Generalized games and Optimal Regulation in Economics and collective action problems. An application to bank regulation during fire sales.

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Abstract

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1 Introduction

Game theory has proven to be a usefool tool in many fields among which Economics, Political sciences, International relations and Social sciences (to name a few) since its introduction by [Morgenstern and Von Neumann, 1953], [Nash, 1950] and [Nash, 1951]. It appears to be the natural mathematical framework one would use to quantify, understand and predict the outcome of a particular economic or social interaction between a set of agents who have a strategical connection with one another. Remarkably, it has also proven to be a fruitful theory to create new ideas and new ways of thinking in collective actions problems: it has permitted the creation of new concepts, take for instance the Nash equilibria or Pareto efficiency, that give a new grasp and point of view in, for instance, Economics or Social sciences (and more broadly in every field it can be applied). It has frequently given birth to new paradigms, new theories that have profoundly marked these different disciplines. In other words, Game theory has not only a power of description of the reality in economics, social and political science, but it has also been the tool of creation of new economic, social and political realities. We refer to [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] for classical textbooks. Modestly, it is the aim of our paper to give a new grasp and propose new concepts about optimal regulation in game theory, that enable the creation of new economic and social realities.

In this article, we consider the point of view of a regulator, for instance a State or a Public policymaker, which has the ability to enforce a law and decisions on a group of N agents interacting with one another in a game theory framework: the goal of the regulator is to maximize a social welfare criterion (or minimize a social cost) for the community of agents. Indeed, it is often the case that when agents act individually in maximizing their own payoff function (or minimizing their own cost) without the help/action of a regulator, the consequences of isolated individual actions can lead to devastating consequences for the whole community of agents.

The Financial crisis of 2007-2008 is a striking example of a situation where the absence of regulation and coordination between banks, and particularly Global Systemically Important Banks (G-SIBs), on the global scale (on certain issues at least) has caused dramatic consequences on the whole financial system and world economies. Such absence of initial regulation has partly been resolved with the Basel Accords issued by the Basel Committee on Banking Supervision (BCBS).

The global environmental crisis we are currently facing and the difficulty in finding a regulation that satisfies all the countries and compatible with the economic challenges and ambitions of every country is another striking problem.

More generally, it is interesting to consider the point of view of a regulator in a framework where N agents are interacting with one another in a game theory framework and to wonder what is the best decision from a regulator point of view. A prerequisite to answer this question is: what is the good criterion of social welfare from the regulatory point of view? And once this criterion has been defined, is there an optimal regulation that maximizes this criterion?

There has been an abundant literature on Optimal Regulation in the past decades, the books [Laffont and Tirole, 1993], [Train et al., 1991], [Laffont and Martimort, 2009] and the article [Laffont, 1994] give a good overview of the research that has been done on this topic. We take a new roadmap in our article in the sense that we consider directly a regulator on a game theory framework: for a given game, we define one or several regulation criteria and we study existence of an optimal regulation for each criterion.

In our framework, the regulator is seeking to maximize a social welfare criterion (or minimize a social cost criterion), the variable is the regulation itself and we are looking for the possible optimal regulation that maximizes (respectively minimizes) this given criterion of social welfare (respectively social cost). Various examples of such criteria are given in the article.

It appears that generalized games seem to be the best tool to model the action of a regulator on a system of agents interacting with one another. Generalized games, also called Generalized Nash Equilibrium Problems (GNEPs for short), have first been introduced by [Arrow and Debreu, 1954], and have gained an increasing attention in operational research over the past decades. In their well-known survey, [Facchinei and Kanzow, 2007] give a global overview of generalized games and remind interesting examples of applications of such games in environment policy or in telecommunication. In the operations research literature more generally, there has been an increasing number of papers on generalized games in recent years offering new methods, existence results or numerical algorithms to find a Nash equilibrium (see for instance [Facchinei et al., 2009], [Aussel and Dutta, 2008], [Fischer et al., 2014]). However, to the best of our knowledge, there has been no paper trying to apply generalized games to optimal regulation. And only a few papers so far are trying to apply generalized games in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020], and see [Kulkarni, 2017] for a short review).

The purpose of our article is to draw and propose an optimal regulation road map for any game theory model where the action of a regulator is relevant: first we consider the different regulation criteria of social welfare that are relevant from the regulator point of view, then we study the existence of an optimal regulation for each regulation criterion. Such a research roadmap gives answers to many questions on the regulatory point of view, among which: is there existence of at least an optimal regulation for each regulation criterion? Is there uniqueness of the optimal regulation? Are some regulation criteria equivalent: are there examples where a criterion A and a criterion B have the same optimal regulation? When can we say that it is equivalent to optimize a regulation criterion A and a regulation criterion B? It is precisely the aim of this paper to answer these different questions.

For instance, we will study the example of a financial regulator trying to implement a regulation (enforcing some laws and taking some decisions) to maintain the stability of the financial system during a financial crisis: what are the good criteria from a financial regulation perspective? Minimize the losses in the financial systems? Minimize the number of banks and companies going bankrupt? Minimize the number of jobs destroyed by the crisis? Can we prove that these minimization problems are equivalent and strongly correlated? Or are they not correlated at all?

Such a complete study gives a global overview of the questions a regulator can wonder and the levers of action at its disposal to complete its social goal and mission: it is the role of the regulator and policymaker to decide what is the good criterion of social welfare for the collective well-being of a community of economic agents, and therefore what is the optimal regulation. Our inquiry gives a radiography of the different mechanisms of action between the regulator and the agents, what is the best policy to enforce for the social welfare of a community of agents and what are the consequences of the decision of a regulator on this community.

The paper is structured as follows: in Section 2 we remind a few definitions and properties about generalized games with individual and shared constraints. In Section 3 we introduce the definitions of a regulation, regulation criterion and optimal regulation on a game. We study the particular case of existence of optimal vector(s). We also introduce the definitions of a surregulation, subregulation and correlated regulation criteria. We give a few examples and properties. In Section 4 we study in detail a model of strategic fire-sales and price-mediated contagion in the banking system inspired from [Braouezec and Wagalath, 2019]: we consider different regulation criteria and we look for the existence of possible optimal regulations. Section 5 concludes the paper.

2 Generalized games with individual or shared constraints

2.1 Generalized games: notations and definitions

We take the following classical notations throughout the paper, which are similar to the ones in cf article Braouezec-Kiani: we consider a Game with N players, we denote $J = \{1, .., N\}$ the set of players, each player $i \in J$ controlling the variable $x_i \in E_i$, with E_i a subset of $\mathbb{R}^{n_i}$; x_i is called the strategy or decision or state of agent i and E_i is called the strategy set. We denote by $x \in E = E_1 \times \ldots \times E_N$ the vector formed by all these decision variables:

$$x := \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \in E \tag{1}$$

which has dimension $n := \sum_{i=1}^{N} n_i$ and such that $E \subset \mathbb{R}^n$. We denote by $x_{-i} \in E_{-i}$ the vector formed by all the players' decision variables except those of player i. To emphasize the i-th player's variables within x, we sometimes write $(x_i, x_{-i}) \in E_i \times E_{-i}$ instead of $x \in E$.

Each player has an objective function $\theta_i : \mathbb{R}^n \to \mathbb{R}$ that depends on both her own variable x_i as well as on the variables x_{-i} of all other players. This mapping θ_i is often called the utility function or payoff function of player i when the agents are seeking to maximize $\theta_i(x_i, x_{-i})$, or it can also be called the loss function or cost function of player i when the agents are seeking to minimize $\theta_i(x_i, x_{-i})$, depending on the particular application in which the Game arises. Throughout the article, if not mentioned otherwise, we will say that θ_i is a payoff function and that each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to maximize $\theta_i(x_i, x_{-i}) = \theta_i(x)$.

A classical Game, or Nash Equilibrium Problem (NEP for short), is the given of the N maximization problems, for $i \in J = \{1, .., N\}$:

$$\text{maximize}_{x_i} \quad \theta_i(x_i, x_{-i}) \text{ subject to } x_i \in E_i \tag{2}$$

and it is defined in a unique way by the 3-uple $(J, E, (\theta_i)_{i\in J})$.

A ganeralized game, or General Nash Equilibrium Problem (GNEP for short), is a Game where each player $i \in J$ needs additionally to satisfy a constraint given by a constraint function $X_i : E_{-i} \to P(E_i)$ (where $P(E_i)$ denotes the set of all subsets of E_i), such that each player's strategy x_i must belong to the set $X_i(x_{-i}) \subset E_i$ that depends on the rival players' strategies and that we call the feasible set or strategy space of player i. For instance, as we will see and develop in the section, the constraint functions X_i, $i \in J$ can be given by a regulator (in economics, political science, international relations, social science...).

If $x \in E$ is such that $x_i \in X_i(x_{-i})$ for a given agent i we will say that the strategy vector x is admissible for agent i or the constraint is satisfied on the individual level for i. And if $x_i \in X_i(x_{-i})$ for all $i \in J$ we will say that the strategy vector x is admissible or the constraint is satisfied on the global level. If there is $i \in J$ such that $x_i \notin X_i(x_{-i})$, then we will say that the strategy vector x is not admissible for agent i or the constraint is not satisfied on the individual level for i, and therefore we will also say that the strategy vector x is not admissible.

Each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to minimize $\theta_i(x_i, x_{-i}) = \theta_i(x)$ and respecting the constraint $x_i \in X_i(x_{-i})$. A generalized game, or GNEP, is therefore the given of the N constrained maximization problems, for $i \in J = \{1, .., N\}$:

$$\text{maximize}_{x_i} \quad \theta_i(x_i, x_{-i}) \text{ subject to } x_i \in X_i(x_{-i}) \tag{3}$$

and it is defined in a unique way by the 4-uple $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Remark 1 *We also notice that a classical game $(J, E, (\theta_i)_{i\in J})$ is a particular case of generalized game with:*

$$\begin{array}{rccc} X_i : & E_{-i} & \to & P(E_i) \\ & x_{-i} & \mapsto & E_i \end{array}$$

Actually, to emphasize the fact that each player has a given exogenous individual constraint function X_i, we often find in the literature (see for instance [Fischer et al., 2014] or Braouezec-Kiani) the denomination generalized game with individual constraints to name such generalized games $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Definition 1 *The 4-uplet $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ is called a generalized game with individual constraints.*

We can also clearly define the set of admissible strategies of a generalized game.

Definition 2 *For a given generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$*, let* K *be the subset of* E *defined as follows.*

$$K = \{\boldsymbol{x} \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} \tag{4}$$

K is called the set of admissible strategies of the generalized game with individual constraints.

The set K represents the set of all the strategies $\boldsymbol{x}$ for which the GNEP with individual constraints is defined for all agents.

