GHENT UNIVERSITY

Faculty of Sciences

Department of Mathematics: Analysis, Logic and Discrete Mathematics

# Contributions to analytic number theory: Zero-density estimates for $L$-functions, almost-prime $k$-tuples, and Tauberian theorems

Bin Chen

Supervisor: Prof. Dr. Jasson Vindas
Co-supervisor: Dr. Gregory Debruyne

Dissertation submitted in fulfillment of the requirements for the degree of Doctor in Science: Mathematics

Academic Year 2023 – 2024

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# Contents

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# Preface

*“Throughout history, mathematicians have been tirelessly seeking patterns in prime numbers, but success has been elusive. We can consider prime numbers as a mystery that eludes human understanding.”*

—Leonhard Euler 1707-1783

This dissertation delves into three pivotal avenues within analytic number theory: multiplicative number theory, sieve theory, and Tauberian theory. The advancements on these subjects, as obtained in this text, are built upon three articles, [10] due to myself, the joint article [11] with G. Debruyne and J. Vindas, and the collaborative preprint [12] with J. Vindas.

The techniques of multiplicative number theory crucially allow one to understand the distribution of primes via the zeros of the corresponding $L$-functions. This ingenious idea traces its origins back to Riemann's celebrated work [70] in 1859, marking the inception of analytic number theory. To prove the prime number theorem, Riemann introduced the complex variable zeta function $\zeta(s)$, which is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re e\ s > 1.$$

The multiplicative nature of the Riemann zeta function is evident through the Euler product:

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad \Re e\ s > 1,$$

where the product runs over all primes $p$. This expression can be interpreted as the analytical manifestation of the fundamental theorem of arithmetic. Moreover, $\zeta(s)$ can be analytically extended to the whole complex plane except for a simple pole at $s = 1$. Until now, the Riemann zeta function remains one of the most important subjects in analytic number theory, primarily due to the renowned Riemann hypothesis (RH). According to this conjecture, all nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re e\ s = \frac{1}{2}$. This property has the profound implication of yielding a *square root cancellation* in the remainder estimate of the Prime Number Theorem. Specifically, under the RH, for any $\varepsilon > 0$, we have[1]

$$\sum_{p\le x} 1 = \int_2^x \frac{\mathrm{d}t}{\log t} + O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}\right).$$

It is worth noting that this estimate is known to be best possible in the sense that the exponent $\frac{1}{2}$ cannot be replaced by a smaller number. In essence, the Riemann Hypothesis suggests that the distribution of prime numbers in our universe is "perfect".

Due to this fascinating conjecture, many mathematicians have explored the zeros of $\zeta(s)$ from various perspectives. One such direction involves establishing estimations of $N(\sigma, T)$, where $N(\sigma, T)$ denotes the number of zeros $\rho = \beta + it$ of the Riemann zeta function $\zeta(s)$ in the rectangle $\sigma \le \beta \le 1$, $|t| \le T$. Indeed, the Riemann Hypothesis is equivalent to the statement that $N(\sigma, T) = 0$ holds for all $\sigma > \frac{1}{2}$ and $T \ge 0$.

Chapter 1 encompasses two new results on zero-density estimates for $L$-functions. The first one is the upper bound

$$N(\sigma, T) \ll_\varepsilon T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon},$$

valid for $279/314 \le \sigma \le 17/18$. This improves on the range $155/174 \le \sigma \le 17/18$ obtained by Ivić [46] in 1980. One of the main ingredients in our proof is Bourgain's dichotomy approach. The dichotomy technique, pioneered by Bourgain [5] in 2000, was initially employed to establish the current record for the range of validity of the density hypothesis for the Riemann zeta function.

As it turns out, many arithmetic results that are derivable from the Riemann hypothesis can also be achieved from an inequality of the form

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon},$$

and this formula has become known as the density hypothesis. During my doctoral research, I integrated Bourgain's dichotomy approach with other classical tools in multiplicative number theory to explore the density hypothesis for $L$-functions associated with holomorphic cusp forms. The other main result in Chapter 1 establishes that $N_f(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}$ holds for $\sigma \ge 1407/1601$, where $N_f(\sigma, T)$ denotes the number of zeros $\rho = \beta + it$ of $L(f, s)$ in the rectangle $\sigma \le \beta \le 1$, $|t| \le T$ and $L(f, s)$ is associated with a cusp Hecke eigenform $f$ of even integral weight. This represents an improvement over a result by Ivić

[1] The notation $f(x) = O_\varepsilon(g(x))$ means that $|f(x)| \le C(\varepsilon)g(x)$ for some constant $C(\varepsilon)$ depending on $\varepsilon$.

[48] from 1989, who had previously established the zero-density estimate in the narrower range $\sigma \geq 53/60$.

Dirichlet $L$-functions play a central role in studying the distribution of primes in arithmetic progressions, so it is natural to investigate the density hypothesis for them. The best known result in this regard was established by Heath-Brown [33] in 1979; he showed

$$\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll_{\varepsilon} (qT)^{2(1-\sigma)+\varepsilon}$$

holds for $\sigma \geq 15/19$, where $N(\sigma, T, \chi)$ represents the number of zeros $\rho = \beta + it$ of $L(s, \chi)$ in the rectangle $\sigma \leq \beta \leq 1$, $|t| \leq T$. In Chapter 2, we will utilize Bourgain's dichotomy to obtain[2] a large value estimate for Dirichlet polynomials $\sum_{N<n\leq 2N} b(n)\chi(n)n^{-it}$ with bounded coefficients $b(n) \leq 1$ and large $N$. Consequently, we will provide an alternative proof of Heath-Brown's result.

In Chapter 3, we study the prime $k$-tuples conjecture which is a significant open problems in prime number theory. It can be viewed as a higher-dimensional version of the twin prime conjecture. This conjecture involves an admissible set $\mathcal{H} = \{h_1, ..., h_k\}$ comprising distinct non-negative integers. Specifically, for every prime $p$, the number of distinct residue classes modulo $p$ occupied by $h_i$ is less than $p$. The prime $k$-tuples conjecture states that there are infinitely many integers $n$ for which all $n + h_i$ are prime. An approach to approximate the prime $k$-tuples conjecture involves establishing an upper bound for the expression

$$\sum_{i=1}^{k} \tau(n + h_i),$$

where $\tau$ stands for the divisor function. The prime $k$-tuples conjecture follows if one has the upper bound $2k$ for infinitely many $n$. In 1997, Heath-Brown [35] was the first to establish an upper bound $\left(\frac{3}{2} + o(1)\right) k^2$ for infinitely many $n$ as $k \to \infty$. Subsequently, Heath-Brown's result was further improved to $(1 + o(1))\, k^2$ by Ho and Tsang [37] in 2016 and to $\left(\frac{3}{4} + o(1)\right) k^2$ by M. Ram Murty and Akshaa Vatwani [64] in 2017. The primary objective of Chapter 3 is to obtain the current best result $\left(\frac{2126}{2853} + o(1)\right) k^2$.

Our work builds upon numerous novel ideas and techniques stemming from recent breakthroughs in sieve theory. Specifically, the multidimensional sieve weight employed is attributed to Maynard [60], Tao [65], Murty and Vatwani [64]. The strategy of utilizing a smoothed version of the higher rank sieve is

[2] The large value estimate, obtained in autumn 2022, is the same as Bourgain's result [6] in 2002, which the author was initially unaware of until receiving Bourgain's paper from Diamond in the spring of 2023.

inspired by Zhang's work on bounded gaps between primes [85]. The concept of smoothing non-smooth test functions is due to Li and Pan's contributions [59]. The estimation of the divisor function in arithmetic progressions to smooth moduli is based on the recent results of Irving [45], Wu, and Xi [81]. By amalgamating these ideas and tools, our problem can be reduced to the following variational problem:

Let $F : [0, \infty)^k \to \mathbb{R}$ be a smooth function supported on the truncated simplex

$$\Delta_k^{[\kappa]}(1) := \{(t_1, \cdots, t_k) \in [0, \kappa]^k : t_1 + \cdots + t_k \le 1\}, \quad \text{for some } \kappa > 0.$$

We are looking for some function $F$ to minimize the quantity $\sum_{m=1}^{k} \frac{\alpha^{(m)}(F)}{I(F)}$, where

$$\alpha^{(m)}(F) = \int_{[0,\infty)^k} t_1 \left( \frac{\partial f(t_1, \cdots, t_k)}{\partial t_m} \right)^2 \mathrm{d}t_1 \cdots \mathrm{d}t_k,$$

$$I(F) = \int_{[0,\infty)^k} (f(t_1, \cdots, t_k))^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_k,$$

and

$$f(t_1, \cdots, t_k) = \frac{\partial^k F(t_1, \cdots, t_k)}{\partial t_1 \cdots \partial t_k}.$$

My contribution to this variational problem involves constructing a sequence of functions $\{F_k\}$ that has not appeared in this context before. This construction leads to

$$\sum_{m=1}^{k} \alpha^{(m)}(F_k)/I(F_k) \to (1/4 + o(1))k^2,$$

as $k \to \infty$. This result is already sufficient for us to obtain the improved arithmetic result.

The final part of our thesis is related to recent advances in Tauberian theory. The aim of Tauberian theorems is to extract asymptotic information for certain objects, such as functions, series, and sequences, from their integral transforms. Nowadays, Tauberian proofs of the Prime Number Theorem are considered to be one of the shortest and most elegant methods available in the literature. Moreover, Tauberian theory has found countless applications in diverse areas of mathematics such as operator theory, partial differential equations, and number theory [19, 55, 75]. The Wiener-Ikehara theorem [80] is a foundational result in complex Tauberian theory. In Chapter 4, we show new versions of the Wiener-Ikehara theorem where only boundary assumptions on the real part of the Laplace transform are imposed. Our results generalize and improve a recent theorem of T. Koga [53]. As an application, we shall give a quick Tauberian proof of Blackwell's renewal theorem in probability theory.

# Chapter 1

# On the density hypothesis for $L$-functions associated with holomorphic cusp forms

*“If you have a question which is generally perceived as unapproachable, it is often you do not even quite know where you have to look to get a solution. From that point of view, we are rather like for you stranded in the desert, hopelessly lost. At the moment you get this inside, all of a sudden you will escape the desert and things open up for you. Then we feel very excited. These are the best moments.”*

—Jean Bourgain 1954-2018

We study in this chapter the range of validity of the density hypothesis for the zeros of $L$-functions associated with cusp Hecke eigenforms $f$ of even integral weight and prove that $N_f(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}$ holds for $\sigma \geq 1407/1601$. This improves upon a result of Ivić, who had previously shown the zero-density estimate in the narrower range $\sigma \geq 53/60$. Our result relies on an improvement of the large value estimates for Dirichlet polynomials based on mixed moment estimates for the Riemann zeta function. The main ingredients in our proof are the Halász-Montgomery inequality, Ivić’s mixed moment bounds for the zeta function, Huxley’s subdivision argument, Bourgain’s dichotomy, and Heath-Brown’s bound for double zeta sums.

## 1.1 Introduction

Zero-density estimates for the Riemann zeta function and $L$-functions play a central role in analytic number theory. They have important arithmetic consequences; see for instance [47, Chap. 12] and [62, Chap. 15] for an overview of applications in prime number theory.

Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + it$ of the Riemann zeta function $\zeta(s)$ in the rectangle $\sigma \leq \beta \leq 1$, $|t| \leq T$. In 1937, Ingham [44] connected estimates of the form $(c, D > 0)$

$$N(\sigma, T) \ll T^{c(1-\sigma)} \log^D T, \qquad \text{uniformly for } \frac{1}{2} \leq \sigma \leq 1, \tag{1.1.1}$$

with the behavior of primes in short intervals. In fact, one can prove that (1.1.1) implies the PNT in the form $\psi(x+h) - \psi(x) = h(1+o(1))$ for $h \gg x^{1+\varepsilon-1/c}$ as $x \to \infty$. Note that the estimate (1.1.1) with $c = 2$ essentially provides the same result as the Riemann hypothesis. As this turns out to be the case for many other arithmetic results that are also obtainable from the Lindelöf hypothesis or the Riemann hypothesis, an inequality of the sort

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}, \tag{1.1.2}$$

has become known as the *density hypothesis*. While a proof that the density hypothesis holds uniformly for $1/2 \leq \sigma \leq 1$ seems to be out of reach by present methods, there has been substantial progress towards maximizing its range of validity. Montgomery showed [61] that the density hypothesis (1.1.2) holds in the range $\sigma \geq 9/10$. The range of validity was subsequently improved (cf. [40, 67, 27, 42, 50]), and the current record is due to Bourgain [5], who showed that (1.1.2) is valid for $\sigma \geq 25/32 = 0.78125$.

It is also natural to consider zero-density estimates for Dirichlet $L$-functions [2, 33, 34, 41, 43, 49, 50] and for $L$-functions associated with modular forms [39, 48, 51, 68, 83]. We are interested in studying the density hypothesis for the latter case. So, let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$ be a holomorphic cusp form of even integral weight $\kappa$ for the full modular group $\mathrm{SL}(2, \mathbb{Z})$ [1]. We assume that $f$ is a Hecke eigenform [18], and that it is normalized, i.e. $a_f(1) = 1$. We set $\lambda_f(n) = a_f(n) n^{-\frac{\kappa-1}{2}}$ and notice that this multiplicative function satisfies $|\lambda_f(n)| \leq \tau(n)$, where $\tau(n)$ is the divisor function, an inequality that was shown by Deligne [17, Thm. 8.2, p. 302] as a consequence of his proof of Weil's conjectures. The $L$-function $L(s, f)$ associated with $f$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \lambda_f(p) p^{-s} + p^{-2s}\right)^{-1}, \quad \Re(s) > 1.$$

A classical result of Hecke [36] establishes that $L(s,f)$ extends to the whole complex plane as an entire function of $s$.

Denote by $N_f(\sigma,T)$ the number of zeros $\rho=\beta+it$ of $L(f,s)$ in the rectangle $\sigma\le\beta\le 1$, $|t|\le T$. In 1989, Ivić [48] showed that $N_f(\sigma,T)\ll T^{2(1-\sigma)+\varepsilon}$ holds for $\sigma\ge 53/60$. The successful establishment of the density hypothesis in the range $\sigma\ge 1/2$ should be one of the key ingredients for obtaining estimates for the asymptotic distribution of $\lambda_f(p)$ for primes in short intervals that are as good as if one were to assume the grand Riemann hypothesis.

The main goal of this chapter is to improve Ivić's result by showing:

**Theorem 1.1.1.** *We have the bound*

$$N_f(\sigma,T)\ll_{f,\varepsilon} T^{2(1-\sigma)+\varepsilon}$$

*for* $\sigma\ge 1407/1601$.

Here $1407/1601\approx 0.8788$, while $53/60=0.8833\ldots$. We now describe the general strategy for our proof of Theorem 1.1.1. The first step, which is standard, is to apply the zero-detection method to divide the zeros of $L(s,f)$ into two categories, the so-called class-I zeros and class-II zeros. The number of class-II zeros is directly estimated by using Good's second moment estimate for $L$-functions associated with holomorphic cusp forms [30]. The innovation of our work is to achieve sharper estimates for the class-I zeros than those obtained by Ivić in [48].

We seek to obtain an upper bound for the class-I zeros by applying the Halász-Montgomery inequality. Ivić's argument is to then directly plug in the mixed moment bounds for the zeta function and to use Huxley's subdivision technique, whence his result is essentially deduced (cf. Remark 1.4.2). Our improvement is based on two aspects. The first one is an optimization of the parameters in the mixed moment estimates. Here we rely on the newly established *exponent pair* $(13/84+\varepsilon, 55/84+\varepsilon)$ due to Bourgain [7]. The second aspect that leads to an additional improvement is the incorporation of a dichotomy technique developed by Bourgain in [5] to achieve the current record of the range of validity of the density hypothesis for the Riemann zeta function. The crucial point is that this technology allows us to apply Heath-Brown's estimate on double zeta sums [32] which is more efficient than the mixed moment bounds in certain ranges. In Bourgain's original paper the dichotomy approach is a bit difficult to follow; one of our goals is to explain the underlying ideas and its advantages more clearly.

In addition to improving the range of validity of the density hypothesis for the zeros of the $L$-functions associated with holomorphic cusp forms, our argument can, with only mild adjustments, also be applied to obtain a zero-density estimate for the Riemann zeta function. In order to further demonstrate the strength of the dichotomy method we prove:

**Theorem 1.1.2.** *There holds*

$$N(\sigma, T) \ll_\varepsilon T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}$$

*for* $279/314 \leq \sigma \leq 17/18$.

This improves on the condition $155/174 \leq \sigma \leq 17/18$ obtained by Ivić ([46]; [47, Thm. 11.2]) in 1980. Observe that $155/174 \approx 0.8908$ and $279/314 \approx 0.8885$.

The chapter is organized as follows. In Section 1.2, as preliminaries, we introduce Good's mean value theorem and Ivić's mixed moment bounds for the Riemann zeta function. In Section 1.3 we recall the classical zero-detection method to divide the zeros into class-I zeros and class-II zeros and explain how to handle the class-II zeros. In Section 1.4 we revisit Ivić's original argument involving mixed moment estimates. In Section 1.5 we study Bourgain's dichotomy in this context; we derive a large value estimate for Dirichlet polynomials from which Theorem 1.1.1 follows. Finally, the proof of Theorem 1.1.2 will be completed in Section 1.6.

We adopt the convention that $\varepsilon$ stands for a small positive quantity. Throughout this chapter, we allow $\varepsilon$ to change by at most a constant factor on places that we do not always specify. We let $\mathbf{1}_E$ denote the indicator function of a set $E$. We use $\ll$ and $\gg$ to denote Vinogradov's notation, while implied constants depend at most on $\varepsilon$ and the cusp form $f$.

## 1.2 Preliminaries

In the section, we introduce Good's second moment estimate for $L$-functions associated with holomorphic cusp forms, recall the definition of an exponent pair, and present Ivić's mixed moment bounds for the Riemann zeta function. These results and concepts play central roles in the subsequent discussion.

### 1.2.1 Good's mean value theorem

We start by stating Good's second moment estimate.

**Theorem 1.2.1** (Good [30]). *Let $L(s, f)$ be the $L$-function associated with a cusp Hecke eigenform $f$ of even integral weight. Then*

$$\int_{-T}^{T} |L(1/2 + it, f)|^2 \, \mathrm{d}t \ll_f T \log T.$$

We now establish the discrete version of Good's mean value theorem.

**Theorem 1.2.2.** *Let $L(s,f)$ be the L-function associated with a cusp Hecke eigenform $f$ of even integral weight. If $|t_r| \le T$, $1 \le r \le R$, and satisfy $|t_r - t_{r'}| \ge 2\log^2 T$ for $r \ne r'$, then*

$$\sum_{r=1}^{R} |L(1/2 + it_r, f)|^2 \ll_f T(\log T)^2.$$

*Proof.* We derive the above sum version from Theorem 1.2.2 along the same lines as it is done for the Riemann zeta function (cf. [47, p. 200 and Lemma 7.1]). Let $s' = \frac{1}{2} + c + it$ and $c = 1/(\log T)$. By employing the Mellin inversion formula for $e^{-n}$, we have

$$(2\pi i)^{-1} \int_{1-i\infty}^{1+i\infty} L^2(s'+w, f)\Gamma(w)\,\mathrm{d}w = \sum_{n=1}^{\infty} \lambda_{f,2}(n) e^{-n} n^{-s'} \ll 1, \qquad (1.2.1)$$

where $\lambda_{f,2}(n) = \sum_{n_1 n_2 = n} = \lambda_f(n_1)\lambda_f(n_2)$. Moving the line of integration in (1.2.1) to $\Re e\, w = -c$ we encounter poles at $w = 0$ with residue $L^2(s', f)$. Recall the classical estimate for $\Gamma(w)$ (cf. [47, Eq. (A. 34)])

$$|\Gamma(\sigma + it)| \sim e^{-\frac{\pi}{2}|t|} |t|^{\sigma - \frac{1}{2}} (2\pi)^{\frac{1}{2}}, \qquad (1.2.2)$$

as $|t| \to \infty$, uniformly in $-\infty < \sigma_1 < \sigma < \sigma_2 < +\infty$. So we have for any real $v$

$$\Gamma(\pm c + iv) \ll e^{-|v|}(c + |v|)^{-1}.$$

Combining this with (1.2.1) we obtain, for $T/3 \le |t| \le 3T$,

$$L^2(s', f) \ll 1 + \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it + iv, f\right)\right|^2 e^{-|v|}(c + |v|)^{-1}\,\mathrm{d}v. \qquad (1.2.3)$$

Note that $L(s,f)$ satisfies the functional equation [36]

$$(2\pi)^{-s}\Gamma\left(s + \frac{k-1}{2}\right) L(s,f) = (-1)^{k/2}(2\pi)^{s-1}\Gamma\left(1 - s + \frac{k-1}{2}\right) L(1-s,f).$$

Together with (1.2.2), thus yields

$$|L(s,f)| \ll |L(1-s,f)||t|^{1-2\sigma},$$

as $|t| \to \infty$, where the implied constant depends on $f$. We therefore have

$$\left|L\left(\frac{1}{2} - c + it\right)\right| \ll \left|L\left(\frac{1}{2} + c + it\right)\right| T^c \ll \left|L\left(\frac{1}{2} + c + it\right)\right|,$$

so that (1.2.3) remains true if $s' = \frac{1}{2} - c + it$. On the other hand by the residue theorem we have, for $s = \frac{1}{2} + it$,

$$L^2(s,f) = (2\pi i)^{-1} \int_{\mathcal{D}} L^2(s+z,f)\Gamma(z)\,\mathrm{d}z,$$

where $\mathcal{D}$ is the rectangle with vertices $\pm c \pm i \log^2 T$. Using [47, Eq. (A. 34)] again, one can find that the integrals over horizontal sides of $\mathcal{D}$ are $o(1)$ as $T \to \infty$. Applying (1.2.3) with $s' = \frac{1}{2} \pm c \pm i(t+u)$, $|u| \le \log^2 T$, we obtain that for $T/2 \le |t| \le T$

$$|L(s,f)|^2 \ll 1 + \int_{-\log^2 T}^{\log^2 T} e^{-u} \left(1 + \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it + iu + iv, f\right)\right|^2 \cdot e^{-|v|}(c+|v|)^{-1}\,\mathrm{d}v\right)(c+|u|)^{-1}\,\mathrm{d}u.$$

To estimate the above expression first note that trivially

$$\int_{-\log^2 T}^{\log^2 T} e^{-u}(c+|u|)^{-1}\,\mathrm{d}u \ll c^{-1} = \log T,$$

and in the remaining integral we make the substitution $v = x - u$ and exchange the order of integration. This gives for $T/2 \le |t| \le T$

$$|L(s,f)|^2 \ll \log T + \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it + ix, f\right)\right|^2 \cdot \left(\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|x-u|)^{-1}(c+|u|)^{-1}\,\mathrm{d}u\right)\mathrm{d}x. \qquad (1.2.4)$$

We claim that

$$\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|x-u|)^{-1}(c+|u|)^{-1}\,\mathrm{d}u \ll c^{-1}e^{-|x|}. \qquad (1.2.5)$$

This claim is obvious when $x = 0$. Now, let us consider the case $x > 0$. Write

$$\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|x-u|)^{-1}(c+|u|)^{-1}\,\mathrm{d}u = \int_{-\infty}^{0} + \int_0^x + \int_x^{\infty} =: I_1 + I_2 + I_3.$$

Then

$$\begin{aligned} I_1 &= \int_{-\infty}^{0} e^{-x}(c+v)^{-1}(c+x+v)^{-1}\,\mathrm{d}v \\ &\ll e^{-x}\left(\int_0^c c^{-2}\,\mathrm{d}v + \int_c^{\infty} v^{-2}\,\mathrm{d}v\right) \ll c^{-1}e^{-x}, \end{aligned}$$

$$I_2 = \int_0^{x/2} + \int_{x/2}^{x} \ll e^{-x}\int_0^{x/2}(c+u)^{-2}\,\mathrm{d}u + \frac{e^{-x}}{2c+x}\log\left(1+\frac{x}{c}\right) \ll c^{-1}e^{-x},$$

and

$$\begin{aligned}I_3 &= \int_x^\infty e^{-2u+x}(c+v)^{-1}(c+u-v)^{-1}\,\mathrm{d}u \\ &\le \frac{1}{2}e^{-x}\int_x^\infty \left((c+u)^{-2}+(c+u-x)^{-2}\right)\,\mathrm{d}u \ll c^{-1}e^{-x}.\end{aligned}$$

This completes the proof of the case $x > 0$. When $x < 0$, notice that

$$\begin{aligned}&\int_{-\infty}^\infty e^{-|u|-|x-u|}(c+|x-u|)^{-1}(c+|u|)^{-1}\,\mathrm{d}u \\ &= \int_{-\infty}^\infty e^{-|u|-|x+v|}(c+|x+v|)^{-1}(c+|v|)^{-1}\,\mathrm{d}v \\ &= \int_{-\infty}^\infty e^{-|u|-|-x-v|}(c+|-x-v|)^{-1}(c+|v|)^{-1}\,\mathrm{d}v.\end{aligned}$$

Hence, the proof of this case follows from the case $x > 0$. We have now verified the claim.

It follows from (1.2.4) and (1.2.5) that for $T/2 \le |t| \le T$

$$\begin{aligned}\left|L\left(\frac{1}{2}+it,f\right)\right|^2 &\ll \log T + \log T\int_{-\infty}^\infty \left|L\left(\frac{1}{2}+it+ix,f\right)\right|^2 e^{-|x|}\,\mathrm{d}x \\ &\ll \log T + \log T\int_{-\log^2 T}^{\log^2 T}\left|L\left(\frac{1}{2}+it+ix,f\right)\right|^2 e^{-|x|}\,\mathrm{d}x. \quad (1.2.6)\end{aligned}$$

Here we used the fact that $L(s,f)$ is polynomially bounded on vertical strips [29, Cor. 3, p. 334] and (1.2.2) to bound the integral $\int_{|x|>\log^2 T} = O(\log T)$. Finally, we conclude from (1.2.6) and Theorem 1.2.1 that

$$\begin{aligned}\sum_{r=1}^R \left|L\left(\frac{1}{2}+it,f\right)\right|^2 &= \left(\sum_{|t_r|\le 2} + \sum_{k=1}^{\log T/\log 2}\ \sum_{\frac{T}{2^k}<|t_r|\le\frac{T}{2^{k-1}}}\right)\left|L\left(\frac{1}{2}+it,f\right)\right|^2 \\ &\ll \log^2 T + \log T\int_{-T-\log^2 T}^{T+\log^2 T}\left|L\left(\frac{1}{2}+ix,f\right)\right|^2\,\mathrm{d}x \\ &\ll T(\log T)^2.\end{aligned}$$

This completes the proof of Theorem 1.2.2. □

### 1.2.2 Exponent pairs

Before we give the definition of an exponent pair, we have to introduce the function class **F** which is closely related to it. For an overview of this subject, we refer the reader to the book of S. W. Graham and G. Kolesnik [31].

**Definition 1.2.3.** Let $N$, $y$, $s$, and $\epsilon$ be positive real numbers with $\epsilon < 1/2$, and let $P$ be a non-negative integer. Define $\mathbf{F}(N,P,s,y,\epsilon)$ to be the set of functions $f$ such that
(1) $f$ is defined and has $P$ continuous derivatives on some interval $[x_1, x_2]$, with $[x_1, x_2] \subseteq [N, 2N]$,
(2) if $0 \le p \le P-1$ and $x_1 \le x \le x_2$ then

$$\left| f^{(p+1)}(x) - \frac{\mathrm{d}^p}{\mathrm{d}x^p}(yx^{-s}) \right| < \epsilon \frac{\mathrm{d}^p}{\mathrm{d}x^p}(yx^{-s}).$$

**Definition 1.2.4** (Exponent pair)**.** Let $a$ and $b$ be real numbers such that $0 \le a \le 1/2 \le b \le 1$. Supose that for every $s > 0$, there is some $P = P(a,b,s)$ and some $\epsilon = \epsilon(a,b,s) < 1/2$ such that for every $N > 0$, every $y > 0$, and every $f \in \mathbf{F}(N,P,s,y,\epsilon)$, the estimate

$$\sum_{x_1 \le n \le x_2} e^{2\pi i f(n)} \ll (yN^{-s})^a N^b + y^{-1} N^s,$$

holds. Here it is also assumed that $f$ is defined on $[x_1, x_2]$ and the implied constant depends only on $a$, $b$, and $s$. We then say that $(a,b)$ is an exponent pair.

**Remark 1.2.5.** Despite this rather technical definition, exponent pairs play a prominent role in analytic number theory, and many deep results are conditional to whether a certain $(a,b)$ is an exponent pair. In the following discussion, the exponent pair we use is $(13/84 + \varepsilon, 55/84 + \varepsilon)$, obtained by Bourgain [7].

### 1.2.3 Ivić's mixed moment bounds for the Riemann zeta function

The following theorem reveals the number of points on the critical line where the Riemann zeta function takes large values can be estimated in terms of a exponent pair $(a,b)$.

**Theorem 1.2.6** (Ivić, [47], Thm 8.2)**.** *Let $(a,b)$ be any exponent pair with $a > 0$, and let $t_1 < \cdots < t_R$ satisfy*

$$|t_r| \le T \quad \textit{for} \quad r = 1, \cdots, R; \qquad |t_r - t_s| \ge 1 \quad \textit{for} \quad 1 \le r \ne s \le R,$$

*and*

$$\left|\zeta\left(\frac{1}{2} + it_r\right)\right| \ge V > 0 \qquad (1 \le r \le R).$$

*Then*

$$R \ll TV^{-6} \log^8 T + T^{(a+b)/a} V^{-2(1+2a+2b)/a} (\log T)^{(3+6a+4b)/a}.$$

As a consequence, we have

**Theorem 1.2.7** (Mixed moment bounds for the Riemann zeta function)**.** *Let* $(a, b)$ *be any exponent pair with* $a > 0$ *and*

$$\zeta_0 = \zeta \mathbf{1}_{\{s:|\zeta(s)|\geq T^{\frac{b}{2+4b-2a}}\}}, \quad \zeta_1 = \zeta \mathbf{1}_{\{s:|\zeta(s)|<T^{\frac{b}{2+4b-2a}}\}}.$$

*Then*

$$\int_0^T \left|\zeta_0\left(\frac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll T^{1+\varepsilon}, \qquad \int_0^T \left|\zeta_1\left(\frac{1}{2}+it\right)\right|^{\frac{2(1+2a+2b)}{a}} \mathrm{d}t \ll T^{\frac{a+b}{a}}.$$

## 1.3 The zero-detection method

Our starting point is a zero-detection method which has become standard by now. As our further analysis heavily uses the concepts that are introduced by this method, we opt, for the convenience of the reader, to briefly recall here the main ideas involved in this technique.

Let $X, Y, T > 1$. We consider an approximate inverse for $L(s, f)$ given by

$$M_X(s, f) = \sum_{n \leq X} \frac{\mu_f(n)}{n^s}.$$

where $\mu_f(n)$ is the multiplicative function for which

$$\mu_f(p^k) := \begin{cases} 1, & \text{if } \ k = 0, 2, \\ -\lambda_f(p), & \text{if } \ k = 1, \\ 0, & \text{if } \ k \geq 3. \end{cases}$$

This gives

$$L(s, f)M_X(s, f) = \sum_{n=1}^{\infty} c_n n^{-s}, \quad \Re s > 1,$$

where $c_n = \sum_{d|n, d\leq X} \mu_f(d)\lambda_f(n/d)$. Observe that $c_1 = 1$, $c_n = 0$ for $1 < n \leq X$ and $|c_n| \ll n^{\varepsilon}$.

Introducing the weight $e^{-n/Y}$ and exploiting the Mellin inversion formula for $e^{-x}$, one finds, for $1/2 < \sigma < 1$,

$$\begin{aligned} e^{-1/Y} + \sum_{n>X} c_n n^{-s} e^{-n/Y} &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z) Y^z L(s+z, f) M_X(s+z, f) \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{1/2-\sigma-i\infty}^{1/2-\sigma+i\infty} \Gamma(z) Y^z L(s+z, f) M_X(s+z, f) \mathrm{d}z \\ &\quad + L(s, f) M_X(s, f), \end{aligned}$$

where we picked up the residue at $z = 0$ while shifting the line of integration[1]. The tail of the sum $\sum_{n>Y\log^2 Y} c_n n^{-s} e^{-n/Y}$ is $o(1)$ as $Y \to \infty$. If $|\Im m\, z| \le T$, then also the tails $|\Im m\, z| \ge \log^2 T$ of the final integral become $o(1)$ as $T \to \infty$ if $X$ is polynomially bounded in $T$, say, in view of the exponential decay on vertical lines of the $\Gamma$ function and the trivial estimate $M_X(1/2+iu) \ll X^{1/2+\varepsilon}$. Therefore, for $Y, T$ sufficiently large, we have

$$L(s,f)M_X(s,f) = 1 + \sum_{X<n\le Y\log^2 Y} c_n n^{-s} e^{-n/Y} + o(1)$$
$$+ \int_{-\log^2 T}^{\log^2 T} \Gamma\left(\frac{1}{2} - \sigma + iu\right) Y^{\frac{1}{2}-\sigma+iu} L\left(\frac{1}{2} + it + iu, f\right) M_X\left(\frac{1}{2} + it + iu, f\right) \mathrm{d}u.$$

Thus, if $\rho = \beta + it$ is a zero of $L(s,f)$ with $1/2 < \sigma \le \beta < 1$ and $|t| \le T$, then either

$$\left|\sum_{X<n\le Y\log^2 Y} c_n n^{-\rho} e^{-n/Y}\right| \gg 1, \tag{1.3.1}$$

or

$$\left|\int_{-\log^2 T}^{\log^2 T} L\left(\frac{1}{2} + i(t+u), f\right) M_X\left(\frac{1}{2} + i(t+u), f\right) Y^{\frac{1}{2}-\beta+iu} \cdot\Gamma\left(\frac{1}{2} - \beta + iu\right) \mathrm{d}u\right| \gg 1. \tag{1.3.2}$$

The zeros for which (1.3.1) holds are referred to as *class-I zeros* while the zeros for which (1.3.2) holds are called *class-II zeros*. As a zero must inevitably belong to one (or both) of these classes we obtain

$$N_f(\sigma, T) \ll (|R_1| + |R_2|)T^{\varepsilon}, \tag{1.3.3}$$

where $R_1 = R_1(X,Y,T)$, resp. $R_2$, is the set of class-I, resp. class-II zeros, and $|R_j|$ denotes their cardinality.

