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NGUYEN Thi Thu

Fonctions de Goldbach généralisées et leurs Asymptotiques

Generalized Goldbach Functions and their Asymptotics

dirigée par Gautami BHOWMIK

Soutenue le 19 novembre 2024 devant le jury composé de:

Gautami BHOWMIK	Maîtresse de Conférences Université de Lille	Directrice de thèse
Alessandro LANGUASCO	Professeur Associé Universita di Padova	Rapporteur
Kohji MATSUMOTO	Professeur Émérite Nagoya University	Rapporteur
Emmanuel ROYER	Professeur Université Clermont Auvergne	Examinateur
Florent JOUVE	Professeur Université de Bordeaux	Examinateur
Lucile DEVIN	Maîtresse de Conférences Université du Littoral Côte d'Opale	Examinatrice
Didier LESESVRE	Maître de Conférences Université de Lille	Invité

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Résumé

Dans cette thèse, nous étudions un sujet important en théorie analytique des nombres, à savoir le problème de Goldbach, qui conjecture que tout nombre pair supérieur à 2 est la somme de deux nombres premiers. Nous nous intéressons particulièrement à deux versions généralisées du cas classique.

Tout d'abord, nous étudions ce problème où les nombres premiers sont en progressions arithmétiques. Il est connu que presque tous les entiers pairs satisfaisant une certaine condition de congruence peuvent s'écrire comme la somme de deux nombres premiers dans des classes de congruence. Nous étudierons la valeur moyenne de la fonction Goldbach pondérée dans les progressions arithmétiques qui est définie comme

$$S(X, q_1, q_2, a_1, a_1) = \sum_{n \leq X} \sum_{\substack{m+l=n \\ m \equiv a_1 (q_1) \\ l \equiv a_2 (q_2)}} \Lambda(m)\Lambda(l),$$

où Λ est la fonction de von Mangoldt.

Similairement au problème original, l'objectif principal est de montrer les termes principaux et le terme d'erreur de $S(X, q_1, q_2, a_1, a_2)$. Elle a été introduite pour la première fois par Rüppel en 2012. En 2017, Suzuki a obtenu une formule asymptotique sous l'hypothèse de Riemann généralisée avec des zéros réels tandis que pour le cas du module commun, Bhowmik-Halupczok-Matsumoto-Suzuki ont prouvé une formule asymptotique où le terme d'erreur est exprimé en termes de zéros de fonctions L.

L'un des objectifs de cette thèse est d'obtenir les résultats pour le cas de modules différents, de nouveau explicitement en termes de zéros de fonctions L de Dirichlet. Les termes principaux étant plus faciles à obtenir, nous nous concentrerons sur l'étude du terme d'erreur qui comprend la recherche d'une bonne borne supérieure et une bonne borne inférieure pour celui-ci. Une difficulté importante pour trouver sa borne supérieure est d'estimer le second moment pour les nombres premiers dans les progressions arithmétiques. Notez que sous l'hypothèse de Riemann généralisée, une telle borne supérieure a été étudiée par Prachar tandis que Goldston et Yildirim ont donné une borne inférieure de cette fonction. Cependant, l'approche ne fonctionne pas bien si nous souhaitons d'abord écrire explicitement le terme d'erreur qui peut ensuite être amélioré sous l'hypothèse de Riemann généralisée. Cela se traduit par le facteur $\log^3 X$ dans le terme d'erreur ainsi nous pouvons récupérer toutes les meilleures estimations connues des valeurs moyennes des représentations de Goldbach dans le cas classique et dans le cas des progressions arithmétiques. De plus, dans ce contexte, nous donnons une formule pondérée de Cesàro dans les progressions arithmétiques.

Nous obtenons en outre une bonne borne inférieure du terme d'erreur de

$S(X, q_1, q_2, a_1, a_2)$, ce qui montre que la formule asymptotique est essentiellement la meilleure possible.

Dans la deuxième partie, nous nous intéressons au nombre de représentations d'un entier comme somme de plusieurs nombres premiers. Nous nous concentrons sur le terme d'erreur des valeurs moyennes. Languasco et Zaccagnini ont énoncé une formule asymptotique avec le terme d'erreur $X^{k-1} \log^k X$ sous l'hypothèse de Riemann. Ici, nous utilisons une autre méthode et améliorons la puissance de log de k à 3 pour $k \geq 2$. Il s'agit d'une amélioration du résultat de Languasco-Zaccagnini pour $k \geq 4$. De plus, nous montrons qu'il existe un résultat oméga dans ce cas.

Il est connu qu'il existe une équivalence entre l'hypothèse de Riemann et d'une bonne estimation de $S(X)$, où

$$S(X) = \sum_{n \leq X} \sum_{m+l=n} \Lambda(m)\Lambda(l).$$

Dans ce travail, nous étendons cette équivalence pour la formule asymptotique des nombres qui peuvent s'écrire comme la somme de $k \geq 2$ nombres premiers.

Abstract

In this thesis, we study an important topic in analytic number theory, which is the Goldbach problem that expects every even number greater than 2 to be the sum of two prime numbers. We are particularly interested in two generalized versions of the classical case.

Firstly, we study this problem where the primes are in arithmetic progressions. It is known that almost all even integers satisfying some congruence condition can be written as the sum of two primes in congruence classes. We will study the average order of the weighted Goldbach function in arithmetic progressions which is defined as

$$S(X, q_1, q_2, a_1, a_1) = \sum_{n \leq X} \sum_{\substack{m+l=n \\ m \equiv a_1 (q_1) \\ l \equiv a_2 (q_2)}} \Lambda(m)\Lambda(l),$$

where Λ is the von Mangoldt function.

