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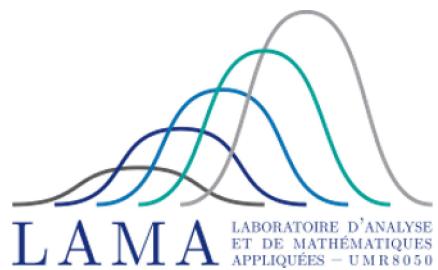
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Distribution de valeurs de la fonction zêta de Riemann

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Résumé - Abstract

Résumé

L'étude de la distribution des valeurs de la fonction zêta de Riemann $\zeta(s)$ peut être remonté au début du XXe siècle lorsque Bohr a montré que pour tout $z \in \mathbb{C}^*$ et $\varepsilon > 0$, il existe une infinité de s avec $1 < \Re s < 1 + \varepsilon$ telle que $\zeta(s) = z$. Plus tard, en 1932, Bohr et Jessen [10] montrèrent que $\log \zeta(\sigma + it)$ a une distribution continue sur le plan complexe pour tout $\sigma > \frac{1}{2}$. Sur la ligne critique $\Re s = \frac{1}{2}$, le théorème de la limite centrale de Selberg [46, 47] indique que $\log |\zeta(\frac{1}{2} + it)|$ se comporte comme une variable aléatoire Gaussienne de moyenne 0 et de variance $\frac{1}{2} \log_2 T$ quand $T \rightarrow \infty$, où t varie dans $[T, 2T]$. Sur la ligne 1 – le bord droit de la bande critique, Granville et Soundararajan [26] ont étudié la distribution de $|\zeta(1 + it)|$, qui est asymptotiquement une fonction double-exponentielle. Ensuite, Lamzouri [33] a considéré la distribution de $\log |\zeta(\sigma + it)|$ avec $\frac{1}{2} < \sigma < 1$ fixe et a également obtenu la fonction de distribution asymptotique. Dans cette thèse, basée sur les résultats établis séparément par Granville et Soundararajan [26], et par Lamzouri [33], nous obtenons des développements d'ordre supérieur des exposants de ces deux fonctions de distribution.

Le problème d'obtenir des grandes valeurs de $|\zeta(\frac{1}{2} + it)|$ a d'abord été étudié par Titchmarsh [51, Theorem 8.12], qui a montré qu'il existe un t arbitrairement grand tel que $|\zeta(\frac{1}{2} + it)| \geq \exp((\log t)^\alpha)$ pour tout $\alpha < \frac{1}{2}$. Le meilleur résultat actuel est dû à de la Bretèche et Tenenbaum [20], qui ont prouvé qu'il existe un t arbitrairement grand tel que $|\zeta(\frac{1}{2} + it)| \geq \exp(\sqrt{2 \log t \log_3 t / \log_2 t})$. Cette borne peut être encore loin de la vraie valeur maximale, en tenant compte que Farmer, Gonek et Hugh [23] ont conjecturé que le maximum devrait être $\exp(\frac{\sqrt{2}}{2} \sqrt{\log t \log_2 t})$.

Outre la ligne critique, il est également intéressant d'étudier les grandes valeurs de $|\zeta(s)|$ sur la ligne 1 et dans la bande $\frac{1}{2} < \Re s < 1$. Sur la ligne 1, l'étude peut remonter à 1925 lorsque Littlewood [36] a montré qu'il existe un t arbitrairement grand tel que $|\zeta(1 + it)| \geq (1 + o(1)) e^\gamma \log_2 t$. Le meilleur résultat actuel est dû à Astleitner, Mahatab et Munsch [3] qui ont prouvé qu'il existe un t arbitrairement grand tel que $|\zeta(1 + it)| \geq e^\gamma (\log_2 t + \log_3 t + O(1))$. Mis à part le terme $O(1)$, leur résultat confirme une conjecture de Granville et Soundararajan [26], qui est basée sur une analyse de la distribution de $|\zeta(1 + it)|$. Dans cette thèse, nous donnons une constante effective c à la place de $O(1)$ dans l'inégalité d'Astleitner, Mahatab et Munsch, ce qui la rapproche de la conjecture de Granville et Soundararajan.

Soit $\frac{1}{2} < \sigma < 1$. En 1928, Titchmarsh [50] montra pour la première fois que pour tout $\varepsilon > 0$, il existe un t arbitrairement grand tel que $\log |\zeta(\sigma + it)| \geq (\log t)^{1-\sigma-\varepsilon}$. En 1977, Montgomery [41] a montré qu'il existe un t arbitrairement grand tel que $\log |\zeta(\sigma + it)| \geq \nu(\sigma) (\log t)^{1-\sigma} / (\log_2 t)^\sigma$ pour une certaine constante $\nu(\sigma)$. Il a également conjecturé que cette valeur est le véritable ordre du maximum de $\log |\zeta(\sigma + it)|$ jusqu'à $\nu(\sigma)$. Toutes les améliorations ultérieures pour ce problème se concentrent sur l'obtention de valeurs plus grandes de $\nu(\sigma)$. En 2011, Lamzouri [33] a donné une valeur conjecturale du maximum de $\nu(\sigma)$. En 2018, Bondarenko et Seip [14] ont considéré les cas quand $\sigma \searrow \frac{1}{2}$ ou $\sigma \nearrow 1$. Dans cette thèse, nous étudions également le premier cas et obtenons une amélioration du résultat de Bondarenko et Seip.

Abstract

The study of the value distribution of the Riemann zeta function $\zeta(s)$ can date back to the early twentieth century when Bohr showed that for any $z \in \mathbb{C}^*$ and $\varepsilon > 0$, there exists s with $1 < \Re s < 1 + \varepsilon$ such that $\zeta(s) = z$ for infinitely many times. Later in 1932, Bohr and Jessen [10] showed that $\log \zeta(\sigma + it)$ has a continuous distribution on the complex plane for any $\sigma > \frac{1}{2}$. On the critical line, Selberg's Central Limit Theorem [46, 47] states that $\log |\zeta(\frac{1}{2} + it)|$ behaves like a complex Gaussian random variable with mean 0 and variance $\frac{1}{2} \log_2 T$ as $T \rightarrow \infty$, where t varies in $[T, 2T]$. On the 1-line, Granville and Soundararajan [26] studied the distribution of $|\zeta(1+it)|$, which is asymptotically a double-exponent function. In the critical strip $\frac{1}{2} < \Re s < 1$, Lamzouri [33] studied the distribution of $\log |\zeta(\sigma + it)|$ with any fixed $\frac{1}{2} < \sigma < 1$ and also got the asymptotic distribution function. In this thesis, based on the results established separately by Granville and Soundararajan [26] and Lamzouri [33], we obtain higher order expansions of the exponents of these two distribution functions.

The problem of getting large values of $|\zeta(\frac{1}{2} + it)|$ was first considered by Titchmarsh [51, Theorem 8.12], who showed that there exists arbitrarily large t such that $|\zeta(\frac{1}{2} + it)| \geq \exp((\log t)^\alpha)$ for any $\alpha < \frac{1}{2}$. The best-known result up to now is due to de la Bretèche and Tenenbaum [20], who showed that there exists arbitrarily large t such that $|\zeta(\frac{1}{2} + it)|$ is as large as $\exp(\sqrt{2 \log t \log_3 t / \log_2 t})$. This bound may be still far from the true maximal value, considering that Farmer, Gonek and Hugh [23] conjectured that the maximum should be $\exp(\frac{\sqrt{2}}{2} \sqrt{\log t \log_2 t})$.

Besides the critical line, it is also interesting to study large values of $|\zeta(s)|$ on the 1-line and in the critical strip $\frac{1}{2} < \Re s < 1$. On the 1-line, it can date back to 1925 when Littlewood [36] showed that there exists arbitrarily large t such that $|\zeta(1+it)| \geq (1+o(1))e^\gamma \log_2 t$. The best-known result up to now is due to Astleitner, Mahatab and Munsch [3] who showed that there exists arbitrarily large t such that $|\zeta(1+it)|$ is as large as $e^\gamma (\log_2 t + \log_3 t + O(1))$. Their result coincides with (up to an error term $O(1)$) a conjecture of Granville and Soundararajan [26], which is based on the analysis of the distribution of $|\zeta(1+it)|$. In this thesis, we give an effective constant c instead of the $O(1)$ in the inequality of Astleitner, Mahatab and Munsch, which makes it closer to the conjecture of Granville and Soundararajan.

Let $\frac{1}{2} < \sigma < 1$. In 1928 Titchmarsh [50] first showed that for any $\varepsilon > 0$ there exists arbitrarily large t such that $\log |\zeta(\sigma + it)| \geq (\log t)^{1-\sigma-\varepsilon}$. In 1977, Montgomery [41] showed that there exists arbitrarily large t such that $\log |\zeta(\sigma + it)|$ can be larger than $\nu(\sigma)(\log t)^{1-\sigma}/(\log_2 t)^\sigma$ for some constant $\nu(\sigma)$. He also conjectured that this value is the maximum of the true order of $\log |\zeta(\sigma + it)|$ up to $\nu(\sigma)$. All the later improvements for this problem focus on getting larger values of $\nu(\sigma)$. In 2011, Lamzouri [33] gave a conjectural largest value of $\nu(\sigma)$. In 2018, Bondarenko and Seip [14] considered the cases when $\sigma \searrow \frac{1}{2}$ or $\sigma \nearrow 1$. In this thesis, we also study the first case and get an improvement of the result of Bondarenko and Seip.

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List of Symbols

$\zeta(s)$	the Riemann zeta function
$\log_j x$	the j -th iterate of the natural logarithm of x
χ	the Dirichlet character
$\omega(n)$	the number of distinct prime factors of n
$\Omega(n)$	the number of all prime factors of n
e	the natural logarithm
γ	the Euler constant
$\Re s$	the real part of s
$\Im s$	the imaginary part of s
\mathbb{C}	the set of complex numbers
\mathbb{R}	the set of real numbers
\mathbb{Z}	the set of integers
\mathbb{N}	the set of natural numbers
\mathbb{N}^+	the set of positive integers
$\mathbb{E}(X)$	the mathematic expectation of X
$\gcd(m, n)$	the greatest common divisor of m and n
$\text{lcm}(m, n)$	the least common multiple of m and n

0 Introduction en français

La fonction zêta de Riemann $\zeta(s)$ joue un rôle central dans la théorie analytique des nombres et a des applications en physique, en théorie des probabilités et en statistique appliquée. L'étude de la distribution des valeurs de $\zeta(s)$ dans la bande critique $0 < \Re s \leq 1$ a une longue histoire. Sur la ligne critique $\Re s = \frac{1}{2}$, nous avons le théorème de la limite centrale de Selberg. Sur la ligne $1 -$ le bord droit de la bande critique, Granville et Soundararajan [26] ont étudié la distribution de $|\zeta(1 + it)|$, qui est asymptotiquement une fonction double exponentielle. Pour la bande $\frac{1}{2} < \Re s < 1$, Lamzouri [33] a étudié la distribution de $\log |\zeta(\sigma + it)|$ avec tout $\frac{1}{2} < \sigma < 1$ fixe et a également obtenu la fonction de distribution asymptotique.

En outre, il est également important d'étudier les valeurs extrêmes de la fonction zêta de Riemann. Sur la ligne 1, le résultat très récent établi par Aistleitner, Mahatab et Munsch [3] dit qu'il existe un t arbitrairement grand tel que $|\zeta(1 + it)|$ puisse être aussi grand que $e^{\gamma}(\log_2 t + \log_3 t + O(1))$. Mis à part le terme $O(1)$, leur résultat confirme une conjecture de Granville et Soundararajan [26], qui est basée sur une analyse de la distribution de $|\zeta(1 + it)|$. Dans la bande $\frac{1}{2} < \Re s < 1$, Montgomery [41] a prouvé qu'il existe un t arbitrairement grand tel que $\log |\zeta(\sigma + it)| \geq (c_m(\sigma) + o(1))(\log t)^{1-\sigma}/(\log t)^{\sigma}$ pour une certaine constante $c_m(\sigma)$ dépendant de σ . De plus, Bondarenko et Seip [14] ont considéré le cas quand $\sigma \searrow \frac{1}{2}$ et ont montré qu'il existe un t arbitrairement grand tel que $\log |\zeta(\sigma + it)|$ est supérieur à $(\frac{\sqrt{2}}{2} + o(1))\sqrt{\log |\sigma - \frac{1}{2}|}(\log t)^{1-\sigma}/(\log t)^{\sigma}$.

Dans les trois sections suivantes, nous présenterons ces sujets en détail.

0.1 Sur la ligne critique

Titchmarsh [51, Théorème 8.12] a montré qu'il existe un t arbitrairement grand tel que pour tout $\alpha < \frac{1}{2}$,

$$|\zeta(\frac{1}{2} + it)| \geq \exp((\log t)^{\alpha}).$$

En 1977, sous l'hypothèse de Riemann (RH), Montgomery [41] a obtenu une meilleure borne inférieure

$$|\zeta(\frac{1}{2} + it)| > \exp \left\{ \frac{1}{20} \sqrt{\frac{\log t}{\log_2 t}} \right\}$$

valable pour un t arbitrairement grand, où \log_j désigne le j -ème logarithme itéré. En 1977, Balasubramanian et Ramachandra [7] ont montré inconditionnellement (sans RH) qu'il existe un t arbitrairement grand tel que

$$|\zeta(\frac{1}{2} + it)| > \exp \left\{ c \sqrt{\frac{\log t}{\log_2 t}} \right\},$$

où $c > 0$ est une constante. En 1986, Balasubramanian [6] a réussi à donner une valeur effective pour cette constante : $c = 0,530\dots$. En 2008, Soundararajan [48] a montré par

la méthode de résonance que $c = 1 + o(1)$. En 2017, Bondarenko et Seip [13] ont fait un progrès significatif en établissant une borne inférieure de type

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| \geq \exp \left\{ (\frac{\sqrt{2}}{2} + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right\}$$

pour $T \rightarrow \infty$. Un an après, ils ont amélioré la constante $\sqrt{2}/2$ en 1. Le meilleur résultat actuel sur les grandes valeurs de $|\zeta(\frac{1}{2} + it)|$ est dû à de la Bretèche et Tenenbaum [20] :

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| \geq \exp \left\{ (\sqrt{2} + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right\}.$$

Notons qu'on obtient au mieux ce résultat en appliquant la méthode des sommes GCD. Le théorème de la limite centrale de Selberg [46, 47] énonce que quand $T \rightarrow \infty$, on a

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log |\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2} \log_2 T}} \geq \tau \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{\tau}^{\infty} e^{-u^2} du.$$

En traitant les maxima locaux de $\log |\zeta(\frac{1}{2} + it)|$ comme des variables aléatoires indépendantes satisfaisant le théorème de la limite centrale de Selberg, en 2007, Farmer, Gonek et Hugh [23] ont conjecturé que

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp \left\{ (\frac{\sqrt{2}}{2} + o(1)) \sqrt{\log T \log_2 T} \right\}.$$

Pour la borne supérieure de $|\zeta(\frac{1}{2} + it)|$, l'hypothèse de Lindelöf énonce que

$$\forall \varepsilon > 0 : \zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^{\varepsilon} \quad (|t| \geq 1).$$

La meilleure borne supérieure actuelle est dûe à Bourgain [17] qui a prouvé que

$$\zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^{\frac{13}{84} + \varepsilon} \quad (|t| \geq 1).$$

Le meilleur record actuel sous l'hypothèse de Riemann est dû à Chandee et Soundararajan [18] qui ont montré que

$$\zeta(\frac{1}{2} + it) \ll \exp \left\{ \frac{c \log |t|}{\log_2 |t|} \right\} \quad (|t| \geq 3),$$

où $c > 0$ est une constante absolue. Pour plus de détails, nous nous référons à [7, 6, 48, 51].

0.2 Sur la ligne 1

Dans cette partie, nous nous concentrerons sur les valeurs de la fonction zêta de Riemann $\zeta(s)$ sur la ligne 1. L'étude des valeurs extrêmes de $|\zeta(1 + it)|$ peut remonter à 1925 lorsque Littlewood [36] a montré qu'il existe un t arbitrairement grand tel que

$$|\zeta(1 + it)| \geq (1 + o(1)) e^{\gamma} \log_2 t.$$

Ceci a été amélioré par Levinson [34], qui en 1972 a prouvé qu'il existe un t arbitrairement grand tel que

$$|\zeta(1 + it)| \geq e^{\gamma} \log_2 t + O(1).$$

En 2006, Granville et Soundararajan [26] ont utilisé l'approximation diophantienne pour prouver

$$\max_{t \in [1, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T - \log_4 T + O(1))$$

pour tout T suffisamment grand. Dans le même article, ils ont aussi étudié la distribution de $|\zeta(1 + it)|$. Pour être plus précis, nous définissons d'abord pour $T > 1$,

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1 + it)| > e^\gamma \tau\}.$$

De plus, posons

$$C_j := \int_0^2 \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t)}{t^2} dt + \int_2^\infty \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt \quad (0.2.1)$$

avec

$$I_0(t) := \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}.$$

Ils ont montré que la formule asymptotique

$$\Phi_T(\tau) = \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + O \left(\frac{1}{\sqrt{\tau}} + \sqrt{\frac{e^\tau}{\log T}} \right) \right\} \right) \quad (0.2.2)$$

a lieu uniformément pour $1 \ll \tau < \log_2 T - 20$. En se basant sur (0.2.2), Granville et Soundararajan [26] ont conjecturé que

$$\max_{t \in [T, 2T]} |\zeta(1 + it)| = e^\gamma (\log_2 T + \log_3 T + C_0 + 1 - \log 2 + o(1)). \quad (0.2.3)$$

La méthode pour prouver (0.2.2) a été adapté pour étudier la distribution des valeurs d'autres fonctions L sur la ligne 1. Toujours dans [26], Granville et Soundararajan ont montré que la distribution des fonctions L de Dirichlet à l'aspect du modulo de caractères a la même forme que (0.2.2). Ce résultat peut être utilisé pour étudier la distribution des grandes sommes de caractères, voir [8] et [9]. En 2003, Granville et Soundararajana [27] ont étudié la distribution des fonctions L de Dirichlet de caractères quadratiques $L(1, \chi_d)$, et résolu une conjecture de Montgomery et Vaughan [41]. En 2007, Wu [54] a amélioré ce résultat de [27] en donnant un développement d'ordre supérieur dans l'exposant de la fonction de distribution. En 2008, Liu, Royer et Wu [38] ont considéré le cas de fonctions L puissance symétriques automorphes. En 2010, Lamzouri [32] a étudié des fonctions L générales qui peuvent couvrir les résultats de [27, 38]. En 2008, Lamzouri [31] généralise (0.2.2) à la distribution conjointe de $\arg \zeta(1 + it)$ et $|\zeta(1 + it)|$.

Inspirés par le résultat de Wu [54], nous cherchons à obtenir une amélioration de (0.2.2), qui présente une expansion d'ordre supérieur dans l'exposant.

Théorème 0.2.1. (Theorem 3.1.1) *Il existe une suite de nombres réels $\{\alpha_j\}_{j \geq 1}$ telle que pour tout entier $J \geq 1$ nous avons*

$$\Phi_T(\tau) = \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + \sum_{j=1}^J \frac{\alpha_j}{\tau^j} + O_J \left(\frac{1}{\tau^{J+1}} + \sqrt{\frac{e^\tau}{\log T}} \right) \right\} \right)$$

uniformément pour $T \rightarrow \infty$ et $1 \ll \tau \leq \log_2 T - 20$, où C_0 est défini comme dans (0.2.1). De plus, $\alpha_1 = 2(1 + C_0 - C_1)$.

Récemment, en 2018, Aistleitner, Mahatab et Munsch [3] ont utilisé la méthode de “résonance longue” pour montrer que

$$\max_{t \in [\sqrt{T}, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T + O(1)).$$

Notons que cela nécessite un plus grand intervalle $[\sqrt{T}, T]$ que $[T, 2T]$ dans (0.2.3) ce qui est typique pour l’application de la méthode de “résonance longue”. Inspirés par leur travail, nous établissons le Théorème 0.2.2 ci-dessous contenant une estimation des valeurs extrêmes dans l’intervalle $[T^\beta, T]$ avec une constante c effective.

Théorème 0.2.2. (Theorem 4.1.1) *Soient $0 < \beta < 1$ et c une constante telle que $c < \log(1 - \beta) - \log_2 4 - 1$. Définissons*

$$Z_\beta(T) := \max_{T^\beta \leq |t| \leq T} |\zeta(1 + it)|.$$

Alors pour T suffisamment grand, nous avons

$$Z_\beta(T) \geq e^\gamma (\log_2 T + \log_3 T + c).$$

Notons que lorsque $\beta = \frac{1}{2}$, nous pouvons choisir la constante $c = -3.6931472$. Cela donne une description du terme d’erreur $O(1)$ dans le résultat par Aistleitner, Mahatab et Munsch. Malgré l’intervalle élargi, le Théorème 0.2.2 est conforme à la Conjecture (0.2.3) qui prédit une constante plus grande $C_0 + 1 - \log 2 = -0.0885469$.

Il est aussi très intéressant d’étudier les valeurs extrêmes des dérivées de la fonction zéta de Riemann. Pour tout $\ell \in \mathbb{N}^+$, définissons

$$Z^{(\ell)}(T) := \max_{t \in [T, 2T]} |\zeta^{(\ell)}(1 + it)|.$$

Outre d’autres résultats, Yang [55] a récemment prouvé que nous avons

$$Z^{(\ell)}(T) \geq \frac{e^\gamma \ell^\ell}{(\ell + 1)^{\ell+1}} \{ \log_2 T - \log_3 T + O(1) \}^{\ell+1}, \quad (0.2.4)$$

uniformément pour $T \rightarrow \infty$ et $\ell \leq (\log T)/(\log_2 T)$. Nous visons à améliorer le facteur constant $\ell^\ell/(\ell + 1)^{\ell+1}$ dans (0.2.4). Nous avons le résultat suivant.

Théorème 0.2.3. (Theorem 5.1.1) *Nous avons*

$$Z^{(\ell)}(T) \geq \frac{e^\gamma}{\ell + 1} (\log_2 T)^{\ell+1} \{ 1 + o(1) \},$$

uniformément pour $T \rightarrow \infty$ et tous les entiers positifs $\ell \leq (\log T)/(\log_2 T)$.

0.3 Dans la bande $\frac{1}{2} < \Re s < 1$

Soit $\sigma \in (\frac{1}{2}, 1)$. En 2011, en appliquant une méthode de Granville et Soundararajan [26] pour étudier la distribution des valeurs de $|\zeta(1 + it)|$, Lamzouri [33] a étudié la distribution des

grandes valeurs de $|\zeta(\sigma + it)|$ lorsque t varie dans $[T, 2T]$. Nous définissons la fonction de distribution par

$$\Phi_T(\tau, \sigma) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : \log |\zeta(\sigma + it)| > \tau\}. \quad (0.3.1)$$

Lamzouri [33] a montré qu'il existe une constante positive $c(\sigma)$, telle que nous avons

$$\Phi_T(\tau, \sigma) = \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \mathfrak{a}_0(\sigma) + O \left(\frac{1}{\sqrt{\log \tau}} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T} \right)^{\sigma - \frac{1}{2}} \right) \right\} \right), \quad (0.3.2)$$

uniformément pour $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma} / \log_2 T$, où

$$\mathfrak{a}_0(\sigma) := \left(\frac{\sigma^2 (1-\sigma)^{1/\sigma-2}}{C_0(\sigma)} \right)^{\frac{\sigma}{1-\sigma}} \quad (0.3.3)$$

avec

$$C_n(\sigma) := \int_0^\infty \frac{(\log t)^n}{t^{1/\sigma+1}} \log I_0(t) dt \quad (n \geq 0). \quad (0.3.4)$$

Nous visons à améliorer la fonction de distribution (0.3.1). Nous avons une expansion d'ordre supérieur dans l'exposant, qui est inspirée par le travail dans [54]. Nous nous référerons à [39] pour des travaux similaires sur les fonctions L attachées aux formulaires de cuspide.

Théorème 0.3.1. (Theorem 3.1.1) *Il existe une suite de polynômes à coefficients réels (dépendants de σ) $\{\mathfrak{a}_n(\sigma, \cdot)\}_{n \geq 0}$ avec $\deg(\mathfrak{a}_n) \leq n$, et une constante $c(\sigma) > 0$, telles que pour tout entier $N \geq 1$, on a*

$$\begin{aligned} & \Phi_T(\tau, \sigma) \\ &= \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathfrak{a}_n(\sigma, \log_2 \tau)}{(\log \tau)^n} + O \left(\left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T} \right)^{\sigma - \frac{1}{2}} \right) \right\} \right) \end{aligned}$$

uniformément pour $T \rightarrow \infty$ et $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma} / \log_2 T$. En particulier, nous avons $\mathfrak{a}_0(\sigma, \cdot) = \mathfrak{a}_0(\sigma)$.

Maintenant nous tournons vers les grandes valeurs de $|\zeta(\sigma + it)|$. En 1928, Titchmarsh [50] a montré que pour tout $\varepsilon > 0$, nous avons

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{(\log t)^{1-\sigma-\varepsilon}} = \infty.$$

En 1972, Levinson [34] a amélioré ce résultat en montrant que pour T suffisamment grand:

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \gg \frac{(\log T)^{1-\sigma}}{\log_2 T}.$$

En 1977, Montgomery [41] a montré que

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \geq \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}, \quad (0.3.5)$$

où $\nu(\sigma) = \frac{1}{20}\sqrt{\sigma - \frac{1}{2}}$, et $\nu(\sigma) = \frac{1}{20}$ sous l'hypothèse de Riemann. Il est conjecturé que la quantité $(\log T)^{1-\sigma}/(\log_2 T)^\sigma$ soit l'ordre de grandeur réel de $\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)|$. Ainsi, la seule amélioration possible de (0.3.4) à laquelle nous pourrions nous attendre est d'obtenir des valeurs plus grandes de $\nu(\sigma)$. En tenant compte de (0.3.1), Lamzouri (voir [33]) a proposé la conjecture suivante :

$$\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)| = (\nu_*(\sigma) + o(1)) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}$$

où $\nu_*(\sigma) := C_0(\sigma)\sigma^{-2\sigma}(1-\sigma)^{\sigma-1}$ et $C_0(\sigma)$ est défini dans (0.3.3). En 2016, en utilisant la méthode de résonance, Aistleitner [1] a amélioré le résultat inconditionnel de Montgomery en montrant (0.3.4) avec $\nu(\sigma) = 0.18(2\sigma - 1)^{1-\sigma}$. En 2018, Bondarenko et Seip [14] ont montré que $\nu(\sigma) \geq 1/(2-2\sigma)$ et ont le comportement asymptotique

$$\nu(\sigma) = \begin{cases} (\frac{\sqrt{2}}{2} + o(1))\sqrt{|\log(\sigma - \frac{1}{2})|} & \sigma \searrow \frac{1}{2}, \\ (1-\sigma)^{-1} + O(|\log(1-\sigma)|) & \sigma \nearrow 1, \end{cases} \quad (0.3.6)$$

pour $\sigma \in (\frac{1}{2} + \frac{1}{\log_2 T}, 1 - \frac{1}{\log_2 T})$. Ici, $\sigma \searrow \frac{1}{2}$ signifie que quand $T \rightarrow \infty$, σ tend vers $\frac{1}{2}$ d'en haut avec $\frac{1}{2} + \frac{1}{\log_2 T} \leq \sigma \leq \frac{3}{4}$ et $\sigma \nearrow 1$ signifie σ tend vers 1 d'en bas avec $\frac{3}{4} \leq \sigma \leq 1 - \frac{1}{\log_2 T}$. En appliquant la méthode de Bretèche et Tenenbaum [20], nous pouvons améliorer la constante $\frac{\sqrt{2}}{2}$ en $\sqrt{2}$ dans la première assertion de (0.3.6).

Théorème 0.3.2. (Corollary 4.1.4) Soit $0 < \beta < 1$. Alors pour $T \rightarrow \infty$ et $\frac{1}{2} + \frac{1}{\log_2 T} < \sigma < \frac{3}{4}$, nous avons

$$\max_{t \in [T^\beta, T]} \log |\zeta(\sigma + it)| \geq \nu_\beta(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma},$$

où

$$\nu_\beta(\sigma) := (\sqrt{2} + o(1))(1-\beta)^{1-\sigma}\sqrt{|\log(\sigma - \frac{1}{2})|}.$$

Par conséquent, lorsque $\beta \searrow 0$, nous pouvons également choisir

$$\nu(\sigma) = (\sqrt{2} + o(1))\sqrt{|\log(\sigma - \frac{1}{2})|}.$$

0.4 Plan de cette thèse

Le Chapitre 0 est une introduction dans laquelle nous présentons nos résultats principaux obtenus dans cette thèse.

Dans le chapitre 1, nous introduisons quelques connaissances préliminaires, y compris certaines propriétés fondamentales de la fonction zêta de Riemann que nous devons utiliser dans les chapitres suivants.

Les Chapitres 2–5 sont le cœur de cette thèse :

Dans le chapitre 2, nous étudions la distribution des grandes valeurs de $|\zeta(1+it)|$.

Dans le chapitre 3, nous étudions la distribution des grandes valeurs de $|\zeta(\sigma+it)|$ dans la bande $\frac{1}{2} < \sigma < 1$.

Dans le chapitre 4 (en collaboration avec Bin Wei), nous étudions séparément les grandes valeurs de la fonction zêta de Riemann sur la ligne 1 et dans la bande $\frac{1}{2} < \Re s < 1$.

Dans le chapitre 5 (en collaboration avec Bin Wei), nous étudions de grandes valeurs des dérivées de la fonction zêta de Riemann sur la ligne 1.

0 Introduction

The Riemann zeta function plays a pivotal role in analytic number theory, and has applications in physics, probability theory, and applied statistics. The study of value distribution of the Riemann zeta function has a long history. On the critical line, we have Selberg's central limit theorem. On the 1-line, Granville and Soundararajan [26] studied the distribution of $|\zeta(1 + it)|$, which is asymptotically a double-exponent function. In the critical strip $\frac{1}{2} < \Re s < 1$, Lamzouri [33] studied the distribution of $\log |\zeta(\sigma + it)|$ with any fixed $\frac{1}{2} < \sigma < 1$ and also got the asymptotic distribution function.

Besides, it is also important to investigate the extreme values of the Riemann zeta function. On the 1-line, the very recent result established by Astleitner, Mahatab and Munsch [3] says that there exists arbitrarily large t such that $|\zeta(1 + it)|$ can be as large as $e^\gamma(\log_2 t + \log_3 t + O(1))$. Their result coincides with a conjecture of Granville and Soundararajan [26], which is based on the analysis of the distribution of $|\zeta(1 + it)|$. In the critical strip $\frac{1}{2} < \Re s < 1$, Montgomery [41] proved that there exists arbitrarily large t such that $\log |\zeta(\sigma + it)| \geq (\nu(\sigma) + o(1))(\log t)^{1-\sigma}/(\log t)^\sigma$ for some constant $\nu(\sigma)$ depending on σ . Moreover, Bondarenko and Seip [14] considered the case when $\sigma \searrow \frac{1}{2}$ and showed that there exists arbitrarily large t such that $\log |\zeta(\sigma + it)|$ can be larger than $(\frac{\sqrt{2}}{2} + o(1))\sqrt{\log |\sigma - \frac{1}{2}|}(\log t)^{1-\sigma}/(\log t)^\sigma$.

In the following three sections, we will introduce these subjects in detail.

0.1 On the critical line

Titchmarsh [51, Theorem 8.12] showed that there exists arbitrarily large t such that for any $\alpha < \frac{1}{2}$

$$|\zeta(\frac{1}{2} + it)| \geq \exp((\log t)^\alpha).$$

In 1977, assuming the Riemann Hypothesis (RH), Montgomery [41] showed that there exists arbitrarily large t such that

$$|\zeta(\frac{1}{2} + it)| > \exp\left\{\frac{1}{20}\sqrt{\frac{\log t}{\log_2 t}}\right\},$$

where \log_j is the j -th iterated logarithm. In 1977, Balasubramanian and Ramachandra [7] showed unconditionally (without RH) there exists arbitrarily large t such that

$$|\zeta(\frac{1}{2} + it)| > \exp\left\{c\sqrt{\frac{\log t}{\log_2 t}}\right\},$$

where c is a constant. In 1986, Balasubramanian [6] gave an effective value of this constant: $c = 0.530\dots$. In 2008, Soundararajan [48] showed by using the resonance method that $c = 1 + o(1)$. In 2017, Bondarenko and Seip [13] made a breakthrough by showing that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| \geq \exp\left\{(\frac{\sqrt{2}}{2} + o(1))\sqrt{\frac{\log T \log_3 T}{\log_2 T}}\right\},$$

as $T \rightarrow \infty$. One year later, they [15] improved the constant $\frac{\sqrt{2}}{2}$ to 1. The best-known result on large values of $|\zeta(\frac{1}{2} + it)|$ is due to de la Bretèche and Tenenbaum [20], who in 2018 showed that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| \geq \exp \left\{ (\sqrt{2} + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right\}.$$

This result was established by using the method of GCD sums. Selberg's central limit theorem [46, 47] says that:

$$\frac{1}{T} \text{meas} \left\{ t \in [T, 2T] : \frac{\log |\zeta(\frac{1}{2} + it)|}{\sqrt{\frac{1}{2} \log_2 T}} \geq \tau \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{\tau}^{\infty} e^{-u^2} du,$$

as $T \rightarrow \infty$. Upon treating the local maxima of $\log |\zeta(\frac{1}{2} + it)|$ as statistically independent variables satisfying Selberg's central limit theorem, in 2007 Farmer, Gonek and Hugh [23] conjectured that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp \{ (\frac{\sqrt{2}}{2} + o(1)) \sqrt{\log T \log_2 T} \}.$$

For the upper bound, the Lindelöf hypothesis states that for any $\varepsilon > 0$

$$\zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^{\varepsilon} \quad (|t| \geq 1).$$

The best-known upper bound is due to Bourgain [17] who proved that

$$\zeta(\frac{1}{2} + it) \ll_{\varepsilon} t^{\frac{13}{84} + \varepsilon} \quad (|t| \geq 1).$$

The best conditional bound (assuming the Riemann hypothesis) is the result of Chandee and Soundararajan [18] that

$$\zeta(\frac{1}{2} + it) \ll \exp \left\{ \frac{c \log t}{\log_2 t} \right\} \quad (|t| \geq 3),$$

for some absolute constant $c > 0$. For further details, we refer to [7, 6, 48, 51].

