

# Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?

Yann BRAOUEZEC\*      Keyvan KIANI†

May 19, 2022

## Abstract

A generalized game is a situation in which interaction between agents occurs not only through their objective function but also through their strategy sets; the strategy set of each agent depends upon the decision of the other agents and is called the individual constraint. As opposed to generalized games with exogenous shared constraint literature pioneered by [Rosen, 1965], we take the *individual constraints as the basic premises* and derive the shared constraint generated from the individual ones, a set  $K$ . For a profile of strategies to be a Nash equilibrium of the game with individual constraints, it must lie in  $K$ . But if, given what the others do, each agent agrees to restrict her choice in  $K$ , something that we call an endogenous shared constraint, this mutual restraint may generate new Nash equilibria. We show that the set of Nash equilibria in endogenous shared constraint contains the set of Nash equilibria in individual constraints. In particular, when there is no Nash equilibrium in individual constraints, there may still exist a Nash equilibrium in endogenous shared constraint. We also prove a few results for a specific class of generalized games that we non-classical games. Finally, we give two economic applications of our results to collective action problems, carbon emission and public good problems.

**Keywords:** Game theory, generalized games, binding agreements, individual and shared constraints, collective action problems

**JEL Classification:** C72, D7, F53

---

\*IESEG School of Management, Univ. Lille, CNRS, UMR 9221 - LEM - Lille Economie Management, F-59000 Lille, France, y.braouezec@ieseg.fr.

†Univ. Lille, CNRS, UMR 9221 - LEM - Lille Economie Management, F-59000 Lille, keyvan.kiani75@gmail.com

# 1 Introduction

It has long been recognized in Economics, Political sciences, International relations and more generally in Social sciences that Game theory is a useful tool to understand (and/or to predict) the outcome of a particular economic or social interaction between a set of agents (see [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] for classical textbooks). A striking feature of most (not to say all) applied game theoretical models is that interaction between agents only takes place through the objective functions (utility or cost function). Given what the others choose, the aim of a given agent is to optimize her objective function by choosing the optimal strategy in a given set assumed to be invariant with respect to the choice of the other agents.

When each agent explicitly faces for instance a common binding constraint, the decision of a given agent may not only impact the objective function of the other agents but also their strategy set. Consider the well-known example of international environmental agreements (such as the Kyoto protocol) in which the total volume of emissions of greenhouse gas must be lower than a given threshold  $\bar{e}$ . From the point of view of a given country  $i$ , given the sum of emissions of greenhouse gas of the other countries  $e_{-i}$ , its total emission  $e_i$  explicitly depends upon the emissions of the other since the strategy set of country  $i$  is equal to  $S_i(e_{-i}) = [0, \bar{e} - e_{-i}]$ . Within such a simple and natural framework with a collective binding constraint, interaction between agents may not only take place through the objective function but also through their strategy set (see [Breton et al., 2006] and [Tidball and Zaccour, 2005] for early economic applications). Games in which the interaction takes place not only through the utility function (or cost function) but also through the strategy sets are called generalized games or Generalized Nash Equilibrium Problem (GNEP for short) (see [Facchinei and Kanzow, 2007], [Fischer et al., 2014]). Throughout this paper, we may call interchangeably GNEPs and generalized games.

Generalized games (GNEPs) are not recent and have received considerable attention in operational research. For instance, in their well-known survey, [Facchinei and Kanzow, 2007] offer a historical overview of GNEPs dating back to the seminal paper of [Arrow and Debreu, 1954] and [Nash, 1950] and [Nash, 1951] and they provide interesting examples of applications of such games in telecommunication or in environmental pollution. In the applied maths literature more generally, there has been an abundant number of articles on GNEPs in recent years either proposing new methods, existence results or numerical algorithms to find a Nash equilibrium (e.g., [Facchinei et al., 2009], [Aussel and Dutta, 2008], [Fischer et al., 2014]). However, to the best of our knowledge, there has been only few papers trying to apply GNEPs in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020]).

A generalized game with *individual constraints* can be naturally defined as a game in which each agent has to satisfy her own individual constraint, that is, each agent  $i$  is required to pick a strategy  $x_i$  in a set which is (possibly) dependent on the strategies picked by the other agents. In this sense, classical games but also GNEPs can be seen as generalized games with individual constraints.

A generalized game with *shared constraint* is a specific category of games with individual constraints and has been introduced for the first time in [Rosen, 1965] although the term shared constraint does not appear<sup>1</sup>. In his seminal paper, [Rosen, 1965] pioneered such games and proves an existence result about concave games. Remarkably, Rosen’s result is true not only on  $E$  defined as the classical Cartesian product of strategy sets but also on any closed and bounded convex subset  $X \subset E$ . Later on in the literature,  $X$  has been called the *shared constraint set*. Such a game is called generalized game with shared constraint in the sense that all the agents share the same constraint, that is, the profile of strategies  $x := (x_1, \dots, x_n)$  must always remain in the shared constraint set  $X$ : given  $x_{-i}$ , each agent  $i$  is required to pick a strategy  $x_i \in E_i$  such that  $x := (x_i, x_{-i}) \in X$ .

The striking feature of the shared constrained approach developed in [Rosen, 1965] and the subsequent literature is that this shared constraint set  $X$  is *exogenously given* and bears no relationship with any possible individual constraints. In [Rosen, 1965], the author never makes use of the notion of an individual constraint. Later on, initiated by [Bensoussan, 1974], [Harker, 1984] and [Harker, 1991] introduced a variational formulation of the equilibrium of a generalized game with shared constraint and the literature on GNEPs now formulate the equilibrium as a (quasi) variational inequality. In its most basic version (see [Fischer et al., 2014] or [Facchinei and Kanzow, 2007] for excellent review papers), the variational formulation involves the partial derivative of the objective function of each agent with respect to her own strategy, which means that the underlying functions (objective function/constraints function) have to be differentiable.

It is the aim of this methodological paper to show the fruitfulness of such generalized games with shared constraints in the Economics of binding constraints<sup>2</sup>. To be very clear, this paper does not aim to contribute to the mathematical literature on generalized games. It is rather to show how, by taking an alternative road-map to the traditional one encountered in the (mathematical) literature on GNEPs with shared constraints, our new methodological approach can be used to provide some foundations to strategic collective actions problems in Economics.

We re-consider in this paper GNEPs with shared constraints and our aim is to shed light on the relationships between the individual constraints on the one hand and the shared constraints on the other hand. Instead of considering a shared constraint set which is *exogenously given* and imposed to the set of agents, as in the literature on GNEPs, we consider these individual constraints as the basic premises of the game and derive an *endogenous shared constraint set* generated precisely by these individual constraints. The building of such a shared constraint set from basic individual constraints is the core of our article and main result, although elementary. From a pure economic point of view, the existing literature on generalized games with shared constraints has two limitations. First, as said, the shared constraint is in general postulated and not generated from the individual constraints<sup>3</sup>. Second, by formulating the equilibrium of the game as a variational inequality, the

---

<sup>1</sup>Historically, generalized games were first introduced by [Debreu, 1952]. In [Harker, 1991] or in [Krawczyk, 2007], the authors note that a number of different names appeared in the literature to define these generalized games, abstract economy, social equilibria games, pseudo-Nash equilibria games or normalized equilibria ([Rosen, 1965]).

<sup>2</sup>It may be the result of an endogenous cooperation between the agents or the result of some constraining regulation.

<sup>3</sup>For instance, in [Tidball and Zaccour, 2005], they consider a game with individual constraints, but, as we shall

characterization of the equilibrium excludes the simplest games in which the strategy set of each agent is a finite set (e.g., the 2-2 games).

We prove in this paper the following simple result for which the economic applications are numerous: the set of Nash equilibria of a generalized game with individual constraints is *included* in the set of Nash equilibria of the generalized game with shared constraint generated from these individual constraints. Interestingly, [Feinstein and Rudloff, 2021] proved independently a similar result (see Theorem 3.2. in their paper) within a more abstract framework. This result, whose proof turns out to be simple, has two basic important consequences.

1. There are situations in which there is no Nash equilibrium in a game with individual constraints while such Nash equilibria exist in the game with shared constraint generated from the individual ones.
2. If there is no Nash equilibrium in shared constraint, then, no Nash equilibrium in the game with individual constraints can exist (the converse is however not true).

From an economic point of view, our result requires some binding agreements or a a constraining regulation; given what the others do, each agent  $i$  agrees or is constrained to pick a strategy not in her basic set of strategy (individual constraint) but in the shared constraint set that results from these individual constraints. Within our generalized game, sharing the constraint means in general that, given what the others do, compared with the primitive individual constraints  $X_i$ , each agent will now have to choose a strategy in a *smaller set*, that is, in  $K_i \subset X_i$ . Some strategies that were available in the game with individual (primitive) constraints are not anymore available in the game with shared constraint. Put it differently, introducing a shared constraint is equivalent to introduce restrictions and this kind of restriction exactly fits the notion of mutual restraint discussed in the well-known book of [Barrett, 2007] entitled *Why cooperate?*. Contrary to what the basic intuition could suggest, this form of mutual restraint typically generates more Nash equilibria and not less.

It is clear that a binding agreement requires some form of cooperation between the agents. In his well-known textbook, [Moulin, 1995] considers three modes of cooperation between a set of agents, direct agreements, decentralized behavior and justice. In this paper, while not explicitly modeled, the mode of cooperation considered is the first one, direct agreements, and can be thought (from a game theoretical point of view) of as the result of preplay communication<sup>4</sup>. These direct binding agreements are particularly important when there is no Nash equilibrium in the game with individual constraints while (at least) one equilibrium exists in the game with shared constraint. We illustrate this idea in the second part of the paper to collective action problems, an externality-pollution and a public good problem. In each model, we show situation in which there is no Nash

---

see, the shared constraint is not derived from the individual constraints.

<sup>4</sup>As observed in [Moulin, 1995], transaction cost is a drawback to this direct agreement mode. If this preplay communication is long and difficult, the transaction cost will be high. This problem of commitment is also discussed in the well-known book of [Ostrom, 1990]

equilibrium in individual constraints while there may be (at least) one Nash equilibrium in shared constraint.

The remainder of the paper is structured as follows. In Section 2, we remind the definitions of generalized games as well as the corresponding notions of Nash equilibria. We then define the notion of generalized game with shared constraint generated from a game with individual constraints, something we call endogenous shared constraint. In section 3, to present our concept of endogenous shared constraint, we offer a gallery of  $2 \times 2$  games, along the lines of [Fishburn and Kilgour, 1990], [Kilgour and Fraser, 1988]. In section 4, we establish the main finding of our paper for generalized games, which states that the set of Nash equilibria of a generalized game with individual constraints is included in the set of Nash equilibria of the generalized game with endogenous shared constraint. Moreover, we prove additional results regarding a specific class of generalized games called non-classical games in which interaction between agents occur only through the constraints. In Section 5, we offer two different collective action problem models which illuminate our results when no Nash equilibrium exists in individual constraints. The first model is a public good problem while the second one is an environment control problem and we show the economic usefulness of the introduction of an endogenous shared constraint in these problems. Section 6 concludes the paper.

## 2 Games with individual and shared constraints

### 2.1 Generalized games with individual constraints

We consider a game with  $N \geq 2$  agents (or players) and we denote  $J = \{1, \dots, N\}$  the set of players. The decision (or control) variable of each player  $i \in J$  is denoted by  $x_i \in E_i$ , where  $E_i$  is a subset of  $\mathbb{R}^{n_i}$  called the strategy set. Let  $E = \prod_{i=1}^N E_i = E_1 \times \dots \times E_N$  and denote by  $x \in E$  the vector formed by all these decision variables (strategies) which has dimension  $n := \sum_{i=1}^N n_i$  so that  $E \subset \mathbb{R}^n$ . As usual in game theory, we denote by  $x_{-i} \in E_{-i}$  the vector formed by all the players' decision variables except those of player  $i$ . To emphasize the  $i$ -th player's strategy, we sometimes write  $(x_i, x_{-i}) \in E_i \times E_{-i}$  instead of  $x \in E$ . Each player  $i$  has an objective function  $\theta_i : \mathbb{R}^n \rightarrow \mathbb{R}$  that depends on both his own decision variables  $x_i$  as well as on the decision variables  $x_{-i}$  of all other players. We denote the objective function of player  $i$  by  $\theta_i(x_i, x_{-i})$ . For a given  $x_{-i} \in E_{-i}$  and depending upon the game, the aim of agent  $i$  may be to maximize or to minimize its objective function  $\theta_i(x_i, x_{-i})$ . In general, when this objective function is a utility (or a payoff) function, it is the aim of the agent to maximize it while when it is a cost (or a loss) function, it is the aim of the agent to minimize it. Throughout the article, unless otherwise specified, we will assume that the objective function is a cost function so that each agent  $i$ , given the other players' strategies  $x_{-i}$ , is seeking a strategy  $x_i$  to minimize  $\theta_i(x_i, x_{-i}) = \theta_i(x)$ . Throughout the paper, we may use interchangeably the terms decision, decision variable and strategy and we only consider the case of pure strategy, that is, the situation in which each agent chooses a strategy with probability one.