Now we can introduce the definition of a Nash equilibrium for a generalized game with individual constraints. A generalized Nash equilibrium is a profile of strategies $\boldsymbol{x}^* = (x_1^*, ..., x_N^*) \in K$ such that no agent i wants to unilaterally deviate from her part of the equilibrium profile $\boldsymbol{x}^*$. However, for $\boldsymbol{x}^* \in E$ to be an equilibrium profile, the constraint of each agent $i \in J$ should be satisfied. The following definition makes clear this constraint.

Definition 3 *The profile of strategies* $\boldsymbol{x}^* \in E$ *is a Nash equilibrium for the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *if for each* $i \in J$ *and each* $x_i \in E_i$ *such that* $x_i \in X_i(x_{-i}^*)$*, it holds true that* $\theta_i(x_i^*, x_{-i}^*) \geq \theta_i(x_i, x_{-i}^*)$*.*

We notice that this definition is well consistent with the definition of a Nash equilibrium in a classical game.

The next subsection gives a few reminders about generalized games with shared constraints and their properties.

2.2 Generalized games with shared constraints

Generalized games with shared constraints were first initiated by Rosen in his famous article [Rosen, 1965] (though the author does not mention explicitly the terminology game with shared constraint in his paper) where he proves quite remarkably an existence result about Nash equilibria for concave n-person games not only on cartesian products of strategy spaces $E = \prod_{i=1}^{N} E_i$, but also on any convex, closed and bounded subset $X \subset E$ (that we call shared constraint), with the assumption that each player's payoff function is continuous and concave in x_i. Such a game is called generalized game with shared constraint in the sense that all the agents share the same constraint. The vector of strategies $x := (x_1, ..., x_n)$ must always remain in the shared constraint set X: given x_i, each agent i is required to pick a strategy $x_i \in E_i$ such that $x := (x_i, x_i) \in X$. Generalized games with shared constraints are a particularly interesting subclass of generalized games as there are many existence results and methods already known for them.

The following definition of generalized game with shared constraint can be found in [Fischer et al., 2014].

Definition 4 *Let $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ be a generalized game with individual constraints. We say that $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ is a generalized game with shared constraint if there exists a subset $X \subset E$ such that for all $i \in J$, $X_i(x_{-i}) = \{x_i, (x_i, x_{-i}) \in X\}$. We will denote this generalized game with shared constraint with the 4-uplet $(J, E, (\theta_i)_{i\in J}, X)$*

Now we can introduce the corresponding definition of a Nash equilibrium for a generalized game with shared constraint.

Definition 5 *$x^* = (x_1^*, .., x_N^*) \in X$ is a Nash equilibrium for the generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$ if and only if:*
$\forall i \in J,\ \forall x_i \in E_i$ such that $(x_i, x_{-i}^*) \in X,\ \theta_i(x_i^*, x_{-i}^*) \geq \theta_i(x_i, x_{-i}^*)$

In the literature, it is quite common to assume that the set X is equal to $X = \{x \in E \mid G(x) \leq 0\}$ where $G : \mathbb{R}^n \to \mathbb{R}$ is a componentwise convex function called the shared (or common) constraint function. This hypothesis is quite convenient for the study of the generalized game, as for instance in [Facchinei and Kanzow, 2010] or [Fischer et al., 2014]. We can also find some articles with hypotheses of the form $X_i(x_{-i}) = \{x_i,\ g_i(x_i, x_{-i}) \leq 0,\ G(x_i, x_{-i}) \leq 0\}$ where $g_i : \mathbb{R}^n \to \mathbb{R}$ represents individual constraints and $G : \mathbb{R}^n \to \mathbb{R}$ the shared ones, as for instance in [Fischer et al., 2014].
We can also remind the result from [Rosen, 1965]:

Theorem 1 *([Rosen, 1965]) Let $(J, E, (\theta_i)_{i\in J}, X)$ be a game with an exogenous shared constraint where θ_i is a payoff function. If the set X is convex, closed and bounded and if each player's payoff function $\theta_i(x_i, x_{-i})$, $i \in J$ is continuous and concave in x_i, then, the generalized game has at least one Nash equilibrium.*

We also note the fact that generalized games with shared constraints are quite convenient because under certain natural hypotheses on X and the cost functions θ_i, $i \in J$, it is very often possible to use a fixed-point theorem (Kakutani, Schauder, Tarski,...) to prove the existence of Nash equilibria.

2.3 Endogenous shared constraint generated from individual constraints

In this subsection, we remind different concepts and results that were introduced in Braouezec-Kiani, and that will be of great help in our inquiry about Optimal Regulations. Given a generalized game with exogenous individual constraints, we will introduce the definition of the generalized game with endogenous shared constraint generated from these individual constraints.

Recall that for the profile $\boldsymbol{x}$ to be a Nash equilibrium, a necessary but not sufficient condition is that $\boldsymbol{x} \in K$, given by equation 4. An endogenous shared constraint generated from individual constraint is the situation where it is required from each agent that, given what the other agents are choosing, i.e., x_{-i}, agent i should pick a strategy x_i such that the profile $\boldsymbol{x} = (x_i, x_{-i})$ lies in K

(which makes economic sense). Given x_{-i}, let $K_i(x_{-i})$ be the set of strategies of agent i defined as follows

$$K_i(x_{-i}) = \{x_i \in X_i(x_{-i}) : \boldsymbol{x} \in K\} \tag{5}$$

Now we can define a generalized game with endogenous shared constraint, that is, a generalized game in which the shared constraint is generated from the individual constraints.

Definition 6 *The* 4*-uplet* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *is called a generalized game with shared constraint generated from the game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

And we can therefore give the definition of a Nash equilibrium in a game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$:

Definition 7 *The profile of strategies* $\boldsymbol{x}^* \in E$ *is a Nash equilibrium for the generalized game with endogenous shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *if for each* $i \in J$ *and each* $x_i \in E_i$ *such that* $x_i \in K_i(x^*_{-i})$*, it holds true that* $\theta_i(x^*_i, x^*_{-i}) \geq \theta_i(x_i, x^*_{-i})$.

In the article, to emphasize that the set K is the shared constraint, we will denote this game with endogenous shared constraints $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ as $(J, E, (\theta_i)_{i\in J}, K)$.

In a game in which the shared constraint is generated from the individual ones, the set of strategies of each agent i in the game with endogenous shared constraint may be *reduced* compared with the game with individual constraints, that is, for x_{-i}, the following inclusion holds

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \quad \forall i \in J \tag{6}$$

In general, the inclusion may be strict for some agents, that is, as long as $X_i(x_{-i})$ is not empty, $K_i(x_{-i}) \subsetneq X_i(x_{-i})$.

The interesting feature of games with shared constraints is that these games may possess *additional equilibrium situations*. It may thus be the case that while there is no Nash equilibrium in the game with individual constraints, a Nash equilibrium exists in the game with shared constraint. This is the main idea of the following result.

Proposition 1 *The set of Nash equilibria of a game with individual constraints is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints, that is, if* $\boldsymbol{x}^* = (x^*_1, ..., x^N_N) \in E$ *is a Nash equilibrium for the game with individual constraints, it is also a Nash equilibrium for its generated game with shared constraint but the converse is not true. In other words, the set of Nash equilibria of* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *is included in the set of the Nash equilibria of the generated game with endogenous shared constraint* $(J, E, (\theta_i)_{i\in J}, K)$*, and this inclusion is not an equality a priori.*

Proof. See Braouezec-Kiani.

This is an interesting result. This means that when looking for the Nash equilibria of a generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, we can first have a look at the Nash equilibria of its generated game with shared constraint $(J, E, (\theta_i)_{i\in J}, K)$, which is a much easier thing to study in general since, as we explained before, we already have many existence results and methods on the generalized games with shared constraint. The Nash equilibria of $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, if they exist, will be among the Nash equilibria of $(J, E, (\theta_i)_{i\in J}, K)$. On the other hand, if $(J, E, (\theta_i)_{i\in J}, K)$ does not have any Nash equilibrium, this means that $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ does not have any Nash equilibrium either.

This has important consequences in regulatory economics as well, as we will see. This means that if a regulator is seeking to get a Nash equilibrium for the collective well-being of a group of agents interacting with one another, this is more likely to happen if the regulation constraint is shared than if the regulation constraint is individual.

This is why we will introduce in Section 3 the concept of regulation with shared constraint generated from a regulation with individual constraints: we consider a situation where economic agents $i \in J$, interacting with one another are trying to maximize a payoff θ_i and a regulator is requiring that each agent satisfies an individual constraint X_i, which we call a regulation with individual constraints. In other words, the regulator asks that $x_i \in X_i(x_{-i})$ for all i. This is clearly a generalized game with individual constraints. The regulator can decide to turn to the regulation with shared constraint generated from this regulation if it asks that for all i, $(x_i, x_{-i}) \in K$, with K the shared constraint set generated from the previous generalized game with individual constraints. We can give the following illustration: let's assume that X_i is of the form $X_i(x_{-i}) = \{x_i, g_i(x_i, x_{-i}) \leq 0\}$ where $g_i : \mathbb{R}^n \to \mathbb{R}$. The shared constraint set generated from these individual constraints is

$$K = \{x = (x_1, .., x^N), \begin{pmatrix} g_1(x) \leq 0 \\ \vdots \\ g_N(x) \leq 0 \end{pmatrix} \} \subset E \tag{7}$$

and K gives us a regulation with shared constraint; an agent i can move from a state x_i to a state y_i only if it sure that for all economic agents i', $g_{i'}(y_i, x_{-i}) \leq 0$ is satisfied.

2.4 An existence result: Shared constraint, Nash equilibrium and maximization of the global payoff (or minimization of the global cost)

We now present an interesting existence result for generalized games with shared constraint that will be useful in Section 3 and 4.

Proposition 2 *Let $(J, E, (\theta_i)_{i\in J}, X)$ be a generalized game with shared constraint. We assume that X is a compact set and we also assume that $\forall i \in J$, θ_i is continuous and of the form $\theta_i(x) = \theta_i(x_i)$. Then this generalized game with shared constraint has at least one Nash equilibrium x^* that maximizes the global payoff $\sum_{i\in J} \theta_i(x)$.*

Proof. See the appendix.

Actually the Nash equilibria x^* for this generalized game coincide exactly with the local maxima of the function $F(x)$.

Obviously, this result is also true if the $\theta_i, i \in J$ are cost functions instead of payoff functions: in this case, there will be a Nash equilibrium that minimizes the global cost and the Nash equilibria x^* for such a generalized game coincide exactly with the local minima of the function $F(x)$.

We have 2 results at the same time: if $(J, E, (\theta_i)_{i \in J}, X)$ is satisfying the assumptions of Proposition 2, there is existence of at least one Nash equilibrium, and moreover there is a Nash equilibirum that maximizes the global payoff for all the agents.

