Asymptotic properties of weighted Goldbach representation function, and its connection with zeta function

Yiyu Tang

June 14, 2020

Sous la direction de Gautami Bhowmik
Laboratoire Paul Painlevé, Université Lille 1

Resumé

La conjecture de Goldbach est l'un des plus vieux problèmes non résolus des mathématieques, qui s'énonce comme suit: Tout nombre entier pair supérieur à 3 peut s'écrire comme la somme de deux nombres premiers.

En d'autres termes, si n est un nombre naturel et nous posons (p et q sont des nombres premiers)

$$g(n) = \sum_{p+q=n} 1, \tag{0.1}$$

alors $g(n) \neq 0$ si n est pair. Dans cet article, au lieu d'étudier $g(n)$ directement, nous considérons une version lisse de $g(n)$:

$$r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k), \tag{0.2}$$

où $\Lambda(n)$ est la fonction de Von Mangoldt. Notre principal intérêt est la propriété asymptotique de

$$\sum_{n \leq X} r_2(n), \tag{0.3}$$

l'estimation du terme d'erreur a un lien avec des zéros de fonction de zeta, qui fait le lien entre le problème original avec l'hypothèse de Riemann.

En outre, nous considérons le cas général:

$$r_k(n) = \sum_{m_1+\ldots+m_k=n} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k), \tag{0.4}$$

et la propriété asymptotique de $\sum_{n \leq X} r_k(n)$, certains résultats ont été prouvés, mais une généralisation du cas particulier $k = 2$ reste à étudier.

L'auteur tient à remercier le Laboratoire Paul Painlevé, qui offre une bourse pour l'auteur de Master 2. En particulier, l'auteur remercie Gautami.Bhowmik, qui donnent ce sujet intéressant et de nombreuses suggestions. L'auteur remercie également Bruno.Martin, qui enseigne le cours de la théorie des nombres analytiques, l'étude erait beaucoup plus difficile sans ces connaissances de base. L'auterur souhaite remercies finalement le jury de soutenance.

Contents

1 Introduction

In mathematics, one of the most famous and oldest problem is Goldbach's conjecture, which was proposed in 1742, during the communications of German mathematician Christian Goldbach and Leonhard Euler. In the language of modern mathematics, the Goldbach's conjecture sates that

$$\textbf{Every even integer greater than 2 can be written as the sum of two primes}. \tag{1.1}$$

With the help of computers, people verified that for all n less than 4×10^{18}, Goldbach's conjecture is true (in 2014). People believe this conjecture is true, just as Euler replied to Goldbach in 1742:

That ... every even number is a sum of two primes, I consider it as a completely certain theorem, but I am not able to prove it.

(1.2)

In this article, instead of studying Goldbach counting function $g(n) = \sum_{p+q=n} 1$, where p and q are prime numbers, we would like to treat a smooth and moderate increasing version, says the *weighted* Goldbach representation function, by using logarithms:

$$r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k), \tag{1.3}$$

where Λ is the Von Mangoldt function to be defined later, we will see that Λ has logarithmic increase, and $r_2(n)$ behaves very similar to $g(n)$.

Our main work in this article is to give an asymptotic formula like the following form:

$$\sum_{n\leq X} r_2(n) = f(X) + O(g(X)), \tag{1.4}$$

two terms $f(X)$ and $g(X)$ are called the **main term** and the **error term** respectively. Furthermore, we will also consider the general case $r_k(n) = \sum_{m_1+\ldots+m_k=n} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k)$, and study some asymptotic properties of $\sum_{n\leq X} r_k(n)$.

In general, main term is easy to obtain, however, if we want to get some good estimates for the error term, then we must deal with non-trivial zeros of the Riemann zeta function. When we talk about non-trivial zeros of zeta function, the **Riemann Hypothesis** is inevitable. This is an interesting point, it connects two most famous unsolved problems in mathematics, Goldbach's conjecture and Riemann Hypothesis, together. Moreover, we will see that some good error term estimates are equivalent to Riemann Hypothesis, this reveals the connection between these two problems to a certain extent.

1.1 Definition and Notation

1.1.1 Some notations

Let f, $g : \mathbb{R}^+ \to \mathbb{C}$ be two functions, we use the notation $f = O(g)$ to mean that there exists a constant C so that for all x,

$$|f(x)| \leq C|g(x)|. \tag{1.5}$$

We will also denote $f \ll g$ to mean that $f = O(g)$.

Similarly, we denote $f = o(g)$ to mean that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \tag{1.6}$$

Moreover, we use the notation

$$f \asymp g \tag{1.7}$$

to mean that $f \ll g$ and $g \ll f$ simultaneously, and $f \sim g$ is to say that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

1.1.2 Some arithmetic functions

The **Von Mangoldt function** is the arithmetic function defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if}\, n = p^k,\, k \geq 1,\, p\, \text{is prime} \\ 0, & \text{otherwise} \end{cases} \tag{1.8}$$

If we decompose n as sum of two integers $n = m + k$, then $\Lambda(m)\Lambda(k) \neq 0$ if and only if m and k are the power of primes, so we get a series $r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k)$. Moreover, when we consider the character function $p(n)$ of primes, i.e. $p(n) = 1$ if n is prime and $p(n) = 0$ otherwise, then Goldbach counting function $g(n) = \sum_{p+q=n} 1$ can be written as

$$g(n) = \sum_{m+k=n} p(m)p(k).$$

We notice that, although not the same thing, the general term $\Lambda(m)\Lambda(k)$ and $p(m)p(k)$ are very similar, they are 0 when m or k is not prime. Therefore, if we know some properties of $r_2(n)$, then we can get some informations of $g(n)$. By this idea, in 1937, Vinogradov considered the partial sum

$$r(n) = \sum_{k_1+k_2+k_3=n} \Lambda(k_1)\Lambda(k_2)\Lambda(k_3),$$

and after an elaborated estimate of $r(n)$, he proved a weak form of Goldbach's conjecture for large integers n, now named as Vinogradov's Theorem:

Theorem 1.1 (Vinogradov)**.** *Any sufficiently large* ***odd*** *integer can be written as a sum of three prime numbers.*

For the proof, see Chapter 26 of [1]. However, we would like to explain how Vinogradov recovered the information of $g_3(n) = \sum_{k+l+m=n} p(k)p(l)p(m)$ from $r(n)$. In fact, Vinogradov proved the following estimate:

$$r(N) = \frac{1}{2}G(N)N^2 + O_A\Big(\frac{N^2}{\log^A N}\Big), \tag{1.9}$$

where A is any positive integer, and

$$G(N) = \prod_{p|N}\Big(1 - \frac{1}{(p-1)^2}\Big)\prod_{p\nmid N}\Big(1 + \frac{1}{(p-1)^3}\Big). \tag{1.10}$$

We notice that if N is odd, then $G(N) \geq \prod_{p\neq 2}(1-(p-1)^{-2}) > 0$ (For some properties of infinite product, see [20], Chapter 5.) We call the infinite product $C_2 = 2\prod_{p\neq 2}(1-(p-1)^{-2}) > 0$ is the **Twin Prime Constant**. Now by Vinogradov's estimate, we have (N is odd and p_i's are prime)

$$\begin{aligned} N^2 \ll r(N) = & \sum_{p_1+p_2+p_3=N} \log p_1 \log p_2 \log p_3 + \sum_{(k_1,k_2,k_3)\neq(1,1,1)} \sum_{p_1^{k_1}+p_2^{k_2}+p_3^{k_3}=N} \log p_1 \log p_2 \log p_3 \\ & \ll \log^3 N \sum_{p_1+p_2+p_3=N} 1 + \sum_{(k_1,k_2,k_3)\neq(1,1,1)} \sum_{p_1^{k_1}+p_2^{k_2}+p_3^{k_3}=N} \log p_1 \log p_2 \log p_3 \\ & = g_3(N)\log^3 N + \sum_{(k_1,k_2,k_3)\neq(1,1,1)} \sum_{p_1^{k_1}+p_2^{k_2}+p_3^{k_3}=N} \log p_1 \log p_2 \log p_3. \end{aligned} \tag{1.11}$$

For the second term, by symmetry, we may assume $k_1 > 1$, we notice that the sum

$$\sum_{(k_1,k_2,k_3)\neq(1,1,1)} \sum_{p_1^{k_1}+p_2^{k_2}+p_3^{k_3}=N} \tag{1.12}$$

is finite. Moreover, consider the sum $\sum_{p^{k_1}\leq N} 1$, firstly we must have $p \leq \sqrt{N}$, by **Prime Number Theorem**, there are at most $\frac{\sqrt{N}}{\log\sqrt{N}}$ such primes (For the prime number theorem, see subsection 1.1.3). Secondly, if $p^{k_1} \leq N$ holds, then $k_1 \leq \log_2 N \ll \log N$. Therefore

$$\sum_{p^{k_1}\leq N} 1 \ll \log N \frac{\sqrt{N}}{\log\sqrt{N}} \ll \sqrt{N}. \tag{1.13}$$

According to the above estimate, we get

$$\sum_{(k_1,k_2,k_3)\neq(1,1,1)} \sum_{p_1^{k_1}+p_2^{k_2}+p_3^{k_3}=N} \log p_1 \log p_2 \log p_3 \ll N^{\frac{3}{2}}\log^3 N. \tag{1.14}$$

This inequality implies $g_3(N) \gg N^2\log^{-3}N + O(\log_2 N)$, which is positive when N sufficiently large. So any large odd integer can be decomposed as a sum of three primes.

The **First Chebyshev function** $\vartheta(x)$ is defined by

$$\vartheta(x) = \sum_{p\leq x}\log p, \tag{1.15}$$

and the **Second Chebyshev function** $\psi(x)$ is defined by the sum of $\Lambda(n)$:

$$\psi(x) = \sum_{n\leq x}\Lambda(n). \tag{1.16}$$

We notice that $\psi(x) = \sum_{p^k\leq x}\log p = \sum_{p\leq x}[\log_p x]\log p$, where $k \in \mathbb{N}\setminus\{0\}$. Therefore, we have the following relationship of ϑ and ψ:

$$\psi(x) = \sum_{k\geq 1}\vartheta(x^{\frac{1}{k}}), \tag{1.17}$$

for each x, the sum is finite because the general term is 0 when $2^k > x$.

1.1.3 Gamma function and Riemann zeta function

We will use zeta function frequently, and in subsection 3.2.1, some properties of complex Gamma function are necessary, so we give some elementary properties of zeta and Gamma function, all of these contents can be found in analytic number theory or complex analysis textbook, like the Chapter 6, 7 of [20], so we omit the proofs and details.

Complex Gamma function In elementary calculus, we have learned the Gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x, \quad s > 0. \tag{1.18}$$

This definition can be extended to $\{s \in \mathbb{C} : Re(s) > 0\}$ directly, just notice that the integral $\int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x$ converges absolutely when $Re(s) > 0$.

Analytic continuation of $\Gamma(s)$ *The definition of $\Gamma(s)$ can be extended to $\{s \in \mathbb{C} : Re(s) > 0\}$, and the integral representation of $\Gamma(s)$ is analytic in this half plane. Moreover, if $Re(s) > 0$, then $\Gamma(s+1) = s\Gamma(s)$. As a consequence, $\Gamma(n+1) = n!$ for $n = 0, 1, 2...$*

By the identity $\Gamma(s+1) = s\Gamma(s)$, we can extend Γ to a meromorphic function on all of $\mathbb{C}$. For example, on the half-plane $\{\Re(s) > -1\}$, we define

$$F_1(s) = \frac{\Gamma(s+1)}{s}. \tag{1.19}$$

Since $\Gamma(s+1)$ is holomorphic in $Re(s) > -1$, we see that F_1 is meromorphic in that half-plane, with only singularity a simple pole at $s = 0$ with residue 1. Moreover, if $Re(s) > 0$, then $F_1(s) = \Gamma(s)$ by previous theorem. So F_1 extends Γ to a meromorphic function on the half-plane $\{Re(s) > -1\}$. Similarly, we can extend $\Gamma(s)$ to the half-plane $\{\Re(s) > -m\}$ with $m \in \mathbb{N}$.

Theorem 1.2. *$\Gamma(s)$ can be extended to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the non-positive integers $s = 0, -1, -2....$ The residue of Γ at $-n$ is $(-1)^n/n!$.*

The **Riemann zeta function** is the function defined by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad \Re(s) > 1. \tag{1.20}$$

Analytic continuation of $\zeta(s)$ For $\Re(s) > 1$, we have

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}[\theta(u) - 1]\mathrm{d}u, \tag{1.21}$$

where $\theta(t) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\pi n^2 t}$ $(t > 0)$ is the *Theta function*. Using the *Poisson summation formula* in Fourier analysis, we get

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\big(\frac{1}{t}\big). \tag{1.22}$$

By this functional equation,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \frac{1}{2}\int_1^\infty \big(u^{-(s+1)/2} + u^{(s/2)-1}\big)\big(\theta(u) - 1\big)\mathrm{d}u, \quad \Re(s) > 1. \tag{1.23}$$

Notice that the integral above defines an entire function in s, because $\theta(u)$ has exponential decay at infinity. Moreover, the right-hand side above remains unchanged if we replace s by $1 - s$, so $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has an analytic continuation to $\mathbb{C}$ with simple poles $s = 0, 1$ and $\xi(s) = \xi(1 - s)$ for all $s \in \mathbb{C}$.

Now we define
$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}, \tag{1.24}$$
then $\zeta(s)$ is an analytic continuation of $\sum_{n\geq 1}\frac{1}{n^s}$. Moreover, by Theorem 1.2, $1/\Gamma(s/2)$ is entire with simple zeros at $0, -2, -4, ...$, so the simple pole of $\xi(s)$ at 0 is canceled with the simple zero of $\Gamma(s/2)$ at 0. In conclusion, we have

*$\zeta(s)$ has a meromorphic continuation in to $\mathbb{C}$, whose only singularity is a simple pole $s = 1$. Moreover $\zeta(-2n) = 0$ for all $n \in \mathbb{N}\setminus\{0\}$, we say that $-2, -4, -6...$ are the **trivial zeros** of ζ.*

Prime Number Theorem, Riemann Hypothesis We have known that $\zeta(-2n) = 0$ for all $n \in \mathbb{N}\setminus\{0\}$, however, the negative even integers are not the only zeros for ζ. The **Riemann Hypothesis** states that:

$$\textbf{The real part of every non-trivial zero of } \zeta \textbf{ is } \frac{1}{2}. \tag{1.25}$$

Like Goldbach's conjecture, people believe that the Riemann Hypothesis is true. Using computers, more than million of non-trivial zeros has been found, and all of them are on the line $\Re(s) = 1/2$. We denote ρ be the non-trivial zeros of ζ, the summation $\sum_\rho$ means take sum over all non-trivial zeros of ζ. Furthermore, considering the zeros of zeta function, we have

Theorem 1.3. *$\zeta(s)$ never vanishes in the half-plane*
$$\{s = \sigma + \mathrm{i}t : \sigma \geq 1\}, \tag{1.26}$$
*and $\{-2n\}_{n\geq 1}$ are the only zeros in the half-plane $\{s = \sigma + \mathrm{i}t : \sigma \leq 0\}$. So all non-trivial zeros lies in the band $\{s = \sigma + \mathrm{i}t : \sigma \in (0, 1),\ t \in \mathbb{R}\}$, we call this band the **critical band**.*

Moreover, there exists a constant $c > 0$, such that ζ never vanishes in the region (See the figure below)
$$\{s = \sigma + \mathrm{i}t : \sigma \geq 1 - \frac{c}{\log(2 + |t|)}\}. \tag{1.27}$$

For the proof, see Chapter II.3 of [21]. A very simple but interesting observation is that **non-trivial zeros are symmetric about the critical line** $\{\Re(s) = 1/2\}$, this is obvious from the functional equation $\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}$ and $\xi(s) = \xi(1 - s)$. Morevover, **non-trivial zeros of zeta function come in conjugate pairs** $\{\rho, \bar{\rho}\}$, this is due to the following theorem:

Theorem 1.4 (Schwarz reflection principle)**.** *Let Ω be an open set of $\mathbb{C}$ that is symmetric with respect to the real line, that is*
$$z \in \Omega \quad \textit{if and only if} \quad \bar{z} \in \Omega \tag{1.28}$$

Let Ω^+ denote the part of Ω that lies in the upper half-plane and Ω^- that part that lies in the lower half-plane. Also, let $I = \Omega \cap \mathbb{R}$. If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I and
$$f^+(x) = f^-(x) \quad \textit{for all } x \in I, \tag{1.29}$$
then the function f defined on Ω by
$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+, \\ f^\pm(z) & z \in I, \\ f^-(z) & z \in \Omega^- \end{cases} \tag{1.30}$$
is holomorphic in Ω.