Similar to the original problem, the main goal is to show the main terms and the error term of $S(X, q_1, q_2, a_1, a_2)$. It was first introduced by Rüppel in 2012. In 2017, Suzuki obtained an asymptotic formula under the Generalized Riemann Hypothesis with real zeros while for the case of common modulus, Bhowmik-Halupczok-Matsumoto-Suzuki proved an asymptotic formula where the error term is in terms of the zeros of L-functions.

One goal of this thesis is to obtain the results for the case of different modulus again explicitly in terms of zeros of Dirichlet L-functions. The main terms are easier to obtain, we will focus on the error term and find a good upper bound and lower bound for it. An important difficulty in finding its upper bound is to estimate the second moment for primes in arithmetic progressions. Note that under the Generalized Riemann Hypothesis, such an upper bound was studied by Prachar while Goldston and Yildirim gave a lower bound of the second moment. However, the approach in does not work well if we wish to first write the error term explicitly which can then be improved under the Generalized Riemann Hypothesis. Here, we use another approach based on the study of the Chebyshev function to prove a good estimation unconditionally. This results in the factor $\log^3 X$ in the error term thus we can recover all the best known estimates of the average orders of Goldbach representations in the classical case and arithmetic progressions. Moreover, in this context, we give a Cesàro weighted formula in arithmetic progressions.

We further obtain a good lower bound of the error term of $S(X, q_1, q_2, a_1, a_2)$, which shows that the asymptotic formula is essentially the best possible.

In the second part, we are interested in the number of representations of an integer as a sum of many primes. We focus on the error term of the average orders. Languasco and Zaccagnini stated an asymptotic formula with the error term is $X^{k-1} \log^k X$ under the Riemann Hypothesis. Here we use another method and

improve the log-power from k to 3 for $k \geq 2$. This is an improvement of Languasco-Zaccagnini's result for $k \geq 4$. Moreover, we show that there is an omega-result in this case.

It is known that there is an equivalence between the Riemann Hypothesis and a good estimate of $S(X)$, where

$$S(X) = \sum_{n \leq X} \sum_{m+l=n} \Lambda(m)\Lambda(l).$$

In this work, we extend this equivalence for the asymptotic formula of the numbers that can be written as the sum of $k \geq 2$ primes.

Structure of the Thesis

My thesis includes 3 Chapters and an Appendix.

Chapter 1: We recall basic definitions, notations and important functions. We explain the original problem, the Goldbach conjecture in the classical case.

Chapter 2: We consider the binary Goldbach problem with summands in arithmetic progressions. Here I want to emphasise on different arithmetic progressions. Moreover, we prove an Omega-result of the error term and then show that the asymptotic formula above is essentially the best possible. This omega result is unconditional. We also deal with averages with a Cesàro weighted, with the constraint that the summands in the Goldbach representations lie in fixed arithmetic progressions. These results are available in the arXiv pre-print [37].

Chapter 3: We study the decomposition of n as a sum of several prime numbers. We can prove an equivalence between the Riemann hypothesis and an average order of the Goldbach representation for several primes. These results are available in the arXiv pre-print [38].

Appendix: We indicate some further directions of work to find bounds for the oscillating terms involving zeros of L-functions in our study of average orders, reduce the gap between the error and omega terms of the asymptotics of Goldbach functions and construct explicit numerical results without error terms for the summatory functions encountered.

Contents

Chapter 1

Introduction

Prime numbers are the building blocks of arithmetic and there and many conjectures has been formulated around them. The study of these problems, whether or not fully solved have given rich tools to number theory.

We cite two famous historical questions. At the start of the 19th century, Gauss and Legendre proposed a conjecture that the number of primes up to x is asymptotic to $x/\log x$, when x tends to infinity, where $\log x$ is the natural logarithm of x. This conjecture was proved independently by Hadamard and De la Vallée Poussin in 1896 using the ideas of Bernhard Riemann. This result is now known as the *Prime Number Theorem*. Another important 19th century result was Dirichlet's theorem on arithmetic progressions, that certain arithmetic progressions contain infinitely many primes.

Some old problems still remain open. One such unsolved problem is the Goldbach conjecture.

Christian Goldbach, a German mathematician, conjectured that every even number is the sum of two primes. Though this conjecture remains unsolved, in the 1930s, Vinogradov proved a weaker form which implies that any sufficiently large odd integer can be written as a sum of three prime numbers.

In this study, we concentrate on Goldbach's conjecture and some of its variants. On 7 June 1742, Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII), in which he proposed the following conjecture:

Every integer that can be written as the sum of two primes can also be written as the sum of as many primes as one wishes until all terms are units.

Goldbach was following the now-abandoned convention of considering 1 to be a prime number. He then proposed a second conjecture in the margin of his letter, which implies the first:

Every integer greater than 2 can be written as the sum of three primes.

Euler replied in a letter dated 30 June 1742 and reminded Goldbach of an earlier conversation they had had:

Every positive even integer can be written as the sum of two primes.

It has been known for a very long time that the conjecture is statistically true and it is empirically supported by calculations for all numbers the threshold of which has gone up from $10,000$ in 1855 [12] to 4×10^{18} [39] in 2014.

This conjecture is equivalent to proving the Goldbach *counting* function

$$g(n) := \sum_{p_1+p_2=n} 1, \tag{1.1}$$

which counts the number of representations of an integer n as the sum of two primes p_1 and p_2, is always positive for even $n > 2$.

