Explicit results on Goldbach Functions

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1 Background

A question posed in 1742, known in its contemporary form as the Goldbach conjecture, asks if every even integer greater than 2 can be expressed as the sum of two prime numbers. Almost three centuries later, the hypothesis remains unproved though it is known to be statistically true and is empirically supported by calculations for all numbers up to 4×10^{18} .

For analytic study, rather than showing that the Goldbach function

$$g(n) = \sum_{\substack{p_1+p_2=n \\ p_i \text{ prime}}} 1$$

is positive for all even n, it is easier to handle the smoother form over powers of prime

$$G(n) = \sum_{\ell+m=n} \Lambda(\ell)\Lambda(m),$$

where Λ is the von Mangoldt function defined to be $\log p$ for a positive power of a prime number p and defined to vanish elsewhere. Hence $g(n)$ can be recovered easily from $G(n)$ by partial summation and if $G(n)$ could be shown to be sufficiently large, i.e. if $G(n) > C\sqrt{n}$, the Goldbach conjecture would be true. However this seems to not yet be within our reach.

It is natural in analytic number theory to obtain information on erratic arithmetic functions via the better-behaved problem on a partial sum, up to some cut-off value. Consider thus the *Goldbach summatory function*

$$S(x) = \sum_{n \leq x} G(n).$$

An asymptotic of the form

$$\sum_{n \leq x} g(n) \sim \frac{x^2}{2\log^2 x}$$

was known since Landau [13] though more precise work on $S(x)$, begun by Fujii [9] almost a century later, required information on zeros of the Riemann zeta function due to Gallagher. The congruence Goldbach function, defined for a, b positive integers coprime to q, as

$$G(n; q, a, b) = \sum_{\substack{\ell+m=n \\ \ell \equiv a,\, m \equiv b \,(\mathrm{mod}\ q)}} \Lambda(\ell)\Lambda(m)$$

together with its summatory function $S(x; q, a, b)$, has been studied more recently see [3]).

We notice that this situation is analogous to the very classical *prime counting function*, denoted by $\pi(x) = \sum_{p \leq x} 1$, the *Chebyshev-ψ function* that counts prime powers, defined by

$$\psi(x) = \sum_{\substack{p^k \leq x, \\ p \text{ is prime}}} \log p = \sum_{n \leq x} \Lambda(n)$$

and the primes in arithmetic progression, whose counting function, for a coprime to q, is defined as

$$\pi(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a (\mathrm{mod}\ q)}} 1,$$

their legendary asymptotics being

$$\pi(x) \sim \frac{x}{\log x},\ \psi(x) \sim x \text{ and } \pi(x;q,a) \sim \frac{1}{\varphi(q)} \frac{x}{\log(x)}$$

respectively.

To obtain precise analytic information on functions involving prime numbers, we encounter the Dirichlet L functions, including the Riemann Zeta function. For the summatory functions it is possible to obtain an expression involving a dominant term, which is computed knowing the order of specific zeros at central points of the associated L functions and to separate a term containing the trivial zeros from an oscillatory term involving the infinitely many non-trivial zeros which can not be studied without assuming some conjectures. And since many plausible hypotheses on zeros of L-functions are out of reach at the moment, it is difficult to treat average orders of the Goldbach functions unconditionally.

Perhaps the most widely known of such conjectures is the *Generalized Riemann Hypothesis* (GRH) and the weaker one on *Siegel zeros.* If χ is a Dirichlet character modulo q and $L(s,\chi)$ the associated Dirichlet L-function, the real part of its non-trivial zeros are expected, according to the GRH, to lie on the line $\Re(s) = \frac{1}{2}$. The existence of a Landau-Siegel zero would serve as a counterexample to the GRH. Such an eventual zero, let us denote it by β_1, would be a real one associated to a unique primitive quadratic Dirichlet character of conductor q_χ with $\beta_1 = 1 - \frac{1}{c(\epsilon)\log q}$ for all $\epsilon > 0$. Siegel's theorem of 1930 asserts that $c(\epsilon) \ll_\epsilon q^\epsilon$. Unfortunately $c(\epsilon)$ cannot be computed effectively for any $\epsilon < 1/2$. There is abundant literature on this topic, for example [17] could be an entry point.