The proof of this theorem can be found in many complex analysis textbook, for example, Chapter 2 of [20], so we omit it.

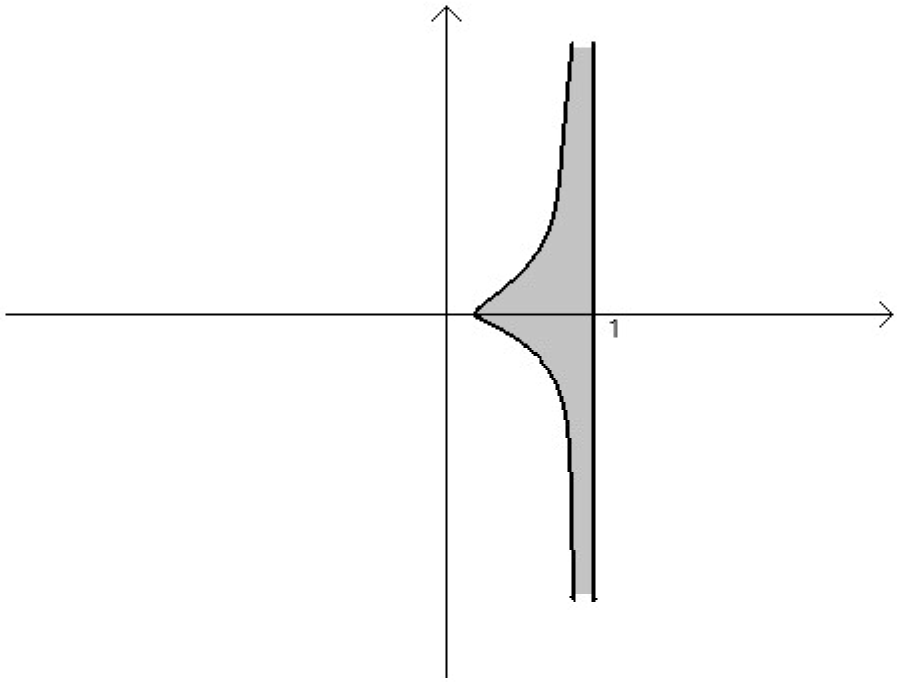

Figure 1: Zero-free region of zeta function in the critical band, notice $1 - \frac{c}{\log(2+|t|)}$ goes to 1 as $|t|$ tends to infinity.

If we denote $\pi(x)$ be the number of primes less than or equal to x, the **Prime Number Theorem** states that:

$$\pi(x) \sim \frac{x}{\log x}, \tag{1.31}$$

and it has a more precise form:

$$\textbf{There exists a positive constant } c > 0\textbf{, such that } \psi(x) = x + O(\mathrm{e}^{-c\sqrt{\log x}}). \tag{1.32}$$

Finally, we give a functional equation which would be used in the proof of Lemma 4.1 (For the proof of this equation, see [20], Chapter 7.)

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n)n^{-s}, \quad \Re(s) > 1, \tag{1.33}$$

summing by parts, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n)n^{-s} = \int_{1^-}^{\infty} u^{-s}\mathrm{d}\psi(u) = s\int_1^{\infty} u^{-s-1}\psi(u)\mathrm{d}u, \quad \Re(s) > 1. \tag{1.34}$$

Notice that we have connected $\zeta(s)$ with $\Lambda(n)$ and $\psi(n)$. As we mentioned before, instead of studying Goldbach's counting function $\sum_{p_1+p_2=n} 1$ directly, we would like to deal with $\sum_{m+k=n} \Lambda(m)\Lambda(k)$. So by using function Λ, now we connect Goldbach's problem with Riemann zeta function. We will see this point later in the proof of our main theorem.

1.2 Main Results

Theorem 1.5. *Let*

$$r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k), \tag{1.35}$$

then Riemann Hypothesis is true, if and only if the following asymptotic property holds:

$$\sum_{n\leq X} r_2(n) = \frac{X^2}{2} + O(X^{3/2}). \tag{1.36}$$

Then we consider the general case, assume $k \geq 2$ be an integer.

Theorem 1.6. *Let*

$$r_k(n) = \sum_{m_1+\ldots+m_k=n} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k), \tag{1.37}$$

if Riemann Hypothesis is true, then the following asymptotic property holds:

$$\sum_{n\leq X} r_k(n) = \frac{X^k}{k!} + O_k(X^{-1/2+k}\log^k X), \tag{1.38}$$

here O_k means that the constant C in section 1.1.1 is depending on k.

1.3 Further Remarks

Remark 1. We must point out that the main work is on the sum $\sum_{n\leq X} r_k(n)$, rather than $r_k(n)$ itself. But for $r_k(n)$, there are also some estimates can be obtained from the **generalized Riemann Hypothesis** of $L(s,\chi)$, for this point, one may see [5, 14].

Remark 2. Theorem 1.5 has a more precise form:

$$\sum_{n\leq X} r(n) = \frac{X^2}{2!} - 2\sum_{\rho}\frac{X^{\rho+1}}{\rho(\rho+1)} + O\Big((X\log X)^{\frac{4}{3}}\Big), \tag{1.39}$$

the error term was reached by A. Fujii in [6]-[8]. In [3], Bhowmik and Schlage-Puchta improved the error term to $O(X\log^5 X)$.

Remark 3. Like the case of $k=2$, Theorem 1.6 has a more precise form:

$$\sum_{n\leq X} r_k(n) = \frac{X^k}{k!} - k\sum_{\rho}\frac{X^{\rho+k-1}}{\rho(\rho+1)\cdots(\rho+k-1)} + O_k(X^{k-1}\log^k X), \tag{1.40}$$

for this point, see [15], Theorem 1.3.

Remark 4. Theorem 1.6 only consider one direction. The author believe that, just like Theorem 1.5, the converse of Theorem 1.6 is also true. This is to say if we admit the asymptotic formula of $\sum_{n\leq X} r_k(n)$ in Theorem 1.6, then the Riemann Hypothesis is true. One may follow the idea in the section 3, however, some estimates (without Riemann Hypothesis) for

$$S_k(x) = \sum_{n\leq x} r_k(n) \tag{1.41}$$

is necessary, just like the step 1 in subsection 3.1, and this is not a trivial work, further study for this is needed and the author is still considering this.

2 Proof of Theorem 1.5, Part 1

In this section, we assume Riemann Hypothesis is always true.

For convenience, we write down the proposition which would be proved in this section:

$$\textbf{Assuming Riemann Hypothesis, then} \sum_{n\leq X} r_2(n) = \frac{X^2}{2} + O(X^{3/2}). \tag{2.1}$$

We use the method in A.Fujii's paper [6], and complete some details for it. Firstly, it suffices to consider the case $X = N \in \mathbb{N}$. The idea of proving the estimate is not difficult: We decompose the sum in several parts time after time, and on each small parts, we get estimates of error terms $\ll N^{\frac{3}{2}}$. The idea is simple, but some estimates are not so trivial and we need more elaborated work, so we will proceed these decompositions by several steps.

Step 1: Getting the main term Let

$$R(y) = -y + \sum_{n\leq y} \Lambda(n), \quad y > 0. \tag{2.2}$$

Then

$$\begin{aligned}\sum_{n\leq N} r_2(n) &= \sum_{m\leq N} \Lambda(m) \sum_{k\leq N-m} \Lambda(k)\\ &= \sum_{m\leq N} \Lambda(m)\Big(N-m+R(N-m)\Big)\\ &= \sum_{m\leq N} \Lambda(m)(N-m) + \sum_{2\leq m\leq N-2} \Lambda(m)R(N-m) - \Lambda(N-1).\end{aligned} \tag{2.3}$$

For the third term, $|\Lambda(N-1)| = O(\log N)$. Summing by parts, we have

$$\begin{aligned}\sum_{m\leq N} m\Lambda(m) =& \psi(x)x\Big|_1^N - \int_1^N \psi(t)\mathrm{d}t\\ =& N\psi(N) - \int_1^N \psi(t)\mathrm{d}t\end{aligned} \tag{2.4}$$

therefore, for the first term, we have

$$\sum_{m\leq N} \Lambda(m)(N-m) = N\psi(N) - \Big(N\psi(N) - \int_1^N \psi(t)\mathrm{d}t\Big) = \int_1^N \psi(t)\mathrm{d}t = \frac{N^2}{2} + O(N^{\frac{3}{2}}), \tag{2.5}$$

the last equality holds due to the following lemma:

Lemma 2.1. *Let $\psi(x)$ be the Chebyshev function, then*

$$\int_2^x \psi(t)\mathrm{d}t = \frac{x^2}{2} + O(x^{\frac{3}{2}}), \tag{2.6}$$

and let $\psi_0(x) = \frac{1}{2}(\psi(x+0)+\psi(x-0))$, then $\psi_0(x) = \psi(x)$ when $x \neq p^m$ for some $m \geq 1$, moreover, we have

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}), \quad x > 1. \tag{2.7}$$

Nextly we come to the error term estimate, for $T \geq 2$, let

$$\sum_\rho \frac{x^\rho}{\rho} = \sum_{|\Im(\rho)|\leq T} \frac{x^\rho}{\rho} + R(x,T) \tag{2.8}$$

where

$$R(x,T) \ll \frac{x}{T}\log^2 xT + \log x. \tag{2.9}$$

For the proof, see [18], Chapter VII, Lemma 8.6 and [12], Chapter 12, Theorem 12.1.

Using the first estimate of Lemma 2.1,

$$\begin{aligned}\sum_{n\le N} r_2(n) &= \sum_{m\le N}\Lambda(m)(N-m) + \sum_{2\le m\le N-2}\Lambda(m)R(N-m) - \Lambda(N-1)\\ &=\frac{N^2}{2} + O(N^{\frac{3}{2}}) + \sum_{2\le m\le N-2}\Lambda(m)R(N-m)\\ &=\frac{N^2}{2} + O(N^{\frac{3}{2}}) + I.\end{aligned} \tag{2.10}$$

Notice that we have finished the first decomposition and get the main term $N^2/2$ and the error term $O(N^{\frac{3}{2}})$. Nextly, according to our idea, now we decompose the last term as some parts, with each parts $\ll N^{\frac{3}{2}}$

Step 2: The estimate of $I \ll N\sqrt{N}$ Write the last term I as

$$\sum_{2\le m\le N-2}\Lambda(N-m)R(m) = \sum_{2\le m\le N-2}\Lambda(N-m)\Big(-m+\psi(m)\Big), \tag{2.11}$$

and using the error term estimate of Lemma 2.1, we get

$$\begin{aligned} I =& \sum_{2\le m\le N-2}\Lambda(N-m)\Big(-\sum_{|\Im(\rho)|\le T}\frac{m^\rho}{\rho} + O\Big(\frac{m}{T}\log^2(mT)\Big) + O(\log m)\Big)\\ =&-\sum_{2\le m\le N-2}\Lambda(N-m)\sum_{|\Im(\rho)|\le T}\frac{m^\rho}{\rho} + \Big\{\sum_{2\le m\le N-2} m\Lambda(N-m)\Big\}O\Big(\frac{1}{T}\log^2(NT)\Big) + \psi(N-2)O(\log N)\\ =&-\sum_{2\le m\le N-2}\Lambda(N-m)\sum_{|\Im(\rho)|\le T}\frac{m^\rho}{\rho} + O\Big(\frac{N^2}{T}\log^2(NT)\Big) + O(N\log N),\end{aligned} \tag{2.12}$$

the last equality holds because we use **Prime Number Theorem** $\psi(x)\sim x$, and the following estimates:

$$\sum_{2\le m\le N-2} m\Lambda(N-m) = O\Big(\sum_{m\le N} m\Lambda(N-m)\Big) = O\Big(\int_2^N \psi(t)\mathrm{d}t\Big) = O\Big(\frac{N^2}{2}\Big), \tag{2.13}$$

In conclusion, we write $I = \sum_{2\le m\le N-2}\Lambda(N-m)\Big(-m+\psi(m)\Big)$ as the form

$$I_1 + O\Big(\frac{N^2}{T}\log^2(NT)\Big) + O(N\log N), \tag{2.14}$$

where $I_1 = -\sum_{2\le m\le N-2}\Lambda(N-m)\sum_{|\Im(\rho)|\le T}\frac{m^\rho}{\rho}$. Here we finished the decomposition second time, with two error term $\ll N^{\frac{3}{2}}$ for some T to be chosen.

Step 3: estimate of $I_1 \ll N\sqrt{N}$ Now we continue our decomposition third time for I_1. **From now on, not only in this step, we always assume that** $1 \ll T \ll N$. By Riemann hypothesis, we can write $\rho = \frac{1}{2} + \mathrm{i}\gamma$, then

$$\begin{aligned} I_1 =& -\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{|\gamma|\le T}\frac{m^{\mathrm{i}\gamma}}{\frac{1}{2}+\mathrm{i}\gamma}\\ =& -2\Im\Big\{\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{0\le\gamma\le T}\frac{m^{\mathrm{i}\gamma}}{\gamma}\Big\} + O\Bigg(\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{0\le\gamma\le T}\frac{m^{\mathrm{i}\gamma}}{\gamma^2}\Bigg)\\ =& -2\Im(I_2) + O(I_3).\end{aligned} \tag{2.15}$$

We consider I_3 at first, summing by parts, and using $\sqrt{a+1}-\sqrt{a}=\big(\sqrt{a+1}+\sqrt{a}\big)^{-1}$

$$\begin{aligned}\sum_{m\le N}\sqrt{N-m}\Lambda(m)&=\sum_{m\le N}\sqrt{N-m}\Big(\psi(m)-\psi(m-1)\Big)\\&=\sum_{m\le N-1}\psi(m)\big(\sqrt{N-m}-\sqrt{N-m-1}\big)\\&\ll\sum_{m\le N-1}\frac{\psi(N)}{\sqrt{N-m}}\\&\ll\psi(N)\sqrt{N}=O(N\sqrt{N}).\end{aligned}\tag{2.16}$$

For the part $\sum_{0\le\gamma\le T}\frac{m^{\mathrm{i}\gamma}}{\gamma^2}$, we need a lemma about the number of zeros of $\zeta(s)$ in a prescribed domain.

Lemma 2.2. *For $T\ge 1$, denote $N(T)$ be the number of zeros of $\zeta(x+\mathrm{i}y)$ in the rectangle*

$$\{(x,y)\in\mathbb{C}:\ x\in(0,1),\ y\in(0,T)\},\tag{2.17}$$

then the following estimate holds

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(\log T).\tag{2.18}$$

In fact, we have a more elaborated estimate:

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+\frac{1}{\pi}\arg\zeta\Big(\frac{1}{2}+\mathrm{i}T\Big)+O\Big(\frac{1}{T}\Big).\tag{2.19}$$

For the proof, see Chapter IX of [22]. We would like to give some explanation about the term $\arg\zeta\big(\frac{1}{2}+\mathrm{i}T\big)$, which is defined by continuity along the polygonal $\mathcal{L}$ joining the point 2, $2+\mathrm{i}T$, $\frac{1}{2}+\mathrm{i}T$.