This is again an interesting result from a regulatory point of view in economics: in a setting where economic agents are trying to maximize (or minimize) a continuous payoff function (respectively cost function) depending on their only actions and are interacting with one another through a shared constraint required by a regulator with a compact constraint set X then, this strategic interaction has a Nash equilibrium that maximizes the global payoff (respectively minimizes the total cost). This means as well that if the same problem is with individual constraints from the regulator and if the shared constraint set generated from the individual constraints is compact, we have a Nash equilibrium that maximizes the global payoff (respectively minimizes the total cost) for the regulation with shared constraint, which is a remarkable fact.

Actually this result is as good as what we would get with a social planner in a planned economy. Assume that instead of having N agents/players (each of which maximizing their payoff function individually, in a non-cooperative way) we would have a social planner controlling the N variables $(x_1, ..., x_N)$ and trying to maximize the global payoff $\sum_{i \in J} \theta_i(x)$. This social planner would exactly get a Nash equilibrium x^* as best solution and in any case could not get a better configuration than such a x^*.

We will see an application of this result in Section 3 with Optimal Regulations and an example in Section 4 in bank regulation.

3 Generalized Games and Optimal Regulation

3.1 Regulation and Regulation criterion

Now that we have introduced generalized games with individual and shared constraints, and reminded a few properties about them, we can turn to the central development of our article, that is defining what is an optimal regulation in a game theory framework. First, we need to define what is a regulation on a set of N agents interacting with one another in a game theory framework, and this concept intuitively boils down to giving a $N-uple$ of individual constraints $(X_i)_{i \in J}$: each agent i has to satisfy the constraint X_i given by a regulator. For instance, see Braouezec-Kiani, in an environment policy problem X_i can be a threshold of emission of greenhouse gaz given by a regulator, or in a public good problem X_i can give the minimum price to pay for agent i ta have the public good built. In Section 4, we will detail an example in bank regulation where X_i is a threshold given to banks by the regulator via the Basel agreements. The definition below makes clear what a regulation in game theory is.

Definition 8 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is called a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$ *and the agents* $i \in J$.

Then, we need to define what is a criterion of well-being - that we will call regulation criterion - for the community of agents that the regulator is seeking to maximize (or minimize) by choosing the right regulation(s). Intuitively, and staying as much general as we can in defining our concepts, a regulation criterion is a function of well-being that the regulator is seeking to maximize (or minimize) for the collective well-being of a group agents. For instance, in a public good problem it can be the minimum price to pay for the community to get the good, or the most comfortable good for the community of agents, etc. In environmental policy, the goal of the regulator is to minimize the emission of greenhouse gases. In Section 4, we will study an example in bank regulation where the regulator is either seeking to minimize the financial losses in the system, or minimize the number of banks going bankrupt. We will also see that in a period of crisis and instability the regulator can be seeking to get a Nash equilibrium as fast as possible which guarantees the end of instability, and therefore will prefer a regulation giving a large number of Nash equilibria that enable a diminution of volatilty and instability (a system of agents in a Nash equilibrium configuration does not move over time). The definition below makes clear the general concept of regulation criterion.

Definition 9 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *We call regulation criterion (or regulation criterion of evaluation) on the Game* $(J, E, (\theta_i)_{i\in J})$ *any application* R *of the form:*

$$\begin{array}{rcccc} R & : & (F(E_{-i}, P(E_i)))^N & \rightarrow & (F, \leq) \\ & & (X_i)_{i\in J} & \mapsto & R((X_i)_{i\in J}) \end{array}$$

with $(F, \leq)$ *a partially ordered set.*

Note that if R_1 and R_2 are two criteria on the Game $(J, E, (\theta_i)_{i\in J})$ with values in $(F, \leq)$ and g is an application from $(F, \leq)^2$ to a partially ordered set $(G, \leq)$, then $g(R_1, R_2)$ is also a criterion on the Game $(J, E, (\theta_i)_{i\in J})$. More generally, if R_1, R_2,..,R_p are p criteria on the Game $(J, E, (\theta_i)_{i\in J})$ with values in $(F, \leq)$ and g is an application from $(F, \leq)^p$ to a partially ordered set $(G, \leq)$, then $g(R_1, R_2, .., R_p)$ is also a criterion on the Game $(J, E, (\theta_i)_{i\in J})$.

We will study 4 examples of regulation criteria in this subsection, two criteria that we will call stability-type criteria (in the sense that the goal of the regulator is to maximize stability) and two payoff-type criteria (in the sense that the goal of the regulator is to maximize the global payoff).

We start with our 2 examples of stability-type criteria.

Example 1 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$.

We can define the regulation criterion:

$$\begin{array}{rcccc} Nash & : & (F(E_{-i},P(E_i)))^N & \rightarrow & (E, \subset) \\ & & (X_i)_{i\in J} & \mapsto & Nash((X_i)_{i\in J}) \end{array}$$

with $Nash((X_i)_{i\in J})$ *the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

This regulation criterion is called the Nash-set stability criterion. If $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *are two regulations such that the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (Y_i)_{i\in J})$ *contains the set of Nash equilibria of the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, *then* $Nash((X_i)_{i\in J}) \subset Nash((Y_i)_{i\in J})$. *We note that* $(E, \subset)$ *is not a totally ordered set.*

Example 2 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and we assume that* E *is a finite set.* $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$ *and since* E *is a finite set, the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *has a finite number of Nash equilibria.*

We can define the regulation criterion:

$$\begin{array}{rcccc} Stab & : & (F(E_{-i},P(E_i)))^N & \rightarrow & (\mathbb{N}, \leq) \\ & & (X_i)_{i\in J} & \mapsto & \text{number of Nash equilibria of } (J,E,(\theta_i)_{i\in J}, (X_i)_{i\in J}) \end{array}$$

This regulation criterion is called the stability criterion. The more Nash equilibria given by the regulation $(X_i)_{i\in J}$ *on the Game* $(J, E, (\theta_i)_{i\in J})$, *the higher* $R((X_i)_{i\in J})$, *and the higher the stability criterion: if* $(X_i)_{i\in J}$ *and* $(Y_i)_{i\in J}$ *are two regulations such that the generalized game* $(J, E, (\theta_i)_{i\in J}, (Y_i)_{i\in J})$ *has more Nash equilibria than the generalized game* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, *then* $Stab((Y_i)_{i\in J}) \geq Stab((X_i)_{i\in J})$.

Proposition 3 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

Then, $Nash((X_i)_{i\in J}) \subset Nash((K_i)_{i\in J})$.

Moreover, if E *is finite set,* $Stab((X_i)_{i\in J}) \leq Stab((K_i)_{i\in J})$.

Proof of Proposition 3

This is a direct consequence of Proposition 1. □

We now study our two examples of payoff-type regulation criteria.

Example 3 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ *is a regulation on the Game* $(J, E, (\theta_i)_{i\in J})$. *In this example we assume that* θ_i *is a payoff function (and not a cost function).*

We can define the regulation criterion:

$$
\begin{array}{rcccc}
T & : & (F(E_{-i},P(E_i)))^N & \rightarrow & (\mathbb{R},\leq) \\
 & & (X_i)_{i\in J} & \mapsto & sup_{x\in K((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x)
\end{array}
$$

with $K((X_i)_{i\in J})$ the set of admissible strategies of the generalized game $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$. $T((X_i)_{i\in J})$ is the supremum of the total payoffs among all the admissible strategies of the generalized game $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$. If this supremum is higher for the regulation $(Y_i)_{i\in J}$ than for the regulation $(X_i)_{i\in J}$, then $T((Y_i)_{i\in J})\geq T((X_i)_{i\in J})$. Of course, if there exists x such that $T((X_i)_{i\in J})=\sum_{i=1}^N \theta_i(x)$ then such a vector x is Pareto Optimal.

Indeed, if θ_i is a cost function instead of a payoff function, we can define this regulation criterion in a symmetric way $T(X_i)_{i\in J})=inf_{x\in K((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x)$, and here is the goal of a regulator is to minimize a global cost instead of maximizing a global payoff.

Proposition 4 *We consider the generalized game with individual constraints $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$ and the generalized game with shared constraint $(J,E,(\theta_i)_{i\in J},(K_i)_{i\in J})$ generated from $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$.*

Then, $T((X_i)_{i\in J})=T((K_i)_{i\in J})$.

Proof of Proposition 4

By definition, $K((X_i)_{i\in J})=K((K_i)_{i\in J})$ and therefore $T((X_i)_{i\in J})=T((K_i)_{i\in J})$. $\square$

Example 4 *We consider the generalized game with individual constraints $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$. $(X_i)_{i\in J}$ is a regulation on the Game $(J,E,(\theta_i)_{i\in J})$. In this example we assume that θ_i is a payoff function (and not a cost function).*

We can define the regulation criterion:

$$
\begin{array}{rcccc}
TN & : & (F(E_{-i},P(E_i)))^N & \rightarrow & (\mathbb{R},\leq) \\
 & & (X_i)_{i\in J} & \mapsto & sup_{x^*\in Nash((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x^*)
\end{array}
$$

with $Nash((X_i)_{i\in J})$ the set of Nash equilibria of the generalized game $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$.

$TN((X_i)_{i\in J})$ is the supremum of the total payoffs among all the Nash equilibria of the generalized game $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$. If this supremum is higher for the regulation $(Y_i)_{i\in J}$ than for the regulation $(X_i)_{i\in J}$, then $TN((Y_i)_{i\in J})\geq TN((X_i)_{i\in J})$. Of course, if there exists x^* such that $TN((X_i)_{i\in J})=\sum_{i=1}^N \theta_i(x^*)$ then such a vector x^* is Pareto Optimal.

Same remark as for Example 3, if θ_i is a cost function instead of a payoff function, we can define this regulation criterion in a symmetric way $TN(X_i)_{i\in J})=inf_{x^*\in Nash((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x^*)$, and here is the goal of a regulator is to minimize a global cost instead of maximizing a global payoff.

Proposition 5 *Given a generalized game with individual constraints $(J,E,(\theta_i)_{i\in J},(X_i)_{i\in J})$, we have that $T((X_i)_{i\in J})\geq TN((X_i)_{i\in J})$*

Proof of Proposition 5

$sup_{x\in K((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x)\geq sup_{x^*\in Nash((X_i)_{i\in J})}\sum_{i=1}^N \theta_i(x^*)$ since $Nash((X_i)_{i\in J})\subset K((X_i)_{i\in J})$.
$\square$

Proposition 6 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.
Then, $TN((X_i)_{i\in J}) \leq TN((K_i)_{i\in J})$.

Proof of Proposition 6

From Proposition 3 we have that $Nash((X_i)_{i\in J}) \subset Nash((K_i)_{i\in J})$ and therefore $TN((X_i)_{i\in J}) \leq TN((K_i)_{i\in J})$. □

Proposition 7 *Consider a generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, X)$ *satisfying the assumptions of Proposition 2, that is* X *is a compact set, and that* $\forall i \in J$, θ_i *is continuous and of the form* $\theta_i(x) = \theta_i(x_i)$ *. For such a game, we have that* $T(X) = TN(X)$.