In a generalized game, each player  $i \in J$  must pick a strategy  $x_i \in X_i(x_{-i}) \subseteq E_i$  where the

set  $X_i(x_{-i})$  explicitly depends upon the rival players' strategies. As in classical games in which the strategy set  $E_i$  of each agent  $i$  is given, in generalized games, the strategy set that we call the individual constraint function  $X_i(x_{-i})$  (or simply the individual constraint) is also exogenously given. To define in full generality the individual constraint which depends upon the decision of others, let  $X_i$  be defined as follows:

$$X_i : E_{-i} \rightarrow \mathcal{P}(E_i) \quad i = 1, 2, \dots, N \quad (1)$$

where  $\mathcal{P}(E_i)$  is the power set of  $E_i$ . It is usual to call  $X_i$  a point-to-set map (or a set-valued map) since it associates a subset of  $E_i$  to each point of  $E_{-i}$ . This means that for a given point  $x_{-i} \in E_{-i}$ , the strategy set also called the individual constraint of agent  $i$  is equal to  $X_i(x_{-i}) \subseteq E_i$ , a subset of  $E_i$  such as an interval. At a more abstract mathematical level,  $X_i(x_{-i})$  may be a finite or countable union of intervals if we think of  $X_i(x_{-i})$  as a Borel set. While interesting, this kind of mathematical generality is not our focus. We want instead to focus on the economic foundation of generalized games. At an economic level, in a strategic interaction with two agents, given the choice  $x_2$  of agent 2, agent 1 may have to choose  $x_1$  in  $E_1$  subject to a constraint of the form  $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$ . Put it differently, the strategy set of agent 1 explicitly depends upon the choice of agent 2. Let us consider two economic examples of such a situation that will be considered in detail in section 5.

- In an environmental problem with externalities, agent 1 (i.e., country 1) may be constrained to choose its emission of greenhouse gas  $x_1$  subject to a constraint of the form  $g_1(x_1, x_2) = x_1 + b_2x_2 \leq r_1$  where  $r_1$  is the maximum emission of country 1 and  $b_2x_2$  is the impact emission of country 2 on country 1.
- In a public good problem, following [Guttman, 1978], each agent  $i \in \{1, 2\}$  may have to choose a flat contribution  $x_1$  plus a matching rate  $bx_2$  (which is a function of the flat contribution of agent 2 where  $b \in (0, 1]$ ) so that  $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$  is the budget constraint of agent 1. A similar budget constraint holds for agent 2.

When thinking about collective action problems, there are various situations in which the decision of a given agent has an impact on the constraint of another agent.

**Definition 1** *The 4-uplet  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  is called a generalized game with individual constraints.*

It should be clear "classical games" encountered in Economics appear as a particular generalized games with individual constraints. When, for each  $i \in J$ ,  $X_i$  is invariant with respect to the choice of the other agents, we are back to a classical game in which the interaction only takes place through the objective functions. In such a case,  $X_i$  reduces to  $E_i$ .

## 2.2 Generalized games with exogenous and endogenous shared constraint: definitions and economic foundations

For what we shall call conventional or classical games, that is, when for each  $i$ ,  $E_i$  is invariant with respect to  $x_{-i}$ , recall that the best response  $BR_i$  of an agent  $i$  is defined as

$$BR_i : E_{-i} \rightarrow E_i$$

$$x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \arg \min_{x_i \in E_i} \theta_i(x_i, x_{-i})\}$$

that is, given a vector of strategies of other players  $x_{-i}$ , the best response gives the set of optimal strategy(s) for agent  $i$ . Unless mentioned otherwise,  $BR_i$  needs not be a singleton and is indeed in general a point-to-set map. As we shall now see, when studying a generalized game, different constraints will inevitably lead to different best response functions.

In a generalized game, as opposed to classical games, it may be the case that for some  $x_{-i}$ , the strategy set of agent  $i$  is simply empty, that is,  $X_i(x_{-i}) = \emptyset$ , which means that the objective function is undefined. As a result, the best response is also undefined so that the equilibrium can not exist. Such an empty set problem never occurs in classical games since the strategy set of each agent  $i$  is invariant with respect to  $x_{-i}$ . When  $x \in E$  is such that  $x_i \in X_i(x_{-i})$  for each agent  $i \in J$ , we say that the profile of strategies  $x$  is *admissible*, which means that it is a candidate point to be an equilibrium of the generalized game.

**Definition 2** For a given generalized game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ , let  $K$  be the subset of  $E$  defined as follows.

$$K = \{x \in E, \forall i \in J, x_i \in X_i(x_{-i})\} \quad (2)$$

$K$  is called the set of admissible strategies of the generalized game with individual constraints.

The set  $K$  represents the set of profiles of strategies  $x$  for which the generalized game with individual constraints is defined for all agents. If  $x$  does not belong to  $K$ , it can not be a Nash equilibrium of the generalized game. Throughout the paper, for the sake of interest, we assume that  $K$  is not empty. For such a generalized game in individual constraint  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ , the best response function of agent  $i$  is defined as

$$BR_i^{Ind} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \arg \min_{x_i \in X_i(x_{-i})} \theta_i(x_i, x_{-i})\} \quad (3)$$

A generalized Nash equilibrium problem (GNEP) is the given of  $N$  constrained optimization problems, that is, for each  $i \in J$ , given  $x_{-i}$ , agent  $i$  optimizes  $\theta_i(x_i, x_{-i})$  subject to  $x_i \in X_i(x_{-i})$ . A Nash equilibrium  $x^* = (x_1^*, \dots, x_N^*) \in K$  of the generalized game thus is such that no agent  $i$  wants to unilaterally deviate from her part of the equilibrium profile  $x^*$  but also such that the constraint of each agent  $i \in J$  is satisfied, i.e.,  $x^* \in K$ . The following definition makes clear this constraint.

**Definition 3** The profile of strategies  $x^* \in E$  is a Nash equilibrium of the generalized game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  if, for each  $i \in J$ ,  $x_i^* \in BR_i^{Ind}(x_{-i}^*)$ .

From the above discussion, a necessary but not sufficient condition on the profile of strategies  $x$  to be a Nash equilibrium is that  $x \in K$ . Since  $K$  is assumed to be not empty, it makes economic sense to require from each agent that given what the other agents are choosing, i.e.,  $x_{-i}$ , agent  $i$

should pick a strategy  $x_i$  such that the profile  $x = (x_i, x_{-i})$  lies in  $K$  if such a  $x_i$  exists. Given  $x_{-i}$ , let  $K_i(x_{-i})$  be the set of strategies of agent  $i$  defined as follows

$$K_i(x_{-i}) = \{x_i \in E_i : x \in K\} \quad (4)$$

We are now in a position to define a generalized game with endogenous shared constraint, that is, a generalized game in which the shared constraint is generated from the individual constraints.

**Definition 4** *The 4-uplet  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$  is called a generalized game with shared constraint generated from the game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ . We call such a game a game with endogenous shared constraint.*

For a generalized game in (endogenous) shared constraint  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$ , the best response function of agent  $i$  is defined as

$$BR_i^{Shd} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \arg \min_{x_i \in K_i(x_{-i})} \theta_i(x_i, x_{-i})\} \quad (5)$$

The best response is similar to the one given for a generalized with individual constraints except that each agent  $i$  is required to choose a strategy in  $K_i(x_{-i})$  rather than in  $X_i(x_{-i})$ . We are now in a position to give the definition of a Nash equilibrium in a game with endogenous shared constraint  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$ .

**Definition 5** *The profile of strategies  $x^* \in E$  is a Nash equilibrium for the generalized game with endogenous shared constraint  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$  if for each  $i \in J$ ,  $x_i^* \in BR_i^{Shd}(x_{-i}^*)$ .*

To see the difference between a game with individual constraints and a game with endogenous shared constraint, assume that  $X_i(x_{-i}) := \{x_i \in E_i : g_i(x) \leq 0\}$  is the strategy set (or constraint) of each agent  $i$  for some function (constraint)  $g_i$ . In the literature, it is standard to describe the constraint using such functions  $g_i$ . Making use of the functions  $g_i$ ,  $i = 1, 2, \dots, N$ , the admissible set  $K$  (see equation (2)) can be rewritten as  $K = \{x \in E, \forall i \in J, g_i(x) \leq 0\}$  so that  $K_i(x_{-i}) := \{x_i \in E_i : \forall j \in J, g_j(x) \leq 0\}$ .

- In the game with individual constraint, given  $x_{-i}$ , each agent optimizes its objective function with respect to  $x_i$  subject to its *own individual constraint*  $g_i(x) \leq 0$ , that is,  $x_i \in X_i(x_{-i})$ .
- In the game with shared constraint generated from the individual ones, given  $x_{-i}$ , each agent optimizes its objective function with respect to  $x_i$  subject to  $g_1(x) \leq 0, g_2(x) \leq 0, \dots, g_N(x) \leq 0$ , that is,  $x_i \in K_i(x_{-i})$ . Given  $x_{-i}$ , when each agent  $i$  chooses a strategy  $x_i$ , she takes not only into account its own constraint but also the *constraints of all the other agents* so that  $(x_i, x_{-i})$  lies in  $K$ .

It should be clear in a game with endogenous shared constraint, the set of strategies of each agent  $i$  may be *reduced* compared to the game with individual constraints, that is,

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \quad \forall i \in J \quad (6)$$



In general, the inclusion may be strict for some agents, that is, as long as  $X_i(x_{-i})$  is not empty,  $K_i(x_{-i}) \subset X_i(x_{-i})$ . In what follows, to emphasize that the set  $K$  is the shared constraint, we may denote the game  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$  as  $(J, E, (\theta_i)_{i \in J}, K)$ .

Let  $\mathcal{N}_{Indiv}$  and  $\mathcal{N}_{Shared}$  be respectively the set of Nash equilibria in individual constraints and with (endogenous) shared constraints and note that it may be the case that both sets are empty.

**Remark 1** *Within our approach, we take the set  $X_i(x_{-i})$  (the individual constraints) as the basic premises and we derive the set  $K$  (the endogenous shared constraint) from these individual constraints. This is in sharp contrast with the literature on generalized games in which the shared constraint is exogenous. Following [Rosen, 1965], authors typically start with a classical game for which  $E = \prod_{i=1}^N E_i$  and what they call the shared constraint is an exogenous (mathematically convenient) prescribed set  $X \subset E$ , that is, the profile of strategies  $x$  must be located in  $X$ , see e.g., [Fischer et al., 2014] for nice review papers.*

In the literature on generalized games, the *exogenous* shared constraint  $X$  (i.e., such that  $X \subset E$ ) thus appears as the basic premise. Given this exogenous shared constraint  $X$ , knowing  $x_{-i}$ , each agent  $i$  thus is required to choose a strategy  $x_i$  in  $X$ , and this means that her strategy set now is defined as  $X_i(x_{-i}) = \{x_i \in E_i : (x_i, x_{-i}) \in X\}$ . As opposed to the approach followed in this paper, the individual constraints  $X_i(x_{-i})$  are derived from the exogenous shared constraint. Let  $(J, E, (\theta_i)_{i \in J}, X)$  be a generalized game with an exogenous shared constraint. The following well-known result about the existence of Nash equilibria for concave n-person games is due to [Rosen, 1965]. Using our terminology, the following result is an existence result for a game with an exogenous shared constraint  $X$ .

**Theorem 1** ([Rosen, 1965]) *Let  $(J, E, (\theta_i)_{i \in J}, X)$  be a game with an exogenous shared constraint where  $\theta_i$  is a payoff function. If the set  $X$  is convex, closed and bounded and if each player's payoff function  $\theta_i(x_i, x_{-i})$ ,  $i \in J$  is continuous and concave in  $x_i$ , then, the generalized game has at least one Nash equilibrium.*

[Rosen, 1965] considers the simple example of a 2-person game in which  $X$  is a compact convex subset of the unit square such as an ellipse. This thus means that given the choice of agent 1, agent 2 has to choose a number between zero and one such that the couple of numbers chosen must be located in the ellipse. In many other papers, the authors consider the  $X$  defined as  $X = \{x \in E \mid G(x) \leq 0\}$ , where  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  is a component-wise convex function called the shared constraint function, an assumption convenient for the mathematical analysis of the generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). An attractive feature of these games with shared constraint is that they are analytically convenient. Under particular assumptions on  $X$  and on the objective functions  $\theta_i$ ,  $i \in J$ , it is often possible to use a fixed-point theorem (Kakutani, Tarski...) to prove the existence of Nash equilibrium.