According to this lemma, if we denote $\{\rho_n=\beta_n+\mathrm{i}\gamma_n\}_n$ be the zeros of ζ in critical band with $\gamma_n>0$, and enumerate it with the increasing order $\gamma_1\le\gamma_2...$, then

$$\gamma_n\sim\frac{2\pi n}{\log n}.\tag{2.20}$$

So

$$\sum_{0\le\gamma\le T}\frac{m^{\mathrm{i}\gamma}}{\gamma^2}\ll\sum_n\frac{\log^2 n}{n^2}<\infty.\tag{2.21}$$

We can also derive this estimate by Lemma 3.2.

Combining these two estimates, we get $I_3=O(N\sqrt{N})$. Nextly, we deal with the term I_2.

Step 4: Estimate of $I_2\ll N^{\frac{3}{2}}$ We write down I_2 again for convenience

$$I_2=\sum_{2\le m\le N-2}\sqrt{m}\Lambda(N-m)\sum_{0\le\gamma\le T}\frac{m^{\mathrm{i}\gamma}}{\gamma}\tag{2.22}$$

Firstly, using Lemma 2.2, we have

$$\begin{aligned}\sum_{0<\gamma\le Y}\frac{1}{\gamma}=&\frac{\log^2 Y}{4\pi}-\frac{\log(2\pi)}{2\pi}\log Y+\int_1^\infty\frac{S(t)}{t^2}\mathrm{d}t-\frac{1+\log(2\pi)}{2\pi}+\frac{7}{8}+\int_1^\infty\frac{\eta(t)}{t^2}\mathrm{d}t+B(Y)\\=&A(Y)+B(Y)\end{aligned}\tag{2.23}$$

where $S(t) = \frac{1}{\pi} \arg\zeta(\frac{1}{2} + \mathrm{i}t)$, $\eta(t)$ varifies $\eta(t) = O(\frac{1}{t})$ for $t > t_0$, and

$$B(Y) = \frac{S(Y)}{Y} - \int_Y^\infty \frac{S(t)}{t^2}\mathrm{d}t + \frac{\eta(Y)}{Y} - \int_Y^\infty \frac{\eta(t)}{t^2}\mathrm{d}t \tag{2.24}$$

Secondly, let u be an integer, if $1 \le u < N$, then

$$\begin{aligned}\sum_{n\le u} \sqrt{n}\Lambda(N-n) &= \sum_{n\le u-1} \sqrt{n}\Lambda(N-n) + \sqrt{u}\Lambda(N-u) \\ &= \sum_{N-u<n\le N-1} \sqrt{N-n}\Lambda(n) + \sqrt{u}\Lambda(N-u) \\ &= \int_{N-u}^{N-1} \sqrt{N-y}\,\mathrm{d}(y + R(y)) + \sqrt{u}\Lambda(N-u),\end{aligned} \tag{2.25}$$

here we use the Riemann-Stieltjes integral, integrating by parts, we get:

$$\begin{aligned}\int_{N-u}^{N-1} \sqrt{N-y}\,\mathrm{d}R(y) =& \sqrt{N-y}R(y)\Big|_{N-u}^{N-1} + \frac{1}{2}\int_{N-u}^{N-1} \frac{R(y)}{\sqrt{N-y}}\,\mathrm{d}y \\ =& R(N-1) - \sqrt{u}R(N-u) + \frac{1}{2}\int_{N-u}^{N-1} \frac{R(y)}{\sqrt{N-y}}\,\mathrm{d}y,\end{aligned} \tag{2.26}$$

therefore,

$$\begin{aligned}\sum_{n\le u} \sqrt{n}\Lambda(N-n) = \int_{N-u}^{N-1} &\sqrt{N-y}\,\mathrm{d}y + R(N-1) - \sqrt{u}R(N-u) \\ &+ \frac{1}{2}\int_{N-u}^{N-1} \frac{R(y)}{\sqrt{N-y}}\,\mathrm{d}y + \sqrt{u}\Lambda(N-u).\end{aligned} \tag{2.27}$$

We put

$$\begin{aligned} C(u) &= \int_{N-u}^{N-1} \sqrt{N-y}\,\mathrm{d}y = \frac{2}{3}(u^{\frac{3}{2}} - 1) \\ D(u) &= R(N-1) + \sqrt{u}\Big(\Lambda(N-u) - R(N-u)\Big) + \frac{1}{2}\int_{N-u}^{N-1} \frac{R(y)}{\sqrt{N-y}}\,\mathrm{d}y.\end{aligned} \tag{2.28}$$

If $u \ge N$, similarly

$$\sum_{n\le u} \sqrt{n}\Lambda(N-n) = \int_1^{N-1} \sqrt{N-y}\mathrm{d}y + \sqrt{N-1} + R(N-1) + \frac{1}{2}\int_1^{N-1} \frac{R(y)}{\sqrt{N-y}}\mathrm{d}y, \tag{2.29}$$

again, we put

$$\begin{aligned} C(u) &= \frac{2}{3}(u^{\frac{3}{2}} - 1) \\ D(u) &= \frac{2}{3}\Big((N-1)^{\frac{3}{2}} - u^{\frac{3}{2}}\Big) + \sqrt{N-1} + R(N-1) + \frac{1}{2}\int_1^{N-1} \frac{R(y)}{\sqrt{N-y}}\mathrm{d}y.\end{aligned} \tag{2.30}$$

Now, by Riemann-Stieltjes integral, we write I_2 as

$$\begin{aligned} I_2 &= \int_1^T \int_1^N u^{\mathrm{i}t}\mathrm{d}(C(u) + D(u))\mathrm{d}(A(t) + B(t)) \\ &= \int_1^T \int_0^{\log N} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}(C(\mathrm{e}^x) + D(\mathrm{e}^x))\mathrm{d}(A(t) + B(t)) \\ &= \int_1^T \int_0^{\log N} -\mathrm{e}^{\mathrm{i}tx}\mathrm{d}C(\mathrm{e}^x)\mathrm{d}A(t) + \int_1^T \int_0^{\log N} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}C(\mathrm{e}^x)\mathrm{d}(A(t) + B(t)) \\ &\quad + \int_1^T \int_0^{\log N} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}(C(\mathrm{e}^x) + D(\mathrm{e}^x))\mathrm{d}A(t) + \int_1^T \int_0^{\log N} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}D(\mathrm{e}^x)\mathrm{d}B(t) \\ &= I_4 + I_5 + I_6 + I_7.\end{aligned} \tag{2.31}$$

We deal with I_4, I_5 and I_6 at first, I_7 is more difficult, so we put it in the next step. Firstly, I_4 is

$$-\int_1^T\int_0^{\log N}\frac{\log t-\log(2\pi)}{2\pi t}\mathrm{e}^{\mathrm{i}tx}\mathrm{e}^{\frac{3}{2}x}\mathrm{d}x\mathrm{d}t\ll N^{\frac{3}{2}}\int_1^T\frac{\log t}{t^2}\mathrm{d}t\ll N^{\frac{3}{2}}. \tag{2.32}$$

Secondly, I_5 is

$$\begin{aligned}\int_1^T\int_0^{\log N}\mathrm{e}^{\mathrm{i}tx}\mathrm{d}C(\mathrm{e}^x)\mathrm{d}(A(t)+B(t))\ll&N^{\frac{3}{2}}\left(\frac{1}{t}\sum_{0<\gamma<t}\gamma^{-1}\Big|_1^T+\int_1^T\frac{1}{t^2}\sum_{0<\gamma<t}\gamma^{-1}\mathrm{d}t\right)\\ \ll&N^{\frac{3}{2}}.\end{aligned} \tag{2.33}$$

Finally, by Fubini's theorem, we write I_6 as

$$\int_0^{\log N}\int_1^T\mathrm{e}^{\mathrm{i}tx}\mathrm{d}A(t)\mathrm{d}(C(\mathrm{e}^x)+D(\mathrm{e}^x))=\int_0^{\log N}\int_1^T\mathrm{e}^{\mathrm{i}tx}\frac{\log t-\log(2\pi)}{2\pi t}\mathrm{d}t\,\mathrm{d}(C(\mathrm{e}^x)+D(\mathrm{e}^x)). \tag{2.34}$$

Then, by the following estimate

$$\int_1^T\mathrm{e}^{\mathrm{i}tx}\frac{\log t-\log(2\pi)}{2\pi t}\mathrm{d}t\ll\min(\frac{1}{x},\log^2T), \tag{2.35}$$

I_6 is bounded by

$$\int_0^1\log^2T\mathrm{d}(C(\mathrm{e}^x)+D(\mathrm{e}^x))+\int_1^{\log N}\frac{1}{x}\mathrm{d}(C(\mathrm{e}^x)+D(\mathrm{e}^x))\ll N^{\frac{3}{2}} \tag{2.36}$$

Step 5: Estimate of $I_7\ll N^{\frac{3}{2}}$ We use the result in [9] (see [9], Lemma 1)

$$I_7\ll T\log N\max_{0<\delta\ll\frac{1}{T}}\Big(\int_0^{\log N}\big(D(\mathrm{e}^{y+\delta})-D(\mathrm{e}^y)\big)^2\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_1^T\Big(B\big(t+(2\log N)^{-1}\big)-B(t)\Big)^2\mathrm{d}t\Big)^{\frac{1}{2}}, \tag{2.37}$$

we denote the two integrals in the right-hand site as I_8 and I_9. I_9 is easy, we just notice that by the definition of $B(y)$

$$B(y)\Big|_t^{t+(2\log N)^{-1}}=\frac{S(y)+\eta(y)}{y}\Big|_t^{t+(2\log N)^{-1}}-\int_t^{t+(2\log N)^{-1}}\frac{S(y)+\eta(y)}{y^2}\mathrm{d}y. \tag{2.38}$$

Notice that $S(y)+\eta(y)=O(1)$, so $B(y)\Big|_t^{t+(2\log N)^{-1}}\ll\frac{1}{t}$, hence $I_9\ll\int_1^T\frac{1}{t^2}\mathrm{d}t=O(1)$.

For I_8, we need a more finer estimate than I_9, because the expressions of $D(u)$ are different in when $u<N$ and $u\geq N$. When $y\in(0,\log N)$, $\mathrm{e}^y\in(1,N)$, so we needn't pay much attention on $D(e^y)$. However, if we solve the inequality $\mathrm{e}^{y+\delta}<N$, then $y<\log N-\delta$, this suggests us to decompose I_8 as $\int_0^{\log N-\delta}+\int_{\log N-\delta}^N=I_{10}+I_{11}$, where

$$\begin{aligned}I_{10}&=\int_0^{\log N-\delta}\left(\sqrt{\mathrm{e}^y}R(N-\mathrm{e}^y)-\sqrt{\mathrm{e}^{y+\delta}}R(N-\mathrm{e}^{y+\delta})+\frac{1}{2}\int_{N-\mathrm{e}^{y+\delta}}^{N-\mathrm{e}^y}\frac{R(u)}{\sqrt{N-u}}\mathrm{d}u\right)^2\mathrm{d}y\\ I_{11}&=\int_{\log N-\delta}^{\log N}\left(\frac{2}{3}\Big((N-1)^{3/2}-\mathrm{e}^{3(y+\delta)/2}\Big)+\frac{1}{2}\int_1^{N-\mathrm{e}^y}\frac{R(u)}{\sqrt{N-u}}\mathrm{d}u+\sqrt{N-1}+\sqrt{\mathrm{e}^y}R(N-\mathrm{e}^y)\right)^2\mathrm{d}y.\end{aligned} \tag{2.39}$$

We deal with I_{11} firstly, I_{10} is more difficult so we put it in the next step. There is an integral term $\int_1^{N-\mathrm{e}^y}$ in I_{11}, which may cause troubles, so we try to control it firstly. We notice that $N-\mathrm{e}^y \in \big(0, N(1-\mathrm{e}^{-\delta})\big)$ when $y \in (\log N - \delta, \log N)$, moreover, by the inequality $\mathrm{e}^x \geq 1-x$ and $\delta \ll 1/T$, we have $1-\mathrm{e}^{-\delta} \leq \delta \ll 1/T$, hence $N-\mathrm{e}^y \ll N/T$. Remember $R(u) = \psi(u) - u$, we obtain

$$\int_1^{N-\mathrm{e}^y} \frac{R(u)}{\sqrt{N-u}} \mathrm{d}u \ll \frac{1}{\sqrt{N-N/T}} \int_1^{\frac{N}{T}} \big(\psi(u) - u\big)\, \mathrm{d}u \ll \frac{(N/T)^{\frac{3}{2}}}{\sqrt{N}}, \tag{2.40}$$

we use Lemma 2.1 in the last equality.

Therefore, we have (Using $(a+b+c)^2 \ll a^2+b^2+c^2$)

$$\begin{aligned} I_{11} \ll & \int_{\log N-\delta}^{\log N} \left(\frac{2}{3}\Big((N-1)^{3/2} - \mathrm{e}^{3(y+\delta)/2}\Big) + \frac{(N/T)^{\frac{3}{2}}}{\sqrt{N}} + \sqrt{N-1} + \sqrt{\mathrm{e}^y} R(N-\mathrm{e}^y) \right)^2 \mathrm{d}y \\ \ll & \frac{1}{T}\left(\frac{N^3}{T^2} + \frac{N^2}{T} \log^4 N \right), \end{aligned} \tag{2.41}$$

some explanation for the last term $\int_{\log N-\delta}^{\log N} \mathrm{e}^y R^2(N-\mathrm{e}^y)\mathrm{d}y$ is necessary, after a change of variable, it's $\int_0^{N(1-\mathrm{e}^{-\delta})} R^2(t)\mathrm{d}t$, and then we use the estimate $R(t) = O(t^{\frac{1}{2}} \log^2 t)$, see Lemma 2.3 for this point.

Step 5: Estimate of $I_{10} \ll N^{\frac{3}{2}}$ Firstly, we write $\sqrt{\mathrm{e}^y} R(N-\mathrm{e}^y) - \sqrt{\mathrm{e}^{y+\delta}} R(N-\mathrm{e}^{y+\delta})$ as following

$$\begin{aligned} & \sqrt{\mathrm{e}^y} R(N-\mathrm{e}^y) - \sqrt{\mathrm{e}^y} R(N-\mathrm{e}^{y+\delta}) + \sqrt{\mathrm{e}^y} R(N-\mathrm{e}^{y+\delta}) - \sqrt{\mathrm{e}^{y+\delta}} R(N-\mathrm{e}^{y+\delta}) \\ = & \sqrt{\mathrm{e}^y}\Big(R(N-\mathrm{e}^y) - R(N-\mathrm{e}^{y+\delta})\Big) - R(N-\mathrm{e}^{y+\delta})(\sqrt{\mathrm{e}^{y+\delta}} - \sqrt{\mathrm{e}^y}). \end{aligned} \tag{2.42}$$

Moreover, $\sqrt{\mathrm{e}^{y+\delta}} - \sqrt{\mathrm{e}^y} = \sqrt{\mathrm{e}^y}(\mathrm{e}^{\frac{\delta}{2}} - 1) \ll \sqrt{\mathrm{e}^y}(\exp \frac{1}{2T} - 1)$, and we have $\mathrm{e}^x - 1 \ll x$ when x is small, hence

$$\sqrt{\mathrm{e}^{y+\delta}} - \sqrt{\mathrm{e}^y} \ll \frac{\sqrt{\mathrm{e}^y}}{T}. \tag{2.43}$$

Then using the inequality $(a+b+c)^2 \ll a^2+b^2+c^2$, we get

$$\begin{aligned} I_{10} \ll & \int_0^{\log N-\delta} \mathrm{e}^y \Big(R(N-\mathrm{e}^y) - R(N-\mathrm{e}^{y+\delta})\Big)^2 \mathrm{d}y + \int_0^{\log N-\delta} \left(\int_{N-\mathrm{e}^{y+\delta}}^{N-\mathrm{e}^y} \frac{R(u)}{\sqrt{N-u}} \mathrm{d}u \right)^2 \mathrm{d}y \\ & + \int_0^{\log N-\delta} \frac{\mathrm{e}^y}{T^2} R(N-\mathrm{e}^{y+\delta})^2 \mathrm{d}y. \end{aligned} \tag{2.44}$$

We denote

$$\begin{aligned} I_{12} &= \int_0^{\log N-\delta} \mathrm{e}^y \Big(R(N-\mathrm{e}^y) - R(N-\mathrm{e}^{y+\delta})\Big)^2 \mathrm{d}y, \\ I_{13} &= \int_0^{\log N-\delta} \left(\int_{N-\mathrm{e}^{y+\delta}}^{N-\mathrm{e}^y} \frac{R(u)}{\sqrt{N-u}} \mathrm{d}u \right)^2 \mathrm{d}y, \\ I_{14} &= \int_0^{\log N-\delta} \frac{\mathrm{e}^y}{T^2} R(N-\mathrm{e}^{y+\delta})^2 \mathrm{d}y. \end{aligned} \tag{2.45}$$

I_{12} is more difficult, so we put it in the next step. To deal with I_{13} and I_{14}, firstly, we need a lemma concerning the growth of $R(u)$.