0.2 On the 1-line

In this part, we focus on the values of the Riemann zeta function $\zeta(s)$ on the 1-line. The study of the extreme values of $|\zeta(1 + it)|$ can date back to 1925 when Littlewood [36] showed that there exists arbitrarily large t for which

$$|\zeta(1 + it)| \geq (1 + o(1)) e^{\gamma} \log_2 t.$$

This was improved by Levinson [34], who in 1972 proved that there exists arbitrarily large t such that

$$|\zeta(1 + it)| \geq e^{\gamma} \log_2 t + O(1).$$

In 2006, Granville and Soundararajan [26] used Diophantine approximation to prove that the lower bound

$$\max_{t \in [1, T]} |\zeta(1 + it)| \geq e^{\gamma} (\log_2 T + \log_3 T - \log_4 T + O(1))$$

holds for sufficiently large T . In the same article, they also studied the distribution of $|\zeta(1 + it)|$. To be more precise, we first define for $T > 1$,

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1 + it)| > e^\gamma \tau\}.$$

Furthermore, let

$$C_j := \int_0^2 \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t)}{t^2} dt + \int_2^\infty \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt, \quad (0.2.1)$$

with

$$I_0(t) := \sum_{n=1}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}.$$

Then they showed that

$$\Phi_T(\tau) = \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + O \left(\frac{1}{\sqrt{\tau}} + \sqrt{\frac{e^\tau}{\log T}} \right) \right\} \right) \quad (0.2.2)$$

holds uniformly for $1 \ll \tau < \log_2 T - 20$. Based on (0.2.2), Granville and Soundararajan [26] conjectured that

$$\max_{t \in [T, 2T]} |\zeta(1 + it)| = e^\gamma (\log_2 T + \log_3 T + C_0 + 1 - \log 2 + o(1)). \quad (0.2.3)$$

The method to prove (0.2.2) was also adjusted to studying the distribution of values on the 1-line of other L -functions. Also in [26], Granville and Soundararajan showed that the distribution of the Dirichlet L -functions in the aspect of the modulo of characters has the same form as (0.2.2). This result can be used to study the distribution of large character sums, see [8] and [9]. In 2003, Granville and Soundararajana [27] studied the distribution of the Dirichlet L -functions of quadratic characters $L(1, \chi_d)$, which proves part of the conjecture of Montgomery and Vaughan in [41]. In 2007, Wu [54] improved this result by giving a higher order expansion in the exponent of the distribution function. In 2008, Liu, Royer and Wu [38] considered the case of symmetric power L -functions. In 2010, Lamzouri [32] studied a generalized L -function which can cover the results of [27, 38]. In 2008 Lamzouri [31] generalize (0.2.2) to the joint distribution of $\arg \zeta(1 + it)$ and $|\zeta(1 + it)|$.

Inspired by the result of Wu [54], we aim to get an improvement of (0.2.2), which presents a higher order expansion in the exponent.

Theorem 0.2.1. (Theorem 3.1.1) *There is a sequence of real numbers $\{\alpha_j\}_{j \geq 1}$ such that for any integer $J \geq 1$ we have*

$$\Phi_T(\tau) = \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + \sum_{j=1}^J \frac{\alpha_j}{\tau^j} + O_J \left(\frac{1}{\tau^{J+1}} + \sqrt{\frac{e^\tau}{\log T}} \right) \right\} \right)$$

uniformly for $T \rightarrow \infty$ and $1 \ll \tau \leq \log_2 T - 20$, where C_0 is defined as in (0.2.1). Moreover, $\alpha_1 = 2(1 + C_0 - C_1)$.

Recently in 2018, Aistleitner, Mahatab and Munsch [3] used the method of “long resonance” to show that

$$\max_{t \in [\sqrt{T}, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T + O(1)).$$

Note that this requires a larger range $[\sqrt{T}, T]$ than $[T, 2T]$ in (0.2.3) which is typical for the application of “long resonance”. Inspired by their work, we establish Theorem 0.2.2 below containing an estimate of extreme values in the range $[T^\beta, T]$.

Theorem 0.2.2. (Theorem 4.1.1) *Let $0 < \beta < 1$ be fixed and c be a constant such that $c < \log(1 - \beta) - \log_2 4 - 1$. Define*

$$Z_\beta(T) = \max_{T^\beta \leq |t| \leq T} |\zeta(1 + it)|.$$

Then for sufficiently large T , we have

$$Z_\beta(T) \geq e^\gamma (\log_2 T + \log_3 T + c).$$

Note that when $\beta = \frac{1}{2}$, we can choose the constant $c = -3.6931472$. This gives a description of the error term $O(1)$ in the result by Aistleitner, Mahatab and Munsch. Despite the enlarged range, Theorem 0.2.2 is in accordance with the conjecture (0.2.3) which predicts a larger constant $C_0 + 1 - \log 2 = -0.0885469$.

It also draws wide interests on the extreme values of the derivatives of the Riemann zeta function. For any $\ell \in \mathbb{N}^+$, denote

$$Z^{(\ell)}(T) := \max_{t \in [T, 2T]} |\zeta^{(\ell)}(1 + it)|.$$

Besides other results, Yang [55] recently proved that we have

$$Z^{(\ell)}(T) \geq \frac{e^{\gamma} \ell^\ell}{(\ell + 1)^{\ell+1}} \{ \log_2 T - \log_3 T + O(1) \}^{\ell+1}, \quad (0.2.4)$$

uniformly for $T \rightarrow \infty$ and $\ell \leq (\log T)/(\log_2 T)$. We aim to improve the constant $\ell^\ell/(\ell + 1)^{\ell+1}$ in (0.2.4). We have the following theorem.

Theorem 0.2.3. (Theorem 5.1.1) *We have*

$$Z^{(\ell)}(T) \geq \frac{e^\gamma}{\ell + 1} (\log_2 T)^{\ell+1} \{ 1 + o(1) \},$$

uniformly for $T \rightarrow \infty$ and all positive integers $\ell \leq (\log T)/(\log_2 T)$.

0.3 In the strip $\frac{1}{2} < \Re s < 1$

Let $\sigma \in (\frac{1}{2}, 1)$. In 2011, Lamzouri [33] studied the distribution of large values of $|\zeta(\sigma + it)|$ as t varies in $[T, 2T]$, by applying a method of Granville and Soundararajan [26] to investigate

the distribution of values of $|\zeta(1 + it)|$. Let T be sufficiently large. Define the distribution function by

$$\Phi_T(\tau, \sigma) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : \log |\zeta(\sigma + it)| > \tau\}. \quad (0.3.1)$$

Lamzouri [33] showed that, there exists a positive constant $c(\sigma)$, such that uniformly in the range $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma}/\log_2 T$ we have

$$\Phi_T(\tau) = \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \mathfrak{a}_0(\sigma) + O \left(\frac{1}{\sqrt{\log \tau}} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T} \right)^{\sigma-\frac{1}{2}} \right) \right\} \right), \quad (0.3.2)$$

where

$$\mathfrak{a}_0(\sigma) := \left(\frac{\sigma^2 (1-\sigma)^{1/\sigma-2}}{C_0(\sigma)} \right)^{\frac{\sigma}{1-\sigma}} \quad (0.3.3)$$

with

$$C_n(\sigma) := \int_0^\infty \frac{(\log t)^n}{t^{1/\sigma+1}} \log I_0(t) dt \quad (n \geq 0). \quad (0.3.4)$$

We aim to improve the asymptotic distribution function (0.3.2). We have a higher order expansion in the exponent, which is inspired by the work in [54]. We refer to [39] for similar work on L -functions attached to cusp forms.

Theorem 0.3.1. (Theorem 3.1.1) *Let $\Phi_T(\tau)$ be defined in (0.3.1). Then there exists a sequence of polynomials with real coefficients $\{\mathfrak{a}_n(\sigma, \cdot)\}_{n \geq 0}$ with $\deg(\mathfrak{a}_n) \leq n$, and a constant $c(\sigma) > 0$, such that for any integer $N \geq 1$, we have*

$$\begin{aligned} & \Phi_T(\tau, \sigma) \\ &= \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathfrak{a}_n(\sigma, \log_2 \tau)}{(\log \tau)^n} + O \left(\left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T} \right)^{\sigma-\frac{1}{2}} \right) \right\} \right) \end{aligned}$$

uniformly for $T \rightarrow \infty$ and $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma}/\log_2 T$. Especially, we have $\mathfrak{a}_0(\sigma, \cdot) = \mathfrak{a}_0(\sigma)$.

Now we turn to the large values of $|\zeta(\sigma + it)|$. In 1928, Titchmarsh [50] showed that for any $\varepsilon > 0$, we have

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{(\log t)^{1-\sigma-\varepsilon}} = \infty.$$

In 1972, Levinson [34] improved it, by showing that for sufficiently large T we have

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \gg \frac{(\log T)^{1-\sigma}}{\log_2 T}.$$

In 1977, Montgomery [41] showed that

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \geq \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}, \quad (0.3.5)$$

where $\nu(\sigma) = \frac{1}{20} \sqrt{\sigma - \frac{1}{2}}$ unconditionally, and $\nu(\sigma) = \frac{1}{20}$ on assuming the Riemann hypothesis. He also conjectured this quantity $(\log T)^{1-\sigma}/(\log_2 T)^\sigma$ is the true order of magnitude of $\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)|$. Thus, the only improvement of (0.3.5) we could expect is to get larger values of $\nu(\sigma)$. In 2011, taking (0.3.2) into account, Lamzouri [33] conjectured :

$$\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)| = (\nu_*(\sigma) + o(1)) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma},$$

where $\nu_*(\sigma) = C_0(\sigma)\sigma^{-2\sigma}(1-\sigma)^{\sigma-1}$, and $C_0(\sigma)$ is defined in (0.3.4). In 2016, using the resonance method, Aistleitner [1] improved Montgomery's unconditional result by showing that (0.3.5) holds for $\nu(\sigma) = 0.18(2\sigma-1)^{1-\sigma}$. In 2018, Bondarenko and Seip [14] showed that $\nu(\sigma) \geq 1/(2-2\sigma)$ and have the asymptotic behavior

$$\nu(\sigma) = \begin{cases} (\frac{\sqrt{2}}{2} + o(1)) \sqrt{|\log(\sigma - \frac{1}{2})|} & \sigma \searrow \frac{1}{2}, \\ (1-\sigma)^{-1} + O(|\log(1-\sigma)|) & \sigma \nearrow 1, \end{cases} \quad (0.3.6)$$

for $\sigma \in (\frac{1}{2} + \frac{1}{\log_2 T}, 1 - \frac{1}{\log_2 T})$. Here $\sigma \searrow \frac{1}{2}$ means σ tends to $\frac{1}{2}$ from above with $\frac{1}{2} + \frac{1}{\log_2 T} \leq \sigma \leq \frac{3}{4}$ and $\sigma \nearrow 1$ means σ tends to 1 from below with $\frac{3}{4} \leq \sigma \leq 1 - \frac{1}{\log_2 T}$, as $T \rightarrow \infty$. By applying the method of de la Bretèche and Tenenbaum [20], we are able to improve the constant $\frac{\sqrt{2}}{2}$ to $\sqrt{2}$ in the first assertion of (0.3.6).

Theorem 0.3.2. (Corollary 4.1.4) *Let $0 < \beta < 1$. For $T \rightarrow \infty$ and $\sigma > \frac{1}{2} + \frac{1}{\log_2 T}$, we have*

$$\max_{t \in [T^\beta, T]} \log |\zeta(\sigma + it)| \geq \nu_\beta(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}$$

holds for a function $c(\sigma)$ which has the asymptotic behavior

$$\nu_\beta(\sigma) = (\sqrt{2} + o(1))(1-\beta)^{1-\sigma} \sqrt{|\log(\sigma - \frac{1}{2})|},$$

as $\sigma \searrow \frac{1}{2}$.

Thus, when $\beta \searrow 0$, we can choose $\nu(\sigma) = (\sqrt{2} + o(1)) \sqrt{|\log(\sigma - \frac{1}{2})|}$.

0.4 Outline of this thesis

Chapter 0 is an introduction in which we present our main results of this thesis.

In Chapter 1 we introduce some preliminary knowledge, including some fundamental properties of the Riemann zeta function we need to use in the following chapters.

Chapters 2–5 are the main parts of this thesis:

In Chapter 2, we study the distribution of large values of $|\zeta(1+it)|$.

In Chapter 3, we study the distribution of large values of $|\zeta(\sigma+it)|$ in the strip $\frac{1}{2} < \sigma < 1$.

In Chapter 4 (joint with Bin Wei), we study large values of the Riemann zeta function on the 1-line and in the strip $\frac{1}{2} < \Re s < 1$ separately.

In Chapter 5 (joint with Bin Wei), we study large values of the derivatives of the Riemann zeta function on the 1-line.

1 Preliminaries

1.1 Basic properties of the Riemann zeta function

For $\Re s > 1$, the Riemann zeta function is defined to be the Dirichlet series with all coefficients equal to 1:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

It has an analytical continuation to the whole complex plane \mathbb{C} except $s = 1$. For $s = 1$, it has a simple pole with residue 1. Also for $\Re s > 1$, it can be written as the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For other values of s , the Riemann zeta function can not be written as the Dirichlet series or the Euler product. However, it can be approximated by the truncated forms, says the Dirichlet polynomials and the short Euler products. More precisely, for a suitable x , we could have

$$\zeta(s) \sim \sum_{n \leq x} \frac{1}{n^s},$$

and for a suitable y , we could have

$$\zeta(s) \sim \zeta(s; y) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For the first kind of approximation, we have the following asymptotic formula.

Proposition 1.1.1. *Let $x > 0$ be a large number and $s = \sigma + it$. Then uniformly in the range $\sigma \geq \sigma_0 > 0$ and $|t| \leq x$, we have*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma}\right).$$

1.2 Zero-density estimates

Let $0 \leq \sigma \leq 1$ and $T > 0$. Define

$$N(\sigma, T) = \#\{\rho = \beta + it : \beta \geq \sigma, |t| \leq T, \zeta(\rho) = 0\}.$$

We have the following estimate

$$N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+\varepsilon},$$

where the \ll constant depends only on ε . The Riemann-von Mangoldt formula implies that

$$N\left(\frac{1}{2}, T\right) \ll T^{1+\varepsilon},$$

while for $\frac{1}{2} < \sigma < 1$ trivially we have $A(\sigma)(1 - \sigma) \leq 1$. The zero-density hypothesis states that

$$A(\sigma) \leq 2 \quad \left(\frac{1}{2} \leq \sigma \leq 1\right),$$

which is a direct result of RH, but is as powerful as RH in using.

The best-known upper bound of $A(\sigma)$ uniformly for $\frac{1}{2} \leq \sigma \leq 1$ should be Ingham's result for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ in 1940:

$$A(\sigma) \leq \frac{3}{2-\sigma},$$

as well as Huxley's result for $\frac{3}{4} \leq \sigma \leq 1$ in 1972:

$$A(\sigma) \leq \frac{3}{3\sigma-1}.$$

Both estimates suggest that the best-known constant upper bound of $A(\sigma)$ is $\frac{12}{5}$ which is attained at $\sigma = \frac{3}{4}$. Many improvements have been established for σ near 1. For example, in 2000 Bourgain [16] showed that the zero-density hypothesis holds for $\sigma > \frac{25}{32}$.

1.3 Resonance method

The resonance method was introduced by Hilberdink and Soundararajan independently to detect large values of the Riemann zeta function. The principle is to find a Dirichlet polynomial $R(t) = \sum_{n \leq N} r(n)n^{it}$ resonating with $\zeta(\sigma + it)$. Then by the inequality

$$\max_{T^\beta \leq t \leq T} |\zeta(\sigma + it)| \geq \frac{\left| \int_{T^\beta}^T \zeta(\sigma + it) |R(t)|^2 w(t) dt \right|}{\int_{T^\beta}^T |R(t)|^2 w(t) dt}, \quad (1.3.1)$$

it remains to choose suitable coefficients $r(n)$ to make the numerator large while the denominator small. The weight function $w(t)$ is to make the dominant contribution arise from the integral in the interval $[T^\beta, T]$. Thus we can extend the integral to the whole real numbers without changing the order of magnitude. Moreover, the weight function $w(t)$ is often chosen to be the adjusted Gaussian

$$w(t) = \phi\left(\frac{t \log T}{T}\right) = \exp\left(-\frac{t^2 (\log T)^2}{2T^2}\right),$$

since the Gaussian $\phi(t) = e^{-t^2/2}$ has a very nice Fourier transform

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}} \phi(t) e^{-it\xi} dt = \sqrt{2\pi} \phi(\xi).$$

For any $n \in \mathbb{N}^+$, we have

$$\int_{\mathbb{R}} \frac{1}{n^{it}} \phi(t) dt = \int_{\mathbb{R}} \phi(t) e^{-it \log n} dt = \widehat{\phi}(\log n) > 0.$$

So if each $r(n)$ is chosen to be positive, then after extending the integral of the numerator in (1.3.1) to \mathbb{R} , each term in the expansion will be positive.

1.4 GCD sums

1.4.1 Definition

Let \mathcal{M} denote a set of positive integers. The GCD sum associated to \mathcal{M} is defined as

$$S_\sigma(\mathcal{M}) := \sum_{m,n \in \mathcal{M}} \frac{\gcd(m,n)^\sigma}{\text{lcm}(m,n)^\sigma} = \sum_{m,n \in \mathcal{M}} \frac{\gcd(m,n)^{2\sigma}}{(mn)^\sigma},$$

and the topic around it is to investigate the upper bound of the quantity

$$\Gamma_\sigma(N) := \sup_{|\mathcal{M}|=N} \frac{S_\sigma(\mathcal{M})}{N}. \quad (1.4.1)$$

When $\sigma = 1$, this was a prize problem by the Dutch Mathematical Society in 1947 suggested by Erdős, and was solved by Gál [25] in 1949. He showed that

$$\Gamma_1(N) \ll (\log_2 N)^2,$$

which makes the GCD sum also known as “Gál-type sums”.

In 2017, Lewko and Radziwiłł [35] used the method of probabilistic models to give a much easier proof of Gál’s theorem as well as determine the implied constant. They proved that

$$\Gamma_1(N) = \left(\frac{1}{\zeta(2)} + o(1) \right) e^{2\gamma} (\log_2 N)^2,$$

as $N \rightarrow \infty$. If we write $\mathbf{c} = (c_1, \dots, c_N) \in \mathbb{C}^N$, and its norm $\|\mathbf{c}\|^2 = \sum_{j=1}^N |c_j|^2$, then the spectral norm of the GCD matrix $(\gcd(m,n)^\sigma / \text{lcm}(m,n)^\sigma)$ is defined by

$$Q_\sigma(\mathcal{M}) := \sup_{\substack{\mathbf{c} \in \mathbb{C}^{|\mathcal{M}|} \\ \|\mathbf{c}\|=1}} \left| \sum_{m,n \in \mathcal{M}} c_m \overline{c_n} \frac{\gcd(m,n)^\sigma}{\text{lcm}(m,n)^\sigma} \right|.$$

In 2015, Aistleitner, Bondarenko and Seip [2] showed that

$$\Gamma_{\frac{1}{2}}(N) \leq \sup_{|\mathcal{M}|=N} Q_{\frac{1}{2}}(\mathcal{M}) \leq (e^2 + 1)(\log N + 2) \max_{n \leq N} \Gamma_{\frac{1}{2}}(n).$$

In this chapter, we focus only on the lower bounds of $\Gamma_\sigma(N)$ for $\frac{1}{2} < \sigma < 1$ for later use. We refer to the paper [20] of de la Bretèche and Tenenbaum for the completely solution of $\Gamma_{\frac{1}{2}}(N)$: as $N \rightarrow \infty$,

$$\Gamma_{\frac{1}{2}}(N) = \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{\frac{\log N \log_3 N}{\log N}} \right\}.$$

1.4.2 Special case of square-free numbers

The following proposition is a simple construction of \mathcal{M} consisting of only square-free integers which was firstly used by Gál [25] to give lower bounds of GCD sums.

Proposition 1.4.1. Let $\frac{1}{2} \leq \sigma \leq 1$, $N = 2^r$, and $\mathcal{M} = \{p_1^{\delta_1} \dots p_r^{\delta_r} : \delta_j = 0, 1 (1 \leq j \leq r)\}$, the square-free numbers with only the first r primes as prime factors, then we have

$$S_\sigma(\mathcal{M}) = N \prod_{1 \leq j \leq r} \left(1 + \frac{1}{p_j^\sigma}\right).$$

Proof. Firstly, we fix $m \in \mathcal{M}$. Without loss of generality, we can assume $m = p_1 p_2 \dots p_l$ with $1 \leq l \leq r$. Then we have

$$\begin{aligned} & \sum_{n \in \mathcal{M}} \frac{\gcd(m, n)^\sigma}{\text{lcm}(m, n)^\sigma} \\ &= \sum_{\delta_1, \dots, \delta_r \in \{0, 1\}} \frac{\gcd(p_1 p_2 \dots p_l, p_1^{\delta_1} \dots p_r^{\delta_r})^\sigma}{\text{lcm}(p_1 p_2 \dots p_l, p_1^{\delta_1} \dots p_r^{\delta_r})^\sigma} = \sum_{\delta_1, \dots, \delta_r \in \{0, 1\}} \left(\frac{p_1^{\delta_1} \dots p_l^{\delta_l}}{p_1 \dots p_l p_{l+1}^{\delta_{l+1}} \dots p_r^{\delta_r}} \right)^\sigma \\ &= \frac{1}{(p_1 \dots p_l)^\sigma} \sum_{\delta_1, \dots, \delta_l \in \{0, 1\}} (p_1^{\delta_1} \dots p_l^{\delta_l})^\sigma \sum_{\delta_{l+1}, \dots, \delta_r \in \{0, 1\}} \frac{1}{(p_{l+1}^{\delta_{l+1}} \dots p_r^{\delta_r})^\sigma} \\ &= \left(\prod_{1 \leq j \leq l} \frac{1}{p_j^\sigma} \right) \left(\prod_{0 \leq j \leq l} (1 + p^\sigma) \right) \prod_{l+1 \leq j \leq r} \left(1 + \frac{1}{p_j^\sigma}\right) = \prod_{1 \leq j \leq r} \left(1 + \frac{1}{p_j^\sigma}\right), \end{aligned}$$

which does not depend on m . We have

$$S_\sigma(\mathcal{M}) = \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{M}} \frac{\gcd(m, n)^\sigma}{\text{lcm}(m, n)^\sigma} = \left(\prod_{1 \leq j \leq r} \left(1 + \frac{1}{p_j^\sigma}\right) \right) \sum_{m \in \mathcal{M}} 1 = N \prod_{1 \leq j \leq r} \left(1 + \frac{1}{p_j^\sigma}\right),$$

since $|\mathcal{M}| = 2^r = N$, which completes the proof. \square

With the help of the above proposition, when $\frac{1}{2} < \sigma < 1$, we can simply get a nontrivial lower bound

$$\begin{aligned} \Gamma_\sigma(N) &\geq \prod_{1 \leq j \leq \log N / \log 2} \left(1 + \frac{1}{p_j^\sigma}\right) = \exp \left(\sum_{1 \leq j \leq \log N / \log 2} \log \left(1 + \frac{1}{p_j^\sigma}\right) \right) \\ &\geq \exp \left(\sum_{1 \leq j \leq \log N / \log 2} \frac{1}{p_j^\sigma} + C \right) \\ &\geq \exp \left(\left\{ \frac{1}{(1-\sigma)(\log 2)^{1-\sigma}} + o(1) \right\} \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right), \end{aligned} \tag{1.4.2}$$

by the definition of $\Gamma_\sigma(N)$ and Lemma 4.2.2.

Gál's identity

Let p_1, \dots, p_r be r different primes. Define

$$\mathcal{M} = \mathcal{M}(r, l) := \{m = p_1^{v_1} \dots p_r^{v_r} : 0 \leq v_j \leq l-1 \quad (1 \leq j \leq r)\}.$$

Gál [25] showed the following identity, which can be seen as a generalization of the case of square-free numbers:

$$S_\sigma(\mathcal{M}) = \prod_{1 \leq j \leq r} \left(l + 2 \sum_{1 \leq v \leq l-1} \frac{l-v}{p_j^{v\sigma}} \right) = |\mathcal{M}| \prod_{1 \leq j \leq r} \left(1 + 2 \sum_{1 \leq v \leq l-1} \left(1 - \frac{v}{l}\right) \frac{1}{p_j^{v\sigma}} \right). \tag{1.4.3}$$

We have

$$\begin{aligned}
1 + 2 \sum_{1 \leq v \leq l-1} \left(1 - \frac{v}{l}\right) \frac{1}{p_j^{v\sigma}} &= 2 \left\{ \sum_{v \geq 0} - \sum_{v \geq l} \right\} \left(1 - \frac{v}{l}\right) \frac{1}{p_j^{v\sigma}} - 1 \\
&= 2 \left(1 - \frac{1}{p_j^\sigma}\right)^{-1} - \frac{2}{lp_j^\sigma} \left(1 - \frac{1}{p_j^\sigma}\right)^{-1} - 1 + O(p_j^{-l\sigma}) \\
&= \left(1 - \frac{1}{p_j^\sigma}\right)^{-1} \left(1 + \frac{1}{p_j^\sigma} - \frac{2}{lp_j^\sigma}\right) + O(p_j^{-l\sigma}).
\end{aligned}$$

Thus (1.4.3) turns to be

$$\begin{aligned}
\frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} &= \prod_{1 \leq j \leq r} \left(\left(1 - \frac{1}{p_j^\sigma}\right)^{-1} \left(1 + \frac{1}{p_j^\sigma} - \frac{2}{lp_j^\sigma}\right) + O(p_j^{-l\sigma}) \right) \\
&= \prod_{1 \leq j \leq r} \left(1 - \frac{1}{p_j^\sigma}\right)^{-2} \prod_{1 \leq j \leq r} \left(\left(1 - \frac{1}{p_j^\sigma}\right) \left(1 + \frac{1}{p_j^\sigma} - \frac{2}{lp_j^\sigma}\right) + O(p_j^{-l\sigma}) \right) \\
&= \prod_{1 \leq j \leq r} \left(1 - \frac{1}{p_j^\sigma}\right)^{-2} \prod_{1 \leq j \leq r} \left(1 - \frac{1}{p_j^{2\sigma}} + O\left(\frac{1}{lp_j^\sigma}\right) + O(p_j^{-l\sigma})\right). \tag{1.4.4}
\end{aligned}$$

Since

$$\prod_{1 \leq j \leq r} \left(1 - \frac{1}{p_j^{2\sigma}}\right) = \frac{1}{\zeta(2)} + o(p_r^{1-2\sigma}),$$

The second product of (1.4.4) can be written as

$$\prod_{1 \leq j \leq r} \left(1 - \frac{1}{p_j^{2\sigma}} + O\left(\frac{1}{lp_j^\sigma}\right) + O(p_j^{-l\sigma})\right) = \frac{1}{\zeta(2)} + o(p_r^{1-2\sigma}) + O\left(\frac{1}{l} \sum_{1 \leq j \leq r} \frac{1}{p_j^\sigma}\right).$$

We choose $l = r + \lfloor \log N \rfloor$ and r such that

$$(r + \log N)^r \leq N < (r + 1 + \log N)^{r+1}.$$

Taking the logarithm of the above inequality, we have

$$r \log(r + \log N) \leq \log N < (r + 1) \log(r + 1 + \log N).$$

It follows that

$$r = (1 + o(1)) \frac{\log N}{\log_2 N}.$$

When $\sigma = 1$, (1.4.4) gives the best bound of Lewko-Radziwiłł [35]

$$\Gamma_1(N) = \max_{|\mathcal{M}|=N} \frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} \geq \left(\frac{1}{\zeta(2)} + o(1)\right) e^{2\gamma} (\log_2 N)^2.$$

However, when $\frac{1}{2} < \sigma < 1$, the choice of \mathcal{M} can not provide a good lower bound for $\Gamma_\sigma(N)$.

1.4.3 Special case of friable numbers

If we restrict \mathcal{M} to be a set of friable numbers, then we can easily give an upper bound for the GCD sums attached to \mathcal{M} .

Proposition 1.4.2. *For $0 < \sigma < 1$, assume $\mathcal{M} \subset S(y)$, where $S(y)$ denotes the set of y -friable numbers. Then we have*

$$\frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} \leq \prod_{p \leq y} \left(1 + \frac{2}{p^\sigma - 1}\right).$$

Proof. Let p_1, p_2, \dots be all the primes in ascending order. Assume $p_k = \max_{m \in \mathcal{M}} P_+(m)$. For any fixed $m \in \mathcal{M}$, assume $m = \prod_{j=1}^k p_j^{u_j}$ with $u_j \geq 0$ for each $1 \leq j \leq k$. We have

$$\begin{aligned} \sum_{n \in \mathcal{M}} \left(\frac{\gcd(m, n)}{\text{lcm}(m, n)} \right)^\sigma &\leq \sum_{v_1 \geq 0} \cdots \sum_{v_k \geq 0} \left(\frac{\gcd(p_1^{u_1} \cdots p_k^{u_k}, p_1^{v_1} \cdots p_k^{v_k})}{\text{lcm}(p_1^{u_1} \cdots p_k^{u_k}, p_1^{v_1} \cdots p_k^{v_k})} \right)^\sigma \\ &= \sum_{v_1 \geq 0} \cdots \sum_{v_k \geq 0} \prod_{j=1}^k \left(\frac{\gcd(p_j^{u_j}, p_j^{v_j})}{\text{lcm}(p_j^{u_j}, p_j^{v_j})} \right)^\sigma = \prod_{j=1}^k \sum_{v_j \geq 0} \left(\frac{\gcd(p_j^{u_j}, p_j^{v_j})}{\text{lcm}(p_j^{u_j}, p_j^{v_j})} \right)^\sigma = \prod_{j=1}^k \sum_{v_j \geq 0} \left(\frac{p_j^{\min(u_j, v_j)}}{p_j^{\max(u_j, v_j)}} \right)^\sigma \\ &= \prod_{j=1}^k \sum_{v_j \geq 0} p_j^{-\sigma|u_j - v_j|} \leq \prod_{j=1}^k \sum_{v_j \geq 0} p_j^{-\sigma|u_j - v_j|} \leq \prod_{j=1}^k \left(1 + 2 \sum_{v \geq 1} p_j^{-\sigma v} \right) = \prod_{j=1}^k \left(1 + \frac{2}{p_j^\sigma - 1} \right). \end{aligned}$$

Since the last product does not depend on m , we have

$$\begin{aligned} S_\sigma(\mathcal{M}) &= \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{M}} \left(\frac{\gcd(m, n)}{\text{lcm}(m, n)} \right)^\sigma \leq \sum_{m \in \mathcal{M}} \prod_{j=1}^k \left(1 + \frac{2}{p_j^\sigma - 1} \right) = \prod_{j=1}^k \left(1 + \frac{2}{p_j^\sigma - 1} \right) \sum_{m \in \mathcal{M}} 1 \\ &= |\mathcal{M}| \prod_{j=1}^k \left(1 + \frac{2}{p_j^\sigma - 1} \right) \leq |\mathcal{M}| \prod_{p \leq y} \left(1 + \frac{2}{p^\sigma - 1} \right), \end{aligned}$$

which completes the proof. \square

1.5 Probabilistic models

1.5.1 Models for the Riemann zeta function

For each prime p , let $X(p)$ be the random variable uniformly distributed on the unit circle. And they are independently identically distributed for all distinct primes. Define the random Euler product

$$\zeta(\sigma, X) := \prod_p \left(1 - \frac{X(p)}{p^\sigma} \right)^{-1},$$

which is almost surely convergent for $\sigma > \frac{1}{2}$. Extending the definition of $X(n)$ for any integer n by

$$X(n) := \prod_{p^v \mid \mid n} X(p)^v,$$

then we have the orthogonality

$$\mathbb{E}(X(m)\overline{X}(n)) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

And for $\sigma > \frac{1}{2}$ we can also write $\zeta(\sigma, X)$ as

$$\zeta(\sigma, X) = \sum_{n \geq 1} \frac{X(n)}{n^\sigma}.$$

It turns out that $\zeta(\sigma, X)$ is a good model for both the Riemann zeta function $\zeta(\sigma + it)$ and the Dirichlet L -function.

Proposition 1.5.1. *Denote the short random Euler product by $\zeta(\sigma, X; y) = \prod_{p \leq y} (1 - X(p)/p^\sigma)^{-1}$. Then for $y \leq (\log T)^2$, and $k \ll \log T/y^{1-\sigma}$, we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it; y)|^k dt = \mathbb{E}(|\zeta(\sigma, X; y)|^k) + O\left(\exp\left(-\frac{\log T}{4 \log y}\right)\right).$$

For $y \leq (\log q)^2$, and $k \ll \log q/y^{1-\sigma}$, we have

$$\frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} |L(\sigma, \chi; y)|^k = \mathbb{E}(|\zeta(\sigma, X; y)|^k) + O\left(\exp\left(-\frac{\log q}{4 \log y}\right)\right).$$

1.5.2 Models for the GCD sum

Using the definitions of X and $\zeta(\sigma, X)$ as in the last section, for any finite set of positive integers \mathcal{M} , we define

$$D(X, \mathcal{M}) := \sum_{m \in \mathcal{M}} X(m).$$

We will see that $\zeta(\sigma, X)^2 D(X, \mathcal{M})^2$ is a good model for $S_\sigma(\mathcal{M})$.