For both of these classes we now consider a (saturated) subset $\tilde{R}_j$ of *well-spaced zeros*; those are subsets of $R_j$ for which the imaginary parts of the zeros are well-spaced in the sense that

$$|t_1 - t_2| \ge T^{\varepsilon}, \tag{1.3.4}$$

for different $\rho_1 = \beta_1 + it_1$ and $\rho_2 = \beta_2 + it_2$ belonging to $\tilde{R}_j$. Since $N_f(\sigma, T+1) - N_f(\sigma, T) \ll \log T$, as follows from e.g. [63, Thm. 3.5, p. 156], one can always select a set of well-spaced zeros $\tilde{R}_j$ such that $|\tilde{R}_j| \gg |R_j|/(T^{\varepsilon}\log T)$. Therefore, the estimate (1.3.3) remains valid if we replace $R_j$ by $\tilde{R}_j$.

[1] The function $L(s,f)$ is polynomially bounded on vertical strips [29, Cor. 3, p. 334], which justifies the switching of contour.

### 1.3.1 The contribution of the class-II zeros

We begin by analyzing the contribution of the well-spaced class-II zeros $\tilde{R}_2$. If we set

$$\left| L\left(\frac{1}{2} + i\gamma_r, f\right) \right| = \max_{-\log^2 T \le u \le \log^2 T} \left| L\left(\frac{1}{2} + it_r + iu, f\right) \right|,$$

where $t_r$ are the imaginary parts of the class-II zeros, then we find

$$1 \ll T^\varepsilon Y^{1/2-\sigma} \left| L\left(\frac{1}{2} + i\gamma_r, f\right) \right|, \quad r = 1, 2, ..., |\tilde{R}_2|,$$

where we have set $X = T^\varepsilon$. We square the above inequality and as the $\gamma_r$ are well-spaced because the class-II zeros are, we may apply (1.2.2) to obtain

$$|\tilde{R}_2| \ll T^\varepsilon Y^{1-2\sigma} \sum_{r \le |\tilde{R}_2|} \left| L\left(\frac{1}{2} + i\gamma_r, f\right) \right|^2 \ll T^{1+\varepsilon} Y^{1-2\sigma}.$$

Therefore, upon choosing $Y = T$, we obtain $|\tilde{R}_2| \ll T^{2(1-\sigma)+\varepsilon}$ and this already concludes the analysis of the class-II zeros.

### 1.3.2 Representative class-I zeros

The rest of the argument is then to bound the contribution of the class-I zeros. First we shall restrict the well-spaced class-I zeros even further. By a dyadic subdivision of the interval $(X, Y \log^2 Y]$, one can find $X \le M < Y \log^2 Y$ such that

$$\left| \sum_{M < n \le 2M} c_n n^{-\rho} e^{-n/Y} \right| \gg \frac{1}{\log Y} \tag{1.3.5}$$

for at least $|\tilde{R}_1| \log 2 / \log(Y \log^2 Y)$ zeros $\rho$ of $\tilde{R}_1$. The elements of $\tilde{R}_1$ that additionally satisfy (1.3.5) are called *representative well-spaced zeros* and this subset will be denoted as $R$. We remark that (1.3.3) remains valid upon replacing $|R_1|$ by $|R|$.

Next, we are going to find some very useful estimates allowing us to bound the size of a set of representative well-spaced zeros in terms of the moduli of certain Dirichlet polynomials. It also turns out that the most problematic range is when $M$ is small; the following argument shall allow us to take care of the range $M < T^{1/2}$ such that the critical range for $M$ becomes $M \approx T^{1/2}$.

Let $\nu$ be a fixed integer and let $A$ be a multiset consisting of elements of $R$. We shall actually set $\nu = 1$ in the proof of Theorem 1.1.1 and $\nu = 2$ for Theorem 1.1.2. We consider an integer power $k$ such that $M^k \le Y^{\nu+\varepsilon} < M^{k+1}$.

Hence, as we have set $X = T^\varepsilon$ and $Y = T$, we deduce $1 \le \nu \le k \ll_\varepsilon 1$ and $Y^{\frac{\nu^2}{\nu+1}} < M^k \le Y^{\nu+\varepsilon}$. Raising (1.3.5) to the power $k$ we find

$$\left| \sum_{M^k < n \le 2^k M^k} c'_n n^{-\rho} \right| \gg (1/\log Y)^k,$$

where $c'_n = \sum_{n_1 n_2 \dots n_k = n} c_{n_1} c_{n_2} \dots c_{n_k} e^{-\frac{n_1+\dots+n_k}{Y}}$. Again $c'_n \ll n^\varepsilon$. A dyadic subdivision of the interval $(M^k, 2^k M^k]$ allows us to find $M^k \le N = N(A) \le 2^k M^k$ for which

$$\begin{aligned} |A| \ll &(\log Y)^k \sum_{\rho\in A} \left| \sum_{N<n\le 2N} c'_n n^{-\rho} \right| \\ =&(\log Y)^k \sum_{\rho\in A} \left| \int_N^{2N} u^{-\beta} \mathrm{d} \left( \sum_{N<n\le u} c'_n n^{-it} \right) \right| \\ =&(\log Y)^k \sum_{\rho\in A} \left| (2N)^{-\beta} \sum_{N<n\le 2N} c'_n n^{-it} + \int_N^{2N} \beta u^{-\beta-1} \sum_{N<n\le u} c'_n n^{-it} \,\mathrm{d}u \right| \\ \ll &(\log Y)^k N^{-\sigma} \max_{N<u\le 2N} \sum_{\rho\in A} \left| \sum_{N<n\le u} c'_n n^{-it} \right|. \end{aligned}$$

If we let $c''_n = 0$ after the point where the above maximum is reached, but $c''_n = c'_n$ otherwise, we obtain

$$|A| \ll N^{-\sigma+\varepsilon} \sum_{\rho\in A} \left| \sum_{N<n\le 2N} c''_n n^{-it} \right|, \quad |A| \ll N^{-2\sigma+\varepsilon} \sum_{\rho\in A} \left| \sum_{N<n\le 2N} c''_n n^{-it} \right|^2,$$

the last inequality being derived from Cauchy-Schwarz. If we now set $b(n) = b(n, A) = \epsilon c''_n$ for a sufficiently small $\epsilon$ such that $|b(n)| \le 1$, we have

$$|A| \ll N^{-\sigma+\varepsilon} \sum_{\rho\in A} \left| \sum_{N<n\le 2N} b(n) n^{-it} \right|, \quad |A| \ll N^{-2\sigma+\varepsilon} \sum_{\rho\in A} \left| \sum_{N<n\le 2N} b(n) n^{-it} \right|^2. \tag{1.3.6}$$

In particular, if one selects $A = R$, we get

$$|R| \ll N^{-\sigma+\varepsilon} \sum_{\rho\in R} \left| \sum_{N<n\le 2N} b(n) n^{-it} \right|. \tag{1.3.7}$$

Our goal in the next section is therefore to realize there is indeed sufficient cancellation in (1.3.7) to extract some non-trivial information.

We do emphasize here again that $N$ and $b(n)$ do depend on the set $A$. Throughout the rest of the chapter, we shall write $b(n)$ and $N$ when we refer to the set $R$. If any other set of representative well-spaced zeros $A$ is considered, we shall explicitly mention the dependence of $b(n)$ and $N$ on $A$. On the other hand we note that $M^k \leq N(A) < 2^k M^k$, therefore $N \ll N(A) \ll N$ for any $A$. Furthermore, as we take $\nu = 1$, one has

$$T^{1/2} \leq N < T^{1+\varepsilon}. \tag{1.3.8}$$

## 1.4 Ivić's Estimate

In this section we deduce a first non-trivial estimate on the number of class-I zeros. The first step is to apply the Halász-Montgomery inequality to realize there is cancellation in (1.3.7). The following lemma is a reformulation of the estimate in [47, Eq. 11.40]. We closely follow here the proof of [47, Thm. 11.2].

**Lemma 1.4.1.** *Let $A \subseteq R$ be a set of representative well-spaced class-I zeros (where we do not allow repetition). For $\ell \in \mathbb{Z}$, define*

$$\Delta_A(\ell) = \#\{(\beta + it, \beta' + it') \in A \times A : |t - t' - \ell| < 1\}.$$

*Then*

$$|A| \ll \left\{ N^{2-2\sigma} + N^{3/4-\sigma} \left[ \sum_{\ell \in \mathbb{Z}} \Delta_A(\ell) \int_{-2\log^2 T}^{2\log^2 T} \left| \zeta\left(\frac{1}{2} + iv + i\ell\right) \right| \,\mathrm{d}v \right]^{\frac{1}{2}} \right\} T^{\varepsilon}. \tag{1.4.1}$$

*Proof.* As $N \asymp N(A)$, the estimate (1.4.1) is equivalent upon replacing $N$ with $N(A)$. Throughout the rest of the proof, however, we shall simply write $N$ for $N(A)$ in order not to overload the notation unnecessarily.

By applying the Halász-Montgomery inequality [62, Lemma 1.7, p. 6] to (1.3.6), we get

$$|A|^2 N^{2\sigma-2\varepsilon} \ll |A| N^2 + N \sum_{r \neq s} |H(it_r - it_s)|, \tag{1.4.2}$$

where $t_r, t_s$ denote the imaginary parts of elements of $A$ and

$$\begin{aligned} H(it) &= \sum_{n=1}^{\infty} (e^{-n/2N} - e^{-n/N}) n^{-it} \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w + it)\Gamma(w)\left((2N)^w - N^w\right) \,\mathrm{d}w. \end{aligned}$$

We switch the contour to the line $\Re e\, w = 1/2$ which is allowed since $\zeta$ is polynomially bounded and $\Gamma$ decays exponentially on vertical strips. We pass over a simple pole at $w = 1 - it$ with residue $O(Ne^{-|t|})$ and our equation becomes

$$H(it) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \zeta\left(w+it\right)\Gamma(w)\left((2N)^w - N^w\right)\, \mathrm{d}w + O\left(Ne^{-|t|}\right).$$

When $|\Im m\, w| \geq \log^2 T$, the tails of the integral contribute $O(N^{1/2}\exp(-\log^2 T)) = o(1)$ as $N \ll T^{1+\varepsilon}$. Therefore,

$$\sum_{r\neq s} |H(it_r - it_s)| \ll N\sum_{r\neq s} e^{-|t_r - t_s|} + o\left(|A|^2\right) + N^{1/2}\int_{-\log^2 T}^{\log^2 T} \sum_{r\neq s}\left|\zeta\left(\frac{1}{2} + it_r - it_s + iv\right)\right| \mathrm{d}v.$$

The first term on the right hand side is $o\left(|A|^2\right)$ as the members of $A$ are well-spaced. Moreover, by the definition of $\Delta_A(\ell)$ , we have

$$\int_{-\log^2 T}^{\log^2 T} \sum_{r\neq s}\left|\zeta\left(\frac{1}{2} + it_r - it_s + iv\right)\right| \mathrm{d}v \ll \sum_{\ell\in\mathbb{Z}} \Delta_A(\ell)\int_{-1-\log^2 T}^{1+\log^2 T}\left|\zeta\left(\frac{1}{2} + i\ell + iv\right)\right| \mathrm{d}v.$$

As $\sigma > 1/2$ we arrive at (1.4.1) after inserting all these estimates in (1.4.2). □

As usual if we do not mention the subscript $A$ for $\Delta_A$ when we are referring to $A = R$.

It thus remains to find adequate estimates for the integral in (1.4.1). For this we shall appeal to moment estimates on the zeta function. Let $B_0, B_1, q_0, q_1$ be positive numbers for which $q_0, q_1 \geq 2$, and $H : [0,\infty) \to [1,\infty)$. We let

$$\zeta_0 = \zeta_{0,T} = \zeta\mathbf{1}_{\{s:|\zeta(s)|\geq H(T)\}}, \quad \zeta_1 = \zeta_{1,T} = \zeta\mathbf{1}_{\{s:|\zeta(s)|< H(T)\}}.$$

In what follows, we rely on an assumption of the form

$$\int_0^T \left|\zeta_0\left(\frac{1}{2}+it\right)\right|^{q_0} \mathrm{d}t \ll T^{B_0}, \quad \int_0^T \left|\zeta_1\left(\frac{1}{2}+it\right)\right|^{q_1} \mathrm{d}t \ll T^{B_1}. \qquad (1.4.3)$$

involving mixed moment estimates for the zeta function.

Recall $\Delta(\ell) = \#\{(\beta + it, \beta' + it') \in R\times R : |t - t' - \ell| < 1\}$. We have $\Delta(\ell) \leq |R|$ because the elements of $R$ are well-spaced and $\sum_{\ell\in\mathbb{Z}}\Delta(\ell) \leq 2|R|^2$

as each couple $(\beta + it, \beta + it')$ can at most contribute to two $\Delta(\ell)$. It follows that for $q > 1$,

$$\sum_{\ell \in \mathbb{Z}} \Delta(\ell)^{\frac{q}{q-1}} \leq \sum_{\ell \in \mathbb{Z}} \Delta(\ell)\Delta(\ell)^{\frac{1}{q-1}} \leq 2|R|^{\frac{2q-1}{q-1}}.$$

With this estimate the integral in (1.4.1) becomes through some applications of Hölder's inequality

$$\begin{aligned}
&\sum_{\ell \in \mathbb{Z}} \Delta(\ell) \int_{-2\log^2 T}^{2\log^2 T} \left|\zeta\left(\frac{1}{2} + i\ell + iv\right)\right| \,\mathrm{d}v \\
&\leq \sum_{j=0}^{1} \left(\sum_{\ell \in \mathbb{Z}} \Delta(\ell)^{\frac{q_j}{q_j-1}}\right)^{\frac{q_j-1}{q_j}} \left\{\sum_{\ell \in \mathbb{Z}} \left[\int_{-2\log^2 T}^{2\log^2 T} \left|\zeta_j\left(\frac{1}{2} + i\ell + iv\right)\right|^{q_j} \,\mathrm{d}v\right]\right\}^{\frac{1}{q_j}} T^{\varepsilon} \\
&\ll |R|^{2-\frac{1}{q_0}} T^{\frac{B_0}{q_0}+\varepsilon} + |R|^{2-\frac{1}{q_1}} T^{\frac{B_1}{q_1}+\varepsilon},
\end{aligned}$$

and thus

$$|R| \ll T^{\varepsilon}\left(N^{2-2\sigma} + T^{B_0} N^{(3-4\sigma)q_0/2} + T^{B_1} N^{(3-4\sigma)q_1/2}\right).$$

We now refine this estimate through Huxley's subdivision argument. Set

$$T_0 = \delta_2 T, \qquad T^{-1} \leq \delta_2 \leq 1, \tag{1.4.4}$$

and let $R_\alpha = \{\rho \in R : \Im m\, \rho \in I_\alpha\}$, with $I_\alpha \subset [-T, T]$ a certain subinterval of length $T_0$. Repeating the above argument with $R_\alpha$ instead of $R$ then gives[2]

$$|R_\alpha| \ll T^{\varepsilon}\left(N^{2-2\sigma} + \delta_2^{B_0} T^{B_0} N^{(3-4\sigma)q_0/2} + \delta_2^{B_1} T^{B_1} N^{(3-4\sigma)q_1/2}\right).$$

Subdividing $R$ into about as many as $\lceil \frac{2}{\delta_2} \rceil$ sets of the form $R_\alpha$ and summing these contributions gives

$$|R| \ll T^{\varepsilon}\left(\delta_2^{-1} N^{2-2\sigma} + \delta_2^{B_0-1} T^{B_0} N^{(3-4\sigma)q_0/2} + \delta_2^{B_1-1} T^{B_1} N^{(3-4\sigma)q_1/2}\right).$$

Now, selecting $\delta_2 = N^{2-2\sigma} T^{2\sigma-2}$ if $N \leq T$ and $\delta_2 = 1$ if $T < N < T^{1+\varepsilon}$ delivers[3]

$$|R| \ll T^{\varepsilon}\left(T^{2-2\sigma} + \sum_{j=0}^{1} T^{(2B_j-2)\sigma+(2-B_j)} N^{(2B_j-2+3q_j/2)-2\sigma(B_j+q_j-1)}\right).$$

[2]The subdivision of $\zeta$ into $\zeta_0$ and $\zeta_1$ is now with respect to $H(T_0)$ instead of $H(T)$.

[3]One can optimize the choice for $\delta_2$ even further here. However, we are in this work only interested into which range the density hypothesis is valid. If one only considers this question there is no advantage in further optimizing $\delta_2$.

We thus obtain $|R| \ll T^{2(1-\sigma)+\varepsilon}$ provided $\sigma \geq 1-q_j/(4B_j+4q_j-4)$ for $j=0,1$ and provided $N$ satisfies

$$N \geq T^{\frac{4B^*\sigma-2B^*}{(4q^*+4B^*-4)\sigma-(3q^*+4B^*-4)}}, \tag{1.4.5}$$

where $(q^*, B^*)$ is the couple $(q_j, B_j)$, $j=0,1$, for which the exponent above is maximal, which in principle may depend on $\sigma$. If $\sigma$ lies in the range where this exponent is less than $1/2$, that is $\sigma \geq (3q^*-4)/(4q^*-4B^*-4)$, we are done as we always have $N \geq T^{1/2}$. In the remaining range $\sigma < (3q^*-4)/(4q^*-4B^*-4)$, we may therefore assume in the sequel that

$$T^{1/2} \leq N < T^{\frac{4B^*\sigma-2B^*}{(4q^*+4B^*-4)\sigma-(3q^*+4B^*-4)}}. \tag{1.4.6}$$

**Remark 1.4.2.** We briefly mention how Ivić arrived at the validity of the density hypothesis in the range $\sigma \geq 53/60$. He selected[4] $q_0 = 6, q_1 = 19, B_0 = 1+\varepsilon$ and $B_1 = 3+\varepsilon$ with $H(T) = T^{2/13}$. With this choice the condition (1.4.5) becomes $N \geq \max\{T^{6(2\sigma-1)/(84\sigma-65)}, T^{(2\sigma-1)/(12\sigma-9)}\}$ and this is always satisfied if $\sigma \geq 53/60$ as the exponents of $T$ are then always smaller than $1/2$.

## 1.5 Bourgain's dichotomy

In this section, inspired by the work of Bourgain [5, 6], we will use the dichotomy method to improve the estimates for the integral terms appearing in (1.4.1). This allows us to obtain a new estimate for the class-I zeros.

### 1.5.1 Lemmas on Dirichlet polynomials

In applying Bourgain's method, we shall require some preliminary lemmas on estimations for Dirichlet polynomials. The first lemma gives an upper estimate for pointwise values of a Dirichlet polynomial in terms of an average of the values near the point. It is a slight modification of [5, Lemma 4.48].

**Lemma 1.5.1.** *Consider the Dirichlet polynomial*

$$F(t) = \sum_{N<n\leq 2N} b_n n^{-it}, \quad t \in \mathbb{R},$$

*where the coefficients $b_n$ satisfy $|b_n| \leq 1$. Then,*

$$|F(t)| \ll 1 + \log N \int_{|v|<\log N} |F(t+v)|\, \mathrm{d}v, \quad N \to \infty.$$

[4] Applying Theorem 1.2.7 with exponent pair $(2/7, 4/7)$.

*Proof.* Let $\psi$ be a smooth function on $\mathbb{R}$ such that $\hat{\psi}$, the Fourier transform of $\psi$, is identically 1 on the interval $[1,2]$ and satisfies

$$|\psi(x)| \ll e^{-|x|^{2/3}}.$$

The existence of such a function $\psi$ is guaranteed by the Denjoy-Carleman theorem.

Let $\psi_\lambda(x) = (1/\lambda)\psi(x/\lambda)$. We have, for $N \geq 2$,

$$\begin{aligned} |F(t)| &= \left| \sum_{N<n\leq 2N} b_n \hat{\psi}\left(\frac{\log n}{\log N}\right) n^{-it} \right| = \left| \sum_{N<n\leq 2N} b_n \hat{\psi}_{(\log N)^{-1}} (\log n)\, n^{-it} \right| \\ &= \left| \int_{\mathbb{R}} F(t+v)\psi_{(\log N)^{-1}}(v)\, \mathrm{d}v \right|. \end{aligned}$$

The result now follows upon realizing that $|\psi_{(\log N)^{-1}}(v)| \ll \log N$ if $|v| < \log N$ and

$$\int_{|v|\geq \log N} \left|F(t+v)\psi_{(\log N)^{-1}}(v)\right| \,\mathrm{d}v \ll N \log N \int_{|v|\geq \log N} e^{-(|v| \log N)^{2/3}} \,\mathrm{d}v \ll 1,$$

as $N \to \infty$ because $|b_n| \leq 1$. □

The next one is a simple estimate due to Bourgain [5, Lemma 3.4] for Dirichlet polynomials over difference sets where the index sets are different.

**Lemma 1.5.2.** *Let $a_n, b_n$ $(1 \leq n \leq N)$ be complex numbers such that $|a_n| \leq b_n$. Let $R, S \subseteq \mathbb{R}$ be two finite sets. Then*

$$\sum_{\substack{t\in R\\ s\in S}} \left| \sum_{n=1}^{N} a_n n^{i(t-s)} \right|^2 \leq \left( \sum_{t,t'\in R} \left| \sum_{n=1}^{N} b_n n^{i(t-t')} \right|^2 \right)^{\frac{1}{2}} \left( \sum_{s,s'\in S} \left| \sum_{n=1}^{N} b_n n^{i(s-s')} \right|^2 \right)^{\frac{1}{2}}.$$

The final lemma is Heath-Brown's estimate on double zeta sums [32, Thm. 1]. It is much deeper and is a crucial ingredient for our argument.

**Lemma 1.5.3.** *Let $R$ be a finite set of well-spaced, cf.* (1.3.4), *points such that $|t| \leq T$ for each $t \in R$. Then*

$$\sum_{t,t'\in R} \left| \sum_{N<n\leq 2N} n^{i(t-t')} \right|^2 \ll T^{\varepsilon}(|R|N^2 + |R|^2 N + N|R|^{5/4}T^{1/2}).$$

### 1.5.2 The dichotomy

Let $T_0$ and $R_\alpha$ be as in Section 1.4, see (1.4.4). We recall that we have set

$$\Delta_\alpha(\ell) = \Delta_{R_\alpha}(\ell) = \#\{(\beta + it, \beta' + it') \in R_\alpha \times R_\alpha : |t - t' - \ell| < 1\}.$$

Let $0 < \delta_1 < 1$ be a parameter to be optimized later. We set

$$\zeta_0 = \zeta_{0,T_0,H} = \zeta \mathbf{1}_{\{s:|\zeta(s)|\geq H(T)\}}, \quad \zeta_1 = \zeta_{1,T_0,H} = \zeta \mathbf{1}_{\{s:1\leq|\zeta(s)|<H(T)\}},$$

and let $B_0, B_1, q_0, q_1$ be the parameters that were introduced in the mixed moment estimates (1.4.3). Note that the definition for $\zeta_1$ is slightly different than in the previous section because of the lower bound $|\zeta| \geq 1$. For each fixed $\alpha$, we distinguish between the following alternatives.

**Case 1.** We have

$$\sum_{\ell\in\mathbb{Z}} \Delta_\alpha(\ell) \int_{|v|<T^\varepsilon} \left|\zeta_0\left(\frac{1}{2} + i\ell + iv\right)\right| \,\mathrm{d}v \leq \delta_1^{2/q_0} T_0^{B_0/q_0} |R_\alpha|^{2-1/q_0} \tag{1.5.1}$$

and

$$\sum_{\ell\in\mathbb{Z}} \Delta_\alpha(\ell) \int_{|v|<T^\varepsilon} \left|\zeta_1\left(\frac{1}{2} + i\ell + iv\right)\right| \,\mathrm{d}v \leq \delta_1^{2/q_1} T_0^{B_1/q_1} |R_\alpha|^{2-1/q_1}. \tag{1.5.2}$$

**Case 2.** Either (1.5.1) or (1.5.2) fails.

We consider a collection of $\lceil 2/\delta_2 \rceil$ sets $R_\alpha$ that cover $R$. We let $\mathcal{I}_0$ be the index set of $\alpha$ for which (1.5.1) fails, $\mathcal{I}_1$ be the index set for which (1.5.2) fails, and $\mathcal{I}_2$ be the index set for which both inequalities hold. Clearly $|R| \ll \sum_{\alpha\in\mathcal{I}_0} R_\alpha + \sum_{\alpha\in\mathcal{I}_1} R_\alpha + \sum_{\alpha\in\mathcal{I}_2} R_\alpha$. An additional constraint on the parameter $\delta_1$ will arise below in the analysis of Case 2.

### 1.5.3 The Case 1 contribution

We suppose here that $|R| \ll \sum_{\alpha\in\mathcal{I}_2} R_\alpha$. Let $\alpha \in \mathcal{I}_2$. Since $|\zeta| = |\zeta_0| + |\zeta_1| + |\zeta \mathbf{1}_{\{s:|\zeta(s)|<1\}}|$, it follows that

$$\sum_{\ell\in\mathbb{Z}} \Delta_\alpha(\ell) \int_{|v|<T^\varepsilon} \left|\zeta\left(\frac{1}{2} + i\ell + iv\right)\right| \,\mathrm{d}v \leq \delta_1^{2/q_0} T_0^{B_0/q_0} |R_\alpha|^{2-1/q_0} + \delta_1^{2/q_1} T_0^{B_1/q_1} |R_\alpha|^{2-1/q_1} + T^\varepsilon |R_\alpha|^2,$$

where the last term is coming from the contribution of $|\zeta| < 1$. Inserting this estimate in (1.4.1) and rearranging $|R_\alpha|$ gives[5], for $\sigma > 3/4$,

$$|R_\alpha| \ll T^\varepsilon \left(N^{2-2\sigma} + \delta_1^2 T_0^{B_0} N^{(3-4\sigma)q_0/2} + \delta_1^2 T_0^{B_1} N^{(3-4\sigma)q_1/2}\right).$$

[5] Note that the term $|R_\alpha|^2$ can never be dominant if $\sigma > 3/4$, as (1.4.1) would then entail $|R_\alpha| \ll N^{3/4-\sigma} |R_\alpha| T^\varepsilon$, which is impossible in view of $N \geq T^{1/2}$.

Replacing $T_0$ by $\delta_2 T$ and summing over the index set $\mathcal{I}_2$ then yields

$$|R| \ll T^\varepsilon \left(\delta_2^{-1} N^{2-2\sigma} + \delta_1^2 \delta_2^{B_0-1} T^{B_0} N^{(3-4\sigma)q_0/2} + \delta_1^2 \delta_2^{B_1-1} T^{B_1} N^{(3-4\sigma)q_1/2}\right). \tag{1.5.3}$$

### 1.5.4 Analysis of Case 2

In this section we consider the case when

$$|R| \ll T^\varepsilon \sum_{\alpha \in \mathcal{I}_0} |R_\alpha|. \tag{1.5.4}$$

The analysis of the case when $|R| \ll \sum_{\alpha \in \mathcal{I}_1} |R_\alpha|$ is analogous. We have incorporated the extra factor $T^\varepsilon$ in (1.5.4) as in some places of the analysis we shall add extra restrictions on the set $\mathcal{I}_0$ and the extra factor $T^\varepsilon$ shall guarantee that (1.5.4) remains valid under these restrictions. We write for simplicity $q$ and $B$ instead of $q_0$ and $B_0$.

First, we translate the failure of (1.5.1) and the dominance of the index set $\mathcal{I}_0$ into a lower bound for the size of a specific multiset of representative well-spaced class-I zeros. In this part we shall perform numerous dyadic decompositions and exploit the mixed moment estimate (1.4.3). Afterwards we apply the analysis of Section 1.3 to find an upper estimate for this multiset in terms of a Dirichlet polynomial which will subsequently be estimated with the technology provided by Lemma 1.5.3. The compatibility of this upper and lower estimate shall then result in an improved estimate on $|R|$.

For $0 < \delta' < 1$ we define the set

$$D_\alpha(\delta') = \{\ell : \delta' |R_\alpha| < \Delta_\alpha(\ell) \le 2\delta' |R_\alpha|\}.$$

As $\sum_\ell \Delta_\alpha(\ell) \le 2|R_\alpha|^2$ we immediately obtain $|D_\alpha(\delta')| \le 2|R_\alpha|/\delta'$. Furthermore, as $\Delta_\alpha(\ell)$ is an integer and $|R_\alpha| \le T_0$ because the points of $R_\alpha$ are well-spaced and $I_\alpha$ has length at most $T_0$, we may through a dyadic argument find $\delta' \in \{2^{-k} | 1 \le k \ll \log T_0\}$ such that

$$\sum_{\ell \in \mathbb{Z}} \Delta_\alpha(\ell) \int_{|v| < T^\varepsilon} \left| \zeta_0 \left( \frac{1}{2} + i\ell + iv \right) \right| \mathrm{d}v$$
$$\ll T_0^\varepsilon \delta' |R_\alpha| \sum_{\ell \in D_\alpha(\delta')} \int_{|v| < T^\varepsilon} \left| \zeta_0 \left( \frac{1}{2} + i\ell + iv \right) \right| \mathrm{d}v.$$

A priori $\delta'$ does depend on $\alpha$, but as there are only $O(\log T)$ possibilities for $\delta'$, the pigeonhole principle asserts that one may select a subset of $\mathcal{I}_0$, which we shall continue to write as $\mathcal{I}_0$, for which the above expression holds for a single $\delta'$ and where (1.5.4) remains valid, possibly with a different value for $\varepsilon$. In conclusion, the parameter $\delta'$ can be chosen independent of $\alpha$.

Exploiting now that (1.5.1) fails, we obtain

$$\sum_{\ell \in D_\alpha(\delta')} \int_{|v|<T^\varepsilon} \left|\zeta_0\left(\frac{1}{2} + i\ell + iv\right)\right| \mathrm{d}v \geq \delta_1^{2/q}(2\delta')^{-1} T_0^{B/q-\varepsilon} |R_\alpha|^{1-1/q}.$$

Next we proceed to narrow the range for the modulus of $\zeta_0$. For $H > 0$, we consider level sets

$$S_{H,T_0} = S_{H,T_0,\varepsilon} = \left\{ |t| \leq T_0 + T^\varepsilon + 1 : H < \left|\zeta_0\left(\frac{1}{2} + it\right)\right| \leq 2H \right\}.$$

As one can cover the support of $\zeta_0(1/2+it)\mathbf{1}_{[-T_0-T^\varepsilon-1, T_0+T^\varepsilon+1]}(t)$ by as many as $O(\log T_0)$ level sets of the form $S_{H,T_0}$ in view[6] of $1 \leq |\zeta_0(1/2+it)| \leq T_0^{1/4}$ (on the support of $\zeta_0$), one can find through another dyadic argument a number $H$ such that

$$\begin{aligned}
&\sum_{\ell \in D_\alpha(\delta')} \int_{|v|<T^\varepsilon} \left|\zeta_0\left(\frac{1}{2} + i\ell + iv\right)\right| \mathrm{d}v \\
&= \sum_{\ell \in D_\alpha(\delta')} \int_{-T_0-T^\varepsilon-1}^{T_0+T^\varepsilon+1} \left|\zeta_0\left(\frac{1}{2} + it\right)\right| \mathbf{1}_{[\ell-T^\varepsilon, \ell+T^\varepsilon]}(t) \mathrm{d}t \\
&\ll \sum_{\ell \in D_\alpha(\delta')} H \log T_0 \int_{-T_0-T^\varepsilon-1}^{T_0+T^\varepsilon+1} \mathbf{1}_{[\ell-T^\varepsilon, \ell+T^\varepsilon] \cap S_{H,T_0}}(t) \mathrm{d}t \\
&\ll T^\varepsilon H \int_{|u| \leq T^\varepsilon} \sum_{\ell \in D_\alpha(\delta')} \mathbf{1}_{S_{H,T_0}}(\ell+u) \mathrm{d}u \\
&= T^\varepsilon H \int_{|u| \leq T^\varepsilon} |(D_\alpha(\delta')+u) \cap S_{H,T_0}| \mathrm{d}u.
\end{aligned}$$

We emphasize here that $|(D_\alpha(\delta')+u) \cap S_{H,T_0}|$ denotes the cardinality of the set (depending on the variable $u$). Again $H$ a priori depends on $\alpha$, but as there are only $O(\log T_0)$ valid choices for $H$, one may select as before a subset of $\mathcal{I}_0$, which we keep denoting as $\mathcal{I}_0$, for which (1.5.4) remains true. Therefore, we may assume without loss of generality that $H$ is independent of $\alpha$.

[6] The last inequality $|\zeta_0(1/2+it)| \ll T_0^{1/4}$ follows from the trivial convexity bound for $\zeta$. Of course there are better estimates available. Here is also why we invoked the additional bound $|\zeta_1| \geq 1$ in the definition of $\zeta_1$. This additional restriction guarantees that also the support of $\zeta_1$ can be covered by $O(\log T_0)$ level sets of the form $S_{H,T_0}$, which is unclear otherwise.

Consider

$$W_\alpha := \int_{|u|\le T^\varepsilon} |(D_\alpha(\delta')+u)\cap S_{H,T_0}|\mathrm{d}u$$
$$= \left(\int_{|u|\le T^\varepsilon} |(D_\alpha(\delta')+u)\cap S_{H,T_0}|\mathrm{d}u\right)^{\frac{q-1}{q}} \left(\int_{S_{H,T_0}} \sum_{\ell\in D_\alpha(\delta')} \mathbf{1}_{(\ell-T^\varepsilon,\ell+T^\varepsilon)}(t)\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\ll T^\varepsilon |D_\alpha(\delta')|^{1-\frac{1}{q}} (\mathfrak{m}(S_{H,T_0}))^{\frac{1}{q}},$$

where $\mathfrak{m}$ stands for the Lebesgue measure and where we have used the trivial estimate $\sum_{\ell\in D_\alpha(\delta')} \mathbf{1}_{(\ell-T^\varepsilon,\ell+T^\varepsilon)}(t) \ll T^\varepsilon$. Hence, via a dyadic argument we can specify $0<\delta''\ll T^\varepsilon$ such that

$$\delta''|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q} < W_\alpha \le 2\delta''|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q}. \tag{1.5.5}$$

Again $\delta''$ can be taken independent of $\alpha$ by an appropriate restriction of the index set $\mathcal{I}_0$ and a constraint on the parameter $\delta_1$; in fact, we shall require[7] from now on that $\delta_1 \gg T^{-c}$ for some $c>0$.