Lemma 2.3. *Let $\theta \in [\frac{1}{2}, 1)$ fixed, then*

$$\psi(x) = x + O(x^\theta \log^2 x) \tag{2.46}$$

if and only if

$$\zeta(s) \neq 0, \quad \textit{for all } \Re(s) > \theta. \tag{2.47}$$

For the proof, one may see [12], Chapter 12, Theorem 12.3. In particular, this lemma shows that the Riemann Hypothesis is equivalent to

$$\psi(x) = x + O(x^{\frac{1}{2}} \log^2 x), \tag{2.48}$$

however, this will never be easier than proving the original form of Riemann Hypothesis.

Using this lemma, I_{13} and I_{14} can be estimated directly:

$$\begin{aligned}
I_{13} \ll & \int_0^{\log N - \delta} \left(\int_{N-\mathrm{e}^{y+\delta}}^{N-\mathrm{e}^y} \frac{\sqrt{u}\log^2 u}{\sqrt{N-u}} \mathrm{d}u \right)^2 \mathrm{d}y \\
\ll & N \log^4 N \int_0^{\log N - \delta} \left(\int_{N-\mathrm{e}^{y+\delta}}^{N-\mathrm{e}^y} \frac{\mathrm{d}u}{\sqrt{N-u}} \right)^2 \mathrm{d}y \\
= & N \log^4 N \int_0^{\log N - \delta} \left(\sqrt{\mathrm{e}^{y+\delta}} - \sqrt{\mathrm{e}^y} \right)^2 \mathrm{d}y \\
\ll & \frac{N \log^4 N}{T^2} \int_0^{\log N - \delta} \mathrm{e}^y \mathrm{d}y \ll \frac{N^2 \log^4 N}{T^2}.
\end{aligned} \tag{2.49}$$

After a change of variable,

$$\begin{aligned}
I_{14} = & \frac{1}{T^2 \delta} \int_0^{N-\mathrm{e}^\delta} R^2(t) \mathrm{d}t \\
\ll & \frac{1}{T^2 \mathrm{e}^\delta} \int_0^{N-\mathrm{e}^\delta} t \log^4 t \mathrm{d}t \ll \frac{N^2 \log^4 N}{T^2}.
\end{aligned} \tag{2.50}$$

Step 6: Estimate of $I_{12} \ll N^{\frac{3}{2}}$ After a change of variable,

$$\begin{aligned}
I_{12} = & \int_{N(1-\mathrm{e}^{-\delta})}^{N-1} \left(R(y\mathrm{e}^\delta + N - N\mathrm{e}^\delta) - R(y) \right)^2 \mathrm{d}y \\
\ll & \int_{N(1-\mathrm{e}^{-\delta})}^{N-1} \left(R(y\mathrm{e}^\delta - N(\mathrm{e}^\delta - 1)) - R(y\mathrm{e}^\delta)) \right)^2 \mathrm{d}y + \int_{N(1-\mathrm{e}^{-\delta})}^{N-1} \left(R(y\mathrm{e}^\delta) - R(y) \right)^2 \mathrm{d}y \\
= & \int_{N(\mathrm{e}^\delta - 1)}^{(N-1)\mathrm{e}^\delta} \left(R(x - N(\mathrm{e}^\delta - 1)) - R(x) \right)^2 \mathrm{d}x + \int_{N(1-\mathrm{e}^{-\delta})}^{N-1} \left(R(y\mathrm{e}^\delta) - R(y) \right)^2 \mathrm{d}y,
\end{aligned} \tag{2.51}$$

the last two integrals $\int_{N(\mathrm{e}^\delta-1)}^{(N-1)\mathrm{e}^\delta}$ and $\int_{N(1-\mathrm{e}^{-\delta})}^{N-1}$ have been treated in Goldston and Montgomery's article, see [10, 19]. Hence

$$I_{12} \ll N^2 \max_{0<\delta\ll\frac{1}{T}} \delta \log^2 \frac{1}{\delta} \ll \frac{N^2 \log^2 N}{T}. \tag{2.52}$$

Conclusion Until now, we have proved $I_8 = I_{10} + I_{11} = I_{11} + I_{12} + I_{13} + I_{14}$, where

$$\begin{aligned}
I_{11} \ll & \frac{1}{T} \left(\frac{N^3}{T^2} + \frac{N^2}{T} \log^4 N \right), \quad & I_{12} \ll \frac{N^2 \log^2 N}{T}, \\
I_{13} \ll & \frac{N^2 \log^4 N}{T^2}, & I_{14} \ll \frac{N^2 \log^4 N}{T^2},
\end{aligned} \tag{2.53}$$

so

$$I_8 \ll \frac{N^3}{T^3} + \frac{N^2}{T^2} \log^4 N + \frac{N^2}{T} \log^2 N. \tag{2.54}$$

In step 5, we have proved $I_9 = O(1)$, and $I_7 = T \log N \max_{0<\delta\ll\frac{1}{T}} \sqrt{I_8} \cdot \sqrt{I_9}$, hence

$$I_7 \ll T \log N \Big(\frac{N^{\frac{3}{2}}}{T^{\frac{3}{2}}} + \frac{N}{T} \log^2 N + \frac{N}{\sqrt{T}} \log N \Big). \tag{2.55}$$

In step 4, we decompose I_2 as $I_4 + I_5 + I_6 + I_7$, and we have also proved $I_4, I_5, I_6 \ll N^{\frac{3}{2}}$, so

$$I_2 \ll I_7 + N^{\frac{3}{2}}. \tag{2.56}$$

In step 3, we write $I_1 = -2\Im(I_2) + O(I_3)$ and prove $I_3 \ll N^{\frac{3}{2}}$, so

$$I_1 \ll I_7 + N^{\frac{3}{2}}. \tag{2.57}$$

In step 2, we write $I = I_1 + O\Big(\frac{N^2}{T} \log^2(NT)\Big) + O(N \log N)$, and notice that $T \ll N$, therefore

$$I \ll \frac{N^2}{T} \log^2(NT) + N^{\frac{3}{2}} + T \log N \Big(\frac{N^{\frac{3}{2}}}{T^{\frac{3}{2}}} + \frac{N}{T} \log^2 N + \frac{N}{\sqrt{T}} \log N \Big), \tag{2.58}$$

now let $T = \sqrt{N} \log^2 N$, then $1 \ll T \ll N$ and $I \ll N^{\frac{3}{2}}$. Finally, we get

$$\sum_{n \le X} r_2(n) = \sum_{n \le N} r_2(n) = \frac{N^2}{2} + O(N^{3/2}). \tag{2.59}$$

3 Proof of Theorem 1.5, Part 2

3.1 Proof of Theorem 1.5, Part 2

We recall the part 2 of Theorem 1.5:

$$\textbf{Assuming } \sum_{n \le X} r_2(n) = \frac{X^2}{2} + O(X^{3/2}), \textbf{ then Riemann Hypothesis is true.} \tag{3.1}$$

Step 1 To prove part 2, we need an asymptotic formula of $S(x) = \sum_{n \le x} r_2(n)$

$$S(x) = \frac{x^2}{2} - \sum_{\rho} \frac{2x^{\rho+1}}{\rho(\rho+1)} + E(x), \tag{3.2}$$

where $B = \sup\{\Re\rho : \zeta(\rho) = 0\}$, $E(x) = O\big(x^{\frac{2+4B}{3}} \log^4(2x)\big)$. We have known that $B \le 1$ by Theorem 1.3, and $B \ge 1/2$ is already known, because there does exist zeros on the line $\{\Re(s) = 1/2\}$, if we consider the positive imaginary parts, then the first non-trivial zero on that line has imaginary parts approximately 14.135.

Step 1 a little bit technical, so we put it in the subsection 3.2.

Step 2 Assuming step 1, we define the corresponding Dirichlet series of $r_2(n)$

$$F(s) = \sum_{n \ge 1} \frac{r_2(n)}{n^s}, \tag{3.3}$$

we notice that

$$r_2(n) = \sum_{m+k=n} \Lambda(m)\Lambda(k) \ll n \log n, \tag{3.4}$$

so when $\Re(s) > 2$, the series converges uniformly and absolutely and $F(s)$ has the integral form

$$F(s) = s \int_1^\infty S(u) u^{-s-1} \mathrm{d}u, \tag{3.5}$$

therefore, $F(s)$ is analytic in $\{s \in \mathbb{C} : \Re(s) > 2\}$ and by step 1

$$\begin{aligned} F(s) =& \frac{s}{2(s-2)} + \sum_{\rho} \frac{r(\rho)s}{(\rho+1)(s-\rho+1)} + s\int_1^{\infty} Er(u)u^{-s-1}\mathrm{d}u + \frac{1}{2} + \sum_{\rho} \frac{r(\rho)}{\rho+1} \\ =& \frac{1}{s-2} + \sum_{\rho} \frac{r(\rho)}{s-\rho-1} + s\int_1^{\infty} Er(u)u^{-s-1}\mathrm{d}u + \Big(\frac{1}{2} + \sum_{\rho} \frac{r(\rho)}{\rho+1}\Big). \end{aligned} \tag{3.6}$$

where $r(\rho) = -\frac{2}{\rho}$. We must point out that due to Lemma 3.2, the series $\sum_{\rho} \frac{r(\rho)}{\rho+1}$ is convergent.

Moreover, $F(s)$ can be extended meromorphically to the half plane $\{s \in \mathbb{C} : \Re(s) > \frac{2+4B}{3}\}$, this is because $Er(u) \ll u^{\frac{2+4B}{3}} \log^4(2u)$.

Step 3 If $B < 1$, then

$$1 + B = \inf\{\sigma_0 \geq \frac{3}{2} : F(s) - \frac{1}{s-2} \text{ is analytic on } \Re(s) > \sigma_0\}. \tag{3.7}$$

Denote the set on the right hand is $\mathscr{A}$. By step 2, $\inf \mathscr{A}$ is at most $\frac{2+4B}{3}$, however, $B \leq 1$, so $\inf \mathscr{A} \leq \frac{2+4B}{3} \leq B + 1$.

For the inverse inequality $B + 1 \leq \inf \mathscr{A}$, we know that this is trivially true when $B = 1/2$, because $\sigma_0 \geq \frac{3}{2}$, so we may assume that $1/2 < B \leq 1$. Now $\max\{\frac{2+4B}{3}, \frac{3}{2}\} < 1 + B$, there exist an $\epsilon > 0$ such that $\max\{\frac{2+4B}{3}, \frac{3}{2}\} < 1 + B - \epsilon$, by the definition of B, there exists a non-trivial zero ρ with $\frac{1}{2} < B - \epsilon < \Re(\rho)$. By the formula of $F(s)$, in the half plane

$$\{s \in \mathbb{C} : \Re(s) > 1 + B - \epsilon > \frac{3}{2}\} \tag{3.8}$$

F has a pole at $\rho + 1$ with residue $-\frac{2}{\rho}$, and we conclude that

$$1 + B - \epsilon \leq \inf\{\sigma_0 \geq \frac{3}{2} : F(s) - \frac{1}{s-2} \text{ is analytic on } \Re(s) > \sigma_0\}, \tag{3.9}$$

finally, let $\epsilon \to 0$.

Step 4 Now we let $D(x) = \sum_{n \leq x} r_2(n) - \frac{x^2}{2}$, then $D(x) \ll_{\epsilon} x^{\frac{3}{2}+\epsilon}$ for all $\epsilon > 0$. Consequently

$$F(s) - \frac{1}{s-2} = s\int_1^{\infty} D(u)u^{-s-1}\mathrm{d}u + \frac{1}{2}, \quad \Re(s) > 2, \tag{3.10}$$

where the right hand side is analytic on $\{s \in \mathbb{C} : \Re(s) > \frac{3}{2}\}$, since $D(u) \ll_{\epsilon} u^{\frac{3}{2}+\epsilon}$. So we conclude $B \leq \frac{1}{2}$ by step 3, hence the Riemann Hypothesis holds.

Step 5 Notice that in Step 3, we assume that $B < 1$, however we only have $B \leq 1$ by Theorem 1.3, so we need to exclude the case $B = 1$. For this, we need a lemma:

Lemma 3.1. *If for some $0 < \delta < 1$, the following asymptotic formula holds:*

$$S(X) = \frac{X^2}{2} + O(X^{2-\delta}), \tag{3.11}$$

then there exists $0 < \delta' < 1$ such that for all non-trivial zeros ρ of ζ function, ρ satisfies

$$\Re(\rho) < 1 - \delta'. \tag{3.12}$$

The proof of this Lemma will be treated in subsection 3.3. By our assumption:

$$\sum_{n \leq X} r_2(n) = \frac{X^2}{2} + O(X^{3/2}), \tag{3.13}$$

just let $\delta = \frac{1}{2}$, then there exists a δ' such that $\Re(\rho) < 1 - \delta' < 1$, so we exclude the case $B = 1$.

3.2 Proof of Step 1 in Theorem 1.5

We follow the method in [4]. [4] considers the general case of $S(x)$ for $L(s,\chi)$, but we only need to deal with ζ, so some proofs in [4] can be simplified. In order to prove the asymptotic formula of $S(x)$, we need some estimates for the non-trivial zeros of zeta function.

Lemma 3.2. *Let $\rho=\beta+\mathrm{i}\gamma$ be non-trivial zeros of $\zeta(s)$, then*

(i) *For any $T\geq 1$*

$$\sum_{|\gamma|\leq T}\frac{1}{|\rho|}\ll \log^2(2T). \tag{3.14}$$

(ii) *For any $T\geq 1$*

$$\sum_{|\gamma|>T}\frac{1}{|\rho|^2}\ll \frac{\log(2T)}{T}, \tag{3.15}$$

particularly, $\sum_\rho \frac{1}{|\rho|^2}<\infty$.

Proof. (i). We may use Lemma 2.2 $N(T)\sim cT\log T$ with $c>0$, remove finite zeros with $0<\gamma\leq 1$, we have

$$\sum_{1<\gamma\leq T}\frac{1}{|\rho|}=\sum_{1\leq n\leq T}\sum_{n<\gamma\leq n+1}\frac{1}{|\rho|}\ll\sum_{1\leq n\leq T}\frac{\log n}{n}\ll\log^2(T), \tag{3.16}$$

so (i) is true by the fact that non-trivial zeros of zeta function come in conjugate pairs.