**Shared constraint as binding agreements.** As discussed in the introduction of this paper, sharing the constraints requires a particular form of cooperation between agents that are called *direct*

*agreements* in [Moulin, 1995] or a constraining regulation. Within our generalized game framework, given what the others do, each agent must agree to restrict their choice in the shared constraint  $K$ , that is, given  $x_{-i}$ , each agent  $i$  must pick a strategy  $x_i$  in  $K_i(x_{-i})$  rather than in  $X_i(x_{-i})$  where  $K_i(x_{-i})$  is included in  $X_i(x_{-i})$ . Given  $x_{-i}$ , it might be the case that the best response of a given agent  $i$  lies in  $X_i(x_{-i})$  and not in  $K_i(x_{-i})$ . As a result, this self-restriction to the set of strategies  $K_i(x_{-i})$  rather than  $X_i(x_{-i})$  requires some binding agreements (or a regulation) that are however not explicitly modeled here. This is identical to the discussion offered in the well-known book of [Ostrom, 1990] in which she notes that one (frequent theoretical) solution to this problem is *coercion*. Roughly speaking, one can make a (perhaps disputable) distinction between two types of coercion, external or internal.

- external (or exogenous) coercion corresponds to the situation in which each agent must comply with law or regulation and this means that there is an external enforcer to use the terminology of ([Ostrom, 1990]). In such a situation, an agent who does not comply with the law or regulation can be sanctioned (i.e., fined) by the external enforcer.
- Internal (or endogenous) coercion corresponds to the situation in which a group of agents (employees, firms, countries or even the overall society itself) must reach an agreement without any external enforcer. This clearly means that such an agreement is based on voluntarism since an agent who breach the (contractual) agreement can not be fined.

Internal coercion thus is an agreement based on self-restriction (i.e., something that must (or must not) be done) which turns out to be very similar to the notion of *mutual restraint* discussed in [Barrett, 2007] (see chapter five). In [Barrett, 2007], the author explicitly considers the case in which agents are countries that seek to supply global public goods such as nuclear non-proliferation or climate change mitigation (e.g., limit carbon emission). For instance, when one considers a set of countries that try to reduce their individual pollution, mutual restraint can be reached through international treaties and these treaties can be thought of as an example of an internal coercion. Given  $x_{-i}$ , the restriction of each agent  $i$  to the set  $K_i(x_{-i})$  can be seen as a possible formalization of the notion of mutual restraint discussed in [Barrett, 2007].

In what follows, we thus make the implicit assumption that agents succeeded to reach binding agreements so each agent (on a voluntary basis given  $x_{-i}$ ) agrees to be restricted to  $K_i(x_{-i})$ . As we shall see, when no equilibrium exists in the game with individual constraints, this mutual restraint might be the unique solution to reach an equilibrium situation. In the last section of this paper, we shall offer two different models of collective actions in which the equilibrium only exists (depending upon parameters) in shared constraint.

### 3 Equilibrium analysis of $2 \times 2$ generalized games

Games with two agents and two strategies called  $2 \times 2$  games are used in many introductory textbooks as a way to explain the basic concepts but also to illustrate how in specific strategic sit-

Cost for each agent	$\theta_A(x_A, x_B)$	$\theta_B(x_A, x_B)$
Constraint for each agent	$g_A(x_A, x_B)$	$g_B(x_A, x_B)$

Table 1: Cost and constraint for a given pair of strategies  $(x_A, x_B)$

uations, individual rationality may differ from the collective rationality (prisoners dilemma). In [DeCanio and Fremstad, 2013], the authors apply  $2 \times 2$  games to international climate negotiation problems where each agent has two strategies, Abate or Pollute. Various games such as the prisoners dilemma and the chicken games are discussed. They also discuss cases in which there is no Nash equilibrium in pure strategy. In [Fishburn and Kilgour, 1990] (see also [Kilgour and Fraser, 1988] and [Walliser, 1988]), they offer a more theoretical analysis to  $2 \times 2$  games.

**A gallery of  $2 \times 2$  generalized games.** Consider the following game with two agents such that each agent  $i \in \{A, B\}$ :

- seeks to minimize a cost function  $\theta_i$ .
- can choose between two strategies 1 and 2:  $E_i = \{1, 2\}$ .
- is subject to the constraint  $g_i(x_A, x_B) \leq c_i$

Generalized  $2 \times 2$  games are more complex than classical  $2 \times 2$  games encountered in economic theory (e.g., [DeCanio and Fremstad, 2013]) in which interaction occurs only through the objective function. In generalized games, whether or not the set strategies is finite, the choice of a given strategy by one agent directly impacts the objective function of the other agents but also their set of strategies (constraints). When one considers simple  $2 \times 2$  generalized games, both the objective function and the constraint thus must be exhibited, as in table 1. In what follows, without loss of generality, we shall assume that  $c_A = c_B = c$ . For instance, in Fig.2, when agent A chooses strategy 1 and agent B chooses strategy 2, for agent A, the cost is equal to  $\theta_A(1, 2) = 0$  and the constraint is equal to  $g_A(1, 2) = 0.5$  while for agent B, the cost is equal to  $\theta_B(1, 2) = 1$  and the constraint is equal to  $g_B(1, 2) = 1.5$ . We consider a simple taxonomy of generalized  $2 \times 2$  games from the point of view of the set of Nash equilibrium in individual and in shared constraint. To make things easier, we consider generalized games with a simple structure: as long as agents are "coordinated", when they choose the same strategy, their cost is equal to one, that is,  $\theta_i(1, 1) = \theta_i(0, 0) = 1$  for  $i \in \{A, B\}$  and their constraint is satisfied. We consider three different games in which we only change the cost or the constraint when agents are not coordinated.

1.  $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared}$ . Consider Fig. 1 and assume that  $c = 1$ . There are two Nash equilibria in individual constraints,  $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$ . Knowing that agent A chooses strategy 1, the best response of agent B is to choose strategy 1. Conversely, knowing that agent B chooses strategy 1, the best response of agent A is to choose strategy 1 so that the pair of strategy  $(1, 1)$  is a Nash equilibrium in individual constraints. The constraint of each agent

		B			
		1		2	
A	1	1	1	2	1
		0.5	0.5	0.5	1.5
	2	1	2	1	1
		1.5	0.5	0.5	0.5

Figure 1:

		B			
		1		2	
A	1	1	1	0	1
		0.5	0.5	0.5	1.5
	2	1	0	1	1
		1.5	0.5	0.5	0.5

Figure 2:

		B			
		1		2	
A	1	1	1	2	0.75
		0.5	1.5	1.5	0.5
	2	1.5	0.5	1	1
		1.5	0.5	0.5	0.5

Figure 3:

	Agent A	1	2
Agent B			
1		(1, Y)	(a, D <sub>A</sub> , D <sub>B</sub> )
2		(b, E <sub>A</sub> , E <sub>B</sub> )	(1, Y)

Table 2: Symmetric payoff game  $2 \times 2$  generalized game with asymmetric constraint

is satisfied so that the pair of strategy (1, 1) is also a Nash equilibrium in shared constraint. A similar analysis can be done for the pair of strategies (2, 2). For such a game, everything is as if the constraint were not an issue since it is always satisfied. Such a game can be thought of as a coordination (generalized) game since there are two identical Pareto optimal Nash equilibria.

2.  $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared}$ . Consider Fig. 2 and assume that the constraint is  $c = 1$ . In such a case, only  $\mathcal{N}_{Indiv} = \{(1, 1)\}$  is a Nash equilibrium in individual constraint. To see this, assume that agent A chooses  $x_A = 1$ . Abstracting the constraint, agent B is indifferent between strategy 1 and 2. However, since  $g_B(1, 1) = 0.5$  and  $g_B(1, 2) = 1.5$ , the constraint of agent B is only satisfied when she/he chooses strategy 1, i.e.,  $X_B(x_A = 1) = \{1\}$ . It thus follows that the best response of agent B is  $BR_B(x_A = 1) = 1$ . Assume now that agent B chooses strategy 1. Since agent A only satisfies the constraint for decision 1, it thus follows that  $BR_A(x_B = 1) = 1$  so that (1, 1) is a Nash equilibrium under individual constraints. Consider now the pair of strategies (2, 2). If agent B chooses  $x_B = 2$ , then, the best response in individual constraint is  $BR_A^{Ind}(x_B = 2) = 1$  and this means that (2, 2) is not a Nash equilibrium under individual constraint. However, the pair of strategies (2, 2) is a Nash equilibrium in shared constraint since the best response in shared constraint are  $BR_A^{Sh}(x_B = 2) = 2$  and  $BR_B^{Sh}(x_A = 2) = 2$ . To see this, it suffices to note that  $X_A(x_B = 2) = \{1, 2\}$  while  $K_A(x_B = 2) = \{2\}$  and  $X_B(x_A = 2) = \{1, 2\}$  while  $K_B(x_A = 2) = \{2\}$ . It thus follows that  $\mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$  while  $\mathcal{N}_{Indiv} = \{(1, 1)\}$
3.  $\mathcal{N}_{Indiv} = \emptyset$  while  $\mathcal{N}_{Shared} \neq \emptyset$ . Along the same line of reasoning, it is easy to see in Fig. 3 that there is no Nash equilibrium in individual constraint while the pair of strategies (2, 2) is a Nash equilibrium in shared constraint. In the game in individual constraints, when agent A chooses strategy 2, agent B chooses strategy 1 so that (2, 2) is not a Nash equilibrium. However, in shared constraint, agent B can not choose strategy 1 because agent A does not satisfy his constraint and his unique is strategy 1, which explains

**On  $2 \times 2$  generalized games with symmetric payoff but asymmetric constraint.** Consider once again the coordination game (see table 2) in which the payoff of each agent is equal to one when they choose the same strategy, i.e., (1, 1) or (2, 2). Without loss of generality, we shall use a dummy variable for the constraint, Y (Yes) when it is satisfied and N (No) when it is not. We thus have four dummy variables,  $D_A, D_B, E_A, E_B$ . In principle, one may derive many simple results

from such a game. We shall consider two. One in which the game reduces to a classical game, i.e., the constraints are irrelevant so that the set of Nash equilibria are identical in individual and in shared constraint and another one in which there is no Nash equilibrium in individual constraint while there is at least one equilibrium in shared constraint.

**Fact 1** . Consider a symmetric payoff  $2 \times 2$  generalized game with asymmetric constraint as given in table 2 and let  $D_A, D_B, E_A, E_B$  be four dummy variables.

1. If  $D_A = D_B = Y$  and  $E_A = E_B = Y$  and  $a > 1, b > 1$ , then,  $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$  and the game reduces to a classical coordination game.
2. If  $D_A = E_A = Y$  and  $E_B = D_B = N$  (or if  $D_A = E_A = N$  and  $E_B = D_B = Y$ ) and  $a < 1, b < 1$ , then,  $\mathcal{N}_{Indiv} = \emptyset$  while  $\mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$

We leave the proof and all the other cases to the interested reader. When the constraint is always satisfied for each agent and when  $a > 1$  and  $b > 1$ , it is in the interest of each agent to be perfectly coordinated and the game reduces to a pure coordination game with two Nash equilibria in individual constraints and in shared constraint since the constraint plays no role. When  $a < 1, b < 1$  and when the constraint is always satisfied for one agent but is not satisfied for the other agent when their decisions are not coordinated, then, there are still two equilibria in shared constraint but no equilibrium in individual constraint.

## 4 Simple theoretical results

### 4.1 An elementary general result

As we shall now see, a striking feature of a game with endogenous shared constraint, compared with the game with individual constraints, is that it may possess *additional Nash equilibria*. We already know that for a profile  $x$  to be a Nash equilibrium,  $x$  must be in  $K$ . But if each agent  $i$ , given  $x_{-i}$ , agrees to choose  $x_i$  such that  $x = (x_i, x_{-i})$  is in  $K$  (i.e.,  $x_i \in K_i(x_{-i})$ ), the set of Nash equilibria may be *larger*.

**Proposition 1** The set of Nash equilibria of a game with individual constraints denoted  $\mathcal{N}_{Indiv}$  is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints, denoted  $\mathcal{N}_{Shared}$  that is,  $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ .

**Proof.** See the appendix.

Independently, [Feinstein and Rudloff, 2021] proved a similar result (see their Theorem 3.2). The basic intuition behind the above proposition is that when agents have less freedom to deviate, this may expand the set of Nash equilibria. While easy to prove from a mathematical point of view, it is important to point out that such a result only makes sense for generalized games in which the shared constraint is derived from the individual ones. Such a result makes no sense in games with

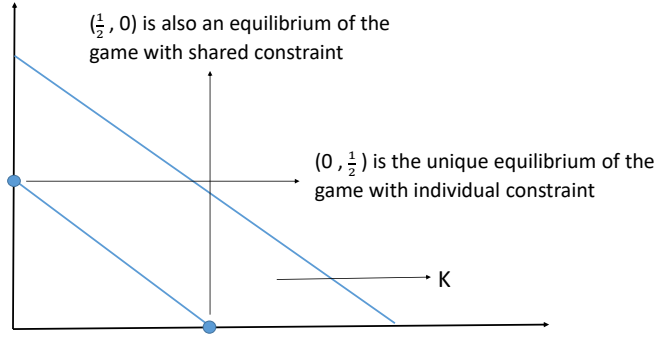


Figure 4: One equilibrium in individual constraints, an infinity of equilibria in shared constraint

exogenous shared constraint. As we shall see, this result is particularly relevant when  $\mathcal{N}_{Indiv} = \emptyset$  while  $\mathcal{N}_{Shared} \neq \emptyset$ . To illustrate the above proposition, that is, to show that  $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ , consider the following game in which the set of strategies have the cardinality of the *continuum*.