Combining all the above inequalities gives

$$T^\varepsilon\delta'' H|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q} \gg \frac{\delta_1^{2/q}}{\delta'} T_0^{B/q}|R_\alpha|^{1-1/q},$$

whence

$$|D_\alpha(\delta')| \gg T^{-\varepsilon}\left(\frac{\delta_1^{2/q}}{\delta'\delta''}\right)^{\frac{q}{q-1}}|R_\alpha|$$

follows because the mixed moment estimate (1.4.3) implies

$$\mathfrak{m}(S_{H,T_0}) \ll H^{-q}T_0^B T^\varepsilon.$$

Thus, using $|D_\alpha(\delta')| \ll |R_\alpha|/\delta'$, we obtain

$$\delta' \gg T^{-\varepsilon}\frac{\delta_1^2}{(\delta'')^q}. \tag{1.5.6}$$

We can derive another lower bound on $|D_\alpha(\delta')|$. Namely, the trivial bound $W_\alpha \ll T^\varepsilon|D_\alpha(\delta')|$ and $W_\alpha \asymp \delta''|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q}$ yield

$$|D_\alpha(\delta')| \gg T^{-\varepsilon}(\delta'')^q\,\mathfrak{m}(S_{H,T_0}) + T^{-\varepsilon}\left(\frac{\delta_1^{2/q}}{\delta'\delta''}\right)^{\frac{q}{q-1}}|R_\alpha|.$$

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[7] We have just derived that $W_\alpha \ll T^\varepsilon|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q} \ll T^{c_2}$ for some $c_2>0$. Going back through the inequalities we also have the lower bound $W_\alpha \gg T^{-\varepsilon}H^{-1}\delta_1^{2/q} \gg T^{-c_1}$ for some $c_1>0$ as $\log H \asymp \log T_0$ and because $\delta_1$ shall later be picked in such a way that $\delta_1 \gg T^{-c}$. Therefore, a dyadic covering for $W_\alpha$ only requires $O(\log T)$ intervals and this enables one to pick a restriction of $\mathcal{I}_0$ such that (1.5.4) remains intact.

Together with (1.5.5) and (1.5.4) this implies

$$\begin{aligned}
\sum_{\alpha\in\mathcal{I}_0}|R_\alpha|W_\alpha &\gg \sum_{\alpha\in\mathcal{I}_0}|R_\alpha|\delta''|D_\alpha(\delta')|^{1-1/q}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q}\\
&\gg T^{-\varepsilon}(\delta'')^q\mathfrak{m}(S_{H,T_0})\left(\sum_{\alpha\in\mathcal{I}_0}|R_\alpha|\right)+T^{-\varepsilon}\frac{\delta_1^{2/q}}{\delta'}\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q}\left(\sum_{\alpha\in\mathcal{I}_0}|R_\alpha|^{\frac{2q-1}{q}}\right)\\
&\gg T^{-\varepsilon}(\delta'')^q|R|\int_{|u|<T^\varepsilon}|(S_{H,T_0}-u)\cap\mathbb{Z}|\,\mathrm{d}u\\
&\qquad+T^{-\varepsilon}\frac{\delta_1^{2/q}\delta_2^{1-1/q}}{\delta'}|R|^{\frac{2q-1}{q}}\int_{|u|<T^\varepsilon}|(S_{H,T_0}-u)\cap\mathbb{Z}|^{1/q}\,\mathrm{d}u
\end{aligned}$$

as Hölder's inequality implies $|R|^{(2q-1)/q}\ll\delta_2^{-(q-1)/q}\sum_{\alpha\in\mathcal{I}_0}|R_\alpha|^{(2q-1)/q}$ and

$$\begin{aligned}
\int_{|u|<T^\varepsilon}|(S_{H,T_0}-u)\cap\mathbb{Z}|^{1/q}\,\mathrm{d}u &\ll\left(\int_{|u|<T^\varepsilon}|(S_{H,T_0}-u)\cap\mathbb{Z}|\mathrm{d}u\right)^{1/q}T^\varepsilon\\
&\ll T^\varepsilon\left(\mathfrak{m}(S_{H,T_0})\right)^{1/q}.
\end{aligned}$$

Recalling the definition of $W_\alpha$, we may therefore find $|u|<T^\varepsilon$ such that the set of integers

$$S=(S_{H,T_0}-u)\cap\mathbb{Z}$$

satisfies

$$\sum_{\alpha\in\mathcal{I}_0}|R_\alpha||D_\alpha(\delta')\cap S|\gg T^{-\varepsilon}(\delta'')^q|R||S|+T^{-\varepsilon}\frac{\delta_1^{2/q}\delta_2^{1-1/q}}{\delta'}|R|^{\frac{2q-1}{q}}|S|^{1/q}. \tag{1.5.7}$$

We keep in mind that we have to multiply by $\delta'\asymp\Delta_\alpha(\ell)/|R_\alpha|$ with $\ell\in D_\alpha(\delta')$ to eliminate $\delta'$ and $\delta''$ from the right-hand side by virtue of (1.5.6). Now that we have established a lower bound for a multiset of class-I zeros, we shift our attention to an upper bound.

We now select $A$ to be the multiset

$$A=\bigcup_{\alpha\in\mathcal{I}_0}\bigcup_{\ell\in S}\bigcup_{t\in\{\Im m\,\rho|\rho\in R_\alpha\}}\{\rho=\beta+it'\in R_\alpha,|t'-(t-\ell)|<1\},$$

where the multiplicity of a zero $\rho$ is according to how many triples $(\alpha,\ell,t)$ produce $\rho$. Therefore, $|A|=\sum_{\alpha\in\mathcal{I}_0}\sum_{\ell\in S}\Delta_\alpha(\ell)$. We now apply the machinery from Section 1.3, in particular (1.3.6), to find a Dirichlet polynomial $F_A(t)=$

$\sum_{N(A)<n\leq 2N(A)} b_n n^{-it}$ with bounded coefficients $b_n$ such that

$$N^{2\sigma}\delta' \sum_{\alpha\in\mathcal{I}_0} |R_\alpha||S\cap D_\alpha(\delta')| \ll N(A)^{2\sigma} \sum_{\alpha\in\mathcal{I}_0}\sum_{\ell\in S} \Delta_\alpha(\ell)$$
$$\ll T^\varepsilon \sum_{\alpha\in\mathcal{I}_0}\sum_{\ell\in S}\sum_{t\in\{\Im m\,\rho:\rho\in R_\alpha\}}\sum_{\substack{t'\in\{\Im m\,\rho:\rho\in R_\alpha\}\\ |t'-(t-\ell)|<1}} |F_A(t')|^2$$
$$\ll T^\varepsilon \sum_{\alpha\in\mathcal{I}_0}\sum_{\ell\in S}\sum_{t\in\{\Im m\,\rho:\rho\in R_\alpha\}}\sum_{\substack{t'\in\{\Im m\,\rho:\rho\in R_\alpha\}\\ |t'-(t-\ell)|<1}} \left(1+\int_{|v|<\log N(A)} |F_A(t'+v)|^2\,\mathrm{d}v\right)$$
$$\ll T^\varepsilon \sum_{\alpha\in\mathcal{I}_0}\sum_{t\in\{\Im m\,\rho:\rho\in R_\alpha\}}\sum_{\ell\in S}\int_{|v|<T^\varepsilon} |F_A(t-\ell+v)|^2\,\mathrm{d}v.$$

In the penultimate transition we applied a Cauchy-Schwarz estimate on Lemma 1.5.1 and in the last step we used that there can only be one $t'$ with a given $t$ as the zeros in $R_\alpha$ are well-spaced and that the term with 1 may be dropped as it only delivers a contribution of at most $T^\varepsilon\sum_{\alpha\in\mathcal{I}_0}\sum_{\ell\in S}\Delta_\alpha(\ell)$ which can never be dominant in view of the estimate on the first line.

The remaining sums and integral are estimated via Lemma 1.5.2 and Lemma 1.5.3. This gives

$$N^{2\sigma}\delta' \sum_{\alpha\in\mathcal{I}_0} |R_\alpha||S\cap D_\alpha(\delta')| \ll T^\varepsilon \int_{|v|<T^\varepsilon} \sum_{\substack{\rho=\beta+it\in R\\ \ell\in S}} |F_A(t-\ell+v)|^2\,\mathrm{d}v$$
$$\ll T^\varepsilon \left(\sum_{\substack{\rho=\beta+it\in R\\ \rho'=\beta'+it'\in R}} \left|\sum_{N(A)<n\leq 2N(A)} n^{i(t-t')}\right|^2\right)^{\frac{1}{2}}$$
$$\cdot \left(\sum_{\ell,\ell'\in S} \left|\sum_{N(A)<n\leq 2N(A)} n^{i(\ell-\ell')}\right|^2\right)^{\frac{1}{2}}$$
$$\ll T^\varepsilon N\left(|R|N+|R|^2+|R|^{5/4}T^{1/2}\right)^{1/2}\left(|S|N+|S|^2+|S|^{5/4}T_0^{1/2}\right)^{1/2}.$$

If we now assume that

$$|R|\leq N,$$

and use the condition $N\geq T^{1/2}$, then we find

$$N^{2\sigma}\delta' \sum_{\alpha\in\mathcal{I}_0} |R_\alpha||S\cap D_\alpha(\delta')| \ll T^\varepsilon N^{3/2}|R|^{5/8}\left(|S|N+|S|^2+|S|^{5/4}T_0^{1/2}\right)^{1/2}.$$

Combining this with the lower bound (1.5.7) and eliminating $\delta', \delta''$ through

(1.5.6), we arrive at

$$N^{2\sigma}\delta_1^2|R||S| + N^{2\sigma}\delta_1^{2/q}\delta_2^{1-1/q}|R|^{\frac{2q-1}{q}}|S|^{1/q}$$
$$\ll (N^2|R|^{5/8}|S|^{1/2} + N^{3/2}|R|^{5/8}|S| + \delta_2^{1/4}N^{3/2}T^{1/4}|R|^{5/8}|S|^{5/8})T^{\varepsilon}.$$

One of the three terms on the right is dominant. We now wish to eliminate $|S|$. This can be done as the exponent of $|S|$ for each term on the right-hand side lies between $1/q$ and 1; note that $q \geq 2$. Therefore, for each term on the right-hand side, $|S|$ can be eliminated by an appropriate interpolation of the two left side terms. After a few calculations one obtains

$$|R| \ll T^{\varepsilon}\left(\delta_1^{-\frac{8}{7}}\delta_2^{-\frac{4}{7}}N^{\frac{16(1-\sigma)}{7}} + \delta_1^{-\frac{16}{3}}N^{\frac{4(3-4\sigma)}{3}} + \delta_1^{-\frac{5}{3}}\delta_2^{-\frac{1}{6}}N^{\frac{2(3-4\sigma)}{3}}T^{1/3}\right). \tag{1.5.8}$$

### 1.5.5 A large value estimate

Collecting the contributions (1.5.3) and (1.5.8) from Cases 1 and 2, we get the following bound for $|R|$:

**Lemma 1.5.4.** *Let $N \geq T^{1/2}$ and $R$ be a set of representative well-spaced class-I zeros. If $B_0, B_1 > 0$ and $q_0, q_1 \geq 2$ are parameters for which* (1.4.3) *holds and*

$$|R| \leq N,$$

*then for any $T^{-c} \ll \delta_1 < 1$ (for some $c > 0$) and $T^{-1} \leq \delta_2 \leq 1$, we have*

$$|R| \ll \Big(\delta_2^{-1}N^{2-2\sigma} + \delta_1^2\delta_2^{B_0-1}T^{B_0}N^{(3-4\sigma)q_0/2} + \delta_1^2\delta_2^{B_1-1}T^{B_1}N^{(3-4\sigma)q_1/2}$$
$$+ \delta_1^{-\frac{8}{7}}\delta_2^{-\frac{4}{7}}N^{\frac{16(1-\sigma)}{7}} + \delta_1^{-\frac{16}{3}}N^{\frac{4(3-4\sigma)}{3}} + \delta_1^{-\frac{5}{3}}\delta_2^{-\frac{1}{6}}N^{\frac{2(3-4\sigma)}{3}}T^{1/3}\Big)T^{\varepsilon}.$$

### 1.5.6 Proof of Theorem 1.1.1

From the analysis of Section 1.3 it only remains to find an estimate for the representative well-spaced class-I zeros and from Section 1.4 we may suppose that (1.4.6) holds and that $\sigma < (3q^* - 4)/(4q^* - 4B^* - 4)$. We also recall the restriction $\sigma \geq 1 - q_j/(4B_j + 4q_j - 4)$, $j = 0, 1$, we encountered in Section 1.4. We consider first the case that $|R| \leq N$. Applying Lemma 1.5.4 with admissible parameters $q_0, q_1, B_0, B_1$ gives

$$|R| \ll (\delta_2^{-1}N^{2-2\sigma} + \delta_1^2\delta_2^{B_0-1}T^{B_0}N^{(3-4\sigma)q_0/2} + \delta_1^2\delta_2^{B_1}T^{B_1-1}N^{(3-4\sigma)q_1/2}$$
$$+ \delta_1^{-\frac{8}{7}}\delta_2^{-\frac{4}{7}}N^{\frac{16(1-\sigma)}{7}} + \delta_1^{-\frac{16}{3}}N^{\frac{4(3-4\sigma)}{3}} + \delta_1^{-\frac{5}{3}}\delta_2^{-\frac{1}{6}}N^{\frac{2(3-4\sigma)}{3}}T^{1/3})T^{\varepsilon}.$$

Let us choose for $T^{1/2} < N \le T$ the parameters[8] $\delta_1$ and $\delta_2$ in such a way that

$$\delta_2^{-1} N^{2-2\sigma} = \delta_1^2 \delta_2^{B-1} T^B N^{(3-4\sigma)q/2} = T^{2(1-\sigma)},$$

where $(q, B)$ is the couple $(q_j, B_j)$, $j = 0, 1$, for which $\delta_2^{B-1} T^B N^{(3-4\sigma)q/2}$ is maximal. This is equivalent to

$$\delta_2 = N^{2-2\sigma} T^{2\sigma-2}, \quad \delta_1 = N^{\frac{(4B-4+4q)\sigma-(4B-4+3q)}{4}} T^{\frac{B-2B\sigma}{2}},$$

whence $T^{-c} < \delta_1 < 1$ (with e.g. $c = \max_{j=0,1}\{7q_j/4 + 3B_j/2\} + 2$) and $T^{-1} \le \delta_2 \le 1$ in view[9] of (1.4.6). Inserting this choice in the estimate for $|R|$ gives, as $N \ge T^{1/2}$, and further imposing the restriction $\sigma \ge 1 - q/(4B + 4q)$, $|R|$ is bounded by

$$\begin{aligned}
&\ll \Big(T^{2(1-\sigma)} + N^{\frac{(8B+6q)-(8B+8q)\sigma}{7}} T^{\frac{(8B-8)\sigma-(4B-8)}{7}} \\
&+ N^{\frac{(16B-4+12q)-(16B+16q)\sigma}{3}} T^{\frac{16B\sigma-8B}{3}} + N^{\frac{(20B+15q)-(20B+20q+8)\sigma}{12}} T^{\frac{(10B-2)\sigma-(5B-4)}{6}}\Big) T^{\varepsilon} \\
&\ll \Big(T^{2(1-\sigma)} + T^{\frac{(3q+8)-(4q-4B+8)\sigma}{7}} + T^{\frac{(6q-2)-(8q-8B)\sigma}{3}} + T^{\frac{(15q+16)-(20q-20B+16)\sigma}{24}}\Big) T^{\varepsilon} \\
&\ll T^{2(1-\sigma)+\varepsilon},
\end{aligned}$$

where the second, third and fourth summand give new restrictions on $\sigma$. Summarizing, for $j = 0, 1$, we obtain the set of constraints

$$\sigma \ge \frac{3q_j - 6}{4q_j - 4B_j - 6}, \ \sigma \ge \frac{3q_j - 4}{4q_j - 4B_j - 3},$$

$$\sigma \ge \frac{15q_j - 32}{20q_j - 20B_j - 32}, \ \sigma \ge \frac{3q_j + 4B_j}{4q_j + 4B_j},$$

provided that also $q_j > B_j + 8/5$ which ensures that the denominators in the above fractions are all positive, as otherwise we would not obtain any range for $\sigma$. The density hypothesis holds under these restrictions[10] for $\sigma$ and if $|R| \le N$.

Now, suppose that $|R| > N$. As $N \ge T^{1/2}$, this implies that $|R| > T^{1/2}$. Select now a subset of representative well-spaced class-I zeros $R'$ such that $|R'| = \lfloor T^{1/2} \rfloor$. Now $|R'| \le N$ and the entire analysis above can be performed for $R'$ to give $|R'| \ll T^{2(1-\sigma)+\varepsilon} \ll T^{1/2-\varepsilon}$, if $\sigma > 3/4$ say, which is impossible

[8] If $T < N < T^{1+\varepsilon}$, we select the same parameters as were $N = T$, that is $\delta_1 = T^{((4q-4)\sigma-(3q+2B-4))/4}$ and $\delta_2 = 1$. The verification of the density hypothesis then becomes the same calculation as for $N = T$ except for some extra factors that can be absorbed in $T^{\varepsilon}$.

[9] One may verify through a monotonicity argument that also $\delta_1 \le 1$ even if $(q, B) \ne (q^*, B^*)$.

[10] The restriction $\sigma \ge (3q_j + 4B_j)/(4q_j + 4B_j)$ is implied by $\sigma \ge (3q_j - 6)/(4q_j - 4B_j - 6)$ if, say, $B_j \ge 3/4$.

(for large enough $T$). Therefore $|R|$ must have been smaller than $N$ to begin with.

It only remains to pick the best possible $q_0, q_1, B_0$ and $B_1$. In order to find admissible values, we are going to appeal to [47, Thm. 8.2, p. 206]. Given an *exponent pair* $(a, b)$, this result guarantees that $q_0 = 6$, $B_0 = 1 + \varepsilon$, $q_1 = 2(1 + 2a + 2b)/a$ and $B_1 = (a + b)/a + \varepsilon$ are admissible values for (1.4.3). The couple $(q_0, B_0) = (6, 1 + \varepsilon)$ subsequently gives the restriction $\sigma \geq 6/7$.

For the other couple $(q_1, B_1)$ it turns out that the restriction $\sigma \geq (3q_1 - 6)/(4q_1 - 4B_1 - 6)$ is the critical one. Rewriting this range in terms of the exponential pair $(a, b)$ gives $\sigma \geq 1 - 1/(3a + 6b + 4)$. Our task is therefore to minimize $a + 2b$. To the best of our knowledge, the exponent pair $(55/194 + \varepsilon, 110/194 + \varepsilon)$ is the best available choice[11] at the moment. This exponent pair is derived from first applying Process A and then Process B on the exponent pair $(13/84 + \varepsilon, 55/84 + \varepsilon)$ that Bourgain established in [7]. Applying Theorem 1.2.7 with exponent pair $(55/194 + \varepsilon, 110/194 + \varepsilon)$ gives $q_1 = 1048/55$, $B_1 = 3 + \varepsilon$ and $H(T) = T^{55/359}$. One ultimately finds that the density hypothesis is valid in the range $\sigma \geq 1407/1601$. This concludes the proof of Theorem 1.1.1.

## 1.6 A zero-density estimate for the Riemann zeta function

In this section we establish the zero-density estimate $N(\sigma, T) \ll T^{\frac{24(1-\sigma)}{30\sigma - 11} + \varepsilon}$ for the Riemann zeta function for a broader range for $\sigma$ than what Ivić initially obtained in [47, Thm. 11.2, Eq. 11.31]. The precise exponent in $T$ comes from an optimization with respect to the specific technology that Ivić employed in his proof; the crucial factors are the exponent pair $(2/7, 4/7)$ from which [47, Thm. 8.2] delivers a bound that was used for the estimation of the class-II zeros, and the specific moments $q_0 = 6$ and $q_1 = 19$ chosen for the mixed moment argument (1.4.3). For specific $\sigma$ in the interval under consideration here, it should be possible to optimize the exponent pair and the mixed moment exponents to obtain a better exponent for $T$ in the final zero-density estimate. This is however not the main focus of the appendix and we decided not to pursue this here. Obtaining the best zero-density estimates for the Riemann

[11] The preprint [76] claims that $(1/4 + \varepsilon, 7/12 + \varepsilon)$ is also an exponent pair, which would deliver a lower value for $a + 2b$. This exponent pair is derived from applying process B on $(1/12 + \varepsilon, 3/4 + \varepsilon)$ and this was, according to the preprint, supposed to have been shown in [71]. However, it is unclear how the exponent pair $(1/12 + \varepsilon, 3/4 + \varepsilon)$ follows from [71, Thm. 1]. Robert also does not claim his result implies that $(1/12 + \varepsilon, 3/4 + \varepsilon)$ would be an exponent pair.

zeta function by selecting the optimal exponent pairs is one of the objectives of the preprint [76]. Our main goal here is to illustrate how Bourgain's method allows one to use Heath-Brown's double zeta sum estimate Lemma 1.5.3 to increase the range of validity of the zero-density estimate

$$N(\sigma,T) \ll T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon},$$

from $0.8908 \approx 155/174 \leq \sigma \leq 17/18$ to $0.8885 \approx 279/314 \leq \sigma \leq 17/18$. As this is the only place where we modify Ivić's argument, we do not achieve a lower exponent for $T$ in the zero-density estimate.

### 1.6.1 Some modifications

As the proof method is very similar to the proof of the density hypothesis in the range $\sigma \geq 1407/1601$ for $L(s,f)$ discussed in detail through this chapter, we only point out the differences. Moreover, since Ivić had already shown Theorem 1.1.2 when $155/174 \leq \sigma \leq 17/18$, we shall only work under the hypothesis $279/314 \leq \sigma < 155/174$.

We employ the same zero-detection method as in Section 1.3, with the obvious changes that $L(s,f)$ is replaced by $\zeta(s)$ and $\mu_f$ by the classical Möbius function. Now in the calculation of $\zeta(s)M_X(s)$, when shifting the line of integration, we encounter an additional pole at $z = 1-s$ which delivers the extra term $M_X(1)Y^{1-s}\Gamma(1-s)$. When $|\Im m\, s| \geq \log^2 T$, this term is however still $o(1)$ as $T \to \infty$. So, if a $\zeta$-zero $\rho = \beta + it$ satisfies $|t| \geq \log^2 T$, it must still be either a class-I zero or a class-II zero. Instead of (1.3.3), we thus obtain

$$N(\sigma,T) \ll (|R_1| + |R_2| + 1)T^{\varepsilon},$$

as $N(1/2, \log^2 T) \ll \log^3 T$, say.

When Ivić handles the class-II zeros $R_2$, he takes $Y = T^{\frac{6}{30\sigma-11}}$ and $\nu = 2$. Therefore (1.3.8) is replaced by

$$T^{\frac{8}{30\sigma-11}} \leq N \leq T^{\frac{12}{30\sigma-11}+\varepsilon}. \tag{1.6.1}$$

As explained above, the estimation of the class-II zeros is slightly different than in the previous section, but following the argument of Ivić [47, Section 11.2], one obtains that $|R_2|$ is bounded by [47, Eq. 11.41],

$$|R_2| \ll (TY^{3-6\sigma} + T^3Y^{19(1/2-\sigma)})T^{\varepsilon} \ll T^{24(1-\sigma)/(30\sigma-11)+\varepsilon}.$$

With the same technology as in Section 1.4 with the mixed moment parameters $q_0 = 6, A_0 = 1+\varepsilon, q_1 = 19, A_1 = 3+\varepsilon$, one finds an estimate for the representative well-spaced class-I zeros [47, Eq. 11.42],

$$|R| \ll (N^{2-2\sigma} + TN^{(65-84\sigma)/6})T^{\varepsilon},$$

which also gives the desired $T^{24(1-\sigma)/(30\sigma-11)+\varepsilon}$ estimate, provided $N$ satisfies

$$N \geq T^{\frac{6}{65-84\sigma}\cdot\frac{35-54\sigma}{30\sigma-11}}.$$

In view of (1.6.1) the above estimate is always valid if $\sigma \geq 155/174$ and this concludes Ivić's argument. For the remaining range, we may thus assume

$$T^{\frac{8}{30\sigma-11}} \leq N \leq T^{\frac{6}{65-84\sigma}\cdot\frac{35-54\sigma}{30\sigma-11}}. \tag{1.6.2}$$

The analysis of Section 1.5 is mostly analogous, except at the end in the treatment of case 2 where instead of the bound $N \geq T^{1/2}$ we shall use $N \geq T^{\frac{8}{30\sigma-11}}$ and instead of $|R| \leq N$ we use the modified

$$T^{24(1-\sigma)/(30\sigma-11)-\varepsilon} \leq |R| \leq N. \tag{1.6.3}$$

This results in the bound

$$\begin{aligned}
&N^{2\sigma}\delta' \sum_{\alpha\in\mathcal{I}_0} |R_\alpha||S\cap D_\alpha(\delta')| \\
&\ll T^\varepsilon N^{3/2}|R|^{1/2}(1+R^{1/4}N^{-1}T^{1/2})^{1/2}\left(|S|N+|S|^2+|S|^{5/4}T_0^{1/2}\right)^{1/2} \\
&\ll T^\varepsilon N^{3/2}|R|^{1/2}(1+R^{1/4}T^{-8/(30\sigma-11)}T^{1/2})^{1/2}\left(|S|N+|S|^2+|S|^{5/4}T_0^{1/2}\right)^{1/2} \\
&\ll T^\varepsilon N^{3/2}|R|^{5/8}T^{\frac{30\sigma-27}{4(30\sigma-11)}}\left(|S|N+|S|^2+|S|^{5/4}T_0^{1/2}\right)^{1/2}.
\end{aligned}$$

The lower inequality for $|R|$ in (1.6.3) was only imposed to guarantee $1 \ll R^{\frac{1}{4}}T^{\frac{-8}{30\sigma-11}}T^{\frac{1}{2}}$. Combining this with the lower inequality (1.5.7), we find

$$\begin{aligned}
N^{2\sigma}\delta_1^2|R||S| + N^{2\sigma}\delta_1^{2/q}\delta_2^{1-1/q}|R|^{\frac{2q-1}{q}}|S|^{1/q} \ll \Big(&N^2|R|^{5/8}|S|^{1/2}T^{\frac{30\sigma-27}{4(30\sigma-11)}} \\
&+ N^{3/2}|R|^{5/8}|S|T^{\frac{30\sigma-27}{4(30\sigma-11)}} + \delta_2^{1/4}N^{3/2}T^{1/4}|R|^{5/8}|S|^{5/8}T^{\frac{30\sigma-27}{4(30\sigma-11)}}\Big)T^\varepsilon.
\end{aligned}$$

With a suitable interpolation to eliminate $|S|$, one then finds after a few calculations

$$\begin{aligned}
|R| \ll T^\varepsilon\Big(&\delta_1^{-\frac{8}{7}}\delta_2^{-\frac{4}{7}}N^{\frac{16(1-\sigma)}{7}}T^{\frac{6(10\sigma-9)}{7(30\sigma-11)}} + \delta_1^{-\frac{16}{3}}N^{\frac{4(3-4\sigma)}{3}}T^{\frac{2(10\sigma-9)}{30\sigma-11}} \\
&+\delta_1^{-\frac{5}{3}}\delta_2^{-\frac{1}{6}}N^{\frac{2(3-4\sigma)}{3}}T^{\frac{1}{3}+\frac{10\sigma-9}{30\sigma-11}}\Big).
\end{aligned}$$

### 1.6.2 The large value estimate

The corresponding large value estimate then becomes

**Lemma 1.6.1.** *Let* $N \geq T^{\frac{8}{30\sigma-11}}$ *and* $R$ *be a set of representative well-spaced class-I zeros. If* $(q_0, A_0)$ *and* $(q_1, A_1)$ *satisfy* (1.4.3) *and*

$$T^{24(1-\sigma)/(30\sigma-11)-\varepsilon} \leq |R| \leq N,$$

*then for any* $T^{-c} \ll \delta_1 < 1$ *(for some* $c > 0$*) and* $T^{-1} \leq \delta_2 \leq 1$*,*

$$\begin{aligned}|R| \ll \big(&\delta_2^{-1} N^{2-2\sigma} + \delta_1^2 \delta_2^{A_0-1} T^{A_0} N^{(3-4\sigma)q_0/2} + \delta_1^2 \delta_2^{A_1-1} T^{A_1} N^{(3-4\sigma)q_1/2} \\ &+ \delta_1^{-\frac{8}{7}} \delta_2^{-\frac{4}{7}} N^{\frac{16(1-\sigma)}{7}} T^{\frac{6(10\sigma-9)}{7(30\sigma-11)}} + \delta_1^{-\frac{16}{3}} N^{\frac{4(3-4\sigma)}{3}} T^{\frac{2(10\sigma-9)}{30\sigma-11}} \\ &+ \delta_1^{-\frac{5}{3}} \delta_2^{-\frac{1}{6}} N^{\frac{2(3-4\sigma)}{3}} T^{\frac{1}{3}+\frac{10\sigma-9}{30\sigma-11}}\big) T^{\varepsilon}.\end{aligned}$$

### 1.6.3 Proof of Theorem 1.1.2

As we already have a bound for the class-II zeros, we are only required to estimate $|R|$. Suppose first that $T^{24(1-\sigma)/(30\sigma-11)-\varepsilon} \leq |R| \leq N$. We apply the large value estimate Lemma 1.6.1 with $(q_0, A_0) = (6, 1+\varepsilon)$ and $(q_1, A_1) = (19, 3+\varepsilon)$. This gives

$$\begin{aligned}|R| \ll (&\delta_2^{-1} N^{2-2\sigma} + \delta_1^2 T N^{9-12\sigma} + \delta_1^2 \delta_2^2 T^3 N^{\frac{19(3-4\sigma)}{2}} + \delta_1^{-\frac{8}{7}} \delta_2^{-\frac{4}{7}} N^{\frac{16(1-\sigma)}{7}} T^{\frac{6(10\sigma-9)}{7(30\sigma-11)}} \\ &+ \delta_1^{-\frac{16}{3}} N^{\frac{4(3-4\sigma)}{3}} T^{\frac{2(10\sigma-9)}{30\sigma-11}} + \delta_1^{-\frac{5}{3}} \delta_2^{-\frac{1}{6}} N^{\frac{2(3-4\sigma)}{3}} T^{\frac{1}{3}+\frac{10\sigma-9}{30\sigma-11}}) T^{\varepsilon}.\end{aligned}$$

We choose the parameters $\delta_1$ and $\delta_2$ in such a way that

$$\delta_2^{-1} N^{2-2\sigma} = \delta_1^2 \delta_2^2 T^3 N^{\frac{19(3-4\sigma)}{2}} = T^{\frac{24(1-\sigma)}{30\sigma-11}}.$$

This is equivalent to

$$\delta_2 = N^{2-2\sigma} T^{\frac{24(\sigma-1)}{30\sigma-11}}, \quad \delta_1 = \delta_2^{-1} N^{\frac{19(4\sigma-3)}{4}} T^{\frac{57(1-2\sigma)}{2(30\sigma-11)}} = N^{\frac{84\sigma-65}{4}} T^{\frac{105-162\sigma}{2(30\sigma-11)}},$$

whence $T^{-c} < \delta_1 < 1$, for $c = 13$, say, and $T^{-1} \leq \delta_2 \leq 1$ in view of (1.6.2). We find, using $N \geq T^{\frac{8}{30\sigma-11}}$,

$$\begin{aligned}|R| &\ll (T^{\frac{24(1-\sigma)}{30\sigma-11}} + T N^{9-12\sigma} + N^{\frac{138-176\sigma}{7}} T^{\frac{612\sigma-378}{7(30\sigma-11)}} + N^{\frac{272-352\sigma}{3}} T^{\frac{452\sigma-298}{30\sigma-11}} \\ &\quad + N^{\frac{345-448\sigma}{12}} T^{\frac{282\sigma-185}{2(30\sigma-11)}+\frac{1}{3}}) T^{\varepsilon} \\ &\ll (T^{\frac{24(1-\sigma)}{30\sigma-11}} + T T^{\frac{8(9-12\sigma)}{30\sigma-11}} + T^{\frac{8(138-176\sigma)}{7(30\sigma-11)}} T^{\frac{612\sigma-378}{7(30\sigma-11)}} + T^{\frac{8(272-352\sigma)}{3(30\sigma-11)}} T^{\frac{452\sigma-298}{30\sigma-11}} \\ &\quad + T^{\frac{2(345-448\sigma)}{3(30\sigma-11)}} T^{\frac{282\sigma-185}{2(30\sigma-11)}+\frac{1}{3}}) T^{\varepsilon} \\ &\ll T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}.\end{aligned}$$

The second, third, fourth and fifth summand give respectively the conditions $\sigma \geq 37/42 \approx 0.8809$, $\sigma \geq 279/314 \approx 0.8885$, $\sigma \geq 605/694 \approx 0.8717$ and $\sigma \geq 659/742 \approx 0.8881$ on the range of validity of this estimate. Therefore the desired zero-density is valid under the condition $T^{24(1-\sigma)/(30\sigma-11)-\varepsilon} \leq |R| \leq N$.

If $|R| \leq T^{24(1-\sigma)/(30\sigma-11)-\varepsilon}$, there is nothing left to prove and if $|R| \geq N$ one may take as before a sufficiently large subset of representative well-spaced class-I zeros to obtain a contradiction as $N \geq T^{\frac{8}{30\sigma-11}} \gg T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}$ in the range under question for $\sigma$. This completes the proof of Theorem 1.1.2.

# Chapter 2

# On the density hypothesis for Dirichlet $L$-functions

*"Cut away half of a rod and keep on halving what is left, and there will be no end to that process."*

—Chuang Tzu (a Chinese philosopher) 369 BC–286 BC

In this chapter, we employ Bourgain's dichotomy to establish a large value estimate for Dirichlet polynomials. As an application, we obtain an alternative proof of Heath-Brown's result concerning the density hypothesis for the zeros of Dirichlet $L$-functions.

## 2.1 Introduction

Zero-density estimates for the Dirichlet $L$-functions are also a central topic in analytic number theory. Let $\chi$ be a Dirichlet character and $L(s,\chi)$ the associated Dirichlet $L$-function. We denote by $N(\sigma,T,\chi)$ the number of zeros $\rho=\beta+it$ of $L(s,\chi)$ in the rectangle $\sigma\leq\beta\leq 1$, $|t|\leq T$. We are interested in the density hypothesis for the Dirichlet $L$-functions:

$$\sum_{\chi \bmod q} N(\sigma,T,\chi) \ll (qT)^{2(1-\sigma)+\varepsilon}. \tag{2.1.1}$$

Let $\sigma_1$ be such that the density hypothesis (2.1.1) holds for $\sigma\geq\sigma_1$. We record the known results as follows, we may take: $\sigma_1\leq 5/6$ (Balasubramanian and Ramachandra [2], Huxley [41], Jutila [49]), $\sigma_1\leq 21/26$ (Jutila [50]), $\sigma_1\leq 4/5+\varepsilon$ (Huxley and Jutila [18]). The best-known result is due to Heath-Brown.