(ii). Similarly, (ii) is followed by

$$\sum_{|\gamma|>T}\frac{1}{|\rho|^2}\ll\sum_{n\geq T}\sum_{n<\gamma\leq n+1}\frac{1}{|\rho|^2}\ll\sum_{n\geq T}\frac{\log n}{n^2}\ll\frac{\log T}{T}. \tag{3.17}$$

□

Recall that $S(x)=\sum_{n\leq x}r_2(n)$, so we can write $S(x)$ as

$$S(x)=\sum_{l\leq x}\Lambda(x)\psi(x-l), \tag{3.18}$$

however, $\psi(x)=x-\sum_{|\Im(\rho)|\leq T}\frac{x^\rho}{\rho}+O(\frac{x}{T}\log^2 x)$, where $T\leq x$ (see Lemma 2.1 or 2.3), hence

$$S(x)=\sum_{l\leq x}\Lambda(l)\Big((x-l)-\sum_{|\Im(\rho)|\leq T}\frac{(x-l)^\rho}{\rho}\Big)+\sum_{l\leq x}\Lambda(l)O\Big(\frac{x-l}{T}\log^2(x-l)\Big). \tag{3.19}$$

For the second term,

$$\sum_{l\leq x}\Lambda(l)O\Big(\frac{(x-l)}{T}\log^2(x-l)\Big)\ll\frac{x}{T}\log^2 x\sum_{l\leq x}\Lambda(l)\ll\frac{x^2}{T}\log^2 x. \tag{3.20}$$

For the first term, we denote

$$\begin{aligned}\Sigma_1&=\sum_{l\leq x}\Lambda(l)(x-l)\\ \Sigma_2&=\sum_{l\leq x}\Lambda(l)\sum_{|\Im(\rho)|\leq T}\frac{(x-l)^\rho}{\rho},\end{aligned} \tag{3.21}$$

then the first term is $\Sigma_1 - \Sigma_2$, by Fubini's theorem, it's easy to verify that

$$\Sigma_1 = \int_0^x \psi(u)\mathrm{d}u. \tag{3.22}$$

Once again, by $\psi(x) = x - \sum_{|\Im(\rho)|\le T} \frac{x^\rho}{\rho} + O(\frac{x}{T}\log^2 x)$, we have

$$\begin{aligned}\Sigma_1 =& \int_0^x \Big\{ u - \sum_{|\Im(\rho)|\le T} \frac{u^\rho}{\rho} + O\Big(\frac{u}{T}\log^2 u\Big)\Big\}\mathrm{d}u \\ =& \frac{x^2}{2} - \sum_{|\Im(\rho)|\le T} \frac{x^{\rho+1}}{\rho(\rho+1)} + O(\frac{x^2}{T}\log^2 x).\end{aligned} \tag{3.23}$$

Σ_2 is much more difficult than Σ_1, we change the double sum at first, then

$$\Sigma_2 = \sum_{|\Im(\rho)|\le T} \frac{1}{\rho} \sum_{l\le x} \Lambda(l)(x-l)^\rho \tag{3.24}$$

for the general term $\frac{1}{\rho}\sum_{l\le x}\Lambda(l)(x-l)^\rho$, by Riemann-Stieltjes integral

$$\begin{aligned}\frac{1}{\rho}\sum_{l\le x}\Lambda(l)(x-l)^\rho =& \frac{1}{\rho}\int_0^x (x-u)^\rho \mathrm{d}\psi(u) \\ =& \frac{1}{\rho}\int_0^x (x-u)^\rho \mathrm{d}\Big(A(u,T) + B(u,T)\Big),\end{aligned} \tag{3.25}$$

where $A(u,T) = u - \sum_{|\Im(\rho)|\le T}\frac{u^\rho}{\rho}$, and $B(u,T) = O\big(\frac{u}{T}\log^2 u\big)$. Then

$$\begin{aligned}\frac{1}{\rho}\int_0^x (x-u)^\rho \mathrm{d}\psi(u) =& \frac{1}{\rho}\int_0^x (x-u)^\rho \mathrm{d}u - \frac{1}{\rho}\sum_{|\Im(\rho')|\le T}\int_0^x (x-u)^\rho u^{\rho'-1}\mathrm{d}u + \frac{1}{\rho}\int_0^x (x-u)^\rho \mathrm{d}B(u,T) \\ =& I_1 - I_2 + I_3,\end{aligned} \tag{3.26}$$

where

$$\begin{aligned} I_1 =& \frac{x^{\rho+1}}{\rho(\rho+1)} \\ I_2 =& \sum_{|\Im(\rho')|\le T} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} x^{\rho+\rho'} \\ I_3 =& \frac{1}{\rho}\,(x-u)^\rho B(u,T)\Big|_{u=0}^{u=x} + \int_0^x (x-u)^{\rho-1}B(u,T)\mathrm{d}u \\ \ll& O\Big(\frac{1}{|\rho|}\frac{x^2}{T}\log^2 x\Big) + \int_0^x (x-u)^{\rho-1}B(u,T)\mathrm{d}u.\end{aligned} \tag{3.27}$$

Here, $\Gamma(\rho)$ is the complex Gamma function, we would like to say something about calculating I_2. After a change of variable $u \mapsto xv$,

$$\int_0^x (x-u)^\rho u^{\rho'-1}\mathrm{d}u = x^{\rho+\rho'}\int_0^1 (1-v)^\rho v^{\rho'-1}\mathrm{d}v = x^{\rho+\rho'}\mathbf{B}(\rho',\rho+1). \tag{3.28}$$

$\mathbf{B}(\rho',\rho+1)$ is the **Beta Function** defined by

$$\mathbf{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t \quad \Re x,\, \Re y > 0. \tag{3.29}$$

Furthermore, we have the following relationship:

$$\mathbf{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.30}$$

so

$$\frac{1}{\rho}\sum_{|\Im(\rho')|\le T}\int_0^x (x-u)^{\rho}u^{\rho'-1}\mathrm{d}u = \sum_{|\Im(\rho')|\le T}\frac{\Gamma(\rho+1)}{\rho}\times\frac{\Gamma(\rho')}{\Gamma(\rho'+\rho+1)}x^{\rho'+\rho} = I_2. \tag{3.31}$$

Therefore,

$$\begin{aligned}\Sigma_2 = &\sum_{|\Im(\rho)|\le T}\frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_{|\Im(\rho)|\le T}\sum_{|\Im(\rho')|\le T}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')}x^{\rho+\rho'}\\ &+\sum_{|\Im(\rho)|\le T}\left(\int_0^x (x-u)^{\rho-1}B(u,T)\mathrm{d}u + O\Big(\frac{1}{|\rho|}\frac{x^2}{T}\log^2 x\Big)\right).\end{aligned} \tag{3.32}$$

However, $\sum_{|\Im(\rho)|\le T} O\Big(\frac{1}{|\rho|}\frac{x^2}{T}\log^2 x\Big) = O(\frac{x^2}{T}\log^4 x)$ by Lemma 3.2 and $T\le x$. If we denote

$$\begin{aligned}\Sigma_3 &= \sum_{|\Im(\rho)|\le T}\sum_{|\Im(\rho')|\le T}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')}x^{\rho+\rho'}\\ \Sigma_4 &= \sum_{|\Im(\rho)|\le T}\int_0^x (x-u)^{\rho-1}B(u,T)\mathrm{d}u.\end{aligned} \tag{3.33}$$

Then

$$S(x) = \frac{x^2}{2} - 2\sum_{|\Im(\rho)|\le T}\frac{x^{\rho+1}}{\rho(\rho+1)} + \Sigma_3 - \Sigma_4 + O\Big(\frac{x^2}{T}\log^4 x\Big). \tag{3.34}$$

Consider the sum $\sum_{|\Im(\rho)|\le T}\frac{x^{\rho+1}}{\rho(\rho+1)}$, we can replace $\sum_{|\Im(\rho)|\le T}$ by $\sum_\rho$, without changing the error term $O(x^2\log^4 x/T)$, in fact:

$$\sum_{|\Im(\rho)|>T}\frac{x^{\rho+1}}{\rho(\rho+1)} \ll \sum_{|\Im(\rho)|>T}\frac{x^2}{|\rho|^2} \ll \frac{\log(2T)}{T}x^2. \tag{3.35}$$

Consequently $\sum_{|\Im(\rho)|\le T} = \sum_\rho - \sum_{|\Im(\rho)|>T} \ll \frac{\log(2T)}{T}x^2 \ll \frac{x^2}{T}\log^4 x$, when $T\le x$. So

$$S(x) = \frac{x^2}{2} - 2\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + \Sigma_3 - \Sigma_4 + O\Big(\frac{x^2}{T}\log^4 x\Big). \tag{3.36}$$

We will prove the following two estimates later:

$$\begin{aligned}\Sigma_3 &\ll x^{2B}T^{\frac{1}{2}}\log^4 x,\\ \Sigma_4 &\ll \frac{x^2}{T}\log^4 x.\end{aligned} \tag{3.37}$$

Now we have

$$S(x) \ll \frac{x^2}{2} - 2\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + x^{2B}T^{\frac{1}{2}}\log^4 x + \frac{x^2}{T}\log^4 x, \tag{3.38}$$

choosing $x^{2B}T^{1/2} = x^2/T$, i.e. $T = x^{4(1-B)/3}$, since $B\ge\frac{1}{2}$, we have $T\le x$, so

$$S(x) \ll \frac{x^2}{2} - 2\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + x^{(2+4B)/3}\log^4 x. \tag{3.39}$$

3.2.1 Estimate for $\Sigma_3 \ll x^{2B}T^{\frac{1}{2}}\log^4 x$

The difficult point of estimate of Σ_3 is $\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')}$, to deal with this, we need the the **complex Stirling's formula** for Gamma function, recall the classical result for $n \in \mathbb{N}$:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+O(n^{-2})\right). \tag{3.40}$$

Lemma 3.3 (Complex Stiriling's formula)**.** *Let $\delta > 0$, and $R_\delta = \{z \in \mathbb{C} : |s| \geq \delta, |\arg s| < \pi - \delta\}$, then*

$$\Gamma(s) = \sqrt{2\pi}s^{s-1/2}\mathrm{e}^{-s}(1+O(1/|s|)). \tag{3.41}$$

For the proof, see [17] Appendix C.

We come back to the estimate of Σ_3. We firstly assume that $|\gamma| \leq |\gamma'|$, where $\rho = \beta + \mathrm{i}\gamma$, $\rho' = \beta' + \mathrm{i}\gamma'$, by Stirling's formula (See [17], Appendix C, (C.19))

$$\Gamma(s) \asymp (|t|+1)^{\sigma-1/2}\mathrm{e}^{-(\pi/2)|t|}, \quad s = \sigma + \mathrm{i}t,\ \sigma \in [0,3],\ |t| \geq 1. \tag{3.42}$$

Case 1. $|\gamma| \leq |\gamma'| \leq 1$**:** In this case, $|\Gamma(1+\rho+\rho')| \asymp 1$, so

$$\begin{aligned}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll& |\rho|^{-1}|\rho'|^{-1} \\ \ll& T^{\frac{1}{2}}|\rho|^{-1}|\rho'|^{-1} \\ \ll& T^{\frac{1}{2}}|\gamma|^{-1}|\gamma'|^{-1}.\end{aligned} \tag{3.43}$$

Case 2. $|\gamma| \leq 1 \leq |\gamma'|$**:** Using Stirling's formula to $\Gamma(\rho')$ and $\Gamma(1+\rho+\rho')$, we get

$$\begin{aligned}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll& |\rho|^{-1}\frac{|\gamma'|^{\beta'-1/2}\mathrm{e}^{-(\pi/2)|\gamma'|}}{(1+|\gamma+\gamma'|)^{\beta+\beta'+1/2}\mathrm{e}^{-(\pi/2)|\gamma+\gamma'|}} \\ \ll& |\rho|^{-1}|\gamma'|^{-\beta-1} \\ \ll& |\rho|^{-1}|\rho'|^{-1} \\ \ll& T^{\frac{1}{2}}|\rho|^{-1}|\rho'|^{-1} \\ \ll& T^{\frac{1}{2}}|\gamma|^{-1}|\gamma'|^{-1}.\end{aligned} \tag{3.44}$$

Case 3-1. $1 \leq |\gamma| \leq |\gamma'| \leq T$**,** $sgn(\gamma) = sgn(\gamma')$**:** Similarly,

$$\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll \frac{|\gamma|^{\beta-1/2}\mathrm{e}^{-(\pi/2)|\gamma|}|\gamma'|^{\beta'-1/2}\mathrm{e}^{-(\pi/2)|\gamma'|}}{(|\gamma+\gamma'|+1)^{\beta+\beta'+1/2}\mathrm{e}^{-(\pi/2)|\gamma+\gamma'|}}, \tag{3.45}$$

in this case, $|\gamma+\gamma'| = |\gamma|+|\gamma'|$, so $\mathrm{e}^{-(\pi/2)|\gamma|}\mathrm{e}^{-(\pi/2)|\gamma'|} = \mathrm{e}^{-(\pi/2)|\gamma+\gamma'|}$, consequently

$$\begin{aligned}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll& \frac{|\gamma|^{\beta-1/2}|\gamma'|^{\beta'-1/2}}{(|\gamma+\gamma'|+1)^{\beta+\beta'+1/2}} \\ \ll& \frac{|\gamma|^{\beta-1/2}|\gamma'|^{\beta'-1/2}}{|\gamma+\gamma'|^{\beta+\beta'+1/2}} \\ \ll& \frac{|\gamma|^{\beta-1/2}}{|\gamma'|^{\beta+1}}.\end{aligned} \tag{3.46}$$

We have assumed that $|\gamma| \leq |\gamma'|$, so $|\gamma|^\beta/|\gamma'|^\beta \leq 1$, hence $|\gamma|^{\beta-1/2}/|\gamma'|^{\beta+1} \ll |\gamma|^{-1/2}|\gamma'|^{-1}$, moreover, by assumption $|\gamma| \leq T$, so $|\gamma|^{-1/2}|\gamma'|^{-1} \ll T^{\frac{1}{2}}|\gamma|^{-1}|\gamma'|^{-1}$. In conclusion,

$$\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll T^{\frac{1}{2}}|\gamma|^{-1}|\gamma'|^{-1} \tag{3.47}$$

Case 3-2. $1 \le |\gamma| \le |\gamma'| \le T$, $sgn(\gamma) = -sgn(\gamma')$**:** Notice that we have assumed that $|\gamma| \le |\gamma'|$, so $|\gamma+\gamma'| = |\gamma'| - |\gamma|$, and

$$\frac{\mathrm{e}^{-(\pi/2)|\gamma|}\mathrm{e}^{-(\pi/2)|\gamma'|}}{\mathrm{e}^{-(\pi/2)|\gamma+\gamma'|}} = \mathrm{e}^{-\pi|\gamma|}. \tag{3.48}$$

Nextly, we estimate the term $(|\gamma+\gamma'|+1)^{-\beta-\beta'-1/2}$, it's

$$\begin{aligned}(1+|\gamma'|-|\gamma|)^{-\beta-\beta'-1/2} =&(1+|\gamma'|)^{-\beta-\beta'-1/2}\Big(\frac{1+|\gamma'|-|\gamma|}{1+|\gamma'|}\Big)^{-\beta-\beta'-1/2}\\ =&(1+|\gamma'|)^{-\beta-\beta'-1/2}\Big(\frac{1+|\gamma'|}{1+|\gamma'|-|\gamma|}\Big)^{\beta+\beta'+1/2}\\ \ll&(1+|\gamma'|)^{-\beta-\beta'-1/2}(1+|\gamma|)^{\beta+\beta'+1/2},\end{aligned} \tag{3.49}$$

the last inequality can be verified directly: $1+|\gamma'| \le (1+|\gamma|)(1+|\gamma'|-|\gamma|)$, because $|\gamma| \le |\gamma'|$. Notice that $\beta, \beta' < 1$, so $\beta+\beta'+1/2 \le 2.5 < \pi$, and

$$(1+|\gamma'|)^{-\beta-\beta'-1/2}(1+|\gamma|)^{\beta+\beta'+1/2} \ll (1+|\gamma'|)^{-\beta-\beta'-1/2}(1+|\gamma|)^{\pi}. \tag{3.50}$$

By the inequality $\mathrm{e}^x \ge 1+x$, we have $(1+|\gamma|)^\pi \le \mathrm{e}^{\pi|\gamma|}$. In conclusion, we get

$$(1+|\gamma'|-|\gamma|)^{-\beta-\beta'-1/2} \ll (1+|\gamma'|)^{-\beta-\beta'-1/2}\mathrm{e}^{\pi|\gamma|}, \tag{3.51}$$

once again

$$\begin{aligned}\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll&\frac{|\gamma|^{\beta-1/2}\mathrm{e}^{-(\pi/2)|\gamma|}|\gamma'|^{\beta'-1/2}\mathrm{e}^{-(\pi/2)|\gamma'|}}{(|\gamma+\gamma'|+1)^{\beta+\beta'+1/2}\mathrm{e}^{-(\pi/2)|\gamma+\gamma'|}}\\ \ll&\frac{|\gamma|^{\beta-1/2}|\gamma'|^{\beta'-1/2}}{(|\gamma+\gamma'|+1)^{\beta+\beta'+1/2}}\mathrm{e}^{-\pi|\gamma|}\\ \ll&|\gamma|^{\beta-1/2}|\gamma'|^{\beta'-1/2}(1+|\gamma'|)^{-\beta-\beta'-1/2}\mathrm{e}^{\pi|\gamma|}\mathrm{e}^{-\pi|\gamma|}\\ =&\frac{|\gamma|^{\beta-1/2}|\gamma'|^{\beta'-1/2}}{(1+|\gamma'|)^{\beta+\beta'+1/2}}\\ \le&\frac{|\gamma|^{\beta-1/2}}{|\gamma'|^{\beta+1}},\end{aligned} \tag{3.52}$$

so we come back to the case $sgn(\gamma) = sgn(\gamma')$.