Instead of studying directly the Goldbach counting function, we consider the corresponding problem for a smoother version using logarithms, that is the *weighted* Goldbach function

$$G(n) := \sum_{m+l=n} \Lambda(m)\Lambda(l), \tag{1.2}$$

where Λ is the von Mangoldt function.

In 1924, Hardy and Littlewood mentioned a conjecture about $G(n)$ that

Conjecture 1.0.1 (Hardy-Littlewood [24]). *The approximation $G(n) \sim J(n)$ holds for even n with*

$$J(n) := nC_2 \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2} \tag{1.3}$$

and $C_2 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)$ is the twin prime constant.

Though the conjecture in its totality seems out of reach at the moment it generates a lot of mathematical activity.

To state asymptotic results, we recall some material.

1.1 Notations

Definition 1.1.1. *Let f, g be two real or complex valued functions. We say that*

i, $f(x) = \mathcal{O}(g(x))$ (or $f(x) \ll g(x)$) if there exists some unspecified constant $C > 0$ and x_0 such that

$$|f(x)| \leq C|g(x)| \text{ for all } x \geq x_0.$$

ii, $f(x) = o(g(x))$ if for every positive constant ϵ there exists x_0 such that

$$|f(x)| \leq \epsilon g(x) \text{ for all } x \geq x_0.$$

iii, $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

iv, $f(x) = \Omega(g(x))$ if

$$\limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| > 0.$$

1.1.2 Arithmetic functions

An *arithmetic function* is a complex-valued function defined on $\mathbb{N}^* = \{1, 2, \dots\}$. The following arithmetic functions are classical.

1. *Euler's totient function*, counting the number of invertible residues modulo n. Noting that

$$\varphi(n) = n \prod_{p|n} (1 - p^{-1}). \tag{1.4}$$

2. The *von Mangoldt function* is defined as

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^\alpha, \alpha \geq 1, p \text{ is prime} \\ 0, & \text{otherwise.} \end{cases}$$

The weighted counting functions

$$\vartheta(x) = \sum_{p \leq x} \log p$$
$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_{n \leq x} \Lambda(n)$$

were introduced by Chebyshev. The functions $\vartheta(x)$ and $\psi(x)$ are approximately equal, and both are closely related to the counting function $\pi(x)$ of the primes. In particular, we have the asymptotic relations (see [35], Corollary 2.5)

$$\vartheta(x) = \psi(x) + \mathcal{O}(x^{1/2}),$$
$$\pi(x) = \frac{\psi(x)}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right).$$

So the asymptotic relations

$$\psi(x) \sim x, \qquad \vartheta(x) \sim x$$

are equivalent to the Prime Number Theorem. We define the *Chebyshev function* ψ *in arithmetic progressions.* Let q be a positive integer, a coprime to q, we have

$$\psi(x, q, a) = \sum_{\substack{n \leq x \\ n \equiv a(q)}} \Lambda(n). \tag{1.5}$$

We next give the partial summation formula, which is a tool frequently used in analytic number theory (see [1, Theorem 4.2]).

Proposition 1.1.3 (Partial summation formula)**.** *Let f be an arithmetic function and g be a continuous function with piecewise continuous derivative on* $[1, \infty)$. *Then*

$$\sum_{n \leq x} f(n) g(n) = F(x) g(x) - \int_1^x F(u) g'(u) du,$$

where F is the summatory function of f,

$$F(x) = \sum_{n \leq x} f(n).$$

The next two results are due to Mertens. These depend on the Chebyshev bound $\psi(x) = \mathcal{O}(x)$ is an essential way. Proofs of these results can be found in ([35, Theorem 2.7]).

Proposition 1.1.4. *The estimates*

$$\sum_{p\le x}\frac{\log p}{p} = \sum_{n\le x}\frac{\Lambda(n)}{n} + \mathcal{O}(1) = \log x + \mathcal{O}(1)$$

hold.

Proposition 1.1.5. *There is a constant* **a** *such that, for $x \ge 2$,*

$$\sum_{p\le x}\frac{1}{p} = \log\log x + \mathbf{a} + \mathcal{O}\left(\frac{1}{\log x}\right).$$

In fact, the constant **a** is known as the Meissel-Mertens constant, $\mathbf{a} \approx 0.261497$.

The next result is of considerable significance in prime number theory for various considerations of a probabilistic nature.

Proposition 1.1.6 (Mertens' product formula)**.** *Let $x \ge 2$, we have*

$$\prod_{n\le x}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma^*}}{\log x},$$

where γ^* denotes the Euler-Mascheroni constant, $\gamma^* \approx 0.5772\ldots$.

1.1.7 Dirichlet L-function

We denote by ζ the Riemann zeta function, defined as

$$\zeta(s) = \sum_{n\ge 1}\frac{1}{n^s}$$

for $\Re(s) > 1$. This functionn is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. The Riemann zeta function plays a pivotal role in analytic number theory. The functional equation, that is

$$\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}, \text{ for all } s,$$

shows that the Riemann zeta function has zeros at $-2, -4, \ldots$. These are called the trivial zeros. The non-trivial zeros have captured far more attention because their study yields important results concerning prime numbers and related objects in number theory. It is known that any non-trivial zero lies in the open strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$, which is called the *critical strip*. The zeta zeros are symmetrical with respect to the real axis and they are also symmetrical about the *critical line* $\Re(s) = \frac{1}{2}$. This brings us to the Riemann Hypothesis.