**Theorem 2.1.1** (Heath-Brown [33]). *We have the bound*

$$\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll_\varepsilon (qT)^{2(1-\sigma)+\varepsilon} \tag{2.1.2}$$

*for* $\sigma \geq 15/19$.

It is natural to ask whether the dichotomy technique developed by Bourgain can be used to investigate the density hypothesis for the Dirichlet $L$-functions to produce better numerical results. The main goal of this chapter is to explore this direction; we ultimately obtain a large value estimate[1] for Dirichlet polynomials and reestablish Theorem 2.1.1.

We adopt the following convention in the use of $\varepsilon$ to denote a small positive quantity, namely that at certain points, which we shall not specify, we shall change $\varepsilon$ by a constant factor. Denote by $\tau(n)$ the number of divisors of $n$. Denote by $\tau_k(n)$ the number of ways of writing $n$ as product of $k$ positive integers, and denote by $\phi(n)$ the Euler totient function. $\sum^*$ denotes summation over primitive character only. $\mathbf{1}_E$ denotes the indicator function of some set $E$. We use $\ll$ and $\gg$ to denote Vinogradov's notation, while implied constants depend at most on $\varepsilon$.

## 2.2 The zero-detection method

Our starting point is the same as in Section 1.3. Using the zero-detection method (see [62, Chapter 12], [47, Chapter 11]), we estimate

$$\sideset{}{^*}\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (|R_1| + |R_2| + 1)(qT)^\varepsilon, \tag{2.2.1}$$

where $R_1$ refers to the class-I zeros $(\rho, \chi)$, $\rho = \beta + it$, $\beta \geq \sigma$, $|t| \leq T$, $\chi$ primitive $(\bmod\ q)$, $L(\rho, \chi) = 0$ for which

$$\left| \sum_{X < n \leq Y \log^2 Y} a(n)\chi(n) n^{-\rho} e^{-n/Y} \right| \geq \frac{1}{6}, \quad |a(n)| \leq \tau(n), \tag{2.2.2}$$

and the class-II zeros $R_2$ satisfy

$$\begin{aligned} &\left| \int_{-\log^2 qT}^{-\log^2 qT} L\left(\frac{1}{2} + i(t+u), \chi\right) M_X\left(\frac{1}{2} + it + u, \chi\right) \right. \\ &\qquad \left. \cdot Y^{\frac{1}{2} - \beta + iu} \Gamma\left(\frac{1}{2} - \beta + iu\right) \mathrm{d}u \right| \geq \frac{1}{6}. \end{aligned} \tag{2.2.3}$$

[1] The large value estimate, obtained in autumn 2022, coincides with Bourgain's result [6] from 2002, of which the author was initially unaware until Diamond provided Bourgain's paper in the spring of 2023.

Here $M_X(s,\chi) = \sum_{n\le X} \frac{\mu(n)\chi(n)}{n^s}$, $X = (qT)^\varepsilon$, and $Y = (qT)^c$ for a suitable chosen exponent $c$. Note that our zeros satisfy

$$|t_1 - t_2| \geq (qT)^\varepsilon.$$

if $\rho_1 = \beta_1 + it_1$ and $\rho_2 = \beta_2 + it_2$ are zeros of the same $L$-function.

### 2.2.1 The contribution of the class-II zeros

We begin by analyzing the contribution of the class-II zeros $R_2$. If we set

$$\left|L\left(\frac{1}{2} + i\gamma_r, \chi\right)\right| = \max_{-\log^2 qT < u \le \log^2 qT} \left|L\left(\frac{1}{2} + it_r + iu, \chi\right)\right|$$

where $t_1, t_2, ..., t_{R_2}$ are imaginary parts of the class-II zeros. Then from (2.2.3) we conclude that

$$1 \ll (qT)^\varepsilon Y^{1/2-\sigma} \left|L\left(\frac{1}{2} + i\gamma_r, \chi_r\right)\right|, \quad r = 1, 2, ..., |R_2|. \tag{2.2.4}$$

Notice that $|t_r - t_s| \gg (qT)^\varepsilon$ for $r \neq s$ ,$\chi_r \neq \chi_s$. Raising (2.2.4) to the power 4, by the fourth moment estimate, we have

$$|R_2| \ll (qT)^\varepsilon Y^{2-4\sigma} \sum_{r \le |R_2|} \left|L\left(\frac{1}{2} + it_r, \chi_r\right)\right|^4 \ll (qT)^{1+\varepsilon} Y^{2-4\sigma}. \tag{2.2.5}$$

Therefore, upon choosing for $Y = (qT)^{1/2+\varepsilon}$, we obtain for $\sigma > 3/4$

$$|R_2| \ll (qT)^{1+\varepsilon} (qT)^{(1/2+\varepsilon)(2-4\sigma)} \ll (qT)^{2(1-\sigma)+\varepsilon}.$$

This concludes the analysis of the class-II zeros.

### 2.2.2 Representative class-I zeros

The rest of the argument is then to bound the contribution of the class-I zeros. First we shall restrict the well-spaced class-I zeros even further. Since each $\rho$ counted by $R_1$ satisfies

$$\left|\sum_{M<n\le 2M} a(n)\chi(n)n^{-\rho}e^{-n/Y}\right| \geq \frac{1}{\log Y} \tag{2.2.6}$$

for at least one of $O(\log Y)$ valus $(qT)^\varepsilon \le M = 2^{-j} Y \log^2 Y$, $j = 1, 2, ...,$ and we can consider representative zeros of those counted by $R_1$ which are $\gg R_1/\log Y$ in number and which satisfy (2.2.6). Denote the set of representative zeros is $R_1'$. We then raise (2.2.6) to power $k$ such that

$$M^k \le Y^{2+\varepsilon} < M^{k+1}, \tag{2.2.7}$$

where $2 \le k \ll_\varepsilon 1$. Then we have

$$Y^{\frac{4}{3}} = (Y^2)^{\frac{2}{2+1}} < (M^{k+1})^{\frac{k}{k+1}} = M^k \le Y^{2+\varepsilon}. \tag{2.2.8}$$

and

$$\left| \sum_{M^k < n \le 2^k M^k} a'(n)\chi(n)n^{-\rho} \right| \ge (1/\log Y)^k, \tag{2.2.9}$$

where $|a'(n)| \ll \tau_{2k}(n) \ll n^\varepsilon$. Let $A$ be a multiset consisting of elements of $R_1'$. Using the estimate (2.2.7), (2.2.9) and partial summation we obtain

$$\begin{aligned} |A| \ll &(\log Y)^k \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le 2N} a'(n)\chi(n)n^{-\rho} \right| \\ =&(\log Y)^k \sum_{(\rho,\chi)\in A} \left| \int_N^{2N} u^{-\beta} \,\mathrm{d} \left( \sum_{N<n\le u} a'(n)\chi(n)n^{-it} \right) \right| \\ =&(\log Y)^k \sum_{(\rho,\chi)\in A} \left| (2N)^{-\beta} \sum_{N<n\le 2N} a'(n)\chi(n)n^{-it} \right. \\ &\qquad\qquad \left. + \int_N^{2N} \beta u^{-\beta-1} \sum_{N<n\le u} a'(n)\chi(n)n^{-it} \,\mathrm{d}u \right| \\ \ll&(\log Y)^k N^{-\sigma} \max_{N<u\le 2N} \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le u} a'(n)\chi(n)n^{-it} \right| \end{aligned}$$

for some $M^k \le N < 2^k M^k$. If we let $a''(n) = 0$ after the point where the above maximum is reached, but $a''(n) = a'(n)$ otherwise, we obtain

$$|A| \ll N^{-\sigma+\varepsilon} \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le 2N} a''(n)\chi(n)n^{-it} \right|,$$

and

$$|A| \ll N^{-2\sigma+\varepsilon} \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le 2N} a''(n)\chi(n)n^{-it} \right|^2,$$

the last inequality being derived from Cauchy-Schwarz. If we now set $b(n) = b(n, A) = \epsilon a''(n)$ for a sufficiently small $\epsilon$ such that $|b(n)| \le 1$, we have

$$|A| \ll N^{-\sigma+\varepsilon} \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le 2N} b(n)\chi(n)n^{-it} \right|, \tag{2.2.10}$$

and

$$|A| \ll N^{-2\sigma+\varepsilon} \sum_{(\rho,\chi)\in A} \left| \sum_{N<n\le 2N} b(n)\chi(n)n^{-it} \right|^2. \tag{2.2.11}$$

In particular, choosing $A = R_1^{'}$, we get

$$|R_1^{'}| \ll N^{-\sigma+\varepsilon} \sum_{(\rho,\chi)\in R_1^{'}} \left| \sum_{N<n\leq 2N} b(n)\chi(n)n^{-it} \right|, \tag{2.2.12}$$

from (2.2.10). We therefore have

$$|R_1| \ll (\log Y)|R_1^{'}| \ll N^{-\sigma+\varepsilon} \sum_{(\rho,\chi)\in R_1^{'}} \left| \sum_{N<n\leq 2N} b(n)\chi(n)n^{-it} \right|. \tag{2.2.13}$$

We do emphasize here again that $N$ and $b(n)$ do depend on the set $A$. Throughout the rest of the chapter, we shall write $b(n)$ and $N$ when we refer to the set $R$. If any other set of the class-I zeros $A$ is considered, we shall explicitly mention the dependence of $b(n)$ and $N$ on $A$. On the other hand we note that $M^k \leq N(A) < 2^k M^k$, therefore $N \ll N(A) \ll N$ for any $A$. Furthermore, as we take $Y = (qT)^{1/2+\varepsilon}$, one has from (2.2.8)

$$T^{\frac{2}{3}+\frac{4\varepsilon}{3}} < N < T^{1+3\varepsilon}. \tag{2.2.14}$$

## 2.3 Jutila and Heath-Brown's argument

In this section we deduce a first non-trivial estimate on the number of class-I zeros by the Halász-Montgomery inequality. The aim of the Halász-Montgomery inequality is to derive distributional properties for Dirichlet polynomials from the $\zeta$ function or its partial sum. The following lemma is a reformulation of the estimate in [33, Lemma 5].

**Lemma 2.3.1.** *Let $\sigma > \frac{3}{4}$, $k$ be any fixed positive integer, and $R_1' = \{(\beta_r + it_r, \chi(r)) : 1 \leq r \leq |R_1'|\}$ be a set of representative class-I zeros. Assume that $N < qT$ in* (2.2.10). *For a character $\chi$, $\ell \in \mathbb{Z}$, define*

$$\Delta(\ell,\chi) = \#\{(\beta+it,\chi,\beta'+it',\chi') \in R_1' \times R_1' : \overline{\chi}\chi' = \chi, |t'-t-\ell| < 1\}. \tag{2.3.1}$$

*Then we have*

$$|R_1'| \ll \Bigg\{ N^{2-2\sigma} + N^{\frac{3k}{4}-k\sigma} \cdot \left[ \int_{-2\log^4 qT}^{2\log^4 qT} \sum_{\chi \bmod q}^{*} \sum_{\ell\in\mathbb{Z}} \Delta_A(\ell,\chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-\frac{1}{2}+iv+i\ell} \right| \mathrm{d}v \right]^{\frac{1}{2}} \Bigg\} (qT)^{\varepsilon} \tag{2.3.2}$$

*for some $M \ll (qT)^{1+\varepsilon}/N$ and $c_n \leq (qT)^{\varepsilon}$.*

*Proof.* We denote

$$H_N(t,\chi)=\sum_{n=1}^{\infty}\left[e^{-(n/2N)^h}-e^{-(n/N)^h}\right]\chi(n)n^{-it}$$

with

$$h=\log^2 qT.$$

By applying the Halász-Montgomery inequality [62, Lemma 1.7, p. 6], from (2.2.10) we get

$$|R_1'|^2N^{2\sigma-2\varepsilon}\ll|R_1'|N^2+N\sum_{r\neq s}|H_N(t_r-t_s,\chi_r\overline{\chi_s})|. \tag{2.3.3}$$

By Huxley's reflection result (cf. [50, Lemma 1] ), for $r\neq s$, we have

$$|H_N(t_r-t_s,\chi_r\overline{\chi_s})|\ll_\varepsilon N^{1/2}q^\varepsilon\int_{-h^2}^{h^2}\left|\sum_{n=1}^{M}\overline{\chi_r}(n)\chi_s(n)n^{-(1/2)+i(t_r-t_s+v)}\right|\mathrm{d}v+1, \tag{2.3.4}$$

where

$$\frac{q(T+h^3)}{N}\ll M\ll\frac{q(T+h^3)}{N}. \tag{2.3.5}$$

Substituting (2.3.4) into (2.3.3), raising the sums to a suitable power $k$, and applying Hölder's inequality give

$$\begin{aligned}
&|R_1'|^2N^{2\sigma-2\varepsilon}\\
&\ll|R_1'|N^2+N|R_1'|^2+N^{3/2}q^\varepsilon\int_{-h^2}^{h^2}\sum_{r\neq s}\left|\sum_{n=1}^{M}\overline{\chi_r}(n)\chi_s(n)n^{-(1/2)+i(t_r-t_s+v)}\right|\mathrm{d}v\\
&\le|R_1'|N^2+N|R_1'|^2\\
&+N^{3/2}q^\varepsilon\int_{-h^2}^{h^2}\left[\sum_{r\neq s}\left|\sum_{n=1}^{M}\overline{\chi_r}(n)\chi_s(n)n^{-(1/2)+i(t_r-t_s+v)}\right|^k\right]^{\frac{1}{k}}|R_1'|^{2(1-\frac{1}{k})}\,\mathrm{d}v\\
&\le|R_1'|N^2+N|R_1'|^2\\
&+N^{3/2}(qT)^\varepsilon|R_1'|^{2-\frac{2}{k}}\int_{-h^2}^{h^2}\left[\sum_{r,s}\left|\left(\sum_{n=1}^{M}\overline{\chi_r}(n)\chi_s(n)n^{-(1/2)+i(t_r-t_s+v)}\right)\right|^k\right]^{\frac{1}{k}}\mathrm{d}v
\end{aligned} \tag{2.3.6}$$

Moreover, by the definition of $\Delta(\ell,\chi)$, the integral in (2.3.6) is then bounded by

$$\int_{-2h^2}^{2h^2} \left[ \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \right]^{\frac{1}{k}} \mathrm{d}v$$

$$\ll (qT)^{\varepsilon} \left[ \int_{-2h^2}^{2h^2} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v \right]^{\frac{1}{k}} \tag{2.3.7}$$

where $|c_n| \leq \tau_k(n) \ll (qT)^{\varepsilon}$. Substitution of (2.3.7) in (2.3.6) gives

$$|R_1'|^2 N^{2\sigma - 2\varepsilon} \ll |R_1'| N^2 + N|R_1'|^2 + N^{3/2}(qT)^{\varepsilon} |R_1'|^{2-\frac{2}{k}}$$
$$\cdot \left[ \int_{-2h^2}^{2h^2} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v \right]^{\frac{1}{k}} \tag{2.3.8}$$

Since $\sigma \geq \frac{3}{4}$, the second term on the right hand side of (2.3.8) can be omitted here. Thus, (2.3.2) follows from (2.3.8). □

It thus remains to find adequate estimates for the integral in (2.3.2). Recall $\Delta(\ell, \chi) = \#\{(\beta + it, \chi, \beta' + it', \chi') \in R_1' \times R_1' : \overline{\chi}\chi' = \chi, |t' - t - \ell| < 1\}$. We have $\Delta(\ell, \chi) \leq |R_1'|$ and $\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi) \leq 2|R_1'|^2$. It follows that

$$\sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi)^2 \leq 2|R_1'|^3.$$

With this estimate the integral in (1.4.1) becomes through some applications of Cauchy-Schwarz

$$\int_{-2h^2}^{2h^2} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v$$
$$\leq \left( \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \Delta(\ell, \chi)^2 \right)^{1/2}$$
$$\cdot \left( \int_{-2h^2}^{2h^2} \left( \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right|^2 \right)^{1/2} \mathrm{d}v \right)$$
$$\ll |R_1'|^{3/2} \left( \int_{-2h^2}^{2h^2} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in \mathbb{Z}} \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right|^2 \mathrm{d}v \right)^{1/2} (qT)^{\varepsilon}$$
$$\ll |R_1'|^{3/2} (qT + M^k)^{1/2} (qT)^{\varepsilon}.$$

In the last step we used the classical mean value estimate on Dirichlet polynomials (cf. Montgomery [62, Thm. 7.4]). However, the aforementioned estimate is insufficient to derive Theorem 2.1.1. In the subsequent section, we will employ Bourgain's dichotomy to achieve a sharper estimate.

## 2.4 Bourgain's dichotomy

### 2.4.1 Lemmas on Dirichlet polynomials

In applying Bourgain's method, we shall require some preliminary lemmas on estimations for Dirichlet polynomials. The first lemma is Jutila's estimate for sums of Dirichlet polynomials over different sets (see [50, Lemma 2]).

**Lemma 2.4.1.** *Let $a_n, b_n, n = 1, ..., N$, be complex numbers such that $|a_n| \leq b_n$. Let $t_r$ be a real number and $\chi_r$ be a Dirichlet character, for $r = 1, ..., R$. Then we have*

$$\sum_{r,s=1}^{R} \left| \sum_{n=1}^{N} a_n \overline{\chi_r}(n) \chi_s(n) n^{i(t_r - t_s)} \right|^2 \leq \sum_{r,s=1}^{R} \left| \sum_{n=1}^{N} b_n \overline{\chi_r}(n) \chi_s(n) n^{i(t_r - t_s)} \right|^2.$$

We also require the following generalization of Lemma 2.4.1. It can also be viewed as a character version of Lemma 1.5.2 in Chapter 1 (cf. [5, Lemma 3.4]).

**Lemma 2.4.2.** *Let $a_n, b_n, n = 1, ..., N$, be complex numbers such that $|a_n| \leq b_n$. Additionally, let $t_r$ and $t'_s$ be real numbers, and $\chi_r$ and $\chi'_s$ be Dirichlet characters for $r = 1, \ldots, R$ and $s = 1, \ldots, S$, respectively. Define sets $E = \{(t_r, \chi_r) : 1 \leq r \leq R\}$ and $F = \{(t'_s, \chi'_s) : 1 \leq s \leq S\}$. Then we have*

$$\begin{aligned} &\sum_{\substack{(t_r,\chi_r)\in E \\ (t'_s,\chi'_s)\in F}} \left| \sum_{n=1}^{N} a_n \overline{\chi_r}(n) \chi'_s(n) n^{i(t_r - t'_s)} \right|^2 \\ &\leq \left( \sum_{\substack{(t_r,\chi_r)\in E \\ (t_{r'},\chi_{r'})\in E}} \left| \sum_{n=1}^{N} b_n \overline{\chi_r}(n) \chi_{r'}(n) n^{i(t_r - t_{r'})} \right|^2 \right)^{\frac{1}{2}} \\ &\qquad \cdot \left( \sum_{\substack{(t'_s,\chi'_s)\in F \\ (t'_{s'},\chi'_{s'})\in F}} \left| \sum_{n=1}^{N} b_n \overline{\chi'_s}(n) \chi'_{s'}(n) n^{i(t'_s - t'_{s'})} \right|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{2.4.1}$$

*Proof.* Expanding the left hand side of (2.4.1), we get

$$\begin{aligned}
&\sum_{n_1,n_2=1}^{N} a_{n_1}\overline{a_{n_2}} \sum_{\substack{(t_r,\chi_r)\in E\\ (t'_s,\chi'_s)\in F}} \overline{\chi_r}(n_1)\chi'_s(n_1)n_1^{i(t_r-t'_s)}\chi_r(n_2)\overline{\chi'_s}(n_2)n_2^{i(t'_s-t_r)}\\
&= \sum_{n_1,n_2=1}^{N} a_{n_1}\overline{a_{n_2}} \left(\sum_{(t_r,\chi_r)\in E} \overline{\chi_r}(n_1)\chi_r(n_2)\left(\frac{n_1}{n_2}\right)^{it_r}\right)\\
&\qquad\cdot\left(\sum_{(t'_s,\chi'_s)\in F} \chi'_s(n_1)\overline{\chi'_s}(n_2)\left(\frac{n_1}{n_2}\right)^{-it'_s}\right)\\
&\le \sum_{n_1,n_2=1}^{N} b_{n_1}b_{n_2}\left|\sum_{(t_r,\chi_r)\in E} \overline{\chi_r}(n_1)\chi_r(n_2)\left(\frac{n_1}{n_2}\right)^{it_r}\right|\\
&\qquad\cdot\left|\sum_{(t'_s,\chi'_s)\in F} \overline{\chi'_s}(n_1)\chi'_s(n_2)\left(\frac{n_1}{n_2}\right)^{it'_s}\right|\\
&\le \left(\sum_{n_1,n_2=1}^{N} b_{n_1}b_{n_2}\left|\sum_{(t_r,\chi_r)\in E} \overline{\chi_r}(n_1)\chi_r(n_2)\left(\frac{n_1}{n_2}\right)^{it_r}\right|^2\right)^{\frac{1}{2}}\\
&\qquad\cdot\left(\sum_{n_1,n_2=1}^{N} b_{n_1}b_{n_2}\left|\sum_{(t'_s,\chi'_s)\in F} \overline{\chi'_s}(n_1)\chi'_s(n_2)\left(\frac{n_1}{n_2}\right)^{it'_s}\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

This resulting bound is the same as the right hand side of (2.4.1). □

The final lemma is Heath-Brown's estimate on double zeta sums [32, Thm. 1], which is a generalization of Lemma 1.5.3. It is much deeper and is a crucial ingredient for our argument.

**Lemma 2.4.3.** *Suppose we are given a set of pairs* $(t_r, \chi(r))$*,* $r = 1, ..., R$*, where* $|t_r| \le T$*,* $\chi$ *primitive* $(\mathrm{mod}\ q)$ *and for* $r \ne s$ *either* $\chi_r \ne \chi_s$ *or* $|t_r - t_s| \ge (qT)^{\varepsilon}$*. Assume that* $N > (qT)^{2/3+\varepsilon}$*. Then*

$$\sum_{r,s=1}^{R}\left|\sum_{N<n\le 2N} \chi_r(n)\overline{\chi_s}(n)n^{i(t_s-t_r)}\right|^2 \le (qT)^{\varepsilon}(RN^2 + R^2N). \tag{2.4.2}$$

## 2.4.2 The dichotomy

Fix a parameter $0 < \delta_1 < 1$ and distinguish the between following alternatives.

**Case 1.** We have

$$\int_{-(qT)^\varepsilon}^{(qT)^\varepsilon} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell\in\mathbb{Z}} \Delta(\ell,\chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v \le \delta_1 |R_1'|^{3/2} (qT+M^k)^{1/2}. \tag{2.4.3}$$

**Case 2.** We have

$$\int_{-(qT)^\varepsilon}^{(qT)^\varepsilon} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell\in\mathbb{Z}} \Delta(\ell,\chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v > \delta_1 |R_1'|^{3/2} (qT+M^k)^{1/2}. \tag{2.4.4}$$

If (2.4.3) holds, then (2.3.2) together with (2.3.5) yields the bound

$$\begin{aligned} |R_1'| &\ll \left(N^{2-2\sigma} + \delta_1^2 N^{3k-4k\sigma}(qT+M^k)\right)(qT)^\varepsilon \\ &\ll \left(N^{2-2\sigma} + \delta_1^2 qT N^{3k-4k\sigma} + \delta_1^2 (qT)^k N^{2k-4k\sigma}\right)(qT)^\varepsilon. \end{aligned} \tag{2.4.5}$$

If (2.4.4) holds, noting that $|R_1'| \ll qT$, we can determine a suitable $\delta'$ such that if

$$D(\delta',\chi) = \{\ell : \delta'|R_1'| < \Delta(\ell,\chi) \le 2\delta'|R_1'|\}, \tag{2.4.6}$$

then

$$\begin{aligned} &\int_{|v|<(qT)^\varepsilon} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell\in\mathbb{Z}} \Delta(\ell,\chi) \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v \\ &\le (qT)^\varepsilon \sideset{}{^*}\sum_{\chi \bmod q} \delta'|R_1'| \sum_{\ell\in D(\delta',\chi)} \int_{|v|<(qT)^\varepsilon} \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v, \end{aligned}$$

and hence

$$\begin{aligned} &\sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell\in D(\delta',\chi)} \int_{|v|<(qT)^\varepsilon} \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+iv+i\ell} \right| \mathrm{d}v \\ &> \frac{\delta_1}{\delta'} |R_1'|^{1/2} (qT+M^k)^{1/2} (qT)^{-\varepsilon}. \end{aligned}$$

We further specify a level set

$$S_{H,T,\chi,\varepsilon} = \left\{ |t| < 2T+1+(qT)^\varepsilon : H < \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+it} \right| \le 2H \right\}$$

such that

$$\sideset{}{^*}\sum_{\chi \bmod q} H \int_{-(qT)^\varepsilon}^{(qT)^\varepsilon} |(D(\delta',\chi)+v) \cap S_{H,T,\chi,\varepsilon}| \;\mathrm{d}v > \frac{\delta_1}{\delta'} |R_1'|^{\frac12} (qT+M^k)^{\frac12} (qT)^{-\varepsilon}. \tag{2.4.7}$$

We shall write $S_{H,T,\chi} = S_{H,T,\chi,\varepsilon}$ for brevity. Note that

$$\sum_{\chi \bmod q}^{*} \sum_{\ell} \Delta(\ell,\chi) \geq \sum_{\chi \bmod q}^{*} \sum_{\ell \in D(\delta',\chi)} \Delta(\ell,\chi) > \sum_{\chi \bmod q}^{*} \sum_{\ell \in D(\delta',\chi)} \delta'|R_1'|.$$

and

$$\sum_{\chi \bmod q}^{*} \sum_{\ell} \Delta(\ell,\chi) \leq 2|R_1'|^2.$$

We have

$$\sum_{\chi \bmod q}^{*} |D(\delta',\chi)| \leq \frac{2|R_1'|}{\delta'}. \tag{2.4.8}$$

Observe that

$$\begin{aligned}\sum_{\chi \bmod q}^{*} H^2 \mathfrak{m}(S_{H,T,\chi}) &\leq \sum_{\chi \bmod q}^{*} \int_{|t|<2T+1+(qT)^{\varepsilon}} \left| \sum_{n=1}^{M^k} c_n \chi(n) n^{-(1/2)+it} \right|^2 \mathrm{d}t \\ &\ll (qT+M^k)(qT)^{\varepsilon} \end{aligned} \tag{2.4.9}$$

and

$$\begin{aligned}&\sum_{\chi \bmod q}^{*} \int_{|v|<(qT)^{\varepsilon}} |(D(\delta',\chi)+v) \cap S_{H,T,\chi}| \,\mathrm{d}v \\ &= \left( \sum_{\chi \bmod q}^{*} \int_{|v|<(qT)^{\varepsilon}} |(D(\delta',\chi)+v) \cap S_{H,T,\chi}| \,\mathrm{d}v \right)^{1/2} \\ &\qquad \cdot \left( \sum_{\chi \bmod q}^{*} \int_{|v|<(qT)^{\varepsilon}} |(D(\delta',\chi)+v) \cap S_{H,T,\chi}| \,\mathrm{d}v \right)^{1/2} \\ &\ll (qT)^{\varepsilon} \left( \sum_{\chi \bmod q}^{*} |D(\delta',\chi)| \right)^{1/2} \left( \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) \right)^{1/2},\end{aligned}$$

where $\mathfrak{m}$ stands for the Lebesgue measure. Hence, we can specify $0 < \delta'' \ll (qT)^{\varepsilon}$ such that

$$\begin{aligned}&\delta'' \left( \sum_{\chi \bmod q}^{*} |D(\delta',\chi)| \right)^{1/2} \left( \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) \right)^{1/2} \\ &< \sum_{\chi \bmod q}^{*} \int_{|v|<(qT)^{\varepsilon}} |(D(\delta',\chi)+v) \cap S_{H,T,\chi}| \,\mathrm{d}v \\ &\leq 2\delta'' \left( \sum_{\chi \bmod q}^{*} |D(\delta',\chi)| \right)^{1/2} \left( \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) \right)^{1/2}. \end{aligned} \tag{2.4.10}$$

From (2.4.10) we get

$$\sum_{\chi \bmod q}^{*} |D(\delta', \chi)| > (qT)^{-\varepsilon}(\delta'')^2 \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}). \tag{2.4.11}$$

It follows from (2.4.7), (2.4.9) and (2.4.10) that

$$\sum_{\chi \bmod q}^{*} |(D(\delta', \chi))| > (qT)^{-\varepsilon} \left(\frac{\delta_1}{\delta'\delta''}\right)^2 |R_1'|. \tag{2.4.12}$$

Thus, we obtain from (2.4.8) and (2.4.12),

$$\delta' > (qT)^{-\varepsilon} \left(\frac{\delta_1}{\delta''}\right)^2. \tag{2.4.13}$$

From (2.4.11) and (2.4.12) we have

$$\sum_{\chi \bmod q}^{*} |(D(\delta', \chi))| > (qT)^{-\varepsilon}(\delta'')^2 \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) + (qT)^{-\varepsilon} \left(\frac{\delta_1}{\delta'\delta''}\right)^2 |R_1'|. \tag{2.4.14}$$

(2.4.10) and (2.4.14) imply

$$\begin{aligned}
&\sum_{\chi \bmod q}^{*} \int_{|v|<(qT)^{\varepsilon}} |R_1'||(D(\delta', \chi)) \cap (S_{H,T,\chi} - v)| \,\mathrm{d}v \\
&> (qT)^{-\varepsilon} \left[ |R_1'| \left( \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) \right) (\delta'')^2 + \frac{\delta_1}{\delta'} |R_1'|^{\frac{3}{2}} \left( \sum_{\chi \bmod q}^{*} \mathfrak{m}(S_{H,T,\chi}) \right)^{\frac{1}{2}} \right]. \\
&> (qT)^{-3\varepsilon} \int_{|v|<(qT)^{\varepsilon}} (\delta'')^2 |R_1'| \sum_{\chi \bmod q}^{*} |(S_{H,T,\chi} - v) \cap \mathbb{Z}| \\
&\qquad + \frac{\delta_1}{\delta'} |R_1'|^{3/2} \left( \sum_{\chi \bmod q}^{*} |(S_{H,T,\chi} - v) \cap \mathbb{Z}| \right)^{\frac{1}{2}} \mathrm{d}v.
\end{aligned} \tag{2.4.15}$$

From (2.4.15) we may find some $|v| < (qT)^{\varepsilon}$ such that the sets

$$S_\chi = (S_{H,T,\chi} - v) \cap \mathbb{Z} \tag{2.4.16}$$

satisfy

$$\begin{aligned}
|R_1'| \sum_{\chi \bmod q}^{*} |D(\delta', \chi) \cap S_\chi| &> (qT)^{-\varepsilon}(\delta'')^2 |R_1'| \sum_{\chi \bmod q}^{*} |S_\chi| \\
&\quad + (qT)^{-\varepsilon} \frac{\delta_1}{\delta'} |R_1'|^{3/2} \left( \sum_{\chi \bmod q}^{*} |S_\chi| \right)^{1/2}.
\end{aligned} \tag{2.4.17}$$

We now select $A$ to be the multiset

$$\bigcup_{\chi \bmod q} \bigcup_{\ell \in S_\chi} \bigcup_{(\beta_s+it_s,\chi_s)\in R_1'} \{(\beta_r+it_r,\chi_r)\in R_1' : \chi_r=\chi_s\overline{\chi}, |t_r-(t_s-\ell)|<1\},$$

where the multiplicity of a zero $(\beta_r+it_r,\chi_r)$ is according to how many triples $(\chi,\ell,\beta_s+it_s,\chi_s)$ produce $\rho$. Therefore, $|A|=\sum^*_{\chi \bmod q}\sum_{\ell\in S_\chi}\Delta(\ell,\chi)$. We now apply the machinery from Section 2.2, in particular (2.2.11), to find Dirichlet polynomials $F_A(t,\chi)=\sum_{N(A)<n\le 2N(A)} b_n\chi(n)n^{-it}$ with bounded coefficients $b_n$ such that

$$\begin{aligned}
N^{2\sigma}\delta'|R_1'| \sum^*_{\chi \bmod q} |S_\chi\cap D(\delta',\chi)| &\ll \sum^*_{\chi \bmod q}\sum_{\ell\in S_\chi} N(A)^{2\sigma}\Delta(\ell,\chi)\\
&\ll (qT)^\varepsilon \sum^*_{\chi \bmod q}\sum_{\ell\in S_\chi}\sum_{\substack{(\beta_s+it_s,\chi_s)\\ 1\le s\le |R_1'|}}\sum_{r\in\{r:\chi_r=\chi_s\overline{\chi},|t_r-(t_s-\ell)|<1\}} |F_A(t_r,\chi_r)|^2\\
&\ll (qT)^\varepsilon \sum^*_{\chi \bmod q}\sum_{\ell\in S_\chi}\sum_{\substack{(\beta_s+it_s,\chi_s)\\ 1\le s\le |R_1'|}}\sum_{r\in\{r:\chi_r=\chi_s\overline{\chi},|t_r-(t_s-\ell)|<1\}}\\
&\qquad\qquad \left(1+\int_{|v|<\log N(A)} |F_A(t_r+v,\chi_r)|^2\,\mathrm{d}v\right)\\
&\ll (qT)^\varepsilon \sum^*_{\chi \bmod q}\sum_{\ell\in S_\chi}\sum_{\substack{(\beta_s+it_s,\chi_s)\\ 1\le s\le |R_1'|}} \int_{|v|<(qT)^\varepsilon} |F_A(t_s-\ell+v,\chi_s\bar{\chi})|^2\,\mathrm{d}v.
\end{aligned}$$

In the penultimate transition we applied a Cauchy-Schwarz estimate on Lemma 1.5.1 and in the last step we used that there can only be one $(t_r,\chi_r)$ with a given $(t_s,\chi_s)$ as the zeros in $R_1'$ are well-spaced and that the term with 1 may be dropped as it only delivers a contribution of at most $(qT)^\varepsilon\sum^*_\chi\sum_{\ell\in S_\chi}\Delta(\ell,\chi)$ which can never be dominant in view of the estimate on the first line.