Proof of Proposition 7

See Proof of Proposition 2. □

3.2 Optimal Regulation

Now that we have defined what a regulation is and what a regulation criterion is, we can easily define a concept of optimal regulation, and study existence of one or several optimal regulation(s) for any game. In this subsection, we assume that the regulator is seeking to maximize a regulation criterion R, function of collective well-being (it is symmetric if we assume that the regulator is seeking to minimize a regulation criterion).

First, we explain what we mean when we say that a regulation $(Y_i)_{i\in J}$ is better than a regulation $(X_i)_{i\in J}$ for the criterion R, which is quite intuitive.

Definition 10 *We say that a regulation* $(Y_i)_{i\in J}$ *is better than a regulation* $(X_i)_{i\in J}$ *for the criterion* R *on the Game* $(J, E, (\theta_i)_{i\in J})$ *if* $R((Y_i)_{i\in J}) \geq R((X_i)_{i\in J})$. *We denote* $(Y_i)_{i\in J} \geq (X_i)_{i\in J}$ *for the criterion* R.

Of course, the definition of "better"' is symmetric if the goal of the regulator is to minimize a social cost rather than maximize a function of social welfare. If the goal of the regulator is to minimize a social cost, we say that a regulation $(Y_i)_{i\in J}$ is better than a regulation $(X_i)_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if $R((Y_i)_{i\in J}) \leq R((X_i)_{i\in J})$.

The corollary below gives us a direct example of a regulation better than another regulation not only for one regulation criterion but actually for four regulation criteria: we proved in the previous subsection that for our 4 examples of regulation criteria, $Nash$, $Stab$, T and TN, we always have that the regulation with endogenous shared constraint is better than the one with individual constraints.

Corollary 1 *We consider the generalized game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ *and the generalized game with shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *generated from* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *Then, from Proposition 3, 4 and 6,* $(K_i)_{i\in J}$ *is a better regulation than* $(X_i)_{i\in J}$ *for the criteria* $Nash$, $Stab$ *(if* E *is a finite set),* T *and* TN.

This corollary is an interesting result in the sense that if a regulator is seeking to maximize one of the 4 criteria $Nash$, $Stab$, T and TN, this regulator should always choose the regulation with endogenous shared constraint rather than the regulation with individual constraints: the regulation with endogenous shared constraint is better than the regulation with individual constraints for these four criteria.

More generally, given a regulation criterion, a natural process of thinking is to compare regulations between one another (when such a comparison is possible) and study if there is one (or several) regulation which is better than the others. Now that we have defined what is a regulation, a regulation criterion, and what we mean when we say that a regulation is better than another for a given criterion, it consequently makes sense to define what is an optimal regulation in the following way.

Definition 11
- *We say that a regulation $(Y_i)_{i\in J}$ is optimal (in the strong sense) for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if for all regulation $(X_i)_{i\in J}$ we have $R((Y_i)_{i\in J}) \geq R((X_i)_{i\in J})$.*
- *We say that a regulation $(Y_i)_{i\in J}$ is optimal in the weak sense for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ if there is no $(X_i)_{i\in J}$ such that $R((X_i)_{i\in J}) > R((Y_i)_{i\in J})$.*

Note that an optimal regulation in the strong sense is also an optimal regulation in the weak sense (but the converse is not true). And note that both notions coincide if $(F, \leq)$ is a totally ordered set. Also note that such optimal regulations may fail to exist. We study a few cases of existence of optimal regulations below.

Theorem 2 *If E is a finite set:*

- *$Stab$, T and TN and always have at least one optimal regulation (in the strong sense).*
- *$Nash$ always have at least one optimal regulation in the weak sense.*

Proof. See the appendix.

We can also state the more general result, which is an extension of Theorem 2:

Theorem 3 *If E is a finite set:*

- *For any criterion R, there exists at least one optimal solution in the weak sense.*
- *For any criterion R such that $(F, \leq)$ is a totally ordered set, there exists an optimal regulation (in the strong sense).*

Proof. See the appendix.

We add the following additional result, which concerns the criteria where the set F is finite:

Theorem 4 *If F is a finite set:*

- *For any criterion R, there exists at least one optimal solution in the weak sense.*
- *For any criterion R such that $(F, \leq)$ is a totally ordered set, there exists an optimal regulation (in the strong sense).*

Proof. See the appendix.

Remark 2 *Interestingly, if E or F can be approximated by a finite set (which is usually the case when modeling real-world problems) the above property is true and we have existence of optimal regulations for the above criteria. A good extension (that we will not develop in this paper) would be to study the limit of sequences of games with finite strategy set to see if we can draw an existence result asymptotically when considering the limit of such a sequence of games.*

We give 2 illustrations below.

Example 5 *We consider a set of N agents driving their car on a highway road with one lane. There is a speed limitation set from the regulator to minimize the number of car accidents, and we assume that this limit is 130km/h. We assume their are two types of drivers on this road, a first category of $N-1$ drivers who like to drive with a speed x_i contained between 80km/h and 120km/h with a preference at 100km/h and a second category of 1 driver who can only drive at a speed x_i between 50km/h and 90km/h with a preference at 70km/h. The actual pace on the road is equal to $min_{i \in J} x_i$. We assume that the cost function for category 1 of drivers is equal to $|100km/h - min_{i \in J} x_i|$ and the cost function for the category 2 of drivers is equal to $|70km/h - min_{i \in J} x_i|$. This game has an infinity of Nash equilibria of the form $(x_1, ..., x_{N-1}, 70)$.*

The total cost is equal to $\sum_{i=1}^{N-1} |100km/h - min_{i \in J} x_i| + |70km/h - min_{i \in J} x_i|$ and the goal of the regulator is to minimize it at the Nash equlibria, so the regulation criterion here is TN. If the regulator asks that their should be a speed minimum of 80km/h on this road then we have a better regulation than the regulation with only the speed limit equal to 130km/h. And the optimal regulation (in the strong sense) for this criterion is that the regulator chooses a minimum speed equal to 90km/h.

Example 6 *We consider a beach which is a portion $[0, w] \times [0, l] \subset \mathbb{R}$, with w the width of the beach in meters and l its length in meters. We have N agents and each agent can choose its location on the beach $(x_i, y_i) \in E_i = [0, w] \times [0, l]$. We assume that there is an ice cream shop located at $(a, b) \in [0, w] \times [0, l]$ on the beach. Each agent sitting somewhere on the beach is seeking to minimize its distance to the ice cream shop $d((x_i, y_i), (a, b))$ and is seeking to maximize its distance to other agents on the beach $d((x_i, y_i), (x_j, y_j)), j \neq i$. So the cost for an agent i is equal $d((x_i, y_i), (a, b)) - \sum_{j \neq i} d((x_i, y_i), (x_j, y_j))$ So the total cost from a regulatory point of view is equal to $\sum_{i=1}^{N} d((x_i, y_i), (a, b)) - \sum_{i=1}^{N} \sum_{j \neq i} d((x_i, y_i), (x_j, y_j))$. A good way to simplify the problem is to* *say that the coordinates of the agents x_i and y_i should be integers $\in \mathbb{N}$. From Theorem 2, we have existence of an optimal regulation for the criteria T and TN, and it is is possible to find these optimal regulations with a simple computation given the data of the model.*

Actually, we will see in the next subsections that it often makes sense to say that a regulation is optimal among a subset of regulations. It is convenient and fruitful to proceed considering if a regulation is optimal among either its surregulations or subregulations.

3.3 Optimal vector

In the particular case where the optimal regulation for a criterion meets with the choice of a vector $\overline{x}$, we say that the game has an optimal vector for this given criterion.

Definition 12 *If there exists an Optimal regulation $(\overline{X_i})_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ which is of the form*

$$\begin{array}{cccc} \overline{X_i}: & E_{-i} & \rightarrow & P(E_i) \\ & x_{-i} & \mapsto & \{\overline{x_i}\} \end{array}$$

with $\overline{x} = (\overline{x_1}, ..., \overline{x_1}) \in E$, then we say that $\overline{x}$ is an optimal vector (either in the strong sense or in the weak sense).

Or equivalently, if there exists a vector of strategies $\overline{x}$ such that the Optimal regulation $(\overline{X_i})_{i\in J}$ for the criterion R on the Game $(J, E, (\theta_i)_{i\in J})$ satisfies $R((\overline{X_i})_{i\in J}) = R(\overline{x})$, we say that $\overline{x}$ is an Optimal vector (either in the strong sense or in the weak sense).

In other words, it is optimal from a regulatory point of view to act as a social planner and choose the vector of strategies $\overline{x}$ to maximize the total payoff (or minimize the total cost) for the agents.

Theorem 5 *If E or F are finite sets, T and TN always have at least one optimal vector (in the strong sense).*

Proof. See the appendix.

For instance, a direct consequence of this Theorem is that for our Example 6 with the beach and ice cream shop, the criteria T and TN always have at least one optimal vector.

3.4 Product regulation, Surregulations and Subregulations

We now introduce 2 new intuitive concepts that are meaningful and essential in the development of our theory. First, the product of two regulations: given two different regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on a game, it makes sense to consider how the mixing of these two different regulations affect economic agents. Such a product regulation means that for all agent i, x_i should satisfy both the regulation $(X_i)_{i\in J}$ and the regulation $(W_i)_{i\in J}$. Take for instance a citizen or company located in a country with different states: this citizen or company lives with both the laws of the country and those of the local state, this is a product regulation. The following definition makes this concept clear.

Definition 13 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$. We define the product regulation of $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$, and we denote $(X_i, W_i)_{i\in J}$ the regulation on the Game $(J, E, (\theta_i)_{i\in J})$ such that each agent $i \in J$ is required to satisfy $x_i \in X_i(x_{-i}) \cap W_i(x_{-i})$.*

This intuitive concept of product regulation is meaningful and essential, as well as the concepts of surregulation and subregulation that we develop below. It is common to hear that a regulation can be stricter or weaker than another one. It makes sense to say that a regulation is stricter than another regulation if the conditions to satisfy the first regulation are stricter than those to satisfy the second one: an economic agent satisfying the first regulation would consequently also satisfy the second regulation. For instance, environmental regulation since the Kyoto protocol is stricter than environmental regulation before the Kyoto protocol; or another example, bank regulation since 2008 is stricter than bank regulation between 2000 and 2007. The definition below makes clear these concepts of surregulation and subregulation.

Definition 14 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$. We say that $(W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$, or that regulation $(W_i)_{i\in J}$ is stronger (or stricter) than regulation $(X_i)_{i\in J}$, if for every agent i and for all x, $W_i(x_{-i}) \subset X_i(x_{-i})$. We will also say that $(X_i)_{i\in J}$ is a subregulation of $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$, or that regulation $(X_i)_{i\in J}$ is weaker (or less strict) than regulation $(W_i)_{i\in J}$.*

For instance, the product of two regulations is obviously a surregulation of both initial regulations.