Hypothesis 1.1.8 (RH)**.** *The real part of every non-trivial zero of the Riemann zeta function is $\frac{1}{2}$.*

We now define the Dirichlet characters and state some useful properties.

Definition 1.1.9. *Let q be a positive integer. A Dirichlet character to the modulo q is the arithmetic function extending a character χ of the group $(\mathbb{Z}/q\mathbb{Z})^*$ by means of the formula*

$$\chi(n) = \begin{cases} \chi(m), & \text{if } n \equiv m(q),\ 1 \le m \le q,\ (m,q) = 1 \\ 0, & \text{if } (n,q) > 1. \end{cases}$$

We denote by $\chi_0(q)$ the principal character modulo q, whose value $\chi(n)$ is always 1 for n coprime to q.
A non-principal character is called quadratic if $\chi^2 = \chi_0$.
A character $\chi(q)$ is termed primitive if it cannot be factored as $\chi = \chi'\chi_0$, where χ' is a character of modulus strictly less than q.

Then we have the following.

Theorem 1.1.10 (Orthogonality of Dirichlet characters)**.** *Let q be a positive integer. We have*

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(n) = \begin{cases} 1 & \text{if } n \equiv 1(q) \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

$$\frac{1}{\varphi(q)} \sum_{\substack{n \\ (n,q)=1}} \chi(n) = \begin{cases} 1 & \text{if } \chi = \chi_0(q), \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

where $\varphi(q)$ is Euler's totient function.

Moreover, for any integers $n, m \ge 1$, we have

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(n)\overline{\chi}(m) = \begin{cases} 1 & \text{if } n \equiv m(q) \text{ and } (m,q) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

For all Dirichlet characters χ, χ', to the modulus q, we have

$$\frac{1}{\varphi(q)} \sum_{1 \le n \le q} \chi(n)\overline{\chi'}(n) = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

Let χ be a Dirichlet character modulo q, then the Dirichlet L-series is defined by the formula

$$L(s,\chi) = \sum_{n \ge 1} \frac{\chi(n)}{n^s}$$

with Euler product

$$L(s,\chi) = \prod \left(1 - \frac{\chi(p)}{p^s}\right).$$

This series and its Euler product converge on the half plane $\{s \in \mathbb{C} : \Re(s) > 1\}$. In particular, $L(s,\chi) \ne 0$ when $\Re(s) > 1$. The function $L(s,\chi)$ can be meromorphically continued to the entire plan. Its only singularity is a simple pole $s = 1$, with residue 1, when $\chi = \chi_0$.

Furthermore, Dirichlet L-functions satisfy a functional equation, which provides a way to analytically continue them throughout the complex plane. Let χ be a primitive character modulo q, the functional equation relates the value of $L(s, \chi)$ to the value of $L(1 - s, \overline{\chi})$,

$$\xi(s, \chi) = W(\chi)\xi(1 - s, \overline{\chi}) \qquad (s \in \mathbb{C})$$

with

$$\xi(s, \chi) := \left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+\delta)} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi), \qquad W(\chi) := \frac{\tau(\chi)}{i^{\delta}\sqrt{q}}, \qquad \delta := \frac{1 - \chi(-1)}{2}.$$

For $\Re(s) < 0$, there are zeros at certain negative integers s, they are the points $-2k$ $(k \geq 0)$ if $\chi(-1) = 1$ and the points $-2k - 1$ $(k \geq 0)$ if $\chi(-1) = -1$. These points are called the *trivial zeros* of $L(s, \chi)$ and the remaining zeros lie in the critical strip $0 \leq \Re(s) \leq 1$ are called the non-trivial zeros. In 1837, Dirichlet mentioned that $L(s, \chi)$ is non-zero at $s = 1$ and moreover, $L(s, \chi) \neq 0$ when $\Re(s) = 1$. The non-trivial zeros are symmetrical about the critical line $\Re(s) = 1/2$, that is, if $L(\rho, \chi) = 0$ then $L(1 - \overline{\rho}, \chi) = 0$ too. If χ is a real character, then the non-trivial zeros are also symmetrical about the real axis, but not if χ is a complex character.

Remark 1.1.11. *If χ is not primitive, we have*

$$L(s, \chi) = L(s, \chi^*) \prod_{\substack{p|q \\ p\nmid q^*}} \left(1 - \frac{\chi^*(p)}{p^s}\right),$$

where $\chi^*(q^*)$ is the primitive character inducing χ.

Similar to the Riemann zeta function, we focus on the non-trivial zeros of L-function. We denote by ρ_χ non-trivial zeros of $L(s, \chi)$ with the real part β_χ and the imaginary part γ_χ,

$$\rho_\chi = \beta_\chi + i\gamma_\chi.$$

We now state the Generalized Riemann Hypothesis.

Hypothesis 1.1.12 (GRH)**.** *The real part of every non-trivial zero of the Dirichlet L-function is $\frac{1}{2}$.*

Remark 1.1.13. *Let $B_\chi = \sup\{\Re(\rho_\chi) : L(s, \rho_\chi) = 0\}$ and $B_q = \sup\{B_\chi : \chi(q)\}$. We then have $\frac{1}{2} \leq B_q \leq 1$. The GRH would give $B_q = \frac{1}{2}$.*

We know that the zeros close to $\Re(s) = 1$ are indeed rare. More precisely, we recall the theorem following ([35, Corollary 11.10]).