The remaining sums and integral are estimated via Lemma 2.4.2 and Lemma

2.4.3. This gives

$$N^{2\sigma}\delta'|R_1'| \sideset{}{^*}\sum_{\chi \bmod q} |S_\chi \cap D(\delta',\chi)|$$

$$\ll (qT)^\varepsilon \int_{|v|<(qT)^\varepsilon} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in S_\chi} \sum_{\substack{(\beta_s+it_s,\chi_s) \\ 1\le s\le |R_1'|}} |F_A(t_s-\ell+v,\chi_s\bar\chi)|^2 \, \mathrm{d}v. \qquad (2.4.18)$$

$$\ll (qT)^\varepsilon \left( \sum_{\substack{(t_s,\chi_s) \\ 1\le s\le |R_1'|}} \sum_{\substack{(t_{s'},\chi_{s'}) \\ 1\le s'\le |R_1'|}} \left| \sum_{N(A)<N\le 2N(A)} \chi_s\overline{\chi_{s'}} n^{i(s'-s)} \right|^2 \right)^{1/2}$$

$$\cdot \left( \sideset{}{^*}\sum_{\chi \bmod q} \sum_{\ell \in S_\chi} \sideset{}{^*}\sum_{\chi' \bmod q} \sum_{\ell' \in S_{\chi'}} \left| \sum_{N(A)<N\le 2N(A)} \chi\overline{\chi'} n^{i(\ell'-\ell)} \right|^2 \right)^{1/2}$$

$$< (qT)^\varepsilon (|R_1'|N^2 + |R_1'|^2 N)^{1/2} \left( N^2 \sideset{}{^*}\sum_{\chi \bmod q} |S_\chi| + N \left( \sideset{}{^*}\sum_{\chi \bmod q} |S_\chi| \right)^2 \right)^{1/2}. \qquad (2.4.19)$$

If we now assume that

$$|R_1'| \le N, \qquad (2.4.20)$$

and let $S = \sum^*_{\chi \bmod q} |S_\chi|$, then we find

$$N^{2\sigma}\delta'|R_1'| \sideset{}{^*}\sum_{\chi \bmod q} |S_\chi \cap D(\delta',\chi)| \ll (qT)^\varepsilon |R_1'|^{1/2} S^{1/2} N^{3/2} (N+S)^{1/2}.$$

Combining this with the lower bound (2.4.17) and eliminating $\delta', \delta''$ through (2.4.13), we obtain the inequality

$$(qT)^\varepsilon |R_1'|^{1/2} S^{1/2} N^{3/2} (N+S)^{1/2} > \delta_1^2 N^{2\sigma} |R_1' S + \delta_1 |R_1'|^{3/2} N^{2\sigma} S^{1/2},$$

which implies one of the following alternatives:

$$(qT)^\varepsilon |R_1'|^{1/2} S^{1/2} N^2 > \delta_1 |R_1'|^{3/2} N^{2\sigma} S^{1/2},$$

and hence

$$|R_1'| < (qT)^\varepsilon \delta_1^{-1} N^{2-2\sigma}, \qquad (2.4.21)$$

or

$$(qT)^\varepsilon |R_1'|^{1/2} S N^{3/2} > \delta_1^2 N^{2\sigma} |R_1'||S|,$$

and hence

$$|R_1'| < (qT)^\varepsilon \delta_1^{-4} N^{3-4\sigma}. \qquad (2.4.22)$$

### 2.4.3 A large value estimate

Collecting the contributions (2.4.5), (2.4.21), and (2.4.22) from Case 1 and Case 2, we establish the following bound for $|R_1'|$:

**Lemma 2.4.4.** *Let* $\sigma > \frac{3}{4}$, $0 < \delta_1 < 1$, $k$ *be fixed positive integer,and* $R_1' = \{(\beta_r + it_r, \chi(r)) : 1 \le r \le |R_1'|\}$ *be the set of representative zeros of class-I zeros. Assume that* $N < qT$ *in* (2.2.10) *and* $|R_1'| \le N$. *Then*

$$|R_1'| \ll \left[\delta_1^{-1} N^{2-2\sigma} + \delta_1^2 qT N^{3k-4k\sigma} + \delta_1^2 (qT)^k N^{2k-4k\sigma} + \delta_1^{-4} N^{3-4\sigma}\right] (qT)^{\varepsilon}.$$

## 2.5 Proof of Theorem 2.1.1

*Proof.* In view of the discussion in Section 2.2, we only need to estimate $|R_1'|$. If $qT \le N < (qT)^{1+\varepsilon}$, we use the mean value estimate. Theorem 7.4 of [62], with $\delta = 1$ yields

$$|R_1'| \ll (qT + N)(1 + \log N) N^{1-2\sigma} \ll N^{2(1-\sigma)+\varepsilon} \ll (qT)^{2(1-\sigma)+\varepsilon}.$$

Thus, we only need to consider case $(qT)^{2/3+\varepsilon} < N < qT$. We apply [33, Lemma 5] with $k = 2$, which yields

$$|R_1'| \ll \left[N^{2-2\sigma} + (qT)^2 N^{4-8\sigma} + (qT)^{2/3} N^{(24-32\sigma)/3}\right] (qT)^{\varepsilon}.$$

Here

$$N^{2-2\sigma} \ll (qT)^{2-2\sigma},$$
$$(qT)^{2/3} N^{(24-32\sigma)/3} \ll (qT)^{(54-64\sigma)/9} \ll (qT)^{2-2\sigma}.$$

We therefore get $|R_1'| \ll (qT)^{2(1-\sigma)+\varepsilon}$ provided $N$ satisfies

$$N \ge (qT)^{\frac{\sigma}{4\sigma-2}}.$$

Note that $(qT)^{\frac{\sigma}{4\sigma-2}} \le N^{2/3}$ when $\sigma \ge 4/5$. We may assume in the sequel that

$$(qT)^{2/3+\varepsilon} < N < (qT)^{\frac{\sigma}{4\sigma-2}}$$

and

$$15/19 \le \sigma < 4/5.$$

We consider first the case that $|R_1'| \le N$. Applying Lemma 1.5.4 with $k = 3$ gives

$$|R_1'| \ll \left[\delta_1^{-1} N^{2-2\sigma} + \delta_1^2 qT N^{9-12\sigma} + \delta_1^{-4} N^{3-4\sigma}\right] (qT)^{\varepsilon}. \tag{2.5.1}$$

Let

$$\delta_1^2 qT N^{9-12\sigma} = (qT)^{2(1-\sigma)}$$

whence $\delta_1 = (qT)^{\frac{1}{2}-\sigma} N^{6\sigma-\frac{9}{2}} < 1$. Inserting this choice in (2.5.1) gives

$$\begin{aligned}
|R_1'| &\ll \left[ (qT)^{\sigma-\frac{1}{2}} N^{\frac{13}{2}-8\sigma} + (qT)^{2(1-\sigma)} + (qT)^{4\sigma-2} N^{21-28\sigma} \right] (qT)^{\varepsilon} \\
&\ll \left[ (qT)^{\sigma-\frac{1}{2}+\frac{13-16\sigma}{2}\times\frac{\sigma}{4\sigma-2}} + (qT)^{2(1-\sigma)} + (qT)^{4\sigma-2+(21-28\sigma)\times\frac{2}{3}} \right] (qT)^{\varepsilon} \\
&\ll (qT)^{2(1-\sigma)+\varepsilon}.
\end{aligned}$$

Now, suppose that $|R_1'| > N$. As $N \geq (qT)^{2/3}$, this implies that $|R_1'| > (qT)^{2/3}$. Select now a subset of representative well-spaced class-I zeros $R'$ such that $|R'| = \lfloor (qT)^{2/3} \rfloor$. Now $|R'| \leq N$ and the entire analysis above can be performed for $R'$ to give $|R'| \ll (qT)^{2(1-\sigma)+\varepsilon} \ll (qT)^{1/2-\varepsilon}$, if $\sigma > 3/4$ say, which is impossible (for large enough $T$). Therefore $|R_1'|$ must have been smaller than $N$ to begin with. This completes the proof of Theorem 2.1.1, since we can include non-primitive characters by applying our estimate for all factors of $q$ and summing. □

# Chapter 3

# On almost-prime $k$-tuples

*"In a handwritten copy that my father had of Carl Stormer's lecture notes, I came across the series known as Leibniz's series: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. It's such a very strange and beautiful relationship that I determined I would read that book in order to find out how this formula came about."*

—Atle Selberg 1917-2007

Let $\tau$ denote the divisor function and $\mathcal{H} = \{h_1, ..., h_k\}$ be an admissible set. In this chapter, we prove that there are infinitely many $n$ for which the product $\prod_{i=1}^{k}(n + h_i)$ is square-free and $\sum_{i=1}^{k} \tau(n + h_i) \leq \lfloor \rho_k \rfloor$, where $\rho_k$ is asymptotic to $\frac{2126}{2853}k^2$. It improves a previous result of M. Ram Murty and A. Vatwani, replacing 3/4 by 2126/2853. The main ingredients in our proof are the higher rank Selberg sieve and Irving-Wu-Xi estimate for the divisor function in arithmetic progressions to smooth moduli.

## 3.1 Introduction

We consider a set $\mathcal{H} = \{h_1, ..., h_k\}$ of distinct non-negative integers. We call such a set admissible if, for every prime $p$, the number of distinct residue classes modulo $p$ occupied by $h_i$ is less than $p$. The following conjecture is one of the greatest open problems in prime number theory.

**Conjecture 3.1.1** (Prime $k$-tuples conjecture)**.** Given an admissible set $\mathcal{H} = \{h_1, ..., h_k\}$, there are infinitely many integers $n$ for which all $n + h_i$ are prime.

The twin prime conjecture follows immediately from this by taking $\mathcal{H} = \{0, 2\}$. Although the Prime $k$-tuples conjecture for $k \geq 2$ is still wide open, many mathematicians succeeded in making partial progress in various directions. One of these directions is the existence of small gaps between primes. In 2013, Zhang [85] showed

$$\liminf_{n\to\infty}(p_{n+1}-p_n) < 7\times 10^7$$

by a refinement of the GPY method [28]. The main ingredient of his proof is a stronger version of the Bombieri-Vinogradov theorem that is applicable when the moduli are smooth numbers. After Zhang's breakthrough, a new higher rank version of the Selberg sieve was developed by Maynard [60] and Tao. This provided an alternative way of proving bounded gaps between primes, but had several other consequences as well since it was more flexible and could show the existence of clumps of many primes in intervals of bounded length (cf. [60, Thm. 1.1]). It is worth mentioning that Zhang's stronger version of the Bombieri-Vinogradov theorem with smooth moduli can be combined with the Maynard-Tao sieve to show there are clumps of primes in shorter intervals of bounded length (cf. [65, Thm. 4(vi)]). This means that a combination of both methods will yield better results than using Maynard-Tao sieve alone. For a further discussion of the progress in this direction, we refer the reader to [65].

Another approximation to the Prime $k$-tuples conjecture is to establish an upper bound for the expression

$$\sum_{i=1}^{k}\tau(n+h_i),$$

where $\tau$ stands for the divisor function . It is clear that the Prime $k$-tuples conjecture follows if one has the upper bound $2k$ for infinitely many $n$. For large $k$, the current best result is

**Theorem 3.1.2** (M. Ram Murty and A. Vatwani [64]). *There exists $\rho_k$ such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \le x$ for which the product $\prod_{i=1}^{k}(n+h_i)$ is square-free and*

$$\sum_{i=1}^{k}\tau(n+h_i) \le \lfloor \rho_k \rfloor.$$

*For large $k$, we have $\rho_k \sim \frac{3}{4}k^2$.*

We record previous results and methods. In 1997, Heath-Brown [35] obtained the above result with $\rho_k \sim \frac{3}{2}k^2$ by using Selberg sieve. In 2006, Ho and Tsang [37] got $\rho_k \sim k^2$ by modifying Heath-Brown's sieve weights. In 2017, M. Ram Murty and Akshaa Vatwani [64] developed a general higher rank Selberg sieve with an additive twist to establish Theorem 3.1.2. We next describe in more detail the aspects that determine the quality of their results.

When we use sieve methods to study the Prime $k$-tuples conjecture, the primary aspect affecting the result is the sieve method itself. Roughly speaking, if a more general form of the sieve weights is used, there is more room to obtain better numerical results. As can be seen, for example, in Maynard's work [60]. Another key aspect is how to deal with the error terms arising from the application of the sieve method. In order to control these error terms, the above results all exploit the divisor function analogue of the Bombieri-Vinogradov theorem. More precisely, let $(a, q) = 1$, and set

$$E(x,q,a) = \sum_{\substack{n\le x\\ n\equiv a \,(\bmod q)}} \tau(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\le x\\ (n,q)=1}} \tau(n), \tag{3.1.1}$$

where $\varphi$ is the Euler totient function. Then for any $A > 0$ and any $\theta < 2/3$,

$$\sum_{q\le x^\theta} \max_{(a,q)=1} |E(x,q,a)| \ll_{A,\theta} \frac{x}{(\log x)^A}. \tag{3.1.2}$$

In fact, (3.1.2) can be deduced from the following result. For $q < x^2$, we have for any $\epsilon > 0$, that

$$|E(x,q,a)| \ll_\epsilon q^{-1/4} x^{1/2+\epsilon}. \tag{3.1.3}$$

This was proved independently by Selberg [74, pp. 234-237] as well as Hooley [38] and Linnik; it is a consequence of the Weil bound for Kloosterman sums. The range of $\theta$ in (3.1.2) determines the value of $\rho_k$ in Theorem 3.1.2. Actually, the proof in [64] gives $\rho_k \sim (2\theta_0)^{-1}k^2$ provided (3.1.2) holds for $0 < \theta < \theta_0$. We remark that the range $\theta < 2/3$ for (3.1.2) is still the best known result although it is reasonable to expect that (3.1.2) should hold for all $\theta < 1$. In 2015, A. J. Irving broke through the barrier $2/3$ under the assumption that $q$ only has small prime factors by using the $q$-analogue of van der Corput's method. More accurately, given $\varepsilon > 0$, Irving [45] (or see Lemma 3.4.2 below) showed that,

$$|E(x,q,a)| \ll_\varepsilon q^{-1} x^{1-\delta'} \tag{3.1.4}$$

for $q < x^{2/3+1/246-\varepsilon}$ provided any prime factor of $q$ does not exceed $x^\eta$, where $\delta'$ and $\eta$ are some positive constants depending on $\varepsilon$. Quite recently, Wu and Xi [81] developed a theory of arithmetic exponent pairs and used it to improve Irving's result by extending the range of $q$ to $q < x^{2/3+55/12756-\varepsilon}$ (see [81, Section 10] or Lemma 3.4.3 below). Note that $55/12756 \approx 1/231.92$.

It is natural to ask whether we can combine the Irving-Wu-Xi estimate and the higher rank Selberg sieve with additive twist to improve Theorem 1.2. This is the main goal of the chapter and we show

**Theorem 3.1.3.** *There exists $\rho_k$ such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \le x$ for which the product $\prod_{i=1}^k (n+h_i)$ is square-free and*

$$\sum_{i=1}^{k} \tau(n+h_i) \le \lfloor \rho_k \rfloor.$$

*For large $k$, we have $\rho_k \sim \frac{2126}{2853}k^2$.*

Observe that $2126/2853 = 0.74518... < 3/4$. Combining the higher rank Selberg sieve with the Irving-Wu-Xi estimate requires some technical adjustments from the traditional literature. A particularly challenging technical task is that we need to construct a new test function, which has not appeared in this context before. Completing various complicated integral estimates of this new function is the novel part of our work. This will be done in Section 5. Finally, we remark that any improvement in the constant 55/12756 occurring in Irving-Wu-Xi estimate would give a corresponding improvement here by our method.

## 3.2 Notation

In this section, we recall the notation and terminology set up from [77]. For a more detailed description, the reader is referred to [64] and [77].

A $k$-tuple of integers $\underline{d} := (d_1, \cdots, d_k)$ is said to be square-free if the product of its components is square-free. For a real number $R$, the inequality $\underline{d} \le R$ means that $\prod_i d_i \le R$. The notation of divisibility among tuples is defined component-wise, that is,

$$\underline{d} | \underline{n} \iff d_i | n_i \text{ for all } 1 \le i \le k.$$

The notation of congruence among tuples, modulo a tuple, is also defined component-wise. On the other hand, we say a scalar $q$ divides the tuple $\underline{d}$ if $q$ divides the product $\prod_i d_i$. When we explicitly write the congruence relation $\underline{d} \equiv \underline{e} \pmod{q}$, we mean that it holds for each component.

A vector function is said to be multiplicative if all its component functions are multiplicative. In this context, we define the function $f(\underline{d})$ as the product of its component (multiplicative) functions, that is,

$$f(\underline{d}) := \prod_{i=1}^{k} f_i(d_i).$$

Similarly, a vector function $v(\underline{d})$ is called additive if all its components $v_i$ are additive, in which case, we define

$$v(\underline{d}) = \sum_{i=1}^{k} v_i(d_i).$$

Some vector functions we will use are the Euler phi function, as well as the lcm and gcd functions. For example,

$$[\underline{d}, \underline{e}] := \prod_{i=1}^{k} [d_i, e_i].$$

When written as the argument of a vector function, $[\underline{d}, \underline{e}]$ will denote the tuple whose components are $[d_i, e_i]$. The meaning of the use will be clear from the context.

We employ the following multi-index notation to denote mixed partial derivatives of a function $F(t)$ on $k$-tuples,

$$F^{(\underline{\alpha})}(\underline{t}) := \frac{\partial^{\alpha} F(t_1, \cdots, t_k)}{(\partial t_1)^{\alpha_1} \cdots (\partial t_k)^{\alpha_k}},$$

for any $k$-tuple $\underline{\alpha}$ with $\alpha := \sum_{j=1}^{k} \alpha_j$.

Given smooth functions $G$ and $H$ with compact support on $\mathbb{R}^k$, we define

$$C(G,H)^{(\underline{a})} := \int_0^{\infty} \cdots \int_0^{\infty} \left( \prod_{j=1}^{k} \frac{t_j^{a_j - 1}}{(a_j - 1)!} \right) G(\underline{t})^{(\underline{a})} H(\underline{t})^{(\underline{a})} \, \mathrm{d}\underline{t}$$

and

$$C(G,H)^{(\underline{a},\underline{b},\underline{c})} := (-1)^{a+b} \int_0^{\infty} \cdots \int_0^{\infty} \left( \prod_{j=1}^{k} \frac{t_j^{c_j - 1}}{(c_j - 1)!} \right) G(\underline{t})^{(\underline{a})} H(\underline{t})^{(\underline{b})} \, \mathrm{d}\underline{t}.$$

The notation $\tau_k(n)$ represents the generalised divisor function , that is, the number of ways of writing $n$ as the product of $k$ positive integers. The number $\gamma$ denotes the Euler's constant. We use $\ll$ to denote Vinogradov's notation. We also use the convention $n \sim N$ to denote $N < n \leq 2N$. Alternatively, $f(x) \sim g(x)$ may also denote that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. The meaning will be clear from the context. The greatest integer less than or equal to $x$ is denoted as $\lfloor x \rfloor$. The dash over the sum means that we sum over $k$-tuples $\underline{d}$ and $\underline{e}$ with $[\underline{d}, \underline{e}]$ square-free and co-prime to $W$. Throughout this chapter, $\delta$ denotes a positive quantity which can be made as small as needed.

## 3.3 Some hypotheses

In this section,we review some of the salient features of the higher rank Selberg sieve discussed in [64] and [77].

Given a set $S$ of $k$-tuples, $S = \{\underline{n} = (n_1, \cdots n_k)\}$, we seek to estimate sums of the form

$$\sum_{\underline{n}\in S} \omega_{\underline{n}} \Big( \sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}} \Big)^2, \tag{3.3.1}$$

where $\omega_{\underline{n}}$ is a 'weight' attached to the tuples $\underline{n}$ and $\lambda_{\underline{d}}$ are parameters to be chosen. Throughout this section, the condition $\underline{n} \in S$ is understood to hold without being explicity stated. We impose the following hypotheses on this sum:

H1. If a prime $p$ divides a tuple $\underline{n}$ such that $p$ divides $n_i$ and $n_j$, with $i \neq j$, then $p$ must lie in some fixed finite set of primes $P_0$.

This hypothesis allows us to perform what is called the '$W$ trick'. That is, we set $W = \prod_{p<D_0} p$, with $D_0$ depending on $S$, such that $p \in P_0$ implies that $p \mid W$. We then fix some tuple of residue classes $\underline{b} \pmod W$ with $(b_i, W) = 1$ for all $i$ and restrict $\underline{n}$ to be congruent to $\underline{b}$ in the sum we are concerned with.

H2'. With $W, \underline{b}$ as in H1, the function $\omega_{\underline{n}}$ satisfies

$$\sum_{\substack{\underline{d}|\underline{n} \\ \underline{n}\equiv \underline{b} \pmod W}} \omega_{\underline{n}} = \frac{X}{f(\underline{d})} + \frac{X^*}{f_*(\underline{d})} v(\underline{d}) + r_{\underline{d}}$$

for some real numbers $X$ and $X^*$ depending on the set $S$, where $f$ and $f_*$ are multiplicative and $v$ is additive.

H3. With $f$ as in H2', the components of $f$ satisfy

$$f_j(p) = \frac{p}{\alpha_j} + O(p^t), \quad \text{with } t < 1,$$

for each prime $p$ and some fixed $\alpha_j \in \mathbb{N}$, $\alpha_j$ independent of $X, k$.

We denote the tuple $(\alpha_1, \cdots, \alpha_k)$ as $\underline{\alpha}$ and the sum of the components $\Sigma_{j=1}^k \alpha_j$ as $\alpha$.

H4'. There exists $\varpi > 0, \eta_0 > 0$ such that

$$\sum_{\substack{[\underline{d},\underline{e}] \leq X^{2/3+\varpi-\epsilon} \\ d_i, e_i \leq X^{\eta_0}\ \forall i}} |r_{[\underline{d},\underline{e}]}| \ll \frac{X}{(\log X)^A},$$

for any $A > 0$, $\epsilon > 0$, as $X \to \infty$. The implied constant may depend on $A$ and $\epsilon$.

H5. Let $v$ be as in H2'. For each $j$, there exists $\beta_j$, such that

$$\sum_p \frac{v_j(p)}{p^{1+\delta}} = \frac{\beta_j}{\delta} + O(1), \qquad \sum_p \frac{|v_j(p)|}{p^{1+\delta}} \ll \frac{1}{\delta}$$

as $\delta \to 0$.

We shall choose $\lambda_{\underline{d}}$ in terms of a fixed symmetric function $F : [0,\infty)^k \to \mathbb{R}$, supported on the truncated simplex

$$\Delta_k^{[\kappa]}(1) := \{(t_1, \cdots, t_k) \in [0,\kappa]^k : t_1 + \cdots + t_k \leq 1\}, \quad \text{for some } \kappa > 0,$$

as

$$\lambda_{\underline{d}} = \mu(\underline{d}) F\left(\frac{\log \underline{d}}{\log R}\right) := \mu(d_1)\cdots\mu(d_k) F\left(\frac{\log d_1}{\log R}, \cdots, \frac{\log d_k}{\log R}\right), \tag{3.3.2}$$

where $R$ is some fixed power of $X$. If $\kappa = 1$, we write $\Delta_k(1) = \Delta_k^{[1]}(1)$ for brevity. Henceforth, we assume $D_0$ (and hence $W$) $\to \infty$ as $X \to \infty$.

## 3.4 Lemmas

In this section we introduce some prerequisite results, some of which are quoted from the literature directly. These lemmas play an important role in the proof of our main theorem in section 5.

Throughout this section, the big oh and little oh notation is understood to be with respect to $X \to \infty$. Moreover, the implied constants may depend on those parameters which are independent of $X$ (such as the function $f$, parameters $A, \alpha_j, \beta_j$, etc.) but not on those quantities which do depend on $X$ (such as $D_0, W, R$).

First recall the main result in [64] which can used to deal with the main term arising from the application of the higher rank Selberg sieve with additive twist.

**Lemma 3.4.1** (M. Ram Murty and A. Vatwani [64, Lemma 4.2])**.** *Set $R$ to be some fixed power of $X$ and let $D_0 = o(\log\log R)$. Let $f$ be a multiplicative vector function and $v$ be an additive vector function satisfying H3 and H5 respectively. Let $G, H$ be smooth functions with compact support. We denote*

$$G\left(\frac{\log \underline{d}}{\log R}\right) := G\left(\frac{\log d_1}{\log R}, \cdots, \frac{\log d_k}{\log R}\right)$$

*and similarly for $H$. Then*

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} v([\underline{d},\underline{e}]) G\left(\frac{\log \underline{d}}{\log R}\right) H\left(\frac{\log \underline{e}}{\log R}\right)$$

*is obtained by (as $R \to \infty$)*

$$(1+o(1))\frac{c(W)}{(\log R)^{\alpha-1}} \sum_{j=1}^{k} \beta_j \alpha_j C_j^*(G,H)^{(\underline{\alpha})} + O\left(\frac{c(W)\log D_0}{(\log R)^{\alpha}}\right),$$

*where,*

$$C_j^*(G,H)^{(\underline{\alpha})} = C(G,H)^{(\underline{\alpha},\underline{\alpha},\underline{\alpha}+e_j)} - C(G,H)^{(\underline{\alpha}-e_j,\underline{\alpha},\underline{\alpha})} - C(G,H)^{(\underline{\alpha},\underline{\alpha}-e_j,\underline{\alpha})},$$

$$c(W) := \frac{W^\alpha}{\varphi(W)^\alpha},$$

*and the tuple* $\underline{\alpha} \pm e_j$ *is* $(\alpha_1, \cdots, \alpha_j \pm 1, \cdots, \alpha_k)$.

Our next two lemmas concern the estimation of the discrepancy of the divisor function in arithmetic progressions to smooth moduli. A notable feature of these estimates is the breakthrough of the $2/3$ barrier.

**Lemma 3.4.2** (A. J. Irving [45, Thm. 1.2])**.** *Let* $E(x,q,a)$ *be as in* (3.1.1). *Suppose that* $\varpi, \eta > 0$ *satisfy*

$$246\varpi + 18\eta < 1.$$

*There exists* $\delta' > 0$*, depending on* $\varpi$ *and* $\eta$*, such that for any* $x^\eta$*-smooth, square-free* $q \le x^{2/3+\varpi}$ *and any* $(a,q)=1$ *we have*

$$E(x,q,a) \ll_{\varpi,\eta} q^{-1}x^{1-\delta'}.$$

**Lemma 3.4.3.** *Let* $E(x,q,a)$ *be defined as in* (3.1.1). *Suppose that* $\varepsilon > 0$. *There exist positive real numbers* $\delta' = \delta'(\varepsilon)$ *and* $\eta = \eta(\varepsilon)$*, such that for any* $q^\eta$*-smooth, square-free* $q \le x^{2/3+55/12756-\varepsilon}$ *and any* $(a,q)=1$ *we have*

$$E(x,q,a) \ll_\varepsilon q^{-1}x^{1-\delta'}.$$

*Proof.* The formulation is slightly different from J. Wu and P. Xi [81, Thm. 1.2]. Following the arguments of Irving [45], it suffices to show, when $N \asymp \sqrt{x}$ and $(h,q)=1$, that

$$\sum_{N<n\le 2N} e\left(\frac{h\bar{n}}{q}\right) \ll \frac{x^{1-\delta'}}{q}, \tag{3.4.1}$$

for some $\delta' > 0$, where $q$ is square-free, $q^\eta$-smooth for some $\eta > 0$, $q \le x^{2/3+55/12756-\varepsilon}$, and $\bar{n}$ is the inverse of $n$ modulo $q$. Using the algorithm for exponent pairs (cf. [31, Section 5]), we select the exponent pair

$$(a,b) = BA^3BA^2BABABA^2\left(\frac{1}{2},\frac{1}{2}\right) = \left(\frac{591}{1535},\frac{808}{1535}\right)$$

and let $\delta' = \varepsilon' = (1+a)\varepsilon/2$. We have (cf. [81, p. 2164, Section 10])

$$\sum_{N<n\le 2N} e\left(\frac{h\bar{n}}{q}\right) \ll_{\varepsilon'} q^a x^{(b-a)/2+\varepsilon'},$$

where $q$ is square-free and $q^\eta$-smooth for some $\eta > 0$ (only depending on $\varepsilon$). This gives (3.4.1) provided $q \le x^{2/3+55/12756-\varepsilon}$. ∎

The following lemma is the analogue of [64, Theorem 4.3] and it can be viewed as a smoothed version of the higher rank Selberg sieve with additive twist. Here a smoothed version means that the sieve weights $\lambda_{\underline{d}}$ are supported on the $\underline{d}$ such that the product $\prod_i d_i$ only has prime factors less than $R^\kappa$. We achieve it by letting each component $d_i$ be smaller than $R^\kappa$.

**Lemma 3.4.4.** *Let $\lambda_{\underline{d}}$ be as in* (3.3.2)*. Suppose hypotheses H1, H2', H4' and H5. We also assume that both functions $f$ and $f_*$ arising from H2' satisfy H3 with $\alpha_j$ and $\alpha_j^*$ respectively. Let $R = X^{\frac{1}{2}(\frac{2}{3}+\varpi)-\delta}$, $\kappa = \frac{2\eta_0}{2/3+\varpi}$ and $D_0 = o(\log\log R)$. Then,*

$$\sum_{\underline{n}\equiv \underline{b} \pmod W} \omega_{\underline{n}} \left(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\right)^2 = (1+o(1))\frac{c(W)X}{(\log R)^\alpha} C(F,F)^{(\underline{\alpha})} + (1+o(1))\frac{c^*(W)X^*}{(\log R)^{\alpha^*-1}} \sum_{j=1}^k \beta_j \alpha_j^* C_j^*(F,F)^{(\underline{\alpha}^*)}$$

*where $C_j^*(F,F)^{(\underline{\alpha}^*)}$ denotes the quantity*

$$C(F,F)^{(\underline{\alpha}^*,\underline{\alpha}^*,\underline{\alpha}^*+e_j)} - C(F,F)^{(\underline{\alpha}^*-e_j,\underline{\alpha}^*,\underline{\alpha}^*)} - C(F,F)^{(\underline{\alpha}^*,\underline{\alpha}^*-e_j,\underline{\alpha}^*)},$$

*and*

$$\alpha^* = \sum_{j=1}^k \alpha_j^*, \qquad c(W) = \frac{W^\alpha}{\varphi(W)^\alpha}, \qquad c^*(W) = \frac{W^{\alpha^*}}{\varphi(W)^{\alpha^*}},$$

*the tuple $\underline{\alpha}^* \pm e_j$ is $(\alpha_1^*, \cdots, \alpha_j^* \pm 1, \cdots, \alpha_k^*)$.*

*Proof.* Our proof follows the argument of [77, Thm. 3.6]. We expand the square, interchanging the order of summation, applying the $W$-trick and finally using $H2'$. We obtain

$$X \sideset{}{'}\sum_{\substack{\underline{d},\underline{e}<R \\ d_i,e_i\le R^\kappa\ \forall i}} \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f([\underline{d},\underline{e}])} + X^* \sideset{}{'}\sum_{\substack{\underline{d},\underline{e}<R \\ d_i,e_i\le R^\kappa\ \forall i}} \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f_*([\underline{d},\underline{e}])} v([\underline{d},\underline{e}]) + O\left(\sideset{}{'}\sum_{\substack{\underline{d},\underline{e}<R \\ d_i,e_i\le R^\kappa\ \forall i}} |\lambda_{\underline{d}}||\lambda_{\underline{e}}||r_{[\underline{d},\underline{e}]}|\right).$$

We have to analyze the two main terms. Substituting our choice (3.3.2) of $\lambda_{\underline{d}}$ into the first term and noting that $F$ is supported on the truncated simplex, we get

$$X \sideset{}{'}\sum_{\underline{d},\underline{e}} \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} F\left(\frac{\log \underline{d}}{\log R}\right) F\left(\frac{\log \underline{e}}{\log R}\right).$$

By [77, Lemma 3.4], the above expression is equal to

$$(1+o(1))\frac{c(W)X}{(\log R)^{\alpha}}C(F,F)^{(\underline{\alpha})}.$$

The second term yields

$$X^{*}\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f_{*}([\underline{d},\underline{e}])}v([\underline{d},\underline{e}])F\left(\frac{\log \underline{d}}{\log R}\right)F\left(\frac{\log \underline{e}}{\log R}\right).$$

By Lemma 3.4.1, this is given by

$$(1+o(1))\frac{c^{*}(W)X^{*}}{(\log R)^{\alpha^{*}-1}}\sum_{j=1}^{k}\beta_j\alpha_j^{*}C_j^{*}(F,F)^{(\underline{\alpha}^{*})}.$$

To complete the proof, we note that the choices of $R$ and $\kappa$ along with H4' ensure that the error term is negligible. □

## 3.5 Application to almost prime $k$-tuples

We recall the definition of an admissible set.

**Definition 3.5.1.** A set $\mathcal{H}=\{h_1,...,h_k\}$ of distinct non-negative integers is said to be admissible if, for every prime $p$, there is a residue class $b_p \pmod p$ such that $b_p \notin \mathcal{H} \pmod p$.

Throughout this section, we work with a fixed admissible set of size $k$, $\mathcal{H}=\{h_1,...,h_k\}$, where $k$ is a sufficiently large integer. First we use the $W$ trick. Set $W=\prod_{p<D_0}p$, by the Chinese remainder theorem, we can find an integer $b$, such that $b+h_i$ is co-prime to $W$ for each $h_i$. We restrict $n$ to be in this fixed residue class $b$ modulo $W$. One can choose $D_0=\log\log\log N$, so that $W\sim(\log\log N)^{1+o(1)}$ by an application of the prime number theorem. We then consider the expressions,

$$S_1=\sum_{\substack{n\sim N\\ n\equiv b \,(\mathrm{mod}\, W)}}\left(\sum_{d_j|n+h_j\forall j}\lambda_{\underline{d}}\right)^2, \tag{3.5.1}$$

$$S_2=\sum_{\substack{n\sim N\\ n\equiv b \,(\mathrm{mod}\, W)}}\left(\sum_{j=1}^{k}\tau(n+h_j)\right)\left(\sum_{d_j|n+h_j\forall j}\lambda_{\underline{d}}\right)^2. \tag{3.5.2}$$

For $\rho$ positive, we denote by $S(N,\rho)$ the quantity

$$\rho S_1 - S_2.$$

The key point of our argument is to show, with an appropriate choice of $\lambda_{\underline{d}}$, that

$$S(N,\rho) > 0$$

for all large $N$. This implies, there are infinitely many integers $n$ such that

$$\sum_{j=1}^{k} \tau(n+h_j) \le \lfloor \rho \rfloor,$$

where $\lfloor \rho \rfloor$ denotes the greatest integer less than or equal to $\rho$.