Example 7 *$(X_i, W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$. It is obviously also a surregulation of $(W_i)_{i\in J}$.*

An endogenous shared constraint generated by exogenous individual constraints provides a natural example of surregulation.

Example 8 *We consider the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and the generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ generated from $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.*

Then, $(K_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$ since by definition for every agent i and for all x, $K_i(x_{-i}) \subset X_i(x_{-i})$.

We prove a few more interesting results below.

Proposition 8 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$ such that $(W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$. Then, the set of admissible strategies of $(J, E, (\theta_i)_{i\in J}, (W_i)_{i\in J})$ is included in the set of admissible strategies of $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.*

Or in other words: $K((W_i)_{i\in J}) \subset K((X_i)_{i\in J})$.

Proof of Proposition 8

$K((W_i)_{i\in J}) = \{x \in E,\ \forall i \in J,\ x_i \in W_i(x_{-i})\} \subset \{x \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} = K((X_i)_{i\in J})$.

$\square$

Proposition 9 *We consider two regulations $(X_i)_{i\in J}$ and $(W_i)_{i\in J}$ on the Game $(J, E, (\theta_i)_{i\in J})$ such that $(W_i)_{i\in J}$ is a surregulation of $(X_i)_{i\in J}$. Then, $T((W_i)_{i\in J}) \leq T((X_i)_{i\in J})$.*

Proof of Proposition 9

$K((W_i)_{i\in J}) \subset K((X_i)_{i\in J})$ from Proposition 8.

Therefore $T((W_i)_{i\in J}) = sup_{x\in K((W_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \leq sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) = T((X_i)_{i\in J})$.

$\square$

Remark 3 *This result is not true for regulation criteria Stab, $Nash$ and TN a priori.*

Corollary 2 *We consider a regulation $(X_i)_{i\in J}$ on a Game $(J, E, (\theta_i)_{i\in J})$. $(X_i)_{i\in J}$ is optimal among all its surregulations for the criterion T.*

We give an example of existence of optimal vector among a set of surregulations which is a corollary of Proposition 2: we proved that under the hypotheses of Proposition 2, there is existence of a Nash equilibrium x^* that maximizes the global payoff among all admissible strategies. Such a x^* is therefore an optimal vector for the criteria T and TN among all the surregulations of X. The proposition below makes clear this result.

Proposition 10 *Consider a generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$ satisfying the assumptions of Proposition 2, that is X is a compact set, and that $\forall i \in J$, θ_i is continuous and of the form $\theta_i(x) = \theta_i(x_i)$. Then such a game admits an optimal vector x^* among all the surregulations of X for the criteria T and TN.*

Proof See Proof of Proposition 2.

We now finish this subsection with an example of regulation which is optimal among a set of subregulations.

Proposition 11 *Let $(J, E, (\theta_i)_{i\in J}, X)$ be a generalized game with shared constraint. Then the regulation X is optimal among all the subregulations $(X_i)_{i\in J}$ such that $K((X_i)_{i\in J}) = X$ for the criteria T, $Nash$ and TN on the Game $(J, E, (\theta_i)_{i\in J})$. Moreover, if E is finite set, this property is also true for the criterion Stab.*

Proof of Proposition 11

If $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ is a generalized game with individual constraints such that $K((X_i)_{i\in J}) = X$ then, we have that $T((X_i)_{i\in J}) = T(X)$. From Proposition Braouezec-Kiani we have that $Nash((X_i)_{i\in J})) \subset Nash(X)$ and therefore we also have $TN((X_i)_{i\in J})) \leq TN(X)$. If E is a finite set, we also have that $Stab((X_i)_{i\in J})) \leq Stab(X)$. $\square$

3.5 Correlated regulation criteria and intersection of optimal solutions

We continue our development with comparing the regulation criteria with one another. The goal of this subsection is to answer to the theoretical question: to what extent is it similar for a regulator to optimize a criterion R_1 and a criterion R_2? Are the different criteria correlated with one another? Do they have common optimal solutions? This is what we inspect here.

Definition 15 *We say that two regulation criteria R_1 and R_2 are positively correlated if for any regulation $(X_i)_{i\in J}$ and $(Y_i)_{i\in J}$ such that $R_1((Y_i)_{i\in J}) \geq R_1((X_i)_{i\in J})$, we also have $R_2((Y_i)_{i\in J}) \geq R_2((X_i)_{i\in J})$. They are negatively correlated if for any regulation $(X_i)_{i\in J}$ and $(Y_i)_{i\in J}$ such that $R_1((Y_i)_{i\in J}) \geq R_1((X_i)_{i\in J})$, we have $R_2((Y_i)_{i\in J}) \leq R_2((X_i)_{i\in J})$.*

Example 9

- *If g is an increasing function then R_1 and $R_2 = g(R_1)$ are positively correlated.*
- *If g is a decreasing function then R_1 and $R_2 = g(R_1)$ are negatively correlated.*

Definition 16 *We say that two regulation criteria R_1 and R_2 are optimally identical if the set of optimal regulations for the criterion R_1 on the Game $(J, E, (\theta_i)_{i\in J})$ is equal to the set of optimal regulations for the criterion R_2 on the Game $(J, E, (\theta_i)_{i\in J})$, either in the strong or the weak sense. We say that the two criteria R_1 and R_2 have a common optimal regulation if the set of optimal regulations for the criterion R_1 and the set of optimal regulations for the criterion R_2 have at least one element in common, either in the strong or the weak sense.*

Note that if R_1 and R_2 are optimally identical then they have a common optical regulation (but the converse is not true).

Example 10 *If g is an increasing function then R_1 and $R_2 = g(R_1)$ are optimally identical.*

Proposition 12 *We consider a Game $(J, E, (\theta_i)_{i\in J})$ with E a finite set. Then the regulations $Nash$ and $Stab$ have at least one common optimal regulation in the weak sense.*

Proof of Proposition 12

Stab has at least one optimal regulation $(Y_i)_{i\in J}$ (in the strong sense). This means that there is no regulation $(X_i)_{i\in J}$such that $Stab((Y_i)_{i\in J}) < Stab((X_i)_{i\in J})$. And for such a $(Y_i)_{i\in J}$, there is no $(X_i)_{i\in J}$ such that $Nash((Y_i)_{i\in J}) \subset Nash((X_i)_{i\in J})$. □

Proposition 13 *We consider a generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$. Then X is a common optimal regulation (in the strong sense) among the subregulations $(X_i)_{i\in J}$ of X such that $K((X_i)_{i\in J}) = X$ for the criteria T, TN and $Nash$ (and also $Stab$ if E is finite).*

Proof of Proposition 13

See Proposition 11. □

We finish this subsection with the example of two criteria that are optimally identical (in the strong sense) if the game satisfies the assumptions of Proposition 2.

Proposition 14 *Consider a generalized game with shared constraint $(J, E, (\theta_i)_{i\in J}, X)$ satisfying the assumptions of Proposition 2, that is X is a compact set, and that $\forall i \in J$, θ_i is continuous and of the form $\theta_i(x) = \theta_i(x_i)$. Then the criteria T and TN are optimally identical (in the strong sense) among all the surregulations of X and their set of optimal regulations is equal to the set of subregulations of the optimal vectors x^* of Nash equilibria that maximize the global payoff given in Proposition 2.*

Proof See Proof of Proposition 2.

Exemple de jeu tel que Nash, T et TN no correlated et no optimally identical 2 à 2.

3.6 Other examples of regulation criteria

We give two additional examples of regulation criteria that can be useful in certain contexts. In particular, these two criteria can be mixed to the four criteria already studied to define new and relevant criteria of collective social well-being from the regulatory point of view.

Example 11 *We consider the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ is a regulation on the Game $(J, E, (\theta_i)_{i\in J})$. In this example we assume that θ_i is a payoff function (and not a cost function).*

We can define the regulation criterion:

$$\begin{array}{rcll} Diff & : & (F(E_{-i},P(E_i)))^N \rightarrow & (\mathbb{R},\leq) \\ & & (X_i)_{i\in J} \mapsto & sup_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) - inf_{x\in K((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x) \end{array}$$

with $K((X_i)_{i\in J})$ the set of admissible strategies of the generalized game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

$Diff((X_i)_{i\in J})$ is the difference between the maximum total payoff among all the admissible strategies of the generalized game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and the minimum total payoff. If this difference is higher for the regulation $(Y_i)_{i\in J}$ than for the regulation $(X_i)_{i\in J}$, then $Diff((Y_i)_{i\in J}) \geq Diff((X_i)_{i\in J})$.

Example 12 *We consider the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. $(X_i)_{i\in J}$ is a regulation on the Game $(J, E, (\theta_i)_{i\in J})$. In this example we assume that θ_i is a payoff function (and not a cost function).*

We can define the regulation criterion:

$$\begin{array}{rcll} DiffNash & : & (F(E_{-i},P(E_i)))^N \rightarrow & (\mathbb{R},\leq) \\ & & (X_i)_{i\in J} \mapsto & sup_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*) - inf_{x^*\in Nash((X_i)_{i\in J})} \sum_{i=1}^{N} \theta_i(x^*) \end{array}$$

with $Nash((X_i)_{i\in J})$ the set of Nash equilibria of the generalized game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$.

$DiffNash((X_i)_{i\in J})$ is the is the difference between the maximum total payoff among all the Nash equilibria of the generalized game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and the minimal total payoff among all the Nash equilibria. If this difference is higher for the regulation $(Y_i)_{i\in J}$ than for the regulation $(X_i)_{i\in J}$, then $DiffNash((Y_i)_{i\in J}) \geq DiffNash((X_i)_{i\in J})$.

3.7 From static games with constraints to dynamic games with constraints

We would like to offer an extension of our theory and results to more general types of games with a time dimension. A frequent critics about static games, with or without constraints, is that they are not always representative of real-world problems in the sense that it is rather rare to meet with real-world problems where the agents act in a single shot and all at the same time. To this extent, static games offer an idealized and simplified modeling of interaction between agents that give a first grasp on the problem one is analyzing. We defeat the critics about static games in the context of our article: the beauty of our theory on Optimal Regulation is that our modeling, framework and results are still true for any dynamic games, either discrete-time or continuous-time, either deterministic or with a stochastic component. In this perspective, our study of problems with static games offer a first ground of study of real-world problems and the results we obtain on this simplified version of the problem we study can be transferred to the dynamic version.