Theorem 1.1.14 (Landau-Page)**.** *There is an absolute constant $c > 0$ such that for any $Q, T \geq 2$, the product $\prod_{q \leq Q} \prod^*_{\chi(q)} L(s, \chi)$ has at most one zero in the region*

$$|t| \leq T, \qquad 1 - \sigma \leq \frac{c}{\log(QT)},$$

where $\prod^*_{\chi(q)}$ denotes a product over all primitive characters $\chi(q)$. If such a zero exists, then it is necessarily real and associated to a unique, quadratic χ mod q.

Such a zero if it exists and the eventual corresponding character are respectively called the *exceptional zero* (or *Landau-Siegel zero*) and the *exceptional character.*

We let $\delta_1(\chi) = 1$ if χ is the exceptional character and 0 otherwise.

1.1.15 The circle method

The *circle method* originated in a paper of Hardy and Ramanujan in 1918 on the partition function $p(n)$, which for each natural number n counts the number of ways of writing n in the shape

$$n = x_1 + x_2 + \cdots + x_s,$$

where $s \in \mathbb{N}$ and $x_1 \geq x_2 \geq \cdots \geq x_s$ are natural numbers.

Hardy and Littlewood were instead interested in Waring's problem.

Conjecture 1.1.16 (E.Waring, 1770)**.** *All natural numbers are the sum of at most 4 squares of natural numbers, or of at most 9 cubes of natural numbers, or of at most 19 fourth powers of natural numbers, and so on.*

Hardy and Littlewood applied the circle method to Waring's problem [22]. Their idea was to write down a power series

$$h_k(z) = \sum_{n=0}^{\infty} z^{n^k},$$

where k is a non-negative integer. Note that this series is absolutely convergent for $\mid z \mid < 1$. We now consider the expression $h_k^s(z)$:

$$\begin{aligned} h_k^s(z) &= \sum_{n_1=0}^{\infty} z^{n_1^k} \sum_{n_2=0}^{\infty} z^{n_2^k} \cdots \sum_{n_s=0}^{\infty} z^{n_s^k} \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_s=0}^{\infty} z^{n_1^k + n_2^k + \cdots + n_s^k} \\ &= \sum_{n=0}^{\infty} R(s,k,n) z^n, \end{aligned}$$

where

$$R(s,k,n) = \#\{(n_1, n_2, \ldots, n_s) \in \mathbb{N}^s : n_1^k + n_2^k + \cdots + n_s^k = n\}.$$

The goal is to obtain a suitable estimate for $R(s,k,n)$, at least when n is large. By Cauchy's integral formula

$$R(s,k,n) = \frac{1}{2\pi i} \int_{\mathcal{C}} h_k^s(z) z^{-n-1} dz,$$

where $\mathcal{C}$ is a circle centered at 0 of radius r, where $0 < r < 1$ is close to 1. We suppose $z = re(\alpha)$ with $e(\alpha) = e^{2\pi i \alpha}$.

To study the order of $R(s,k,n)$, they split the circle $|z| = r$ into two sets; the *major arcs* corresponding to roots of unity of low order, and the complement, called the *minor arcs*. Integration over the major arcs gave a main term together with an error term that they could handle well by choosing the major arcs suitably. This left the harder problem of bounding the contribution to the final error term from the integral over the minor arcs. One remarkable application of the circle method is Rademacher's exact formula for the partition function ([2], Chapter 5).

Moreover, Hardy and Littlewood discovered that their method could also be applied to the binary and ternary Goldbach problems [23], [24].

1.2 Exceptional sets

With the definitions of $G(n)$, $J(n)$ in (1.2) and (1.3), Hardy and Littlewood [24] proved that

$$\sum_{n\leq x}(G(n)-J(n))^2 \ll x^{5/2+\epsilon} \tag{1.10}$$

under the GRH. Let now

$$E(x) = \#\{n \leq x, n \in 2\mathbb{N} : n \neq p_1 + p_2\}$$

denote the size of the exceptional set depending on x, i.e., the number of even integers up to x which do not satisfy the Goldbach conjecture.

From the bounds for the second moment (1.10) we are able to deduce estimates

$$E(x) \ll x^{1/2+\epsilon},$$

for any $\epsilon > 0$. This result was improved by Goldston [17] to

$$E(x) \ll x^{1/2}\log^3 x.$$

In 1975, Montgomery and Vaughan [34] showed unconditionally that there exists a positive effectively computable constant $\delta > 0$ such that, for all large $x = x(\delta)$, one has

$$E(x) \ll x^{1-\delta}. \tag{1.11}$$

Lu [32] showed that $\delta = 0.121$ is admissible in (1.11) and recently Pintz [40] obtained that $\delta = 0.28$ is admissible.

1.3 Average orders

We now return to the weighted Goldbach function

$$G(n) := \sum_{m+l=n} \Lambda(m)\Lambda(l).$$

We remark that the function $g(n)$, which is defined in (1.1), can be recovered from $G(n)$ by the use of the partial summation formula.

Now we define its *average order*, that is

$$S(X) := \sum_{n\leq X} G(n).$$

The study of which is a common practice in analytic number theory. A first asymptotic result was obtained by Landau in 1900 [26], he proved that

$$\sum_{n\le X} g(n) \sim \frac{X^2}{2\log^2 X}, \text{ as } X \to \infty,$$

which is equivalent to

$$S(X) \sim \frac{X^2}{2}, \text{ as } X \to \infty.$$

This was later refined by Fujii under the assumption of the Riemann Hypothesis.