Proposition 1.5.2. *We have*

$$\mathbb{E}(|\zeta(\sigma, X) D(X, \mathcal{M})|^2) = \zeta(2\sigma) S_\sigma(\mathcal{M}).$$

Proof. Expanding the square, we have

$$|\zeta(\sigma, X) D(X, \mathcal{M})|^2 = \sum_{l, k \geq 1} \frac{X(l)\overline{X}(k)}{(lk)^\sigma} \sum_{m, n \in \mathcal{M}} X(m)\overline{X}(n) = \sum_{l, k \geq 1} \sum_{m, n \in \mathcal{M}} \frac{X(lm)\overline{X}(kn)}{(lk)^\sigma}.$$

So by the orthogonality of X , we have

$$\begin{aligned} \mathbb{E}(|\zeta(\sigma, X) D(X, \mathcal{M})|^2) &= \mathbb{E}\left(\sum_{l, k \geq 1} \sum_{m, n \in \mathcal{M}} \frac{X(lm)\overline{X}(kn)}{(lk)^\sigma}\right) = \sum_{l, k \geq 1} \sum_{m, n \in \mathcal{M}} \frac{\mathbb{E}(X(lm)\overline{X}(kn))}{(lk)^\sigma} \\ &= \sum_{l, k \geq 1} \sum_{\substack{m, n \in \mathcal{M} \\ lm = kn}} \frac{1}{(lk)^\sigma} = \sum_{m, n \in \mathcal{M}} \sum_{\substack{l, k \geq 1 \\ lm = kn}} \frac{1}{(lk)^\sigma}. \end{aligned}$$

For any fixed m, n , the relation $lm = kn$ implies

$$l \frac{m}{\gcd(m, n)} = k \frac{n}{\gcd(m, n)},$$

while

$$\gcd\left(\frac{m}{\gcd(m, n)}, \frac{n}{\gcd(m, n)}\right) = 1.$$

Thus we must have

$$l = j \frac{n}{\gcd(m, n)} \quad \text{and} \quad k = j \frac{m}{\gcd(m, n)},$$

for some positive integer j . Then we have

$$\sum_{m,n \in \mathcal{M}} \sum_{\substack{l,k \geq 1 \\ lm=kn}} \frac{1}{(lk)^\sigma} = \sum_{m,n \in \mathcal{M}} \sum_{j \geq 1} \left(\frac{\gcd(m, n)^2}{j^2 mn} \right)^\sigma = \sum_{j \geq 1} \frac{1}{j^{2\sigma}} \sum_{m,n \in \mathcal{M}} \frac{\gcd(m, n)^{2\sigma}}{(mn)^\sigma} = \zeta(2\sigma) S_\sigma(\mathcal{M}).$$

This completes the proof. \square

1.5.3 Models for $L(1, \chi_d)$

Since the real primitive character χ_d only takes values ± 1 and 0, the model X in the last two sections is not in accordance with χ_d . We define a new random variable $X(p)$ for each prime p :

$$X(n) = \begin{cases} \pm 1 & \text{Prob} = \frac{p}{2(p+1)}, \\ 0 & \text{Prob} = \frac{1}{p+1}. \end{cases}$$

And again we extend the definition to any integers completely multiplicatively

$$X(n) = \prod_{p^v || n} X(p)^v.$$

Then we similarly define

$$L(\sigma, X) = \prod_p \left(1 - \frac{X(p)}{p^\sigma} \right)^{-1},$$

and it will be a good model for $L(\sigma, \chi_d)$.

Proposition 1.5.3. *Denote the short random Euler product by $L(\sigma, X; y) = \prod_{p \leq y} (1 - X(p)/p^\sigma)^{-1}$. Then for $y \leq (\log x)^2$, and $k \ll \log x \log_2 y / y^{1-\sigma}$, we have the following asymptotic formula*

$$\frac{1}{x/\zeta(2)} \sum_{|d| \leq x} {}^\flat L(\sigma + it, \chi_d; y)^k dt = \mathbb{E}(L(\sigma, X; y)^k) + O\left(\exp\left(-\frac{\log x \log_2 y}{40 \log y}\right)\right).$$

2 Distribution of $|\zeta(1 + it)|$

2.1 Background

The study of the value distribution of the Riemann zeta function $\zeta(s)$ can date back to the early twentieth century when Bohr showed that for any $z \in \mathbb{C}^*$ and $\varepsilon > 0$, there are infinitely many s 's with $1 < \Re s < 1 + \varepsilon$ such that $\zeta(s) = z$. Later in 1932, Bohr and Jessen [10] showed that $\log \zeta(\sigma + it)$ has a continuous distribution on the complex plane for any $\sigma > \frac{1}{2}$.

The values on the 1-line have much significance. For example, the fact that $\zeta(1 + it) \neq 0$ implies the prime number theorem. The extreme values of $\zeta(1 + it)$ has been widely investigated. In 1925, Littlewood [36] showed that there exists arbitrarily large t for which

$$|\zeta(1 + it)| \geq \{1 + o(1)\} e^\gamma \log_2 t.$$

Here and throughout, we denote by γ the Euler constant and by \log_j the j -th iterated logarithm. In 1972, Levinson [34] improved the error term from $o(\log_2 t)$ to $O(1)$, by showing that there exists arbitrarily large t such that

$$|\zeta(1 + it)| \geq e^\gamma \log_2 t + O(1).$$

In 2006, Granville and Soundararajan [26] got a much stronger result

$$\max_{t \in [1, T]} |\zeta(1 + it)| \geq e^\gamma \{\log_2 T + \log_3 T - \log_4 T + O(1)\}$$

holds for sufficiently large T . Then in 2019, Aistleitner, Mahatab, and Munsch [3] canceled the term $\log_4 T$:

$$\max_{t \in [\sqrt{T}, T]} |\zeta(1 + it)| \geq e^\gamma \{\log_2 T + \log_3 T + O(1)\}. \quad (2.1.1)$$

This bound is best possible up to the error term $O(1)$, since in [26], Granville and Soundararajan conjectured that

$$\max_{t \in [T, 2T]} |\zeta(1 + it)| = e^\gamma \{\log_2 T + \log_3 T + C_0 + 1 - \log 2 + o(1)\}, \quad (2.1.2)$$

where C_0 is some absolute constant (see (2.1.5) below). This conjecture was based on some analysis on the following distribution function they introduced in [26]: define for $T > 1$,

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1 + it)| > e^\gamma \tau\}. \quad (2.1.3)$$

Then they proved the asymptotic formula in the logarithm of the distribution function

$$\Phi_T(\tau) = \exp\left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{1 + O\left(\frac{1}{\sqrt{\tau}} + \sqrt{\frac{e^\tau}{\log T}}\right)\right\}\right) \quad (2.1.4)$$

valid uniformly for $1 \leq \tau < \log_2 T - 20$, where

$$C_j := \int_0^2 \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t)}{t^2} dt + \int_2^\infty \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt \quad (2.1.5)$$

and

$$I_0(t) := \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}. \quad (2.1.6)$$

The distribution function (2.1.4) describes the frequency with which each large value is attained. Obviously, the maximum of the range of τ is much less than the large value (2.1.1). However, if (2.1.4) were to persist to the end of the viable range, then we could get (2.1.2).

The method to prove (2.1.4) was also adjusted to apply to the distribution of values on the 1-line of other L -functions. Also in [26], Granville and Soundararajan showed that the distribution of the Dirichlet L -functions in the aspect of the modulo of characters has the same form as (2.1.4). This result can be used to study the distribution of large character sums, see [8] and [9]. In 2003, Granville and Soundararajan [27] studied the distribution of the Dirichlet L -functions of quadratic characters $L(1, \chi_d)$, which proves part of Montgomery and Vaughan's conjecture in [41]. In 2007, Wu [54] improved this result by giving a high order expansion in the exponent of the distribution function. In 2008, Liu, Royer and Wu [38] considered the case of symmetric power L -functions. In 2010, Lamzouri [32] studied a generalized L -function which can cover the results of [27, 38]. Again concerning the Riemann zeta function, in 2008 Lamzouri [31] generalize (2.1.4) to the joint distribution of $\arg \zeta(1+it)$ and $|\zeta(1+it)|$.

Inspired by the result of Wu [54], the aim of this chapter is to get an improvement of (2.1.4), which presents a higher order expansion in the exponent.

Theorem 2.1.1. *There is a sequence of real numbers $\{\alpha_j\}_{j \geq 1}$ such that for any integer $J \geq 1$ we have*

$$\Phi_T(\tau) = \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + \sum_{j=1}^J \frac{\alpha_j}{\tau^j} + O_J \left(\frac{1}{\tau^{J+1}} + \sqrt{\frac{e^\tau}{\log T}} \right) \right\} \right)$$

uniformly for $T \rightarrow \infty$ and $1 \leq \tau \leq \log_2 T - 20$, where C_0 is defined as in (2.1.5). Moreover, $\alpha_1 = 2(1 + C_0 - C_1)$.

Our main new ingredient for the proof of Theorem 2.1.1 is Proposition 2.5.1 below, which gives a better approximation of the distribution function of the short Euler products:

$$\Phi_T(\tau; y) := \frac{1}{T} \text{meas} \{ t \in [T, 2T] : |\zeta(1+it; y)| > e^\gamma \tau \}, \quad (2.1.7)$$

where

$$\zeta(s; y) := \prod_{p \leq y} (1 - p^{-s})^{-1}. \quad (2.1.8)$$

For this, it is necessary to improve Theorem 3 of [26] (see Propositions 2.3.1 and 2.4.1 below).

2.2 Preliminary lemmas

Let $k \geq 1$ be a positive integer. Define $d_k(n)$ by the relation

$$\zeta(s)^k = \sum_{n \geq 1} d_k(n) n^{-s} \quad (\Re s > 1). \quad (2.2.1)$$

Firstly, we quote the following asymptotic formulae of sums attached to the divisor function $d_k(n)$ and the Bessel function $I_0(t)$ to show their correlation.

Lemma 2.2.1. *For any prime p and positive integer k , we have*

$$\sum_{\nu \geq 0} \frac{d_k(p^\nu)^2}{p^{2\nu}} = I_0\left(\frac{2k}{p}\right) \exp\left\{O\left(\frac{k}{p^2}\right)\right\}, \quad (2.2.2)$$

$$\frac{\min(1, p/k)}{50} \left(1 - \frac{1}{p}\right)^{-2k} \leq \sum_{\nu \geq 0} \frac{d_k(p^\nu)^2}{p^{2\nu}} \leq \left(1 - \frac{1}{p}\right)^{-2k}, \quad (2.2.3)$$

where $I_0(t)$ is the Bessel function as defined in (2.1.6).

Proof. This is [26, Lemma 4]. \square

We need to approximate Riemann zeta function by its short Euler product. The following lemma shows that when $\zeta(s)$ has no zero in a good region, it can be approximated well by its short Euler product.

Lemma 2.2.2. *Let $y \geq 2$ and $|t| \geq y + 3$ be real numbers. Let $\frac{1}{2} \leq \sigma_0 < 1$ and suppose that the rectangle $\{z : \sigma_0 < \Re z \leq 1, |\Im z - t| \leq y + 2\}$ is free of zeros of $\zeta(z)$. Then for any $\sigma_0 < \sigma \leq 2$ and $|\xi - t| \leq y$, we have*

$$|\log \zeta(\sigma + i\xi)| \ll \log |t| \log(e/(\sigma - \sigma_0)).$$

Further for $\sigma_0 < \sigma \leq 1$, we have

$$\log \zeta(\sigma + it) = \sum_{n=2}^y \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O\left(\frac{\log |t|}{(\sigma_1 - \sigma_0)^2} y^{\sigma_1 - \sigma}\right),$$

where $\sigma_1 = \min\left(\sigma_0 + \frac{1}{\log y}, \frac{\sigma + \sigma_0}{2}\right)$.

Proof. See [26, Lemma 1]. \square

In order to approximate the Riemann zeta function $\zeta(s)$ by its truncated Euler product $\zeta(s; y)$ defined by (2.1.8), we need the following evaluation for moments of the sum over complex power of primes between two large numbers.

Lemma 2.2.3. *Let $\{b(p)\}_{p \text{ primes}}$ be a complex sequence. Then we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{b(p)}{p^{it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} |b(p)|^2 \right)^k + \frac{1}{T^{2/3}} \left(\sum_{y \leq p \leq z} |b(p)| \right)^{2k}$$

uniformly for $T \geq 8$, $2 \leq y \leq z \leq T^{1/3}$ and all integers $1 \leq k \leq \log T/(3 \log z)$, where the implied constant is absolute.

Proof. This is [26, Lemma 3]. \square

The following lemma is an approximation of $\zeta(s)$ by $\zeta(s; y)$.

Lemma 2.2.4. *Let $T \geq 2$ and $y \geq \log T$. Then we have*

$$\zeta(1 + it) = \zeta(1 + it; y) \left\{ 1 + O\left(\frac{\sqrt{(\log T)/y}}{\log_2 T}\right) \right\}$$

for all $t \in [T, 2T]$ except for a set of measure at most $O(T \exp\{-(\log T)/(50 \log_2 T)\})$.

Proof. This is essentially [26, Proposition 1] while we erase restrictions of the upper bound of y . In fact, the truncated Euler product with larger length would provide a better approximation of the zeta function. Without loss of generality, we assume $y \leq (\log T)^{100}$. Firstly we use Lemma 2.2.2 to approximate $\zeta(1 + it)$ by $\zeta(1 + it; z)$ with z relatively large. Choosing $z = (\log T)^{100}$, then by Lemma 2.2.2 we have

$$\zeta(1 + it) = \zeta(1 + it; z)(1 + O(1/\log T))$$

for all $t \in [T, 2T]$ but at most a set of measure of $T^{4/5}$. Then we use Lemma 2.2.3 to approximate $\zeta(1 + it; z)$ by $\zeta(1 + it; y)$ since

$$\zeta(1 + it; z) = \zeta(1 + it; y) \exp\left(\sum_{y \leq p \leq z} \left(\frac{1}{p^{1+it}} + O\left(\frac{1}{p^2}\right)\right)\right).$$

Choosing $k = [\log T/(3 \log z)] = [\log T/(300 \log_2 T)]$ and $x(p) = 1/p$, then by Lemma 2.2.3 we have

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right|^{2k} dt &\ll \left(k \sum_{y \leq p \leq z} \frac{1}{p^2} \right)^k + T^{-2/3} \left(k \sum_{y \leq p \leq z} \frac{1}{p} \right)^{2k} \\ &\ll \left(\frac{2k}{y \log y} \right)^k + T^{-1/3}. \end{aligned}$$

By the choice of the value of k , we have

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right|^{2k} dt &\ll \left(\frac{\log T}{150y \log y \log_2 T} \right)^k + T^{-1/3} \\ &\ll \left(\frac{\sqrt{\log T}}{12\sqrt{y \log y \log_2 T}} \right)^{2k} + T^{-1/3} \\ &\ll \left(\frac{\sqrt{\log T}}{12\sqrt{y \log_2 T}} \right)^{2k} + T^{-1/3}, \end{aligned}$$

where the last inequality holds since $y \geq \log T$. Thus

$$\left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right| \leq \frac{\sqrt{\log T}}{\sqrt{y \log_2 T}}$$

for all $t \in [T, 2T]$ except for a set of measure $12^{-2k}T = T \exp(-\log 12 \log T / (150 \log_2 T)) \leq T \exp(-\log T / (49 \log_2 T))$. Combining this with the first step, Lemma 2.2.4 follows. \square

2.3 An asymptotic development

The integer $n \geq 1$ is called y -friable if the largest prime factor $P(n)$ of n is less than y ($P(1) = 1$ by convention). Denote by $S(y)$ the set of y -friable integers and define

$$D_k(y) := \sum_{n \in S(y)} d_k(n)^2 / n^2. \quad (2.3.1)$$

The aim of this section is to prove the following proposition, which is an improvement of the second part of Theorem 3 in [26]. Our improvement is double: a higher order expansion in the exponent and a larger domain of y .

Proposition 2.3.1. *Let $A > 1$ be a positive number and let $J \geq 0$ be an integer. We have*

$$D_k(y) = \prod_{p \leq k} \left(1 - \frac{1}{p}\right)^{-2k} \exp\left(\frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j}{(\log k)^j} + O_{A,J}\left(\frac{1}{(\log k)^{J+1}} + \frac{k}{y}\right)\right\}\right)$$

uniformly for $k \geq 2$ and $2k \leq y \leq k^A$, where the C_j is defined as in (2.1.5) and the constant implied depends on A and J only.

Proof. Firstly we have trivially

$$\prod_{\sqrt{k} < p \leq y} \exp\left\{O\left(\frac{k}{p^2}\right)\right\} = e^{O(\sqrt{k})}, \quad \prod_{p \leq \sqrt{k}} \frac{\min(1, p/k)}{50} = e^{O(\sqrt{k})}. \quad (2.3.2)$$

Secondly, by Lemma 2.2.1, we can write

$$D_k(y) = \prod_{p \leq \sqrt{k}} \left(1 - \frac{1}{p}\right)^{-2k} \prod_{\sqrt{k} < p \leq y} I_0\left(\frac{2k}{p}\right) e^{O(\sqrt{k})} = \prod_{p \leq k} \left(1 - \frac{1}{p}\right)^{-2k} \Pi_1 \Pi_2 e^{O(\sqrt{k})}, \quad (2.3.3)$$

where

$$\Pi_1 := \prod_{\sqrt{k} < p \leq k} \left(1 - \frac{1}{p}\right)^{2k} I_0\left(\frac{2k}{p}\right) \quad \text{and} \quad \Pi_2 := \prod_{k < p \leq y} I_0\left(\frac{2k}{p}\right).$$

In order to evaluate Π_1 , we apply the formula $\log(1+t) = t + O(t^2)$ ($|t| \leq \frac{1}{2}$) and the first estimate in (2.3.2) to obtain

$$\Pi_1 = \exp\left\{ \sum_{\sqrt{k} < p \leq k} \left(\log I_0\left(\frac{2k}{p}\right) - \frac{2k}{p} \right) + O(\sqrt{k}) \right\}.$$

Recall the prime number theorem

$$\pi(u) := \sum_{p \leq u} 1 = \int_2^u \frac{du}{\log u} + O(ue^{-2c\sqrt{\log u}}).$$

Then we can derive that

$$\begin{aligned} \sum_{\sqrt{k} < p \leq k} \left(\log I_0\left(\frac{2k}{p}\right) - \frac{2k}{p} \right) &= \int_{\sqrt{k}}^k \left(\log I_0\left(\frac{2k}{u}\right) - \frac{2k}{u} \right) d\pi(u) \\ &= \int_{\sqrt{k}}^k \frac{\log I_0(2k/u) - 2k/u}{\log u} du + O(ke^{-c\sqrt{\log k}}). \end{aligned} \quad (2.3.4)$$

By putting $t = 2k/u$ and using the fact that

$$\frac{1}{1-t} = \sum_{j=0}^J t^j + O_J(t^{J+1}) \quad (|t| \leq \frac{1}{2}),$$

we can derive that the integral in (2.3.4) is equal to

$$\frac{2k}{\log k} \int_2^{2\sqrt{k}} \frac{\log I_0(t) - t}{t^2(1 - \frac{\log(t/2)}{\log k})} dt = \frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j^*(k)}{(\log k)^j} + O_J\left(\frac{C_{J+1}^*(k)}{(\log k)^{J+1}}\right) \right\}, \quad (2.3.5)$$

where

$$C_j^*(k) := \int_2^{2\sqrt{k}} \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt.$$

Since $\log I_0(t) = t + O(\log t)$ for $t \geq 2$, we have

$$\int_{2\sqrt{k}}^{\infty} \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt \ll \int_{\sqrt{k}}^{\infty} \frac{(\log t)^{j+1}}{t^2} dt \ll_j \frac{(\log k)^{j+1}}{\sqrt{k}},$$

which implies that

$$C_j^*(k) = C_j^* + O_j\left(\frac{(\log k)^{j+1}}{\sqrt{k}}\right) \quad \text{with} \quad C_j^* := \int_2^{\infty} \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t) - t}{t^2} dt.$$

Combining this with (2.3.4) and (2.3.5), we obtain

$$\sum_{\sqrt{k} < p \leq k} \left(\log I_0\left(\frac{2k}{p}\right) - \frac{2k}{p} \right) = \frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j^*}{(\log k)^j} + O_J\left(\frac{1}{(\log k)^{J+1}}\right) \right\}.$$

Therefore we derive that

$$\Pi_1 = \exp \left(\frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j^*}{(\log k)^j} + O_J\left(\frac{1}{(\log k)^{J+1}}\right) \right\} \right). \quad (2.3.6)$$

For Π_2 , by the prime number theorem, we have similarly

$$\begin{aligned} \sum_{k < p \leq y} \log I_0\left(\frac{2k}{p}\right) &= \int_k^y \log I_0\left(\frac{2k}{u}\right) \frac{du}{\log u} + O(k e^{-c\sqrt{\log k}}) \\ &= \frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j^{**}}{(\log k)^j} + O\left(\frac{k}{y}\right) \right\} + O(k e^{-c\sqrt{\log k}}), \end{aligned}$$

where

$$C_j^{**} := \int_0^2 \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t)}{t^2} dt.$$

Here we have used the inequality $\log I_0(t) \ll t^2$ for $0 \leq t < 2$ to evaluate the truncated integral that

$$\int_0^{2k/y} \left(\log \frac{t}{2} \right)^j \frac{\log I_0(t)}{t^2} dt \ll \int_0^{k/y} (\log t)^j dt \ll_{A,j} \frac{k(\log k)^j}{y}.$$

Therefore we derive that

$$\Pi_2 = \exp \left(\frac{2k}{\log k} \left\{ \sum_{j=0}^J \frac{C_j^{**}}{(\log k)^j} + O_{A,J} \left(\frac{k}{y} \right) \right\} \right). \quad (2.3.7)$$

Now Proposition 2.3.1 follows from (2.3.3), (2.3.6) and (2.3.7) with $C_j = C_j^* + C_j^{**}$. \square

2.4 Moments of the short Euler products

Let $\zeta(s; y)$ and $D_k(y)$ be defined as in (2.1.8) and (2.3.1), respectively. In this section, we shall evaluate the $2k$ -th moment of $\zeta(1 + it; y)$ by proving the following proposition. This is essentially the first part of Theorem 3 in [26]. The main difference is a slightly enlarged length of the short Euler products, which is important for the proof of Theorem 2.1.1.

Proposition 2.4.1. *Let $A > 0$ be a constant. Then we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(1 + it; y)|^{2k} dt = D_k(y) \left\{ 1 + O_A \left(\exp \left\{ - \frac{\log T}{2(\log_2 T)^4} \right\} \right) \right\}$$

uniformly for

$$\begin{cases} T \geq T_0(A), \\ e^2 \log T \leq y \leq (\log T)(\log_2 T)^A, \\ k \in \mathbb{N} \cap [2, (\log T)/(e^{10} \log(y/\log T))], \end{cases} \quad (2.4.1)$$

where the implied constant and the constant $T_0(A)$ depend on A only.

We show that for k and y in (2.4.1), the diagonal terms lead to the main term, while the off-diagonal terms only contribute to the error term. For this, we need to establish a preliminary lemma.

2.4.1 A preliminary lemma

If $2 \leq k \leq 10^6$, we write $\mathcal{I}_0 = (k, y]$, $\mathcal{I}_1 = (1, k]$ and $J = 0$. When $k > 10^6$, we take $J = \lfloor 4(\log_2 k)/\log 2 \rfloor$ and divide $(1, y]$ into $J+2$ intervals

$$(1, y] = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{J+1},$$

where $\mathcal{I}_0 := (k, y]$, $\mathcal{I}_j := (k/2^j, k/2^{j-1}]$ ($1 \leq j \leq J$) and $\mathcal{I}_{J+1} := (1, k/2^J] \subset (1, 2k/(\log_2 k)^4]$. For each $j \in \{0, 1, \dots, J+1\}$, we use $S(\mathcal{I}_j)$ to represent the set of all positive integers which have prime divisors only in \mathcal{I}_j ($1 \in S(\mathcal{I}_j)$ by convention). Recall that $S(y)$ is the set of y -friable integers. Thus

$$n \in S(y) \Leftrightarrow n \stackrel{\text{uniquely}}{=} n_0 \cdots n_{J+1} \text{ with } n_j \in S(\mathcal{I}_j) \ (0 \leq j \leq J+1). \quad (2.4.2)$$

Set

$$D_{k,j} := \sum_{h \in S(\mathcal{I}_j)} d_k(h)^2/h^2 \quad (2.4.3)$$

such that

$$D_k(y) = D_{k,0} D_{k,1} \cdots D_{k,J+1}. \quad (2.4.4)$$

We have the following lemma.

Lemma 2.4.2. Let $G_0 := T^{1/5}$ and $G_j := T^{1/(5j^2)}$ ($j \geq 1$). Then we have

$$\sum_{g \in S(\mathcal{I}_j), g > G_j} \frac{2^{\omega(g)}}{g} \sum_{h \in S(\mathcal{I}_j)} \frac{d_k(gh)d_k(h)}{h^2} \leq D_{k,j} \exp\left(-\frac{\log T}{(\log_2 T)^4}\right) \quad (2.4.5)$$

for (T, y, k) in (2.4.1) and $0 \leq j \leq J + 1$, where $\omega(n)$ denotes the number of distinct prime factors of n .

Proof. This is essentially [26, Lemma 5]. The difference is that the upper bound of y is shifted from $(\log T)(\log_2 T)^4$ to $(\log T)(\log_2 T)^A$ with arbitrary $A > 0$. Note that this change is harmless to the range of k , since whether $y = (\log T)(\log_2 T)^4$ or $y = (\log T)(\log_2 T)^A$, it does not influence the upper bound of k :

$$k \leq \frac{\log T}{e^{10} \log(y/\log T)} \leq \frac{\log T}{e^{10} \log(e^2 \log T/\log T)} = \frac{\log T}{2e^{10}}. \quad (2.4.6)$$

So this makes no difference so that we can follow Granville and Soundararajan's procedure. Here we reproduce their proof for convenience of the reader. Denote by $\mathfrak{S}_k(\mathcal{I}_j)$ the member on the left-hand side of (2.4.5).

Firstly we consider the case of $1 \leq j \leq J + 1$. For $\delta = 1/(2^{j/2} \log k)$, by Rankin's trick and exchanging the order of the sums, we have

$$\mathfrak{S}_k(\mathcal{I}_j) \leq \frac{1}{G_j^\delta} \sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(g)}}{g^{1-\delta}} \sum_{h \in S(\mathcal{I}_j)} \frac{d_k(gh)d_k(h)}{h^2} = \frac{1}{G_j^\delta} \sum_{h \in S(\mathcal{I}_j)} \frac{d_k(h)}{h^{1+\delta}} \sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(g)} d_k(gh)}{(gh)^{1-\delta}}. \quad (2.4.7)$$

The inner sum is over part of $S(\mathcal{I}_j)$, so for any $h \in S(\mathcal{I}_j)$ we have

$$\sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(g)} d_k(gh)}{(gh)^{1-\delta}} \leq \sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(gh)} d_k(gh)}{(gh)^{1-\delta}} \leq \sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(g)} d_k(g)}{g^{1-\delta}}.$$

Thus

$$\begin{aligned} \mathfrak{S}_k(\mathcal{I}_j) &\leq \frac{1}{G_j^\delta} \sum_{h \in S(\mathcal{I}_j)} \frac{d_k(h)}{h^{1+\delta}} \sum_{g \in S(\mathcal{I}_j)} \frac{2^{\omega(g)} d_k(g)}{g^{1-\delta}} \\ &= \frac{1}{G_j^\delta} \prod_{p \in \mathcal{I}_j} \left(1 - \frac{1}{p^{1+\delta}}\right)^{-k} \left(2 \left(1 - \frac{1}{p^{1-\delta}}\right)^{-k} - 1\right) \\ &\leq \frac{1}{G_j^\delta} \prod_{p \in \mathcal{I}_j} 2 \left(1 - \frac{1}{p}\right)^{-2k} \Xi_j(p)^{-k} \end{aligned} \quad (2.4.8)$$

with

$$\Xi_j(p) := \left(1 - \frac{1}{p^{1+\delta}}\right) \left(1 - \frac{1}{p^{1-\delta}}\right) \left(1 - \frac{1}{p}\right)^{-2} = 1 - \frac{p(p^\delta + p^{-\delta} - 2)}{(p-1)^2}.$$

Noticing that $p \in \mathcal{I}_j$ with $1 \leq j \leq J + 1$, we have $p \leq k$. Thus by the first inequality in (2.2.3) of Lemma 2.2.1, we can derive that

$$\prod_{p \in \mathcal{I}_j} \left(1 - \frac{1}{p}\right)^{-2k} \leq D_{k,j} \prod_{p \in \mathcal{I}_j} \frac{50k}{p}, \quad (2.4.9)$$

while

$$\begin{aligned}\Xi_j(p) &= 1 - \frac{2p}{(p-1)^2} \sum_{n=1}^{\infty} \frac{(\delta \log p)^{2n}}{(2n)!} \\ &\geq 1 - \frac{2p(\delta \log p)^2}{(p-1)^2} \sum_{n=1}^{\infty} \frac{2^{-j(n-1)}}{(2n)!} \geq 1 - \frac{c(2\delta \log p)^2}{p}.\end{aligned}\tag{2.4.10}$$

with $c := \sum_{n=1}^{\infty} \frac{2^{-(n-1)}}{(2n)!} < \frac{16}{31}$. Combining (2.4.8)–(2.4.10) and using the inequality $-\log(1-t) \leq \sqrt{3}t$ ($0 \leq t \leq 2^{-\frac{1}{2}}c$), we can derive that

$$\mathfrak{S}_k(\mathcal{I}_j) \leq D_{k,j} \exp \left(\log \frac{1}{G_j^\delta} + \sum_{p \in \mathcal{I}_j} \left(\log \frac{100k}{p} + \frac{\sqrt{3}k(2\delta \log p)^2}{p} \right) \right). \tag{2.4.11}$$

We choose $\delta = 1/(2^{j/2} \log k)$. For $1 \leq j \leq J+1$, we have

$$\begin{aligned}\sum_{p \in \mathcal{I}_j} \left(\log \frac{100k}{p} + \frac{\sqrt{3}k(2\delta \log p)^2}{p} \right) &\leq \left(\log \left(\frac{100k}{k/2^j} \right) + \frac{\sqrt{3}k(2\delta \log(k/2^j))^2}{k/2^j} \right) \frac{2(k/2^{j-1})}{\log(k/2^{j-1})} \\ &\leq (j \log 2 + 2 \log 10 + 4\sqrt{3}) \frac{4k}{2^j \log k} \leq \frac{50jk}{2^j \log k} \\ &\leq \frac{25j \log T}{2^j e^{10} \log k} \leq \frac{25 \log T}{2^{j/2} j^2 e^6 \log k}\end{aligned}$$

thanks to (2.4.6) and the inequality $j^3 \leq e^4 2^{j/2}$ ($j \geq 1$). Thus

$$\begin{aligned}\log \frac{1}{G_j^\delta} + \sum_{p \in \mathcal{I}_j} \left(\log \frac{100k}{p} + \frac{\sqrt{3}k(2\delta \log p)^2}{p} \right) &\leq -\frac{1-125e^{-6}}{5j^2 2^{j/2}} \cdot \frac{\log T}{\log k} \\ &\leq -\frac{1-125e^{-6}}{5J^2 2^{J/2}} \cdot \frac{\log T}{\log k} \\ &\leq -\frac{\log T}{(\log k)^4} \leq -\frac{\log T}{(\log_2 T)^4}\end{aligned}\tag{2.4.12}$$

for (T, y, k) in (2.4.1) and $1 \leq j \leq J+1$. Inserting (2.4.12) into (2.4.11), then $1 \leq j \leq J+1$ we have

$$\mathfrak{S}_k(\mathcal{I}_j) \leq D_{k,j} \exp \left(-\frac{\log T}{(\log_2 T)^4} \right).$$

Thus the lemma in this case follows.

When $j = 0$, by Rankin's trick and the trivial inequality $d_k(gh) \leq d_k(g)d_k(h)$, we have for any $0 < \delta < 1$

$$\mathfrak{S}_k(\mathcal{I}_0) \leq \frac{D_{k,0}}{G_0^\delta} \sum_{g \in S(\mathcal{I}_0)} \frac{2^{\omega(g)} d_k(g)}{g^{1-\delta}} = \frac{D_{k,0}}{G_0^\delta} \prod_{p \in \mathcal{I}_0} \left(2 \left(1 - \frac{1}{p^{1-\delta}} \right)^{-k} - 1 \right). \tag{2.4.13}$$

For $k < p \leq y$, we have the upper bound

$$2 \left(1 - \frac{1}{p^{1-\delta}} \right)^{-k} - 1 \leq 2 \exp \left(\frac{2k}{p^{1-\delta}} \right) - 1 \leq \exp \left(\frac{3k}{p^{1-\delta}} \right) \leq \exp \left(\frac{3ky^\delta}{\log k} \cdot \frac{\log p}{p} \right).$$

Inserting this into (2.4.13) and using [49, Theorem I.1.7] in form

$$\sum_{k < p \leq y} \frac{\log p}{p} \leq \log \left(\frac{25y}{k} \right),$$

we can derive that

$$\mathfrak{S}_k(\mathcal{I}_0) \leq \frac{D_{k,0}}{G_0^\delta} \exp \left(\frac{3ky^\delta}{\log k} \log \left(\frac{25y}{k} \right) \right).$$

Taking $\delta = 1/(10 \log_2 T)$ and noticing that $t \mapsto (t/\log t) \log(25y/t)$ is increasing in \mathcal{I}_0 , we deduce, for (T, y, k) in (2.4.1),

$$\begin{aligned} \frac{ky^\delta}{\log k} \log \left(\frac{25y}{k} \right) &\leq \frac{e^{1/5}(\log T)/(e^{10} \log(y/\log T))}{\log((\log T)/(e^{10} \log(y/\log T)))} \log \left(\frac{25e^{10}y}{\log T} \log \left(\frac{y}{\log T} \right) \right) \\ &\leq \frac{10e^{1/5} \log T}{e^{10} \log((\log T)/(e^{10} \log(y/\log T)))} \\ &\leq \frac{20e^{1/5} \log T}{e^{10} \log_2 T}. \end{aligned}$$

Inserting this into the preceding inequality, we have

$$\mathfrak{S}_k(\mathcal{I}_0) \leq D_{k,0} \exp \left(-\delta \log G_0 + \frac{20e^{1/5} \log T}{e^{10} \log_2 T} \right)$$

for (T, y, k) in (2.4.1). Now the result of Lemma 2.4.2 follows by recalling that $G_0 = T^{1/5}$. \square

2.4.2 Proof of Proposition 2.4.1

As in [26], we shall prove a more general result: *Let $A > 0$ be a constant and $\mathcal{R} \subset \{0, 1, \dots, J+1\}$. Then we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(1+it; \mathcal{R})|^{2k} dt = D_k(\mathcal{R}) \left\{ 1 + O_A \left(\exp \left\{ -\frac{\log T}{2(\log_2 T)^4} \right\} \right) \right\} \quad (2.4.14)$$

uniformly for (T, y, k) in (2.4.1), where

$$\mathcal{I}_{\mathcal{R}} := \bigcup_{r \in \mathcal{R}} \mathcal{I}_r, \quad \zeta(s; \mathcal{R}) := \prod_{p \in \mathcal{I}_{\mathcal{R}}} (1 - p^{-s})^{-1}, \quad D_k(\mathcal{R}) := \sum_{n \in S(\mathcal{I}_{\mathcal{R}})} d_k(n)^2/n^2 \quad (2.4.15)$$

and the implied constant depends on A only.

We shall prove (2.4.14) by induction on the cardinal of \mathcal{R} . The case of $\mathcal{R} = \emptyset$ (i.e. $|\mathcal{R}| = 0$) is trivial, since $\zeta(s; \emptyset) = 1 = D_k(\emptyset)$. Now we suppose that (2.4.14) holds for all proper subset of \mathcal{R} and prove that it is true for \mathcal{R} .