Let  $J = \{1, 2\}$  be the set of agents and let  $E_1 = E_2 = [0, 1]$  be the strategy st of each agents so that  $E = [0, 1] \times [0, 1]$  is the strategy space. As usual, the objective function of each agent is to minimize a cost function. The cost functions are  $\theta_1(x_1, x_2) = x_1 + x_2$  and  $\theta_2(x_1, x_2) = x_2 - x_1$ . Assume now that the individual constraint for agent 1 is  $g_1(x_1, x_2) = x_1 + x_2 \leq 1$  while it is equal to  $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$  for agent 2. In this example, the set  $K$  is defined as

$$K = \left\{ (x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1 \text{ and } x_1 + x_2 \geq \frac{1}{2} \right\} \quad (7)$$

It is not difficult to see that  $K$  is a compact and convex set (see Fig. 4) and that the profile of strategies  $(0, \frac{1}{2}) \in K$  is a Nash equilibrium of the game with individual constraints. Consider now the profile of strategies  $(\frac{1}{2}, 0) \in K$ .

- In the game with individual constraints,  $(\frac{1}{2}, 0) \in K$  is *not* a Nash equilibrium. To see this, it suffices to note that when  $x_2 = 0$ , the best response of agent 1 is 0 and the constraint of agent 1 is fulfilled since  $g_1(x_1, x_2) = x_1 + x_2 \leq 1$ . Since  $\frac{1}{2}$  is not the best response,  $(\frac{1}{2}, 0) \in K$  thus is not an Nash equilibrium.
- In the game with shared constraint,  $(\frac{1}{2}, 0) \in K$  is a Nash equilibrium. To see this, note that if  $x_1 = \frac{1}{2}$ , then, the best response of agent 2 is 0. This choice of 0 minimizes the cost of agent 2 and satisfies the constraint of agent 2 since  $\frac{1}{2} + 0 \geq \frac{1}{2}$  but also the constraint of agent 1 since  $\frac{1}{2} + 0 \leq 1$ . If  $x_2 = 0$ , the best response of agent 1 is now to choose  $\frac{1}{2}$  because agent 1 takes into account the constraint of agent 2, i.e.,  $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$ . As opposed to the game with individual constraints, agent 1 can not choose 0. As a result,  $(\frac{1}{2}, 0)$  is a Nash equilibrium for the game with endogenous shared constraint. Actually all vectors  $(x_1, x_2)$  such that  $x_1 + x_2 = \frac{1}{2}$  are Nash equilibria in shared constraint. Indeed, if

$x_1 + x_2 = \frac{1}{2}$  both constraints are satisfied and no agent has an incentive and possibility to move to a better state. Let us prove that the converse is true. Consider  $(x_1, x_2)$  such that  $1 \geq x_1 + x_2 > \frac{1}{2}$  then one of the two agents will have the incentive and possibility to decrease its cost. Therefore the set of Nash equilibria in endogenous shared constraint for this game is equal to:  $\mathcal{N}_{Shared} = \{(x_1, x_2) \in E, x_1 + x_2 = \frac{1}{2}\}$ .

- Let us now show that  $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$ . From Proposition 1 we have that  $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared} = \{(x_1, x_2) \in E, x_1 + x_2 = \frac{1}{2}\}$ . Consider  $(x_1, x_2) \in \mathcal{N}_{Shared}$  such that  $x_1 > 0$ . Then agent 1 would have an incentive and possibility to take  $x_1 = 0$  instead and therefore  $(x_1, x_2)$  would not be a Nash equilibrium in individual constraints. Therefore,  $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$ .

This is an example of game in which there is a single equilibrium in individual constraints and an infinity of equilibria in shared constraint. As already said, by requiring from each agent  $i$  to choose a strategy in  $K_i(x_{-i}) \subseteq X_i(x_{-i})$ , this may *expand* the set of Nash equilibria. Proposition 1 thus yields the two following basic insights.

1. There may be situations in which there is no Nash equilibrium in the game with individual constraints, that is,  $\mathcal{N}_{Indiv} = \emptyset$  while there exists a Nash equilibrium in the game with endogenous shared constraint (generated from the individual ones), that is  $\mathcal{N}_{Shared} \neq \emptyset$ .
2. If no Nash equilibrium exists in the game with endogenous shared constraint, that is, if  $\mathcal{N}_{Shared} = \emptyset$ , no Nash equilibrium exists in the game with individual constraints (the converse is however not true), that is,  $\mathcal{N}_{Indiv} = \emptyset$ .

From a mathematical point of view, it is difficult to find conditions under which a Nash equilibrium exists in individual constraints. It is however not difficult to see when such a Nash equilibrium does not exist, that is, when  $\mathcal{N}_{Indiv} = \emptyset$ . Assume one moment that the Nash equilibrium in shared constraint  $x^* \in \mathcal{N}_{Shared}$  is unique. Further assume that for each  $i$ , the best response in individual constraint  $BR_i^{Ind}(x_{-i}^*)$  of a given agent  $i$  (given what the others do ( $x_{-i}^*$ ) at equilibrium) is well defined as a unique point in  $X_i(x_{-i}^*)$ . If there exists  $i \in J$  such that  $BR_i^{Ind}(x_{-i}^*) \in X_i(x_{-i}^*) \setminus K_i(x_{-i}^*)$ , then  $\mathcal{N}_{Indiv} = \emptyset$ . In such a case in which there exists an agent with an incentive to choose a strategy  $BR_i^{Ind}(x_{-i}^*)$  which is outside  $K_i(x_{-i}^*)$ , the resulting profile of strategy is outside  $K$ , that is,  $(BR_i^{Ind}(x_{-i}^*), x_{-i}^*) \notin K$ .

While proposition 1 is elementary to prove from a mathematical point of view, it is interesting to note that it does not require the underlying functions to be differentiable, as opposed to the variational formulation of the Nash equilibrium in generalized games<sup>5</sup>. Proposition 1 thus can also be applied to generalized games in which the set of strategies of each agent is finite, for which the objective function needs not be a continuous function. In classical game theory—games for which the

---

<sup>5</sup>It should also be pointed out that the solutions of a variational inequality do not necessarily coincide with the Nash equilibrium of the generalized game, that is, a Nash equilibrium needs not be the solution of a variational inequality, see e.g., [Fischer et al., 2014], theorem 2.2 p. 525.



set of strategies  $E$  can be written as the "full" Cartesian product of individual strategy set  $E_i$ —there is a fairly important body of literature<sup>6</sup>, initiated by [Dasgupta and Maskin, 1986], that prove the existence of a Nash equilibrium (in pure strategy) in discontinuous games. In this paper, we take an alternative road map and we submit the idea that when a Nash equilibrium does not exist in a game with individual constraints, its existence can be obtained simply by sharing the constraints rather than weakening the assumptions on the objective functions. Let us provide such an example in which  $\mathcal{N}_{Indiv} = \emptyset$  while  $\mathcal{N}_{Shared} \neq \emptyset$ .

Let  $J = \{1, 2, \dots, N\}$  be the set of agents. For each  $i \in J$ , the strategy set of agent  $i$  is  $E_i = [0, 1]$  so that  $E = [0, 1]^N$ . Assume moreover that the characteristics of the agents are as follows.

1. Each agent  $i \in J$  has a cost function (to be minimized) equal to  $\theta_i(x_i) = x_i$ .
2. Each agent  $i \in J$  has the following constraint function.
  - If  $x_j \geq 0.9$  for every  $j \in J \setminus \{i\}$ , then,  $X_i(x_{-i}) = [\frac{1}{2}, 1]$ .
  - If  $x_j < 0.9$  for at least one  $j \in J \setminus \{i\}$ , then,  $X_i(x_{-i}) = \emptyset$ .

This game thus differs from classical ones encountered in economic theory in that the interaction only occurs through the set of strategies but not through the objective functions. From the specification of the game, if a given agent  $i$ , with  $i \neq j$  chooses a number  $x_i \geq 0.9$ , the cost of agent  $i$  is simply equal to the number  $x_i$  chosen. If there is one agent  $j$  who chooses a number  $x_j < 0.9$ , with  $i \neq j$  then, the set of strategies of a given agent is empty and the objective function thus is undefined<sup>7</sup>. Before discussing the outcome of the game with individual constraints, let us consider the game with shared constraint generated from the individual constraints. In this game with shared constraint, the set of strategies of each agent  $i \in J$  is equal to  $K_i(x_{-i}) = [0.9, 1]$  (see equation 4) so that  $K$  is not empty and equal to  $K = [0.9, 1]^N$ . Since agents minimize a cost function, the profile of strategies  $x^* = (0.9, \dots, 0.9)$  thus is the *unique* Nash equilibrium of the game with shared constraint  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$ . Therefore, from proposition 1, if the game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  has a Nash equilibrium, it must necessarily be  $x^* = (0.9, \dots, 0.9)$ . But  $x^* = (0.9, \dots, 0.9)$  is *not* a Nash equilibrium of the game with individual constraints. If  $x_{-i} = (0.9, \dots, 0.9)$ , the best response of agent  $i$  is 0.5 and not 0.9, which means that  $x^* = (0.9, \dots, 0.9)$  is not a Nash equilibrium of the game with individual constraints. When one agents picks a number lower than 0.9, the objective functions are undefined for the other agents and this means that there is no Nash equilibrium in such a game with individual constraints.

---

<sup>6</sup>We refer the reader to the introductory paper by [Reny, 2016] in the 2016-symposium of *Economic Theory*

<sup>7</sup>One can think of this abstract game theoretic framework to represent a tax competition problem in a fiscal union (see for instance [Zodrow, 2003]). State members may be committed to choose a tax rate greater than a given threshold. However, if one member state breaches the commitment, the problem becomes undefined for the other member states in that their strategy set is empty.

### Types of interaction in games

	Strategy set	Yes	No
Objective function		Generalized game	Classical game
Yes		Non-Classical game	Non-strategic decision problem
No			

Table 3: Dependence through the objective function and/or through the strategy sets

#### 4.2 Existence and characterization results for non-classical games

**Classification.** In a generalized game, (see table 3), interaction occurs not only through the objective function but also through the strategy sets, that is, the objective function of a given agent  $i \in J$  depends upon the decisions of the other players, i.e.,  $\theta_i(x_i, x_{-i})$  but the set of strategy of each agent  $i$  also depends upon the decisions of the other agents, that is, each agent  $i$  must choose  $x_i \in X_i(x_{-i})$  where  $X_i(x_{-i})$  is a subset of  $E_i$ . In table 2, we offer a fruitful classification of games, encountered in Economics and Operations Research literature, through the way interaction is introduced. While not in the table, one can also distinguish generalized games with an exogenous shared constraint from generalized games with endogenous shared constraint, the subject of this paper.

In table 3, for what we call *classical games*, those encountered in most economic textbooks and papers in economic theory (see e.g. classic textbooks [Fudenberg and Tirole, 1991], [Moulin, 1986], [Myerson, 2013], [Osborne and Rubinstein, 1994]), interaction occurs through the objective functions but not through the constraints. For what we call *non-classical games*, interaction occurs through the constraints but not through the objective functions. While these non-classical games are fairly natural when one thinks to collective action problems, their application in the Economic literature is still limited at the moment. In the future, these kind of generalized games could be widely applied. Finally, when there is no interaction at all, that is, when the objective function only depends upon the decision variable of agent  $i$ , that is  $\theta_i(x_i)$  and when  $E_i$  is exogenously given (e.g., it is a compact set), this gives rise a non-strategic decision problem, the field of decision-theory (as opposed to game theory).

**Existence and characterization results for non-classical games.** We now provide a simple existence result which can be thought as a generalization of Rosen’s theorem for non-classical game. To the best of our knowledge, such non-classical game appear for the first time [Braouezec and Wagalath, 2019] and more recently in [Braouezec and Kiani, 2022] in a financial stress test framework. A related more general framework is provided in [Banerjee and Feinstein, 2021] although the authors do not explicitly formulate their problem as a generalized game.

**Proposition 2** Consider a non-classical game  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  such that for each  $i \in J$ ,  $\theta_i(x) = \theta_i(x_i)$  and assume that for each  $i \in J$ ,  $\theta_i$  is continuous and that  $K$  is compact. The following results hold.

1. The game with shared constraint  $(J, E, (\theta_i)_{i \in J}, K)$  admits at least one Nash equilibrium.
2. The set of Nash equilibria in shared constraint  $\mathcal{N}_{Shared}$  exactly coincides with the set of minimizers of the total cost function  $\sum_{i \in J} \theta_i(x_i)$  on  $K$  and each Nash equilibrium is Pareto optimal.