Indeed, consider the generalized game with constraint $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ and now consider that the strategy x_i chosen by agent i is also a function of time t and, therefore the strategy $x_i(t)$ of agent i can be reevaluated over time, either at some dates given by a deterministic law or at dates given by a law with a stochastic component, either at discrete times or continuous times. The goal of each agent is still to optimize its objective function $\theta_i(x(t))$. We observe that the definition of a Nash equilibrium for such a game is identical to the one in a static game and all the definitions and results of Section 2 and 3 can be extended to such dynamic games. The study of such dynamic generalized games offer enough substance for an article itself dedicated to this topic and we will not go further in this direction in the present article.

We also note that the introduction of the time dimension or a stochastic dimension in our study can also give birth to additional relevant definitions of regulation criteria: for instance, a regulator could be seeking to minimize the time before the agents get to a Nash equlibrium or another given configuration, or in a stochastic context the goal of a regulator could be to maximize the expected total payoff... And there are many other considerations that we will not develop here but that can also be the topic of discussion of a future article.

4 An application: strategic fire-sales in the banking system and optimal regulation

In this Section, we study applications of the theory and results developed in Section 2 and 3 to answer a problem in bank regulation during fire sales. Study of contagion of financial losses between financial institutions during a period of financial crisis has become a hot topic in financial regulation research since the global crisis of 2007-2008. In particular, Global Systemically Important Banks (G-SIBs), have raised an increasing attention over the years due to the risk of implosion of the financial system in case of bankruptcy of such a Bank. See [Brunnermeier, 2009], [Krishnamurthy, 2010] and [Glasserman and Young, 2016] for review papers. A salient feature of the financial crisis of 2007-2008 is the role played by financial markets, which have been a vector of not only direct contagion through contractual links between institutions, but also indirect contagion of losses and/or bankruptcy between banks and financial institutions, through asset prices. See for instance [Clerc et al., 2016]. A financial institution hit by a shock during a financial crisis may be forced to sell an important quantity of assets to rebalance its portfolio and capital ratio, what we call fire sales, and this can cause a major decrease in the prices of the assets sold (through price impact) which therefore impacts the balance sheets of other banks.

In this article we study a model of contagion between banks through the prices of assets in the financial markets inspired from [Braouezec and Wagalath, 2019]. First we remind and summarize the model developed in their article: they present the situation of p banks which need to sell some of their assets after a financial shock to re-balance their risk-based capital ratio over a regulatory threshold (see the initial article for full details). Then, we take the regulator point of view and study different regulation criteria, and existence of possible optimal regulations: what is the relevant criterion to optimize for a regulator who seeks the collective welfare of the community of agents? For instance, a regulator can either choose to minimize the losses in the financial system or to minimize the number of financial institutions going bankrupt. Do we have existence of an optimal regulation for these two regulation criteria?

4.1 The model

4.1.1 Banks' balance-sheets and regulatory constraints

We take same notations as in [Braouezec and Wagalath, 2019]. We consider a set $B = \{1, 2, ..., p\}$ of $p \geq 2$ banks that can invest in a risky asset and in cash.

For each bank i, we denote by $v_i > 0$ the amount of cash (in dollars) and by $q_i P_t > 0$ the value (in dollars) of risky assets, where q_i is the quantity of risky assets held by the bank and P_t is the market price of the risky asset at a given date t. Let D_i be the sum of the value of deposits and/or debt securities, that have been issued by bank i.

The balance-sheet of the bank at time t is as follows.

Balance-sheet of bank i at time t

Assets	Liabilities
Cash: v_i	Debt: D_i
Risky assets: $q_i P_t$	Equity: $E_{i,t}$
$A_{i,t}$	$E_{i,t} + D_i$

We make the assumption that the risky asset is a financial security issued by a non-financial institution whose price is quoted on financial markets.

From the Basel accords on banking regulation, banks are required to hold enough capital as a *percentage* of their risk-weighted assets (RWA). Within our model, since there is a single risky asset, the risk-weighted asset of bank i is simply equal to

$$\mathrm{RWA}_{i,t} = \alpha_i q_i P_t \tag{8}$$

where α_i is the risk weight of bank i associated to the risky asset. Note that α_i may vary across banks.

We define the risk-based capital ratio (RBC) $\theta_{i,t}$ for a given bank i at time t:

$$\theta_{i,t} := \frac{E_{i,t}}{\mathrm{RWA}_{i,t}} \tag{9}$$

For the sake of interest, we assume that all banks are solvent at date t, that is, $E_{i,t} > 0$ for all $1 \leq i \leq p$.

We denote by θ_{min} the minimum capital ratio imposed by the regulator, which is equal to 8%. For the sake of interest, we shall assume that, at date t, all banks comply with the regulatory constraint:

$$\theta_{i,t} \geq \theta_{min} \text{ for each } i = 1, 2, ..., p \tag{10}$$

4.1.2 Impact of an exogenous shock on banks' capital ratios

Assume that a shock on the risky asset occurs at date t^+ and denote $\Delta \in (0, 1)$ the size of the adverse shock in percentage of P_t. The price of the risky asset at time t^+ thus is equal to

$$P_{t^+} = P_t(1 - \Delta) \tag{11}$$

At time t^+, right after the shock, the balance-sheet of bank i is given as follows.

Balance-sheet at time t^+

Assets	Liabilities
Cash: v_i	Debt: D_i
Risky assets: $q_i P_t(1 - \Delta)$	Equity: E_{i,t^+}
A_{i,t^+}	$E_{i,t^+} + D_i$

It is the role of equity to absorb the shock, i.e., the loss which is equal to $q_i P_t \Delta$ in dollars. The RBC of bank i at date t^+ is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{\mathrm{RWA}_{i,t^+}} = \frac{\max\{E_{i,t} - q_i P_t \Delta; 0\}}{\alpha_i q_i P_t(1 - \Delta)} \tag{12}$$

In the remainder of this paper, we work under the following assumption.

Assumption 1 *At date t, each bank's equity is lower than the size of its risky assets, that is, for all* $1 \leq i \leq p$*:*

$$E_{i,t} < q_i P_t \tag{13}$$

This assumption is natural in the banking system as, in practice, banks' equities typically do not exceed 20% of their risky assets.

Lemma 1 *Under Assumption 1, the bank's RBC after the shock is a decreasing function of the shock size* Δ.

See [Braouezec and Wagalath, 2019] for the proof. A given bank i may thus be in one of the three following situations, depending on the size of the shock Δ:

1. solvent and complying with regulatory capital requirement, that is $\theta_{i,t^+}(\Delta) \geq \theta_{min}$
2. solvent but not complying with regulatory capital requirement, that is $0 < \theta_{i,t^+}(\Delta) < \theta_{min}$
3. insolvent, that is $\theta_{i,t^+}(\Delta) = 0$, which is equivalent to $E_{i,t} - q_i P_t \Delta \leq 0$

4.1.3 Endogenous fire sales and feedback effects

Since Δ is a common shock, it affects the balance-sheet of *all* banks that hold the risky asset and may leave some of them undercapitalized. Banks that do not comply with the regulatory capital constraints consequently need to restore their capital ratio above the minimum required θ_{min} by selling assets and decrease the denominator of the risk-based capital ratio.

Such forced sales are usually called "fire sales". We will also assume that, as in most models (e.g., [Elliott et al., 2014, Caccioli et al., 2014]), a bank which is unable to restore its capital ratio above θ_{min} is fully liquidated at date $t+1$.

We denote by $x_i \in [0,1]$ the proportion of risky assets sold by bank i at date $t+1$, in reaction to the shock Δ at date t^+. When bank i does not need to liquidate assets, then $x_i = 0$. On the contrary, when the shock Δ is such that bank i is insolvent or unable to restore its capital ratio above θ_{min}, then it is fully liquidated and $x_i = 1$. The volume (in shares) of liquidation by bank i is denoted by $x_i q_i$ and $\sum_{i \in B} x_i q_i$ denotes the total volume of fire sales in the banking system at date $t+1$.

Fire sales obviously impact the price of the asset at date $t+1$ and we assume here this price impact to be linear. We introduce the asset market depth Φ which is a linear measure of the asset liquidity [Kyle and Obizhaeva, 2016]. In [Cont and Wagalath, 2016], it is shown that the relevant quantity to capture the magnitude of feedback effects is $\frac{\sum_{i \in B} q_i}{\Phi}$.

The asset price at date $t+1$ thus depends on the vector of liquidations $\boldsymbol{x}(\Delta, \Phi) := \boldsymbol{x} = (x_1, x_2, ..., x_p) \in [0,1]^p$, and this vector of liquidation depends on both the shock Δ and the market depth Φ.

Assumption 2 *The price of the risky asset at time $t+1$ is equal to*

$$
\begin{aligned}
P_{t+1}(\boldsymbol{x}, \Phi) &= P_t (1 - \Delta) \left(1 - \frac{\sum_{i \in B} x_i q_i}{\Phi}\right) && (14) \\
\frac{Q_{tot}}{\Phi} &< 1 && (15) \\
\text{where } Q_{tot} &= \sum_{i \in B} q_i && (16)
\end{aligned}
$$

At time $t+1$, the balance-sheet of bank i that sold a portion x_i of the risky asset is given below:

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + x_i q_i P_{t+1}(\boldsymbol{x}, \Phi)$	Debt: D_i
Risky asset: $(1 - x_i) q_i P_{t+1}(\boldsymbol{x}, \Phi)$	Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + q_i P_{t+1}(\boldsymbol{x}, \Phi)$	$E_{i,t+1} + D_i$

where $P_{t+1}(\boldsymbol{x}, \Phi)$ is given in Assumption (2). Let $E_{i,t+1}(\boldsymbol{x})$ be the total capital at time $t+1$ after deleveraging. From the above balance-sheet, we have that

$$E_{i,t+1}(\boldsymbol{x}, \Delta) = \max\left\{E_{i,t} - q_i P_t \left(\Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi}(1 - \Delta)\right); 0\right\} \tag{17}$$

and note that it is a *decreasing* function of x_i due to the existence of a price impact. The regulatory capital ratio of bank i at time $t+1$ (i.e., after deleveraging) thus is equal to

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) = \frac{E_{i,t+1}(\boldsymbol{x})}{\alpha_i q_i P_{t+1}(\boldsymbol{x}, \Phi)(1 - x_i)} \tag{18}$$

with the natural convention that $\theta_{i,t+1}(\boldsymbol{x}, \Delta) = 0$ when $x_i = 1$ and when $E_{i,t+1} = 0$.

We can also introduce the concept of the *implied shock*:

$$\Delta(\boldsymbol{x}) := \Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi}(1 - \Delta) \tag{19}$$

associated to the vector of liquidation $\boldsymbol{x}$ such that the price of the risky asset at date $t+1$ can be written as follows

$$P_{t+1}(\boldsymbol{x}, \Phi) = P_t(1 - \Delta(\boldsymbol{x})) \tag{20}$$

Assumption 3 *Each bank $i = 1, 2, ..., p$ rebalances its portfolio of assets (i.e., deleverage) in order to minimize $x_i \in [0, 1]$ subject to the constraint*

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) \geq \theta_{min} \tag{21}$$

If the constraint can not be satisfied for some $x_i \in [0, 1)$, then bank i is insolvent and is costlessly liquidated at time $t+1$ so that $x_i = 1$.