Combining these cases and the symmetry of ρ and ρ', we have for all non-trivial zeros ρ, ρ' with $|\gamma|, |\gamma'| \le T$, the estimate

$$\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll T^{\frac{1}{2}}|\gamma|^{-1}|\gamma'|^{-1} \tag{3.53}$$

always holds.

Moreover, we notice that $\beta \in (0,1)$ when ρ is a non-trivial zero, so $|\rho| \le 1+|\gamma| \ll |\gamma|$. Therefore:

$$\frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} \ll T^{\frac{1}{2}}|\rho|^{-1}|\rho'|^{-1}. \tag{3.54}$$

Using this estimate, we get

$$
\begin{aligned}
\Sigma_3 = & \sum_{|\Im(\rho)|\le T} \sum_{|\Im(\rho')|\le T} \frac{\Gamma(\rho)\Gamma(\rho')}{\Gamma(1+\rho+\rho')} x^{\rho+\rho'} \\
\ll & \sum_{|\Im(\rho)|\le T} \sum_{|\Im(\rho')|\le T} T^{\frac{1}{2}} |\rho|^{-1} |\rho'|^{-1} x^{\beta+\beta'} \\
\ll & x^{2B} T^{\frac{1}{2}} \sum_{|\Im(\rho)|\le T} \sum_{|\Im(\rho')|\le T} |\rho|^{-1} |\rho'|^{-1} \\
\ll & x^{2B} T^{\frac{1}{2}} \Big(\sum_{|\Im(\rho)|\le T} |\rho|^{-1} \Big)^2 \\
\ll & x^{2B} T^{\frac{1}{2}} \log^4 x, \quad \text{when } T \le x,
\end{aligned}
\tag{3.55}
$$

the last $\ll$ comes from Lemma 3.2.

3.2.2 Estimate for $\Sigma_4 \ll \frac{x^2}{T} \log^4 x$

We give a lemma at first, it would be used during our estimate. The proof can be found in [11, 13].

Lemma 3.4. *[Gonek-Landau Formula] In 1911, the German mathematician E.Landau proved the following estimate: for all $x, T \ge 1$, $\rho = \beta + \mathrm{i}\gamma$ be non-trivial zeros of $\zeta(s)$*

$$
\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi} \Lambda(x) + O(\log T). \tag{3.56}
$$

This estimate was improved by S.M.Gonek in 1993, now named as ***Gonek-Landau formula****:*

$$
\begin{aligned}
\sum_{0<\gamma\le T} x^{\rho} = & -\frac{T}{2\pi} \Lambda(x) + O(x \log 2xT \log\log 3x) + O\bigg(\log x \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\bigg) \\
& + O(\log 2T \min\{T, \log^{-1} x\}),
\end{aligned}
\tag{3.57}
$$

where $\Lambda(x) = 0$ when $x \notin \mathbb{N}$, and $\langle x\rangle$ is the distance from x to the nearest prime power other than x itself, i.e.

$$
\langle x\rangle = \min\{|x - p^k| : p \text{ prime},\ k \ge 1,\ p^k \ne x\}. \tag{3.58}
$$

Particularly,

$$
\begin{aligned}
\sum_{|\gamma|\le T} x^{\rho} = & -\frac{T}{\pi} \Lambda(x) + O(x \log 2xT \log\log 3x) + O\bigg(\log x \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\bigg) \\
& + O(\log 2T \min\{T, \log^{-1} x\})
\end{aligned}
\tag{3.59}
$$

since non-trivial zeros of zeta function come in conjugate pairs $\{\rho, \bar{\rho}\}$.

Now we come back to the proof of $\Sigma_4 \ll \frac{x^2}{T} \log^4 x$. For convenience, we write down Σ_4 again

$$
\begin{aligned}
\Sigma_4 = & \sum_{|\Im(\rho)|\le T} \int_0^x (x-u)^{\rho-1} B(u,T) \mathrm{d}u \\
= & \sum_{|\Im(\rho)|\le T} \int_0^x u^{\rho-1} B(x-u,T) \mathrm{d}u, \quad B(u,T) \ll \frac{u}{T} \log^2 u.
\end{aligned}
\tag{3.60}
$$

We write the integral as $\int_0^x = \int_0^3 + \int_3^x$. For the first part, notice that $\rho = \beta + \mathrm{i}\gamma$ with $\beta < 1$, so

$$\begin{aligned}\sum_{|\Im(\rho)|\le T} \int_0^3 u^{\rho-1} B(x-u,T)\mathrm{d}u \ll & \sum_{|\Im(\rho)|\le T} \frac{x}{T}\log^2 x \int_0^3 u^{\beta-1}\mathrm{d}u \\ \ll & \frac{x}{T}\log^2 x \sum_{|\Im(\rho)|\le T} \frac{1}{\beta}.\end{aligned} \tag{3.61}$$

Now we denote $\rho = \beta + \mathrm{i}\gamma$ to be a non-trivial zero with $|\gamma| < T$, then $1-\rho$ is also a non-trivial zero $|\Im(1-\rho)| = |-\gamma| < T$ (See the paragraph before Theorem 1.4), by this observation, we can write $\sum_{|\Im(\rho)|\le T} \frac{1}{\beta}$ as

$$\sum_{|\Im(\rho)|\le T} \frac{1}{1-\beta}. \tag{3.62}$$

By Theorem 1.3, if $\beta + \mathrm{i}\gamma$ is a non-trivial zero, then

$$\beta < 1 - \frac{c}{\log(2+|\gamma|)}, \tag{3.63}$$

therefore,

$$\begin{aligned}\sum_{|\Im(\rho)|\le T} \frac{1}{1-\beta} \ll & \sum_{|\Im(\rho)|\le T} \log(2+|\gamma|) \\ \ll & \log(2+T) \sum_{|\Im(\rho)|\le T} 1 \\ \ll & \log(2+T)\, T\log T,\end{aligned} \tag{3.64}$$

the last $\ll$ comes from Lemma 2.2. In conclusion, we get

$$\begin{aligned}\sum_{|\Im(\rho)|\le T} \int_0^3 u^{\rho-1} B(x-u,T)\mathrm{d}u \ll & \frac{x}{T}\log^2 x \sum_{|\Im(\rho)|\le T} \frac{1}{\beta} \\ \ll & \frac{x}{T}\log^2 x \times T\log T\log(2+T) \\ \ll & x\log^4 x,\end{aligned} \tag{3.65}$$

the last $\ll$ holds because we have assumed that $T \le x$.

For the second part

$$\sum_{|\Im(\rho)|\le T} \int_3^x u^{\rho-1} B(x-u,T)\mathrm{d}u = \int_3^x \Big(\sum_{|\Im(\rho)|\le T} u^{\rho-1} \Big) B(x-u,T)\mathrm{d}u, \tag{3.66}$$

by Gonek-Landau's formula, when $3 \le u \le x$, $T \le x$,

$$\sum_{|\Im(\rho)|\le T} u^{\rho} \ll u\log(uT)\log\log u + T\log u \ll u\log^2 x + x\log x, \quad u \notin \mathbb{N}. \tag{3.67}$$

Furthermore, $\mathbb{N}$ has Lebesgue measure 0 in $\mathbb{R}$, so $\int_{[3,x]} = \int_{[3,x]\setminus\mathbb{N}}$, and

$$\int_3^x \Big(\sum_{|\Im(\rho)|\le T} u^{\rho-1} \Big) B(x-u,T)\mathrm{d}u \ll \int_3^x u^{-1}(u\log^2 x + x\log x)\frac{x-u}{T}\log^2(x-u)\mathrm{d}u. \tag{3.68}$$

By a direct calculation:

$$\int_3^x \frac{x-u}{T}\log^2 x\log^2(x-u)\mathrm{d}u \le \frac{\log^2 x}{T}\int_3^x (x-u)\log^2(x-u)\mathrm{d}u \ll \frac{x^2\log^4 x}{T}, \tag{3.69}$$

and integral by parts, we get

$$\int_3^x u^{-1}x\frac{x-u}{T}\log x\log^2(x-u)\mathrm{d}u = \frac{x\log x}{T}\int_3^x u^{-1}(x-u)\log^2(x-u)\mathrm{d}u \ll \frac{x^2\log^4 x}{T}, \tag{3.70}$$

combining these two estimates, we get

$$\int_3^x \Big(\sum_{|\Im(\rho)|\le T} u^{\rho-1}\Big)B(x-u,T)\mathrm{d}u \ll \frac{x^2\log^4 x}{T}. \tag{3.71}$$

Finally,

$$\Sigma_4 = \int_0^x \sum_{|\Im(\rho)|\le T}(x-u)^{\rho-1}B(u,T)\mathrm{d}u = \int_0^3 + \int_3^x, \tag{3.72}$$

with $\int_0^3 \ll x\log^4 x$ and $\int_3^x \ll \frac{x^2\log^4 x}{T}$, so we get (Remember that by our assumption $T \le x$)

$$\Sigma_4 \ll \frac{x^2}{T}\log^4 x. \tag{3.73}$$

3.3 Proof of Lemma 3.1

We use the method in [2], recall the condition:

$$S(X) = \frac{X^2}{2} + O(X^{2-\delta}), \quad \text{for some } \delta \in (0,1), \tag{3.74}$$

and our goal is to find a $\delta' \in (0,1)$ such that for all ρ,

$$\Re(\rho) < 1-\delta'. \tag{3.75}$$

Let $|z| < 1$, consider the power series:

$$F(z) = \sum_{n\ge 1}\Lambda(n)z^n. \tag{3.76}$$

We have $F(z)$ is analytic in the disc $|z| < 1$, because $\Lambda(n) \le \log n$, moreover,

$$F^2(z) = \sum_{n\ge 1} r_2(n)z^n, \text{ and } \frac{F^2(z)}{1-z} = \sum_{n\ge 1} S(n)z^n, \tag{3.77}$$

recalling the definition of $S(x)$ in step 1: $S(x) = \sum_{n\le x} r_2(n)$.

By our assumption:

$$\sum_{n\ge 1} S(n)z^n = \sum_{n\ge 1}\Big(\frac{n^2}{2} + O(n^{2-\delta})\Big)z^n = \sum_{n\ge 0}\frac{n^2}{2}z^n + O\Big(\sum_{n\ge 0} n^{2-\delta}|z|^n\Big). \tag{3.78}$$

Using $\frac{1}{1-z} = 1 + z + z^2 + ...$, an easy calculation gives

$$\sum_{n\ge 0}\frac{n^2}{2}z^n = \frac{1}{(1-z)^3} - \frac{3}{2(1-z)^2} + \frac{1}{2(1-z)}. \tag{3.79}$$

For the error term, we need an asymptotic formula for Γ (See [17], Appendix C)

$$\Gamma(\alpha) = \lim_{n\to\infty}\frac{n^\alpha n!}{\alpha(\alpha+1)\cdots(\alpha+n)}, \quad \alpha > 0. \tag{3.80}$$

Let $\alpha = 2-\delta$, we get

$$\Gamma(2-\delta) \sim \frac{1}{2-\delta} \times \frac{n^{2-\delta} n!}{(3-\delta)\cdots(n+2-\delta)}, \tag{3.81}$$

which implies (Using $s\Gamma(s) = \Gamma(s+1)$)

$$\frac{(3-\delta)\cdots(n+2-\delta)}{n!} \sim \frac{n^{2-\delta}}{(2-\delta)\Gamma(2-\delta)} = \frac{n^{2-\delta}}{\Gamma(3-\delta)}. \tag{3.82}$$

So we have

$$\sum_{n\geq 0} n^{2-\delta}|z|^n \ll \sum_{n\geq 0} \frac{(3-\delta)\cdots(n+2-\delta)}{n!}|z|^n = (1-|z|)^{3-\delta}, \tag{3.83}$$

the equality comes from the power series expansion.

Now we get

$$\begin{aligned}\sum_{n\geq 1} S(n)z^n =& \frac{1}{(1-z)^3} - \frac{3}{2(1-z)^2} + \frac{1}{2(1-z)} + O\Big(\frac{1}{(1-|z|)^{3-\delta}}\Big) \\ =& \frac{1}{(1-z)^3} + O\Big(\frac{1}{(1-|z|)^{3-\delta}}\Big),\end{aligned} \tag{3.84}$$

the second equality holds because $|1-z| \geq 1-|z|$, and $3-\delta > 2$. So

$$F(z)^2 = (1-z)\sum_{n\geq 1} S(n)z^n = \frac{1}{(1-z)^2} + O\Big(\frac{|1-z|}{(1-|z|)^{3-\delta}}\Big). \tag{3.85}$$

We consider the circle $|z| = R = 1-1/N$, where N is a large positive integer, and rewrite F^2 as

$$F(z)^2 = (1-z)\sum_{n\geq 1} S(n)z^n = \frac{1}{(1-z)^2} + O\big(|1-z|N^{3-\delta}\big). \tag{3.86}$$

We introduce the kernel function $K(z)$:

$$K(z) = z^{-N-1}\frac{1-z^N}{1-z}, \quad |z| < 1, \tag{3.87}$$

then $K(z) \ll |1-z|^{-1}$ when $|z| = R$ (We use $\lim_n(1+\frac{1}{n})^n = \mathrm{e}$). Moreover, we have (by direct calculation)

$$\begin{aligned}\psi(N) =& \frac{1}{2\pi \mathrm{i}}\int_{|z|=R} F(z)K(z)\mathrm{d}z \\ =& N + \frac{1}{2\pi \mathrm{i}}\int_{|z|=R} \Big(F(z) - \frac{1}{1-z}\Big)K(z)\mathrm{d}z.\end{aligned} \tag{3.88}$$

Case 1. If $|1-z| < N^{\frac{\delta}{3}-1}$, then $|1-z|N^{3-\delta} < |(1-z)|^{-2}$, and taking square root we obtain

$$F(z) = \frac{1}{1-z} + O(|1-z|^2 N^{3-\delta}). \tag{3.89}$$

Set $\{|z| = R\} \cap \{|1-z| < N^{\frac{\delta}{3}-1}\}$ is a minor arc, we denote it as l, then

$$\int_{|z|=R} = \int_l + \int_{l^c}, \tag{3.90}$$

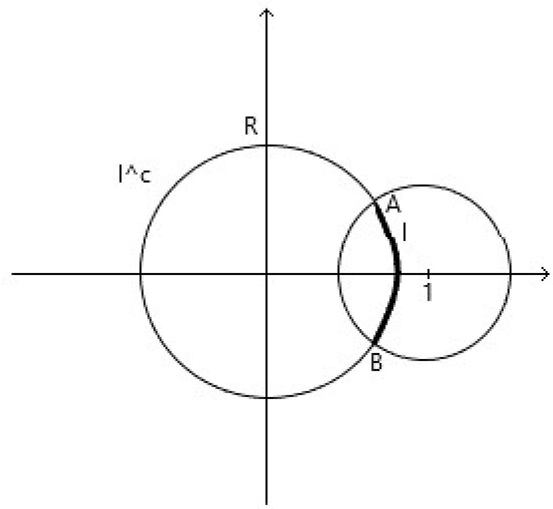

Figure 2: The minor arc and major arc

and

$$\begin{aligned}\int_l \Big(F(z) - \frac{1}{1-z}\Big)K(z)\mathrm{d}z \ll & \int_l |1-z|^2 N^{3-\delta}\big|K(z)\big|\,\mathrm{d}|z| \\ \ll & \int_l |1-z| N^{3-\delta}\mathrm{d}|z| \\ \ll & N^{2-\frac{2\delta}{3}}\, length(l) \\ \ll & N^{1-\frac{\delta}{3}},\end{aligned} \tag{3.91}$$

we use $|K(z)| \ll |1-z|^{-1}$ in the second $\ll$, and for the third $\ll$, we use $|1-z| < N^{\frac{\delta}{3}-1}$ when $z \in l$.