Firstly, in view of (2.4.2), we have

$$\begin{aligned} \frac{1}{T} \int_T^{2T} |\zeta(1+it; \mathcal{R})|^{2k} dt &= \sum_{m, n \in S(\mathcal{I}_{\mathcal{R}})} \frac{d_k(m)d_k(n)}{mn} \frac{1}{T} \int_T^{2T} \left(\frac{n}{m} \right)^{it} dt \\ &= \sum_{\substack{m_r, n_r \in S(\mathcal{I}_r) \\ r \in \mathcal{R}}} \left(\prod_{r \in \mathcal{R}} \frac{d_k(m_r)d_k(n_r)}{m_r n_r} \right) \frac{1}{T} \int_T^{2T} \left(\prod_{r \in \mathcal{R}} \frac{n_r}{m_r} \right)^{it} dt. \end{aligned} \quad (2.4.16)$$

Denote

$$g_j = \frac{\text{lcm}(m_j, n_j)}{\text{gcd}(m_j, n_j)} \quad \text{and} \quad h_j = \text{gcd}(m_j, n_j).$$

Using the principle of inclusion-exclusion, we divide the sum in (2.4.16) into two parts

$$\sum_{\substack{m_r, n_r \in S(\mathcal{I}_r) \\ g_r \leq G_r \\ \forall r \in \mathcal{R}}} + \sum_{\substack{\mathcal{W} \subset \mathcal{R} \\ \mathcal{W} \neq \emptyset}} (-1)^{|\mathcal{W}|} \sum_{\substack{m_r, n_r \in S(\mathcal{I}_r), \forall r \in \mathcal{R} \\ g_w > G_w, \forall w \in \mathcal{W}}} \quad (2.4.17)$$

where the G_j is defined as in Lemma 2.4.2.

In the first sum, the case $g_r = 1$ ($r \in \mathcal{R}$) counts the diagonal terms and leads to the main term

$$\sum_{n \in S(\mathcal{I}_{\mathcal{R}})} d_k(n)^2 / n^2 = D_k(\mathcal{R}).$$

Otherwise, we have $\prod_{r \in \mathcal{R}} (m_r/n_r) \neq 1$. Therefore by $g_r \leq G_r$ we have

$$\left| \log \prod_{r \in \mathcal{R}} \frac{m_r}{n_r} \right| = \left| \log \prod_{r \in \mathcal{R}} \frac{m_r/h_r}{n_r/h_r} \right| \geq \log \left(1 + \prod_{r \in \mathcal{R}} g_r^{-1} \right) \gg \prod_{r \in \mathcal{R}} G_r^{-1}.$$

Thus in these terms we have

$$\left| \frac{1}{T} \int_T^{2T} \left(\prod_{r \in \mathcal{R}} \frac{m_r}{n_r} \right)^{it} dt \right| \ll \frac{1}{T} \left| \log \prod_{r \in \mathcal{R}} \frac{m_r}{n_r} \right|^{-1} \ll \frac{1}{T^{2/5}}.$$

Therefore the sum over these terms is

$$\ll \frac{1}{T^{2/5}} \sum_{m, n \in S(\mathcal{I}_{\mathcal{R}})} \frac{d_k(m)d_k(n)}{mn} \ll \frac{1}{T^{2/5}} \prod_{p \in \mathcal{I}_{\mathcal{R}}} \left(1 - \frac{1}{p} \right)^{-2k}.$$

By (2.2.3) in Lemma 2.2.1 and the inequality $-\log(1-t) \leq 2t$ ($0 \leq t \leq \frac{1}{2}$), we have

$$\begin{aligned} \prod_{p \in \mathcal{I}_{\mathcal{R}}} \left(1 - \frac{1}{p} \right)^{-2k} &\leq \prod_{\substack{p \in \mathcal{I}_{\mathcal{R}} \\ p \leq k}} \left(\frac{50k}{p} \sum_{\nu \geq 0} \frac{d_k(p^{\nu})^2}{p^{2\nu}} \right) \prod_{\substack{p \in \mathcal{I}_{\mathcal{R}} \\ k < p \leq y}} \left(1 - \frac{1}{p} \right)^{-2k} \\ &\leq D_k(\mathcal{R}) \exp \left(\sum_{p \leq k} \log \frac{50k}{p} + \sum_{k < p \leq y} \frac{4k}{p} \right) \\ &\leq D_k(\mathcal{R}) \exp \left(\frac{10k}{\log k} \log \frac{25y}{k} \right). \end{aligned}$$

Therefore, the contribution of the first sum in (2.4.17) is

$$\sum_{\substack{m_r, n_r \in S(\mathcal{I}_r) \\ g_r \leq G_r \\ \forall r \in \mathcal{R}}} = D_k(\mathcal{R}) + O \left(\frac{D_k(\mathcal{R})}{T^{2/5}} \exp \left(\frac{12k}{\log k} \log \frac{25y}{k} \right) \right) = D_k(\mathcal{R}) \left\{ 1 + O \left(\frac{1}{T^{1/3}} \right) \right\} \quad (2.4.18)$$

for (T, y, k) as in (2.4.1).

Now consider the second sum in (2.4.17). For a given non-empty subset $\mathcal{W} \subset \mathcal{R}$, we have

$$\sum_{\substack{m_r, n_r \in S(\mathcal{I}_r) \\ \forall r \in \mathcal{R} \setminus \mathcal{W}}} \left(\prod_{r \in \mathcal{R} \setminus \mathcal{W}} \frac{d_k(m_r)d_k(n_r)}{m_r n_r} \right) \left(\prod_{r \in \mathcal{R} \setminus \mathcal{W}} \frac{n_r}{m_r} \right)^{it} = |\zeta(1 + it; \mathcal{R} \setminus \mathcal{W})|^{2k}.$$

where $\zeta(s; \mathcal{R} \setminus \mathcal{W})$ is defined as in (2.4.15). Thus the inner sum in (2.4.17) gives

$$\begin{aligned} & \sum_{\substack{m_r, n_r \in S(\mathcal{I}_r), \forall r \in \mathcal{R} \\ g_w > G_w, \forall w \in \mathcal{W}}} \left(\prod_{r \in \mathcal{R}} \frac{d_k(m_r)d_k(n_r)}{m_r n_r} \right) \frac{1}{T} \int_T^{2T} \left(\prod_{r \in \mathcal{R}} \frac{n_r}{m_r} \right)^{it} dt \\ &= \sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w \\ \forall w \in \mathcal{W}}} \left(\prod_{w \in \mathcal{W}} \frac{d_k(m_w)d_k(n_w)}{m_w n_w} \right) \frac{1}{T} \int_T^{2T} \left(\prod_{w \in \mathcal{W}} \frac{n_w}{m_w} \right)^{it} |\zeta(1 + it; \mathcal{R} \setminus \mathcal{W})|^{2k} dt, \end{aligned}$$

which is bounded by

$$\sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w \\ \forall w \in \mathcal{W}}} \left(\prod_{w \in \mathcal{W}} \frac{d_k(m_w)d_k(n_w)}{m_w n_w} \right) \frac{1}{T} \int_T^{2T} |\zeta(1 + it; \mathcal{R} \setminus \mathcal{W})|^{2k} dt. \quad (2.4.19)$$

Observe that the integral does not depend on w , so we can change the order of sum and integral. Further, we have

$$\sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w \\ \forall w \in \mathcal{W}}} \left(\prod_{w \in \mathcal{W}} \frac{d_k(m_w)d_k(n_w)}{m_w n_w} \right) = \prod_{w \in \mathcal{W}} \sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w}} \frac{d_k(m_w)d_k(n_w)}{m_w n_w}. \quad (2.4.20)$$

For any multiplicative function f , we have $f(m)f(n) = f(\text{lcm}(m, n))f(\gcd(m, n))$. While the number of (m, n) such that $\gcd(m, n)/\text{lcm}(m, n) = g$, $\gcd(m, n) = h$ is $2^{\omega(g)}$. Thus we derive that

$$\begin{aligned} \sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w}} \frac{d_k(m_w)d_k(n_w)}{m_w n_w} &= \sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w}} \frac{d_k(g_w h_w)d_k(h_w)}{g_w h_w^2} \\ &= \sum_{\substack{m_w, n_w \in S(\mathcal{I}_w) \\ g_w > G_w}} \frac{2^{\omega(g_w)} d_k(g_w h_w)d_k(h_w)}{g_w h_w^2}. \end{aligned} \quad (2.4.21)$$

Therefore by Lemma 2.4.2, this is bounded by $D_k(\mathcal{W}) \exp(-(\log T)/(\log_2 T)^4)$.

Summarizing (2.4.19)-(2.4.21), we deduce that the second sum in (2.4.17) is bounded by

$$\sum_{\substack{\mathcal{W} \subset \mathcal{R} \\ \mathcal{W} \neq \emptyset}} D_k(\mathcal{W}) \left(\frac{1}{T} \int_T^{2T} |\zeta(1 + it; \mathcal{R} \setminus \mathcal{W})|^{2k} dt \right) \exp \left(- \frac{\log T}{(\log_2 T)^4} \right). \quad (2.4.22)$$

According to the induction hypothesis, it follows that

$$\frac{1}{T} \int_T^{2T} |\zeta(1 + it; \mathcal{R} \setminus \mathcal{W})|^{2k} dt \ll D_k(\mathcal{R} \setminus \mathcal{W}). \quad (2.4.23)$$

Then noticing that $D_k(\mathcal{W})D_k(\mathcal{R} \setminus \mathcal{W}) = D_k(\mathcal{R})$, (2.4.22) is bounded by

$$2^{L+1}D_k(\mathcal{R}) \exp\left(-\frac{\log T}{(\log_2 T)^4}\right) \ll D_k(\mathcal{R}) \exp\left(-\frac{\log T}{2(\log_2 T)^4}\right) \quad (2.4.24)$$

since $2^{L+1} \ll \exp(2 \log_2 k) \ll \exp(2 \log_3 T)$. Now the desired result follows from (2.4.18) and (2.4.24). \square

2.5 Proof of Theorem 2.1.1

Firstly we recall the definition of the short Euler products

$$\zeta(s; y) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1}$$

and the definition of its distribution function:

$$\Phi_T(\tau; y) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1 + it; y)| > e^\gamma \tau\}.$$

The aim of this section is to prove the following result.

Proposition 2.5.1. *Let $A > 0$ be any constant and let $J \geq 1$ be an integer, and ε satisfying (2.5.10). Then we have*

$$\Phi_T(\tau + \varepsilon; y) \leq \exp\left(-\frac{2e^{\tau - C_0 - 1}}{\tau} \left\{1 + \sum_{j=1}^J \frac{\alpha_j}{\tau^j} + O_J\left(\frac{1}{\tau^{J+1}} + \frac{e^\tau}{y}\right)\right\}\right) \leq \Phi_T(\tau - \varepsilon; y) \quad (2.5.1)$$

uniformly for

$$T \geq T_0(A), \quad e^2 \log T \leq y \leq (\log T)(\log_2 T)^A, \quad 2 \leq \tau \leq \log_2 T - 20, \quad (2.5.2)$$

where the α_j and C_0 are the same as in Theorem 2.1.1, $T_0(A)$ is a positive constant depending on A and the implied constant depends on A and J at most.

2.5.1 Two preliminary lemmas

In the following lemma, we will see the correlation between the distribution function and the moments of the short Euler products:

Lemma 2.5.2. *Let $A > 0$ be any constant and let $J \geq 1$ be an integer. Then we have*

$$\int_0^\infty \Phi_T(t; y) t^{2\kappa-1} dt = \frac{(\log \kappa)^{2\kappa}}{2\kappa} \exp\left(\frac{2\kappa}{\log \kappa} \left\{\sum_{j=0}^J \frac{C_j}{(\log \kappa)^j} + O_{A,J}\left(\frac{\kappa}{y} + \frac{1}{(\log \kappa)^{J+1}}\right)\right\}\right) \quad (2.5.3)$$

uniformly for

$$T \geq 2, \quad e^2 \log T \leq y \leq (\log T)(\log_2 T)^A, \quad 2 \leq \kappa \leq (\log T)/(e^{10} \log(y/\log T)), \quad (2.5.4)$$

where the C_j are defined as in (2.1.5) and the implied constant depends on A and J at most.

Proof. For any $\kappa > 0$, we have

$$\begin{aligned} \int_0^\infty \Phi_T(u; y) u^{2\kappa-1} du &= \frac{1}{T} \int_0^\infty \left(\int_{\substack{T \\ e^{-\gamma} |\zeta(1+it;y)| > u}}^{2T} 1 dt \right) u^{2\kappa-1} du \\ &= \frac{1}{T} \int_T^{2T} \left(\int_0^{e^{-\gamma} |\zeta(1+it;y)|} u^{2\kappa-1} du \right) dt \\ &= \frac{1}{T} \int_T^{2T} \frac{1}{2\kappa} (e^{-\gamma} |\zeta(1+it;y)|)^{2\kappa} dt, \end{aligned}$$

i.e.

$$2\kappa \int_0^\infty \Phi_T(t; y) t^{2\kappa-1} dt = \frac{e^{-2\kappa\gamma}}{T} \int_T^{2T} |\zeta(1+it;y)|^{2\kappa} dt. \quad (2.5.5)$$

Now (2.5.3) follows from Propositions 2.3.1 and 2.4.1 when κ is an integer.

Next let $\kappa \notin \mathbb{N}$ be a real number verifying (2.5.4). There is a unique integer k verifying (2.5.4) such that $k-1 < \kappa < k$. The formula (2.5.5) with $\kappa = \frac{1}{2}$ and [26, Theorem 3] imply that

$$\int_0^\infty \Phi_T(u; y) du = \frac{e^{-\gamma}}{T} \int_T^{2T} |\zeta(1+it;y)| dt \leq e^{-\gamma} \left(\frac{1}{T} \int_T^{2T} |\zeta(1+it;y)|^4 dt \right)^{1/4} \ll 1.$$

Now for any $b > a > 0$, by the Hölder inequality, it follows that

$$\int_0^\infty \Phi_T(t; y) t^a dt \leq \left(\int_0^\infty \Phi_T(t; y) dt \right)^{1-a/b} \left(\int_0^\infty \Phi_T(t; y) t^b dt \right)^{a/b}.$$

Thus there are two absolute positive constants C and D such that

$$\begin{aligned} \int_0^\infty \Phi_T(t; y) t^a dt &\leq C \left(\int_0^\infty \Phi_T(t; y) t^b dt \right)^{a/b}, \\ \int_0^\infty \Phi_T(t; y) t^b dt &\geq \left(D \int_0^\infty \Phi_T(t; y) t^a dt \right)^{b/a}. \end{aligned}$$

Applying the first inequality with $(a, b) = (2\kappa - 1, 2k - 1)$ and the second inequality with $(a, b) = (2k - 3, 2\kappa - 1)$ respectively, we can obtain that

$$\left(D \int_0^\infty \Phi_T(t; y) t^{2(k-1)-1} dt \right)^{\frac{2\kappa-1}{2k-3}} \leq \int_0^\infty \Phi_T(t; y) t^{2\kappa-1} dt \leq C \left(\int_0^\infty \Phi_T(t; y) t^{2k-1} dt \right)^{\frac{2\kappa-1}{2k-1}}.$$

On the other hand, setting $f(u) := \frac{2u}{\log u} \sum_{j=0}^J \frac{C_j}{(\log u)^j}$, then $f'(u) = -\frac{2}{\log u} \sum_{j=0}^J \frac{jC_j}{(\log u)^{j+1}}$. Thus

$$f(k-1) = f(\kappa) + O(1) \quad \text{and} \quad f(k) = f(\kappa) + O(1). \quad (2.5.6)$$

Now we can obtain (2.5.3) for $\kappa \notin \mathbb{N}$ by substituting (2.5.3) for integers $k-1$ and k and by using (2.5.6). This completes the proof of Lemma 2.5.2. \square

Lemma 2.5.3. Let $\{a_j\}_{j \geq 0}$ be a sequence of real numbers and $J \geq 0$ be an integer. If

$$\tau = \log \kappa + a_0 + \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} + O_J\left(\frac{1}{(\log \kappa)^{J+1}}\right) \quad (k \rightarrow \infty), \quad (2.5.7)$$

then there is a sequence of real numbers $\{b_j\}_{j \geq 0}$ such that

$$\log \kappa = \tau + b_0 + \sum_{j=1}^J \frac{b_j}{\tau^j} + O_J\left(\frac{1}{\tau^{J+1}}\right) \quad (\tau \rightarrow \infty). \quad (2.5.8)$$

Further we have $b_0 = -a_0$ and $b_1 = -a_1$.

Proof. We shall reason by recurrence on J . Taking $J = 0$ in (2.5.7), we have

$$\tau = \log \kappa + a_0 + O\left(\frac{1}{\log \kappa}\right) \quad (k \rightarrow \infty).$$

From this we easily deduce that

$$\log \kappa = \tau - a_0 + O\left(\frac{1}{\tau}\right) \quad (\tau \rightarrow \infty).$$

This is (2.5.8) with $J = 0$ and $b_0 = -a_0$. Suppose that

$$\tau = \log \kappa + \sum_{j=0}^{J+1} \frac{a_j}{(\log \kappa)^j} + O_J\left(\frac{1}{(\log \kappa)^{J+3}}\right) \quad (k \rightarrow \infty). \quad (2.5.9)$$

Clearly this implies (2.5.7). Thus according to the hypothesis of recurrence, (2.5.8) holds. Using (2.5.8) and (2.5.9); we can derive that

$$\begin{aligned} \log \kappa &= \tau - a_0 + \sum_{j=1}^{J+1} \frac{a_j}{(\log \kappa)^j} + O_J\left(\frac{1}{\tau^{J+2}}\right) \\ &= \tau - a_0 + \sum_{j=1}^{J+1} \frac{a_j}{\tau^j} \left\{ 1 + \sum_{d=1}^{J+2-j} \frac{b_{d-1}}{\tau^d} + O_J\left(\frac{1}{\tau^{J+2-j}}\right) \right\}^{-j} + O_J\left(\frac{1}{\tau^{J+2}}\right). \end{aligned}$$

This implies the required result via the Taylor development of $(1-t)^{-j}$. \square

2.5.2 Proof of Proposition 2.5.1

Let $\varepsilon \in [c(\log \kappa)^{-J-1}, 9c(\log \kappa)^{-J-1}]$ be a parameter to be chosen later, where c is a large constant. Without loss of generality, we can suppose

$$\varepsilon \leq (\log \kappa)^{-J}, \quad \varepsilon^2 \leq (\log \kappa)^{-J-1} \quad (2.5.10)$$

for $k \geq k_0$, where $\kappa_0 = \kappa_0(c)$ is a constant depending c . Put $K = \kappa e^\varepsilon$. Noticing that $(\frac{t}{\tau+\varepsilon})^{2K-2\kappa} \geq 1$ for $t \geq \tau + \varepsilon$, we have

$$2\kappa \int_{\tau+\varepsilon}^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt \leq (\tau + \varepsilon)^{2\kappa-2K} \left(2K \int_0^{\infty} \Phi_T(t; y) t^{2K-1} dt \right).$$

From this and Lemma 2.5.2, we deduce that

$$\frac{\int_{\tau+\varepsilon}^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt}{\int_0^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt} \leq \exp \left\{ 2(g(K, \tau) - g(\kappa, \tau)) + O_J \left(\frac{\kappa^2}{y} + \frac{\kappa}{(\log \kappa)^{J+3}} \right) \right\} \quad (2.5.11)$$

uniformly for (T, y, κ) in (2.5.4) above, where

$$g(\kappa, \tau) := -\kappa \log \left(\frac{\tau + \varepsilon}{\log \kappa} \right) + \frac{\kappa}{\log \kappa} \sum_{j=0}^{J+1} \frac{C_j}{(\log \kappa)^j}. \quad (2.5.12)$$

Let $\tau_0 = \tau_0(c, J)$ be a suitable constant depending on c and J . For $\tau_0 \leq \tau \leq \log_2 T - 20$, take $\kappa = \kappa_\tau$ such that

$$\tau = \log \kappa + a_0 + \sum_{j=1}^{J+1} \frac{a_j}{(\log \kappa)^j}, \quad (2.5.13)$$

where the $a_j = a_j(C_0, \dots, C_j)$ are constants to be determined later. Our choice of τ_0 guarantees that $\tau \geq \tau_0 \Rightarrow \kappa \geq \kappa_0$, which guarantees that

$$\varepsilon \leq 9c(\log \kappa)^{-J-1} \leq (\log \kappa)^{-J} \quad \text{and} \quad \varepsilon^2 \leq 81c^2(\log \kappa)^{-2J-2} \leq (\log \kappa)^{-2J-1}. \quad (2.5.14)$$

These bounds will be used often and all implied constants in the O -symbol is independent of c . In view of (2.5.13) and the Taylor formula, we can write

$$\begin{aligned} g(\kappa, \tau) &= -\kappa \log \left(1 + \frac{a_0 + \varepsilon}{\log \kappa} + \frac{1}{\log \kappa} \sum_{j=1}^{J+1} \frac{a_j}{(\log \kappa)^j} \right) + \frac{\kappa}{\log \kappa} \sum_{j=0}^{J+1} \frac{C_j}{(\log \kappa)^j} \\ &= -\kappa \left(\frac{a_0 - C_0 + \varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log \kappa)^2} + \sum_{j=2}^{J+1} \frac{a_j - a_j^* - C_j}{(\log \kappa)^{j+1}} \right) + O_J \left(\frac{1}{(\log \kappa)^{J+3}} \right), \end{aligned}$$

where the $a_j^* = a_j^*(a_0, \dots, a_{j-1})$ are constants depending on a_0, \dots, a_{j-1} . Take

$$a_0 = C_0 + 1, \quad a_1 = C_0^2 + C_0 + C_1 + 2, \quad a_j = a_j^*(a_0, \dots, a_{j-1}) + C_j \quad (2 \leq j \leq J+1).$$

Thus

$$g(\kappa, \tau) = -\kappa \left(\frac{1 + \varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log \kappa)^2} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right).$$

Let

$$\mathfrak{T} := \log K + a_0 + \sum_{j=1}^{J+1} \frac{a_j}{(\log K)^j},$$

then

$$g(K, \mathfrak{T}) = -K \left(\frac{1 + \varepsilon}{\log K} + \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log K)^2} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right).$$

From these, we easily deduce that

$$\begin{aligned} g(K, \mathfrak{T}) - g(\kappa, \tau) &= -K \left(\frac{1 + \varepsilon}{\log K} - \frac{1 + \varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log K)^2} - \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log \kappa)^2} \right) \\ &\quad - (K - \kappa) \left(\frac{1 + \varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1 + a_0 \varepsilon}{(\log \kappa)^2} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right). \end{aligned}$$

Using (2.5.14), a simple computation shows that

$$\begin{aligned}
g(K, \mathfrak{T}) - g(\kappa, \tau) &= K \left(\frac{(1+\varepsilon)\varepsilon}{(\log K) \log \kappa} + \frac{(a_1 - a_0^2 - C_1 + a_0\varepsilon)\varepsilon \log(K\kappa)}{(\log K)^2 (\log \kappa)^2} \right) \\
&\quad - (K - \kappa) \left(\frac{1+\varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1 + a_0\varepsilon}{(\log \kappa)^2} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right) \\
&= \frac{e^\varepsilon \varepsilon \kappa}{(\log \kappa)^2} - (e^\varepsilon - 1)\kappa \left(\frac{1+\varepsilon}{\log \kappa} + \frac{a_1 - a_0^2 - C_1}{(\log \kappa)^2} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right). \tag{2.5.15}
\end{aligned}$$

On the other hand, in view of (2.5.12), we have

$$\frac{\partial g}{\partial \tau}(\kappa, \tau) = -\frac{\kappa}{\tau + \varepsilon}.$$

Thus we have, for some $\eta_\kappa \in (\tau, \mathfrak{T})$,

$$\begin{aligned}
g(K, \tau) - g(K, \mathfrak{T}) &= \frac{\partial g}{\partial \tau}(K, \eta_\kappa)(\tau - \mathfrak{T}) \leq \frac{\varepsilon K}{\tau + \varepsilon} \left\{ 1 + O_J \left(\frac{1}{(\log \kappa)^2} \right) \right\} \\
&= \frac{\varepsilon e^\varepsilon \kappa}{\log \kappa} \left(1 - \frac{a_0}{\log \kappa} \right) + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right). \tag{2.5.16}
\end{aligned}$$

Writing

$$g(K, \tau) - g(\kappa, \tau) = g(K, \mathfrak{T}) - g(\kappa, \tau) + g(K, \tau) - g(K, \mathfrak{T})$$

and using (2.5.15) and (2.5.16), we can derive that

$$\begin{aligned}
&g(K, \tau) - g(\kappa, \tau) \\
&\leq -((a_1 - a_0^2 - C_1)(e^\varepsilon - 1) + C_0 \varepsilon e^\varepsilon) \frac{\kappa}{(\log \kappa)^2} - (e^\varepsilon - 1 - \varepsilon) \frac{\kappa}{\log \kappa} + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right) \\
&\leq -(a_1 - a_0^2 - C_1 + C_0) \frac{\varepsilon \kappa}{(\log \kappa)^2} + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right) \\
&= -\frac{\varepsilon \kappa}{(\log \kappa)^2} + O_J \left(\frac{\kappa}{(\log \kappa)^{J+3}} \right),
\end{aligned}$$

thanks to the choice of $a_1 = a_0^2 + C_1 - C_0 + 1 = C_0^2 + C_0 + C_1 + 2$. Thus the inequality (2.5.11) can be written as

$$\frac{\int_{\tau+\varepsilon}^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt}{\int_0^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt} \leq \exp \left\{ -\frac{2\varepsilon \kappa}{(\log \kappa)^2} + O_J \left(\frac{\kappa^2}{y} + \frac{\kappa}{(\log \kappa)^{J+3}} \right) \right\}$$

for $\tau_0 \leq \tau \leq \log_2 T - 20$ and $\kappa = \kappa_\tau$. This implies that

$$\int_{\tau+\varepsilon}^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt \leq \frac{1}{4} \int_0^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt, \tag{2.5.17}$$

provided the constant c is suitably large and $y \geq \kappa (\log \kappa)^{J+3}$. Similarly

$$\int_0^{\tau-\varepsilon} \Phi_T(t; y) t^{2\kappa-1} dt \leq \frac{1}{4} \int_0^{\infty} \Phi_T(t; y) t^{2\kappa-1} dt. \tag{2.5.18}$$

From (2.5.17) and (2.5.18), we deduce that

$$\frac{1}{2} \int_0^\infty \Phi_T(t; y) t^{2\kappa-1} dt \leq \int_{\tau-\varepsilon}^{\tau+\varepsilon} \Phi_T(t; y) t^{2\kappa-1} dt \leq \int_0^\infty \Phi_T(t; y) t^{2\kappa-1} dt.$$

Combining this with Lemma 2.5.2 leads to

$$\begin{aligned} & \int_{\tau-\varepsilon}^{\tau+\varepsilon} \Phi_T(t; y) t^{2\kappa-1} dt \\ &= \frac{(\log \kappa)^{2\kappa}}{2\kappa} \exp \left(\frac{2\kappa}{\log \kappa} \left\{ \sum_{j=0}^J \frac{C_j}{(\log \kappa)^j} + O_J \left(\frac{\kappa}{y} + \frac{1}{(\log \kappa)^{J+1}} \right) \right\} \right) \end{aligned} \quad (2.5.19)$$

uniformly for (T, y, κ) in (2.5.3) above and (2.5.4).

On the other hand, in view of the fact that $\Phi_T(t; y)$ is decreasing in t , we have

$$\Phi_T(\tau + \varepsilon; y)(\tau - \varepsilon)^{2\kappa-1} \leq 2k \int_{\tau-\varepsilon}^{\tau+\varepsilon} \Phi_T(t; y) t^{2\kappa-1} dt \leq \Phi_T(\tau - \varepsilon; y)(\tau + \varepsilon)^{2\kappa-1} \quad (2.5.20)$$

Since $\tau = \log \kappa + \sum_{j=0}^{J+1} a_j / (\log \kappa)^j$ and $\varepsilon \asymp (\log \kappa)^{-J-1}$, it follows that

$$\begin{aligned} \left(\frac{\tau \pm \varepsilon}{\log \kappa} \right)^{-2k} \frac{\tau \pm \varepsilon}{2\kappa} &= \exp \left(-2\kappa \log \left\{ 1 + \sum_{j=0}^J \frac{a_j}{(\log \kappa)^{j+1}} + O \left(\frac{1}{(\log \kappa)^{J+3}} \right) \right\} \right) \\ &= \exp \left(-\frac{2\kappa}{\log \kappa} \left\{ a_0 + \sum_{j=1}^J \frac{\tilde{a}_j}{(\log \kappa)^j} + O \left(\frac{1}{(\log \kappa)^{J+1}} \right) \right\} \right). \end{aligned} \quad (2.5.21)$$

where the \tilde{a}_j are constants ($\tilde{a}_1 = a_1 + a_0^2$). From (2.5.19)–(2.5.21), we can deduce that

$$\Phi_T(\tau + \varepsilon; y) \leq \exp \left(-\frac{2\kappa}{\log \kappa} \left\{ 2 + \sum_{j=1}^J \frac{\tilde{a}_j - C_j}{(\log \kappa)^j} + O_J \left(\frac{\kappa}{y} + \frac{1}{(\log \kappa)^{J+1}} \right) \right\} \right) \leq \Phi_T(\tau - \varepsilon; y).$$

Since $\tau = \log \kappa + \sum_{j=0}^J a_j / (\log \kappa)^j$, we can apply Lemma 2.4.2 to write

$$\frac{2\kappa}{\log \kappa} = \frac{2e^{\tau-C_0-1+\sum_{j=1}^J b_j/\tau^j}}{\tau + \sum_{j=0}^J b_j/\tau^j} = \frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + \sum_{j=1}^J \frac{b'_j}{\tau^j} + O \left(\frac{1}{\tau^{J+1}} \right) \right\}$$

and

$$\sum_{j=1}^J \frac{\tilde{a}_j - C_j}{(\log \kappa)^j} = \sum_{j=1}^J \frac{\tilde{a}_j - C_j}{(\tau + \sum_{\ell=0}^J b_\ell/\tau^\ell)^j} = \sum_{j=1}^J \frac{c_j}{\tau^j} + O \left(\frac{1}{\tau^{J+1}} \right).$$

Combining (2.5.19)–(2.5.21), we obtain

$$\Phi_T(\tau + \varepsilon; y) \leq \exp \left(-\frac{2e^{\tau-C_0-1}}{\tau} \left\{ 1 + \sum_{j=1}^J \frac{a_j}{\tau^j} + O_J \left(\frac{1}{\tau^{J+1}} + \frac{e^\tau}{y} \right) \right\} \right) \leq \Phi_T(\tau - \varepsilon; y) \quad (2.5.22)$$

with

$$\begin{aligned} a_1 &= 2b'_1 + c_1 = 2(b_1 - b_0) + \tilde{a}_1 - C_1 = 2(-a_1 + a_0) + a_1 + a_0^2 - C_1 \\ &= 2a_0 - a_1 + a_0^2 - C_1 = 2a_0 - (a_0^2 + C_1) + a_0^2 - C_1 = 2a_0 - 2C_1 = 2(1 + C_0 - C_1). \end{aligned}$$

2.5.3 End of the proof of Theorem 2.1.1

By Lemma 2.2.4, we can derive that

$$\Phi_T(\tau) = \Phi_T(\tau + O(\varepsilon + \eta); y) + O(\exp(-(\log T)/(50 \log_2 T))) \quad (2.5.23)$$

with $\eta := \sqrt{(\log T)/y}$. Combining (2.5.22) and (2.5.23), we can obtain

$$\begin{aligned} & \Phi_T(\tau) \\ &= \exp\left(-\frac{2e^{\tau-C_0-1}}{\tau}\left\{1 + \sum_{j=1}^J \frac{a_j}{\tau^j} + O_J\left(\frac{1}{\tau^{J+1}} + \sqrt{\frac{\log T}{y}}\right)\right\}\right) + O\left(\exp\left(-\frac{\log T}{50 \log_2 T}\right)\right). \end{aligned}$$

This implies the required result by choosing $y = \min\{(\log T)\tau^{2J+2}, (\log T)^2/e^{10+\tau}\}$. \square

3 Distribution of $|\zeta(\sigma + it)|$ in the strip

$$\frac{1}{2} < \sigma < 1$$

3.1 Background

Throughout this section, σ will denote any fixed number in $(\frac{1}{2}, 1)$, $\zeta(s)$ the Riemann zeta function and \log_j the j -th iterated logarithm. Firstly we make a brief review of the extreme values of $|\zeta(\sigma + it)|$ as t varies. In 1928, Titchmarsh [50] showed that for any $\varepsilon > 0$, we have

$$\limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{(\log t)^{1-\sigma-\varepsilon}} = \infty.$$

In 1972, Levinson [34] replaced $(\log t)^\varepsilon$ by $\log_2 t$, by showing that for sufficiently large T we have

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \gg \frac{(\log T)^{1-\sigma}}{\log_2 T}.$$

In 1977, Montgomery [41] showed that

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \geq \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}, \quad (3.1.1)$$

where $\nu(\sigma) = \frac{1}{20}(\sigma - \frac{1}{2})^{\frac{1}{2}}$ unconditionally, and $\nu(\sigma) = \frac{1}{20}$ on assuming the Riemann hypothesis. This quantity $(\log T)^{1-\sigma}/(\log_2 T)^\sigma$ is conjectured to be the true order of magnitude of $\max_{t \in [0, T]} \log |\zeta(\sigma + it)|$. More precisely, we believe the following inequality holds:

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \asymp_\sigma \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}.$$

Thus, the only improvement of (3.1.1) we could expect is to get larger values of $\nu(\sigma)$. We refer to [1, 14].

In 2011, applying a method of Granville and Soundararajan [26] to investigate the distribution of values of $|\zeta(1 + it)|$, Lamzouri [33] studied the distribution of large values of $|\zeta(\sigma + it)|$ as t varies in $[T, 2T]$. Let T be sufficiently large. Define the distribution function by

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : \log |\zeta(\sigma + it)| > \tau\}. \quad (3.1.2)$$

Then he [33] showed that there exists a positive constant $c(\sigma)$ such that

$$\Phi_T(\tau) = \exp\left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \mathfrak{a}_0 + O\left(\frac{1}{\sqrt{\log \tau}} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T}\right)^{\sigma - \frac{1}{2}}\right)\right\}\right) \quad (3.1.3)$$

uniformly in the range $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma}/\log_2 T$, where \mathfrak{a}_0 will be defined later in (3.1.4). Despite the maximum of the range of τ being much less than (3.1.1), the distribution

function (3.1.3) has more significance. If (3.1.3) were to persist to the end of the viable range, then we could get a conjectural value of $\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)|$. More precisely, we have Lamzouri's conjecture (see [33]):

$$\max_{t \in [T, 2T]} \log |\zeta(\sigma + it)| = \{c(\sigma) + o(1)\} \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}$$

holds for $T \rightarrow \infty$, where

$$c(\sigma) := \frac{C_0}{\sigma^{2\sigma}(1-\sigma)^{1-\sigma}}$$

and C_0 will be defined in (3.3.2). Note that this conjecture also implies the upper bound of $|\zeta(\sigma + it)|$. For more work concerning it, we refer to [18, 19, 24, 44, 51].