Theorem 1.3.1 (Fujii [14])**.** *Suppose that the RH is true. Then we have*

$$S(X) = \frac{X^2}{2} + H(X) + \widetilde{E}(X),$$

where $H(X) = -2\sum_\rho \frac{X^{\rho+1}}{\rho(\rho+1)}$ *is the sum over the non-trivial zeros* ρ *of the Riemann zeta function, counted with multiplicity and* $\widetilde{E}(X) \ll (X\log X)^{4/3}$*.*

Granville proved the same result and unconditionally stated [20, 21], that

$$\widetilde{E}(X) \ll X^{\frac{2+4B}{3}} \log^2 X,$$

where

$$B = \sup\{\Re(\rho) : \zeta(\rho) = 0\}. \tag{1.12}$$

Egami and Matsumoto expected the error term to be $\mathcal{O}(X^{1+\epsilon})$ ([13, Conjecture 2.2]). In fact, this conjecture was proved by Bhowmik and Schlage-Puchta [7] who showed in 2010 that the error term is $\mathcal{O}(X\log^5 X)$ under RH. This result was improved in 2012 by Languasco-Zaccagnini [28] to

$$\widetilde{E}(X) \ll X\log^3 X. \tag{1.13}$$

Using different methods later Goldston-Yang [19] in 2017 and Goldston-Suriajaya [18] in 2023 to obtained the same result.

Remark 1.3.2. *Assuming the RH, the order of magnitude of* $H(X)$ *is* $\mathcal{O}(X^{3/2})$*, while unconditionally, it is*

$$H(X) \ll X^{1+B},$$

where B *is defined in* (1.12)*.*

Moreover, Bhowmik and Schlage-Puchta [7] also proved an omega-result showing that the asymptotic result is essentially the best possible.

Theorem 1.3.3 ([7])**.** *We have*

$$S(X) = \frac{X^2}{2} + H(X) + \Omega(X\log\log X).$$

Interestingly obtaining a good average order is actually equivalent to the Riemann Hypothesis. It is a combination of the results of Bhowmik-Halupczok-Matsumoto-Suzuki's paper [5] and Bhowmik-Ruzsa's paper [4].

Theorem 1.3.4. *The RH is equivalent to the estimate*

$$S(X) = \frac{X^2}{2} + \mathcal{O}(X^{3/2+o(1)}),$$

for X sufficiently large.

After this brief and largely incomplete introduction to the state of the art about Goldbach conjecture related results, we start to describe our work.

Chapter 2

Goldbach Representations in Arithmetic Progressions

In this chapter, we study the case in which the prime summands in $g(n)$ lie in arithmetic progressions. Analogously to the original problem, it is known that almost all even integers satisfying some congruence conditions can be written as the sum of two primes in congruence classes.

Here again, the conjecture is known to be almost always true. In fact, we consider the following exceptional set:

$$E_{q,a,b}(x) = \#\{n \leq x, n \in 2\mathbb{N}, n \equiv a + b(q) : n \neq p_1 + p_2, p_1 \equiv a(q), p_2 \equiv b(q)\}.$$

This function can be estimated as

$$E_{q,a,b}(x) \ll \frac{x^{1-\delta}}{\varphi(q)}$$

for a computable positive constant δ and all $q < x^{\delta}$ (see [31]). Moreover there exists $D > 0$ such that for all but $\ll \log^D x$ prime number $q \leq x^{5/24-\epsilon}$ and all integers a, b such that $(ab, q) = 1$,

$$E_{q,a,b}(x) \ll \frac{x}{q \log^D x},$$

see [3]. Such an exceptional set was recently studied by Martin [33] using heuristic arguments and numerical data.

Let q_1, q_2 be positive integers and $1 \leq a_1 < q_1, 1 \leq a_2 < q_2$ be positive integers such that $(a_1, q_1) = 1$, $(a_2, q_2) = 1$. We consider the function

$$G(n, q_1, q_2, a_1, a_2) := \sum_{\substack{m+l=n \\ m \equiv a_1(q_1) \\ l \equiv a_2(q_2)}} \Lambda(m)\Lambda(l),$$

whose summatory function is defined as

$$S(X, q_1, q_2, a_1, a_2) := \sum_{n \leq X} G(n, q_1, q_2, a_1, a_2).$$

This function was first introduced by Rüppel [42] in 2012. Suzuki [44] in 2017 obtained an asymptotic formula under the Generalized Riemann Hypothesis with real zeros while for the case of common modulus, Bhowmik-Halupczok-Matsumoto-Suzuki [5] proved an asymptotic formula where the error term is in terms of the zeros of L-functions. In general, we expect to obtain similar results for the case of different modulus.

2.1 Some preliminaires

With the notations of 1.1 of Chapter 1, we first evoke some lemmas for sums over non-trivial zeros.

Lemma 2.1.1 ([35], Theorem 10.17). *For any $T \geq 0$,*

$$\sum_{T \leq |\gamma_\chi| \leq T+1} 1 \ll \log(q(T+2)).$$

Lemma 2.1.2 ([5], Lemmas 2 and 3). *For any $T \geq 1$, we have*

$$\sum_{|\gamma_\chi| \leq T} \frac{1}{|\rho_\chi|} \ll \log^2(2qT) + \delta_1(\chi) q^{1/2} \log^2 q,$$

$$\sum_{|\gamma_\chi| > T} \frac{1}{|\rho_\chi|^2} \ll \frac{\log(qT)}{T},$$

$$\sum_{\rho_\chi} \frac{1}{|\rho_\chi(\rho_\chi + 1)|} \ll \log^2(2qT) + \delta_1(\chi) q^{1/2} \log^2 q,$$

where $\delta_1(\chi) = 1$ if χ is the exceptional character and $\delta_1(\chi) = 0$ otherwise.