Case 2. If $|1-z| \geq N^{\frac{\delta}{3}-1}$, then by Cauchy-Schwarz inequality:

$$\bigg|\int_{l^c}\Big(F(z) - \frac{1}{1-z}\Big)K(z)\mathrm{d}z\bigg|^2 \leq \bigg(\int_{l^c}\Big|F(z)-\frac{1}{1-z}\Big|^2\mathrm{d}|z|\bigg)\bigg(\int_{l^c}|K(z)|^2\mathrm{d}|z|\bigg). \tag{3.92}$$

For the first integral, we recall the definition of F, and using the power series of $(1-z)^{-1}$:

$$\begin{aligned}\int_{l^c}\Big|F(z)-\frac{1}{1-z}\Big|^2\mathrm{d}|z| \leq & \int_0^{2\pi}\sum_n\sum_m(\Lambda(n)-1)(\Lambda(m)-1)R^{m+n}\mathrm{e}^{\mathrm{i}(n-m)t}\mathrm{d}t \\ = & 2\pi\sum_{n\geq 1}(\Lambda(n)-1)^2\Big(1-\frac{1}{N}\Big)^{2n} \\ = & O(N\log N).\end{aligned} \tag{3.93}$$

For the second integral, we have:

$$\int_{l^c}|K(z)|^2\mathrm{d}|z| \leq \int_{|z|=R,\,|1-z|\geq N^{3/\delta-1}}\frac{1}{|1-z|^2}\mathrm{d}|z|, \tag{3.94}$$

if we let $z = Re^{\mathrm{i}t}$ and $|1 - Re^{\mathrm{i}t_0}| = N^{\delta/3-1}$, then

$$\int_{|z|=R,\,|1-z|\geq N^{3/\delta-1}} \frac{1}{|1-z|^2}\mathrm{d}|z| = 2\int_{t_0}^{\pi} \frac{\mathrm{d}t}{|1-Re^{\mathrm{i}t}|^2}. \tag{3.95}$$

On the arc l^c, we have

$$|1 - Re^{\mathrm{i}t}|^2 = (1-R)^2 + 4R\sin^2\frac{t}{2}. \tag{3.96}$$

Recall that $R = 1 - 1/N$, so we have

$$(1-R)^2 + 4R\sin^2\frac{t}{2} < N^{-2} + 4\sin^2\frac{t}{2} < N^{-2} + t^2, \tag{3.97}$$

and

$$(1-R)^2 + 4R\sin^2\frac{t}{2} > \frac{t^2}{3}. \tag{3.98}$$

Therefore,

$$2\int_{t_0}^{\pi} \frac{\mathrm{d}t}{|1-Re^{\mathrm{i}t}|^2} \ll \int_{t_0}^{\pi} t^{-2}\mathrm{d}t < t_0^{-1} \ll N^{1-\delta/3}, \tag{3.99}$$

as a consequence, $\int_{l^c} |K(z)|^2\mathrm{d}|z| \ll N^{1-\delta/3}$.

Now we have

$$\left(\int_{l^c} \left|F(z) - \frac{1}{1-z}\right|^2 \mathrm{d}|z|\right)^{\frac{1}{2}} \left(\int_{l^c} |K(z)|^2\mathrm{d}|z|\right)^{\frac{1}{2}} \ll N^{1-\delta/6}\log^{\frac{1}{2}} N. \tag{3.100}$$

Combining the estimates $\int_l + \int_{l^c}$, we have

$$\psi(N) - N = N^{1-\delta/6}\log^{\frac{1}{2}} N. \tag{3.101}$$

We can choose $\delta' = \delta/6$ by Lemma 2.1.

4 Proof of Theorem 1.6, Part 1

4.1 Proof of the main theorem

We recall the part 1 of Theorem 1.6: Let

$$r_k(n) = \sum_{m_1+\ldots+m_k=n} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k), \tag{4.1}$$

and assuming Riemann Hypothesis, then

$$\sum_{n\leq X} r_k(n) = \frac{X^k}{k!} + O_k(X^{-1/2+k}\log^k X). \tag{4.2}$$

Similarly, we only need to consider the case $X = N \in \mathbb{N}$. We may proceed by induction in k, when $k = 2$, it is nothing but

$$\sum_{n\leq N} r_2(n) = \frac{N^2}{2} + O(N^{3/2}\log^2 N). \tag{4.3}$$

Now assuming the asymptotic formula is ture for $2, \ldots, k-1$, then for k

$$\sum_{n\leq N} r_k(n) = \sum_{m_1+\ldots+m_k\leq N} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_k), \tag{4.4}$$

if we let $m_1 + m_2 + ... + m_{k-1} = m'$, then

$$\sum_{n\le N} r_k(n) = \sum_{m_k\le N} \Lambda(m_k) \sum_{m'\le N-m_k} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_{k-1}). \tag{4.5}$$

We notice that by assumption:

$$\sum_{m'\le N-m_k} \Lambda(m_1)\Lambda(m_2)\cdots\Lambda(m_{k-1}) = \frac{(N-m_k)^{k-1}}{(k-1)!} + O_{k-1}\Big((N-m_k)^{-3/2+k}\log^{k-1}(N-m_k)\Big), \tag{4.6}$$

therefore

$$\sum_{k\le N} r_k(n) = \sum_{m_k\le N} \Lambda(m_k)\frac{(N-m_k)^{k-1}}{(k-1)!} + O_{k-1}\Big(\sum_{m_k\le N} \Lambda(m_k)(N-m_k)^{-3/2+k}\log^{k-1}(N-m_k)\Big). \tag{4.7}$$

We denote

$$\begin{aligned} I =& \sum_{i\le N} \Lambda(i)\frac{(N-i)^{k-1}}{(k-1)!}, \\ II =& O\Big(\sum_{i\le N} \Lambda(i)(N-i)^{-3/2+k}\log^{k-1}(N-i)\Big). \end{aligned} \tag{4.8}$$

Step 1: Estimate of II We deal with II firstly. In fact, we have

$$\Lambda(i) \le \log N, \text{ and } \log^{k-1}(N-i) \ll \log^{k-1} N, \tag{4.9}$$

since $i \le N$. Consequently,

$$\begin{aligned} \sum_{i\le N} \Lambda(i)(N-i)^{-3/2+k}\log^{k-1}(N-i) \ll& \sum_{i\le N} (N-i)^{-3/2+k}\log^k N \\ \le& N\cdot N^{-3/2+k}\log^k N \\ =& N^{-1/2+k}\log^k N. \end{aligned} \tag{4.10}$$

Therefore $II \ll N^{-1/2+k}\log^k N$.

Step 2: Estimate of I Let

$$\psi_j(x) = \frac{1}{j!}\sum_{m\le x} (x-m)^j \Lambda(m), \tag{4.11}$$

we need a lemma on ψ_j:

Lemma 4.1. *For $j \ge 1$, ψ_j has the asymptotic property:*

$$\psi_j(x) = \frac{x^{j+1}}{(j+1)!} - \sum_\rho \frac{x^{\rho+j}}{\rho(\rho+1)\cdots(\rho+j)} - \frac{\zeta'(0)}{\zeta(0)}\frac{x^j}{j!} + O_j(x^{j-\frac{3}{4}}). \tag{4.12}$$

We will prove the above lemma in the next subsection, assuming this lemma, let $j = k-1$ and $x = N$, we get

$$I = \frac{N^k}{k!} - \sum_\rho \frac{N^{\rho+k-1}}{\rho(\rho+1)\cdots(\rho+k-1)} - \frac{\zeta'(0)}{\zeta(0)}\frac{N^{k-1}}{(k-1)!} + O(N^{k-\frac{7}{4}}). \tag{4.13}$$

Notice that we have assumed Riemann Hypothesis is true, so $|N^{\rho+k-1}| = N^{-1/2+k}$, therefore

$$\begin{aligned} I \ll & \frac{N^k}{k!} + \sum_{\rho} \frac{N^{-1/2+k}}{\big|\rho(\rho+1)\cdots(\rho+k-1)\big|} + O(N^{k-1}) + O(N^{k-\frac{7}{4}}) \\ = & \frac{N^k}{k!} + O(N^{-1/2+k}). \end{aligned} \tag{4.14}$$

We must point out that the series $\sum_{\rho} \frac{1}{|\rho(\rho+1)\cdots(\rho+k-1)|}$ is convergent, just notice that when $\rho = \frac{1}{2}+\mathrm{i}\beta$, and $j = 1,\, 2 \ldots,\, k-1$, we have $|\rho + j| = |1/2 + j + \mathrm{i}\beta| \asymp 1/2 + j + |\beta| \asymp |\rho| + |\beta|$, theorefore

$$|\rho(\rho+1)\cdots(\rho+k-1)| \asymp |\rho| \big(|\rho|+1\big)\cdots\big(|\rho|+k-1\big) \geq |\rho|^2, \tag{4.15}$$

and finally, $\sum_{\rho} |\rho|^{-2}$ is convergent by Lemma 3.2.

Combining step 1 and 2, we get

$$\sum_{n\leq N} r_k(n) = I + II = \frac{N^k}{k!} + O_k(N^{-1/2+k}\log^k N). \tag{4.16}$$

4.2 Proof of the Lemma 4.1

Recall that $\psi_j(x) = \frac{1}{j!}\sum_{m\leq x}(x-m)^j\Lambda(m)$, we consider the simplest case ψ_1 at first, which can be expressed by the following integral:

Integral representation of ψ_1 For all $c > 1$,

$$\psi_1(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'(s)}{\zeta(s)}\Big)\mathrm{d}s, \tag{4.17}$$

here the integral is over the vertical line $\Re(s) = c$.

To prove this proposition, we need a lemma of contour integral:

Lemma 4.2. *If $c > 0$, then*

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{a^s}{s(s+1)} = \begin{cases} 0 & \text{if } 0 < a \leq 1, \\ 1 - 1/a & \text{if } a \geq 1. \end{cases} \tag{4.18}$$

Proof. We notice that $|a^s| = a^c$, so the integral converges. Firstly we assume that $a \geq 1$, and let

$$f(s) = \frac{a^s}{s(s+1)}, \tag{4.19}$$

then $\mathrm{res}_{s=0} f = 1$ and $\mathrm{res}_{s=-1} f = -1/a$. For $T > 1 + c$, consider the contour $\Gamma(T)$ shown in the Figure. The contour consists of the vertical segment $S(T)$ from $c - \mathrm{i}T$ to $c + \mathrm{i}T$, and of the left-hand half-circle $C(T)$ centered at c of radius T. Since $T > 1 + c$, 0 and -1 are contained in the interior of $\Gamma(T)$, by the residue formula:

$$\frac{1}{2\pi\mathrm{i}}\int_{\Gamma(T)} f(s)\mathrm{d}s = 1 - 1/a. \tag{4.20}$$

Since

$$\int_{\Gamma(T)} f(s)\mathrm{d}s = \int_{S(T)} f(s)\mathrm{d}s + \int_{C(T)} f(s)\mathrm{d}s, \tag{4.21}$$

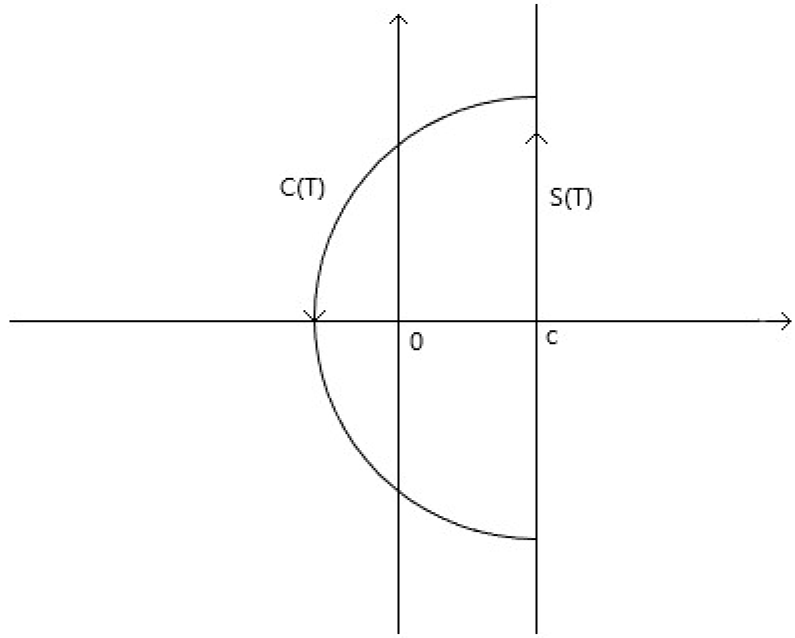

Figure 3: The contour $\Gamma(T)$

it suffices to prove that the integral over $C(T)$ tends to 0 when T goes to infinity. Notice that if $s = \sigma + \mathrm{i}t \in C(T)$, then for large T we have

$$|s(s+1)| \geq \frac{T^2}{2}, \tag{4.22}$$

moreover, $|\mathrm{e}^{\beta s}| \leq \mathrm{e}^{\beta c}$ because $\sigma \leq c$. Therefore

$$\Big| \int_{C(T)} f(s)\mathrm{d}s \Big| \leq \frac{C}{T^2} 2\pi T \to 0 \quad \text{as } T \to \infty. \tag{4.23}$$

The case $0 < a \leq 1$ is similar, we only need to change $C(T)$ to be the right-hand half-circle, and notice that there are no poles in the interior of $\Gamma(T)$. $\square$

Now we come to the proof of the integral representation of ψ_1, just recall $-\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n) n^{-s}$, and observe (with $a = x/n$)

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \mathrm{d}s =& x \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(x/n)^s}{s(s+1)} \mathrm{d}s \\ =& x \sum_{n=1}^{\infty} \Lambda(n) \Big(1 - \frac{n}{x} \Big) \\ =& \psi_1(x), \end{aligned} \tag{4.24}$$

In the first equality, we change the integral $\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}$ and the sum $\sum_{n=1}^{\infty}$, this is because $\sum_n \Lambda(n) n^{-s}$ converges to $-\frac{\zeta'(s)}{\zeta(s)}$ uniformly when $\Re(s) = c > 1$.