In this section, we aim to improve the asymptotic distribution function (3.1.3). We have a higher order expansion in the exponent, which is inspired by the work in [54].

Theorem 3.1.1. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed real number. Let $\Phi_T(\tau)$ be defined in (3.1.2). Then there exists a sequence of polynomials with real coefficients $\{\mathbf{a}_n(\cdot)\}_{n \geq 0}$ with $\deg(\mathbf{a}_n) \leq n$, and a constant $c(\sigma) > 0$, such that for any integer $N \geq 1$, we have*

$$\Phi_T(\tau) = \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathbf{a}_n(\log_2 \tau)}{(\log \tau)^n} + O \left(\left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{\log T} \right)^{\sigma-\frac{1}{2}} \right) \right\} \right)$$

uniformly for $T \rightarrow \infty$ and $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma}/\log_2 T$, where the implied constant depend on N and σ . Especially, we have

$$\mathbf{a}_0 := \left(\frac{\sigma^{2\sigma}}{C_0^\sigma (1-\sigma)^{2\sigma-1}} \right)^{1/(1-\sigma)} \quad (3.1.4)$$

with C_0 defined in (3.3.2).

The main new ingredient for the proof of Theorem 3.1.1 is Proposition 3.4.1 below, which gives a better approximation of the distribution function of the short Euler products:

$$\Phi_T(\tau; y) := \frac{1}{T} \text{meas} \{ t \in [T, 2T] : \log |\zeta(\sigma + it; y)| > \tau \}, \quad (3.1.5)$$

where

$$\zeta(\sigma + it; y) := \prod_{p \leq y} \left(1 - \frac{1}{p^{\sigma+it}} \right)^{-1}.$$

We refer to [39] for similar work on L -functions attached to cusp forms.

3.2 Preliminary lemmas

Firstly, we will show the relationship between sums attached to the divisor function and the Bessel function by two asymptotic formulas. These will be used in the progress of calculating the moments of the short Euler products for the Riemann zeta function and the Dirichlet L -functions. One should pay attention that here k is not necessarily an integer.

The modified Bessel function $I_0(t)$ of order 0 is defined by

$$I_0(t) := \int_0^1 \exp(t \cos(2\pi\theta)) d\theta = \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2}. \quad (3.2.1)$$

It's not difficult to see that

$$\log I_0(t) \ll t^2 \quad (0 \leq t < 1), \quad (3.2.2)$$

$$\log I_0(t) \ll t \quad (t \geq 1), \quad (3.2.3)$$

$$(\log I_0(t))' \ll \min\{1, |t|\}. \quad (3.2.4)$$

Lemma 3.2.1. Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed real number. For any prime p and positive number k , we have

$$\sum_{\nu \geq 0} \frac{d_{k/2}(p^\nu)^2}{p^{2\nu\sigma}} = I_0\left(\frac{k}{p^\sigma}\right) \exp\left\{O_\sigma\left(\frac{k}{p^{2\sigma}}\right)\right\}, \quad (3.2.5)$$

$$\sum_{\nu \geq 0} \frac{d_{k/2}(p^\nu)^2}{p^{2\nu\sigma}} = \exp\left\{O_\sigma\left(\frac{k}{p^\sigma}\right)\right\} \quad (p \leq k^{1/\sigma}), \quad (3.2.6)$$

where the implied constants depend on σ only.

Proof. See also of [26, Lemma 4]. Writing $e(\theta) := e^{2\pi i\theta}$, then

$$\begin{aligned} \left|1 - \frac{e(\theta)}{p^\sigma}\right|^{-k} &= \left(1 - \frac{e(\theta)}{p^\sigma}\right)^{-k/2} \left(1 - \frac{e(-\theta)}{p^\sigma}\right)^{-k/2} \\ &= \sum_{\nu \geq 0} \sum_{\nu' \geq 0} \frac{d_{k/2}(p^\nu) d_{k/2}(p^{\nu'}) e((\nu - \nu')\theta)}{p^{(\nu + \nu')\sigma}}. \end{aligned}$$

Thus we can derive that

$$\begin{aligned} \sum_{\nu \geq 0} \frac{d_{k/2}(p^\nu)^2}{p^{2\nu\sigma}} &= \int_0^1 \left|1 - \frac{e(\theta)}{p^\sigma}\right|^{-2(k/2)} d\theta \\ &= \int_0^1 \left(1 - \frac{2\cos(2\pi\theta)}{p^\sigma} + \frac{1}{p^{2\sigma}}\right)^{-k/2} d\theta \\ &= \int_0^1 \exp\left(-\frac{k}{2} \log\left(1 - \frac{2\cos(2\pi\theta)}{p^\sigma} + \frac{1}{p^{2\sigma}}\right)\right) d\theta. \end{aligned}$$

This implies (3.2.5) thanks to the formula $\log(1 + t) = t + O(t^2)$ ($|t| \leq 2^{-\frac{1}{2}}$), and (3.2.6) follows from (3.2.5) and (3.2.3) immediately. \square

Lemma 3.2.2. We have

$$\sum_{p \leq x} \frac{1}{p^\sigma} = \frac{x^{1-\sigma}}{(1-\sigma)\log x} + O\left(\frac{x^{1-\sigma}}{(1-\sigma)^2(\log x)^2}\right)$$

uniformly for $x \rightarrow \infty$ and $\frac{1}{2} < \sigma < 1$, where the implied constant is absolute.

Proof. This is equation (2.1) of [33]. See also [14, Lemma 6], [43, Lemma 3.1], and [8, Lemma 3.3]. \square

We need to approximate Riemann zeta function $\zeta(s)$ by its short Euler product. The following lemma shows that when $\zeta(s)$ has no zero in a good region, it can be approximated well by its short Euler product.

Lemma 3.2.3. *Let $\sigma_0 \in [\frac{1}{2}, 1)$ be a fixed number. Let $y \geq 2$ and $|t| \geq y+3$ be real numbers and suppose that the rectangle $\{z : \sigma_0 < \Re z \leq 1 \text{ and } |\Im z - t| \leq y+2\}$ is free of zeros of $\zeta(z)$. Then for any $\sigma_0 < \sigma \leq 2$ and $|\xi - t| \leq y$, we have*

$$|\log \zeta(\sigma + i\xi)| \ll (\log |t|) \log(e/(\sigma - \sigma_0)).$$

Further for $\sigma_0 < \sigma \leq 1$, we have

$$\log \zeta(\sigma + it) = \sum_{n=2}^y \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O\left(\frac{\log |t|}{(\sigma_1 - \sigma_0)^2} y^{\sigma_1 - \sigma}\right),$$

where $\sigma_1 := \min(\sigma_0 + (\log y)^{-1}, \frac{1}{2}(\sigma + \sigma_0))$. The implied constants depend on σ_0 at most.

Proof. See [26, Lemma 1]. \square

With the help of Lemma 3.2.3, as well as a result of zero density estimate for the Riemann zeta-function $\zeta(s)$, we can approximate $\zeta(s)$ by its short Euler product mostly often. Of course, here the short Euler product is a bit “long”, that means y needs to be relatively large. Otherwise, the error term will be too large to make sense.

Lemma 3.2.4. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed number and $0 < a(\sigma) < \frac{1}{2}(\sigma - \frac{1}{2}) < 2/(\sigma - \frac{1}{2}) < A(\sigma)$. Then for*

$$T \rightarrow \infty \quad \text{and} \quad (\log T)^{A(\sigma)} \leq y \leq T^{a(\sigma)}$$

the asymptotic formula

$$\log \zeta(\sigma + it) = \sum_{n=2}^y \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O(y^{-\frac{1}{2}(\sigma - \frac{1}{2})} (\log y)^2 \log T)$$

holds for all $t \in [T, 2T]$ except for a set of measure at most $O(T^{1-\frac{1}{2}(\sigma - \frac{1}{2})} y (\log T)^5)$, where the implied constants depend on σ at most.

Proof. This is essentially [26, Lemma 2] while we restrict $(\log T)^{A(\sigma)} \leq y \leq T^{a(\sigma)}$ such that both the error term $O(y^{-\frac{1}{2}(\sigma - \frac{1}{2})} (\log y)^2 \log T)$ and the measure $T^{1-\frac{1}{2}(\sigma - \frac{1}{2})} y (\log T)^5$ make sense. We replace the term $O(y^{-\frac{1}{2}(\sigma - \frac{1}{2})} (\log T)^3)$ in [26, Lemma 2] by $O(y^{-\frac{1}{2}(\sigma - \frac{1}{2})} (\log y)^2 \log T)$. The proof has no difference from that of [26, Lemma 2]. \square

In order to approximate $\zeta(s)$ by its “shorter” Euler product, we need the following moment evaluation for the sum over complex power of primes between two large numbers y and z , where y can be relatively smaller.

Lemma 3.2.5. Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed number. Then we have

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{\sigma+it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} \frac{1}{p^{2\sigma}} \right)^k + \frac{1}{T^{\frac{1}{3}}}$$

for $2 \leq y \leq z$ and all integers $1 \leq k \leq (\log T)/(3 \log z)$, where the implied constant depends on σ at most.

Proof. This is [33, Lemma 4.2]. \square

Using Lemma 2.4, we can give a generalization of Lemma 2.3. Here y can be as small as $\log T$.

Lemma 3.2.6. Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed number, and let $c_j(\sigma)$ be some suitable positive constants depending on σ . Let $T \geq 2$, $\log T \leq y \leq (\log T)^{2/(\sigma-\frac{1}{2})}$ and $c_1(\sigma)(\log_2 T / \log T)^2 \leq \lambda \leq (\log T)^{\frac{1}{2}} / (y^{\sigma-\frac{1}{2}} \log_2 T)$. Then we have

$$|\log \zeta(\sigma + it) - \log \zeta(\sigma + it; y)| \leq 2\lambda$$

for all $t \in [T, 2T]$ except for a set of measure at most $O(T \exp(-4e^{-1}(\sigma - \frac{1}{2})\lambda^2 y^{2\sigma-1} \log y))$.

Proof. Noticing that

$$\sum_{p \leq z, p^\nu > z} \frac{1}{\nu p^{\nu\sigma}} \leq \sum_{p \leq z, \nu \geq 2} \frac{(p^\nu/z)^{\sigma-\frac{1}{2}}}{\nu p^{\nu\sigma}} = \frac{1}{z^{\sigma-\frac{1}{2}}} \sum_{p \leq z, \nu \geq 2} \frac{1}{\nu p^{\nu/2}} \ll \frac{1}{z^{\sigma-\frac{1}{2}}} \sum_{p \leq z} \frac{1}{p} \ll \frac{\log_2 z}{z^{\sigma-\frac{1}{2}}},$$

we can write

$$\begin{aligned} \sum_{2 \leq n \leq z} \frac{\Lambda(n)}{n^{\sigma+it} \log n} &= \sum_{p^\nu \leq z} \frac{1}{\nu p^{\nu(\sigma+it)}} = \sum_{p \leq z} \sum_{\nu \geq 1} \frac{1}{\nu p^{\nu(\sigma+it)}} + O\left(\frac{\log_2 z}{z^{\sigma-\frac{1}{2}}}\right) \\ &= \log \zeta(1 + it; z) + O\left(\frac{\log_2 z}{z^{\sigma-\frac{1}{2}}}\right). \end{aligned}$$

Using this and Lemma 3.2.4 with $y = z = (\log T)^{6/(\sigma-\frac{1}{2})}$, we obtain

$$\log \zeta(1 + it; z) + O\left(\frac{\log_2 z}{z^{\sigma-\frac{1}{2}}}\right) = \log \zeta(1 + it) + O\left(\frac{(\log z)^2 \log T}{z^{\frac{1}{2}(\sigma-\frac{1}{2})}}\right)$$

i.e.

$$\zeta(1 + it) = \zeta(1 + it; z) \left\{ 1 + O\left(\left(\frac{\log_2 T}{\log T}\right)^2\right) \right\} \quad (3.2.7)$$

for all $t \in [T, 2T]$ but at most a set of measure of

$$T^{1-\frac{1}{2}(\sigma-\frac{1}{2})} z (\log T)^5 \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}. \quad (3.2.8)$$

Then we use Lemma 3.2.5 to approximate $\zeta(\sigma + it; z)$ by $\zeta(\sigma + it; y)$ since

$$\zeta(\sigma + it; z) = \zeta(\sigma + it; y) \exp \left(\sum_{y \leq p \leq z} \left\{ \frac{1}{p^{\sigma+it}} + O\left(\frac{1}{p^{2\sigma}}\right) \right\} \right).$$

Choosing

$$k = \lfloor (4e^{-1}(\sigma - \frac{1}{2})\lambda^2 y^{2\sigma-1} \log y) \rfloor,$$

which satisfies the condition in Lemma 3.2.5, then by this lemma we have

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{\sigma+it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} \frac{1}{p^{2\sigma}} \right)^k + \frac{1}{T^{\frac{1}{3}}} \ll \left(\frac{k}{(\sigma - \frac{1}{2})y^{2\sigma-1} \log y} \right)^k + \frac{1}{T^{\frac{1}{3}}}.$$

So the frequency of $t \in [T, 2T]$ such that $|\log \zeta(\sigma + it; z) - \log \zeta(\sigma + it; y)| > 2\lambda$ is less than

$$\frac{1}{T} \int_T^{2T} \left| \frac{1}{2\lambda} \sum_{y \leq p \leq z} \frac{1}{p^{\sigma+it}} \right|^{2k} dt \ll \left(\frac{k}{4(\sigma - \frac{1}{2})\lambda^2 y^{2\sigma-1} \log y} \right)^k + (2\lambda)^{-2k} T^{-\frac{1}{3}}. \quad (3.2.9)$$

Since $\lambda > c_1(\sigma)(\log_2 T / \log T)^2$, we have

$$\begin{aligned} & |\log \zeta(\sigma + it) - \log \zeta(\sigma + it; y)| \\ & \geq |\log \zeta(\sigma + it; z) - \log \zeta(\sigma + it; y)| - |\log \zeta(\sigma + it; z) - \log \zeta(\sigma + it)| \\ & \geq 2\lambda + O((\log_2)^2 / (\log T)^2) > \lambda. \end{aligned}$$

By (3.2.8) and (3.2.9), the frequency of $t \in [T, 2T]$ such that $|\log \zeta(\sigma + it) - \log \zeta(\sigma + it; y)| > 2\lambda$ is less than, thanks to our choice of k ,

$$\begin{aligned} & \ll \left(\frac{k}{4(\sigma - \frac{1}{2})\lambda^2 y^{2\sigma-1} \log y} \right)^k + \frac{1}{(2\lambda)^{2k} T^{\frac{1}{3}}} + \frac{1}{T^{\frac{1}{4}(\sigma - \frac{1}{2})}} \\ & \ll e^{-k} + (2\lambda)^{-2k} T^{-\frac{1}{4}(\sigma - \frac{1}{2})}. \end{aligned}$$

This implies the required result, since our hypothesis on (λ, y) garanties

$$(2\lambda)^{-2k} T^{-\frac{1}{4}(\sigma - \frac{1}{2})} \leq T^{-\frac{1}{8}(\sigma - \frac{1}{2})} \leq \exp(-4e^{-1}(\sigma - \frac{1}{2})\lambda^2 y^{2\sigma-1} \log y).$$

Combining this with the first step, Lemma 3.2.6 follows. \square

3.3 Moments of the short Euler products

In this section, we will evaluate the k -th moment of the short Euler product $\zeta(\sigma + it; y)$ by proving the following proposition, which is important for the proof of Theorem 3.1.1. It has a higher order expansion in the exponent, which is an improvement of equation (4.2) in [33].

Proposition 3.3.1. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed constant and let N be a non-negative integer. Then we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it; y)|^k dt = \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^N \frac{C_n}{(\log k)^n} + O \left(\frac{1}{(\log k)^{N+1}} + \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} \right) \right\} \right)$$

uniformly for

$$T \geq 3 \quad \text{and} \quad ky^{1-\sigma} \leq \frac{1}{8}(1-\sigma) \log T, \quad (3.3.1)$$

where

$$C_n := \int_0^\infty \frac{(\log t)^n}{t^{1/\sigma+1}} \log I_0(t) dt \quad (n \geq 0) \quad (3.3.2)$$

and $I_0(t)$ is the Bessel function given by (3.2.1). Especially, we have $C_0 > 0$.

The integer $n \geq 1$ is called y -friable if the largest prime factor $P(n)$ of n is less than y ($P(1) = 1$ by convention). Denote by $S(y)$ the set of y -friable integers. We will show that, in the expansion of the k -th moment of $\zeta(\sigma + it; y)$, the diagonal terms lead to the main term, while the off-diagonal terms contribute to the error term. Again we strengthen that, k is not necessarily an integer.

Lemma 3.3.2. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed constant. Then we have*

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it; y)|^k dt = \sum_{n \in S(y)} \frac{d_{k/2}(n)^2}{n^{2\sigma}} + O\left(\exp\left(-\frac{\log T}{4 \log y}\right)\right),$$

uniformly for (T, y, k) in (3.3.1), where the implied constant depends on σ only.

Proof. This is a special case of Proposition 4.1 of [33]. \square

Now we are ready to prove Proposition 3.3.1.

Proof of Proposition 3.3.1. In view of Lemma 3.3.2, it is sufficient to show that

$$\sum_{n \in S(y)} \frac{d_{k/2}(n)^2}{n^{2\sigma}} = \exp\left(\frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^N \frac{C_n}{(\log k)^n} + O\left(\frac{1}{(\log k)^{N+1}} + \left(\frac{k^{1/\sigma}}{y}\right)^{2\sigma-1}\right) \right\}\right). \quad (3.3.3)$$

Firstly, we note that (3.3.3) is trivial if $y \leq k^{1/\sigma}$. In fact, since the divisor function is multiplicative, by (3.2.6) of Lemma 3.2.1 we have

$$\sum_{n \in S(y)} \frac{d_{k/2}(n)^2}{n^{2\sigma}} = \prod_{p \leq y} \sum_{\nu \geq 0} \frac{d_{k/2}(p^\nu)^2}{p^{2\nu\sigma}} = \exp\left\{O\left(\sum_{p \leq k^{1/\sigma}} \frac{k}{p^\sigma}\right)\right\} = \exp\left\{O\left(\frac{k^{1/\sigma}}{\log k}\right)\right\}.$$

Now we treat the case of $y > k^{1/\sigma}$. As before, by Lemma 3.2.1 we can write

$$\begin{aligned} \sum_{n \in S(y)} \frac{d_{k/2}(n)^2}{n^{2\sigma}} &= \prod_{p \leq k^{1/(2\sigma)}} \exp\left\{O\left(\frac{k}{p^\sigma}\right)\right\} \prod_{k^{1/(2\sigma)} < p \leq y} I_0\left(\frac{k}{p^\sigma}\right) \exp\left\{O\left(\frac{k}{p^{2\sigma}}\right)\right\} \\ &= \exp\left\{O_{\sigma, N}\left(\frac{k^{1/\sigma}}{(\log k)^{N+2}}\right)\right\} \prod_{k^{1/(2\sigma)} < p \leq y} I_0\left(\frac{k}{p^\sigma}\right), \end{aligned} \quad (3.3.4)$$

where the last equation holds since

$$\sum_{p \leq k^{1/(2\sigma)}} \frac{k}{p^\sigma} \ll k \frac{(k^{1/(2\sigma)})^{1-\sigma}}{\log k^{1/(2\sigma)}} \ll \frac{k^{\frac{1}{2}+1/(2\sigma)}}{\log k} \ll_{\sigma, N} \frac{k^{1/\sigma}}{(\log k)^{N+2}}$$

and

$$\sum_{k^{1/(2\sigma)} < p \leq y} \frac{k}{p^{2\sigma}} \ll \frac{k^{1/(2\sigma)}}{\log k} \ll_{\sigma, N} \frac{k^{1/\sigma}}{(\log k)^{N+2}}.$$

Next we evaluate the second factor on the right-hand side de (3.3.4). Taking the logarithm of this factor and using the prime number theorem, we have

$$\log \prod_{k^{1/(2\sigma)} < p \leq y} I_0\left(\frac{k}{p^\sigma}\right) = \int_{k^{1/(2\sigma)}}^y \log I_0\left(\frac{k}{u^\sigma}\right) d\pi(u) = \mathcal{M} + \mathcal{E}, \quad (3.3.5)$$

where

$$\mathcal{M} := \int_{k^{1/(2\sigma)}}^y \log I_0\left(\frac{k}{u^\sigma}\right) \frac{du}{\log u}, \quad \mathcal{E} := \int_{k^{1/(2\sigma)}}^y \log I_0\left(\frac{k}{u^\sigma}\right) dO(ue^{-c\sqrt{\log u}}).$$

In view of (3.2.2) and (3.2.3), we always have $\log I_0(t) \ll t^2$ ($t \geq 0$). Thus using this bound and (3.2.4), we can derive that

$$\begin{aligned} \mathcal{E} &= \log I_0\left(\frac{k}{u^\sigma}\right) O(ue^{-c\sqrt{\log u}}) \Big|_{k^{1/(2\sigma)}}^y - \int_{k^{1/2\sigma}}^y \left(\log I_0\left(\frac{k}{u^\sigma}\right) \right)' O(ue^{-c\sqrt{\log u}}) du \\ &\ll \left(\frac{k}{y^\sigma}\right)^2 \frac{y}{e^{c'\sqrt{\log y}}} + \frac{k^{\frac{1}{2}+1/(2\sigma)}}{e^{c'\sqrt{\log k}}} + k \int_{k^{1/(2\sigma)}}^{k^{1/\sigma}} \frac{e^{-c\sqrt{\log u}}}{u^\sigma} du + k^2 \int_{k^{1/\sigma}}^y \frac{e^{-c\sqrt{\log u}}}{u^{2\sigma}} du \\ &\ll \left(\frac{k^{1/\sigma}}{y}\right)^{2\sigma-1} k^{1/\sigma} e^{-c'\sqrt{\log y}} + k^{\frac{1}{2}+1/(2\sigma)} e^{-c'\sqrt{\log k}} + k^{1/\sigma} e^{-c'\sqrt{\log k}} \\ &\ll \frac{k^{1/\sigma}}{\log k} \left(\left(\frac{k^{1/\sigma}}{y}\right)^{2\sigma-1} \frac{\log k}{e^{c'\sqrt{\log y}}} + \frac{k^{-(1/\sigma-1)/2} \log k}{e^{c'\sqrt{\log k}}} \right). \end{aligned} \tag{3.3.6}$$

This is acceptable, since $y \geq k^{1/\sigma}$.

In order to calculate the main term of (3.3.5), setting $t = k/u^\sigma$, and integrating by substitution, then we have

$$\mathcal{M} = k^{1/\sigma} \int_{k/y^\sigma}^{k^{1/2}} \frac{\log I_0(t)}{t^{1/\sigma+1} \log(k/t)} dt = \frac{k^{1/\sigma}}{\log k} \int_{k/y^\sigma}^{k^{1/2}} \frac{\log I_0(t)}{t^{1/\sigma+1}} \frac{1}{1 - \log t / \log k} dt.$$

For $k/y^\sigma \leq t \leq k^{1/2}$, we can write

$$\frac{1}{1 - \log t / \log k} = \sum_{n=0}^N \frac{(\log t)^n}{(\log k)^n} + O_{\sigma, N}\left(\frac{(\log t)^{N+1}}{(\log k)^{N+1}}\right).$$

Thus

$$\mathcal{M} = \frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^N \frac{C_n(k, y)}{(\log k)^n} + O\left(\frac{1}{(\log k)^{N+1}}\right) \right\},$$

where

$$C_n(k, y) := \int_{k/y^\sigma}^{k^{1/2}} \frac{(\log t)^n}{t^{1/\sigma+1}} \log I_0(t) dt$$

and we have used (3.2.2)-(3.2.3) to bound

$$\int_{k/y^\sigma}^{k^{1/2}} \frac{(\log t)^{N+1}}{t^{1/\sigma+1}} \log I_0(t) dt \ll \int_{k/y^\sigma}^1 \frac{(\log t)^{N+1}}{t^{1/\sigma-1}} dt + \int_1^{k^{1/2}} \frac{(\log t)^{N+1}}{t^{1/\sigma}} dt \ll_{\sigma, N} 1.$$

On the other hand, we enlarge the integral interval to $(0, \infty)$, and use the definition of C_n , then the main term of the last formula is

$$C_n(k, y) = C_n - \tilde{C}_n$$

where

$$\begin{aligned}
\tilde{C}_n &:= \left(\int_0^{k/y^\sigma} + \int_{k^{\frac{1}{2}}}^{\infty} \right) \frac{(\log t)^n}{t^{1/\sigma+1}} \log I_0(t) dt \\
&\ll \int_0^{k/y^\sigma} \frac{(-\log t)^n}{t^{1/\sigma-1}} dt + \int_{k^{\frac{1}{2}}}^{\infty} \frac{(\log t)^n}{t^{1/\sigma}} dt \\
&\ll_{\sigma, N} \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} (\log(y^\sigma/k))^n + \frac{(\log k)^n}{k^{(1/\sigma-1)/2}},
\end{aligned}$$

thanks to (3.2.2)-(3.2.3). It follows that

$$\begin{aligned}
\frac{\tilde{C}_n}{(\log k)^n} &\ll \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} \left(\frac{\log(y^\sigma/k)}{\log k} \right)^n + \frac{1}{k^{(1/\sigma-1)/2}} \\
&\ll \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} + \frac{1}{(\log k)^{N+1}},
\end{aligned}$$

since $\left(\frac{\log(y^\sigma/k)}{\log k} \right)^n \ll_{\sigma, N} 1$ if $y \leq k^{2/\sigma}$ and otherwise we have

$$\left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} \left(\frac{\log(y^\sigma/k)}{\log k} \right)^n \leq \left(\frac{k^{1/\sigma}}{k^{2/\sigma}} \right)^{2\sigma-1} \left(\frac{\log((k^{2/\sigma})^\sigma/k)}{\log k} \right)^n \ll \frac{1}{(\log k)^{N+1}}.$$

Thus

$$\mathcal{M} = \frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^N \frac{C_n}{(\log k)^n} + O_{\sigma, N} \left(\frac{1}{(\log k)^{N+1}} + \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} \right) \right\}. \quad (3.3.7)$$

Now the required (3.3.3) follows from (3.3.4), (3.3.5), (3.3.6) and (3.3.7). \square

3.4 Proof of Theorem 3.1.1

Recall that we have define the short Euler products by

$$\zeta(\sigma + it; y) := \prod_{p \leq y} \left(1 - \frac{1}{p^{\sigma+it}} \right)^{-1},$$

and its distribution function

$$\Phi_T(\tau; y) := \frac{1}{T} \text{meas} \{ t \in [T, 2T] : \log |\zeta(\sigma + it; y)| > \tau \}.$$

In this section, we aim to prove the following proposition.

Proposition 3.4.1. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed constant and let $N \geq 1$ be an integer and let $c_0 = c_0(\sigma, N)$ be a large positive constant depending on (σ, N) . Then we have*

$$\Phi_T((1 + \varepsilon_0)\tau; y) \leq \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathbf{a}_n(\log_2 \tau)}{(\log \tau)^n} + O(\varepsilon_0) \right\} \right) \leq \Phi_T((1 - \varepsilon_0)\tau; y)$$

uniformly for

$$T \rightarrow \infty, \quad \log T \leq y \leq (\log T)^2, \quad 1 \ll \tau \leq \frac{c(\sigma)}{\log_2 T} \left(\frac{\log T}{y^{1-\sigma}} \right)^{\frac{1-\sigma}{\sigma}}, \quad (3.4.1)$$

where $c(\sigma)$ is a positive constant depending only on σ ,

$$\varepsilon_0 = \varepsilon_0(\tau, y) = c_0 \left\{ \left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} + \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{y} \right)^{\sigma - \frac{1}{2}} \right\}, \quad (3.4.2)$$

the polynomials $a_n(\cdot)$ is the same as in Theorem 3.1.1 and the implied constant is absolute.

3.4.1 Two preliminary lemmas

The following lemma relates the moments of the short Euler products to the distribution function.

Lemma 3.4.2. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed constant. For any non-negative integer N , we have*

$$\int_{-\infty}^{\infty} \Phi_T(t; y) k e^{kt} dt = \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^N \frac{C_n}{(\log k)^n} + O \left(\frac{1}{(\log k)^{N+1}} + \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1} \right) \right\} \right)$$

uniformly for (T, y, k) in (3.3.1), where C_n is defined in (3.3.2) and the implied constant depends only on N and σ .

Proof. Since

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_T(u; y) k e^{ku} du &= \frac{1}{T} \int_{-\infty}^{\infty} \left(\int_T^{2T} 1 dt \right) k e^{ku} du \\ &= \frac{1}{T} \int_T^{2T} \left(\int_{-\infty}^{\log |\zeta(\sigma+it; y)|} k e^{ku} du \right) dt \\ &= \frac{1}{T} \int_T^{2T} |\zeta(\sigma+it; y)|^k dt, \end{aligned}$$

the required result of Lemma 3.4.2 follows from Proposition 3.3.1 immediately. \square

Lemma 3.4.3. *Let $\sigma \in (\frac{1}{2}, 1)$ be a fixed constant. Let $\{a_n\}_{n \geq 0}$ be a sequence of real numbers and $N \geq 0$ be an integer. If*

$$\tau = \frac{k^{1/\sigma-1}}{\sigma \log k} \sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n} \quad (k \rightarrow \infty), \quad (3.4.3)$$

then there is a sequence of polynomials $\{b_n(\cdot)\}_{n \geq 0}$ with $\deg(b_n) \leq n$ and $b_0 = \frac{\sigma}{1-\sigma}$ such that

$$\log k = (\log \tau) \left\{ \sum_{n=0}^N \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O \left(\left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} \right) \right\}, \quad (3.4.4)$$

where the implied constant depends on the sequence $\{a_n\}_{n \geq 0}$ and N .

Proof. We prove it by recurrence. Firstly, taking logarithm of both sides in (3.4.3), we have

$$\log \tau = \frac{1-\sigma}{\sigma} \log k - \log \sigma - \log_2 k + \log \left(\sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n} \right). \quad (3.4.5)$$

From this we derive that

$$\log(\tau \log \tau) = \frac{1-\sigma}{\sigma} \log k + O_\sigma(1)$$

and

$$\log k = \frac{\sigma}{1-\sigma} \log(\tau \log \tau) + O_\sigma(1) = \frac{\sigma}{1-\sigma} (\log \tau) \left(1 + \frac{\log_2 \tau + O_\sigma(1)}{\log \tau} \right), \quad (3.4.6)$$

which is the case for $N = 0$.

