{j=1}^n q_{ij}P_j(\Delta_j+\epsilon(\Phi_j,x))(1-\Delta_j))}{\theta_{i,min}}$$

Let $X_{ij} = q_{ij}P_j(1-\Delta_j)x_{ij}$ and $C_i = \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j) - \frac{1}{\theta_{i,min}}(E_{i,t} - \sum_{j=1}^n q_{ij}P_j\Delta_j)$. Equation (10) is therefore equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\epsilon_j(\Phi_j,x) + \frac{\sum_{j=1}^n q_{ij}P_j\epsilon_j(\Phi_j,x)(1-\Delta_j))}{\theta_{i,min}}$$

Define $\eta_j(\Phi_j,x_{-i}) = \epsilon_j(\Phi_j,x) - \frac{x_{ij}q_{ij}}{\Phi_j}$. Equation (10) is equivalent to:

$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\frac{x_{ij}q_{ij}}{\Phi_j} + \frac{\sum_{j=1}^n q_{ij}P_j\frac{x_{ij}q_{ij}}{\Phi_j}(1-\Delta_j))}{\theta_{i,min}} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

which is in turn equivalent to

$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} - \sum_{j=1}^n \frac{\alpha_{ij}q_{ij}}{\Phi_j}X_{ij} + \sum_{j=1}^n X_{ij}\frac{q_{ij}}{\Phi_j\theta_{i,min}} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

To get a more compact expression, let

$$\eta(\Phi,x_{-i}) = -\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$$

It thus follows that (10) is equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x) - \frac{\alpha_{ij}q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j\theta_{i,min}})X_{ij} = C_i + \eta(\Phi,x_{-i})$$

Letting now $\delta_j(\Phi_j,x) = \epsilon_j(\Phi_j,x) - \frac{\alpha_{ij}q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j\theta_{i,min}}$, equation (10) is finally equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\delta_j(\Phi_j,x))X_{ij} = C_i + \eta(\Phi,x_{-i})$$

an expression, up to functions in $\frac{1}{\Phi_j}$, identical to equation (7) and note that $\delta_j(\Phi_j, x) \leq \delta_j(\Phi_j, 1)$ for all $x \in [0,1]^{np}$.

By assumption, $\alpha_{i1} > \alpha_{i2} > ... > \alpha_{in}$, i.e., for all $j \in [2, .., n]$, $\frac{\alpha_{ij}}{\alpha_{ij-1}} < 1$ so that there exists $\delta^0_{ij} > 0$ such that $\frac{\alpha_{ij}}{\alpha_{ij-1}} = 1 - \delta^0_{ij}$. We know that for all i and all j, $\lim_{\Phi_j = \infty} \delta_{ij}(\Phi_j, 1) = 0$. There thus exists $\Phi^{i,0}_j$ such that for all $\Phi_j \geq \Phi^{i,0}_j$, $\delta_{ij}(\Phi_j, 1) < \delta^0_{ij}$. As a result, for all $\Phi^i \geq \Phi^{i,0} = (\Phi^{i,0}_1, .., \Phi^{i,0}_n)$ and all $x \in [0,1]^{np}$, we have

$$\alpha_{i1}(1 - \delta_{i1}(\Phi^i_1, x)) > \alpha_{i2}(1 - \delta_{i2}(\Phi^i_2, x)) > ... > \alpha_{in}(1 - \delta_{in}(\Phi^i_n, x))$$

Since bank i is seeking to minimize $\sum_{j=1}^n X_{ij}$ we are back in Proposition 1, that is, it is optimal to sell assets by decreasing risk weights. For this result to be true for all banks, it suffices to take $\Phi^0_j = \sup_i \Phi^{i,0}_j$, and $\Phi \geq \Phi^0 = (\Phi^0_1, .., \Phi^0_n)$. $\square$

7 Appendix: calculation of Φ^0 and Φ'

In this section, we propose to establish the values of Φ^0 and Φ'.

- Φ^0 :

  δ^0_{ij} is defined such that $\frac{\alpha_{ij}}{\alpha_{ij-1}} = 1 - \delta^0_{ij}$.

  We solve the inequation $\delta_{ij}(\Phi_j, 1) < \delta^0_{ij}$:

  $\delta_{ij}(\Phi_j, 1) < \delta^0_{ij} \iff \epsilon_j(\Phi_j, 1) - \frac{\alpha_{ij} q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j \theta_{i,min}} < \delta^0_{ij}$

  $\iff \frac{\sum_{k \in \mathcal{B}} q_{kj}}{\Phi_j} - \frac{\alpha_{ij} q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j \theta_{i,min}} < 1 - \frac{\alpha_{ij}}{\alpha_{ij-1}}$

  We can thus take: $\Phi^{i,0}_j = \frac{\sum_{k \in \mathcal{B}} q_{kj} - \alpha_{ij} q_{ij} + \frac{q_{ij}}{\theta_{i,min}}}{1 - \frac{\alpha_{ij}}{\alpha_{ij-1}}}$

  And $\Phi^0_j = \sup_i \Phi^{i,0}_j = \sup_i \frac{\sum_{k \in \mathcal{B}} q_{kj} - \alpha_{ij} q_{ij} + \frac{q_{ij}}{\theta_{i,min}}}{1 - \frac{\alpha_{ij}}{\alpha_{ij-1}}}$ for all j, which gives us Φ^0.

- Φ'

  To find Φ', we need to solve:

  $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1 - x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x) > 0$, for all i and j.

  We note that :

  $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1 - x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x) > \alpha_{ij} N(1)[1 - \frac{\sum_{k \in B} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(0)$

  We solve the inequation:

  $\alpha_{ij} N(1)[1 - \frac{\sum_{k \in B} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(0) > 0 \iff \Phi_j > \frac{q_{ij} D(0) + \sum_{k \in B} q_{kj}}{\alpha_{ij} N(1)}$

  $\iff \Phi_j > \frac{q_{ij} \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) + \sum_{k \in B} q_{kj}}{\alpha_{ij}[E_i - \sum_{j=1}^n q_{ij} P_j \times (\Delta_j + \frac{\sum_{k \in \mathcal{B}} q_{kj}}{\Phi_j}(1 - \Delta_j))]}$

  We can thus take:

  $\Phi'_j = \sup_i \frac{q_{ij} \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) + \sum_{k \in B} q_{kj}}{\alpha_{ij}[E_i - \sum_{j=1}^n q_{ij} P_j \times (\Delta_j + \frac{\sum_{k \in \mathcal{B}} q_{kj}}{\Phi_j}(1 - \Delta_j))]}$ for all j, which gives us Φ'.

Note importantly that the two market depth Φ^0 and Φ' only depend upon the input of the model. It would be easy to consider numerical examples within a two assets two banks framework.

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