The asymptotic formula for $S_1$ was already derived in [77, Lemma 4.2]. We proceed to derive an asymptotic formula for $S_2$.

### 3.5.1 An asymptotic formula for $S_2$

We write

$$S_2 = \sum_{m=1}^{k} S_2^{(m)}, \qquad S_2^{(m)} = \sum_{\substack{n\sim N \\ n\equiv b \,(\mathrm{mod}\, W)}} \tau(n+h_m) \left( \sum_{d_j \mid n+h_j \forall j} \lambda_{\underline{d}} \right)^2 .$$

In this subsection we obtain an asymptotic formula for $S_2^{(m)}$.

**Lemma 3.5.2.** *Assume* $0 < \varpi < \frac{55}{12756}$. *Let* $\varepsilon = \frac{55}{12756} - \varpi$, $\eta = \eta(\varepsilon)$ *be defined as in Lemma 3.4.3,* $\eta_0 = \eta/2$, $\kappa = \frac{2\eta_0}{2/3+\varpi}$. *With* $\lambda_{\underline{d}}$ *chosen as in* (3.3.2) *and* $R = N^{\frac{1}{2}(\frac{2}{3}+\varpi)-\delta}$, *we have as* $N \to \infty$,

$$\begin{aligned} S_2^{(m)} :=& \sum_{\substack{n\sim N \\ n\equiv b \,(\mathrm{mod}\, W)}} \tau(n+h_m) \left( \sum_{d_j \mid n+h_j \forall j} \lambda_{\underline{d}} \right)^2 \\ =& (1+o(1)) \frac{W^{k-1}}{\varphi(W)^k} \frac{N}{(\log R)^k} \left( \frac{\log N}{\log R} \alpha^{(m)} - \beta_1^{(m)} - 4\beta_2^{(m)} \right), \end{aligned}$$

*with*

$$\alpha^{(m)} = \int_{\Delta_k(1)} t_m \left( F^{(\underline{1}+e_m)}(\underline{t}) \right)^2 \mathrm{d}\underline{t},$$

$$\beta_1^{(m)} = \int_{\Delta_k(1)} t_m^2 \left( F^{(\underline{1}+e_m)}(\underline{t}) \right)^2 \mathrm{d}\underline{t},$$

*and*

$$\beta_2^{(m)} = \int_{\Delta_k(1)} t_m F^{(\underline{1}+e_m)}(\underline{t}) F^{(\underline{1})}(\underline{t}) \,\mathrm{d}\underline{t}.$$

*Proof.* Following the same argument as in [64, Lemma 5.10], we can show that H1, H2', H3 and H5 holds. The variables in H2' satisfy

$$X = \frac{\varphi(W)}{W^2} N \left( \log N + 2\gamma - 1 + \sum_{p\mid W} \frac{2\log p}{p-1} \right), \quad X^* = -\frac{\varphi(W)}{W^2} N,$$

$$f(\underline{d}) = f_*(\underline{d}) = \frac{\varphi(d_m)}{d_m\tau(d_m)} \prod_{p|d_m} \left(\frac{2p}{2p-1}\right) \prod_{j=1}^{k} \frac{d_j^2}{\varphi(d_j)},$$

$$v(\underline{d}) = \log d_m - \sum_{p|d_m} \frac{\log p}{2p-1} - \sum_{j\neq m} \sum_{p|d_j} \frac{2\log p}{p-1},$$

$$r_{\underline{d}} = E'(N,q,a) + O(d_m^{1/2} q^{\epsilon-1}\sqrt{N}),$$

where

$$q = W\prod_{j=1}^{k} d_j,$$

$$E'(N,q,a) = \tau(\delta)\sum_{d|\delta} \frac{\mu(d)}{\tau(d)} E(N/\delta d, q', a_d), \tag{3.5.3}$$

with $\delta = (a,q), q' = q/\delta, a_d \equiv a\overline{\delta d} \,(\mathrm{mod}\, q')$. Here $\overline{\delta d}$ is the inverse of $\delta d$ modulo $q'$ and $a$ is some integer depending on $b$, $m$, $\underline{d}$, and $W$. H3 holds for $f$ and $f_*$ with

$$\alpha_j = \alpha_j^* = \begin{cases} 1 & \text{if } j = 1, \cdots, k, \quad j \neq m, \\ 2 & \text{if } j = m. \end{cases} \tag{3.5.4}$$

H5 holds for the additive function $v$ with $\beta_j$, given by

$$v_j(p) = -\frac{2\log p}{p-1} \quad \text{for} \quad j\neq m, \qquad v_m(p) = \log p - \frac{\log p}{2p-1},$$

and

$$\beta_j = \begin{cases} 0 & \text{if } j = 1, \cdots, k, \quad j \neq m, \\ 1 & \text{if } j = m. \end{cases} \tag{3.5.5}$$

We give details to verify H4'. In fact, it suffices to show that for any $A > 0, \epsilon > 0$,

$$\sideset{}{'}\sum_{\substack{[\underline{d},\underline{e}]\le N^{2/3+\varpi-\epsilon} \\ d_j,e_j\le N^{\eta_0}\ \forall j}} |E'(N,q,a)| + O\left( \sideset{}{'}\sum_{\substack{[\underline{d},\underline{e}]\le N^{2/3+\varpi-\epsilon} \\ d_j,e_j\le N^{\eta_0}\ \forall j}} [d_m,e_m]^{1/2} q^{\epsilon-1}\sqrt{N} \right) \ll \frac{N}{(\log N)^A}, \tag{3.5.6}$$

where $q = W\prod_j [d_j,e_j]$. Denoting $\prod_{j\neq m}[d_j,e_j]$ as $[\underline{d},\underline{e}]_m$, we have

$$\sideset{}{'}\sum_{\substack{[\underline{d},\underline{e}]\le N^{2/3+\varpi-\epsilon} \\ d_i,e_i\le N^{\eta_0}\ \forall j}} [d_m,e_m]^{1/2} q^{\epsilon-1} \ll \sideset{}{'}\sum_{[\underline{d},\underline{e}]\le N^{2/3+\varpi}} [d_m,e_m]^{1/2} q^{\epsilon-1}$$

$$\ll W^{\epsilon-1} \sum_{[d_m,e_m]\le N^{2/3+\varpi}} [d_m,e_m]^{\epsilon-1/2} \sum_{\substack{[\underline{d},\underline{e}]_m\le N^{2/3+\varpi} \\ [\underline{d},\underline{e}]_m \text{ square-free}}} ([\underline{d},\underline{e}]_m)^{\epsilon-1}.$$

Using [77, Proposition 3.1] and partial summation along with the fact that the average order of $\tau_3(n)$ is $(\log n)^2$, we get

$$\sum_{[d_m,e_m]\le N^{2/3+\varpi}} [d_m,e_m]^{\epsilon-1/2} \ll \sum_{r\le N^{2/3+\varpi}} r^{\epsilon-1/2}\tau_3(r) \ll (N^{2/3+\varpi})^{\epsilon+1/2}(\log N)^2.$$

Similarly,

$$\sum_{\substack{[\underline{d},\underline{e}]_m\le N^{2/3+\varpi}\\ [\underline{d},\underline{e}]_m \text{ square-free}}} ([\underline{d},\underline{e}]_m)^{\epsilon-1} \ll \sum_{r\le N^{2/3+\varpi}} r^{\epsilon-1}\tau_{3(k-1)}(r) \ll (N^{2/3+\varpi})^{\epsilon}(\log N)^{3k}.$$

As $\epsilon$ can be arbitrarily small and $W \ll (\log\log N)^2$, we obtain,

$$\sideset{}{'}\sum_{\substack{[\underline{d},\underline{e}]\le N^{2/3+\varpi-\epsilon}\\ d_j,e_j\le N^{\eta_0}\ \forall j}} [d_m,e_m]^{1/2}q^{\epsilon-1}\sqrt{N} \ll N^{\epsilon'+(2/3+\varpi)/2}\sqrt{N},$$

for any $\epsilon'>0$. As $2/3+\varpi<1$, this term is indeed of the order of $N(\log N)^{-A}$ for any $A>0$ as required. We now only need to consider the first term of (3.5.6). It can be bounded by

$$\sum_{\substack{q\le WN^{2/3+\varpi-\epsilon}\\ q\,\mid\,\prod_{p\le N^{\eta_0}} p}} \tau_{3k}(q)\max_{a \pmod q}|E'(N,q,a)|$$

$$\ll \left(\sum_{q\le N^{2/3-\epsilon}} + \sum_{\substack{N^{2/3-\epsilon}<q\le N^{2/3+\varpi-\epsilon}\\ q\,\mid\,\prod_{p\le N^{\eta_0}} p}}\right)\mu(q)^2\tau_{3k}(q)\max_{a \pmod q}|E'(N,q,a)| \quad (3.5.7)$$

We first deal with the first term of (3.5.7). Note that (3.5.3) gives

$$E'(N,q,a) = \tau(\delta)\sum_{d\mid\delta}\frac{\mu(d)}{\tau(d)}E\left(\frac{N}{\delta d},\frac{q}{\delta},a_d\right),$$

where $\delta=(a,q)$. If $\left(\frac{N}{\delta d}\right)^2 > \frac{q}{\delta}$, we find by (3.1.3)

$$\left|E\left(\frac{N}{\delta d},\frac{q}{\delta},a_d\right)\right| \ll \left(\frac{q}{\delta}\right)^{-\frac14}\left(\frac{N}{d\delta}\right)^{\frac12+\frac{\epsilon}{4}} \ll q^{-\frac14}N^{\frac12+\frac{\epsilon}{4}} \ll \frac{N}{q},$$

provided $q<N^{\frac23-\frac{\epsilon}{3}}$. If $\left(\frac{N}{\delta d}\right)^2 \le \frac{q}{\delta}$, we use the trivial bound to obtain

$$\left|E\left(\frac{N}{\delta d},\frac{q}{\delta},a_d\right)\right| \le \left|\sum_{\substack{n\le\frac{N}{\delta d}\\ n\equiv a_d \pmod{q\delta^{-1}}}}\tau(n)\right| + \frac{1}{\varphi(q\delta^{-1})}\left|\sum_{\substack{n\le\frac{N}{\delta d}\\ (n,q\delta^{-1})=1}}\tau(n)\right|$$

$$\le N^{\epsilon}+\frac{N}{\delta d}\log N\frac{\log q\delta^{-1}}{q\delta^{-1}} \ll \frac{N\log^2 N}{q},$$

where we used the estimate $\frac{1}{\varphi(n)} \ll \frac{\log n}{n}$. Therefore, we obtain

$$|E'(N,q,a)| \ll \tau(\delta)^2 \frac{N\log^2 N}{q} \ll \tau(q)^2 \frac{N\log^2 N}{q}$$

for $q \leq N^{2/3-\epsilon}$. On the other hand, [64, Thm. 5.9] gives

$$\sum_{q\leq N^\theta} \mu(q)^2 \max_{y\leq N} \max_{a \,(\mathrm{mod}\, q)} |E'(y,q,a)| \ll \frac{N}{(\log N)^{A'}}$$

for any $A' > 0$ and $\theta < 2/3$. Hence, by Cauchy-Schwarz, the first term of (3.5.7) is bounded by

$$\begin{aligned}
&\sum_{q\leq N^{2/3-\epsilon}} \mu(q)^2 \tau_{3k}(q) \max_{a \,(\mathrm{mod}\, q)} |E'(N,q,a)| \\
&\ll \left( \sum_{q\leq N^{2/3-\epsilon}} \mu(q)^2 \tau_{3k}(q)^2 \tau(q)^2 \frac{N\log^2 N}{q} \right)^{\frac{1}{2}} \left( \sum_{q\leq N^{2/3-\epsilon}} \mu(q)^2 \max_{a \,(\mathrm{mod}\, q)} |E'(N,q,a)| \right)^{\frac{1}{2}} \\
&\ll \frac{N}{(\log N)^A} \qquad (3.5.8)
\end{aligned}$$

for any $A > 0$, as $N \to \infty$.

We now turn to the second term of (3.5.7). In order to estimate $E'(N,q,a)$ for $N^{2/3-\epsilon} < q \leq N^{2/3+\varpi-\epsilon}$, $q \mid \prod_{p\leq N^{\eta_0}} p$, we consider three cases. If $d \geq N^{\epsilon/2}$, the crude bound gives

$$\begin{aligned}
\left| E\left(\frac{N}{\delta d}, \frac{q}{\delta}, a_d\right) \right| &\leq \left| \sum_{\substack{n\leq \frac{N}{\delta d} \\ n\equiv a_d \,(\mathrm{mod}\, q\delta^{-1})}} \tau(n) \right| + \frac{1}{\varphi(q\delta^{-1})} \left| \sum_{\substack{n\leq \frac{N}{\delta d} \\ (n,q\delta^{-1})=1}} \tau(n) \right| \\
&\ll N^{\frac{\epsilon}{4}} \left( \frac{N}{\delta d} \cdot \frac{\delta}{q} + 1 \right) + \frac{\delta}{q} \cdot \frac{N}{\delta d} \cdot N^{\frac{\epsilon}{4}} \\
&\ll \frac{N^{1-\frac{\epsilon}{4}}}{q}. \qquad (3.5.9)
\end{aligned}$$

If $d < N^{\epsilon/2}, \delta > N^{4\varpi}$, we obtain by (3.1.3),

$$\left| E\left(\frac{N}{\delta d}, \frac{q}{\delta}, a_d\right) \right| \ll \left(\frac{q}{\delta}\right)^{-\frac{1}{4}} \left(\frac{N}{\delta d}\right)^{\frac{1}{2}+\epsilon} \ll \left(\frac{q}{\delta}\right)^{-\frac{1}{4}} \left(\frac{N}{\delta}\right)^{\frac{1}{2}+\epsilon} \ll q^{-\frac{1}{4}} N^{\frac{1}{2}-\varpi-4\varpi\epsilon+\epsilon}. \qquad (3.5.10)$$

Finally, if $d < N^{\epsilon/2}, \delta \leq N^{4\varpi}$, we have

$$\frac{q}{\delta} \leq \frac{N^{\frac{2}{3}+\varpi-\epsilon}}{\delta} \leq \left(\frac{N}{\delta d}\right)^{\frac{2}{3}+\varpi} = \left(\frac{N}{\delta d}\right)^{\frac{2}{3}+\frac{55}{12756}-\varepsilon},$$

and

$$N^{\eta_0} = N^{\frac{\eta}{2}} \le \left(\frac{N^{\frac{2}{3}-\epsilon}}{N^{4\varpi}}\right)^{\eta} \le \left(\frac{q}{\delta}\right)^{\eta}.$$

As $\epsilon$ can be made arbitrarily small. We obtain by Lemma 3.4.3,

$$\left|E\left(\frac{N}{\delta d}, \frac{q}{\delta}, a_d\right)\right| \ll_{\varpi} \left(\frac{N}{\delta d}\right)^{1-\delta'} \frac{\delta}{q} \ll \frac{N^{1-\delta'}\delta^{\delta'}}{q} \ll \frac{N^{1-\delta'+4\varpi\delta'}}{q}, \qquad (3.5.11)$$

where $\delta'$ is some positive number depending on $\varpi$. Hence, it follows by (3.5.3), (3.5.9), (3.5.10), and (3.5.11) that

$$\sum_{\substack{N^{2/3-\epsilon} < q \le N^{2/3+\varpi-\epsilon/2} \\ q \mid \prod_{p \le N^{\eta_0}} p}} \tau_{3k}(q) \max_{a \,(\mathrm{mod}\, q)} |E'(N,q,a)|$$

$$\ll \sum_{N^{2/3-\epsilon} < q \le N^{2/3+\varpi-\epsilon/2}} \tau_{3k}(q)\tau(\delta)^2 \left(\frac{N^{1-\frac{\epsilon}{4}}}{q} + \frac{N^{\frac{1}{2}-\varpi-4\varpi\epsilon+\epsilon}}{q^{\frac{1}{4}}} + \frac{N^{1-\delta'+4\varpi\delta'}}{q}\right)$$

$$\ll \frac{N}{(\log N)^A} \qquad (3.5.12)$$

for any $A > 0$, as $N \to \infty$. This concludes the verification of H4'.

As the choice of $D_0$ gives

$$\frac{\log D_0}{\log R} = o(1),$$

we are now in a position to apply Lemma 3.4.4. The remaining argument is the same as [64, Lemma 5.10]. This completes the proof. □

Noting that our choice of $\lambda_{\underline{d}}$ satisfies the conditions of [77, Lemma 4.2], we combine the above lemma with the asymptotic formula for $S_1$ obtained in [77, Lemma 4.2].

**Lemma 3.5.3.** *Assume* $0 < \varpi < \frac{55}{12756}$. *Let* $\varepsilon = \frac{55}{12756} - \varpi$, $\eta = \eta(\varepsilon)$ *be defined as in Lemma 3.4.3,* $\eta_0 = \eta/2$, $\kappa = \frac{2\eta_0}{2/3+\varpi}$. *With* $\lambda_{\underline{d}}$ *chosen as in* (3.3.2) *and* $R = N^{\frac{1}{2}(\frac{2}{3}+\varpi)-\delta}$, *we have as* $N \to \infty$,

$$\begin{aligned} S(N,\rho) :=& \rho S_1 - \sum_{m=1}^{k} S_2^{(m)} \\ =& (1+o(1)) \frac{W^{k-1}}{\varphi(W)^k} \frac{N}{(\log R)^k} \left(\rho I(F) - \frac{\alpha^*}{(\frac{1}{2}(\frac{2}{3}+\varpi)-\delta)} + \beta_1^* + 4\beta_2^*\right), \end{aligned}$$

*with*

$$\alpha^* = \sum_{i=1}^{k} \alpha^{(m)} = k\alpha^{(k)}, \quad \beta_1^* = \sum_{i=1}^{k} \beta_1^{(m)} = k\beta_1^{(k)}, \quad \beta_2^* = \sum_{i=1}^{k} \beta_2^{(m)} = k\beta_2^{(k)},$$

*and*

$$I(F) = \int_{\Delta_k(1)} (F^{(\underline{1})}(\underline{t}))^2 \, \mathrm{d}\underline{t}.$$

### 3.5.2 The choice of the test function

Now, we adopt some ideas from [60] and [59] to choose a suitable smooth function $F$. Let $T = \frac{k}{\log\log k}$. Define the function $g\colon [0,\infty) \to \mathbb{R}$ by

$$g(t) := \begin{cases} e^{-\frac{t}{2}}\left(1 - \frac{t}{T}\right), & \text{if } t \le T, \\ 0, & \text{if } t > T, \end{cases} \tag{3.5.13}$$

and the simplex set

$$\Delta_k(r) := \{(t_1, \cdots, t_k) \in [0,\infty)^k : t_1 + \cdots + t_k \le r\}.$$

Let $h_1(t_1,\cdots,t_k)\colon [0,\infty)^k \to \mathbb{R}$ be a smooth function with $|h_1(t_1,\cdots,t_k)| \le 1$ such that

$$h_1(t_1,\cdots,t_k) = \begin{cases} 1, & \text{if } (t_1,\cdots,t_k) \in \Delta_k(1-\delta_1), \\ 0, & \text{if } (t_1,\cdots,t_k) \notin \Delta_k(1), \end{cases} \tag{3.5.14}$$

where $\delta_1 > 0$ is a small constant to be chosen soon. Furthermore, we may assume that

$$\left|\frac{\partial h_1}{\partial t_j}(t_1,\cdots,t_k)\right| \le \frac{1}{\delta_1} + 1 \tag{3.5.15}$$

for each $(t_1,\cdots,t_k) \in \Delta_k(1) \setminus \Delta_k(1-\delta_1)$ and $1 \le j \le k$.

Let $h_2(t)\colon [0,\infty) \to \mathbb{R}$ be a smooth function with $|h_2(t)| \le 1$ such that

$$h_2(t) = \begin{cases} 1, & \text{if } 0 \le t \le T - \delta_2, \\ 0, & \text{if } t > T, \end{cases} \tag{3.5.16}$$

where $\delta_2 < 1$ is a small positive constant to be chosen later. We may also assume that

$$|h_2'(t)| \le \frac{1}{\delta_2} + 1 \tag{3.5.17}$$

for each $T - \delta_2 \le t \le T$. Finally, we define the function $F\colon [0,\infty)^k \to \mathbb{R}$ by

$$F(\underline{t}) = (-1)^k \int_{t_1}^{\infty} \cdots \int_{t_k}^{\infty} h_1(\underline{t}) \prod_{j=1}^{k} h_2(kt_j) g(kt_j)\, \mathrm{d}\underline{t}, \quad \text{for } \underline{t} \in [0,\infty)^k. \tag{3.5.18}$$

As $h_1(\underline{t}) \prod_{j=1}^{k} h_2(kt_j) g(kt_j)$ is a smooth function supported on $\Delta_k^{[\frac{T}{k}]}(1)$, we obtain that $F(\underline{t})$ is also a smooth function supported on $\Delta_k^{[\frac{T}{k}]}(1)$ and

$$F^{(\underline{1})}(\underline{t}) = h_1(\underline{t}) \prod_{j=1}^{k} h_2(kt_j) g(kt_j). \tag{3.5.19}$$

In view of Lemma 3.5.3, our main goal for the remainder of this section becomes to estimate $\alpha^{(k)}$, $\beta_1^{(k)}$, $\beta_2^{(k)}$ and $I(F)$.

**Remark 3.5.4.** The idea of choosing the derivative of the test function $F$ to be of the form (3.5.19) is due to Maynard [60, Section 7]. Our introduction of smooth functions $h_1$ and $h_2$ here is inspired by the work of H. Li and H. Pan [59].

Before proceeding further, we mention some numerical results for integrals related to the function $g$:

$$\int_0^T g(t)^2\,\mathrm{d}t = 1 - 2(T + e^{-T} - 1)T^{-2}, \tag{3.5.20}$$

$$\int_0^T tg'(t)^2\,\mathrm{d}t = \frac{1}{4} - \frac{Te^{-T} + e^{-T} - 1}{2T^2}, \tag{3.5.21}$$

$$\int_0^T tg(t)^2\,\mathrm{d}t = 1 + (6 - 4T - 2Te^{-T} - 6e^{-T})T^{-2}. \tag{3.5.22}$$

### 3.5.3 An upper bound for $\alpha^{(k)}$

Throughout the remainder of this article, any constants implied by the notion $O$ or $\ll$ are absolute. We have

$$\begin{aligned} F^{(\underline{1}+e_k)}(\underline{t}) =& \frac{\partial F^{(\underline{1})}}{\partial t_k}(\underline{t}) \\ =& \frac{\partial h_1}{\partial t_k}(\underline{t}) \prod_{j=1}^{k} h_2(kt_j)g(kt_j) + kh_1(\underline{t})h_2'(kt_k)g(kt_k) \prod_{j=1}^{k-1} h_2(kt_j)g(kt_j) \\ &+ kh_1(\underline{t})h_2(kt_k)g'(kt_j) \prod_{j=1}^{k-1} h_2(kt_j)g(kt_j) \\ =:& I_1 + I_2 + I_3. \end{aligned} \tag{3.5.23}$$

By the definition of $h_1$, we find

$$\begin{aligned} &\int_{\Delta_k(1)} I_1^2 t_k d\underline{t} = \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} t_k \left(\frac{\partial h_1}{\partial t_k}(\underline{t})\right)^2 \prod_{j=1}^{k} e^{-kt_j}\left(1 - \frac{kt_j}{T}\right)^2 h_2(kt_j)^2\,\mathrm{d}\underline{t} \\ &\le \left(1 + \frac{1}{\delta_1}\right)^2 \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} t_k e^{-kt_k}\left(1 - \frac{kt_k}{T}\right)^2 h_2(kt_k)^2 \prod_{j=1}^{k-1} h_2(kt_j)^2 g(kt_j)^2\,\mathrm{d}\underline{t} \\ &\le \frac{1}{ke}\left(1 + \frac{1}{\delta_1}\right)^2 \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k-1} h_2(kt_j)^2 g(kt_j)^2\,\mathrm{d}\underline{t}. \end{aligned} \tag{3.5.24}$$

In the last step we used $\max_{u\geq 0} ue^{-ku} = \frac{1}{ke}$. Let $r = t_1 + \cdots + t_k$, it follows

$$\int_{\Delta_k(1)\backslash\Delta_k(1-\delta_1)} \prod_{j=1}^{k-1} h_2^2(kt_j)g^2(kt_j)\,\mathrm{d}\underline{t}$$
$$\leq \int_{\Delta_{k-1}(1)} \left(\int_{1-\delta_1}^{1} dr\right) \prod_{j=1}^{k-1} g(kt_j)^2 h_2(kt_j)^2\,\mathrm{d}t_1\cdots\,\mathrm{d}t_{k-1} \leq \frac{\delta_1 \Upsilon^{k-1}}{k^{k-1}}, \quad (3.5.25)$$

where $\Upsilon = \int_0^T g(t)^2\,\mathrm{d}t$. We conclude that

$$\int_{\Delta_k(1)} I_1^2 t_k\,\mathrm{d}\underline{t} \ll \frac{1}{k\delta_1}\cdot\frac{\Upsilon^{k-1}}{k^{k-1}}. \quad (3.5.26)$$

from (3.5.24) and (3.5.25). Regarding the upper bound for $\int_{\Delta_k(1)} I_2^2 t_k\,\mathrm{d}\underline{t}$, we find

$$\int_{\Delta_k(1)} I_2^2 t_k\,\mathrm{d}\underline{t} \leq \left(1+\frac{1}{\delta_2}\right)^2 \int_{\Delta_{k-1}(1)} \left(\prod_{j=1}^{k-1} g(kt_j)^2 h_2(kt_j)^2\right)$$
$$\cdot\left(\int_{(T-\delta_2)/k}^{T/k} t_k k^2 e^{-kt_k}\left(1-\frac{kt_k}{T}\right)^2 dt_k\right)\mathrm{d}t_1\cdots\,\mathrm{d}t_{k-1}$$
$$\leq \left(1+\frac{1}{\delta_2}\right)^2 \int_{\Delta_{k-1}(1)} \left(\prod_{j=1}^{k-1} g(kt_j)^2 h_2(kt_j)^2\right)$$
$$\cdot\frac{\delta_2}{k}\cdot\frac{T}{k}\cdot k^2 e^{-(T-\delta_2)}\left(\frac{\delta_2}{T}\right)^2\mathrm{d}t_1\cdots\,\mathrm{d}t_{k-1}$$
$$\ll \frac{\delta_2}{Te^T}\cdot\frac{\Upsilon^{k-1}}{k^{k-1}}. \quad (3.5.27)$$

In the second inequality, we used the trivial bound for the second integral. Now, we obtain

$$\int_{\Delta_k(1)} I_3^2 t_k\,\mathrm{d}\underline{t} = \int_{\Delta_k(1)} t_k k^2 h_1(\underline{t})^2 h_2(kt_k)^2 g'(kt_k)^2 \prod_{j=1}^{k-1} h_2(kt_j)^2 g(kt_j)^2\,\mathrm{d}\underline{t}$$
$$\leq k^2 \int_0^\infty t_k h_2(kt_k)^2 g'(kt_k)^2\,\mathrm{d}t_k \prod_{j=1}^{k-1}\int_0^\infty h_2(kt_j)^2 g(kt_j)^2\,\mathrm{d}t_j$$
$$\leq k^2 \int_0^{\frac{T}{k}} t_k g'(kt_k)^2\,\mathrm{d}t_k \prod_{j=1}^{k-1}\int_0^{\frac{T}{k}} g(kt_j)^2\,\mathrm{d}t_j$$
$$= \frac{\Upsilon^{k-1}}{k^{k-1}}\int_0^T t g'(t)^2 dt. \quad (3.5.28)$$

From the Cauchy-Schwarz inequality, (3.5.26), (3.5.27), (3.5.28), and (3.5.21) we deduce

$$\begin{aligned}\alpha^{(k)} &= \int (I_1+I_2+I_3)^2 t_k \, \mathrm{d}\underline{t} \\ &= \int (I_1^2+I_2^2+I_3^2+2I_1I_2+2I_1I_3+2I_2I_3) t_k \, \mathrm{d}\underline{t} \\ &\leq \int I_1^2 t_k d\underline{t} + \int I_2^2 t_k \, \mathrm{d}\underline{t} + \int I_3^2 t_k \, \mathrm{d}\underline{t} + 2\left(\int I_1^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} \left(\int I_2^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} \\ &\quad + 2\left(\int I_1^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} \left(\int I_3^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} + 2\left(\int I_2^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} \left(\int I_3^2 t_k \, \mathrm{d}\underline{t}\right)^{\frac{1}{2}} \\ &\leq \frac{\Upsilon^{k-1}}{k^{k-1}} \int_0^T t g'(t)^2 \, \mathrm{d}t + O\left(\left(\frac{1}{k\delta_1} + \sqrt{\frac{\delta_2}{k\delta_1 T e^T}} + \frac{1}{\sqrt{k\delta_1}} + \sqrt{\frac{\delta_2}{Te^T}}\right) \frac{\Upsilon^{k-1}}{k^{k-1}}\right).\end{aligned} \tag{3.5.29}$$

Now, selecting $\delta_1 = \frac{\sqrt{\log k}}{k}$ and observing that $T = \frac{k}{\log\log k}$ and $\delta_2 < 1$ gives that

$$\alpha^{(k)} \leq \frac{\Upsilon^{k-1}}{k^{k-1}} \int_0^T t g'(t)^2 \, \mathrm{d}t + O\left(\frac{1}{(\log k)^{\frac{1}{4}}} \cdot \frac{\Upsilon^{k-1}}{k^{k-1}}\right). \tag{3.5.30}$$

### 3.5.4 An upper bound for $\beta_1^{(k)}$ and $\beta_2^{(k)}$

Recall

$$\beta_1^{(k)} = \int_{\Delta_k(1)} t_k^2 (F^{(\underline{1}+e_k)}(\underline{t}))^2 \, \mathrm{d}\underline{t} = \int_{\Delta_k(1)} t_k^2 (I_1+I_2+I_3)^2 \, \mathrm{d}\underline{t}. \tag{3.5.31}$$

By an analogous argument as in the estimation of $\alpha^{(k)}$, it is not hard to see the main contribution of right hand side of (3.5.31) comes from $\int_{\Delta_k(1)} I_3^2 t_k^2 \, \mathrm{d}\underline{t}$. Hence,

$$\begin{aligned}\beta_1^{(k)} &\ll \int_{\Delta_k(1)} I_3^2 t_k^2 \, \mathrm{d}\underline{t} \\ &= \int_{\Delta_k(1)} t_k^2 k^2 h_1(\underline{t})^2 h_2(kt_k)^2 g'(kt_k)^2 \prod_{j=1}^{k-1} h_2(kt_j)^2 g(kt_j)^2 \, \mathrm{d}\underline{t} \\ &\leq \frac{\Upsilon^{k-1}}{k^k} \int_0^T t^2 g'(t)^2 \, \mathrm{d}t \ll \frac{\Upsilon^{k-1}}{k^k}.\end{aligned} \tag{3.5.32}$$

In the last step we used the fact $\int_0^T t^2 g'(t)^2 \, \mathrm{d}t = O(1)$, as $k \to \infty$. This can be readily seen by noting that the integrand is an exponentially decreasing function. Similarly, we have

$$\beta_2^{(k)} = \int_{\Delta_k(1)} t_k F^{(\underline{1}+e_k)}(\underline{t}) F^{(\underline{1})}(\underline{t}) \, \mathrm{d}\underline{t} = \int_{\Delta_k(1)} t_k (I_1+I_2+I_3) F^{(\underline{1})}(\underline{t}) \, \mathrm{d}\underline{t}. \tag{3.5.33}$$

The main contribution of the right hand side of (3.5.33) comes from

$$\int_{\Delta_k(1)} t_k I_3 F^{(\underline{1})}(\underline{t})\, \mathrm{d}\underline{t}.$$

Therefore, we obtain

$$\begin{aligned} \beta_2^{(k)} &\ll \int_{\Delta_k(1)} t_k I_3 F^{(\underline{1})}(\underline{t})\, \mathrm{d}\underline{t} \\ &= \int_{\Delta_k(1)} t_k k h_1^2(\underline{t}) h_2(kt_k)^2 g'(kt_k) g(kt_k) \prod_{j=1}^{k-1} h_2(kt_j)^2 g(kt_j)^2\, \mathrm{d}\underline{t} \\ &\le \frac{\Upsilon^{k-1}}{k^k} \int_0^T t g'(t) g(t)\, \mathrm{d}t \ll \frac{\Upsilon^{k-1}}{k^k}. \end{aligned} \tag{3.5.34}$$

### 3.5.5 Lower bound for $I(F)$

In this subsection we derive a lower bound for $I(F)$. Our argument is inspired by [60, Section 7].