**Proof.** See the appendix.

Note that from the above result, one cannot infer any result on the existence of Nash equilibria in individual constraints. Note that in proposition 2, we make no assumption regarding the evolution of  $\theta_i$  with respect to  $x_i \in \mathbb{R}^d$  with  $d \geq 1$ . We shall now show an example in which the non-classical game satisfies the assumptions of Proposition 2, and therefore the game with endogenous shared constraint  $(J, E, (\theta_i)_{i \in J}, K)$  has at least one Nash equilibrium, but the initial game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  may or may not have a Nash equilibrium. The following result holds for non-classical games with positive linear constraints that will be illustrated in Section 5.1. Note importantly that in the next result, the objective function is typically a profit function to be maximized.

**Proposition 3** Consider a non-classical game  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$  such that for each  $i \in J$ ,  $E_i = \mathbb{R}_+$  and let  $S_i \in \mathbb{R}^+$  be the constraint of agent  $i$ . Assume that the payoff function  $\theta_i(x) = \theta_i(x_i)$  is an increasing, continuous and concave function of  $x_i$  and that the constraint of agent  $i$ ,  $l_i(x) = \sum_{j \in J} a_{ij}x_j$  is a linear function of  $x_j$  where  $a_{ij} > 0$  for each  $i$  and  $j$ , i.e.,  $X_i(x_{-i}) = \{x_i \in E_i, l_i(x) \leq S_i\}$ . Let  $A = (A_{i,j})_{i,j \in J}$  be a matrix and  $S = (S_i)_{i \in J}$  be a column vector.

1) Game in individual constraints:

- If the linear system  $Ax = S$  admits no solution in  $\mathbb{R}_+^N$ , then there is no Nash equilibrium in individual constraints.
- If the matrix  $A$  is invertible, then, there exists a unique solution  $x^*$  to the linear system and if  $x^*$  is in  $\mathbb{R}_+^N$ , there exists a unique Nash equilibrium of the form  $x^* = A^{-1}S$ . If  $x^*$  is not in  $\mathbb{R}_+^N$ , there is no Nash equilibrium.
- If the matrix  $A$  is not invertible and the linear system  $Ax = S$  admits at least one solution  $x^{*,0} \in \mathbb{R}_+^N$ , then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria in individual constraints given by  $x^{*,0} + \ker(A) \cap \mathbb{R}_+^N$ .

2) Game in shared constraint:

- $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$  (this is a direct consequence of Proposition 1)
- There always exist at least one Nash equilibrium.

**Proof.** See the Appendix.

**Remark 2** This result is obviously true for the symmetric case of a game with agents seeking to minimize a cost function  $Cost_i(x) = Cost_i(x_i)$  decreasing continuous and convex function of  $x_i$  and with individual constraints  $X_i(x_{-i}) = \{x_i \in E_i, l_i(x) \geq S_i\}$  with  $l_i(x) = \sum_{j \in J} a_{ij}x_j$ .

## 5 Applications to collective action problems

We shall now discuss two interesting applications of games with endogenous shared constraint, one applied to an environmental problem and another applied to a public good problem. A striking feature of these two economic applications is that a Nash equilibrium may not exist in individual constraints while it always exists in endogenous shared constraint. The first example is applied to an environmental problem (it is indeed formulated as a non-classical game) and the existence result can be proved either with the fundamental theorem of [Rosen, 1965] (see theorem 1 in this paper) or with Proposition 2. In the second example, applied to the financing problem of a public good, we consider a generalized game and show that when the dispersion of the optimal contribution of each agent is too large, no Nash equilibrium exists in individual constraints while a Nash equilibrium (indeed many) exists in endogenous shared constraint. From a mathematical point of view, to ease the analysis, we shall focus on the case in which the individual constraints are uni-dimensional.

### 5.1 Limiting global warming

Given the critical nature of the subject, the limitation of global warming of the earth, there is a large (game theoretical based) literature on the subject (see e.g., [Hoel and Schneider, 1997], [Carraro and Siniscalco, 1993], [Barrett, 2001]). We refer the reader to [Missfeldt, 1999], which is a survey of game theoretic models of trans-boundary pollution but note that this review paper do not mention generalized games. We found only few papers on the subject, [Tidball and Zaccour, 2005] and [Krawczyk, 2005] that analyze the pollution problem as a generalized game. In these models, the choice variable of a given country  $x_i$  is typically its pollution level (i.e., measured by the volume of emission of greenhouse gas) which, in the simplest case, is modeled as a linear function of its production. Such an environmental problem is interesting but challenging because the production of a given country typically generates *negative externalities* (i.e., pollution) to all the other countries.

In [Tidball and Zaccour, 2005], as seen earlier, they consider a model where each country seeks to maximize a profit function of the type  $w_i(x_1, \dots, x_n) = f_i(x_i) - d_i(x_1 + \dots + x_n)$ , where  $x_i$  represent the emissions of country  $i$  and are assumed to be proportional to the production of country  $i$ ,  $f_i(x_i)$  is a non-negative, twice-differentiable, concave and increasing function and the damage cost due to all the countries is denoted by a convex twice-differentiable increasing cost function  $d_i(x_1 + \dots + x_n)$ . They consider three types of problems which correspond to three types of constraint:

- A *Generalized Nash equilibrium problem with individual constraints* where each agent is seeking to maximize  $w_i(x_1, \dots, x_n) = f_i(x_i) - d_i(x_1 + \dots + x_n)$  subject to the constraint  $x_i \leq T_i$  with  $T_i$  an exogenous given upper bound threshold on emissions.
- A *cooperative scenario*, where agents agree to jointly maximize the sum of their profit function:  $\max_{x_1, \dots, x_n} \sum_{i=1}^n (f_i(x_i) - d_i(x_1 + x_2 + \dots + x_n))$  subject to  $\sum_{i=1}^n x_i \leq \sum_{i=1}^n T_i$ .
- A *Generalized Nash equilibrium problem with an exogenous shared constraint* where each agent is seeking to maximize  $w_i(x_1, \dots, x_n) = f_i(x_i) - d_i(x_1 + x_2 + \dots + x_n)$  subject to the constraint

$$\sum_{i=1}^n x_i \leq \sum_{i=1}^n T_i.$$

The purpose of [Tidball and Zaccour, 2005] is to characterize and compare the solutions of these three different scenarios. They show that the Nash equilibrium in individual constraints may be better than the Nash equilibrium with (exogenous) shared constraint. In their framework, the shared constraint is actually not generated from the individual constraints<sup>8</sup>

In the same vein, [Krawczyk, 2005] proposes another model where three players  $j = 1, 2, 3$  located along a river are engaged in an economic activity at a chosen level  $x_j$  and their joint production externalities must satisfy environmental constraints set by a local authority. It is assumed that player  $j$  has a level of pollution  $e_j x_j$ , where  $e_j$  is the emission coefficient of player  $j$ . The pollution is expelled into the river and reaches a monitoring station in the amount of  $\sum_{j=1}^3 \delta_{jl} e_j x_j$  where  $\delta_{jl}$  is the decay-and-transportation coefficient from player  $j$  to location  $l$ , and it is assumed that there are two monitoring stations,  $l = 1, 2$ , and the local authority has set maximum pollutant concentration levels  $C_l$ . It gives the following Generalized Nash equilibrium problem with shared constraint: each player  $j$  is supposed to maximize its net profit  $\phi_j(x) = d_1 - d_2(x_1 + x_2 + x_3) - (c_{1j} + c_{2j}x_j)x_j$  (where  $d_1, d_2, c_{1j}, c_{2j}$  are given constants) under the shared constraint  $\sum_{j=1}^3 \delta_{jl} e_j x_j \leq C_l, l = 1, 2$ . [Krawczyk, 2005] proves existence of a unique Nash equilibrium under reasonable assumptions and exhibits an algorithms that converges towards the solution.

Inspired by [Tidball and Zaccour, 2005] and [Krawczyk, 2005], we now introduce a new environmental model formulated in terms of generalized with individual constraints and with endogenous shared constraint.

Consider the following model with  $N \geq 2$  countries/economies. Let  $x_i \in \mathbb{R}_+$  be the quantity of non-renewable energy (i.e., fossil energy) chosen by country  $i$  where the carbon emissions  $C_i$  by country  $i \in \{1, 2, \dots, N\}$  are proportional to  $x_i$  (similar to [Tidball and Zaccour, 2005] and [Krawczyk, 2005]), that is  $C_i = c_i \times x_i$ , where  $c_i > 0$  (and note that  $E = \mathbb{R}_+^N$ ). Let  $f_i(x_i)$  be the profit function of country  $i$  as a function of  $x_i$  where  $f_i$  is an increasing continuous and concave function. On this aspect, we differ from [Tidball and Zaccour, 2005] and [Krawczyk, 2005] since we assume that the payoff function of a country only depends on  $x_i$ , the quantity of non-renewable energy chosen by  $i$  while the constraint depends upon  $x = (x_1, x_2, \dots, x_N)$ . More specifically, we assume that each country  $i$  has to satisfy an individual linear constraint of the form:

$$g_i(x) := a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i \quad (8)$$

where  $a_{ji}$  is a coefficient measuring how the use of fossil fuels by country  $j$  is impacting the environment of country  $i$  and  $S_i$  is a threshold determined for each country  $i$ . The constraint  $S_i$  may

---

<sup>8</sup>The individual constraint of country  $i$  does not depend upon the choices of the other countries, for each  $i \in J$ ,  $X_i(x_{-i}) = [0, T_i]$  no matter what  $x_{-i}$  is. It thus follows that  $K = \prod_{i \in J} X_i$  while the shared constraint is given by the set  $S := \{x \in K : \sum_{i \in J} x_i \leq \sum_{i \in J} T_i\}$ . The game with shared constraint proposed by [Tidball and Zaccour, 2005] thus is *not* generated from the game with individual constraints since  $x$  can be in  $S$  but not in  $K$ . To see this within a numerical example, assume that  $N = 2$  and that  $\bar{x}_1 = 3$  and  $\bar{x}_2 = 2$  so that  $X_1 = [0, 3]$  while  $X_2 = [0, 2]$ . The point  $(2, 3) \notin X_1 \times X_2$  while  $(2, 3) \in S$ .

come from a regulatory institution. Note that in case in which  $j = i$ , the coefficient  $a_{ii}$  measures how the use of fossil fuels by country  $i$  is impacting its own environment. Given  $x_{-i}$ , the aim of country  $i$  is to maximize  $f_i(x_i)$  subject to  $g_i(x_i, x_{-i}) \leq S_i$  and note that, using our classification of games, such a strategic interaction is an example of a non-classical game.

The aim is now to study the possible existence of a Nash equilibrium for this game with individual constraints and its generated game with (endogenous) shared constraint. As seen earlier, it may indeed be the case that there is no Nash equilibrium for the game with individual constraints. To see this, assume that  $N = 2$ . Consider country 1 and assume that  $a_{11} = c_1 = 1$  and that  $a_{21} = 0$  so that its constraint is given by  $x_1 \leq S_1$ . Consider now country 2 and assume that  $a_{22} = c_2 = 1$  and assume now that  $a_{12} = 1$ . From equation (8), it thus follows that the constraint of country 2 is given by  $x_2 + x_1 \leq S_2$ . Assume now that  $S_1 > S_2$  and that this generalized game with individual constraints has at least a Nash equilibrium  $(x_1^*, x_2^*)$ . Such an equilibrium must satisfy  $x_1^* = S_1$ . But in such a case, no matter what  $x_2 \geq 0$  is, the constraint of country 2 is never satisfied which means that there is no Nash equilibrium in individual constraints.

We shall now derive a more general result about the (non) existence of a Nash equilibrium in this environmental game theoretical model with individual constraints. From the individual constraint given by equation (8), for a profile of strategy  $x \in \mathbb{R}_+^N$  to be Nash equilibrium, the constraints must be satisfied, that is,  $x \in \mathbb{R}_+^N$  must be such that  $g_i(x) \leq S_i$  for  $i = 1, 2, \dots, N$ . From the monotonicity of the profit function of each country  $i$ , the constraint of each country will be binding at equilibrium  $x^* = (x_1^*, \dots, x_N^*)$ , which means that for  $i = 1, 2, \dots, N$ ,  $g_i(x^*) = S_i$ . The problem reduces to the analysis of a linear system of the form  $A^\top \text{diag}(c)X = S$  where  $A$  is the matrix of the coefficient  $(a_{ij})_{i,j=1,2,\dots,N}$  ( $\top$  indicates the transpose),  $\text{diag}(c) = \text{diag}(c_1, \dots, c_N)$ ,  $x = (x_1, \dots, x_N) \in \mathbb{R}_+^N$  and  $S$  is the vector formed by  $S_i$ ,  $i = 1, 2, \dots, N$ . If  $x^* = (x_1^*, \dots, x_N^*)$  is a Nash equilibrium for this game with individual constraints, it must satisfy the linear system  $A^\top \text{diag}(c)x^* = S$ , that is:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{22} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix}^\top \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_N \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ \dots \\ x_N^* \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_N \end{pmatrix} \quad (9)$$

where  $x^* \in \mathbb{R}_+^N$ .

The next result exhibits conditions under which there is no Nash equilibrium in individual constraints. In particular, when the above linear system has no solution in  $\mathbb{R}_+^N$ , then, no Nash equilibrium of the game with individual constraints can exist. However, in such a case in which the linear system has no solution, a Nash equilibrium of the game with endogenous shared constraint still exists.