Lemma 2.1.3. *Let $\rho_\chi = \gamma_\chi + i\beta_\chi$ be a non-trivial zero of $L(s, \chi)$ such that $|\gamma_\chi| > 1$. Then we have*

$$\sum_{\gamma'_\chi} \frac{1}{|\gamma'_\chi|(1 + |\gamma_\chi - \gamma'_\chi|)} \ll \frac{\log^2(q|\gamma_\chi|)}{|\gamma_\chi|}, \tag{2.1}$$

$$\sum_{\gamma'_\chi} \frac{1}{(1 + |\gamma_\chi - \gamma'_\chi|)^2} \ll \log(q|\gamma_\chi|), \tag{2.2}$$

where $\sum_{\gamma'_\chi}$ denotes a sum over the non-trivial zeros of Dirichlet L-function associated to χ modulo q.

Proof. Firstly, to prove (2.1), we split into two cases.

Case 1: If $|\gamma'_\chi| > 2|\gamma_\chi|$. Then we have

$$|\gamma_\chi - \gamma'_\chi| \geq |\gamma'_\chi| - |\gamma_\chi| > \frac{|\gamma'_\chi|}{2}.$$

Thus, by using Lemma 2.1.2, we obtain

$$\sum_{\substack{\gamma'_\chi \\ |\gamma'_\chi| > 2|\gamma_\chi|}} \frac{1}{|\gamma'_\chi|(1 + |\gamma_\chi - \gamma'_\chi|)} \ll \sum_{\substack{\gamma'_\chi \\ |\gamma'_\chi| > 2|\gamma_\chi|}} \frac{1}{|\gamma'_\chi|^2} \ll \frac{\log(q|\gamma_\chi|)}{|\gamma_\chi|}.$$

Case 2: If $|\gamma'_\chi| \leq 2|\gamma_\chi|$, we have

$$|\gamma_\chi - \gamma'_\chi| \leq |\gamma'_\chi| + |\gamma_\chi| \leq 3|\gamma_\chi|.$$

This implies

$$\begin{aligned}\sum_{\substack{\gamma'_\chi\\|\gamma'_\chi|\le 2|\gamma_\chi|}}\frac{1}{|\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)} &\le \sum_{\substack{\gamma'_\chi\\|\gamma_\chi-\gamma'_\chi|\le 3|\gamma_\chi|}}\frac{1}{|\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)}\\
&= \sum_{\substack{\gamma'_\chi\\|\gamma_\chi-\gamma'_\chi|\le 1}}\frac{1}{|\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)} + \sum_{\substack{\gamma'_\chi\\1<|\gamma_\chi-\gamma'_\chi|\le 3|\gamma_\chi|}}\frac{1}{|\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)}\\
&\le \sum_{\substack{\gamma'_\chi\\|\gamma_\chi-\gamma'_\chi|\le 1}}\frac{1}{|\gamma'_\chi|} + \sum_{1<n\le 3|\gamma_\chi|}\frac{1}{n}\sum_{\substack{\gamma'_\chi\\n<|\gamma_\chi-\gamma'_\chi|\le n+1}}\frac{1}{|\gamma'_\chi|}.\end{aligned}$$

By using Lemma 2.1.1, we can estimate the last sum to be

$$\ll \frac{\log(q|\gamma_\chi|)}{|\gamma_\chi|}\left(1+\sum_{1<n\le 3|\gamma_\chi|}\frac{1}{n}\right) \ll \frac{\log^2(q|\gamma_\chi|)}{|\gamma_\chi|}.$$

So we obtain the stated result. And similarly, we also can prove (2.2). □

We now recall the Vinogradov-Korobov zero free region in a suitable form.

Lemma 2.1.4. *Let $q\ge 3$, and let χ (q) be a Dirichlet character. For any $\delta>0$, there is some constant $c_1(\delta)$ such that the Dirichlet L-function does not vanish in the region*

$$\sigma \ge 1-\frac{c_1(\delta)}{\max(q^\delta,(\log X)^{2/3}(\log\log X)^{1/3})},\quad \textit{and } |t|\le X.$$

The result in Lemma 2.1.4 is a consequence of Theorem 1.1 [25] and Siegel's theorem ([35], Corollary 11.15).

2.2 Asymptotic formula of the average order

Now we prove an asymptotic formula for $S(X,q_1,q_2,a_1,a_2)$, which can be stated as follows.

Theorem 2.2.1. *For any $\epsilon>0$*

$$\begin{aligned}S(X,q_1,q_2,a_1,a_2) &= \frac{X^2}{2\varphi(q_1)\varphi(q_2)}+\frac{1}{\varphi(q_2)}H(X,q_1,a_1)+\frac{1}{\varphi(q_1)}H(X,q_2,a_2)\\
&\quad+\mathcal{O}(X^{B^*_{q_1}+B^*_{q_2}}\log X\log(q_1X)\log(q_2X)),\end{aligned}\tag{2.3}$$

where the implicit constant depends on q_1,q_2, and

$$H(X,q,a) = -\frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{\rho_\chi}\frac{X^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)},$$

$$B^*_q = B^*_q(X)=\min(B_q,1-\eta),\quad \eta=\eta_q(X)=\frac{c_1(\epsilon)}{\max(q^\epsilon,(\log X)^{2/3}(\log\log X)^{1/3})},$$

with some small constant $c_1(\epsilon)>0$.