4.2 Strategic fire sales and Regulation criteria

Now that we have reminded the model developed in the article [Braouezec and Wagalath, 2019], we can turn to the study from a regulator point of view of different regulation criteria, existence of possible optimal regulation, and possible correlation between different criteria.

We easily note that these strategic fire-sales with 1 asset are an example of generalized game with individual constraints:

- $J = \{1, ..., p\}$ a set of p banks
- $\forall i$, $E_i = [0, 1]$ and $E = [0, 1]^p$

- a cost function that we will denote f_i here $\begin{array}{rccc} f_i & : & E & \to & \mathbb{R} \\ & & x & \mapsto & x_i \end{array}$

- $\begin{array}{lcccc} \text{A regulation } X_i & : & [0,1]^{p-1} & \to & P([0,1]) \\ & & x_{-i} & \mapsto & \{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq \theta_{min}\} \cup \{1\} \end{array}$

To avoid any confusion, we keep the notation θ_i of the article [Braouezec and Wagalath, 2019] for the RBC ratio and we will denote the cost function f_i in our presentation for this section.

In this section, we will consider the 3 criteria studied in Section 3 $Nash$ (a stability criterion), T, and TN (two cost criteria; obviously, as $f_i(x) = x_i$ here is a cost function, we adapt the corresponding definition of the criteria T and TN), plus two additional regulation criteria that we will call bankruptcy criteria:

$$\begin{array}{rcccc} NB & : & (F(E_{-i},\mathrm{P}(\mathrm{E}_i)))^N & \to & ([0,\mathrm{N}],\leq) \\ & & (\mathrm{X}_i)_{i\in J} & \mapsto & \text{maximum}_{x\in K((X_i)_{i\in J})}\text{number of solvent banks(x) at } t+1 \end{array}$$

and

$$\begin{array}{rcccc} NBN & : & (F(E_{-i},\mathrm{P}(\mathrm{E}_i)))^N & \to & ([0,\mathrm{N}],\leq) \\ & & (\mathrm{X}_i)_{i\in J} & \mapsto & \text{maximum}_{x\in Nash((X_i)_{i\in J})}\text{number of solvent banks(x) at } t+1 \end{array}$$

NB measures the minimum number of banks going bankrupt among all the admissible strategies of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ and it is equal to min_x $N-$ the number of banks going bankrupt for the strategy vector x.

NBN measures the minimum number of banks going bankrupt among all the Nash equilibria of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ and it is equal to min_{x^*} $N-$ the number of banks going bankrupt for the Nash equilibrium x^*.

Note that we immediately have that for any regulation $(X_i)_{i\in J}$, $NBN((X_i)_{i\in J}) \leq NB((X_i)_{i\in J})$.

It is the goal of our inquiry to study these 5 regulation criteria and to see if there is existence of optimal regulations and optimal vectors.

We also note that we immediately have the following results of existence since the set F is finished for NB and NBN.

Proposition 15 *The criteria NB and NBN have at least one optimal regulation and one optimal vector.*

4.3 Regulation with individual constraints or endogenous shared constraint

It is proven in [Braouezec and Wagalath, 2019] that for such a game with individual constraints, there is always existence of Nash equilibria thanks to Tarski's theorem.

We want to go further and consider the regulation with endogenous shared constraint generated from this regulation with individual constraints. We already know that the regulation with shared constraint is at least better than the initial regulation with individual constraints for criteria $Nash$, T and TN. We will moreover prove that with such a regulation in place, there is existence of a Nash equilbrium that minimizes the total cost and diffusion of the assets $\sum_{i=1}^{p} x_i$ in the banking system.

First, we introduce the generated shared constraint set:

$$K = \{x \in [0,1]^p;\ \forall i \in \{1,...,p\} : x_i \in [0,1[\ and\ \theta_i(x_i, x_{-i}) \geq 8\%,\ or\ x_i = 1\} \tag{22}$$

Proposition 16 *K is a compact set.*

Proof See the Appendix.

So the regulation with shared constraint we introduced gives a generalized game with shared constraint that satisfies all the assumptions of Proposition 2 and therefore, as a corollary, such strategic fire-sales have a Nash equilibrium that minimizes the total cost and loss diffusion in the banking system if the regulator takes the regulation with shared constraint generated from the initial regulation with individual constraints, which is a new interesting result.

Corollary 3 *The generalized game with shared constraint $(J, E, (f_i)_{i\in J}, K)$ admits at least one Nash equilibrium that minimizes the global cost $\sum_{i\in J} x_i$ on K: there exists a Nash equilibrium $x^* \in E$ such that $\sum_{i\in J} x_i^* = min_{x\in K} \sum_{i=1}^{N} x_i$. Therefore the regulation with shared constraint K is optimal among all the surregulations of K for the criteria T and TN: in particular it is better than the regulation with individual constraints for the criteria T and TN, $K \leq (X_i)_{i\in J}$ for these two criteria. And obviously, from Proposition 7, $T(K) = TN(K)$. Moreover, such a vector x^* is an optimal vector (in the strong sense) for the criteria T and TN.*

We will go further in our study and actually prove that this remarkable Nash equilibrium in shared constraint is also a Nash equilibrium in individual constraints.

First, we remind this intuitive assumption:

Assumption 4 *We make the following natural assumptions:*

- *we assume that the ratio θ_i of a given bank i is an increasing function of the quantity of assets its sells. Or in other words: for all $i \in J$, $\theta_i(x)$ is an increasing function of x_i.*

- *we assume that the ratio θ_i of a given bank i is a decreasing function of the quantity of assets sold by other banks $j \in J$. Or in other words: for all $i \in J$, for all $j \neq i$, $\theta_i(x)$ is a decreasing function of x_j.*

Lemma 2 *If x^* is a Nash equilibrium of the generalized game $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ then for all i such that $x_i^* < 1$, we have that $\theta_i(x^*) = 8\%$.*

Proof See the appendix.

Proposition 17 *The generalized game with individual constraints* $(J, E, (f_i)_{i\in J}, (X_i)_{i\in J})$ *admits at least one Nash equilibrium that minimizes the global cost* $\sum_{i\in J} x_i$ *on* $K((X_i)_{i\in J})$*: there exists a Nash equilibrium* $x^* \in E$ *such that* $\sum_{i\in J} x_i^* = min_{x\in K((X_i)_{i\in J})} \sum_{i=1}^N x_i$*. Therefore,* $T((X_i)_{i\in J}) = TN((X_i)_{i\in J})$*, the regulation* $(X_i)_{i\in J}$ *is as good as the regulation* K *for the criteria* T *and* TN *and it is also optimal among all the surregulations of* $(X_i)_{i\in J}$ *for the criteria* T *and* TN*. Moreover, such a vector* x^* *is an optimal vector (in the strong sense) for the criteria* T *and* TN*.*

Proof See the Appendix.

Corollary 4 T *and* TN *have a common optimal regulation (in the strong sense) on the Game* $(J, E, (f_i)_{i\in J})$ *among the subregulations* $(X_i)_{i\in J}$ *of* X *such that* $K((X_i)_{i\in J}) = K$*. The criteria* T *and* TN *are optimally identical (in the strong sense) among all the surregulations of* K *and their set of optimal regulations is equal to the set of surregulations of the optimal vectors* x^* *of Nash equilibria that maximize the global payoff given in Proposition 2.*

This is a new result that does not appear in [Braouezec and Wagalath, 2019]: they prove that Nash equilibria are ranked and that it is optimal to choose the smallest Nash equilibrium, we moreover prove that the smallest Nash equilibrium minimizes the total cost among all the admissible strategies of the game $(J, E, (f_i)_{i\in J})$. This Nash equilibrium is clearly the preferred solution of regulatory public institutions who seek to minimize the global losses in the financial system.

But what if the regulator is seeking to minimize the number of banks going bankrupt, or equivalently maximize the number of solvent banks? We answer to this question in the next subsection. We also note that we have the following result for the criterion NBN.

Proposition 18 K *is a better regulation than* $(X_i)_{i\in J}$ *for the criterion* NBN*.*

Proof See the Appendix.

4.4 Comparison of optimal regulations and optimal vectors for the different criteria

We know that we have existence of optimal regulations and optimal vectors for the criteria T, TN, NB and NBN. Now, one can wonder: are these optimal regulations/vectors identical? Is it the same from a regulatory point of view to say that a regulator is seeking to minimize the losses in the system or to minimize the number of banks going bankrupt? This is an important question from a regulatory point of view.

To answer this question, one should look for:

- The set $\{x_T^*\}$ of optimal vectors for the criterion T.
- The set $\{x_{TN}^*\}$ of optimal vectors for the criterion TN.
- The set $\{x_{NB}\}$ of optimal vectors for the criterion NB.

- The set $\{x_{NBN}\}$ of optimal vectors for the criterion NBN.

and look at the intersections of these four sets.

We consider two ways to describe these four sets:

- either it is analytically convenient to find these four sets, given the data in our stress-test model.
- or if it's not the case, one can write do the following approximation and compute the following algorithm. Given the data in the model, we make the approximation that the set E is finite (which is actually the case in real life since x_i takes its value in a finite set in real life), and we compute an algorithm that tests all vectors $x \in K$ and find the optimal vectors for each criterion.

We can easily see that these vectors do not always coincide if for instance we have a world with one very large bank and (N-1) tiny banks with a quantity of assets which is for instance divided by 100. When these sets do not coincide and therefore optimizing a given criterion leads to a different result than optimzing another criterion, the regulator should wonder: what criterion of social welfare should be chosen, given that optimizing the different criteria is not equivalent? It is interesting to think of these questions retrospectively with the financial crisis of 2007-2008 and figure out if the decisions taken were optimal for the different regulation criteria.

We end up this section with considering a new possibility from a regulatory point of view during stress-tests when enabling that the threshold $\theta_{i,min}$ given to each bank is variable.

4.5 Regulation with variable $\theta_{i,min}$

We go further in our inquiry by assuming that the thresholds $\theta_{i,min}$ asked by the regulator are also variable, so we get new variable regulations.

We consider the family of following regulations $((X_{i,s_i})_{s_i>0})_{i\in J}$ on the Game $(J, E, (f_i)_{i\in J}$:

$$\begin{array}{rccc} X_{i,s_i} & : & [0,1]^{p-1} & \rightarrow & P([0,1]) \\ & & x_{-i} & \mapsto & \{x_i \in [0,1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i\} \cup \{1\} \end{array}$$

with $s_i > 0$. Note that we do not impose that all banks have the same ratio threshold s and the ratio threshold s_i can be customized by the regulator for each bank depending its size and own intrinsic characteristics.