**Proposition 4** *The game with individual constraints has (at least) one Nash equilibrium  $x^*$  if and only if the linear system as defined in equation (9) admits at least one solution  $x^* \in \mathbb{R}_+^N$ . If the linear*

system given by equation (9) admits no solution in  $\mathbb{R}_+^N$ , then, the game with individual constraints has no Nash equilibrium.

**Proof.** See the appendix.

In appendix, we give a complete characterization of the existence or not of Nash equilibria for the game with individual constraints. It is interesting at this stage to compare our framework with the one offered in [Tidball and Zaccour, 2005] and [Krawczyk, 2005]. Within our approach, as opposed to the two mentioned papers, a Nash equilibrium does not always exist in individual constraints. However, as we shall see, a Nash equilibrium in shared constraint always exists. In [Tidball and Zaccour, 2005], they show that Nash equilibria in individual constraints may be better in some sense than the ones in shared constraint but, as already discussed, their shared constraint is exogenous since it is not generated from the individual ones.

Let us now consider the game with shared constraint generated from our game with individual constraints in which each country  $i$  is seeking to maximize  $f_i(x_i)$  subject to the endogenous shared constraint defined as  $g_k(x) := a_{kk} \times c_k \times x_k + \sum_{j \neq k} a_{jk} \times c_j \times x_j \leq S_k, k = 1, \dots, N$ . From proposition 1, we know that the Nash equilibria of the game with shared constraint contains the Nash equilibria of the game with individual constraints. But Nash equilibria of the game with individual constraints may not always exist, as seen in proposition 4. In the next result, we show that there always exists Nash equilibria for our game with endogenous shared constraint. Actually, there are two ways to prove and see this result: the first way thanks to the existence result by Rosen for  $n$ -person concave games, and the second way thanks to Proposition 2. We provide two proofs in appendix.

**Proposition 5** *The game with a shared constraint generated from the individual ones always has at least one Nash equilibrium. In particular, a Nash equilibrium in the game with endogenous shared constraint exists whether the linear system given by equation (9) admits a solution or not.*

**Proof.** See the appendix.

Proposition 5 clearly shows that a Nash equilibrium always exists in the game with shared constraint generated from the individual constraints while it may fail to exist in the game with individual constraints. By reinforcing the environmental constraints, i.e., by sharing the constraints, each country takes into account the individual constraints of the other countries, and this binding agreement always generates at least one Nash equilibrium. As already discussed, this interesting feature is counter intuitive at first glance: by putting more constraints on each country (by considering an endogenous shared constraint rather than individual constraints), one may generate a larger set of Nash Equilibria.

## 5.2 Contributing to a public good

We now consider a model of collective action with  $N \geq 2$  agents, similar to [Guttman, 1978] and to [Cornes and Hartley, 2007] (see also [Buchholz et al., 2011]) but in which each agent faces an

individual constraint (see [Sandler, 2015] for a review paper on collective action models). Each agent  $i$  is asked to make a voluntary contribution to finance a public investment project such as a bridge, a street lighting or any public infrastructure equipment (water, gas, internet...) subject to an individual constraint. For concreteness, we assume that the higher the total contributions, the higher the quality of the underlying investment. Following the standard terminology introduced in [Guttman, 1978] (see also ([Buchholz et al., 2011])), let  $x_i \in E_i := \mathbb{R}_+$  be the flat (or direct) contribution of agent  $i$  and let  $b \in [0, 1]$  the known percentage of the sum of the flat contributions of all the other agents, which thus defines here indirect contribution. The indirect contribution of a given agent  $i$  is by definition equal to  $b \sum_{j \neq i} x_j$ , so that the total contribution of agent  $i$  is equal to  $x_i + b \sum_{j \neq i} x_j$ . Following [Guttman, 1978], the utility of each agent  $i$  is assumed to be equal to

$$U_i(x_i, x_{-i}) = v_i(x_i + b \sum_{j \neq i} x_j) - (x_i + b \sum_{j \neq i} x_j) \quad (10)$$

where the function  $v_i(\cdot)$  measures the willingness-to-pay of an agent  $i$  for the public project and depends upon her own contribution but also upon the contributions of all the other agents. As in the literature on collective actions and aggregative games ([Guttman, 1978], [Cornes and Hartley, 2007] [Chen and Zeckhauser, 2018], [Cornes and Hartley, 2012], see also [Cornes, 2016] for a recent review paper), the willingness-to-pay  $v_i(\cdot)$  depends upon  $x_{-i}$  only through the sum of the contributions of the other agents. Given a profile of strategies  $x = (x_i, x_{-i})$ , let

$$z_i = x_i + b \sum_{j \neq i} x_j \quad (11)$$

be the total contribution of agent  $i$  and note that the utility function of agent  $i$  can be written as a function of the sole scalar  $z_i$ .

$$U_i(x_i, x_{-i}) = v_i(z_i) - z_i \quad (12)$$

We shall assume that  $v_i(z_i)$  is a twice continuously differentiable increasing and strictly concave function of  $z_i$  (with  $v_i(0) = 0$ ) so that  $U_i$  is also a strictly concave function<sup>9</sup>. An example of a functional form for  $v_i(z_i)$  is  $v_i(z_i) = \xi_i \sqrt{z_i}$  where  $\xi_i$  is a positive scalar.

Regarding now the individual constraints, it can be formulated as a *budget constraint* or as a *reservation utility*.

- The budget constraint of agent  $i$  can be given as an exogenous revenue of agent  $i$ ,  $r_i$ , which means that it must be the case that  $z_i \leq r_i$ .
- The reservation utility constraint can be given as an exogenous reservation utility of agent  $i$ ,  $\bar{u}_i$ , which means that it must be the case that  $U_i(x_i, x_{-i}) \geq \bar{u}_i$ . The threshold  $\bar{u}_i$  can be interpreted as a minimal quality for the public good according to agent  $i$ .

---

<sup>9</sup>Strict concavity is actually not required. What is actually required is only the single-peakedness of  $U$  with respect to  $z_i$ , that is, the strict quasi-concavity of  $U$ . Such a single-peakedness assumption is fairly natural and standard (see [Guttman, 1978], see also [Greenberg and Weber, 1993]).



The optimization problem of a given agent  $i$  can be formulated as a utility maximization problem subject to a budget constraint and/or subject to a reservation utility constraint. Let

- $\mathcal{B} \neq \emptyset$  be the subset of  $J$  that are subject to a budget constraint.
- $\mathcal{U} \neq \emptyset$  be the subset of  $J$  that are subject to a utility constraint.

where the  $\mathcal{B}$  and  $\mathcal{U}$  are assumed to form a partition of  $J$ , that is,  $\mathcal{B} \cup \mathcal{U} = J$ , with  $\mathcal{B} \cap \mathcal{U} = \emptyset$ . Agents of the group  $\mathcal{U}$  may be interpreted as those agents that have other investment opportunities and are only be subject to a utility constraint. Agents of the group  $\mathcal{B}$  with no other opportunities are subject to a budget constraint. Since each agent  $i$  is endowed with a utility function assumed to be a concave function of  $z_i$ , it makes thus sense to consider her *ideal contribution*  $z_i^*$  to the public project.

Let  $z_i^* = \arg \max_{z_i \geq 0} U_i(z_i) := v_i(z_i) - z_i$  subject to a budget utility constraint if  $i \in \mathcal{B}$  and a reservation utility constraint if  $i \in \mathcal{U}$ . Given the assumptions on the utility function,  $z_i^*$  is unique and for the sake of interest, we assume that  $z_i^* > 0$  for each  $i \in J$ .

Let  $\bar{z}_i$  be the critical threshold of each agent  $i \in J$ . If  $i \in \mathcal{B}$ , then  $\bar{z}_i = r_i$ . Consider now agent  $i \in \mathcal{U}$ . Let  $\underline{z}_i$  and  $\bar{z}_i$ , with  $\underline{z}_i < \bar{z}_i$ , be the critical thresholds of each agent  $i \in \mathcal{U}$ . If  $i \in \mathcal{U}$ , then  $U_i(\underline{z}_i) = v_i(\underline{z}_i) - \underline{z}_i = \bar{u}_i = U_i(\bar{z}_i) = v_i(\bar{z}_i) - \bar{z}_i$ . As before, let us denote  $BR_i^{Ind}$  the best response function in individual constraints. The following fact holds and is an elementary consequence of the definition of  $z_i^*$ .

**Fact 2** *The best response of agent  $i \in J$  is given below.*

- If  $b \sum_{j \neq i} x_j \leq z_i^*$ , then,  $BR_i^{Ind}(x_{-i}) = z_i^* - b \sum_{j \neq i} x_j > 0$
- If  $b \sum_{j \neq i} x_j \in (z_i^*, \bar{z}_i)$ , then,  $BR_i^{Ind}(x_{-i}) = 0$
- If  $b \sum_{j \neq i} x_j > \bar{z}_i$  then, either  $BR_i^{Ind}(x_{-i}) = \emptyset$

The fact that  $BR_i(x_{-i}) = \emptyset$  depends upon the type of constraint. When  $X_i(x_{-i}) = \emptyset$  for  $i \in \mathcal{B}$ , this simply means that this agent must pay *more* than her own revenue, which is impossible. As a result  $BR_i(x_{-i}) = \emptyset$ . One may interpret this as an exclusion, similar to the *violation of the no-bankruptcy condition* in aggregative games (see e.g., [Buchholz et al., 2011]). The situation is similar for agents of the group  $\mathcal{U}$ . When  $b \sum_{j \neq i} x_j > \bar{z}_i$  for an agent  $i \in \mathcal{U}$ , her utility will be *lower* than her reservation utility. As a result  $BR_i(x_{-i}) = \emptyset$ .

Consider an agent  $i \in \mathcal{B}$ . Depending upon the willingness-to-pay and the revenue, an agent  $i$  may be such that its optimal contribution  $z_i^*$  may be equal to  $r_i$  or may be lower than  $r_i$ . When  $z_i^* < r_i$ , the budget constraint is not binding and this occurs when  $z_i^*$  solves the unconstrained maximization of  $U_i(z_i)$ . When  $z_i^* = r_i$ , the constraint is binding. Consider now an agent  $j \in \mathcal{U}$  with a high  $z_j^*$ . Assume that given  $x_i$  with  $i \in \mathcal{B}$ , the best response of  $j$  equal to  $BR_j(x_i) := x_j = z_j^* - bx_i$  is higher than  $\frac{r_i}{b}$ . In such a case,  $bx_j > r_i = \bar{z}_i$ . From fact 2,  $BR_i(x_j) = \emptyset$ , that is, no matter the

choice of agent  $i$ , agent  $j$  will always choose a contribution so high that  $i$  is left with an empty set. Agent  $j$  has a dominant strategy which implies that  $BR_i(x_j) = \emptyset$ . The following result provides a simple illustration of this when  $N = 2$  but nothing is changed in the general case of an arbitrary number  $N$  of agents.

**Proposition 6** *Let  $b \in (0, 1]$  and  $J = \{1, 2\}$  where the group  $\mathcal{B} = \{1\}$  and the group  $\mathcal{U} = \{2\}$ .*

(i) *A sufficient condition for the non-existence of a Nash equilibrium in individual constraints is  $z_2^* > (b + \frac{1}{b})r_1$ .*

(ii) *Assume moreover that  $z_1^* = r_1 < z_2^*$  and that  $z_1^* \geq bz_2$  and let  $\bar{x}_1 := \frac{r_1 - bz_2}{(1 - b^2)}$ . Then, for any  $\theta \in [0, 1]$ , the pair of strategies  $(x_1^* = \theta\bar{x}_1, x_2^* = \frac{r_1 - \theta\bar{x}_1}{b})$  is a Nash equilibrium in shared constraint.*

**Proof.** See the appendix.

When agent 1 faces a budget constraint such that  $z_1^* = r_1$ , the existence of a Nash equilibrium in individual constraints critically depends upon the heterogeneity (or the dispersion) of the set of ideal contributions  $(z_i^*)_{i \in \{1, 2\}}$ . If this dispersion is too high, that is, if  $\frac{z_2^*}{z_1^*}$  is greater than a critical threshold, then, a Nash equilibrium in individual constraints does not exist. For any  $\theta \in [0, 1]$ , the pair  $(x_1^* = \theta\bar{x}_1, x_2^* = \frac{z_1^* - \theta\bar{x}_1}{b})$  is a Nash equilibrium and is such that  $x_1^* + bx_2^* = z_1^*$ , it thus follows that for any  $\theta \in (0, 1)$ , the Nash equilibrium is Pareto optimal.

**Corollary 1** *Each Nash equilibrium of the game with shared constraint is Pareto optimal.*

Within our particular model of collective action, when a Nash equilibrium in the game with individual constraint does not exist, there still exists a *continuum* of Nash equilibria in the game with shared constraint that are all Pareto-optimal.