Now assume we already have

$$\log k = (\log \tau) \left\{ \sum_{n=0}^m \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+1}\right) \right\},$$

for some $m < N$. Inserting this into (3.4.5), it follows that

$$\begin{aligned} \log \tau &= \frac{1-\sigma}{\sigma} \log k - \log \sigma - \log \left((\log \tau) \left\{ \sum_{n=0}^m \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+1}\right) \right\} \right) \\ &\quad + \log \left(\sum_{n=0}^{N+1} a_n \left((\log \tau) \left\{ \sum_{n=0}^m \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+1}\right) \right\} \right)^{-n} \right), \end{aligned}$$

from which we derive that

$$\begin{aligned} \frac{1-\sigma}{\sigma} \log k &= (\log \tau) \left\{ 1 + \frac{\log \sigma + \log_2 \tau}{\log \tau} + \frac{1}{\log \tau} \log \left\{ \sum_{n=0}^m \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+1}\right) \right\} \right. \\ &\quad \left. - \frac{1}{\log \tau} \log \left(\sum_{n=0}^{N+1} a_n \left((\log \tau) \left\{ \sum_{n=0}^m \frac{b_n(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+1}\right) \right\} \right)^{-n} \right) \right\}. \end{aligned}$$

By expansion of the log-terms, we can obtain

$$\frac{1-\sigma}{\sigma} \log k = (\log \tau) \left\{ \sum_{n=0}^{m+1} \frac{b_n^*(\log_2 \tau)}{(\log \tau)^n} + O\left(\left(\frac{\log_2 \tau}{\log \tau}\right)^{m+2}\right) \right\}$$

with some polynomials $b_n^*(\log_2 \tau)$ of $\deg(b_n^*) \leq n$ and of $b_0^* = 1$. Thus Lemma 3.4.3 follows from recurrence. \square

3.4.2 Proof of Proposition 3.4.1

Let $\{a_n\}_{n \geq 0}$ be a real sequence depending on σ , which will be chosen later. It is clear that there is a large constant $t_0 = t_0(\sigma)$ such that the function

$$t \mapsto \frac{t^{1/\sigma-1}}{\sigma \log t} \sum_{n=0}^{N+1} \frac{a_n}{(\log t)^n}$$

is strictly increasing on $[t_0, \infty)$. Thus we choose a unique k such that

$$\tau = \frac{k^{1/\sigma-1}}{\sigma \log k} \sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n}. \quad (3.4.7)$$

Noticing that (3.4.6) and (3.4.1) imply that

$$ky^{1-\sigma} \ll_\sigma (\tau \log \tau)^{\frac{\sigma}{1-\sigma}} y^{1-\sigma} \ll_\sigma \frac{\log T}{y^{1-\sigma}} y^{1-\sigma} \leq \frac{1}{8}(1-\sigma) \log T,$$

we can apply Lemma 3.4.2 to write

$$\int_{-\infty}^{\infty} \Phi_T(t; y) k e^{kt} dt = \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} + O_{\sigma, N}(R_{2N+2}(k, y)) \right\} \right), \quad (3.4.8)$$

where

$$R_{2N+2}(k, y) := \frac{1}{(\log k)^{2N+2}} + \left(\frac{k^{1/\sigma}}{y} \right)^{2\sigma-1}.$$

We choose

$$\varepsilon = A \left(\frac{1}{(\log k)^{N+1}} + \left(\frac{k^{1/\sigma}}{y} \right)^{\sigma-\frac{1}{2}} \right) \in (0, 10^{-2022}) \quad (k \geq k_0), \quad (3.4.9)$$

where $A = A(\sigma, N)$ and $k_0 = k_0(\sigma, N)$ are large constants depending on (σ, N) , and let

$$k_1 := (1 + \varepsilon)k, \quad k_2 := (1 - \varepsilon)k, \quad \tau_1 := \left(1 + \frac{\varepsilon}{2\sigma}\right)\tau, \quad \tau_2 := \left(1 - \frac{\varepsilon}{2\sigma}\right)\tau.$$

When $t \leq \tau_2$, we have

$$kt \leq (k - k_2)(\tau_2 - t) + kt = (k - k_2)\tau_2 + k_2t = \varepsilon k \tau_2 + k_2t.$$

Thus

$$\int_{-\infty}^{\tau_2} e^{kt} \Phi_T(t; y) dt \leq e^{\varepsilon k \tau_2} \int_{-\infty}^{\infty} e^{k_2 t} \Phi_T(t; y) dt. \quad (3.4.10)$$

Using (3.4.8) and noticing that $R_{2N+2}(k_2, y) \ll_{\sigma, N} R_{2N+2}(k, y)$, we have

$$\int_{-\infty}^{\infty} e^{k_2 t} \Phi_T(t; y) dt = \exp \left(\frac{k_2^{1/\sigma}}{\log k_2} \left\{ \sum_{n=0}^{2N+1} \frac{C_n}{(\log k_2)^n} + O_{\sigma, N}(R_{2N+2}(k, y)) \right\} \right).$$

Inserting this into (3.4.10) and using the definition of τ_2 with (3.4.7), then we have

$$\int_{-\infty}^{\tau_2} e^{kt} \Phi_T(t; y) dt \leq \exp \left(\frac{k^{1/\sigma}}{\log k} \{ \mathcal{S}_1 + \mathcal{S}_2 + O_{\sigma, N}(R_{2N+2}(k, y)) \} \right), \quad (3.4.11)$$

where

$$\mathcal{S}_1 := \frac{\varepsilon}{\sigma} \left(1 - \frac{\varepsilon}{2\sigma} \right) \sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n},$$

$$\mathcal{S}_2 := \frac{(1-\varepsilon)^{1/\sigma}}{1 + \log(1-\varepsilon)/\log k} \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} \frac{1}{(1 + \log(1-\varepsilon)/\log k)^n}.$$

The first part \mathcal{S}_1 can be calculated easily, using the choice of ε , as

$$\mathcal{S}_1 = \frac{\varepsilon}{\sigma} \sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n} - \frac{\varepsilon^2}{2\sigma^2} a_0 + o_{\sigma,N}(R_{2N+2}(k, y)). \quad (3.4.12)$$

In order to calculate the second part \mathcal{S}_2 , we take Taylor series for $\log(1-\varepsilon)$, use the geometric series formula, and put all infinitesimal of higher order than R_{2N+2} into the error term, then we have

$$\begin{aligned} \mathcal{S}_2 &= \left(1 - \frac{\varepsilon}{\sigma} + \frac{\varepsilon^2}{2\sigma} \left(\frac{1}{\sigma} - 1\right)\right) \left(1 + \frac{\varepsilon}{\log k}\right) \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} \left(1 + \frac{n\varepsilon}{\log k}\right) + o_{\sigma,N}(R_{2N+2}(k, y)) \\ &= \left(1 - \frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\log k} + \frac{\varepsilon^2}{2\sigma^2} - \frac{\varepsilon^2}{2\sigma}\right) \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} \left(1 + \frac{n\varepsilon}{\log k}\right) + o_{\sigma,N}(R_{2N+2}(k, y)). \end{aligned}$$

We separate the same part as in the exponent of (3.4.12) from the above formula, and again put all the infinitesimal of higher order than R_{2N+2} into the error term, then we can write

$$\mathcal{S}_2 = \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} + \frac{\varepsilon^2}{2\sigma^2} C_0 - \frac{\varepsilon^2}{2\sigma} C_0 + -\frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\sigma} C_0 \sum_{n=1}^{N+1} \frac{\sigma n C_{n-1} - C_n}{(\log k)^n} + o_{\sigma,N}(R_{2N+2}(k, y)). \quad (3.4.13)$$

Combining (3.4.12) and (3.4.13), we have

$$\begin{aligned} \mathcal{S}_1 + \mathcal{S}_2 &= \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} + \frac{\varepsilon}{\sigma} \sum_{n=1}^{N+1} \frac{a_n + \sigma n C_{n-1} - C_n}{(\log k)^n} \\ &\quad + \left(\frac{\varepsilon}{\sigma} - \frac{\varepsilon^2}{2\sigma^2}\right)(a_0 - C_0) - \frac{\varepsilon^2}{2\sigma} C_0 + o_{\sigma,N}(R_{2N+2}(k, y)). \end{aligned}$$

Choosing $a_0 = C_0$ and $a_n = C_n - \sigma n C_{n-1}$ for $n \geq 1$, we find that

$$\mathcal{S}_1 + \mathcal{S}_2 = \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} - \frac{\varepsilon^2}{2\sigma} C_0 + o_{\sigma,N}(R_{2N+2}(k, y)).$$

Inserting this into (3.4.11), and using (3.4.8), then we have

$$\begin{aligned} \int_{-\infty}^{\tau_2} e^{kt} \Phi_T(t; y) dt &\leq \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ -\frac{\varepsilon^2}{2\sigma} C_0 + \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} + O_{\sigma,N}(R_{2N+2}(k, y)) \right\} \right) \\ &= \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ -\frac{\varepsilon^2}{2\sigma} C_0 + O_{\sigma,N}(R_{2N+2}(k, y)) \right\} \right) \int_{-\infty}^{\infty} e^{kt} \Phi_T(t; y) dt. \end{aligned}$$

By the choice of the value of ε ($A = A(\sigma, N)$ is a suitably large constant), and $C_0 > 0$, we can obtain

$$\int_{-\infty}^{\tau_2} e^{kt} \Phi_T(t; y) dt \leq \frac{1}{4} \int_{-\infty}^{\infty} e^{kt} \Phi_T(t; y) dt.$$

Similarly, we have

$$\int_{\tau_1}^{\infty} e^{kt} \Phi_T(t; y) dt \leq \frac{1}{4} \int_{-\infty}^{\infty} e^{kt} \Phi_T(t; y) dt.$$

Thus combining the above two inequalities we have

$$\frac{1}{2} \int_{-\infty}^{\infty} e^{kt} \Phi_T(t; y) dt \leq \int_{\tau_2}^{\tau_1} e^{kt} \Phi_T(t; y) dt \leq \int_{-\infty}^{\infty} e^{kt} \Phi_T(t; y) dt.$$

So thanks to (3.4.8), we can get the asymptotic formula for the integral over (τ_2, τ_1) :

$$\int_{\tau_2}^{\tau_1} e^{kt} \Phi_T(t; y) dt = \exp \left(\frac{k^{1/\sigma}}{\log k} \left\{ \sum_{n=0}^{2N+1} \frac{C_n}{(\log k)^n} + O(R_{2N+2}(k, y)) \right\} \right). \quad (3.4.14)$$

On the other hand, since $\Phi_T(t; y)$ is decreasing in t , we have

$$(\tau_1 - \tau_2) e^{k\tau_2} \Phi_T(\tau_1; y) \leq \int_{\tau_2}^{\tau_1} e^{kt} \Phi_T(t; y) dt \leq (\tau_1 - \tau_2) e^{k\tau_1} \Phi_T(\tau_2; y).$$

By the choice of the values of τ_1 and τ_2 , the above inequality is

$$\frac{\varepsilon\tau}{\sigma} e^{k\tau(1-\frac{\varepsilon}{2\sigma})} \Phi_T((1 + \frac{\varepsilon}{2\sigma})\tau; y) \leq \int_{\tau_2}^{\tau_1} e^{kt} \Phi_T(t; y) dt \leq \frac{\varepsilon\tau}{\sigma} e^{k\tau(1+\frac{\varepsilon}{2\sigma})} \Phi_T((1 - \frac{\varepsilon}{2\sigma})\tau; y). \quad (3.4.15)$$

In view of (3.4.7), it is easy to see that

$$\begin{aligned} \frac{\sigma}{\varepsilon\tau} e^{-k\tau(1\pm\frac{\varepsilon}{2\sigma})} &= \exp \left(\log \left(\frac{\sigma}{\varepsilon\tau} \right) - k\tau \{1 + O(\varepsilon)\} \right) \\ &= \exp \left(- \frac{k^{1/\sigma}}{\sigma \log k} \left\{ \sum_{n=0}^{N+1} \frac{a_n}{(\log k)^n} + O(\varepsilon) \right\} \right). \end{aligned}$$

Combining this with (3.4.14) and (3.4.15), it follows that

$$\frac{\sigma}{\varepsilon\tau} e^{-k\tau(1\pm\frac{\varepsilon}{2\sigma})} \int_{\tau_2}^{\tau_1} e^{kt} \Phi_T(t; y) dt = \exp \left(- \frac{k^{1/\sigma}}{\sigma \log k} \left\{ \sum_{n=0}^N \frac{a_n - \sigma C_n}{(\log k)^n} + O(\varepsilon) \right\} \right).$$

Back to (3.4.15), we get

$$\Phi_T((1 + \frac{\varepsilon}{2\sigma})\tau; y) \leq \exp \left(- \frac{k^{1/\sigma}}{\sigma \log k} \left\{ \sum_{n=0}^N \frac{a_n - \sigma C_n}{(\log k)^n} + O(\varepsilon) \right\} \right) \leq \Phi_T((1 - \frac{\varepsilon}{2\sigma})\tau; y). \quad (3.4.16)$$

Recall that Lemma 3.4.3 and (3.4.6) give

$$\log k = \log \tau \left\{ \sum_{n=0}^N \frac{b_n (\log_2 \tau)}{(\log \tau)^n} + O \left(\left(\frac{\log_2 \tau}{\log \tau} \right)^{N+1} \right) \right\}$$

and

$$\log k = \frac{\sigma}{1 - \sigma} \log(\tau \log \tau) + O_\sigma(1).$$

With the help of these formulas, after some computations of Taylor's expansions we easily see that there are a sequence of polynomials $\{\mathfrak{a}_n(\cdot)\}_{n \geq 0}$ ^{*} with $\deg(\mathfrak{a}_n) \leq n$ and a positive constant $c_0 = c_0(\sigma, N)$ depending on (σ, N) such that

$$\frac{k^{1/\sigma}}{\sigma \log k} \left\{ \sum_{n=0}^N \frac{a_n - \sigma C_n}{(\log k)^n} + O(\varepsilon) \right\} = (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathfrak{a}_n(\log_2 \tau)}{(\log \tau)^n} + O(\varepsilon_0) \right\} \quad (3.4.17)$$

and

$$\varepsilon/(2\sigma) \leq \varepsilon_0, \quad (3.4.18)$$

where ε_0 is given as in (3.4.2). Inserting (3.4.17) into (3.4.16) and using the fact that the function $t \mapsto \Phi_T(t; y)$ is decreasing with (3.4.18), we obtain the required result. This completes the proof. \square

3.4.3 End of the proof of Theorem 3.1.1

Let

$$\eta := c_0 \left(\frac{(\tau \log \tau)^{\frac{1}{1-\sigma}}}{y} \right)^{\sigma - \frac{1}{2}},$$

where $c_0 = c_0(\sigma, N)$ be a large positive constant given as in Proposition 3.4.1. Applying Lemma 3.2.6 with $\lambda = \eta \tau$, we can obtain

$$\Phi_T(\tau) = \Phi_T(\tau(1 \pm \eta); y) + O \left(\exp \left\{ -(4e)^{-1} (\sigma - \frac{1}{2}) c_0^2 \frac{\log y}{\log \tau} (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \right\} \right). \quad (3.4.19)$$

On the other hand, noticing that $\eta \leq \varepsilon_0$ and that $\Phi_T(t; y)$ is decreasing in t , (3.4.19) and Proposition 3.4.1 imply that

$$\begin{aligned} \Phi_T(\tau) &= \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathfrak{a}_n(\log_2 \tau)}{(\log \tau)^n} + O(\varepsilon_0) \right\} \right) \\ &\quad + O \left(\exp \left\{ -(4e)^{-1} (\sigma - \frac{1}{2}) c_0^2 \frac{\log y}{\log \tau} (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \right\} \right) \\ &= \exp \left(-(\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \left\{ \sum_{n=0}^N \frac{\mathfrak{a}_n(\log_2 \tau)}{(\log \tau)^n} + O(\varepsilon_0) \right\} \right) \Delta(\tau, y), \end{aligned} \quad (3.4.20)$$

uniformly for

$$T \rightarrow \infty, \quad \log T \leq y \leq (\log T)^2, \quad 1 \ll \tau \leq \frac{c(\sigma)}{\log_2 T} \left(\frac{\log T}{y^{1-\sigma}} \right)^{\frac{1-\sigma}{\sigma}},$$

where

$$\Delta(\tau, y) := 1 + O \left(\exp \left\{ -(4e)^{-1} (\sigma - \frac{1}{2}) c_0^2 \frac{\log y}{\log \tau} (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} + O_{\sigma, N}((\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}}) \right\} \right).$$

Since c_0 is suitably large, we have, with choice of $y = \log T$,

$$\Delta(\tau, \log T) = 1 + O \left(\exp \left\{ -(8e)^{-1} (\sigma - \frac{1}{2}) c_0^2 (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \right\} \right)$$

*The value of \mathfrak{a}_0 follows easily from $b_0 = \frac{\sigma}{1-\sigma}$ in Lemma 3.4.3.

$$\begin{aligned}
&= \exp \left(\exp \left\{ - (8e)^{-1} (\sigma - \frac{1}{2}) c_0^2 (\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \right\} \right) \\
&= \exp \left\{ O((\tau \log^\sigma \tau)^{\frac{1}{1-\sigma}} \varepsilon_0(\tau, \log T)) \right\}
\end{aligned}$$

uniformly for $T \rightarrow \infty$ and $1 \ll \tau \leq c(\sigma)(\log T)^{1-\sigma}/\log_2 T$. Inserting this into (3.4.20), we obtain the result of Theorem 3.1.1. \square

4 Large values of $|\zeta(\sigma + it)|$

4.1 Background

In this section, we investigate the extreme values of the Riemann zeta function $\zeta(s)$ in the strip $\frac{1}{2} < \Re s \leq 1$. The study of the values on the 1-line can date back to 1925 when Littlewood [36] showed that there exists arbitrarily large t for which

$$|\zeta(1 + it)| \geq (1 + o(1))e^\gamma \log_2 t.$$

Here and throughout, we denote by \log_j the j -th iterated logarithm and by γ the Euler constant. This was improved by Levinson [34], who in 1972 proved that there exists arbitrarily large t such that

$$|\zeta(1 + it)| \geq e^\gamma \log_2 t + O(1).$$

In 2006, Granville and Soundararajan [26] used Diophantine approximation to prove that the lower bound

$$\max_{t \in [1, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T - \log_4 T + O(1))$$

holds for sufficiently large T . Based on a distribution function which exhibits slightly smaller values, they also proposed a strong conjecture that

$$\max_{t \in [T, 2T]} |\zeta(1 + it)| = e^\gamma (\log_2 T + \log_3 T + C_0 + 1 - \log 2 + o(1)), \quad (4.1.1)$$

for some constant $C_0 = -0.3953997$, which provides a precise description of the extreme values. For the upper bound, the best unconditional result is established by Vinogradov [52] who proved that

$$|\zeta(1 + it)| \ll (\log t)^{\frac{2}{3}}.$$

In recent years, the resonance method has been extensively developed, which can detect extreme values of the Riemann zeta function more effectively. It was first used by Voronin [53] in 1988, and developed by Soundararajan [48] in 2008 and Hilberdink [30] in 2009 separately. In 2018, Aistleitner, Mahatab and Munsch [3] used a variant “long resonance” to show that

$$\max_{t \in [\sqrt{T}, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T + O(1)). \quad (4.1.2)$$

Note that this requires a larger range $[\sqrt{T}, T]$ than $[T, 2T]$ in (4.1.1), which is typical in the application of “long resonance”. Their method can also apply to a class of generalized L -functions (see [21]). Inspired by their work, the aim of this section is to get an improved lower bound of large values, which presents an explicit description of the error term $O(1)$. For any $\beta \in (0, 1)$, we define

$$Z_\beta(\sigma, T) := \max_{T^\beta \leq |t| \leq T} |\zeta(\sigma + it)|.$$

Then we have the following theorem.

Theorem 4.1.1. Let $0 < \beta < 1$ be fixed and c be a constant such that $c < \log(1 - \beta) - \log_2 4 - 1$. Then we have

$$Z_\beta(1, T) \geq e^\gamma (\log_2 T + \log_3 T + c)$$

holds for sufficiently large T ,

When $\beta = \frac{1}{2}$, we can choose the constant $c = -2.0197814$. This gives a description of the error term $O(1)$ in the result (4.1.2) by Aistleitner, Mahatab and Munsch. When β tends to 0, we can further choose $c = -1.32663426$. Despite the enlarged range, This is comparable with the conjecture (4.1.1) which predicts a larger constant $C_0 + 1 - \log 2 = -0.0885469$. Theorem 4.1.1 is also in accordance with the results on the Dirichlet L -functions, due to Aistleitner, Mahatab, Munsch and Peyrot [4].

Now we turn our attention to the values of the Riemann zeta function in the strip $\frac{1}{2} < \Re s < 1$. For any fixed $\sigma \in (\frac{1}{2}, 1)$, Titchmarsh [50] in 1928 showed that for any $\varepsilon > 0$ there exists arbitrarily large t such that

$$\log |\zeta(\sigma + it)| \geq (\log t)^{1-\sigma-\varepsilon}.$$

In 1972, Levinson [34] improved this result by showing that for large T there exists a positive c such that

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \geq c \frac{(\log T)^{1-\sigma}}{\log_2 T}.$$

In 1977, Montgomery [41] showed that

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| \geq \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}, \quad (4.1.3)$$

where $\nu(\sigma) = \frac{1}{20} \sqrt{\sigma - \frac{1}{2}}$ unconditionally, and $\nu(\sigma) = \frac{1}{20}$ under the Riemann hypothesis. In 2016, using the resonance method, Aistleitner [1] improved Montgomery's unconditional result by showing that (4.1.3) holds for $\nu(\sigma) = 0.18(2\sigma - 1)^{1-\sigma}$.

For the upper bound, Richert [44] in 1967 proved that

$$|\zeta(\sigma + it)| \leq At^{B(1-\sigma)^{3/2}} (\log t)^{2/3} = At^{B(1-\sigma)^{3/2}+\varepsilon}$$

holds for some absolute A and B . Successive improvements of Richert's bound have reduced the admissible size of A and B . We refer to Cheng [19] and Ford [24] for more details. Under the Riemann Hypothesis, we could have a much better upper bound (see [33] and [51])

$$\log |\zeta(\sigma + it)| \ll \frac{(\log t)^{2-2\sigma}}{\log_2 t}. \quad (4.1.4)$$

It is conjectured that the true order of the magnitude of $\max_{t \in [0, T]} \log |\zeta(\sigma + it)|$ corresponds to the lower bound (4.1.3) rather than the upper bound (4.1.4). In 2011, based on the probabilistic model, Lamzouri [33] gave an explicit conjectural value of $c(\sigma)$, claiming that

$$\max_{t \in [0, T]} \log |\zeta(\sigma + it)| = c(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma},$$

holds for

$$c(\sigma) = \frac{1}{\sigma^{2\sigma}(1-\sigma)^{1-\sigma}} \int_0^\infty \frac{\log I_0(t)}{t^{1/\sigma+1}} dt, \quad (4.1.5)$$

where I_0 is the modified Bessel function of order 0.

In 2018, Bondarenko and Seip [14] made celebrated improvement on this topic. They proved that there exists a function $\nu(\sigma)$ which is bounded below by $1/(2-2\sigma)$ and has the asymptotic behavior

$$\nu(\sigma) = \begin{cases} (\frac{\sqrt{2}}{2} + o(1))\sqrt{|\log(\sigma - \frac{1}{2})|} & \sigma \searrow \frac{1}{2}, \\ (1-\sigma)^{-1} + O(|\log(1-\sigma)|) & \sigma \nearrow 1. \end{cases} \quad (4.1.6)$$

Here $\sigma \searrow \frac{1}{2}$ means σ tends to $\frac{1}{2}$ from above with $\frac{1}{2} + \frac{1}{\log_2 T} \leq \sigma \leq \frac{3}{4}$ and $\sigma \nearrow 1$ means σ tends to 1 from below with $\frac{3}{4} \leq \sigma \leq 1 - \frac{1}{\log_2 T}$, as $T \rightarrow \infty$. Then for $\frac{1}{2} + \frac{1}{\log_2 T} \leq \sigma \leq \frac{3}{4}$,

$$\max_{t \in [\sqrt{T}, T]} \log |\zeta(\sigma + it)| \geq \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}, \quad (4.1.7)$$

and for $\frac{3}{4} \leq \sigma \leq 1 - \frac{1}{\log_2 T}$,

$$\max_{t \in [T/2, T]} \log |\zeta(\sigma + it)| \geq \log_3 T + c + \nu(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma},$$

where c is an absolute constant. In this section, we aim to improve their first result (4.1.7). This is accomplished by deriving a lower bound for the maximum of Gál-type sums, which is a kind of certain greatest common divisor (GCD) sums of the form

$$S_\sigma(\mathcal{M}) := \sum_{m,n \in \mathcal{M}} \frac{\gcd(m,n)^\sigma}{\text{lcm}(m,n)^\sigma} = \sum_{m,n \in \mathcal{M}} \frac{\gcd(m,n)^{2\sigma}}{(mn)^\sigma}. \quad (4.1.8)$$

For any positive integer N , we denote

$$\Gamma_\sigma(N) := \sup_{|\mathcal{M}|=N} \frac{S_\sigma(\mathcal{M})}{N}. \quad (4.1.9)$$

A brief historic description on Gál-type sums will be presented in §4.4. By adapting the argument of de la Bretèche and Tenenbaum [20] about $\Gamma_{\frac{1}{2}}(N)$, we have the following theorem concerning the lower bound of $\Gamma_\sigma(N)$.

Theorem 4.1.2. *As $\sigma \searrow \frac{1}{2}$, we have*

$$\Gamma_\sigma(N) \geq \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{|\log(\sigma - \frac{1}{2})|} \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right\}.$$

The following theorem sets up the relation between large values of the Riemann zeta function and the Gál-type sums.

Theorem 4.1.3. *For $0 < \beta < 1$ and $\sigma \in (\frac{1}{2} + \frac{1}{\log_2 T}, 1)$, we have*

$$Z_\beta(\sigma, T) \gg \sqrt{\Gamma_\sigma(T^{1-\beta})},$$

where the implied constant is absolute.

As a direct deduction of Theorem 4.1.2, we get the following lower bound for the Riemann zeta function.

Corollary 4.1.4. *Let $T \rightarrow \infty$ and $\frac{1}{2} + \frac{1}{\log_2 T} < \sigma < \frac{3}{4}$. Then we have that*

$$\max_{t \in [T^\beta, T]} \log |\zeta(\sigma + it)| \geq \nu_\beta(\sigma) \frac{(\log T)^{1-\sigma}}{(\log_2 T)^\sigma}$$

holds for a function $\nu_\beta(\sigma)$ which has the asymptotic behavior

$$\nu_\beta(\sigma) = (\sqrt{2} + o(1))(1 - \beta)^{1-\sigma} \sqrt{|\log(\sigma - \frac{1}{2})|},$$

as $\sigma \searrow \frac{1}{2}$.

Therefore when $\beta = \frac{1}{2}$ and $\sigma \searrow \frac{1}{2}$, we improved the result of Bondarenko and Seip by a factor 2^σ . It is also worthy noting that Lamzouri's conjecture (4.1.5) predicts $c(\sigma) \sim (2\sigma - 1)^{-\frac{1}{2}}$ as $\sigma \searrow \frac{1}{2}$.

For the sake of completeness, we also mention the large values of the Riemann zeta function on the critical line. For the lower bound, de la Bretèche and Tenenbaum [20] have shown that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| \geq \exp \left\{ (\sqrt{2} + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right\},$$

improving earlier results made by Bondarenko and Seip [13, 15]. For the upper bound, the Lindelöf hypothesis states that for any $\varepsilon > 0$

$$\zeta(\frac{1}{2} + it) \ll t^\varepsilon,$$

while the best-known upper bound is due to Bourgain [17] who proved that

$$\zeta(\frac{1}{2} + it) \ll t^{\frac{13}{84} + \varepsilon}.$$

However, we have the conjectural value due to Farmer, Gonek and Hugh [23] which asserts that

$$\max_{t \in [0, T]} |\zeta(\frac{1}{2} + it)| = \exp\{(\frac{\sqrt{2}}{2} + o(1)) \sqrt{\log T \log_2 T}\}.$$

For much earlier work, we refer to [6, 7, 18, 48, 51].

This chapter is organized as follows. In §4.2, we introduce some preliminary lemmas. We establish the approximation of the Riemann zeta function by its truncated Euler product. In §4.3, we discuss the large values of the Riemann zeta function on 1-line and establish Theorem 4.1.1. In §4.4, we make a brief review on the Gál-type sums and then prove Theorem 4.1.2. Finally in §4.5, we connect Gál-type sums to the values of the Riemann zeta function, and establish Theorem 4.1.3.

4.2 Preliminary lemmas

In this section, we introduce some preliminary lemmas. We start with Mertens' formula with an explicit error term.

Lemma 4.2.1. *Let $x > 1000$, then we have*

$$\frac{1}{e^\gamma \log x} \left(1 - \frac{1}{2(\log x)^2}\right) \leq \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \leq \frac{1}{e^\gamma \log x} \left(1 + \frac{1}{2(\log x)^2}\right).$$

Proof. See Theorem 7 of [45]. □

The following lemma plays a key role in the proof of Theorem 4.1.2.

Lemma 4.2.2. *Let x be large, then we have*

$$\sum_{p \leq x} \frac{1}{p^\sigma} = (1 + o(1)) \frac{x^{1-\sigma}}{(1-\sigma) \log x}.$$

Proof. This is essentially Lemma 6 of [14]. The only adjustment lies in the use of prime number theory, where we replace the lower bound by the asymptotic formula

$$\pi(x) = (1 + o(1)) \frac{x}{\log x}.$$

For analogous statements, see also Lemma 3.1 of [43], equation (2.1) of [33], and Lemma 3.3 of [8]. □

The following lemma can be seen as a generalization of the greatest common divisors to rational numbers.

Lemma 4.2.3. *Let $a, a', b, b' \in \mathbb{N}^+$ such that $\gcd(a, b) = 1$ and $\gcd(a', b') = 1$. Then for any $N \in \mathbb{N}^+$ satisfying $b|N$ and $b'|N$, we have*

$$\frac{1}{N} \gcd\left(\frac{Na}{b}, \frac{Na'}{b'}\right) = \frac{\gcd(a, a')}{\text{lcm}(b, b')},$$

and

$$\frac{1}{N} \text{lcm}\left(\frac{Na}{b}, \frac{Na'}{b'}\right) = \frac{\text{lcm}(a, a')}{\gcd(b, b')}.$$

As a direct inference, we have

$$\frac{\gcd(Na/b, Na'/b')}{\text{lcm}(Na/b, Na'/b')} = \frac{\gcd(a, a')}{\text{lcm}(a, a')} \frac{\gcd(b, b')}{\text{lcm}(b, b')}.$$

Proof. Since $\text{lcm}(b, b')|N$, we may write

$$\gcd\left(\frac{Na}{b}, \frac{Na'}{b'}\right) = \frac{N \gcd(a, a')}{\text{lcm}(b, b')} \gcd\left(\frac{\text{lcm}(b, b')}{b} \frac{a}{\gcd(a, a')}, \frac{\text{lcm}(b, b')}{b'} \frac{a'}{\gcd(a, a')}\right)$$

$$= \frac{N \gcd(a, a')}{\text{lcm}(b, b')} \gcd\left(\frac{b'}{\gcd(b, b')} \frac{a}{\gcd(a, a')}, \frac{b}{\gcd(b, b')} \frac{a'}{\gcd(a, a')}\right).$$

Then the first assertion follows by the assumptions of co-primeness and the second one follows by the simple fact that

$$\gcd\left(\frac{Na}{b}, \frac{Na'}{b'}\right) \cdot \text{lcm}\left(\frac{Na}{b}, \frac{Na'}{b'}\right) = \frac{N^2 aa'}{bb'} = \frac{N^2 \gcd(a, a') \text{lcm}(a, a')}{\gcd(b, b') \text{lcm}(a, a')}.$$

□

For convenience, we denote the truncated Euler product of the Riemann zeta function by

$$\zeta(s; y) := \prod_{p \leq y} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The following lemma approximates $\zeta(s)$ on the 1-line by its truncated Euler product.

Lemma 4.2.4. *Let T be a large positive number and $0 < \beta < 1$ be fixed. Put $Y = \exp((\log T)^{1/\beta})$. Then we have*

$$\zeta(1 + it) = \zeta(1 + it; Y) \left(1 + O\left(\frac{1}{(\log T)^{1/\beta}}\right)\right)$$

for any $T^\beta \leq |t| \leq T$.

Proof. Let $s = 1 + it$ with $T^\beta \leq |t| \leq T$. Firstly, when $\Re w > 1$, we have

$$\log \zeta(w) = \log \prod_p \left(1 - \frac{1}{p^w}\right)^{-1} = \sum_p \sum_{\nu \geq 1} \frac{1}{\nu p^{\nu w}} = \sum_{n \geq 2} \frac{\Lambda(n)}{n^w \log n},$$

where $\Lambda(n)$ is the von Mangoldt function. Set $\sigma_1 = 1/\log Y$, $\sigma_0 = -c/\log T$ for suitably positive constant $c = c(\beta) > 0$ and $T_0 = T^\beta/2$. Denote the contour joining $\sigma_1 - iT_0$, $\sigma_1 + iT_0$, $\sigma_0 + iT_0$, $\sigma_0 - iT_0$ and $\sigma_0 + iT_0$ by Γ , i.e.,

$$\oint_{\Gamma} = \int_{\sigma_1 - iT_0}^{\sigma_1 + iT_0} + \int_{\sigma_1 + iT_0}^{\sigma_0 + iT_0} + \int_{\sigma_0 + iT_0}^{\sigma_0 - iT_0} + \int_{\sigma_0 - iT_0}^{\sigma_1 - iT_0}.$$

Since $\log \zeta(1 + it + w) Y^w$ is analytic inside Γ , by Cauchy's integral formula we have

$$\frac{1}{2\pi i} \oint_{\Gamma} \log \zeta(1 + it + w) \frac{Y^w}{w} dw = \log \zeta(1 + it). \quad (4.2.1)$$

Then by Perron's formula ([49, Corollary II.2.1] with $s = 1 + it$, $\sigma_a = \alpha = 1$, $\kappa = 1/\log Y$ and $B(x) = 1$), we have

$$\frac{1}{2\pi i} \int_{\sigma_1 - iT_0}^{\sigma_1 + iT_0} \log \zeta(1 + it + w) \frac{Y^w}{w} dw = \sum_{2 \leq n \leq Y} \frac{\Lambda(n)}{n^{1+it} \log n} + O\left(\frac{\log T}{T_0}\right).$$

On the other hand, noticing that

$$\sum_{p \leq Y, p^{\nu} > Y} \frac{1}{\nu p^{\nu}} \ll \frac{1}{\log Y} \sum_{p \leq Y, \nu \geq 2} \frac{\log p}{p^{\nu}} \ll \frac{1}{\log Y},$$

we can write

$$\begin{aligned} \sum_{2 \leq n \leq Y} \frac{\Lambda(n)}{n^{1+it} \log n} &= \sum_{p^\nu \leq Y} \frac{1}{\nu p^{\nu(1+it)}} = \sum_{p \leq Y} \sum_{\nu \geq 1} \frac{1}{\nu p^{\nu(1+it)}} + O\left(\frac{1}{\log Y}\right) \\ &= \log \zeta(1+it; Y) + O\left(\frac{1}{\log Y}\right). \end{aligned}$$

Inserting this into the preceding formula, we get

$$\frac{1}{2\pi i} \int_{\sigma_1 - iT_0}^{\sigma_1 + iT_0} \log \zeta(1+it+w) \frac{Y^w}{w} dw = \log \zeta(1+it; Y) + O\left(\frac{1}{(\log T)^{1/\beta}}\right). \quad (4.2.2)$$

For the other three integrals, in view of bounds of $\log \zeta(w)$ in the zero-free region (see [49, Theorem II.3.16]), typically we have

$$\left(\int_{\sigma_1 + iT_0}^{\sigma_0 + iT_0} + \int_{\sigma_0 + iT_0}^{\sigma_0 - iT_0} \right) \log \zeta(1+it+w) \frac{Y^w}{w} dw \ll \frac{\log T}{T_0},$$

and

$$\int_{\sigma_0 + iT_0}^{\sigma_0 - iT_0} \log \zeta(1+it+w) \frac{Y^w}{w} dw \ll \frac{(\log T)^2}{\exp(c(\log T)^{(1/\beta)-1})}.$$

Thus the lemma follows from (4.2.1), (4.2.2) and these two bounds. \square

4.3 Extreme values of $|\zeta(1+it)|$: Proof of Theorem 4.1.1

Choose B such that $e^{c+1} < B < (1-\beta)/\log 4$ and set $X := B \log T \log_2 T$. Denote by $\mathcal{S}(X)$ the set of all X -friable numbers. Let $a_n = a(n)$ be the completely multiplicative function supported on $\mathcal{S}(X)$ with $a_p = 1-p/X$ for $p \leq X$ and $a_p = 0$ otherwise. Define the resonator

$$R(t) = \sum_{n \in \mathcal{S}(X)} a_n n^{it} = \prod_{p \leq X} (1 - a_p p^{it})^{-1}.$$

Then by the prime number theorem, for $t \in \mathbb{R}$ we have

$$\log |R(t)| \leq \sum_{p \leq X} \log(1 - a_p)^{-1} = \pi(X) \log X - \theta(X),$$

where $\theta(x)$ is the Chebyshev function. It is well known that

$$\pi(x) \log x - \theta(x) = \{1 + o(1)\} \frac{x}{\log x}$$

and thus we have

$$|R(t)| \leq T^{B+o(1)} \quad (t \in \mathbb{R}). \quad (4.3.1)$$

Set the weight function to be the Gaussian function $\phi(t) = e^{-t^2}$ which satisfies

$$\widehat{\phi}(x) = \int_{\mathbb{R}} \phi(t) e^{-itx} dt = \sqrt{2\pi} \phi(x).$$

Denote

$$M_1(R; T) := \int_{T^\beta \leq |t| \leq T} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt,$$

and

$$M_2(R; T) := \int_{T^\beta \leq |t| \leq T} \zeta(1 + it) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt.$$

Then clearly we have

$$Z_\beta(T) \geq \frac{|M_2(R; T)|}{M_1(R; T)}. \quad (4.3.2)$$

Put $Y := \exp((\log T)^{1/\beta})$ as in Lemma 4.2.4. It follows that

$$M_2(R; T) = \left\{ 1 + O\left(\frac{1}{(\log T)^{1/\beta}}\right) \right\} \int_{T^\beta \leq |t| \leq T} \zeta(1 + it; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt.$$