Now we have found an integral representation of ψ_1, furthermore, notice that for all $j \geq 1$,

$$\psi_{j+1}(x) = \int_0^x \psi_j(u)\mathrm{d}u. \tag{4.25}$$

So

$$\begin{aligned}\psi_2(x) =& \int_0^x \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{u^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \mathrm{d}s\, \mathrm{d}u \\ =& \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \int_0^x u^{s+1}\mathrm{d}u\, \mathrm{d}s \\ =& \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{s+2}}{s(s+1)(s+2)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \mathrm{d}s.\end{aligned} \tag{4.26}$$

Here we change two integrals $\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}$ and $\int_0^x$. In fact, when $\Re(s) = c$, $|-\zeta'(s)/\zeta(s)| \leq \sum_n \frac{\Lambda(n)}{n^c} := C$ is uniformly bounded in s, and one can verify easily that

$$\begin{aligned}\left| \int_0^x \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{u^{s+1}}{s(s+1)} \Big(-\frac{\zeta'(s)}{\zeta(s)} \Big) \mathrm{d}s\, \mathrm{d}u \right| \ll& C \int_0^x \int_0^\infty \frac{u^{c+1}}{(t+c)\,(t+c+1)} \mathrm{d}t\, \mathrm{d}u \\ \ll& \int_0^x u^{c+1} \int_0^\infty \frac{1}{1+t^2} \mathrm{d}t\, \mathrm{d}u < \infty.\end{aligned} \tag{4.27}$$

Therefore, by Fubini's theorem, two integrals $\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \int_0^x$, $\int_0^x \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}$ exist and are the same.

By induction in j, we obtain

$$\psi_j(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)} \Big(-\frac{\zeta'(\omega)}{\zeta(\omega)} \Big) \mathrm{d}\omega, \tag{4.28}$$

where $c > 1$ is fixed.

Similar to the proof of Lemma 4.2, we choose a rectangle $Rct(T)$, with the vertices $c \pm \mathrm{i}T$ and $-3/4 \pm \mathrm{i}T$. However, we may need some conditions on T. Notice that by Lemma 2.2:

$$N(T+1) - N(T-1) \ll \log T. \tag{4.29}$$

If $|\Im(\rho) - T| \geq 1$, then $|\Im(\rho) - T| \geq 1 \gg \log^{-1} T$. Conversely, if $|\Im(\rho) - T| \leq 1$, then there are at most $O(\log T)$ such zeros. So among the imaginary parts $\Im(\rho)$ of these zeros, there must be a gap of length $\gg \log^{-1} T$. Hence by varying T by a bounded amount less than 1, we can assume that

$$|\Im(\rho) - T| \gg \log^{-1} T, \quad \text{for all } \rho. \tag{4.30}$$

The singularities of $\omega \mapsto -\frac{x^{\omega+j}\zeta'(\omega)}{\omega(\omega+1)\cdots(\omega+j)\zeta(\omega)}$ which lie in the interior of $Rct(T)$ are 0, 1 and non-trivial zeros ρ with $|\Im(\rho)| < T$. We have the following cases:

Case 1: When $\omega = \rho$, we denote $\mathrm{ord}(\rho)$ be the order of ρ. Then near ρ, ζ can be written as $\zeta(\omega) = (\omega - \rho)^{\mathrm{ord}(\rho)} h(\omega)$, where $h(\omega)$ is holomorphic and never vanishes near ρ, therefore

$$\frac{\zeta'(\omega)}{\zeta(\omega)} = \frac{\mathrm{ord}(\rho)}{\omega - \rho} + \frac{h'(\omega)}{h(\omega)}, \quad \text{near } \rho, \tag{4.31}$$

on the other hand $\omega \mapsto -\frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)}$ is analytic at ρ, therefore

$$\mathrm{res}_{\omega=\rho} \Big(\omega \mapsto -\frac{x^{\omega+j}\zeta'(\omega)}{\omega(\omega+1)\cdots(\omega+j)\zeta(\omega)} \Big) = -\frac{x^{\rho+j}\mathrm{ord}(\rho)}{\rho(\rho+1)\cdots(\rho+j)}. \tag{4.32}$$

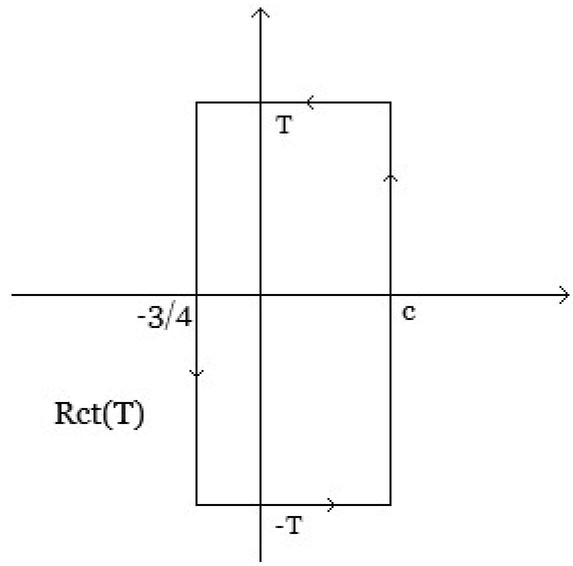

Figure 4: The contour $Rct(T)$

Case 2: When $\omega = 0$, $\zeta'(\omega)/\zeta(\omega)$ is analytic at 0, and $\omega \mapsto -\frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)}$ has a simple pole at 0, therefore

$$\operatorname{res}_{\omega=0}\left(\omega \mapsto -\frac{x^{\omega+j}\zeta'(\omega)}{\omega(\omega+1)\cdots(\omega+j)\zeta(\omega)}\right) = -\frac{\zeta'(0)}{\zeta(0)}\frac{x^j}{j!}. \tag{4.33}$$

Case 3: When $\omega = 1$, $\zeta(\omega)$ has a simple pole at 1. Like case 1, we write $\zeta(\omega) = \frac{g(\omega)}{\omega-1}$, and obtain

$$\operatorname{res}_{\omega=1}\left(\omega \mapsto -\frac{x^{\omega+j}\zeta'(\omega)}{\omega(\omega+1)\cdots(\omega+j)\zeta(\omega)}\right) = \frac{x^{j+1}}{(j+1)!}. \tag{4.34}$$

Combining these cases and using the residue theorem, we get

$$\frac{1}{2\pi\mathrm{i}}\int_{Rct(T)} \frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)}\left(-\frac{\zeta'(\omega)}{\zeta(\omega)}\right)\mathrm{d}\omega = \frac{x^{j+1}}{(j+1)!} - \sum_{\rho,\, |\Im(\rho)|<T} \frac{x^{\rho+j}}{\rho(\rho+1)\cdots(\rho+j)} - \frac{\zeta'(0)}{\zeta(0)}\frac{x^j}{j!}, \tag{4.35}$$

here the sum $\sum_{\rho,\, \Im(\rho)<T}$ takes multiples into account, so $\operatorname{ord}(\rho)$ doesn't appear in the sum.

We write the contour integral as

$$\int_{Rct(T)} = \int_{c-\mathrm{i}T}^{c+\mathrm{i}T} + \int_{c+\mathrm{i}T}^{-3/4+\mathrm{i}T} + \int_{-3/4+\mathrm{i}T}^{-3/4-\mathrm{i}T} + \int_{-3/4-\mathrm{i}T}^{c-\mathrm{i}T}, \tag{4.36}$$

and denote $I_1 = \int_{c+\mathrm{i}T}^{-3/4+\mathrm{i}T}$, $I_2 = \int_{-3/4-\mathrm{i}T}^{c-\mathrm{i}T}$, now we prove I_1, I_2 goes to 0 as T tends to infinity. We need two estimates. (Remember we have assumed that $|\Im(\rho) - T| \gg \log^{-1} T$ for all ρ.)

Estimate 1: If $\omega = \sigma \pm \mathrm{i}T$ with $\sigma \in [-1, 2]$, we have:

$$\omega(\omega+1)\cdots(\omega+j) \gg T^{j+1}. \tag{4.37}$$

Estimate 2: If $\omega = \sigma \pm \mathrm{i}T$ with $\sigma \in [-1, 2]$, then

$$\frac{\zeta'(\omega)}{\zeta(\omega)} \ll \log^2 T. \tag{4.38}$$

Estimate 1 is almost obvious so we omit it, for estimate 2, see [12], Chapter 12, (12.20). Combining these two estimates, we obtain

$$\begin{aligned} I_1 =& \int_{c+\mathrm{i}T}^{-3/4+\mathrm{i}T} \frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)} \Big(-\frac{\zeta'(\omega)}{\zeta(\omega)} \Big) \mathrm{d}\omega \\ \ll& \int_{-3/4}^{c} \frac{x^{\sigma+j}}{T^{j+1}} \log^2 T \mathrm{d}\sigma \\ =& O\Big(\frac{\log^2 T}{T^{j+1}} \Big). \end{aligned} \tag{4.39}$$

Similarly, after change T to $-T$, we also have $I_2 = O\Big(\frac{\log^2 T}{T^{j+1}}\Big)$.

Now we have

$$\frac{1}{2\pi \mathrm{i}} \int_{Rct(T)} = \frac{1}{2\pi \mathrm{i}} \Big(\int_{c-\mathrm{i}T}^{c+\mathrm{i}T} + \int_{-3/4+\mathrm{i}T}^{-3/4-\mathrm{i}T} \Big) + O\Big(\frac{\log^2 T}{T^{j+1}} \Big), \tag{4.40}$$

and

$$\frac{1}{2\pi \mathrm{i}} \int_{Rct(T)} = \frac{x^{j+1}}{(j+1)!} - \sum_{\rho,\, |\Im(\rho)| < T} \frac{x^{\rho+j}}{\rho(\rho+1)\cdots(\rho+j)} - \frac{\zeta'(0)}{\zeta(0)} \frac{x^j}{j!}. \tag{4.41}$$

Let $T \to \infty$ and recall $\psi_j = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}$, we get

$$\begin{aligned} \psi_j(x) =& \frac{x^{j+1}}{(j+1)!} - \sum_{\rho} \frac{x^{\rho+j}}{\rho(\rho+1)\cdots(\rho+j)} - \frac{\zeta'}{\zeta}(0) \frac{x^j}{j!} \\ & - \frac{1}{2\pi \mathrm{i}} \int_{-3/4-\mathrm{i}\infty}^{-3/4+\mathrm{i}\infty} \frac{x^{\omega+j}}{\omega(\omega+1)\cdots(\omega+j)} \Big(-\frac{\zeta'(\omega)}{\zeta(\omega)} \Big) \mathrm{d}\omega. \end{aligned} \tag{4.42}$$

To handle the integral $\int_{-3/4-\mathrm{i}\infty}^{-3/4+\mathrm{i}\infty}$, we need an estimate of ζ'/ζ on the line $\Re(s) = -3/4$.

Estimate 3: Let $\omega = -3/4 + \mathrm{i}t$, then for all $t \in \mathbb{R}$, we have

$$\frac{\zeta'(\omega)}{\zeta(\omega)} \ll \log(|t| + 2). \tag{4.43}$$

Estimate 4: Let $\omega = -3/4 + \mathrm{i}t$, then for all $t \in \mathbb{R}$, we have

$$\omega(\omega+1)\cdots(\omega+j) \gg (|t|+2)^{j+1}. \tag{4.44}$$

For estimate 3, see [1], Chapter 13. Estimate 4 is almost obvious so we omit the proof.

Using these estimates, we obtain

$$\int_{-3/4-\mathrm{i}\infty}^{-3/4+\mathrm{i}\infty} \ll x^{j-3/4} \int_{-\infty}^{\infty} \frac{\log(|t|+2)}{(|t|+2)^{j+1}} \mathrm{d}t = O_j(x^{j-3/4}), \tag{4.45}$$

and finally,

$$\psi_j(x) = \frac{x^{j+1}}{(j+1)!} - \sum_{\rho} \frac{x^{\rho+j}}{\rho(\rho+1)\cdots(\rho+j)} - \frac{\zeta'(0)}{\zeta(0)} \frac{x^j}{j!} + O_j(x^{j-\frac{3}{4}}). \tag{4.46}$$

References

[1] H, Davenport (2000). *Multiplicative Number Theory (3rd. ed)*. Berlin: Springer.

[2] G, Bhowmik; I, Z, Ruzsa (2018). *Average Goldbach and the Quasi-Riemann Hypothesis*, Analysis Mathematica, **44** (2018), 51-56.

[3] G, Bhowmik; J, -C, Schlage-Puchta (2010). *Mean representation number of integers as the sum of primes*, Nagoya. Math. J, **200** (2010), 27-33.

[4] G, Bhowmik; K, Halupczok; K, Matsumoto; Y, Suzuki (2019). *Goldbach Representation inn Arithmetic Progressions and zeros of Dirichlet L-functions*, Mathematika, **65** (2019), 57-97.

[5] J, B, Friedlander; D, A, Goldston (1997). *Sums of Three or More Primes*, Trans. Amer. Math. Soc. **349** (1997), 287-310.

[6] A, Fujii (1991). *An additive problem of prime numbers*, Acta. Arith **58** (1991), 173-179.

[7] A, Fujii (1991). *An additive problem of prime numbers II*, Proc. Japan Acad. Ser. A Math. Sci, **67** (1991), 248-252

[8] A, Fujii (1991). *An additive problem of prime numbers III*, Proc. Japan Acad. Ser. A Math. Sci, **67** (1991), 278-283.

[9] P, X, Gallagher (1989). *A double sum over primes and zeros of the zeta function*, Number Theory, Trace Formula and Discrete Groups (1989): 229-240.

[10] D, A, Goldston; H, L, Montgomery (1987). *Pair correlation of zeros and primes in short intervals*, Analytic Number Theory and Diophantine Problems (1987), 183-203.

[11] S, M, Gonek (1993). *An explicit formula of Landau and its applications to the theory of the zeta function*, Contemp. Math **143** (1993), 395-413.

[12] A, Ivić (1985). *The Riemann zeta function: Theory and Applications*. Mineola, New York: Dover Publications, INC.

[13] E, Landau (1911). *Über die Nullstellen der Zetafunction*, Math. Ann. **71** (1911), 548-564.

[14] A, Languasco; A, Zaccagnini (2012). *Sums of many primes*, Journal of Number Theory **132** (2012), 1265-1283.

[15] A, Languasco; A, Zaccagnini (2012). *The number of Goldbach Representations of an integer*, Proc. Am. Math. Soc **140** (2012), 795-804.

[16] A, Martin; Z, Günter (2009). *Proof from THE BOOK (4th. ed)*. Berlin, New York: Springer-Verlag.

[17] H, L, Montgomery; R, C, Vaughan (2006). *Multiplicative Number Theory. Vol I*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

[18] K, Prachar (1957). *Primzahlverteilung*. Berlin, Göttingen, Heidelberg: Springer-Verlag.

[19] B, Saffari; R, C, Vaughan (1977). *On the fractional parts of x/n and related sequences*, Ann. Inst. Fourier(Grenoble) **27** (2) (1977), 1-30.

[20] E, Stein; R, Shakarchi (2003). *Complex Analysis*. Princeton University Press.

[21] G, Tenenbaum (2015). *Introduction to Analytic and Probabilistic Number Theory (3rd. ed)*. Graduate Studies in Mathematics, Volume 163. American Mathematical Society.

[22] E, C, Titchmarsh; D, R, Heath-Brown (1986). *The Theory of the Riemann zeta-function (revised 2nd. ed)*. Oxford: Clarendon Press.