Our analysis also reveals an important property of collective actions. If one only looks at the equilibrium in individual constraints when for instance  $z_i^* = r_i$  for each  $i \in \mathcal{B}$ , the existence of such an equilibrium in individual constraints critically depends upon the dispersion of the ideal contribution  $(z_i^*)_{i \in J}$ . This thus suggests that for a collective action to be possible (in individual constraints), agents of the group  $J$  must have homogenous ideal contributions, something not required for the shared constraint problem. In the general case with  $N \geq 2$  agents, let

$$K = \{x \in \mathbb{R}_+^N : x_i + b \sum_{j \neq i} x_j \leq r_i \ \forall i \in \mathcal{B} \text{ and } U_k(x_k + b \sum_{j \neq k} x_j) \geq \bar{u}_k \ \forall k \in \mathcal{U}\}$$

As in proposition 6, as long as the dispersion of  $(z_i^*)_{i \in J}$  is "too high",  $K$  will be empty and no Nash equilibrium in individual constraints exists. But in shared constraint, assuming for simplicity that  $z_i^* = r_i$ , the non-vacuity of  $K$  will *only* depend upon the ideal contribution of agents of  $\mathcal{B}$ . In case in which  $K$  is a compact and convex subset of  $\mathbb{R}_+^N$ , from theorem 1 ([Rosen, 1965]), a Nash

equilibrium in the game with shared constraint always exist (since the utility functions are concave) while a Nash equilibrium in individual constraint might not exist.

Throughout the discussion, we made the assumption that for each  $i \in \mathcal{B}$ ,  $z_i^* = r_i$ , which means that the budget constraint is always binding. It would also be possible to consider the case in which  $z_i^* < r_i$ . If one let  $z_{i,0} := \inf_{z_i} U(z_i) = 0$ , there are thus two possibilities, either  $z_{i,0} < r_i$  or  $z_{i,0} > r_i$ . In the first case, this means that when  $b \sum_{j \neq i} x_j$  is slightly higher than  $z_{i,0}$ , the strategy set of agent  $i$  is not empty while its utility is negative. If as in [Greenberg and Weber, 1993] however in a different framework, we allow such an agent to reject the project if its utility becomes negative, it suffices to consider the constraint  $c_i$  of agent  $i$  given as  $c_i = \min\{r_i, z_{i,0}\}$  and nothing is changed.

## 6 Conclusion

In this paper, we presented the notions of generalized games with individual constraints, generalized games with shared constraint, and generalized games with an endogenous shared constraint generated from individual constraints. We proved a simple yet interesting and rich result regarding the existence of Nash equilibria for a generalized game with an endogenous shared constraint generated from individual ones, that is the Nash equilibria of a generalized game with individual constraints are included in the set of Nash equilibria of the generalized game with endogenous shared constraint. We then studied different applications of this result, among which a public good problem and an environment control problem. A number of interesting investigations remains to be done. For instance, under which condition(s) no Nash equilibrium exist in individual constraint while it exists in endogenous shared constraint? When there is more than one equilibrium, does one of them Pareto-dominates the others?

## 7 Appendix

**Proof of Proposition 1.** If  $x^* = (x^{*1}, \dots, x^{*N}) \in E$  is a Nash equilibrium for the game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ , then:

- $\forall i \in J, x_i^* \in X_i(x_{-i}^*)$  so that  $x^* \in K$ .
- $\forall i \in J, \forall x_i \in X_i(x_{-i}^*), \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$  so  $\forall i \in J, \forall x_i \in E_i$  such that  $(x_i, x_{-i}^*) \in K, \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$ .

It thus follows that if the point  $x^*$  is a Nash equilibrium of the game with individual constraints  $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ , it is also a Nash equilibrium for the game with shared constraint  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$  but the converse is however not true. If the point  $x^*$  is a Nash equilibrium of the game with shared constraint generated from the individual constraints  $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$ , it may be the case that the profile of strategies  $x^* \in E$  is not a Nash equilibrium of the game with individual constraint because there may exist  $i \in J$  such that the best response  $x_i^* := BR_i(x_{-i}^*) \in X_i(x_{-i}^*)$  is such that  $x^* \notin K$  where  $x^* = (x_i^*, x_{-i}^*)$ . Examples in which a profile of strategies which is a Nash equilibrium in shared constraint but not a Nash equilibrium in individual constraints have been given.  $\square$

### Proof of Proposition 2

Consider the application:

$$\begin{aligned} \theta &: K \rightarrow \mathbb{R} \\ x &\mapsto \sum_{i \in J} \theta_i(x_i) \end{aligned}$$

Since  $K$  is compact and  $\theta$  is continuous, there exists at least one  $x^* \in K$  that minimizes  $\theta$ ,  $\theta(x^*) = \min_{x \in K} \theta(x)$ . Let us prove that  $x^*$  is a Nash equilibrium. If this is was not the case, there would exist an agent  $j \in J$  and a strategy  $x_j \in E_j$  such that  $(x_j, x_{-j}^*) \in K$  and  $\theta_j(x_j) < \theta_j(x_j^*)$  so that  $\theta_j(x_j) + \sum_{i \neq j} \theta_i(x_i) < \sum_{i \in J} \theta_i(x_i^*)$  and this would contradict the definition of  $x^*$ . Therefore  $x^*$  is a Nash equilibrium and all Nash equilibria of  $(J, E, (\theta_i)_{i \in J}, K)$  are minimizers of  $\theta$ . As a result, a given Nash equilibrium  $x^*$  is Pareto optimal.  $\square$

### Proof of Proposition 3

1) Assume that this game has a Nash equilibrium in individual constraints  $x^* = (x_1^*, \dots, x_N^*)$ . Such a Nash equilibrium must satisfy  $\sum_j a_{ij}x_j^* = S_i$  for all  $i$ . Indeed, assume that for a given  $i$  we have  $\sum_j a_{ij}x_j^* < S_i$ , then agent  $i$  could still increase its payoff  $\theta_i(x_i)$  with a  $x_i$  higher than  $x_i^*$  still satisfying the constraint, and therefore  $x^*$  would not be a Nash equilibrium. Therefore a Nash equilibrium  $x^* = (x_1^*, \dots, x_N^*)$  for this problem with individual constraints must satisfy:  $AX^* = S$  with  $X^* \in \mathbb{R}_+^N$ , with

And this has a solution if and only if the linear system  $Ax = S$  has a solution  $x^* \in \mathbb{R}_+^N$ .

- If  $A$  is invertible, we have an explicit formula for the Nash equilibrium candidate vector  $X^* = A^{-1}S$

- If  $A$  is not invertible and the linear system  $Ax = S$  admits one solution  $X^{*,0} \in \mathbb{R}_+^N$ , then  $Ax = Ax^{*,0}$ , which is equivalent to  $A(x - x^{*,0}) = 0$ , which is equivalent to  $x = (x^{*,0} + Ker(A)) \cap \mathbb{R}_+^N$ , and there are infinitely many Nash equilibria.
- If the linear system  $Ax = S$  admits no solution in  $\mathbb{R}_+^N$ , then there is no Nash equilibrium.

2) We provide two different proofs of this result:

- First proof :

It is easy to see that the game always satisfies the assumptions of the existence result of Rosen for  $n$ -person concave games (see theorem 1 in the text). Indeed, the payoff functions  $\theta_i(x_i)$  are continuous and concave, and the shared constraint space  $K$  is clearly a convex compact space:

$$K = \{x \in \mathbb{R}_+^N : \sum_{j \in J} a_{ij}x_j \leq S_i, i = 1, 2, \dots, N\}$$

Therefore, there always exist a Nash equilibrium for the game with shared constraint.

- Second proof :

This is a non-classical game,  $K$  is a compact set and the cost functions are continuous, therefore this game satisfies the assumptions of Proposition 2, and a Nash equilibrium in shared constraint always exist. Moreover,  $\mathcal{N}_{Shared}$  coincides exactly with the minimizers of the total cost function  $\sum_{i \in J} \theta_i(x_i)$  on  $K$  and all its elements are Pareto optimal.

□

#### Proof of proposition 4.

We shall give here a result more detailed than the one stated in the text. In what follows, we actually offer a complete characterization of the existence or non-existence of the Nash equilibrium in individual strategies. The environmental problem formulated as a game with individual constraints has a Nash equilibrium if and only if the linear system  $A^T \text{diag}(c)X = S$  ( $A^T$  denotes the transpose of  $A$ ) admits at least one solution  $X^* \in \mathbb{R}_+^N$ .

#### PropositionA 4

- If the linear system  $A^T \text{diag}(c)X = S$  admits no solution in  $\mathbb{R}_+^N$ , then there is no Nash equilibrium.
- If the matrix  $A$  is invertible, then, there exists a unique solution  $X^*$  to the linear system, and if  $X^*$  is in  $\mathbb{R}_+^N$ , there exists a unique Nash equilibrium of the form:

$$x^* = \left( x_1^* = \frac{1}{c_1} \sum_{j=1}^N b_{1j}S_j, \dots, x_i^* = \frac{1}{c_i} \sum_{j=1}^N b_{ij}S_j, \dots, x_N^* = \frac{1}{c_N} \sum_{j=1}^N b_{Nj}S_j \right)$$

with  $(B_{i,j})_{i,j \in J} = (A^T)^{-1}$ . If  $X^*$  is not in  $\mathbb{R}_+^N$ , there is no Nash equilibrium.

- If the matrix  $A$  is not invertible and the linear system  $A^T \text{diag}(c)X = S$  admits at least one solution  $X^{*,0} \in \mathbb{R}_+^N$ , then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria given by  $(X^{*,0} + \text{Ker}(A^T \text{diag}(c))) \cap \mathbb{R}_+^N$ .

#### Proof of proposition A 4

Let's assume that this game has a Nash equilibrium  $x^* = (x_1^*, \dots, x_N^*)$ . Such a Nash equilibrium must satisfy  $\sum_j a_{ji} \times c_j \times x_j^* = S_i$  for all  $i$ . Indeed, assume that for a given  $i$  we have  $\sum_j a_{ji} \times c_j \times x_j^* < S_i$ , then country  $i$  could still increase its profit  $f_i(x_i)$  with a  $x_i$  higher than  $x_i^*$  still satisfying the constraint, and therefore  $x^*$  would not be a Nash equilibrium. Therefore a Nash equilibrium  $x^* = (x_1^*, \dots, x_N^*)$  for this problem with individual constraints must satisfy:  $A^T \text{diag}(c)X^* = S$ .

And this has a solution if and only if the linear system  $A^T \text{diag}(c)X = S$  has a solution  $X^* \in \mathbb{R}_+^N$ .

- If  $A$  is invertible, we have an explicit formula for the Nash equilibrium candidate vector since:

$$\text{diag}(c)X^* = (A^T)^{-1} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_N \end{pmatrix}$$

If we denote  $B = (A^T)^{-1}$ , we have:

$$\text{diag}(c)X^* = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \dots & \dots & \dots & \dots \\ b_{N1} & b_{N2} & \dots & b_{NN} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_N \end{pmatrix}$$

And for all  $i$   $x_i^* = \frac{1}{c_i} \sum_{j=1}^N b_{ji} S_j$ , and we have a unique Nash equilibrium candidate  $x^* = (x_1^* = \frac{1}{c_1} \sum_{j=1}^N b_{1j} S_j, \dots, x_i^* = \frac{1}{c_i} \sum_{j=1}^N b_{ij} S_j, \dots, x_N^* = \frac{1}{c_N} \sum_{j=1}^N b_{Nj} S_j)$ .

- If  $A$  is not invertible and the linear system  $A^T \text{diag}(c)X = S$  admits one solution  $X^{*,0} \in \mathbb{R}_+^N$ , then  $A^T \text{diag}(c)X = A^T \text{diag}(c)X^{*,0}$ , which is equivalent to  $A^T \text{diag}(c)(X - X^{*,0}) = 0$ , which is equivalent to  $X = (X^{*,0} + \text{Ker}(A^T \text{diag}(c))) \cap \mathbb{R}_+^N$ , and there are infinitely many Nash equilibria.
- If the linear system  $A^T \text{diag}(c)X = S$  admits no solution in  $\mathbb{R}_+^N$ , then there is no Nash equilibrium.

□

#### Proof of proposition 5

The proof of this result is similar to the second part of the proof of Proposition 3.