Using (4.3.1) and the trivial bound

$$|\zeta(1 + it; Y)| \ll \log Y = (\log T)^{1/\beta} \quad (t \in \mathbb{R}),$$

we can deduce that

$$\left| \int_{|t| \leq T^\beta} \zeta(1 + it; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \right| \leq T^{2B+\beta+o(1)},$$

and

$$\left| \int_{|t| \geq T} \zeta(1 + it; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \right| \ll T^{2B+o(1)} \int_T^\infty e^{-(t \log T/T)^2} dt \ll 1.$$

Thus by denoting

$$I_2(R; T) = \int_{\mathbb{R}} \zeta(1 + it; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt,$$

we have

$$M_2(R; T) = \left\{ 1 + O\left(\frac{1}{(\log T)^{1/\beta}}\right) \right\} \{ I_2(R; T) + O(T^{2B+\beta-1+o(1)}) \}.$$

On the other hand, we have

$$M_1(R; T) \leq \int_{\mathbb{R}} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt =: I_1(R; T),$$

Therefore by (4.3.2), we derive that

$$Z_\beta(T) \geq \left\{ 1 + O\left(\frac{1}{(\log T)^{1/\beta}}\right) \right\} \frac{I_2(R; T) + O(T^{2B+\beta-1+o(1)})}{I_1(R; T)}. \quad (4.3.3)$$

We first estimate the ratio $I_2(R; T)/I_1(R; T)$. For $I_1(R; T)$, we have

$$\begin{aligned} I_1(R; T) &= \int_{\mathbb{R}} \sum_{m, n \in \mathcal{S}(X)} a_m a_n \left(\frac{m}{n} \right)^{\frac{it}{2}} \phi \left(\frac{t \log T}{T} \right) dt \\ &= \frac{T}{\log T} \sum_{m, n \in \mathcal{S}(X)} a_m a_n \widehat{\phi} \left(\frac{T}{\log T} \log \left(\frac{n}{m} \right) \right) \geq \widehat{\phi}(0) \frac{T}{\log T} \sum_{n \in \mathcal{S}(X)} a_n^2, \end{aligned} \quad (4.3.4)$$

thanks to the positivity of a_n and $\widehat{\phi}(t)$. Similarly, for $I_2(R; T)$ we have

$$I_2(R; T) = \frac{T}{\log T} \sum_{l \in \mathcal{S}(Y)} \sum_{m, n \in \mathcal{S}(X)} \frac{a_m a_n}{l} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{nl}{m} \right).$$

By sifting the terms with $l|m$ and noticing $Y > X$, we have that

$$\begin{aligned} I_2(R; T) &\geq \frac{T}{\log T} \sum_{l, n, k \in \mathcal{S}(X)} \frac{a_{kl} a_n}{l} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{n}{k} \right) \\ &= \frac{T}{\log T} \sum_{l \in \mathcal{S}(X)} \frac{a_l}{l} \sum_{k, n \in \mathcal{S}(X)} a_k a_n \widehat{\phi} \left(\frac{T}{\log T} \log \frac{n}{k} \right). \end{aligned}$$

Therefore, we can deduce that

$$\frac{I_2(R; T)}{I_1(R; T)} \geq \sum_{l \in \mathcal{S}(X)} \frac{a_l}{l} = \prod_{p \leq X} \left(1 - \frac{a_p}{p} \right)^{-1} = \prod_{p \leq X} \left(1 - \frac{1}{p} \right)^{-1} \prod_{p \leq X} \left(\frac{p-1}{p-a_p} \right).$$

For the first product, we use Lemma 4.2.1 to derive that

$$\prod_{p \leq X} \left(1 - \frac{1}{p} \right)^{-1} \geq e^\gamma \log X \left(1 - \frac{1}{2(\log X)^2} \right).$$

For the second one, since

$$-\log \prod_{p \leq X} \left(\frac{p-1}{p-a_p} \right) \leq \sum_{p \leq X} \frac{p}{(p-1)X} \leq \left\{ 1 + O \left(\frac{1}{\log X} \right) \right\} \frac{1}{\log X},$$

we have

$$\prod_{p \leq X} \left(\frac{p-1}{p-a_p} \right) \geq 1 - \frac{1}{\log X} + O \left(\frac{1}{(\log X)^2} \right).$$

It follows that

$$\frac{I_2(R; T)}{I_1(R; T)} \geq e^\gamma \left\{ \log X - 1 + O \left(\frac{1}{\log X} \right) \right\}. \quad (4.3.5)$$

It remains to exclude the influence of the error term in (4.3.3). In view of (4.3.4), we deduce that

$$\log \sum_{n \in \mathcal{S}(X)} a_n^2 = \log \prod_{p \leq X} (1 - a_p^2)^{-1} = \log \prod_{p \leq X} (1 - (1 - p/X)^2)^{-1}$$

$$= 2\pi(X) \log X - \theta(X) - \sum_{p \leq X} \log(2X - p). \quad (4.3.6)$$

By partial summation formula, we can derive that

$$\sum_{p \leq X} \log(2X - p) = \pi(X) \log X + \int_1^{X/(\log X)^2} \frac{\pi(t)}{2X - t} dt + \int_{X/(\log X)^2}^X \frac{\pi(t)}{2X - t} dt. \quad (4.3.7)$$

For the first integral, we have

$$\int_1^{X/(\log X)^2} \frac{\pi(t)}{2X - t} dt \ll \frac{X}{(\log X)^3} \int_1^{X/(\log X)^2} \frac{dt}{2X - t} \ll \frac{X}{(\log X)^5}.$$

For the second one, the interval guarantees that

$$\pi(t) = \frac{t}{\log X} \left\{ 1 + O\left(\frac{\log_2 X}{\log X}\right) \right\}.$$

Thus we have

$$\int_{X/(\log X)^2}^X \frac{\pi(t)}{2X - t} dt = \{2 \log 2 - 1 + o(1)\} \frac{X}{\log X}.$$

Combining with (4.3.6) and (4.3.7), we obtain that

$$\log \sum_{n \in S(X)} a_n^2 = \{2 - 2 \log 2 + o(1)\} \frac{X}{\log X},$$

which implies by (4.3.4) that

$$I_1(R; T) \gg T^{(2-2 \log 2)B+1+o(1)}.$$

In view of (4.3.3), for the error term we have

$$\frac{T^{2B+\beta-1+o(1)}}{I_1(R; T)} \leq \frac{T^{2B+\beta+o(1)}}{T^{(2-2 \log 2)B+1+o(1)}} = T^{B \log 4 - (1-\beta) + o(1)},$$

where the last exponent is negative and thus admissible. Substituting this and (4.3.5) into (4.3.3), we deduce that

$$Z_\beta(T) \geq e^\gamma (\log_2 T + \log_3 T + c),$$

where $c < \log B - 1$. This completes the proof. \square

4.4 Lower bound for $\Gamma_\sigma(N)$ as $\sigma \searrow \frac{1}{2}$: Proof of Theorem 4.1.2

4.4.1 A brief review on Gál-type sums

In this subsection, we focus on the lower bounds of $\Gamma_\sigma(N)$ defined in (4.1.9) and prove Theorem 4.1.2. The study of $\Gamma_\sigma(N)$ arises naturally in metric Diophantine approximation. When $\sigma = 1$, this was a prize problem posed by the Dutch Mathematical Society in 1947 on Erdős's suggestion. Gál [25] investigated the problem in 1949 and proved that

$$\Gamma_1(N) \ll (\log_2 N)^2.$$

Thereafter, the GCD sums (4.1.8) are also known as “Gál-type sums”. In 2017, Lewko and Radziwiłł [35] used the method of probabilistic models to give a much easier proof of Gál's theorem. They further determined the implied constant and proved that as $N \rightarrow \infty$, one has

$$\Gamma_1(N) = \left\{ \frac{e^{2\gamma}}{\zeta(2)} + o(1) \right\} (\log_2 N)^2.$$

Let \mathcal{M} be a finite set of integers. For general σ , define the spectral norm of the GCD matrix $(\gcd(m, n)^\sigma / \text{lcm}(m, n)^\sigma)_{(m,n) \in \mathcal{M}^2}$ as

$$Q_\sigma(\mathcal{M}) := \sup_{\substack{\mathbf{c} \in \mathbb{C}^{|\mathcal{M}|} \\ \|\mathbf{c}\|_2=1}} \left| \sum_{m,n \in \mathcal{M}} c_m \overline{c_n} \frac{\gcd(m, n)^\sigma}{\text{lcm}(m, n)^\sigma} \right|,$$

where $\mathbf{c} := (c_1, \dots, c_N) \in \mathbb{C}^N$ and its norm $\|\mathbf{c}\|_2 := \sum_{j=1}^N |c_j|^2$. Then in 2015, Aistleitner, Bondarenko and Seip [2] showed that

$$\Gamma_{\frac{1}{2}}(N) \leq \sup_{|\mathcal{M}|=N} Q_{\frac{1}{2}}(\mathcal{M}) \leq (e^2 + 1)(\log N + 2) \max_{n \leq N} \Gamma_{\frac{1}{2}}(n).$$

Recently, de la Bretèche and Tenenbaum [20] gave asymptotic formulas for $\Gamma_{\frac{1}{2}}(N)$. Namely, they proved that

$$\Gamma_{\frac{1}{2}}(N) = \exp \left(\{2\sqrt{2} + o(1)\} \sqrt{\frac{\log N \log_3 N}{\log N}} \right), \quad (4.4.1)$$

as $N \rightarrow \infty$. For further details about GCD sums, we refer to [2, 11, 12, 35].

Since Theorem 4.1.2 is a lower bound of $\Gamma_\sigma(N)$, we only need to construct a set of integers \mathcal{M} with $|\mathcal{M}| \leq N$ such that

$$\frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} \geq \exp \left(\{2\sqrt{2} + o(1)\} \sqrt{|\log(\sigma - \frac{1}{2})|} \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right)$$

for $\frac{1}{2} + \frac{1}{\log_2 N} \leq \sigma < 1$ and $N \rightarrow \infty$. Next we prove this by adapting the method of de la Bretèche and Tenenbaum [20].

4.4.2 Construction of the set \mathcal{M}

Let $\alpha \in (1, +\infty)$, $\eta \in (0, +\infty)$, $f \in (1, e]$ and $\lambda \in (0, 1)$ be some parameters. For $1 \leq j \leq J := \lfloor (\sigma - \frac{1}{2})^{-\lambda} \rfloor$, define

$$I_j := (f^j (\log N) \log_2 N, f^{j+1} (\log N) \log_2 N].$$

Then for each interval I_j , we have that

$$P_j := \sum_{p \in I_j} 1 = (f - 1) f^j (\log N) \left\{ 1 + O_\varepsilon \left(\frac{j + \log_3 N}{\log_2 N} \right) \right\}, \quad (4.4.2)$$

where we have supposed that $f \geq 1 + ((\log N) \log_2 N)^{-5/12+\varepsilon}$ such that the prime number theorem in short intervals holds and the implied constant depends on ε only. On the other hand we note that the hypothesis $\sigma \geq \frac{1}{2} + 1/\log_2 N$ guarantees $J \leq (\log_2 N)^\lambda = o(\log_2 N)$.

Let $N_j = \prod_{p \in I_j} p$ and $\omega(\cdot)$ counts the number of different prime factors. Set

$$u_j := \left\lfloor \frac{\eta(\log N)^{1-\sigma}}{j^{f^j(\sigma-\frac{1}{2})} \sqrt{\log J} (\log_2 N)^\sigma} \right\rfloor, \quad v_j := \left\lfloor \frac{\alpha \log N}{j^2 \log J} \right\rfloor.$$

Then we define

$$\mathcal{M}_j := \left\{ m = \frac{N_j a}{b} : ab \mid N_j \text{ and } \omega(a), \omega(b) \leq v_j \right\}$$

and

$$\mathcal{M} := \prod_{1 \leq j \leq J} \mathcal{M}_j = \left\{ m = \prod_{1 \leq j \leq J} m_j : m_j \in \mathcal{M}_j \ (1 \leq j \leq J) \right\}.$$

Now we evaluate the cardinal of \mathcal{M} . For $1 \leq j \leq J$, we have

$$|\mathcal{M}_j| = \sum_{\substack{0 \leq k \leq v_j \\ 0 \leq \ell \leq v_j}} \binom{P_j}{k} \binom{P_j - k}{\ell}. \quad (4.4.3)$$

For fixed k , since P_j is much larger than v_j , for $0 \leq \ell \leq v_j$ we can deduce that

$$\binom{P_j - k}{v_j} = \frac{(P_j - k - v_j) \cdots (P_j - k - (\ell + 1))}{v_j \cdots (\ell + 1)} \binom{P_j - k}{\ell} \geq 2^{v_j - \ell} \binom{P_j - k}{\ell}.$$

Therefore, we have

$$\sum_{0 \leq \ell \leq v_j} \binom{P_j - k}{\ell} \leq \sum_{0 \leq \ell \leq v_j} 2^{-(v_j - \ell)} \binom{P_j - k}{v_j} \leq 2 \binom{P_j - k}{v_j}. \quad (4.4.4)$$

On the other hand, we have

$$\binom{P_j}{k} \binom{P_j - k}{v_j} = \binom{P_j}{v_j} \binom{P_j - v_j}{k}.$$

Thus, by (4.4.3) and (4.4.4), we deduce that

$$|\mathcal{M}_j| \leq 2 \sum_{0 \leq k \leq v_j} \binom{P_j}{k} \binom{P_j - k}{v_j} = 2 \binom{P_j}{v_j} \sum_{0 \leq k \leq v_j} \binom{P_j - v_j}{k} \leq 4 \binom{P_j}{v_j} \binom{P_j - v_j}{v_j}. \quad (4.4.5)$$

Using the Euler-Maclaurin formula, we have

$$|\mathcal{M}| \leq \prod_{1 \leq j \leq J} 4 \binom{P_j - v_j}{v_j} \binom{P_j}{v_j} \leq \prod_{1 \leq j \leq J} 4 \left(\frac{e(P_j - v_j)}{v_j} \right)^{v_j} \left(\frac{eP_j}{v_j} \right)^{v_j} \leq \prod_{1 \leq j \leq J} 4 \left(\frac{eP_j}{v_j} \right)^{2v_j}.$$

Therefore, by (4.4.2) and the definitions of v_j and J , we have

$$\log |\mathcal{M}| \leq \sum_{1 \leq j \leq J} 2v_j \{j \log f + O(\log j + \log_4 N)\} \leq 2\alpha \log f \log N + o(\log N),$$

and consequently we obtain that

$$|\mathcal{M}| \leq N^{2\alpha \log f + o(1)}. \quad (4.4.6)$$

4.4.3 Proof of Theorem 4.1.2

Note that $ab|N_j$ implies $\gcd(a, b) = 1$ since N_j is square-free. By Lemma 4.2.3, we have

$$S_\sigma(\mathcal{M}_j) = \sum_{\substack{a, a' | N_j \\ \omega(a), \omega(a') \leq v_j}} \frac{\gcd(a, a')^\sigma}{\operatorname{lcm}(a, a')^\sigma} \sum_{\substack{b, b' | N_j \\ \gcd(b, a) = \gcd(b', a') = 1 \\ \omega(b), \omega(b') \leq v_j}} \frac{\gcd(b, b')^\sigma}{\operatorname{lcm}(b, b')^\sigma}. \quad (4.4.7)$$

Denote the inner sum by $\tilde{S}_\sigma = \tilde{S}_\sigma(a, a')$, then we have

$$\begin{aligned} \tilde{S}_\sigma &= \sum_{\substack{D, b, b' | N_j \\ \gcd(b, a) = \gcd(b', a') = 1 \\ \gcd(b, b') = D \\ \omega(b), \omega(b') \leq v_j}} \frac{D^{2\sigma}}{(bb')^\sigma} = \sum_{\substack{d | N_j \\ \gcd(d, aa') = 1 \\ \omega(d) \leq v_j}} \varphi_{2\sigma}(d) \sum_{\substack{b, b' | N_j \\ \gcd(b, a) = \gcd(b', a') = 1 \\ d | \gcd(b, b') \\ \omega(b), \omega(b') \leq v_j}} \frac{1}{(bb')^\sigma} \\ &= \sum_{\substack{d | N_j \\ \gcd(d, aa') = 1 \\ \omega(d) \leq v_j}} \frac{\varphi_{2\sigma}(d)}{d^{2\sigma}} \sum_{\substack{B, B' | N_j \\ \gcd(B, ad) = \gcd(B', a'd) = 1 \\ \omega(B), \omega(B') \leq v_j - \omega(d)}} \frac{1}{(BB')^\sigma}, \end{aligned}$$

where $\varphi_{2\sigma}(d)$ is the Euler's totient function of order 2σ , satisfying $\sum_{d|D} \varphi_{2\sigma}(d) = D^{2\sigma}$. By the definition of $\varphi_{2\sigma}(d)$, for $d|N_j$ we have

$$\frac{\varphi_{2\sigma}(d)}{d^{2\sigma}} = \prod_{p|d} \left(1 - \frac{1}{p^{2\sigma}}\right) \geq \prod_{p|N_j} \left(1 - \frac{1}{p^{2\sigma}}\right) \gg 1. \quad (4.4.8)$$

Consequently, we derive that

$$\tilde{S}_\sigma \gg \sum_{d | N_j} \sum_{\substack{B, B' | N_j \\ \gcd(d, aa') = 1 \\ \gcd(B, ad) = \gcd(B', a'd) = 1 \\ \omega(d) \leq v_j \\ \omega(B), \omega(B') \leq v_j - \omega(d)}} \frac{1}{(BB')^\sigma}.$$

Substitute this into (4.4.7). For the sum over a and a' , we follow a similar procedure and derive that

$$S_\sigma(\mathcal{M}_j) \gg \sum_{\substack{c | N_j \\ \omega(c) \leq v_j}} \sum_{\substack{A, A' | N_j \\ \gcd(A, c) = \gcd(A', c) = 1 \\ \omega(A), \omega(A') \leq v_j - \omega(c)}} \frac{1}{(AA')^\sigma} \sum_{\substack{d | N_j \\ \gcd(d, AA'c) = 1 \\ \omega(d) \leq v_j}} \sum_{\substack{B, B' | N_j \\ \gcd(B, Acd) = \gcd(B', A'cd) = 1 \\ \omega(B), \omega(B') \leq v_j - \omega(d)}} \frac{1}{(BB')^\sigma}$$

$$= \sum_{\substack{c|N_j \\ \omega(c) \leq v_j}} \sum_{\substack{d|N_j \\ \omega(d) \leq v_j}} \sum_{\substack{A|N_j \\ \omega(A) \leq v_j - \omega(c)}} \frac{1}{A^\sigma} \sum_{\substack{A'|N_j \\ \omega(A') \leq v_j - \omega(c)}} \frac{1}{A'^\sigma} \sum_{\substack{B|N_j \\ \omega(B) \leq v_j - \omega(d)}} \frac{1}{B^\sigma} \sum_{\substack{B'|N_j \\ \omega(B') \leq v_j - \omega(d)}} \frac{1}{B'^\sigma}. \quad (4.4.9)$$

We calculate from inside successively. Since each term is positive, we can sift a suitable subset. Therefore we restrict c, d such that $\omega(c) = \omega(d) = v_j - u_j$, and A, A' such that $\omega(A) = \omega(A') = u_j$. Then the inner sum turns to

$$\sum_{\substack{B'|N_j \\ \gcd(B', A'cd) = 1 \\ \omega(B') \leq u_j}} \frac{1}{B'^\sigma} \geq \frac{1}{u_j!} \left(\sum_{\substack{p \in I_j \\ p \nmid A'cd}} \frac{1}{p^\sigma} \right)^{u_j} = \frac{1}{u_j!} \left(\sum_{p \in I_j} \frac{1}{p^\sigma} - \sum_{\substack{p \in I_j \\ p \mid A'cd}} \frac{1}{p^\sigma} \right)^{u_j}. \quad (4.4.10)$$

For the factorial, we use Stirling's formula

$$u_j! = \exp(u_j \log u_j - u_j + O(\log u_j)) = \left((1 + o(1)) \frac{u_j}{e} \right)^{u_j}.$$

For the sums, by Lemma 4.2.2 we have

$$\sum_{p \in I_j} \frac{1}{p^\sigma} = (1 + o(1)) \frac{(f^{1-\sigma} - 1)f^{j(1-\sigma)}(\log N)^{1-\sigma}}{(1-\sigma)(\log_2 N)^\sigma}.$$

Since $\omega(A'cd) \leq \omega(A') + \omega(c) + \omega(d) \leq 2v_j$, we have

$$\sum_{\substack{p \in I_j \\ p \mid A'cd}} \frac{1}{p^\sigma} \leq \sum_{p \mid A'cd} \frac{1}{p^\sigma} \leq \frac{2v_j}{(f^j \log N \log_2 N)^\sigma} \leq \frac{2\alpha(\log N)^{1-\sigma}}{j^2 f^{j\sigma} \log J (\log_2 N)^\sigma} \ll \frac{1}{j^2 f^j \log J} \sum_{p \in I_j} \frac{1}{p^\sigma}.$$

Note that $J \rightarrow \infty$ as $\sigma \searrow \frac{1}{2}$. Therefore, in (4.4.10) we have

$$\sum_{\substack{B'|N_j \\ \gcd(B', A'cd) = 1 \\ \omega(B') \leq u_j}} \frac{1}{B'^\sigma} \geq \left((1 + o(1)) \frac{e(f^{1-\sigma} - 1) j f^{j/2} \sqrt{\log J}}{\eta(1-\sigma)} \right)^{u_j}.$$

We can play similar trick on sums over B, A', A in (4.4.9) successively and therefore

$$S_\sigma(\mathcal{M}_j) \gg \left((1 + o(1)) \frac{e(f^{1-\sigma} - 1) j f^{j/2} \sqrt{\log J}}{\eta(1-\sigma)} \right)^{4u_j} \sum_{\substack{c|N_j \\ \omega(c) = v_j - u_j}} \sum_{\substack{d|N_j \\ \gcd(d,c) = 1 \\ \omega(d) = v_j - u_j}} 1. \quad (4.4.11)$$

Trivially, we have

$$\sum_{\substack{c|N_j \\ \omega(c) = v_j - u_j}} \sum_{\substack{d|N_j \\ \gcd(d,c) = 1 \\ \omega(d) = v_j - u_j}} 1 \geq \binom{P_j}{v_j - u_j} \binom{P_j - v_j + u_j}{v_j - u_j} \geq \binom{P_j}{v_j} \binom{P_j - v_j}{v_j} \left(\frac{v_j}{P_j} \right)^{2u_j}.$$

Therefore, by (4.4.5) we have

$$\sum_{\substack{c|N_j \\ \omega(c) \leq v_j - u_j \\ \omega(d) \leq v_j - u_j}} \sum_{d|N_j} 1 \geq \frac{|\mathcal{M}_j|}{4} \left(\frac{v_j}{P_j} \right)^{2u_j} = \frac{|\mathcal{M}_j|}{4} \left((1 + o(1)) \frac{\sqrt{\alpha}}{j(f-1)^{\frac{1}{2}} f^{j/2} \sqrt{\log J}} \right)^{4u_j},$$

Combined with (4.4.11), we obtain that

$$\frac{S_\sigma(\mathcal{M}_j)}{|\mathcal{M}_j|} \gg \left((1 + o(1)) \frac{e\sqrt{\alpha}(f^{1-\sigma} - 1)}{\eta(1-\sigma)\sqrt{f-1}} \right)^{4u_j}.$$

By the definition of u_j , we have

$$\begin{aligned} \sum_{j \leq J} u_j &= \frac{4\eta(\log N)^{1-\sigma}}{\sqrt{\log J}(\log_2 N)^\sigma} \sum_{j \leq J} \frac{1}{jf^{j(\sigma-\frac{1}{2})}} + O(J) \\ &= (1 + o(1)) \frac{4\eta\sqrt{\log J}(\log N)^{1-\sigma}}{f^{J(\sigma-\frac{1}{2})}(\log_2 N)^\sigma} \\ &= (1 + o(1)) \frac{4\eta\sqrt{\lambda|\log(\sigma-\frac{1}{2})|}(\log N)^{1-\sigma}}{f^{(\sigma-\frac{1}{2})^{1-\lambda}}(\log_2 N)^\sigma}. \end{aligned}$$

Therefore, by taking product over j , we obtain

$$\frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} \geq \exp \left\{ (1 + o(1)) \frac{4\eta\sqrt{\lambda|\log(\sigma-\frac{1}{2})|}(\log N)^{1-\sigma}}{f^{(\sigma-\frac{1}{2})^{1-\lambda}}(\log_2 N)^\sigma} \log \left(\frac{e\sqrt{\alpha}(f^{1-\sigma} - 1)}{\eta(1-\sigma)\sqrt{f-1}} \right) \right\}.$$

Write

$$H := \frac{4\eta\sqrt{\lambda}}{f^{(\sigma-\frac{1}{2})^{1-\lambda}}} \log \left(\frac{e\sqrt{\alpha}(f^{1-\sigma} - 1)}{\eta(1-\sigma)\sqrt{f-1}} \right).$$

Note that (4.4.6) implies we need to restrict $2\alpha \log f \leq 1$. To get large value of H , we set

$$f \rightarrow 1^+, \quad 2\alpha \log f \rightarrow 1^-, \quad \eta \rightarrow \frac{\sqrt{2}}{2}, \quad \lambda \rightarrow 1^-.$$

Then H can be sufficiently close to $2\sqrt{2}$ and we finally derive that

$$\Gamma_\sigma(N) \geq \exp \left\{ (2\sqrt{2} + o(1)) \sqrt{|\log(\sigma-\frac{1}{2})|} \frac{(\log N)^{1-\sigma}}{(\log_2 N)^\sigma} \right\}.$$

4.5 Convolution method: Proof of Theorem 4.1.3

Let \mathcal{M} be a set of positive integers with cardinal $|\mathcal{M}| = N = T^\kappa$ where $0 < \kappa < 1$ to be chosen. Define

$$\mathcal{M}_j = \mathcal{M} \cap \left[\left(1 + \frac{\log T}{T} \right)^j, \left(1 + \frac{\log T}{T} \right)^{j+1} \right).$$

For $\mathcal{J} = \{j \geq 0 : \mathcal{M}_j \neq \emptyset\}$, let

$$\mathcal{M}' = \{m_j = \min \mathcal{M}_j : j \in \mathcal{J}\}.$$

Then we define the resonator

$$R(t) = \sum_{m \in \mathcal{M}'} r(m)m^{it},$$

where $r(m_j) = |\mathcal{M}_j|^{\frac{1}{2}}$. Trivially we have

$$|R(t)| \leq R(0) = \sum_{m \in \mathcal{M}'} r(m) \leq \left(\sum_{m \in \mathcal{M}'} 1 \right)^{\frac{1}{2}} \left(\sum_{m \in \mathcal{M}'} r(m)^2 \right)^{\frac{1}{2}} \leq |\mathcal{M}'|^{\frac{1}{2}} |\mathcal{M}|^{\frac{1}{2}} \leq N.$$

Let $0 < \varepsilon < 1$, for $u \in \mathbb{R}$, we take

$$K(u) := \frac{\sin^2(\varepsilon u \log T)}{\pi u^2 \varepsilon \log T},$$

which satisfies

$$\hat{K}(\xi) = \max \left(1 - \frac{|\xi|}{2 \log(T^\varepsilon)}, 0 \right). \quad (4.5.1)$$

Write

$$\mathfrak{Z}_\sigma(t, u) := \zeta(\sigma + it + iu)\zeta(\sigma - it + iu)K(u).$$

Then we define

$$M_1(R, T) := \int_{T^\beta \leq |t| \leq T} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt.$$

$$M_2(R, T) := \int_{2T^\beta \leq |t| \leq T/2} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) \int_{|u| \leq |t|/2} \mathfrak{Z}_\sigma(t, u) du dt.$$

Since $K(\cdot)$ is bounded by 1, clearly we have

$$Z_\beta(\sigma, T)^2 \geq \frac{|M_2(R, T)|}{M_1(R, T)}. \quad (4.5.2)$$

As in §3, we approximate $M_1(R, T)$ and $M_2(R, T)$ by their relative full integral, i.e., by

$$I_1(R, T) = \int_{\mathbb{R}} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt,$$

$$I_2(R, T) = \int_{\mathbb{R}} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) \int_{\mathbb{R}} \mathfrak{Z}_\sigma(t, u) du dt.$$

Lemma 4.5.1. *For $I'(R, T)$, we have*

$$M_1(R, T) \leq I'(R, T) \ll \frac{T |\mathcal{M}|}{\log T},$$

where the implied constant is absolute.

Proof. The first inequality is trivial. Further we have

$$I_1(R, T) = \frac{T}{\log T} \sum_{i,j \in \mathcal{J}} r(m_i)r(m_j) \widehat{\phi} \left(\frac{T}{\log T} \log \frac{m_j}{m_i} \right).$$

In the sum, the diagonal terms contribute

$$\widehat{\phi}(0) \sum_{i \in \mathcal{J}} r(m_i)^2 = \widehat{\phi}(0) \sum_{i \in \mathcal{J}} |\mathcal{M}_i| = \widehat{\phi}(0) |\mathcal{M}'| \leq \widehat{\phi}(0) |\mathcal{M}|. \quad (4.5.3)$$

For the off-diagonal terms, we divide the sum according to the values of $|i - j|$ and have

$$\begin{aligned} \sum_{\substack{i,j \in \mathcal{J} \\ i \neq j}} r(m_i)r(m_j) \widehat{\phi} \left(\frac{T}{\log T} \log \frac{m_j}{m_i} \right) &= \sum_{l \geq 1} \sum_{\substack{i,j \in \mathcal{J} \\ |i-j|=l}} r(m_i)r(m_j) \widehat{\phi} \left(\frac{T}{\log T} \log \frac{m_j}{m_i} \right) \\ &\leq \sum_{l \geq 1} \sum_{\substack{i,j \in \mathcal{J} \\ |i-j|=l}} r(m_i)r(m_j) \widehat{\phi} \left(\frac{T}{\log T} \log \left(1 + \frac{\log T}{T} \right)^{l-1} \right). \end{aligned}$$

Recall that ϕ is rapidly decay. Thus this is bounded by

$$\sum_{l \geq 1} \sum_{\substack{i,j \in \mathcal{J} \\ |i-j|=l}} r(m_i)r(m_j) \leq \sum_{i \in \mathcal{J}} r(m_i)^2 \leq |\mathcal{M}|.$$

This combined with (4.5.3) proves the lemma. \square

Lemma 4.5.2. *For $I(R, T)$, we have*

$$I_2(R, T) = M_2(R, T) + O(|\mathcal{M}| T^{\beta+\kappa} \log T),$$

where the implied constant is absolute.

Proof. By the definition, we have that

$$\begin{aligned} I_2(R, T) - M_2(R, T) &= \left(\int_{|t|<2T^\beta} \int_{\mathbb{R}} + \int_{2T^\beta \leq |t| \leq T/2} \int_{|u|>|t|/2} + \int_{|t|>T/2} \int_{\mathbb{R}} \right) |R(t)|^2 \phi \left(\frac{t \log T}{T} \right) \mathfrak{Z}_\sigma(t, u) du dt. \end{aligned}$$

Denote the three integrals on the right-hand side by $D_1(R, T)$, $D_2(R, T)$ and $D_3(R, T)$ respectively. We prove that each is bounded by $O(|\mathcal{M}| T^{\beta+\kappa} \log T)$.

Firstly, recall that

$$|\zeta(\sigma + it)| \ll (1 + |t|)^{\frac{1}{3}(1-\sigma)}.$$

Therefore, we have

$$\int_{|t|<2T^\beta} \int_{|u|\leq T^\beta} |\mathfrak{Z}_\sigma(t, u)| du dt \leq \int_{|t|<2T^\beta} \int_{|u|\leq T^\beta} |\zeta(\sigma + it)|^2 du dt \ll T^\beta \log T,$$

and

$$\int_{|t|<2T^\beta} \int_{|u|>T^\beta} |\mathfrak{Z}_\sigma(t, u)| du dt \leq \int_{|t|<2T^\beta} \int_{|u|>T^\beta} \frac{(|t| + |u|)^{\frac{1}{3}(1-\sigma)}}{u^2} du dt \ll T^\beta.$$

Consequently, we derive that

$$D_1(R, T) \leq N^2 \int_{|t| < 2T^\beta} \left(\int_{|u| \leq T^\beta} + \int_{|u| > T^\beta} \right) |\mathfrak{Z}_\sigma(t, u)| du dt \ll |\mathcal{M}| T^{\beta+\kappa} \log T.$$

Secondly, we have that $D_2(R, T)$ is bounded by

$$\int_{T^\beta \leq |t| \leq T} |R(t)|^2 \phi \left(\frac{t \log T}{T} \right) \int_{|u| > |t|/2} \frac{(|t| + |u|)^{\frac{1}{3}(1-\sigma)}}{u^2} du dt \leq T^{-\beta/2} I'(R, T),$$

and thus is admissible by Lemma 4.5.1. Finally, since ϕ decays rapidly, we have

$$D_3(R, T) \ll N^2 = |\mathcal{M}| T^\kappa.$$

This completes the proof. \square

Using Lemma 4.5.1 and Lemma 4.5.2, we establish from (4.5.2) that

$$\frac{|M_2(R, T)|}{M_1(R, T)} \gg \frac{I_2(R, T)}{T|\mathcal{M}|} \log T + O(T^{\beta+\kappa-1}(\log T)^2). \quad (4.5.4)$$

To estimate $I(R, T)$, we need to deal with the convolution of $K(u)$ with $\zeta(s)$. Here we quote the following lemma due to de la Bretèche and Tenenbaum [20, Lemma 5.3].

Lemma 4.5.3. *Let $\sigma \in (-\infty, 1)$. Suppose $K(z)$ is analytic in the strip $\Im z \in [\sigma - 2, 0]$, satisfying*

$$\sup_{\sigma-2 \leq y \leq 0} |K(x + iy)| \ll \frac{1}{1+x^2}.$$

Then for any $t \neq 0$ we have

$$\int_{\mathbb{R}} \mathfrak{Z}_\sigma(t, u) du = \sum_{k, l \geq 1} \frac{\widehat{K}(\log kl)}{k^{\sigma+it} l^{\sigma-it}} - 2\pi \zeta(1 - 2it) K(i(\sigma + it) - i) - 2\pi \zeta(1 + 2it) K(i(\sigma - it) - i).$$

We expand $K(u)$ analytically continuously to the whole complex plane, satisfying the conditions of Lemma 4.5.3. Therefore, we may deduce that

$$I_2(R, T) = I_{2,1}(R, T) - I_{2,2}(R, T) - I_{2,3}(R, T),$$

where

$$\begin{aligned} I_{2,1}(R, T) &= \int_{\mathbb{R}} |R(t)|^2 \phi \left(\frac{t \log T}{T} \right) \sum_{k, l \geq 1} \frac{\widehat{K}(\log kl)}{k^{\sigma+it} l^{\sigma-it}} dt, \\ I_{2,2}(R, T) &= 2\pi \int_{\mathbb{R}} \zeta(1 - 2it) K(i(\sigma + it) - i) |R(t)|^2 \phi \left(\frac{t \log T}{T} \right) dt, \\ I_{2,3}(R, T) &= 2\pi \int_{\mathbb{R}} \zeta(1 + 2it) K(i(\sigma - it) - i) |R(t)|^2 \phi \left(\frac{t \log T}{T} \right) dt. \end{aligned}$$

For $I_{2,2}(R, T)$ and $I_{2,3}(R, T)$, we have

$$I_{2,2}(R, T) + I_{2,3}(R, T) \ll \frac{|\mathcal{M}| T^\kappa}{\log T} \int_{\mathbb{R}} \frac{|\zeta(1 \pm 2it)|}{1+t^2} \phi \left(\frac{t \log T}{T} \right) dt \ll \frac{|\mathcal{M}| T^\kappa}{\log T}, \quad (4.5.5)$$

since $K(\mathrm{i}(\sigma \pm it) - \mathrm{i}) \ll 1/((1+t^2)\log T)$. Thus $I_{2,2}(R, T)$ and $I_{2,3}(R, T)$ are admissible as error terms.