Proposition 19 *If $0 < s_i \leq s'_i$, then for any $x_{-i} \in E_{-i}$, we have $X_{i,s'_i}(x_{-i}) \subset X_{i,s_i}(x_{-i})$.*

Proof See the Appendix.

Corollary 5 *If **s** and **s'** in $(\mathbb{R}_+^*)^N$ are such that for all $i \in J, 0 < s_i \leq s'_i$, we have that $K((X_{i,s'_i})_{i\in J}) \subset K((X_{i,s_i})_{i\in J})$.*

Corollary 6 *If **s** and **s'** in $(\mathbb{R}_+^*)^N$ are such that for all $i \in J, 0 < s_i \leq s'_i$, we have that:*

- $T((X_{i,s'_i})_{i\in J}) \geq T((X_{i,s_i})_{i\in J})$
- $TN((X_{i,s'_i})_{i\in J}) \geq TN((X_{i,s_i})_{i\in J})$
- $NB((X_{i,s'_i})_{i\in J}) \leq NB((X_{i,s_i})_{i\in J})$

Therefore $(X_{i,s_i})_{i\in J}$ is a better regulation than $(X_{i,s'_i})_{i\in J}$ for the criteria T, TN and NB. And therefore T and TN are positively correlated on the family of regulation $(X_{i,s_i})_{i\in J, s_i>0}$, and T and NB (and also TN and NB) are negatively correlated on the family of regulation $(X_{i,s_i})_{i\in J, s_i>0}$.

Proof See the Appendix.

Corollary 7 *The regulation $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$ is optimal (in the strong sense) for T, TN and NB among the family of regulations $(X_{i,s_i})_{i\in J, s_i\geq 0}$. Therefore these 3 criteria are optimally identical (in the strong sense) among the family of regulations $(X_{i,s_i})_{i\in J, s_i\geq 0}$.*

Therefore in a period of financial crisis following a financial shock our study provides the result that it is optimal for a regulator who is seeking to maximize either T, TN or NB to forget the RBC ratio for a while and choose the regulation $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$. In other words, a regulator who is seeking to minimize the losses in the system or the number of banks going bankrupt had better choose such a regulation $(X_{i,0})_{i\in J}$.

We also have the following result for the regulation criterion NBN:

Proposition 20 *There exists an optimal regulation for NBN among the family of regulations $(X_{i,s_i})_{i\in J, s_i>0}$.*

Proof See the Appendix.

One can wonder if the optimal regulation for NBN is same as NB and equal to $X_{i,0}(x_{-i}) = \{x_i \in [0,1(, \theta_{i,t+1}(x_i, x_{-i}) > 0\} \cup \{1\}$.

And same as in Subsection 4.4, one can wonder if there is existence of Optimal vectors x_T^*, x_{TN}^*, x_{NB} and x_{NBN} and if our four criteria are optimally identical or have at least some common optimal regulation/vector. The method to answer such questions is similar to the one in Subsection 4.4:

- either it is analytically convenient to find the four sets of optimal regulations and possible optimal vectors, given the data in our stress-test model.
- or if it's not the case, one can write the following approximation and compute the following algorithm. Given the data in the model, we make the approximation that the set E is finite (which is once again actually the case in real life since x_i takes its value in a finite set and also we note that the price of the asset is in a finite set of prices), and we compute an algorithm that tests all vectors $x \in K$ and find the optimal vectors for each criterion.

This gives us again a process of comparison of the different optimal regulations/vectors and the criteria.

5 Conclusion

6 Appendix

Proof of Proposition 2

We consider the application:

$$\begin{array}{rcll} F & : & E & \to & \mathbb{R} \\ & & x & \mapsto & \sum_{i\in J}\theta_i(x)=\sum_{i\in J}\theta_i(x_i) \end{array}$$

F is continuous on our compact set $X \subset E$ therefore F has a maximum $x^* \in X$, and x^* is a Nash equilibrium. Indeed if there was $i \in J$ and $y_i \in E_i$ such that $(y_i, x^*_{-i}) \in X$, and $\theta_i(y_i, x^*_{-i}) > \theta_i(x^*_i, x^*_{-i})$, we would have $F(y_i, x^*_{-i}) > F(x^*_i, x^*_{-i}) = F(x)$, which is not possible. □

Proof of Theorem 2

- $Stab$, T and TN are applications from a finite set of regulations to $(\mathbb{R}, \leq)$, therefore they each admit a maximum $(Y_i)_{i\in J}$ and such a maximum is an optimal regulation in the strong sense.

- $Nash$ is an application from a finite set of regulations to $(E, \subset)$, therefore there exists at least one $(Y_i)_{i\in J}$ such that there is no $(X_i)_{i\in J}$ with $Nash((Y_i)_{i\in J}) \subsetneq Nash((X_i)_{i\in J})$. □

Proof of Theorem 3

- R is an application from a finite set of regulations to $(F, \leq)$, therefore there exists at least one $(Y_i)_{i\in J}$ such that there is no $(X_i)_{i\in J}$ with $R((Y_i)_{i\in J}) < R((X_i)_{i\in J})$.

- R is an application from a finite set of regulations to a totally ordered set $(F, \leq)$, therefore R has at least one maximum $(Y_i)_{i\in J}$ and such a maximum is an optimal regulation in the strong sense. □

Proof of Theorem 4

- R is an application to a finite set $(F, \leq)$, therefore there exists at least one $(Y_i)_{i\in J}$ such that there is no $(X_i)_{i\in J}$ with $R((Y_i)_{i\in J}) < R((X_i)_{i\in J})$.

- R is an application to a finite set $(F, \leq)$ which is totally ordered, therefore R has at least one maximum $(Y_i)_{i\in J}$ and such a maximum is an optimal regulation in the strong sense. □

Proof of Theorem 5

If E and F are finite set then then the maxima for T and TN are reached at some vectors that are optimal. □

Proof of Proposition 16

- K is clearly a bounded set.

- We want to prove that K is a closed set. Let $(x_m)_{m\in\mathbb{N}} = (x_{1,m}, ..., x_{p,m})_{m\in\mathbb{N}} \in K^{\mathbb{N}}$ be a sequence which converges to a given $x_\infty \in [0,1]^p$. We will show that $x_\infty \in K$.

  Let $i \in \{1, .., p\}$.

– either $x_{i,\infty} = 1$

– or $x_{i,\infty} \in [0, 1($ and there exists $\epsilon > 0$ such that $B(x_{i,\infty}, \epsilon) \subset [0, 1] \setminus \{1\}$ and there exists $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0, x_{i,m} \in B(x_{i,\infty}, \epsilon)$. Therefore $\forall m \geq m_0, g_i(x_m) \geq 8\%$. And since g_i is continuous on $B(x_{i,\infty}, \epsilon)$ (see [Braouezec and Wagalath, 2019]), we have $g_i(x_\infty) \geq 8\%$

And this is true for all $i \in \{1, .., p\}$, so K is a closed set.

Therefore K is a closed bounded set of $[0, 1]^p$, so K is a compact set. $\square$

Proof of Lemma 2

From Assumption 4, if $\theta_i(x^*) > 8\%$, then there would exist $y_i < x_i^*$ such that $(y_i, x_{-i}^*) \in K$, which is not possible.

Proof of Proposition 17

Similar to the proof of Proposition 2, we consider the application:

$$\begin{array}{rccc} F & : & E & \to & \mathbb{R} \\ & & x & \mapsto & \sum_{i \in J} f_i(x) = \sum_{i \in J} f_i(x_i) = \sum_{i \in J} x_i \end{array}$$

F is continuous on our compact set $K \subset E$ therefore F has a minimum $x^* \in X$. Let's prove that x^* is a Nash equilibrium.

Indeed let's assume that there is $i \in J$ and $y_i \in E_i$ such that $y_i \in X(x_{-i}^*)$, and $f_i(y_i, x_{-i}^*) = y_i < x_i^* = f_i(x_i, x_{-i}^*)$.

- Either $x_i^* < 1$ and therefore since θ_i is an increasing function of x_i from Assumption 4 we have that $\theta_i(y_i, x_{-i}^*) < \theta_i(x_i^*, x_{-i}^*) = 8\%$, which is not possible.

- Either $x_i^* = 1$ and therefore since for all $j \neq i$, $\theta_j(x)$ is a decreasing function of x_i, we have that $\theta_j(y_i, x_{-i}^*) \geq \theta_j(x_i^*, x_{-i}^*)$ for all j such that $x_j < 1$. And therefore, for all $j \neq i$ such that $x_j < 1$ we have that $x_{*,j} \in X_j(y_i, x_{*,-i,-j})$, and this is also the case if j is such that $x_j = 1$. Therefore, for all $j \neq i$, $x_{*,j} \in X_j(y_i, x_{*,-i,-j})$, and therefore (y_i, x_{-i}^*) is an admissible strategy and $(y_i, x_{-i}^*) \in K$. But we would have $F(y_i, x_{-i}^*) < F(x^{*,i}, x_{-i}^*) = F(x)$, which is not possible.

Therefore x_* is a Nash equilibrium of the generalized game $(J, E, (f_i)_{i \in J}, (X_i)_{i \in J})$. $\square$

Proof of Proposition 18 From Proposition 1 we have that $Nash((X_i)_{i \in J}) \subset Nash(K)$, and therefore $NBN(K) \geq NBN((X_i)_{i \in J})$.

Proof of Proposition 19

If $s_i \leq s_i'$, from Assumption 4 we have that $\theta_i(x_i, x_{-i})$ is an increasing function of x_i and therefore $\{x_i \in [0, 1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i'i\} \subset \{x_i \in [0, 1),\ \theta_{i,t+1}(x_i, x_{-i}) \geq s_i\}$, so that $X_{i,s_i'}(x_{-i}) \subset X_{i,s_i}(x_{-i})$. $\square$

Proof of Corollary 6

$X_{i,s'_i}(x_{-i}) \subset X_{i,s_i}(x_{-i})$ for any $x_{-i} \in E_{-i}$ therefore $K((X_{i,s'_i})_{i\in J}) \subset K((X_{i,s_i})_{i\in J})$ and consequently, $T((X_{i,s'_i})_{i\in J}) \leq T((X_{i,s_i})_{i\in J})$ and $NB((X_{i,s'_i})_{i\in J}) \leq NB((X_{i,s_i})_{i\in J})$. From Proposition 17, $T((X_{i,s'_i})_{i\in J}) = TN((X_{i,s'_i})_{i\in J})$ and $T((X_{i,s_i})_{i\in J}) = TN((X_{i,s_i})_{i\in J})$ and therefore $TN((X_{i,s'_i})_{i\in J}) \leq TN((X_{i,s_i})_{i\in J})$ $\square$

Proof of Proposition 20

For the criterion NBN, the set F is equal to $\{0, ..., N\}$ and is therefore finite so we can apply Theorem 4. $\square$

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