### Proof of proposition 6

Part *i*). Assume that  $x_1 = r_1$ . The best response of 2,  $x_2$ , is equal to  $z_2^* - br_1$ . As seen in the paragraph before stating Proposition 6, if  $z_2^* - br_1 > \frac{r_1}{b}$ , then,  $x_2 > r_1$  so that the best response of agent 1 is empty. Solving  $z_2^* - br_1 > \frac{r_1}{b}$  yields part (*i*)  $\square$

Part *ii*). Let  $K = \{(x_1, x_2) \in \mathbb{R}_2^+ : x_1 + bx_2 \leq r_1 \text{ and } U_2(x_1, x_2) \geq \bar{u}_2\}$ . Let us denote  $BR_i^S$  the best response of agent  $i$  in shared constraint. For a Nash equilibrium with shared constraint to exist, we must first prove the non-vacuity of  $K$ , which in turn is equivalent to the existence of a solution that solves the following system.

$$x_1 + bx_2 \leq r_1 \quad (13)$$

$$U_2(x_2 + bx_1) \geq \bar{u}_2 \quad (14)$$

We know from lemma 2 that there exists two critical thresholds  $\underline{z}_2$  and  $\bar{z}_2$  defined as  $\underline{z}_2 = \inf_{z_2} U_2(z_2) = \bar{u}_2$  and  $\bar{z}_2 = \sup_{z_2} U_2(z_2) = \bar{u}_2$ . It thus follows that equation (14) is equivalent to  $z_2 \in [\underline{z}_2, \bar{z}_2]$ , where  $z_2 = x_2 + bx_1$ . For  $K$  to be non-empty, it thus suffices to find a solution to the following system

$$x_1 + bx_2 \leq r_1 \quad (15)$$

$$\underline{z}_2 \leq x_2 + bx_1 \leq \bar{z}_2 \quad (16)$$

First let us prove that (16) reduces to  $\underline{z}_2 \leq x_2 + bx_1 \leq z_2^*$ . Indeed, if we had  $x_2 + bx_1 > z_2^*$ , then this would imply  $x_2 + bx_1 > (b + \frac{1}{b})r_1$ , which would imply  $bx_2 > r_1 = \bar{z}_1$ , which would imply  $BR_1^S(x_2) = \emptyset$ . So inequation (16) becomes

$$\underline{z}_2 \leq x_2 + bx_1 \leq z_2^* \quad (17)$$

Solving inequation (15) in  $x_2$  yields  $x_2 \leq \frac{r_1 - x_1}{b}$  while solving inequation (17) yields  $x_2 \geq \underline{z}_2 - bx_1$ . For a solution in  $x_2$  to exist,  $\frac{r_1 - x_1}{b}$  must be higher than  $\underline{z}_2 - bx_1$ . Solving  $\frac{r_1 - x_1}{b} \geq \underline{z}_2 - bx_1$  yields  $r_1 - b\underline{z}_2 \geq x_1(1 - b^2)$ . Then a necessary condition is  $r_1 - b\underline{z}_2 \geq 0$ . Assuming that  $r_1 - b\underline{z}_2 \geq 0$ ,  $x_1$  must be in  $[0, \frac{r_1 - b\underline{z}_2}{(1 - b^2)}]$ . Let  $\bar{x}_1 := \frac{r_1 - b\underline{z}_2}{(1 - b^2)}$  be the maximal value of  $x_1$  and assume that  $x_1 = \theta\bar{x}_1$  for some  $\theta \in [0, 1]$ .

Consider  $x_2^*$  such that  $\theta\bar{x}_1 + bx_2^* = r_1 = z_1^*$ , which means that  $x_2^* = \frac{r_1 - \theta\bar{x}_1}{b}$ .

Given  $x_2^* = \frac{r_1 - \theta\bar{x}_1}{b}$ , agent 1 will choose  $x_1$  such that  $x_1 + bx_2^* = z_1^*$  so that  $BR_1^S(\frac{r_1 - \theta\bar{x}_1}{b}) = x_1^* = \theta\bar{x}_1$ .

Let us prove that  $x_2^* = BR_2^S(\theta\bar{x}_1)$ . Given  $\theta\bar{x}_1$ , agent 2 is seeking to maximize  $x_2$  subject to the two constraints (15) and (17).  $x_2^*$  is the maximum solution with (15). If the best response in shared constraint  $BR_2^S(\theta\bar{x}_1)$  of agent 2 was higher than  $x_2^*$  then this would kill the constraint (15).

Now, let us prove that  $\underline{z}_2 < x_2^* + b\theta\bar{x}_1 < z_2^*$ .

$x_2^* + b\theta\bar{x}_1 = \frac{r_1 - \theta\bar{x}_1}{b} + b\theta\bar{x}_1$ . This reaches its maximum when  $\theta = 0$  and the maximum value is equal to  $\frac{r_1}{b}$ , which is lower than  $z_2^*$ . It reaches its minimum value when  $\theta = 1$  and the minimum value is equal to  $\frac{r_1 - \bar{x}_1}{b} + b\bar{x}_1 = \frac{r_1}{b} + \frac{b^2 - 1}{b} \bar{x}_1 = \frac{r_1}{b} + \frac{b^2 - 1}{b} \frac{r_1 - b\underline{z}_2}{1 - b^2} = \frac{r_1}{b} - \frac{r_1}{b} + \underline{z}_2 = \underline{z}_2$ .

Therefore  $z_2 < x_2^* + b\theta\bar{x}_1 < z_2^*$  is always satisfied so  $BR_2^S(\theta\bar{x}_1) = x_2^*$ .

Therefore, for all  $\theta \in [0, 1]$ , the pair of strategies  $(x_1^* = \theta\bar{x}_1, x_2^* = \frac{r_1 - \theta\bar{x}_1}{b})$  is a Nash equilibrium in shared constraint.  $\square$



## References

- [Arrow and Debreu, 1954] Arrow, K. J. and Debreu, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica*, 22(3):265–290.
- [Aussel and Dutta, 2008] Aussel, D. and Dutta, J. (2008). Generalized nash equilibrium problem, variational inequality and quasiconvexity. *Operations Research Letters*, 36:461–464.
- [Banerjee and Feinstein, 2021] Banerjee, T. and Feinstein, Z. (2021). Price mediated contagion through capital ratio requirements with vwap liquidation prices. *European Journal of Operational Research*, 295(3):1147–1160.
- [Barrett, 2001] Barrett, S. (2001). International cooperation for sale. *European economic review*, 45(10):1835–1850.
- [Barrett, 2007] Barrett, S. (2007). *Why cooperate?: the incentive to supply global public goods*. Oxford University Press.
- [Bensoussan, 1974] Bensoussan, A. (1974). Points de nash dans le cas de fonctionnelles quadratiques et jeux différentiels linéaires à  $n$  personnes. *SIAM Journal on Control and Optimization*, 12(3):460.
- [Brams, 2011] Brams, S. J. (2011). *Game theory and politics*. Courier Corporation.
- [Braouezec and Kiani, 2022] Braouezec, Y. and Kiani, K. (2022). A generalized nash equilibrium problem arising in banking regulation: An existence result with tarski’s theorem.
- [Braouezec and Wagalath, 2019] Braouezec, Y. and Wagalath, L. (2019). Strategic fire-sales and price-mediated contagion in the banking system. *European Journal of Operational Research*, 274(3):1180–1197.
- [Breton et al., 2006] Breton, M., Zaccour, G., and Zahaf, M. (2006). A game-theoretic formulation of joint implementation of environmental projects. *European Journal of Operational Research*, 168:221–239.
- [Buchholz et al., 2011] Buchholz, W., Cornes, R., and Rübhelke, D. (2011). Interior matching equilibria in a public good economy: An aggregative game approach. *Journal of Public Economics*, 95(7-8):639–645.
- [Carraro and Siniscalco, 1993] Carraro, C. and Siniscalco, D. (1993). Strategies for the international protection of the environment. *Journal of public Economics*, 52(3):309–328.
- [Chen and Zeckhauser, 2018] Chen, C. and Zeckhauser, R. (2018). Collective action in an asymmetric world. *Journal of Public Economics*, 158:103–112.
- [Cornes, 2016] Cornes, R. (2016). Aggregative environmental games. *Environmental and resource economics*, 63(2):339–365.

- [Cornes and Hartley, 2007] Cornes, R. and Hartley, R. (2007). Aggregative public good games. *Journal of Public Economic Theory*, 9(2):201–219.
- [Cornes and Hartley, 2012] Cornes, R. and Hartley, R. (2012). Fully aggregative games. *Economics Letters*, 116(3):631–633.
- [Dasgupta and Maskin, 1986] Dasgupta, P. and Maskin, E. (1986). The existence of equilibrium in discontinuous economic games, i: Theory. *The Review of economic studies*, 53(1):1–26.
- [Debreu, 1952] Debreu, G. (1952). A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences*, 38(10):886–893.
- [DeCanio and Fremstad, 2013] DeCanio, S. J. and Fremstad, A. (2013). Game theory and climate diplomacy. *Ecological Economics*, 85:177–187.
- [Elfoutayeni et al., 2012] Elfoutayeni, Y., Khaladi, M., and Zegzouti, A. (2012). A generalized nash equilibrium for a bioeconomic problem of fishing. *Studia Informatica Universalis*, 10:186–204.
- [Facchinei et al., 2009] Facchinei, F., Fischer, A., and Piccialli, V. (2009). Generalized nash equilibrium problems and newton methods. *Mathematical Programming*, 117:163–194.
- [Facchinei and Kanzow, 2007] Facchinei, F. and Kanzow, C. (2007). Generalized nash equilibrium problems. *4or*, 5:173–210.
- [Facchinei and Kanzow, 2010] Facchinei, F. and Kanzow, C. (2010). Generalized nash equilibrium problems. *Annals of Operations Research*, 175:177–211.
- [Feinstein and Rudloff, 2021] Feinstein, Z. and Rudloff, B. (2021). Characterizing and computing the set of nash equilibria via vector optimization. *arXiv preprint arXiv:2109.14932*.
- [Fischer et al., 2014] Fischer, A., Herrich, M., and Klaus, S. (2014). Generalized nash equilibrium problems-recent advances and challenges. *Pesquisa Operacional*, 34(3):521–558.
- [Fishburn and Kilgour, 1990] Fishburn, P. C. and Kilgour, D. M. (1990). Binary  $2 \times 2$  games. *Theory and Decision*, 29(3):165–182.
- [Fudenberg and Tirole, 1991] Fudenberg, D. and Tirole, J. (1991). *Game theory*. MIT press.
- [Greenberg and Weber, 1993] Greenberg, J. and Weber, S. (1993). Stable coalition structures with a unidimensional set of alternatives. *Journal of Economic Theory*, 60(1):62–82.
- [Guttman, 1978] Guttman, J. M. (1978). Understanding collective action: matching behavior. *The American Economic Review*, 68(2):251–255.
- [Harker, 1984] Harker, P. T. (1984). A variational inequality approach for the determination of oligopolistic market equilibrium. *Mathematical Programming*, 30(1):105–111.

- [Harker, 1991] Harker, P. T. (1991). Generalized nash games and quasi-variational inequalities. *European journal of Operational research*, 54(1):81–94.
- [Hoel and Schneider, 1997] Hoel, M. and Schneider, K. (1997). Incentives to participate in an international environmental agreement. *Environmental and Resource economics*, 9(2):153–170.
- [Kilgour and Fraser, 1988] Kilgour, D. M. and Fraser, N. M. (1988). A taxonomy of all ordinal  $2 \times 2$  games. *Theory and decision*, 24(2):99–117.
- [Krawczyk, 2007] Krawczyk, J. (2007). Numerical solutions to coupled-constraint (or generalised nash) equilibrium problems. *Computational Management Science*, 4(2):183–204.
- [Krawczyk, 2005] Krawczyk, J. B. (2005). Coupled constraint nash equilibria in environmental games. *Resource and Energy Economics*, 27(2):157–181.
- [Le Cadre et al., 2020] Le Cadre, H., Jacquot, P., Wan, C., and Alasseur, C. (2020). Peer-to-peer electricity market analysis: From variational to generalized nash equilibrium. *European Journal of Operational Research*, 282:753–771.
- [Missfeldt, 1999] Missfeldt, F. (1999). Game-theoretic modelling of transboundary pollution. *Journal of Economic Surveys*, 13(3):287–321.
- [Moulin, 1986] Moulin, H. (1986). *Game theory for the social sciences*. NYU press.
- [Moulin, 1995] Moulin, H. (1995). *Cooperative microeconomics: a game-theoretic introduction*, volume 313. Princeton University Press.
- [Myerson, 2013] Myerson, R. B. (2013). *Game theory*. Harvard university press.
- [Nash, 1950] Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49.
- [Nash, 1951] Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics*, 54:286–295.
- [Osborne and Rubinstein, 1994] Osborne, M. J. and Rubinstein, A. (1994). *A course in game theory*. MIT press.
- [Ostrom, 1990] Ostrom, E. (1990). *Governing the commons: The evolution of institutions for collective action*. Cambridge university press.
- [Reny, 2016] Reny, P. J. (2016). Introduction to the symposium on discontinuous games. *Economic Theory*, 61(March):423–429.
- [Rosen, 1965] Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33(3):520–534.
- [Sandler, 2015] Sandler, T. (2015). Collective action: fifty years later. *Public Choice*, 164(3-4):195–216.

- [Schelling, 1980] Schelling, T. C. (1980). *The strategy of conflict*. Harvard university press.
- [Tidball and Zaccour, 2005] Tidball, M. and Zaccour, G. (2005). An environmental game with coupling constraints. *Environmental Modeling & Assessment*, 10(2):153–158.
- [Walliser, 1988] Walliser, B. (1988). A simplified taxonomy of  $2 \times 2$  games. *Theory and Decision*, 25(2):163–191.
- [Zodrow, 2003] Zodrow, G. R. (2003). Tax competition and tax coordination in the european union. *International tax and public finance*, 10(6):651–671.