**Proposition 3.5.5.** *For every $\epsilon > 0$, there exists $\delta_2 = \delta_2(\epsilon) > 0$, such that*

$$\begin{aligned} I(F) > \frac{\Upsilon^{k-1}}{k^k} &\left(1 - \frac{T}{k(1 - T/k - \mu)^2}\right) \int_0^\infty g(u)^2\, \mathrm{d}u - \epsilon \\ &+ O\left(\frac{e^{\sqrt{\log k}} (\log k)^{\frac{9}{2}}}{\sqrt{k-1}} \frac{\Upsilon^{k-1}}{k^k}\right), \end{aligned} \tag{3.5.35}$$

*where*

$$\mu = \frac{\int_0^\infty u g(u)^2\, \mathrm{d}u}{\int_0^\infty g(u)^2\, \mathrm{d}u} \qquad \text{and} \qquad \Upsilon = \int_0^T g(t)^2\, \mathrm{d}t.$$

*Proof.* Recall the definition of $I(F)$ and our test function $F$, we have

$$I(F) = \int_{\Delta_k(1)} (F^{(\underline{1})}(\underline{t}))^2\, \mathrm{d}\underline{t} = \int_{\Delta_k(1)} h_1(\underline{t})^2 \prod_{j=1}^{k} h_2(kt_j)^2 g(kt_j)^2\, \mathrm{d}\underline{t}.$$

In view of Lebesgue's dominated convergence theorem, for every $\epsilon > 0$, we can take $\delta_2 = \delta_2(\epsilon)$ sufficiently small, such that

$$I(F) > \int_{\Delta_k(1)} h_1(\underline{t})^2 \prod_{j=1}^{k} g(kt_j)^2\, \mathrm{d}\underline{t} - \epsilon. \tag{3.5.36}$$

By the definition of $h_1(\underline{t})$, it follows that

$$I(F) > \int_{\Delta_k(1)} \prod_{j=1}^{k} g(kt_j)^2\, \mathrm{d}\underline{t} - \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k} g(kt_j)^2\, \mathrm{d}\underline{t} - \epsilon. \tag{3.5.37}$$

Thus, to prove (3.5.35), it suffices to establish

$$\int_{\Delta_k(1)} \prod_{j=1}^{k} g(kt_j)^2 \,\mathrm{d}\underline{t} \geq \frac{\Upsilon^{k-1}}{k^k}\left(1 - \frac{T}{k(1-T/k-\mu)^2}\right)\int_0^\infty g(u)^2 \,\mathrm{d}u, \quad (3.5.38)$$

and

$$\int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k} g(kt_j)^2 \,\mathrm{d}\underline{t} \ll \frac{e^{\sqrt{\log k}}(\log k)^{\frac{9}{2}}}{\sqrt{k-1}} \frac{\Upsilon^{k-1}}{k^k}. \quad (3.5.39)$$

We begin with (3.5.39). It is clear that

$$\begin{aligned}
\int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k} g(kt_j)^2 \,\mathrm{d}\underline{t} &= \int_{\substack{\Delta_k(1)\setminus\Delta_k(1-\delta_1) \\ \underline{t}\in[0,T/k]^k}} \prod_{j=1}^{k} e^{-kt_j}\left(1-\frac{kt_j}{T}\right)^2 \mathrm{d}\underline{t} \\
&\leq \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} e^{-k(\sum_{j=1}^{k} t_j)} \,\mathrm{d}\underline{t} \\
&\leq e^{-k(1-\delta_1)} \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} 1 \,\mathrm{d}\underline{t} \\
&\leq e^{-k(1-\delta_1)} \int_{\Delta_{k-1}(1)} \int_{1-\delta_1}^{1} \mathrm{d}r \,\mathrm{d}t_1 \cdots \mathrm{d}t_{k-1} \\
&= \frac{e^{-k(1-\delta_1)}\delta_1}{(k-1)!}. \quad (3.5.40)
\end{aligned}$$

In the penultimate step, we changed the variable by $r = t_1 + \cdots + t_k$. Combining Stirling's formula with (3.5.40), we obtain

$$\int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k} g(kt_j)^2 \,\mathrm{d}\underline{t} \ll \frac{e^{k\delta_1}\delta_1}{\sqrt{k-1}(k-1)^{k-1}}. \quad (3.5.41)$$

From (3.5.20), we find

$$\Upsilon = \int_0^T g(t)^2 \,\mathrm{d}t \geq 1 - \frac{2}{T}. \quad (3.5.42)$$

As $T = \frac{k}{\log\log k}$, one has

$$\begin{aligned}
\Upsilon^{k-1} \geq \left(1 - \frac{2\log\log k}{k}\right)^{k-1} &= e^{(k-1)\log\left(1-\frac{2\log\log k}{k}\right)} \\
&\geq e^{(k-1)\frac{-4\log\log k}{k}} \geq \frac{1}{(\log k)^4}. \quad (3.5.43)
\end{aligned}$$

Combining (3.5.41) with (3.5.43) then gives, as $\delta_1 = \frac{\sqrt{\log k}}{k}$,

$$\begin{aligned}
\frac{k^k}{\Upsilon^{k-1}} \int_{\Delta_k(1)\setminus\Delta_k(1-\delta_1)} \prod_{j=1}^{k} g(kt_j)^2 \,\mathrm{d}\underline{t} &\ll \frac{e^{k\delta_1}\delta_1 k^k (\log k)^4}{\sqrt{k-1}(k-1)^{k-1}} \\
&\ll \frac{e^{\sqrt{\log k}}(\log k)^{\frac{9}{2}}}{\sqrt{k-1}}, \quad (3.5.44)
\end{aligned}$$

and the claim (3.5.35) follows.

Now we show (3.5.38). Since squares are nonnegative, we restrict the outer integral to $\sum_{j=2}^k t_j \le 1 - T/k$ and find

$$\begin{aligned}\int_{\Delta_k(1)} \prod_{j=1}^k g(kt_j)^2 \, \mathrm{d}\underline{t} \ge & \underset{\substack{t_2,\cdots,t_k \ge 0 \\ \sum_{j=2}^k t_j \le 1-T/k}}{\int \cdots \int} \int_0^{T/k} \prod_{j=1}^k g(kt_j)^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \, \mathrm{d}t_k \\ =& I' - E, \end{aligned} \tag{3.5.45}$$

where

$$I' = \underset{t_2,\cdots,t_k \ge 0}{\int \cdots \int} \int_0^{T/k} \prod_{j=1}^k g(kt_j)^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \, \mathrm{d}t_k = \left( \int_0^\infty g(kt)^2 \, \mathrm{d}t \right)^k = \frac{\Upsilon^k}{k^k}, \tag{3.5.46}$$

$$\begin{aligned} E = & \underset{\substack{t_2,\cdots,t_k \ge 0 \\ \sum_{j=2}^k t_j > 1-T/k}}{\int \cdots \int} \int_0^{T/k} \prod_{j=1}^k g(kt_j)^2 \, \mathrm{d}t_1 \, \mathrm{d}t_2 \cdots \, \mathrm{d}t_k \\ =& k^{-k} \left( \int_0^\infty g(u)^2 \, \mathrm{d}u \right) \underset{\substack{u_2,\cdots,u_k \ge 0 \\ \sum_{j=2}^k u_j > k-T}}{\int \cdots \int} \prod_{j=2}^k g(u_j)^2 \, \mathrm{d}u_2 \cdots \, \mathrm{d}u_k. \end{aligned} \tag{3.5.47}$$

We can check the choice of $g$ satisfies

$$\mu = \frac{\int_0^\infty ug(u)^2 \, \mathrm{d}u}{\int_0^\infty g(u)^2 \, \mathrm{d}u} < 1 - \frac{T}{k}. \tag{3.5.48}$$

Actually, from (3.5.20) and (3.5.22), we have

$$\begin{aligned} \mu =& \frac{\int_0^\infty ug(u)^2 \, \mathrm{d}u}{\int_0^\infty g(u)^2 \, \mathrm{d}u} = \frac{1 + (6 - 4T - 2Te^{-T} - 6e^{-T})T^{-2}}{1 - 2(T + e^{-T} - 1)T^{-2}} \\ =& 1 - \frac{2T + 2Te^{-T} + 4e^{-T} - 4}{1 - 2(T + e^{-T} - 1)T^{-2}}. \end{aligned} \tag{3.5.49}$$

Since $T = \frac{k}{\log \log k}$, it follows

$$1 - \mu - \frac{T}{k} = \frac{2T + 2Te^{-T} + 4e^{-T} - 4}{1 - 2(T + e^{-T} - 1)T^{-2}} - \frac{T}{k} \gg 1. \tag{3.5.50}$$

Let $\Theta = (k - T)/(k - 1) - \mu > 0$. If $\sum_{j=2}^k u_j > k - T$, then $\sum_{j=2}^k u_j > (k-1)(\mu + \Theta)$, and so we have

$$1 \le \Theta^{-2} \left( \frac{1}{k-1} \sum_{j=2}^k u_j - \mu \right)^2. \tag{3.5.51}$$

Since the right hand side of (3.5.51) is nonnegative for all $u_j$, we can obtain an upper bound for $E$ if we multiply the integrand by $\Theta^{-2}\left(\sum_{j=2}^{k} u_j/(k-1)-\mu\right)^2$ and then drop the requirement that $\sum_{j=2}^{k} u_j > k-T$. We find

$$E \le \Theta^{-2}k^{-k}\left(\int_0^\infty g(u)^2\,\mathrm{d}u\right)\int_0^\infty \cdots \int_0^\infty \left(\frac{\sum_{j=2}^{k} u_j}{k-1}-\mu\right)^2 \cdot \left(\prod_{j=2}^{k} g(u_j)^2\right)\mathrm{d}u_2\cdots\,\mathrm{d}u_k. \tag{3.5.52}$$

Expanding out the inner square and calculating all the terms which are not of the form $u_j^2$ gives

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty \left(\frac{2\sum_{2\le i<j\le k} u_iu_j}{(k-1)^2}-\frac{2\mu\sum_{j=2}^{k} u_j}{k-1}+\mu^2\right)\left(\prod_{j=2}^{k} g(u_j)^2\right)\mathrm{d}u_2\cdots\,\mathrm{d}u_k\\
&=\frac{k-2}{k-1}\mu^2\Upsilon^{k-1}-2\mu^2\Upsilon^{k-1}+\mu^2\Upsilon^{k-1}\\
&=\frac{-\mu^2\Upsilon^{k-1}}{k-1}.
\end{aligned} \tag{3.5.53}$$

For the $u_j^2$ terms, we see that $u_j^2 g(u_j)^2 \le Tu_jg(u_j)^2$ in view of the support of $g$. Hence,

$$\begin{aligned}
\int_0^\infty \cdots \int_0^\infty u_j^2\left(\prod_{i=2}^{k} g(u_i)^2\right)\mathrm{d}u_2\cdots\,\mathrm{d}u_k &\le T\Upsilon^{k-2}\int_0^\infty u_jg(u_j)^2\,\mathrm{d}u_j\\
&=\mu T\Upsilon^{k-1}.
\end{aligned} \tag{3.5.54}$$

It follows from (3.5.52), (3.5.53), and (3.5.54) that

$$\begin{aligned}
E &\le \Theta^{-2}k^{-k}\left(\int_0^\infty g(u)^2\,\mathrm{d}u\right)\left(\frac{\mu T\Upsilon^{k-1}}{k-1}-\frac{\mu^2\Upsilon^{k-1}}{k-1}\right)\\
&\le \left(\frac{\Theta^{-2}\mu Tk^{-k}\Upsilon^{k-1}}{k-1}\right)\left(\int_0^\infty g(u)^2\,\mathrm{d}u\right).
\end{aligned} \tag{3.5.55}$$

Since $(k-1)\Theta^2 \ge k(1-T/k-\mu)^2$ and $\mu \le 1$, from (3.5.55) we obtain

$$E \le \left(\frac{Tk^{-k-1}\Upsilon^{k-1}}{(1-T/k-\mu)^2}\right)\left(\int_0^\infty g(u)^2\,\mathrm{d}u\right). \tag{3.5.56}$$

From (3.5.45), (3.5.46), (3.5.56) we conclude (3.5.38). The proof of the proposition is now complete. □

### 3.5.6 Completion of the proof of Theorem 1.3

*Proof.* Recall that supp $F(\underline{t}) \subset \Delta_k^{[\frac{T}{k}]}(1)$ and $T/k = 1/\log\log k$. We can find a sequence $\{\varpi_k\}_{k=0}^{\infty} \subset (0, 55/12756)$ with $\lim_k \varpi_k = 55/12756$ and a real number $K$, such that

$$\frac{\eta(\varepsilon_k)}{2/3 + \varpi_k} \geq \frac{1}{\log\log k}, \tag{3.5.57}$$

for $k > K$, where $\varepsilon_k = 55/12756 - \varpi_k$ and the function $\eta(\cdot)$ is defined as in Lemma 3.4.3. Applying Lemma 3.5.3 with $\varpi = \varpi_k$, we get $S(N, \rho) > 0$ for all large $N$, provided

$$\rho > \frac{k\alpha^{(k)}}{(\frac{1}{2}(\frac{2}{3} + \varpi_k) - \delta)I(F)} - \frac{k\beta_1^{(k)}}{I(F)} - \frac{4k\beta_2^{(k)}}{I(F)}. \tag{3.5.58}$$

Plugging the estimates for $\alpha^{(k)}$, $\beta_1^{(k)}$, $\beta_2^{(k)}$ and $I(F)$ (see (3.5.30), (3.5.32), (3.5.34), and Proposition 3.5.5) into the right hand side of (3.5.58) yields

$$\begin{aligned}&\frac{k\alpha^{(k)}}{(\frac{1}{2}(\frac{2}{3} + \varpi_k) - \delta)I(F)} - \frac{k\beta_1^{(k)}}{I(F)} - \frac{4k\beta_2^{(k)}}{I(F)} \\ &\leq \frac{(\frac{1}{2}(\frac{2}{3} + \varpi_k) - \delta)^{-1}\frac{\Upsilon^{k-1}}{k^{k-2}}\int_0^T tg'(t)^2\,\mathrm{d}t + O\left(\frac{1}{(\log k)^{\frac{1}{4}}} \cdot \frac{\Upsilon^{k-1}}{k^{k-2}}\right)}{\frac{\Upsilon^{k-1}}{k^k}\left(1 - \frac{T}{k(1-T/k-\mu)^2}\right)\int_0^T g(t)^2\,\mathrm{d}t - \epsilon + O\left(\frac{e^{\sqrt{\log k}}(\log k)^{\frac{9}{2}}}{\sqrt{k-1}}\frac{\Upsilon^{k-1}}{k^k}\right)}.\end{aligned} \tag{3.5.59}$$

We choose $\delta_2 > 0$ sufficiently small such that this estimate becomes

$$\leq \frac{(\frac{1}{2}(\frac{2}{3} + \varpi_k) - \delta)^{-1}\frac{\Upsilon^{k-1}}{k^{k-2}}\int_0^T tg'(t)^2\,\mathrm{d}t + O\left(\frac{1}{(\log k)^{\frac{1}{4}}} \cdot \frac{\Upsilon^{k-1}}{k^{k-2}}\right)}{\frac{\Upsilon^{k-1}}{k^k}\left(1 - \frac{T}{k(1-T/k-\mu)^2}\right)\int_0^T g(t)^2\,\mathrm{d}t + O\left(\frac{e^{\sqrt{\log k}}(\log k)^{\frac{9}{2}}}{\sqrt{k-1}}\frac{\Upsilon^{k-1}}{k^k}\right)}. \tag{3.5.60}$$

Since (3.5.50) and $T/k = 1/\log\log k$, we have

$$\frac{T}{k(1 - T/k - \mu)^2} = o(1), \quad \text{as } k \to \infty. \tag{3.5.61}$$

It follows from (3.5.20) and (3.5.21) that

$$\frac{\int_0^T tg'(t)^2\,\mathrm{d}t}{\int_0^T g(t)^2\,\mathrm{d}t} \to \frac{1}{4}, \quad \text{as } k \to \infty. \tag{3.5.62}$$

Combining (3.5.60), (3.5.61), (3.5.62), and $\lim_k \varpi_k = 55/12756$, gives that

$$\begin{aligned} &\frac{k\alpha^{(k)}}{(\frac{1}{2}(\frac{2}{3}+\varpi_k)-\delta)I(F)} - \frac{k\beta_1^{(k)}}{I(F)} - \frac{4k\beta_2^{(k)}}{I(F)} \\ &\leq \frac{\int_0^T tg'(t)^2\,\mathrm{d}t}{(\frac{1}{2}(\frac{2}{3}+\varpi_k)-\delta)\int_0^T g(t)^2\,\mathrm{d}t}k^2 + o(k^2) \\ &= \frac{1}{\frac{4}{3}+2\varpi_k-2\delta}k^2 + o(k^2) \\ &= \frac{1}{\frac{4}{3}+2\cdot\frac{55}{12756}-2\delta}k^2 + o(k^2), \end{aligned} \tag{3.5.63}$$

as $k \to \infty$. Since $\delta$ can be made arbitrarily small, we can take

$$\rho_k = \frac{1}{\frac{4}{3}+2\cdot\frac{55}{12756}}k^2 + o(k^2) = \frac{2126}{2853}k^2 + o(k^2). \tag{3.5.64}$$

Finally, we remark that the number of integers $\leq x$ that satisfy the requirements of this theorem is $\gg x(\log\log x)^{-1}(\log x)^{-k}$. It can be deduced by using the same argument as in [64, Thm. 5.13]. The proof of Theorem 3.1.3 is now complete. □

# Chapter 4

# Generalizations of Koga's version of the Wiener-Ikehara theorem

*"Any useful logic must concern itself with Ideas with a fringe of vagueness and a Truth that is a matter of degree."*

—Norbert Wiener 1894-1964

In this chapter, we establish new versions of the Wiener-Ikehara theorem where only boundary assumptions on the real part of the Laplace transform are imposed. Our results generalize and improve a recent theorem of T. Koga [53]. As an application, we give a quick Tauberian proof of Blackwell's renewal theorem.

## 4.1 Introduction

The Wiener-Ikehara theorem [80] is a foundational result in complex Tauberian theory. Originally devised to significantly simplify an early result of Landau [57] and so deliver one of the quickest deductions of the prime number theorem, it has found countless applications in diverse areas of mathematics such as operator theory, partial differential equations, and number theory. The interested reader is referred to the books [19, Chapter 10], [55, Chapter III], and [75, Chapter II.7] for excellent accounts on this and related complex Tauberian theorems; see also the articles [9, 15, 16, 26, 53, 69, 84] for some developments during the last decade.

In one of its many forms, the Wiener-Ikehara theorem states that if a non-decreasing function $S$ has convergent Laplace transform

$$\mathcal{L}\{S;s\} = \int_0^\infty e^{-sx} S(x) \mathrm{d}x$$

for $\Re e\, s > 1$ and if there is a constant $a \in \mathbb{R}$ such that the analytic function

$$G(s) = \mathcal{L}\{S;s\} - \frac{a}{s-1} \tag{4.1.1}$$

admits $L^1_{loc}$-boundary behavior on the whole boundary line $1 + i\mathbb{R}$, then

$$S(x) \sim a e^x \qquad \text{as } x \to \infty. \tag{4.1.2}$$

Naturally, the hypothesis of $L^1_{loc}$-boundary behavior covers the case of continuous extension, and in particular that of analytic continuation. On the other hand, we point out that the boundary requirements on the Laplace transform can further be taken to a minimum if one employs the so-called local pseudofunction boundary behavior (cf. [15, 16, 56]). The pseudofunction approach plays a major role in modern complex Tauberian theory (cf. [55, Chapter III]).

Very recently [53], Koga has obtained an interesting generalization of this version of the Wiener-Ikehara theorem with $L^1_{loc}$-boundary behavior, where only the boundary properties of the real part of the Laplace transform are needed. His result also weakens the non-decreasing hypothesis on $S$ to log-linear slow decrease, a Tauberian condition that was introduced and studied in [15, 84] and that is intimately connected with exact Wiener-Ikehara theorems, that is, complete Laplace transform characterizations of the asymptotic behavior (4.1.2). We call a function $S$ *log-linearly slowly decreasing* (at $\infty$) if for each $\varepsilon$ there are $h, x_0 > 0$ such that

$$\frac{S(y) - S(x)}{e^x} \geq -\varepsilon \qquad \text{for } x \leq y \leq x + h \text{ and } x \geq x_0.$$

Koga's main motivation to establish a novel version of the Wiener-Ikehara theorem was to provide a Dirichlet series generalization of the Kolmogorov-Erdős-Feller-Pollard renewal theorem [21, 54] (cf. [24, Sections XIII.3 and XIII.11]). Moreover, he also obtained a Tauberian theorem for power series and applied it to give a new proof of the classical quoted renewal theorem. Upon a minor reformulation (cf. Remark 4.6.2), Koga's Tauberian theorem for Laplace transforms reads:

**Theorem 4.1.1** ([53, Thm. 2]). *Let $S \in L^1_{loc}[0,\infty)$ be log-linearly slowly decreasing and satisfy*

$$\int_1^\infty \frac{|S(x)|}{x^2 e^x} \, \mathrm{d}x < \infty. \tag{4.1.3}$$

*Let* $U(s) = \Re e\, \mathcal{L}\{S; s\}$. *Assume there are* $\lambda > 0$ *and* $g \in L^1(-\lambda, \lambda)$ *such that*

$$U(\sigma + it) \geq g(t), \qquad \text{for a.e. } t \in (-\lambda, \lambda) \text{ and } \sigma \in (1, 2]. \tag{4.1.4}$$

*If in addition* $U$ *has* $L^1_{loc}$*-boundary behavior on the boundary open subset* $1 + i(\mathbb{R} \setminus \{0\})$, *namely, if there is* $f \in L^1_{loc}(\mathbb{R} \setminus \{0\})$ *such that on any finite interval* $I$ *not containing the origin we have*

$$\lim_{\sigma \to 1^+} \int_I |U(\sigma + it) - f(t)| \mathrm{d}t = 0, \tag{4.1.5}$$

*then* (4.1.2) *must hold for some constant* $a \in \mathbb{R}$.

The aim of this chapter is to extend Koga's theorem by further considering other useful hypotheses for the boundary behavior of $\Re e\, \mathcal{L}\{S; s\}$ near $s = 1$. In addition, our results considerably improve Koga's ones, in the sense that we have been able to replace the somehow unnatural condition (4.1.3) (and its analog for power series) by just requirements on the Laplace transform. Notice that (4.1.3) always implies (4.1.7) below with $k = 2$, so that the next theorem, our first main result, includes Theorem 4.1.1 as its particular instance, and more generally also shows that one can relax (4.1.3) in Theorem 4.1.1 to

$$\int_1^\infty \frac{|S(x)|}{x^k e^x} \mathrm{d}x < \infty \qquad \text{for some } k \in \mathbb{N}; \tag{4.1.6}$$

see also Remark 4.6.1. (It is worth noting that these integral growth conditions could still be significantly weakened, see (4.6.6) and (4.6.7) in Remark 4.6.4.)

**Theorem 4.1.2** (Laplace transforms)**.** *Let* $S \in L^1_{loc}[0, \infty)$ *be log-linearly slowly decreasing and have convergent Laplace transform on* $\Re e\, s > 1$. *Suppose that the harmonic function* $U(s) = \Re e\, \mathcal{L}\{S; s\}$ *has* $L^1_{loc}$*-boundary behavior on* $1 + i(\mathbb{R} \setminus \{0\})$ *and that there is some* $\lambda > 0$ *such that one of the following three conditions holds:*

*(B.1) there is* $c \geq 0$ *such that* $U(\sigma + it) \geq -c$, *for* $t \in (-\lambda, \lambda)$ *and* $\sigma \in (1, 2]$;

*(B.2)* $\displaystyle\sup_{1<\sigma<2} \int_{-\lambda}^{\lambda} |U(\sigma + it)| \mathrm{d}t < \infty$;

*(B.3) there are* $g \in L^1(-\lambda, \lambda)$ *and* $k \in \mathbb{N}$ *such that* (4.1.4) *holds and*

$$U(\sigma + it) = O\left((\sigma - 1)^{-k}\right), \qquad \text{for } t \in (-\lambda, \lambda) \text{ and } \sigma \in (1, 2]. \tag{4.1.7}$$

*Then*

$$S(x) \sim a e^x \qquad \text{as } x \to \infty, \tag{4.1.8}$$

*where* $a \in \mathbb{R}$ *is in fact given by*

$$a = \lim_{\sigma \to 1^+} (\sigma - 1) U(\sigma). \tag{4.1.9}$$

When $S$ is non-decreasing, it is clearly automatically log-linearly slowly decreasing. In this case however, it is more natural to work with its Laplace-Stieltjes transform $\mathcal{L}\{\mathrm{d}S; s\} = \int_{0^-}^{\infty} e^{-sx}\mathrm{d}S(x)$ instead of the Laplace transform of the function. We shall show the following version of our Tauberian theorem for Laplace-Stieltjes transforms. Working with this new formulation has great practical value as in certain situations it is easier to apply than Theorem 4.1.2. In fact, we shall exemplify its usefulness in Section 4.2 by giving a quick Tauberian proof of Blackwell's renewal theorem [4] and also a simpler treatment of Koga's renewal theorem for Dirichlet series [53, Thm. 5].

**Theorem 4.1.3** (Laplace-Stieltjes transforms)**.** *Let $S$ be log-linearly decreasing and of local bounded variation on $[0,\infty)$ with convergent Laplace-Stieltjes transform on $\Re e\, s > 1$. Suppose that the hypotheses of Theorem 4.1.2 are satisfied with $U(s) = \Re e\, \mathcal{L}\{\mathrm{d}S; s\}$ instead of $\Re e\, \mathcal{L}\{S; s\}$. Then (4.1.8) and (4.1.9) still hold true.*

We shall also prove the next Tauberian theorem for power series, which improves upon [53, Thm. 3].

**Theorem 4.1.4** (Power series)**.** *Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic on the unit disc $\mathbb{D}$ with real coefficients $\{c_n\}_{n=0}^{\infty}$. Suppose that the harmonic function $U(z) = \Re e\, F(z)$ has $L^1_{loc}$-boundary behavior on $\partial\mathbb{D} \setminus \{1\}$ and there is some $\theta_0 \in (0,\pi)$ such that one of the following three conditions holds:*

*(b.1) there is $c \geq 0$ such that $U(re^{i\theta}) \geq -c$, for $\theta \in (-\theta_0, \theta_0)$ and $r \in [0,1)$;*

*(b.2)* $\displaystyle\sup_{0<r<1} \int_{-\theta_0}^{\theta_0} |U(re^{\theta})|\mathrm{d}\theta < \infty$;

*(b.3) there are $g \in L^1(-\theta_0, \theta_0)$ and $k \in \mathbb{N}$ such that*

$$U(re^{i\theta}) = O\left((1-r)^{-k}\right), \qquad \text{for } \theta \in (-\theta_0, \theta_0) \text{ and } r \in [0,1) \tag{4.1.10}$$

*and*

$$U(re^{i\theta}) \geq g(\theta), \qquad \text{for a.e. } \theta \in (-\theta_0, \theta_0) \text{ and } r \in [0,1). \tag{4.1.11}$$

*Then $\{c_n\}_{n=0}^{\infty}$ is convergent. In particular, its limit is given by*

$$\lim_{n\to\infty} c_n = \lim_{r\to 1^-} (1-r)U(r). \tag{4.1.12}$$

The plan of this chapter is as follows. We discuss in Section 4.2 how Theorem 4.1.3 and Theorem 4.1.4 can be applied to renewal theory [25, Chapter XI]; our applications emphasize the role of the assumptions (B.1) and (b.1) in the corresponding cases, which make the theorems relatively simple to apply. In Section 4.3 we obtain a slight extension of the exact Wiener-Ikehara Tauberian theorem [15, Thm. 3.6], where we shall show that (4.1.2) holds if and only if $S$ is log-linearly slowly decreasing, its Laplace transform converges for $\Re e\, s > 1$, and the real part of the function $G$ given by (4.1.1) has so-called local pseudo-function boundary behavior on $1 + i\mathbb{R}$. Section 4.4 is devoted to the proofs of Theorem 4.1.2 and Theorem 4.1.3; our approach there will be to reduce them to the exact Wiener-Ikehara theorem from Section 4.3. Theorem 4.1.4 will be shown in Section 4.5. Finally, we close the chapter with some remarks and further extensions of our Tauberian theorems, which will be discussed in Section 4.6.

## 4.2 Application: Renewal theorems

Before showing our new versions of the Wiener-Ikehara theorem, we illustrate their usefulness with some applications. Our first application is to probability theory. We will give in this section a quick simple Tauberian proof of a fundamental result in renewal theory, namely, the renewal theorem [25].

Let $\mathrm{d}P$ be a probability measure[1] on $[0, \infty)$ that is continuous at the origin, namely, $P(0) = 0$. Its renewal function $Q$ is determined by the convolution equation

$$\mathrm{d}Q = \delta + \mathrm{d}Q * \mathrm{d}P, \tag{4.2.1}$$

where hereafter $\delta$ stands for the Dirac delta measure concentrated at 0 and $*$ stands for additive convolution of measures. In fact, the solution to (4.2.1) is given by the convergent[2] series $\mathrm{d}Q = \sum_{n=0}^{\infty} \mathrm{d}P^{*n}$.

We shall distinguish two cases for $\mathrm{d}P$. We say that it is *lattice* if there is $\alpha > 0$ such that $\mathrm{d}P$ is concentrated on $\alpha\mathbb{N} = \{\alpha, 2\alpha, 3\alpha \dots\}$ (when $\alpha$ is maximal we call it its *span*); otherwise, we shall call $\mathrm{d}P$ *non-lattice*.

**Theorem 4.2.1** (The renewal theorem [4, 21, 54])**.** *If* $\mathrm{d}P$ *is non-lattice, then, for each* $h > 0$,

$$Q(h + x) - Q(x) \to \frac{h}{\int_0^\infty x \, \mathrm{d}P(x)}, \qquad x \to \infty. \tag{4.2.2}$$

*For lattice* $\mathrm{d}P$ *with span* $\alpha > 0$, *the relation* (4.2.2) *holds for all* $h = n\alpha$, $n \in \mathbb{N}$.

[1]All measures considered in this chapter are locally finite Borel measures and their primitives are normalized to be right continuous and supported on the same interval as the measure when applicable.

[2]Unlike $\mathrm{d}P$, the measure $\mathrm{d}Q$ might not be a finite, the convergence is thus interpreted in e.g. the space of Radon measures.

*Proof.* We divide the proof into the corresponding two cases.

*Non-lattice* $\mathrm{d}P$ (Blackwell's renewal theorem). Let $S(x)=\int_{0^-}^{x}e^{u}\mathrm{d}Q(u)$, so that $F(s):=\mathcal{L}\{\mathrm{d}S;s\}=\mathcal{L}\{\mathrm{d}Q;s-1\}$. Laplace transforming (4.2.1), we obtain

$$F(s)=\frac{1}{1-G(s)},\qquad \Re e\ s>1. \tag{4.2.3}$$

with $G(s)=\mathcal{L}\{\mathrm{d}P;s-1\}$. The function $G(s)$ clearly extends continuously to the boundary line $\Re e\, s=1$, and, with the exception of $s=1$, we have $G(s)\neq 1$ for all other points of $\{s:\Re e\ s\geq 1\}$ (since otherwise $\mathrm{d}P$ would necessarily be lattice). We conclude that $F$ has a continuous extension to $\{1+it:\ t\neq 0\}$ and in particular has $L^1_{loc}$-behavior on this boundary subset. Furthermore,

$$\Re e\ F(s)=\frac{1-\int_{0^-}^{\infty}e^{-(1-\sigma)x}\cos(tx)\mathrm{d}P(x)}{|1-G(s)|^2}>0,\qquad \sigma=\Re e\ s>1.$$

Theorem 4.1.3 then yields $S(x)\sim ae^{x}$ for some $a$. To compute $a$, we employ (4.2.3):

$$a=\lim_{\sigma\to 1^+}(\sigma-1)F(\sigma)=\lim_{y\to 0^+}\frac{y}{1-\int_{0^-}^{\infty}e^{-yx}\mathrm{d}P(x)}=\frac{1}{\int_0^{\infty}x\mathrm{d}P(x)}.$$

Writing $\theta(x)=e^{-x}S(x)-a=o(1)$, noticing that $\mathrm{d}Q(x)=e^{-x}\mathrm{d}S(x)$, and integrating by parts, we obtain,

$$Q(x)=a(x+1)+\theta(x)+\int_0^x\theta(u)\mathrm{d}u.$$

We therefore have $Q(x+h)-Q(x)=ah+\theta(x+h)-\theta(x)+\int_x^{x+h}\theta(u)\mathrm{d}u$, whence (4.2.2) follows at once.

*Lattice* $\mathrm{d}P$ (the Kolmogorov-Erdős-Feller-Pollard renewal theorem). In this case $\mathrm{d}P(x)=\sum_{n=0}^{\infty}p_n\delta(x-n\alpha)$ and $\mathrm{d}Q(x)=\sum_{n=0}^{\infty}q_n\delta(x-n\alpha)$ with $q_0=1$, $p_0=0$, and $\sum_{n=1}^{\infty}p_n=1$. Furthermore, these non-negative sequences are linked by the convolution relation

$$q_n=\sum_{k=1}^{n}p_kq_{n-k},\qquad n\geq 1. \tag{4.2.4}$$

Since we assumed $\alpha$ to be maximal, we have that $1=\gcd\{n:p_n\neq 0\}$, which implies that $G(re^{i\theta})\neq 1$ for all $\theta\in[-\pi,\pi]\setminus\{0\}$. Here $G$ stands for the power series $G(z)=\sum_{n=1}^{\infty}p_nz^n$, which is continuous on the closed unit disc. Due to (4.2.4), we obtain $F(z)=\sum_{n=0}^{\infty}q_nz^n=(1-G(z))^{-1}$. As in the previous the case, we also have $\Re e\ F(z)>0$ for all $z\in\mathbb{D}$ (since $\Re e\, G(z)<1$ on $\mathbb{D}$). Hence, Theorem 4.1.4 allows us to conclude that

$$\lim_{n\to\infty}q_n=\lim_{r\to 1^-}\frac{1-r}{1-\sum_{n=1}^{\infty}r^np_n}=\frac{1}{\sum_{n=1}^{\infty}np_n},$$

which completes the proof of the renewal theorem. □

We can also give a simpler proof than Koga's original one for his version of the renewal theorem for Dirichlet series. The symbol $\star$ below stands for the Dirichlet convolution [75] of two arithmetic functions, while $e$ denotes the identity of this convolution, namely, the arithmetic function $e : \mathbb{N} \to \mathbb{R}$ given by $e(1) = 1$ and $e(n) = 0$ for $n \geq 2$.

**Theorem 4.2.2** ([53, Thm. 5]). *Let $g : \mathbb{N} \to [0, \infty)$ be such that $g(1) = 0$, $\sum_{n=2}^{\infty} g(n)/n = 1$, and no set $\{d^k : k \in \mathbb{N}\}$ with $d \geq 2$ entirely contains $\{n : g(n) \neq 0\}$. If $f$ is defined through $f = e + f \star g$, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n) = \frac{1}{\sum_{n=2}^{\infty} \frac{g(n)\log n}{n}} .$$

*Proof.* We set $S(x) = \sum_{n\leq e^x} f(n)$, so that $F(s) = \mathcal{L}\{\mathrm{d}S; s\} = \sum_{n=1}^{\infty} f(n)/n^s$. The familiar properties of Dirichlet series and $f = e + f \star g$ yield (4.2.3) with now $G$ given by the Dirichlet series of $g$, i.e., $G(s) = \sum_{n=2}^{\infty} g(n)/n^s$, which continuously extends to $\Re e\, s = 1$. In view of [53, Lemma 11], the assumption on $\{n : g(n) \neq 0\}$ implies that $G(1 + it) \neq 1$ for $t \neq 0$. Obviously, $\Re e\, F(s) > 0$ on the entire open half-plane $\Re e\, s > 1$. An application of Theorem 4.1.3 thus shows that

$$\lim_{x\to\infty} \frac{S(x)}{e^x} = \lim_{\sigma\to 1^+} (\sigma - 1)F(\sigma) = \lim_{\sigma\to 1^+} \frac{(\sigma - 1)}{1 - G(\sigma)} = \frac{1}{\sum_{n=2}^{\infty} \frac{g(n)\log n}{n}} .$$

□

## 4.3 Exact Wiener-Ikehara theorem revisited

One of the exact Wiener-Ikehara theorems from [15] states that for $S \in L^1_{loc}[0, \infty)$ to satisfy $S(x) \sim ae^x$ is necessary and sufficient that $S$ is log-linearly slowly decreasing, its Laplace transform converges for $\Re e\, s > 1$, and

$$G(s) = \mathcal{L}\{S; s\} - \frac{a}{s - 1}$$

has local pseudofunction boundary behavior on $1 + i\mathbb{R}$. We wish to replace $G$ by its real part in this characterization. This will be done in fact in Corollary 4.3.2 below, but before we move on, let us briefly recall what is meant by local pseudofunction boundary behavior.