Then there are the folkloric assumptions on *Linear Independence of zeros* which states that the imaginary parts of the non-trivial zeros of L-functions are linearly independent over the rationals and the *Non-Coincidence of zeros* which expects that different primitive Dirichlet characters with the same modulus to not have a common non-trivial zero with the same multiplicity. If we relax the condition on multiplicity we get the *Distinct Zero conjecture* which predicts that for any given conductor any two distinct Dirichlet L-functions associated to it do not have a common non-trivial zero, except for a possible multiple zero at $s = 1/2$.

As an example of a conditional Goldbach result, we note that assuming the *Riemann Hypothesis* (RH), the average order of $G(n)$ can be written as

$$S(x) = \frac{x^2}{2} - H(x) + O(\log^3 x)$$

where the oscillating term $H(x) = \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$ is a sum over ρ, the non-trivial zeros of the Riemann zeta function [15, 6], whereas an unconditional form

$$S(x) = \frac{x^2}{2} + O(x^{3/2+\epsilon})$$

for all positive ϵ would already imply the RH [3].

Similarly for the congruence Goldbach function, for a, b coprime to q, we state the following simplified summatory expression without the oscillating term

$$S(x;q,a,b) = \frac{x^2}{2\phi(q)^2} + O(x^{1+B_q})$$

where B_q depends on the non-trivial zeros of the associated Dirichlet L-functions.

On the other hand, for a coprime to an odd q , if the *Distinct Zero Conjecture* is assumed be true, the asymptotic formula

$$S(x;q,a,a) = \frac{x^2}{2\varphi(q)^2} + O_{q,\varepsilon}(x^{3/2+\varepsilon})$$

for any $\varepsilon > 0$ is equivalent to the GRH for the functions $L(s,\chi)$ with any character χ mod q [3].

2 Explicit Versions

We notice that many of the results above have implied constants, some of these can de determined, and are called *effective* others which can not be determined and called *ineffective*. Siegel's Theorem, cited above, is a classical example that involves the second type of constant and Goldbach sums that depend on this theorem also give rise to such constants. The numbers behind the effective constants are often not specified and when they are, we obtain *explicit* numeric values, which answer questions like which cut-off points can be used. As a prototype, note that the original form of the Prime Number Theorem

$$\pi(x) \sim \frac{x}{\log x}$$

of 1896 was made effective by Rosser and Schoenfeld in 1976 and states that under the RH

$$|\pi(x) - \frac{x}{\log x}| \leq \frac{1}{\sqrt{8\pi}}\sqrt{x}\log x$$

for all $x > 2657$.

Such explicit results around Goldbach functions been little investigated.

One rare explicit consideration of the Goldbach function is that of Fujii[8] proved that under the linear independence conjecture on the first 70 zeros on the critical line of the Riemann zeta function the oscillating term

$$R(x) = \Re \sum_{\gamma>0} \frac{x^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}$$

with the sum over ordinates of the non-trivial zeros of $\zeta(s)$, the inequalities

$$R(x) > 0.012, \quad R(x) < -0.012$$

hold for an unbounded sequence of positive real numbers x. More recently [19] Mossinghof and Trudgian improved the above bounds to

$$R(x) < -0.022978, \quad R(x) > 0.021030$$

under the RH and that these oscillations are almost optimal. Though the authors assume linear independence of imaginary parts of zeros of $\zeta(s)$, in fact a weaker form called *N-independence* is enough to establish large oscillations for sums of arithmetic functions. A similar classical example (op. cit) is that the Mertens' conjecture $M(x) = \sum_{n\leq x}\mu(n) \leq x^{1/2}$ implies infinitely many linear dependences over the rationals among the ordinates of the zeros of the zeta function on the critical line in the upper half plane.

To study explicit results on the Goldbach functions, we could build upon existing explicit results on zeros of L-functions. Here we quote some of these.