For $I_{2,1}(R, T)$, each term is nonnegative, so we have

$$\begin{aligned} I_{2,1}(R, T) &= \frac{T}{\log T} \sum_{m,n \in \mathcal{M}'} r(m)r(n) \sum_{k,l \geq 1} \frac{\widehat{K}(\log kl)}{k^\sigma l^\sigma} \widehat{\phi}\left(\frac{T}{\log T} \log \frac{mk}{nl}\right) \\ &\geq \frac{T}{\log T} \sum_{m,n \in \mathcal{M}'} r(m)r(n) \sum_{kl \leq T^\varepsilon} \frac{\widehat{K}(\log kl)}{k^\sigma l^\sigma} \widehat{\phi}\left(\frac{T}{\log T} \log \frac{mk}{nl}\right). \end{aligned}$$

Note that by (4.5.1), $kl \leq T^\varepsilon$ implies $\widehat{K}(\log kl) \geq \frac{1}{2}$. Therefore

$$I_{2,1}(R, T) \geq \frac{T}{2\log T} \sum_{kl \leq T^\varepsilon} \frac{1}{k^\sigma l^\sigma} \sum_{m,n \in \mathcal{M}'} r(m)r(n) \widehat{\phi}\left(\frac{T}{\log T} \log \frac{mk}{nl}\right).$$

For the inner sum, for fixed k and l , we have

$$\begin{aligned} \sum_{m,n \in \mathcal{M}'} r(m)r(n) \widehat{\phi}\left(\frac{T}{\log T} \log \frac{mk}{nl}\right) &\geq \sum_{i,j \in \mathcal{J}} \sum_{\substack{m \in \mathcal{M}_i, n \in \mathcal{M}_j \\ mk = nl}} \widehat{\phi}\left(\frac{T}{\log T} \log \frac{m_i k}{m_j l}\right) \\ &= \sum_{i,j \in \mathcal{J}} \sum_{\substack{m \in \mathcal{M}_i, n \in \mathcal{M}_j \\ mk = nl}} \widehat{\phi}\left(\frac{T}{\log T} \log \frac{m_i n}{m_j m}\right) \gg \sum_{\substack{m,n \in \mathcal{M} \\ mk = nl}} 1, \end{aligned}$$

since $1 \leq m/m_i \leq 1 + \log T/T$ and $1 \leq n/m_j \leq 1 + \log T/T$. Thus we have

$$I_{2,1}(R, T) \gg \frac{T}{\log T} \sum_{\substack{m,n \in \mathcal{M} \\ mk = nl}} \sum_{kl \leq T^\varepsilon} \frac{1}{k^\sigma l^\sigma}. \quad (4.5.6)$$

Write $m = m' \gcd(m, n)$ and $n = n' \gcd(m, n)$. Then the relation $mk = nl$ implies

$$L = \frac{k}{n'} = \frac{l}{m'},$$

for some integer L and

$$kl = L^2 m' n' = L^2 \frac{\operatorname{lcm}(m, n)}{\gcd(m, n)}.$$

Therefore in (4.5.6), we have

$$\begin{aligned} I_{2,1}(R, T) &\gg \frac{T}{\log T} \sum_{m,n \in \mathcal{M}} \sum_{\substack{\operatorname{lcm}(m,n) \\ \operatorname{gcd}(m,n)}} \frac{1}{L^{2\sigma}} \left(\frac{\operatorname{gcd}(m, n)}{\operatorname{lcm}(m, n)} \right)^\sigma \\ &\gg \frac{T}{\log T} \sum_{\substack{m,n \in \mathcal{M} \\ \frac{\operatorname{lcm}(m,n)}{\operatorname{gcd}(m,n)} \leq T^\varepsilon}} \left(\frac{\operatorname{gcd}(m, n)}{\operatorname{lcm}(m, n)} \right)^\sigma = \frac{T}{\log T} S_\sigma(\mathcal{M}) - \frac{T}{\log T} \sum_{\substack{m,n \in \mathcal{M} \\ \frac{\operatorname{lcm}(m,n)}{\operatorname{gcd}(m,n)} > T^\varepsilon}} \left(\frac{\operatorname{gcd}(m, n)}{\operatorname{lcm}(m, n)} \right)^\sigma. \end{aligned}$$

By Rankin's trick, we have

$$\sum_{\substack{m,n \in \mathcal{M} \\ \frac{\operatorname{lcm}(m,n)}{\operatorname{gcd}(m,n)} > T^\varepsilon}} \left(\frac{\operatorname{gcd}(m, n)}{\operatorname{lcm}(m, n)} \right)^\sigma \leq \frac{1}{T^{(\sigma - \frac{1}{2})\varepsilon}} \sum_{m,n \in \mathcal{M}} \left(\frac{\operatorname{gcd}(m, n)}{\operatorname{lcm}(m, n)} \right)^{\frac{1}{2}} = \frac{1}{T^{(\sigma - \frac{1}{2})\varepsilon}} S_{\frac{1}{2}}(\mathcal{M}).$$

By using (4.4.1), we have that this is bounded by

$$\frac{|\mathcal{M}|}{T^{(\sigma-\frac{1}{2})\varepsilon}} \exp\left((2\sqrt{2} + o(1))\sqrt{\frac{\kappa \log T \log_3 T}{\log_2 T}}\right) = o(|\mathcal{M}|),$$

Since $\sigma \geq \frac{1}{2} + \frac{1}{\log_2 T}$ and we can choose $\varepsilon = \frac{1}{2022}$. So we have

$$I_{2,1}(R, T) \gg \frac{T}{\log T} S_\sigma(\mathcal{M}).$$

Combining this with (4.5.4) and (4.5.5), we have

$$\frac{|M_2(R, T)|}{M_1(R, T)} \gg \frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} + T^{\beta+\kappa-1} (\log T)^2.$$

Choose $\kappa = 1 - \beta$, by (4.5.2) we have

$$Z_\beta(\sigma, T)^2 \geq \frac{|M_2(R, T)|}{M_1(R, T)} \gg \frac{S_\sigma(\mathcal{M})}{|\mathcal{M}|} + (\log T)^2,$$

and thus by taking maximum of both sides over $|\mathcal{M}| = N$ we have

$$Z_\beta(\sigma, T) \gg \sqrt{\Gamma_\sigma(T^{1-\beta})}.$$

4.6 Long resonance method

In this section, we will use the long resonance method to detect large values of the Riemann zeta function in the strip $\frac{1}{2} < \Re s < 1$ without using the conclusions of GCD sums. The following lemma states that for almost all $t \in [0, T]$, $\zeta(\sigma + it)$ can be approximated by its short Euler product.

Lemma 4.6.1. *Assume the zero-density estimates $N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)+o(1)}$. Then we have the short Euler product approximation of the Riemann zeta function for $\frac{1}{2} < \sigma < 1$ and $0 < \lambda < 1$ fixed:*

$$\zeta(\sigma + it) = \zeta(\sigma + it; Y) \left(1 + O\left(\frac{\log |t| (\log Y)^2}{Y^{\lambda(\sigma-\frac{1}{2})}}\right) \right),$$

for all $|t| \in [0, T]$ except for a set \mathcal{E} of measure at most $YT^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+o(1)}$.

Proof. This follows directly from Lemma 3.2.3 by choosing $\sigma_0 = \lambda/2 + (1 - \lambda)\sigma$. \square

For $X = b \log T \log_2 T$ with $b = b(\sigma)$ positive to be determined later, we define the resonator

$$R(t) := \prod_{p \leq X} (1 - a_p p^{it})^{-1} = \sum_{n \in S(X)} a_n n^{it},$$

where

$$a_p = 1 - \left(\frac{p}{X}\right)^\sigma \quad \text{for } p \leq X, \quad \text{and} \quad a_p = 0 \quad \text{for } p > X,$$

and we extend a_n such that it is completely multiplicative. Since

$$\log R(0) = \log \prod_{p \leq X} \left(\frac{X}{p} \right)^\sigma = \sigma(\pi(X) \log X - \theta(X)) = \sigma(1 + o(1)) \frac{X}{\log X},$$

we have

$$|R(t)| \leq R(0) \leq T^{b\sigma+o(1)}. \quad (4.6.1)$$

Choosing $Y = (\log T)^{3/(\lambda(\sigma - \frac{1}{2}))}$, then we have

$$\zeta(\sigma + it) = \zeta(\sigma + it; Y) \left(1 + O\left(\frac{1}{\log T}\right) \right),$$

for all $t \in [-T, T] \setminus \mathcal{E}$ with

$$\text{meas } \mathcal{E} \leq T^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+o(1)}. \quad (4.6.2)$$

Since

$$\zeta(\sigma + it; Y) = \prod_{p \leq Y} \left(1 - \frac{1}{p^{\sigma+it}} \right)^{-1} = \exp \left(- \sum_{p \leq Y} \log \left(1 - \frac{1}{p^{\sigma+it}} \right) \right) = \exp \left(\sum_{p \leq Y} \frac{1}{p^{\sigma+it}} + O(1) \right),$$

we define

$$\mathfrak{S}_\sigma(t; Y) := \sum_{p \leq Y} \frac{1}{p^{\sigma+it}}.$$

Clearly we have

$$\mathfrak{S}_\sigma(t; Y) \ll T^{o(1)}$$

by the choice of Y . The relation

$$\begin{aligned} Z_\beta(\sigma, T) &:= \max_{T^\beta \leq |t| \leq T} |\zeta(\sigma + it)| \geq \max_{|t| \in [T^\beta, T] \setminus \mathcal{E}} |\zeta(\sigma + it)| \\ &\gg \max_{|t| \in [T^\beta, T] \setminus \mathcal{E}} |\zeta(\sigma + it; Y)| \gg \exp \left(\max_{|t| \in [T^\beta, T] \setminus \mathcal{E}} \Re \mathfrak{S}_\sigma(t; Y) \right) \end{aligned}$$

shows that it suffices to determine the maximum inside the exponent. Define

$$M_2(R, T) := \int_{|t| \in [T^\beta, T] \setminus \mathcal{E}} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt.$$

Then by (4.6.1) and (4.6.2) we have

$$\begin{aligned} M_2(R, T) &= \int_{|t| \in [T^\beta, T]} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt - \int_{|t| \in \mathcal{E}} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \\ &= \int_{|t| \in [T^\beta, T]} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt + O((\text{meas } \mathcal{E}) R(0)^2 T^{o(1)}) \\ &= \int_{|t| \in [T^\beta, T]} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt + T^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+2b\sigma+o(1)}. \end{aligned}$$

To extend the integral to the whole \mathbb{R} , we need the following evaluations. By (4.6.1) we have

$$\left| \int_{|t| < T^\beta} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \right| \leq R(0)^2 T^{\beta+o(1)} \leq T^{\beta+2b\sigma+o(1)}.$$

The rapid decay of $\phi(\cdot)$ implies

$$\int_{|t| > T} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \ll 1.$$

Thus by

$$\int_{\mathbb{R}} = \int_{|t| \in [T^\beta, T]} + \int_{|t| < T^\beta} + \int_{|t| > T},$$

we have

$$\begin{aligned} M_2(R, T) &= \int_{\mathbb{R}} \mathfrak{S}_\sigma(t; Y) |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt + O(T^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+2b\sigma+o(1)} + T^{\beta+2b\sigma+o(1)}) \\ &:= I_2(R, T) + O(T^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+2b\sigma+o(1)} + T^{\beta+2b\sigma+o(1)}) \\ &= I_2(R, T) + E, \end{aligned}$$

where we have denoted the sum of the two error terms by $E = E(\sigma, \lambda, \beta, a, b)$. Note that $I_2(R, T) \in \mathbb{R}^+$. Now since

$$M_1(R, T) := \int_{|t| \in [T^\beta, T] \setminus \mathcal{E}} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt \leq \int_{\mathbb{R}} |R(t)|^2 \phi\left(\frac{t \log T}{T}\right) dt := I_1(R, T),$$

we have

$$\max_{|t| \in [T^\beta, T] \setminus \mathcal{E}} \Re e \mathfrak{S}_\sigma(t; Y) \geq \frac{\Re e M_2(R, T)}{M_1(R, T)} = \frac{\Re e I_2(R, T) + E}{M_1(R, T)} \geq \frac{I_2(R, T)}{I_1(R, T)} + \frac{E}{I_1(R, T)}.$$

We have the trivial lower bound for $I_1(R, T)$

$$\begin{aligned} I_1(R, T) &= \int_{\mathbb{R}} \sum_{m, n \in S(X)} a_m a_n \left(\frac{m}{n}\right)^{it} \phi\left(\frac{t \log T}{T}\right) dt \\ &= \sum_{m, n \in S(X)} a_m a_n \frac{T}{\log T} \widehat{\phi}\left(\frac{T}{\log T} \log \frac{n}{m}\right) \\ &\geq \widehat{\phi}(0) \frac{T}{\log T} \sum_{n \in S(X)} a_n^2 = \widehat{\phi}(0) \frac{T}{\log T} \prod_{p \leq X} (1 - a_p^2)^{-1}. \end{aligned} \tag{4.6.3}$$

We have

$$\log \prod_{p \leq X} (1 - a_p^2)^{-1} = 2\sigma\pi(X) \log X - \sigma\theta(X) - \sum_{p \leq X} \log(2X^\sigma - p^\sigma).$$

Since

$$\sum_{p \leq X} \log(2X^\sigma - p^\sigma) = \sigma\pi(X) \log X + \left(\sigma \int_0^1 \frac{dt}{2t^{-\sigma} - 1} + o(1)\right) \frac{X}{\log X},$$

writing

$$c(\sigma) := \int_0^1 \frac{dt}{2t^{-\sigma} - 1},$$

we have

$$\log \prod_{p \leq X} (1 - a_p^2)^{-1} = \sigma \{1 + c(\sigma) + o(1)\} \frac{X}{\log X} = b\sigma \{1 + c(\sigma) + o(1)\} \log T,$$

and then by (4.6.3), we have

$$I_1(R, T) \geq T^{b\sigma(1+c(\sigma))+1+o(1)}.$$

Thus

$$\frac{E}{I_1(R, T)} \leq T^{A(\lambda/2+(1-\lambda)\sigma)(1-\lambda/2-(1-\lambda)\sigma)+b\sigma(1-c(\sigma))-1+o(1)} + T^{\beta+b\sigma(1-c(\sigma))-1+o(1)}.$$

Assume

$$A(\lambda/2 + (1 - \lambda)\sigma)(1 - \lambda/2 - (1 - \lambda)\sigma) + b\sigma(1 - c(\sigma)) - 1 < 0 \quad (4.6.4)$$

and

$$\beta + b\sigma(1 - c(\sigma)) - 1 < 0. \quad (4.6.5)$$

Then we only need to bound the ratio $I_2(R, T)/I_1(R, T)$. Since

$$\begin{aligned} I_2(R, T) &= \sum_{\substack{p \leq Y \\ m, n \in S(X)}} \frac{a_m a_n}{p^\sigma} \int_{\mathbb{R}} \left(\frac{m}{np} \right)^{it} \phi \left(\frac{t \log T}{T} \right) dt = \sum_{\substack{p \leq Y \\ m, n \in S(X)}} \frac{a_m a_n}{p^\sigma} \frac{T}{\log T} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{np}{m} \right) \\ &\geq \sum_{p, m, n \in S(X)} \frac{a_m a_n}{p^\sigma} \frac{T}{\log T} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{np}{m} \right) \geq \sum_{\substack{p, m, n \in S(X) \\ p|m}} \frac{a_m a_n}{p^\sigma} \frac{T}{\log T} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{np}{m} \right) \\ &= \sum_{p, n \in S(X)} \sum_{k \in S(X)} \frac{a_{kp} a_n}{p^\sigma} \frac{T}{\log T} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{n}{k} \right) \\ &= \sum_{p \leq X} \frac{a_p}{p^\sigma} \sum_{k, n \in S(X)} a_k a_n \frac{T}{\log T} \widehat{\phi} \left(\frac{T}{\log T} \log \frac{n}{k} \right) = I_1(R, T) \sum_{p \leq Y} \frac{a_p}{p^\sigma}, \end{aligned}$$

we have

$$\begin{aligned} \frac{I_2(R, T)}{I_1(R, T)} &\geq \sum_{p \leq X} \frac{a_p}{p^\sigma} = \sum_{p \leq X} \frac{1 - (p/X)^\sigma}{p^\sigma} = \sum_{p \leq X} \frac{1}{p^\sigma} - \frac{\pi(X)}{X^\sigma} \\ &= (1 + o(1)) \frac{X^\sigma}{(1 - \sigma) \log X} = \left(\frac{b^\sigma}{1 - \sigma} + o(1) \right) \frac{(\log T)^\sigma}{(\log_2 T)^{1-\sigma}}, \end{aligned}$$

by Lemma 4.2.2 and the choice of X . Now we treat the inequality (4.6.4) and (4.6.5). Set $\beta = 0$, then we only need to do with (4.6.4). Since $\lambda/2 + (1 - \lambda)\sigma$ can be as close to σ as possible, for any fixed small $\delta = \delta(\sigma)$. Then (4.6.5) implies

$$b < \frac{1 - A(\sigma)(1 - \sigma)}{\sigma(1 - c(\sigma))} - \delta/2.$$

Thus we can choose

$$b = \frac{1 - A(\sigma)(1 - \sigma)}{\sigma(1 - c(\sigma))} - \delta.$$

We use Ingham's bound $A(\sigma) \leq 3/(2 - \sigma)$, then we have

$$b = \frac{2\sigma - 1}{\sigma(2 - \sigma)(1 - c(\sigma))} - \delta.$$

So at last, we have obtained

$$\max_{t \in [0, T]} \geq \exp \left\{ \left(\frac{b^\sigma}{1 - \sigma} + o(1) \right) \frac{(\log T)^\sigma}{(\log_2 T)^{1-\sigma}} \right\},$$

where b is as above.

5 Large values of $|\zeta^{(\ell)}(1 + it)|$

5.1 Background

The study of the extreme values of the Riemann zeta function has a long history. Over the past decades, quite a few of results has been established. The values on the critical line $\sigma = \frac{1}{2}$ was first considered by Titchmarsh [51], who showed that there exists arbitrarily large t such that for any $\alpha < \frac{1}{2}$ we have $|\zeta(\frac{1}{2} + it)| \geq \exp((\log t)^\alpha)$. We refer to [41, 7, 6, 13, 14, 20] for results in progression. For the critical strip $\frac{1}{2} < \Re s < 1$, it was also Titchmarsh [50] who first showed that for any $\varepsilon > 0$ and fixed $\sigma \in (\frac{1}{2}, 1)$, there exists arbitrarily large t such that $|\zeta(\sigma + it)| \geq \exp\{(\log t)^{1-\sigma-\varepsilon}\}$. We refer to [34, 41, 1, 14, 22] on this topic. The study of the values on the 1-line can date back to 1925 when Littlewood [36] showed that there exists arbitrarily large t for which $|\zeta(1 + it)| \geq \{1 + o(1)\}e^\gamma \log_2 t$. Here and throughout, we denote by \log_j the j -th iterated logarithm and by γ the Euler constant. For further results, we refer to [34, 26, 3, 22].

It also draws wide interests on the extreme values of the derivatives of the Riemann zeta function. For any $\ell \in \mathbb{N}^+$, denote

$$Z^{(\ell)}(T) := \max_{t \in [T, 2T]} |\zeta^{(\ell)}(1 + it)|.$$

Besides other results, Yang [55] recently proved that if T is sufficiently large, then uniformly for $\ell \leq (\log T)/(\log_2 T)$, we have

$$Z^{(\ell)}(T) \geq e^\gamma \frac{\ell^\ell}{(\ell+1)^{\ell+1}} \{ \log_2 T - \log_3 T + O(1) \}^{\ell+1}. \quad (5.1.1)$$

In this section, we aim to improve the constant $\ell^\ell/(\ell+1)^{\ell+1}$ in (5.1.1). We have the following theorem.

Theorem 5.1.1. *For $T \rightarrow \infty$ and $\ell \leq (\log T)/(\log_2 T)$, we have*

$$Z^{(\ell)}(T) \geq \frac{e^\gamma}{\ell+1} (\log_2 T)^{\ell+1} \{1 + o(1)\}.$$

Remark 1. Recently, we [22] proved that

$$\max_{t \in [\sqrt{T}, T]} |\zeta(1 + it)| \geq e^\gamma (\log_2 T + \log_3 T + c),$$

where c is a computable constant. Granville and Soundararajan [26] predicted that this is still true for $\max_{t \in [T, 2T]} |\zeta(1 + it)|$. These results seems stronger than that of Theorem 5.1.1 with $\ell = 0$. The reason is that after taking derivatives of the Riemann zeta function, we are no longer able to make use of the multiplicativity of its Dirichlet coefficients as previously did. Nevertheless, Theorem 5.1.1 remains a generalization of Littlewood's initial bound (see [36]).

Both Yang's proof and ours employ the resonance method used by Bondarenko and Seip [14]. For a large x , we take

$$\mathcal{P} := \prod_{p \leqslant x} p^{b-1} \quad \text{and} \quad \mathcal{M} := \{n \in \mathbb{N} : n \mid \mathcal{P}\}. \quad (5.1.2)$$

The key ingredient of the proof is a weighted reciprocal sum in the form

$$\mathcal{S}(x; \ell) := \sum_{m \in \mathcal{M}} \sum_{k \mid m} \frac{(\log k)^\ell}{k},$$

where $\ell \geqslant 0$ is an integer. In [55], Yang divided \mathcal{M} as well as \mathcal{P} into two subsets, according to whether $p \leqslant x^{\ell/(\ell+1)}$. Our choice is to give a finer division. Specifically, let $J \geqslant 1$ be a positive integer. For $0 \leqslant j \leqslant J$, denote

$$\mathcal{M}_j := \left\{ m \in \mathbb{N} : m \mid \prod_{p \leqslant x^{j/J}} p^{b-1} \right\}. \quad (5.1.3)$$

Thus we divide the set \mathcal{M} into J subsets:

$$\mathcal{M} = \bigsqcup_{j=1}^J (\mathcal{M}_j \setminus \mathcal{M}_{j-1}).$$

By this trick, we are able to enlarge the estimate of $\mathcal{S}(x; \ell)$ with a factor $(1 + 1/\ell)^\ell$. We summarize it as the following proposition.

Proposition 5.1.1. *Under the previous notation, we have*

$$\frac{1}{|\mathcal{M}|} \mathcal{S}(x; \ell) \geqslant \frac{e^\gamma}{\ell+1} \left\{ 1 + O\left(\frac{1}{J} + \frac{J \log_2 x}{b} + \frac{J^2}{\log x}\right) \right\} (\log x)^{\ell+1},$$

uniformly for $x \geqslant 3$, $b \geqslant 1$, $J \geqslant 1$, $\ell \geqslant 0$ where the implied constant is absolute.

5.2 Proof of Proposition 5.1.1

The following asymptotic formula plays a key role in the proof of Proposition 5.1.1.

Lemma 5.2.1. *We have*

$$\prod_{p \leqslant x} \sum_{\nu=0}^{b-1} \left(1 - \frac{\nu}{b}\right) \frac{1}{p^\nu} = \left\{ 1 + O\left(\frac{\log_2 x}{b} + \frac{1}{\log x}\right) \right\} e^\gamma \log x$$

uniformly for $x \geqslant 3$ and $b \geqslant 1$, where the implied constants are absolute.

Proof. See also [55, Eq. (15)] and [14, page 129]. For a fixed prime p , we have

$$\sum_{\nu=0}^{b-1} \left(1 - \frac{\nu}{b}\right) \frac{1}{p^\nu} = \left(\sum_{\nu \geqslant 0} - \sum_{\nu \geqslant b} \right) \left(1 - \frac{\nu}{b}\right) \frac{1}{p^\nu}$$

$$= \left(1 - \frac{1}{b(p-1)}\right) \left(1 - \frac{1}{p}\right)^{-1} + O\left(\frac{1}{p^b}\right).$$

Therefore, we can deduce that

$$\prod_{p \leq x} \sum_{\nu=0}^{b-1} \left(1 - \frac{\nu}{b}\right) \frac{1}{p^\nu} = \left\{1 + O\left(\frac{1}{b} \sum_{p \leq x} \frac{1}{p-1}\right)\right\} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}.$$

Then the lemma follows by Mertens' formula

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \left\{1 + O\left(\frac{1}{\log x}\right)\right\} e^\gamma \log x,$$

and the fact that $\sum_{p \leq x} \frac{1}{p-1} \ll \log_2 x$. \square

Now we are prepared to prove Proposition 5.1.1. By the construction, the set \mathcal{M} is divisor-closed which means $k | m$, $m \in \mathcal{M}$ implies $k \in \mathcal{M}$. Then by the definition of \mathcal{M}_j in (5.1.3), we have

$$\sum_{m \in \mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} = \sum_{i=1}^J \sum_{k \in \mathcal{M}_j \setminus \mathcal{M}_{j-1}} \frac{(\log k)^\ell}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1.$$

Note that $k \in \mathcal{M}_j \setminus \mathcal{M}_{j-1}$ implies $k \geq x^{(j-1)/J}$. Therefore

$$\sum_{m \in \mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} \geq (\log x)^\ell \sum_{j=1}^J \left(\frac{j-1}{J}\right)^\ell \sum_{k \in \mathcal{M}_j \setminus \mathcal{M}_{j-1}} \frac{1}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1. \quad (5.2.1)$$

For each $0 \leq i \leq J$, we rewrite $m = m_1 m_2$ where m_1 and m_2 lie in \mathcal{M}_i and $\mathcal{M} \setminus \mathcal{M}_i$ accordingly. Then we have

$$\sum_{k \in \mathcal{M}_i} \frac{1}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1 = \sum_{m_1 \in \mathcal{M}_i} \sum_{\substack{k|m_1 \\ k|m}} \frac{1}{k} \sum_{m_2 \in \mathcal{M} \setminus \mathcal{M}_i} 1.$$

For the sum over m_1 , we have

$$\sum_{m_1 \in \mathcal{M}_i} \sum_{k|m_1} \frac{1}{k} = \prod_{p \in \mathcal{M}_i} \sum_{\substack{m_1 | p^{b-1} \\ k|m_1}} \frac{1}{k} = \prod_{p \leq x^{i/J}} \sum_{\nu=0}^{b-1} \frac{b-\nu}{p^\nu}.$$

For the sum over m_2 , clearly we have

$$\sum_{m_2 \in \mathcal{M} \setminus \mathcal{M}_i} 1 = \prod_{x^{i/J} < p \leq x} b.$$

Therefore, we deduce that

$$\sum_{k \in \mathcal{M}_i} \frac{1}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1 = b^{\pi(x)} \prod_{p \leq x^{i/J}} \sum_{\nu=0}^{b-1} \frac{1}{p^\nu} \left(1 - \frac{\nu}{b}\right).$$

Note that $b^{\pi(x)} = |\mathcal{M}|$. By Lemma 5.2.1, we have

$$\sum_{k \in \mathcal{M}_i} \frac{1}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1 = \frac{i}{J} |\mathcal{M}| \left\{ 1 + O\left(\frac{\log_2 x}{b} + \frac{J}{\log x}\right) \right\} e^\gamma \log x. \quad (5.2.2)$$

In view of (5.2.2), by taking difference of \mathcal{M}_{j-1} and \mathcal{M}_j , we obtain

$$\sum_{k \in \mathcal{M}_j \setminus \mathcal{M}_{j-1}} \frac{1}{k} \sum_{\substack{m \in \mathcal{M} \\ k|m}} 1 = \frac{|\mathcal{M}|}{J} \left\{ 1 + O\left(\frac{J \log_2 x}{b} + \frac{J^2}{\log x}\right) \right\} e^\gamma \log x.$$

Inserting this into (5.2.1), we have

$$\sum_{m \in \mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} \geq \frac{|\mathcal{M}|}{J} \sum_{j=1}^J \left(\frac{j-1}{J}\right)^\ell \left\{ 1 + O\left(\frac{J \log_2 x}{b} + \frac{J^2}{\log x}\right) \right\} e^\gamma (\log x)^{\ell+1}.$$

Now Proposition 5.1.1 follows supplied that

$$\frac{1}{J} \sum_{j=1}^J \left(\frac{j-1}{J}\right)^\ell = \frac{1}{\ell+1} + O\left(\frac{1}{J}\right).$$

While this is trivial by the integral inequalities

$$\frac{1}{J} \sum_{j=1}^J \left(\frac{j-1}{J}\right)^\ell \leq \int_0^1 u^\ell du \leq \frac{1}{J} \sum_{j=1}^J \left(\frac{j-1}{J}\right)^\ell + \frac{1}{J}.$$

5.3 Proof of Theorem 5.1.1

We start with the following lemma, which helps approximate the derivatives of the Riemann zeta function by the Dirichlet polynomials.

Lemma 5.3.1. *For $T \rightarrow \infty$, $T \leq t \leq 2T$ and $\ell \leq (\log T)/(\log_2 T)$, we have that*

$$(-1)^\ell \zeta^{(\ell)}(1+it) = \sum_{n \leq T} \frac{(\log n)^\ell}{n^{1+it}} + O((\log_2 T)^\ell),$$

where the implied constant is absolute.

Proof. This is [55, Lemma 1], where we have taken $\sigma = 1$ and $\varepsilon = (\log_2 T)^{-1}$ as Yang did. See also [51, Theorem 4.11]. \square

To employ the resonance method, we choose the same weight function $\phi(\cdot)$ as that used by Soundararajan [48, page 471]. Thus let $\phi(t)$ be a smooth function compactly supported in $[1, 2]$, such that $0 \leq \phi(t) \leq 1$ always and $\phi(t) = 1$ for $t \in (5/4, 7/4)$. Then the Fourier transform of ϕ satisfies $\widehat{\phi}(u) \ll_\alpha |u|^{-\alpha}$ for any integer $\alpha \geq 1$.

For sufficiently large T , we set

$$x = \frac{\log T}{3 \log_2 T} \quad \text{and} \quad b = \lfloor \log_2 T \rfloor.$$

Furthermore, we take \mathcal{P} and \mathcal{M} as (5.1.2). Note that $\mathcal{P} \leq \sqrt{T}$ by the prime number theorem. Then we define the resonator

$$R(t) := \sum_{m \in \mathcal{M}} m^{it}.$$

Denote

$$\begin{aligned} M_1(R, T) &:= \int_{\mathbb{R}} |R(t)|^2 \phi\left(\frac{t}{T}\right) dt, \\ M_2(R, T) &:= \int_{\mathbb{R}} (-1)^\ell \zeta^{(\ell)}(1+it) |R(t)|^2 \phi\left(\frac{t}{T}\right) dt. \end{aligned}$$

Since $\text{supp}(\phi) \subset [1, 2]$, we have that

$$Z^{(\ell)}(T) \geq \frac{|M_2(R, T)|}{M_1(R, T)}. \quad (5.3.1)$$

For $M_1(R, T)$, we have

$$M_1(R, T) = \sum_{m, n \in \mathcal{M}} \int_{\mathbb{R}} \left(\frac{m}{n}\right)^{it} \phi\left(\frac{t}{T}\right) dt = T \sum_{m, n \in \mathcal{M}} \widehat{\phi}(T \log(n/m)).$$

When $m \neq n$, the choice of \mathcal{P} guarantees that $|\log(n/m)| \gg 1/\sqrt{T}$, and consequently

$$\widehat{\phi}(T \log(n/m)) \ll \frac{1}{T^2}. \quad (5.3.2)$$

Thus the off-diagonal terms contributes

$$T \sum_{\substack{m, n \in \mathcal{M} \\ m \neq n}} \widehat{\phi}(T \log(n/m)) \ll \frac{1}{T} |\mathcal{M}|^2.$$

Therefore, we derive that

$$M_1(R, T) = T \widehat{\phi}(0) |\mathcal{M}| + O\left(\frac{1}{T} |\mathcal{M}|^2\right). \quad (5.3.3)$$

For $M_2(R, T)$, by Lemma 5.3.1 we have

$$\begin{aligned} M_2(R, T) &= \int_{\mathbb{R}} \left(\sum_{k \leq T} \frac{(\log k)^\ell}{k^{1+it}} \right) |R(t)|^2 \phi\left(\frac{t}{T}\right) dt + O((\log_2 T)^\ell M_1(R, T)) \\ &= T \sum_{k \leq T} \frac{(\log k)^\ell}{k} \sum_{m, n \in \mathcal{M}} \widehat{\phi}(T \log(kn/m)) + O((\log_2 T)^\ell M_1(R, T)). \end{aligned} \quad (5.3.4)$$

Similar to (5.3.2), for $kn \neq m$ we have

$$\widehat{\phi}(T \log(kn/m)) \ll \frac{1}{T^2}.$$

Consequently, we obtain that

$$\sum_{k \leq T} \frac{(\log k)^\ell}{k} \sum_{\substack{m, n \in \mathcal{M} \\ kn \neq m}} \widehat{\phi}(T \log(kn/m)) \ll \frac{(\log T)^{\ell+1}}{T^2} |\mathcal{M}|^2.$$

Inserting this into (5.3.4), we have

$$M_2(R, T) = T\widehat{\phi}(0) \sum_{m \in \mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} + O\left(\frac{(\log T)^{\ell+1}}{T} |\mathcal{M}|^2\right) + O\left((\log_2 T)^\ell M_1(R, T)\right).$$

Combining this with (5.3.1) and (5.3.3), we derive that

$$Z^{(\ell)}(T) \geq \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \sum_{k|m} \frac{(\log k)^\ell}{k} + O\left((\log_2 T)^\ell\right). \quad (5.3.5)$$

Then the theorem follows by taking $J = \lfloor \frac{1}{2} \log_3 T \rfloor$ in Proposition 5.1.1. \square

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