In this and the next sections, we shall make use of Schwartz distribution theory. Our notation for calculus with distributions is as in the standard textbooks [78] or [23]; in particular, we make use of dummy variables of evaluation to facilitate our manipulations. As usual, $\mathcal{D}(I)$ stands for the space smooth test functions with compact supports on an open set $I \subset \mathbb{R}$, while $\mathcal{D}'(I)$ is the space of distributions on $I$. We say that $f \in \mathcal{D}'(I)$ is a local *pseudofunction* if for every $\varphi \in \mathcal{D}(I)$ the (distributional) Fourier transform of $\varphi f$ (which is entire by the Paley-Wiener theorem) is a continuous function that vanishes at $\pm\infty$. We then write $f \in \mathrm{PF}_{loc}(I)$. Note that $L^1_{loc}(I) \subset \mathrm{PF}_{loc}(I)$, thanks to the Riemann-Lebesgue lemma. In what follows we exploit that $\mathcal{D}'$ and $PF_{loc}$ are both (fine) sheaves, which allows us to work with localizations.

Let $I \subset \mathbb{R}$ be open. A harmonic function $U$ on $\Re e\ s > 1$ is said to have distributional boundary values on $1 + iI$ if there is $u \in \mathcal{D}'(I)$ such that

$$\lim_{\sigma \to 1^+} U(\sigma + it) = u(t) \qquad \text{in } \mathcal{D}'(I),$$

that is, if for each test function $\varphi \in \mathcal{D}(I)$,

$$\lim_{\sigma \to 1^+} \int_{-\infty}^{\infty} U(\sigma + it)\varphi(t)\mathrm{d}t = \langle u(t), \varphi(t) \rangle.$$

We say that $U$ has local pseudofunction boundary behavior on $1 + iI$ if it has distributional boundary values there and its boundary distribution $u \in \mathrm{PF}_{loc}(I)$. We refer to [22, 58] for the theory of boundary values of harmonic functions in distribution spaces (the article [58] actually deals with the general case of distributional boundary values for zero solutions of partially hypoelliptic constant coefficient partial differential operators, such as the Laplacian in our case). We point out that the harmonic function $U$ has distributional boundary values on $1 + iI$ if and only if for each compact $K \subset I$ one can find $k = k(K)$ such that

$$U(\sigma + it) = O((\sigma - 1)^{-k})$$

for $t \in K$ and, say, $1 < \sigma < 2$.

The next lemma is our most important technical tool in this section.

**Lemma 4.3.1.** *Let $U$ be a real-valued harmonic function on the half-plane $\Re e\ s > 1$ and let $I \subseteq \mathbb{R}$ be open. If $U$ has local pseudofunction boundary behavior on the boundary set $1 + iI$, so does any harmonic conjugate to $U$.*

*Proof.* Let $V$ be a harmonic conjugate to $U$. Note that since $U$ has distributional boundary values, then $V$ should also admit a boundary distribution[3]. It suffices to see that the analytic function $F = U + iV$ has local pseudofunction boundary behavior on $1 + iI$. We first show this under the additional assumption $U(\bar{s}) = U(s)$. Using the Cauchy-Riemann equations, we see that

[3]This is easily seen for a harmonic function on the unit disc $U(re^{i\theta}) = \sum_{n \in \mathbb{Z}} c_n r^{|n|} e^{in\theta}$, because having distributional boundary values in this case becomes equivalent to $\{c_n\}_{n \in \mathbb{Z}}$ being of at most polynomial growth (see e.g. [22]). Since our assertion is local, the general case follows by applying conformal maps mapping boundary segments into disc arcs.

$(V(s) - V(\bar{s}))/2$ must also be harmonic conjugate to $U$. Therefore, dropping a constant summand, we may assume that $V$ satisfies $V(\bar{s}) = -V(s)$. We might also assume that $I$ is symmetric about the origin. Set

$$u(t) = \lim_{\sigma\to 1^+} U(\sigma + it) \in \mathrm{PF}_{loc}(I) \qquad \text{and} \qquad f(t) = \lim_{\sigma\to 1^+} F(\sigma + it) \in \mathcal{D}'(I).$$

By [15, Proposition 2.1], $\langle f(t), \varphi(t)e^{iht}\rangle = o(1)$ as $h \to -\infty$, for each $\varphi \in \mathcal{D}(I)$ and our assumption is $\langle u(t), \varphi(t)e^{iht}\rangle = o(1)$ as $|h| \to \infty$. We also notice that $u$ is an even distribution, while $f(t) - u(t)$ is odd. For $h > 0$ and $\varphi \in \mathcal{D}(I)$ real-valued and even,

$$\begin{aligned}\langle f(t), \varphi(t)e^{iht}\rangle &= \langle f(t), \varphi(t)(e^{iht} + e^{-iht})\rangle + o(1)\\ &= \langle u(t), \varphi(t)(e^{iht} + e^{-iht})\rangle + o(1) = o(1)\end{aligned}$$

as $h \to \infty$. Likewise, for $\varphi \in \mathcal{D}(I)$ real-valued and odd,

$$\langle f(t), \varphi(t)e^{iht}\rangle = \langle u(t), \varphi(t)(e^{iht} - e^{-iht})\rangle + o(1) = o(1) \qquad \text{as } h \to \infty.$$

Decomposing an arbitrary test function into real and imaginary parts, and then each of them into the sum of their even and odd parts, we obtain that $\langle f(t), \varphi(t)e^{iht}\rangle = o(1)$ as $|h| \to \infty$ for each $\varphi \in \mathcal{D}(I)$, namely, $f \in \mathrm{PF}_{loc}(I)$.

A small variant of the above argument also applies when $U$ satisfies $U(s) = -U(\bar{s})$. Finally, the general case follows from these two particular ones by writing $U(s) = (U(s) + U(\bar{s}))/2 + (U(s) - U(\bar{s}))/2$. □

Lemma 4.3.1 and [15, Thm. 3.6] together thus yield:

**Corollary 4.3.2.** *Let $S \in L^1_{loc}[0, \infty)$. Then, $S(x) \sim ae^x$ holds if and only if $S$ is log-linearly slowly decreasing, its Laplace transform is convergent on $\Re e\, s > 1$, and the harmonic function*

$$\Re e \left(\mathcal{L}\{S; s\} - \frac{a}{s-1}\right) \tag{4.3.1}$$

*admits local pseudofunction boundary behavior on the whole line $\Re e\, s = 1$.*

**Remark 4.3.3.** Let $S$ be of local bounded variation, so that $\mathcal{L}\{\mathrm{d}S; s\} = s\mathcal{L}\{S; s\}$. Since smooth functions are multipliers for local pseudofunctions, $\mathcal{L}\{S; s\} - a/(s - 1)$ has local pseudofunction boundary behavior on a given boundary subset if and only if $\mathcal{L}\{\mathrm{d}S; s\} - a/(s - 1)$ does it. Employing Lemma 4.3.1 once more, we might replace (4.3.1) by the hypothesis that the real part of $\mathcal{L}\{\mathrm{d}S; s\} - a/(s - 1)$ has local pseudofunction boundary behavior on $\Re e\, s = 1$.

**Remark 4.3.4.** Lemma 4.3.1 highlights a key advantage of the local pseudofunction approach over $L^1_{loc}$-boundary behavior. In fact, it is well-known that if a harmonic function has $L^1_{loc}$-boundary behavior, the distributional boundary values of its harmonic conjugate functions do not necessarily belong to $L^1_{loc}$ (see e.g. [52, p. 73])

## 4.4 Proof of Theorem 4.1.2 (and Theorem 4.1.3)

We are now ready to show Theorem 4.1.2. We will do so with the aid of Corollary 4.3.2. Observe that it suffices to prove (4.1.8), since once this is established (4.1.9) automatically holds by the familiar real Abelian result for Laplace transforms.

We first claim that any of our assumptions imply that $U$ admits a boundary distribution on $1+i\mathbb{R}$. This is actually our hypothesis away from the boundary point $s=1$, where we even have the stronger $L^1_{loc}$-boundary behavior. So, we must then still establish the existence of a boundary distribution in a boundary neighborhood of 1. We notice that the bound (4.1.7) is actually equivalent to it (cf. Section 4.3). Let us verify that (4.1.7) also remains valid under either assumption (B.1) or (B.2). Let $\Phi : \mathbb{D} \to \Omega$ be a conformal equivalence between the unit disc and a region $\Omega \subset \{s : \Re e\, s > 1\}$ whose boundary is a smooth Jordan curve that meets the line $\Re e\, s = 1$ in a closed segment containing $s=1$ and inside the interval $1+i(-\lambda,\lambda)$. Set $V = U \circ \Phi$. If (B.1) holds, then $V(z)+c$ is a non-negative harmonic function. On the other hand, if (B.2) is satisfied, $V$ belongs to the harmonic Hardy space $h^1(\mathbb{D})$, as inferred from[4] [20, Thm. 10.1, p. 168] and [66, Thm. 3.5, p. 48]. Hence [20, Thm. 1.1, p. 2], $V$ is a Poisson-Stieltjes integral, whence we readily obtain the bound

$$V(z) = O\left(\frac{1}{1-|z|}\right), \qquad |z| < 1.$$

Since $\Phi$ extends to a diffeomorphism[5] (by [66, Thm. 3.5, p. 48] again) between complex neighborhoods of an arc of $\partial\mathbb{D}$ and a segment $1+iI$ containing 1, we also have

$$U(\sigma+it) = O\left(\frac{1}{\sigma-1}\right), \qquad \sigma+it \in (1,2]\times I.$$

Let now $u(t) = \lim_{\sigma\to 1^+} U(\sigma+it)$ in $\mathcal{D}'(\mathbb{R})$. By assumption $u=f$ on $\mathbb{R}\setminus\{0\}$ with $f \in L^1_{loc}(\mathbb{R}\setminus\{0\})$. If either (B.1) or (B.3) hold, then $u-g$ should be a non-negative measure on $(-\lambda,\lambda)$, where we set $g=-c$ in the case of condition (B.1). When (B.2) holds, $u$ is a measure on $(-\lambda,\lambda)$, as follows from

[4] Thm. 10.1 is only stated for analytic functions in [20, p. 168], but the proof given there applies to harmonic functions as well.

[5] Actually, conformally extends to a complex neighborhood of 1 as an application of the Schwarz reflection principle for analytic arcs yields.

the Banach-Alaoglu theorem (or Helly's selection principle as better known in the Lebesgue-Stieltjes measure context), because (B.2) tells us that $\{U(\sigma+i\cdot) : 1<\sigma<2\}$ is bounded in the dual of $C[-\lambda,\lambda]$. Summarizing, in every case we have shown that the distribution $u$ is a Radon measure on $\mathbb{R}$. Using the Lebesgue decomposition of $u$ (Lebesgue-Radon-Nikodym theorem [73, p. 121]), we conclude that $f \in L^1_{loc}(\mathbb{R})$ and that $f$ is its absolutely continuos part, while its singular part must have point support at 0. Hence, $u = f + \pi a\delta$ for some constant $a \in \mathbb{R}$, where as usual $\delta$ stands for the Dirac delta distribution. Finally, using the well-known formula (cf. [23, Eq. (2.17), p. 58])

$$\lim_{\sigma\to 1^+} \frac{1}{\sigma - 1 + it} = \frac{-i}{t - i0} = \pi\delta(t) - i \,\text{p.v.}\left(\frac{1}{t}\right),$$

we deduce from Corollary 4.3.2 that $S(x) \sim ae^x$ as $x \to \infty$, because (4.3.1) has boundary value distribution $f \in L^1_{loc}(\mathbb{R}) \subset PF_{loc}(\mathbb{R})$ . This completes the proof of Theorem 4.1.2.

The proof of Theorem 4.1.3 is exactly the same as the one we just gave, but now making use of Remark 4.3.3.

## 4.5 The power series case: proof of Theorem 4.1.4

The proof of Theorem 4.1.4 is similar to that of Theorem 4.1.2, but simpler since we can avoid using Corollary 4.3.2 via a more direct argument. It suffices to show that $\{c_n\}_{n=0}^{\infty}$ converges to some finite limit, because then necessarily (4.1.12) should hold due to Abel's classical limit theorem for power series. As in Section 4.4, the assumptions imply that $U(re^{i\theta})$ converges distributionally to a boundary measure, which is absolutely continuous with respect to the Lebesgue measure off $2\pi\mathbb{Z}$. Therefore, there are $f \in L^1[-\pi,\pi]$ and $a \in \mathbb{R}$ such that

$$\lim_{r\to 1^-} U(re^{i\theta}) = a\pi\delta(\theta) + f(\theta)$$

in, say, the dual of $C^\infty[-\pi,\pi]$. Now, for $n > 0$,

$$\begin{aligned} c_n r^n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\theta})e^{-in\theta}\mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\theta})(e^{-in\theta} + e^{in\theta})\mathrm{d}\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} U(re^{i\theta})\cos n\theta \,\mathrm{d}\theta, \end{aligned}$$

where we have used that $\{c_k\}_{k=0}^{\infty}$ is real. Taking $r \to 1^-$,

$$c_n = a + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \,\mathrm{d}\theta = a + o(1) \qquad \text{as } n \to \infty,$$

in view of the Riemann-Lebesgue lemma.

## 4.6 Further extensions and concluding remarks

We end this chapter with some remarks.

**Remark 4.6.1.** Theorem 4.1.1 is also covered by (B.2). In fact, suppose that (4.1.4) and (4.1.6) are satisfied. Let $\varphi$ be a real-valued even non-negative smooth function with support on $(-\lambda, \lambda)$ such that $\varphi(t) = 1$ for $t \in [-\lambda/2, \lambda/2]$. Let $\widehat{\varphi}(x) = \int_{-\infty}^{\infty} \varphi(t)e^{-itx}\mathrm{d}x$, so that $\widehat{\varphi}$ is a Schwartz function. Then, since $U(\sigma + it) - g(t) \geq 0$ in the considered range,

$$\begin{aligned}\int_{-\lambda/2}^{\lambda/2} |U(\sigma+it)|\mathrm{d}t &\leq \int_{-\lambda/2}^{\lambda/2} |g(t)|\mathrm{d}t + \int_{-\infty}^{\infty} (U(\sigma+it) - g(t))\varphi(t)\mathrm{d}t \\ &\leq 2\int_{-\infty}^{\infty} |g(t)|\varphi(t)\mathrm{d}t + \Re e \int_{-\infty}^{\infty} \mathcal{L}\{S;(\sigma+it)\}\varphi(t)\mathrm{d}t \\ &= 2\int_{-\infty}^{\infty} |g(t)|\varphi(t)\mathrm{d}t + \int_0^{\infty} e^{-x}S(x)e^{-(\sigma-1)x}\widehat{\varphi}(x)\mathrm{d}x \\ &\leq 2\int_{-\infty}^{\infty} |g(t)|\varphi(t)\mathrm{d}t + \int_0^{\infty} e^{-x}|S(x)||\widehat{\varphi}(x)|\mathrm{d}x < \infty.\end{aligned}$$

Similarly, (b.2) is always implied by (4.1.11) and

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n^k} < \infty \qquad \text{for some } k \in \mathbb{N}.$$

**Remark 4.6.2.** Koga originally stated his Tauberian theorem (cf. [53, Thm. 2]) by only imposing the boundary requirements for the Laplace transform on a sequence tending to $1^+$. More precisely, in addition to (4.1.3), he assumes[6] the existence of $\sigma_n \to 1^+$ such that

$$\lim_{n\to\infty} \int_I |U(\sigma_n + it) - f(t)|\mathrm{d}t = 0, \tag{4.6.1}$$

for some $f \in L^1_{loc}(\mathbb{R} \setminus \{0\})$ and any finite interval $I$ not containing the origin, and

$$U(\sigma_n + it) \geq g(t), \qquad \text{for a.e. } t \in (-\lambda, \lambda) \text{ and } n \in \mathbb{N}, \tag{4.6.2}$$

for some $\lambda > 0$ and $g \in L^1(-\lambda, \lambda)$.

We have however that (4.6.1) is equivalent to (4.1.5) in our case. In fact, since (4.1.3), or more generally (4.1.6) which allows us to view $e^{-x}S(x)$ as a tempered distribution, ensures [78, Section 6.6.9, p. 100] that $\mathcal{L}\{S; s\}$ has distributional boundary values on $\Re e\, s > 1$, the relation (4.1.5) might be inferred

[6] His formulation of (4.6.1) in [53, Thm. 2] is slightly different, but equivalent in view of the well known completeness of the $L^1$-spaces.

from (4.6.1) by using a standard localization argument together with the (distributional) Schwarz reflection principle [72, p.3, Thm. C] (see also [14, 58] for generalized reflection principles).

Although (4.6.2) appears to be weaker than (4.1.4), Koga's original set of hypotheses are also covered by Theorem 4.1.2. To see his, the same argument employed in Remark 4.6.1 clearly yields, say,

$$\sup_{n\in\mathbb{N}} \int_{-2\lambda/3}^{2\lambda/3} |U(\sigma_n + it)|\mathrm{d}t < \infty,$$

which in turn (via localization and the reflection principle once more) implies that (B.2) holds with $\lambda$ replaced by, say, $\lambda/2$.

**Remark 4.6.3.** As Koga, we could also have only assumed in Theorem 4.1.2 and Theorem 4.1.3 that (4.1.4) just holds on a sequence $\sigma = \sigma_n \to 1^+$. Exactly the same argument given in Section 4.4 would then still yield that the boundary distribution of $U$ is a Radon measure. This comment also applies to Theorem 4.1.4.

**Remark 4.6.4.** In this article we have chosen to work within the framework of distributional boundary values because distribution theory should be well-known to most analysts. We could also have worked with ultradistributions, which allows us to relax the assumption (4.1.7) in (B.3) to

$$U(\sigma + it) = O\left(e^{\eta\left(\frac{1}{\sigma-1}\right)}\right), \qquad \text{for } t \in (-\lambda, \lambda) \text{ and } \sigma \in (1, 2], \tag{4.6.3}$$

where $\eta : [0, \infty) \to [1, \infty)$ is non-decreasing and satisfies the mild integral growth condition

$$\int_1^\infty \frac{\log \eta(x)}{x^2}\mathrm{d}x < \infty. \tag{4.6.4}$$

Likewise, Theorem 4.1.4 also holds if we replace (4.1.10) in the hypotheses (b.3) by the weaker growth condition

$$U(re^{i\theta}) = O\left(e^{\eta\left(\frac{1}{1-r}\right)}\right), \qquad \text{for } \theta \in (-\theta_0, \theta_0) \text{ and } r \in [0, 1). \tag{4.6.5}$$

In particular, (4.6.3) holds with $\eta(y) = 1 + \sup_{x>0}(\omega(x) - x/y)$ if $\omega : [0, \infty) \to [0, \infty)$ is non-decreasing and concave with

$$\int_1^\infty \frac{\omega(x)}{x^2}\mathrm{d}x < \infty \tag{4.6.6}$$

and if

$$\int_1^\infty |S(x)|e^{-x-\omega(x)}\mathrm{d}x < \infty. \tag{4.6.7}$$

(To see that (4.6.4) holds, one can apply [13, Theorems 1.8 (i), (ii) and 1.10 (i), (ii)].) Similarly, (4.6.5) holds with this $\eta$ if

$$\sum_{n=0}^{\infty} |c_n| e^{-\omega(n)} < \infty. \tag{4.6.8}$$

For example, $\omega(x) = x^{\alpha/(1+\alpha)}$ with $\alpha > 0$ gives rise to $\eta(y) \asymp y^{\alpha}$.

We refer to [14, 79] for the theory of boundary values of harmonic functions in spaces of ultradistributions[7].

Finally, we mention that the comments from Remark 4.6.1 also apply to (4.6.7) and (4.6.8), due to the non-quasianalytic condition (4.6.6), which, by Beurling's theorem [3], guarantees the existence of cut-off functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\widehat{\varphi}(x) = O(e^{-\omega(x)})$.

**Remark 4.6.5.** Sometimes one is just interested in deducing an upper bound

$$S(x) = O(e^x) \tag{4.6.9}$$

from relatively mild regularity boundary properties of the Laplace transform. For instance, such criteria play an important role in abstract analytic number theory (see e.g. [19, Chapter 11] for applications to Beurling primes). The following extension of [15, Proposition 3.1] (cf. [19, Thm. 10.1]) could then be useful in that respect. We call $S$ *log-linearly boundedly decreasing* if there is $h > 0$ such that

$$\liminf_{x\to\infty} \inf_{y\in[x,x+h]} \frac{S(y) - S(x)}{e^x} > -\infty.$$

We also use the notation $A_c(\mathbb{R})$ for the subspace of compactly supported elements of the Wiener algebra, namely, those compactly supported continuous functions such that their Fourier transforms belong to $L^1(\mathbb{R})$.

**Proposition 4.6.6.** *Let $S \in L^1_{loc}[0,\infty)$ and let $\varphi \in A_c(\mathbb{R}) \setminus \{0\}$ be even real-valued and have non-negative Fourier transform. Then, (4.6.9) holds if and only if $S$ is log-linearly boundedly decreasing, has convergent Laplace transform on $\Re e\, s > 1$, and there is a sequence $\sigma_n \to 1^+$ such that*

$$\mathfrak{I}_{\varphi}(h) = \lim_{n\to\infty} \int_{-\infty}^{\infty} (\Re e \mathcal{L}\{S; \sigma_n + it\})\, \varphi(t) \cos ht \,\mathrm{d}t$$

*exists for each $h > 0$ and $\mathfrak{I}_{\varphi}(h) = O(1)$ as $h \to \infty$.*

*Proof.* That the conditions are necessary is easy to verify. Let us show their sufficiency for (4.6.9). Set $\widehat{\varphi}(x) = \int_{-\infty}^{\infty} \varphi(t) e^{-itx} \mathrm{d}x$ and $\Delta(x) = e^{-x} S(x)$. As

[7] To be able to apply those results under (4.6.4), one needs to pass through the majorization results [13, Thm. 1.10, p. 92], [8, Lemma 8.5, p. 233], and [14, Lemma 2.4, p. 664].

shown inside the proof of [15, Proposition 3.1], $\Delta(x) = O(1)$ as $x \to \infty$ would follow from the log-linear bounded decrease if we show that $(\Delta * \widehat{\varphi})(h) = O(1)$ as $h \to \infty$. Also, it is shown there that we may assume without any loss of generality that $\Delta \geq 0$. Thus (the use of Parseval's relation is justified by [15, Lemma 3.4])

$$\begin{aligned} 0 &\leq \int_0^\infty e^{(1-\sigma_n)x}\Delta(x)\widehat{\varphi}(x-h)\mathrm{d}x \\ &\leq \int_0^\infty e^{(1-\sigma_n)x}\Delta(x)\left((\widehat{\varphi}(x-h)+\widehat{\varphi}(x+h)\right)\mathrm{d}x \\ &= \int_{-\infty}^\infty \mathcal{L}\{S;\sigma_n+it\}\varphi(t)\left(e^{iht}+e^{-iht}\right)\mathrm{d}t \\ &= 2\int_{-\infty}^\infty \Re e\mathcal{L}\{S;\sigma_n+it\}\,\varphi(t)\cos ht\,\mathrm{d}t. \end{aligned}$$

Applying the Beppo Levi theorem, $0 \leq (\Delta * \widehat{\varphi})(h) \leq 2\mathfrak{I}_\varphi(h) = O(1)$, $h \to \infty$. □

For example, Proposition 4.6.6 could have been used in Section 4.4 to directly show that $U(s)$ has distributional boundary values under the hypothesis (B.2) without having to pass through the conformal map argument, because once $T(x) = e^{-x}S(x) = O(1)$, the Laplace transform $\mathcal{L}\{S;s\}$ tends to the (distributional) Fourier transform of $T$ on $\Re e\, s = 1$.

# Summary

This doctoral dissertation contributes to the field of analytic number theory by proving several new results in multiplicative number theory, sieve theory, and Tauberian theory. These new results are obtained by combining recent breakthroughs from many mathematicians in related topics, new theories and techniques, classical tools of analytic number theory, and the author's new ideas. The thesis presents some new theorems, and even improves upon results that have been maintained for decades.

In Chapter 1, we investigate the density hypothesis for $L$-functions associated with holomorphic cusp forms. By leveraging the dichotomy technique developed by Bourgain in 2000, along with various classical tools in multiplicative number theory such as the Halász-Montgomery inequality, Ivić's mixed moment bounds for the zeta function, Huxley's subdivision argument, and Heath-Brown's bound for double zeta sums, we establish the following theorem.

**Theorem.** The inequality

$$N_f(\sigma, T) \ll_{f,\varepsilon} T^{2(1-\sigma)+\varepsilon},$$

holds for $\sigma \geq 1407/1601$.

This signifies an advancement over a result established by Ivić in 1989, where the zero-density estimate $N_f(\sigma, T) \ll_{f,\varepsilon} T^{2(1-\sigma)+\varepsilon}$ was previously established only for the narrower range $\sigma \geq 53/60$. With only mild adjustments of our method, we obtain a zero-density estimate for the Riemann zeta function. The zero-density estimate

$$N(\sigma, T) \ll_{\varepsilon} T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}$$

holds for $279/314 \leq \sigma \leq 17/18$. This extends beyond the range $155/174 \leq \sigma \leq 17/18$ established by Ivić in 1980.

We study the density hypothesis for Dirichlet $L$-functions in Chapter 2. Specifically, we employ Bourgain's dichotomy to establish a large value estimate

for Dirichlet polynomials. As an application, we derive an alternative proof of the theorem obtained by Heath-Brown in 1979.

Chapter 3 is focused on employing high-dimensional sieve theory to investigate the prime $k$-tuples conjecture. Let $\mathcal{H} = \{h_1, ..., h_k\}$ be an admissible set. We prove the following theorem.

**Theorem.** There exists $\rho_k$ such that there are $\gg x(\log\log x)^{-1}(\log x)^{-k}$ integers $n \leq x$ for which the product $\prod_{i=1}^{k}(n+h_i)$ is square-free and

$$\sum_{i=1}^{k} \tau(n+h_i) \leq \lfloor \rho_k \rfloor.$$

For large $k$, we have $\rho_k \sim \frac{2126}{2853}k^2$.

This improves upon a previous result by M. Ram Murty and A. Vatwani from 2017, where 3/4 is replaced by 2126/2853. The key components of our proof involve the higher rank Selberg sieve and the Irving-Wu-Xi estimate for the divisor function in arithmetic progressions to smooth moduli.

Our final chapter is dedicated to establishing new versions of the Wiener-Ikehara theorem where only boundary assumptions on the real part of the Laplace transform are imposed. More precisely, let $S \in L^1_{loc}[0,\infty)$ be log-linearly slowly decreasing and have convergent Laplace transform on $\Re e\, s > 1$. Suppose that the harmonic function $U(s) = \Re e\, \mathcal{L}\{S; s\}$ has $L^1_{loc}$-boundary behavior on $1 + i(\mathbb{R} \setminus \{0\})$ and that there is some $\lambda > 0$ such that one of the following three conditions holds:

(B.1) there is $c \geq 0$ such that $U(\sigma+it) \geq -c$, for $t \in (-\lambda, \lambda)$ and $\sigma \in (1, 2]$;

(B.2) $\displaystyle\sup_{1<\sigma<2} \int_{-\lambda}^{\lambda} |U(\sigma+it)| \mathrm{d}t < \infty$;

(B.3) there are $g \in L^1(-\lambda, \lambda)$ and $k \in \mathbb{N}$ such that

$$U(\sigma+it) \geq g(t), \qquad \text{for a.e. } t \in (-\lambda, \lambda) \text{ and } \sigma \in (1, 2],$$

and

$$U(\sigma+it) = O\left((\sigma-1)^{-k}\right), \qquad \text{for } t \in (-\lambda, \lambda) \text{ and } \sigma \in (1, 2].$$

Then

$$S(x) \sim ae^{x} \qquad \text{as } x \to \infty,$$

where $a \in \mathbb{R}$ is in fact given by

$$a = \lim_{\sigma \to 1^+} (\sigma - 1)U(\sigma).$$

This result generalizes and improves upon a recent theorem by T. Koga obtained in 2021. As an application, we provide a quick Tauberian proof of Blackwell's renewal theorem in probability theory.

# Nederlandstalige samenvatting

Deze doctoraatsthesis levert een bijdrage aan de analytische getaltheorie door verschillende nieuwe resultaten te bewijzen in de multiplicatieve getaltheorie, zeeftheorie en Tauberse theorie. Deze nieuwe resultaten werden verkregen door het combineren van recente doorbraken van veel wiskundigen in aanverwante onderwerpen, nieuwe theorieën en technieken, klassieke methodes uit de analytische getaltheorie en nieuwe ideeën van de auteur. De thesis beschrijft enkele nieuwe stellingen en verbetert zelfs sommige resultaten die al tientallen jaren worden gehandhaafd.

In Hoofdstuk 1 onderzoeken we de dichtheidshypothese voor $L$-functies die geassocieerd zijn met holomorfe cuspvormen. Door gebruik te maken van de tweedelingstechniek van Bourgain uit 2000, samen met andere klassieke technieken uit de multiplicatieve getaltheorie, zoals de ongelijkheid van Halász-Montgomery, Ivić's gemengde-momentgrenzen voor de zetafunctie, Huxleys verdelingsargument en Heath-Browns afschatting voor dubbele zetasommen, tonen we de volgende stelling aan.

**Theorem.** De ongelijkheid

$$N_f(\sigma, T) \ll_{f,\varepsilon} T^{2(1-\sigma)+\varepsilon},$$

geldt voor $\sigma \geq 1407/1601$.

Dit verbetert een resultaat van Ivić uit 1989, waarbij hij de afschatting $N_f(\sigma, T) \ll_{f,\varepsilon} T^{2(1-\sigma)+\varepsilon}$ voor de nulpunten van de $L$-functie enkel in de regio $\sigma \geq 53/60$ kon aantonen. Met slechts kleine aanpassingen aan onze methode verkrijgen we ook een afschatting voor de nulpunten van de Riemann-zetafunctie. We tonen aan dat

$$N(\sigma, T) \ll_{\varepsilon} T^{\frac{24(1-\sigma)}{30\sigma-11}+\varepsilon}$$

geldt in de regio $279/314 \leq \sigma \leq 17/18$. Dit vergroot de regio $155/174 \leq \sigma \leq 17/18$ die Ivić bekwam in 1980.

We bestuderen de dichtheidshypothese voor Dirichlet-$L$-functies in Hoofdstuk 2. Specifiek gebruiken we Bourgains tweedeling om een afschatting voor de hoeveelheid grote waarden van Dirichletveeltermen te bekomen. Als toepassing geven we een alternatief bewijs voor de stelling van Heath-Brown uit 1979.

In Hoofdstuk 3 richten we ons op het gebruik van hoogdimensionale zeefmethodes om het $k$-priemtupelvermoeden te onderzoeken. Zij $\mathcal{H} = \{h_1, ..., h_k\}$ een toegestane verzameling. We tonen de volgende stelling aan.

**Theorem.** Er bestaan $\rho_k$ zodat er $\gg x(\log\log x)^{-1}(\log x)^{-k}$ gehele getallen $n \leq x$ bestaan waarvoor het product $\prod_{i=1}^{k}(n+h_i)$ kwadraatvrij is en

$$\sum_{i=1}^{k} \tau(n+h_i) \leq \lfloor \rho_k \rfloor,$$

waarbij $\rho_k \sim \frac{2126}{2853}k^2$ als $k$ nadert naar oneindig.

Dit is een verbetering van een eerder resultaat van M. Ram Murty en A. Vatwani uit 2017, waarbij 3/4 vervangen is door 2126/2853. De belangrijkste ingrediënten van ons bewijs zijn de Selbergzeef van hogere rang en de afschatting van Irving, Wu en Xi voor de delerfunctie in rekenkundige rijen waarvan het verschil enkel kleine priemfactoren bevat.

Ons laatste hoofdstuk is gewijd aan het aantonen van nieuwe versies van de stelling van Wiener en Ikehara, waarbij alleen grensvoorwaarden voor het reële deel van de Laplace-transformatie worden opgelegd. Zij meer precies $S \in L^1_{loc}[0,\infty)$ een log-lineair traag dalende functie met convergente Laplace-transformatie op $\Re e\ s > 1$. Veronderstel dat de harmonische functie $U(s) = \Re e\ \mathcal{L}\{S; s\}$ $L^1_{loc}$-grensgedrag vertoont op $1 + i(\mathbb{R} \setminus \{0\})$ en dat er een $\lambda > 0$ bestaat zodat aan een van de volgende drie voorwaarden wordt voldaan:

(B.1) er is $c \geq 0$ zodat $U(\sigma+it) \geq -c$, voor alle $t \in (-\lambda,\lambda)$ en $\sigma \in (1,2]$;

(B.2) $\displaystyle\sup_{1<\sigma<2} \int_{-\lambda}^{\lambda} |U(\sigma+it)| \mathrm{d}t < \infty$;

(B.3) er bestaan $g \in L^1(-\lambda,\lambda)$ en $k \in \mathbb{N}$ zodat

$$U(\sigma+it) \geq g(t), \qquad \text{voor bijna alle } t \in (-\lambda,\lambda) \text{ en } \sigma \in (1,2],$$

en

$$U(\sigma+it) = O\left((\sigma-1)^{-k}\right), \qquad \text{voor } t \in (-\lambda,\lambda) \text{ en } \sigma \in (1,2].$$

Dan is

$$S(x) \sim ae^x \qquad \text{zodra } x \to \infty,$$

waarbij $a \in \mathbb{R}$ gelijk is aan

$$a = \lim_{\sigma \to 1^+} (\sigma - 1)U(\sigma).$$

Dit resultaat veralgemeent en verbetert een recente stelling van T. Koga uit 2021. Als toepassing geven we een kort Taubers bewijs van Blackwells hernieuwingsstelling uit de waarschijnlijkheidsleer.

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