Languasco[14], recently obtained numerical constants for on the region of possible existence of the Siegel zero for primes $4.10^5 < q \leq 10^7$ and for even quadratic Dirichlet characters. Morrill and Trudgian [18] made explicit a method of Pintz to obtain $1 - \frac{0.933}{\log q} < \beta_1 < 1$. In another paper [2] it is proved that $\beta_1 \leq 1 - \frac{40}{\sqrt{q}\log^2 q}$, $q \geq 3$. At about the same time, Bordignon [5] [4] improved such bounds using explicit estimates for $L'(\sigma,\chi)$ for σ close to unity.

Earlier computational work of Platt [20], who verified the GRH for primitive characters to height upto $\max(108/q, A.107/q)$ with A depending on the parity of the conductor, show that there is no exceptional zero for primes $q \leq 4 \cdot 10^5$.

These are in line with the bounds $\beta_1 > 1 - \frac{1}{1-R\log(\max(q,q|t|,10)}$ with $R = 5.60$. [11], or $R = 9.46$ [16].

The Vinogradov-Korobov type estimates of the zero-free regions of the Riemann zeta function

$$\sigma > 1 - \frac{c}{\log|t|^{2/3}(\log\log|t|)^{1/3}},$$

with a positive constant c, have been studied for a very long time (It is possible to take $c = 1/57.54, |t| \geq 3$, see Ford's arXiv account 1910.08205) and very recently Khale[12] established, for the first time a similar result for L-functions, with a dependence on the conductor, for $|\Im(s)| \geq 10$, i.e. with explicit c and c_q in

$$\sigma > 1 - \frac{1}{c_q \log q + c\log|t|^{2/3}(\log\log|t|)^{1/3}}.$$

Another question relevant to the study of the explicit Goldbach problem is that of recent explicit results obtained on primes in arithmetic progression obtained unconditionally that for all $q \geq 3$ and for x larger than $\exp(8\sqrt{q}\log^3 q)$,

$$|\pi(x;q,a) - \frac{1}{\varphi(q)}\frac{x}{\log x}| \leq \frac{1}{160}\frac{x}{\log^2 x}$$

in [1] and under the GRH in [7]. To arrive at the last, the authors derived many explicit results on the sum of the von Mangoldt function, on spacing of zeros of the L-functions and on oscillating factors.

In a more general context, we can describe the splitting of primes in a given Galois extension of a field. Thus, for example, the classical Chebotarev density theorem predicts such a density by computing the set of conjugacy classes of the associated group. For L/K a finite Galois extension with Galois group G, and C a union of conjugacy classes of G, with the Chebyshev's function ψ_C counts the number of non-ramified prime ideals of norm up to x with a von Mangoldt type weight; it states that $\psi_C(x) \sim \frac{|C|}{|G|}x$ as x tends to infinity.

Grenié and Molteni [10] have recently obtained an explicit conditional version for the above asymptotic form. They show that, under the GRH, the density result can be expressed with their dependence on the degree of extension n_L and discriminant Δ_L. Their auxiliary results with $\Delta_L \leq q^{\varphi(q)}$ could provide, conditional to the GRH, appropriate estimates in the Goldbach setting.

In work under preparation with Anne-Maria Ernvall-Hytönen and Neea Palojärvi we are adapting existing asymptotic results on the Goldbach functions, in particular those that appear in [3], to get numeric results instead of error terms.

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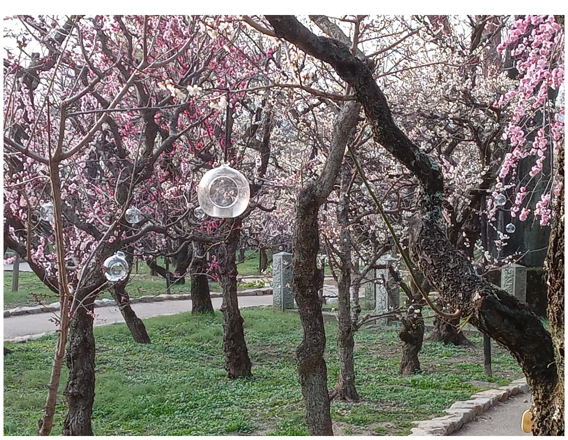

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