Remark 2.2.2. *For the unconditional order of magnitude of $H(X,q,a)$, we have*

$$H(X,q,a) \ll \frac{X^{1+B_q}}{\varphi(q)} \sum_{\chi(q)} \sum_{\rho_\chi} \frac{1}{|\rho_\chi(\rho_\chi+1)|}.$$

By using Lemma 2.1.2, we obtain

$$\begin{aligned} H(X,q,a) &\ll \frac{X^{1+B_q}}{\varphi(q)} \sum_{\chi(q)} \left(\log^2 q + \delta_1(\chi) q^{1/2} \log^2 q\right) \\ &\ll_q X^{1+B_q}. \end{aligned}$$

Assuming the GRH, we can obtain

$$H(X,q,a) \ll_q X^{3/2}.$$

Corollary 2.2.3. *Assuming the GRH, we have*

$$\begin{aligned} S(X,q_1,q_2,a_1,a_2) = &\frac{X^2}{2\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)} H(X,q_1,a_1) + \frac{1}{\varphi(q_1)} H(X,q_2,a_2) \\ &+ \mathcal{O}(X \log X \log(q_1 X) \log(q_2 X)), \end{aligned} \tag{2.4}$$

where the implicit constant depends on q_1, q_2.

Remark 2.2.4. *Assuming the GRH, we can see that Corollary 2.2.3 is similar to [44, Theorem 1.1].*

When $q_1 = q_2 = q$, we replace $S(X,q_1,q_2,a_1,a_2)$ by $S(X,q,a_1,a_2)$. Then we obtain the following corollary of Theorem 2.2.1.

Corollary 2.2.5. *Let a_1, a_2 be positive integers with $(a_1a_2, q) = 1$, we have*

$$\begin{aligned} S(X,q,a_1,a_2) = &\frac{X^2}{2\varphi^2(q)} - \frac{1}{\varphi^2(q)} \sum_{\chi(q)} (\overline{\chi}(a_1) + \overline{\chi}(a_2)) \sum_{\rho_\chi} \frac{X^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \\ &+ \mathcal{O}(X^{2B_q^*} \log X \log^2(qX)), \end{aligned}$$

where the implicit constant depends on q.

This result improves the upper bound $\log^5(qX)$ of [5, Theorem 2] by a factor $\frac{\log^3(qX)}{\log X}$. When we take $q=1$, we recover the best known result for $S(X)$.

Now, we put $z = re(\alpha)$, $r = e^{-1/X}$, $|\alpha| \le 1/2$. Let q be a positive integer, $1 \le a < q$ be an integer coprime to q and $1 \le y \le X$, let us consider the power serieses as was done by Hardy-Littlewood [23]:

$$F_{a,q}(z) := \sum_{\substack{n \ge 1 \\ n \equiv a(q)}} \Lambda(n) z^n,$$

$$I_q(z) := \frac{1}{\varphi(q)} \sum_{n \ge 1} z^n, \qquad I(y,z) := \sum_{n \le y} z^n.$$

Their choice was natural because $F_{a,q}(z)$ is a generating function of primes in arithmetic progression with additive property and we expect it to be close to $I_q(z)$ by Dirichlet's theorem.

We will show that $S(X,q_1,q_2,a_1,a_2)$ can be expressed in terms of $F_{a_1,q_1}(z)$, $F_{a_2,q_2}(z)$ in the following theorem.

Theorem 2.2.6. *We have*

$$S(X,q_1,q_2,a_1,a_2) = \frac{X^2}{2\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)}H(X,q_1,a_1) + \frac{1}{\varphi(q_1)}H(X,q_2,a_2) + \mathcal{O}(X(\log^2(q_1X)+\log^2(q_2X))) + E(X,q_1,q_2,a_1,a_2),$$

where

$$E(X,q_1,q_2,a_1,a_2) = \int_0^1 (F_{a_1,q_1}(z) - I_{q_1}(z))(F_{a_2,q_2}(z) - I_{q_2}(z)) I\left(X, \frac{1}{z}\right) d\alpha, \quad (2.5)$$

and the implicit constant depends on q_1, q_2*.*

To prove Theorem 2.2.1, we use the circle method to compute the sum. First we will prove Theorem 2.2.6.

2.2.7 Proof of Theorem 2.2.6

In this part we consider the generating function

$$F_{a,q}(z) := \sum_{\substack{n\geq 1 \\ n\equiv a(q)}} \Lambda(n) z^n = \sum_{\substack{n\geq 1 \\ n\equiv a(q)}} \Lambda(n) e^{-n/X} e(n\alpha).$$

We first note that

$$(F_{a_1,q_1}(z) - I_{q_1}(z))(F_{a_2,q_2}(z) - I_{q_2}(z)) = F_{a_1,q_1}(z)F_{a_2,q_2}(z) - F_{a_1,q_1}(z)I_{q_2}(z) - F_{a_2,q_2}(z)I_{q_1}(z) + I_{q_1}(z)I_{q_2}(z). \quad (2.6)$$

From the definitions of $I_q(z)$ and $F_{a,q}(z)$, we calculate

$$F_{a_1,q_1}(z)F_{a_2,q_2}(z) = \sum_{\substack{m\geq 1 \\ m\equiv a_1(q_1)}} \Lambda(m) z^m \sum_{\substack{k\geq 1 \\ k\equiv a_2(q_2)}} \Lambda(k) z^k = \sum_{n\geq 2} G(n,q_1,q_2,a_1,a_2) z^n,$$

$$F_{a_1,q_1}(z)I_{q_2}(z) = \frac{1}{\varphi(q_2)} \sum_{\substack{m,k\geq 1 \\ m\equiv a_1(q_1)}} \Lambda(m) z^{m+k} = \frac{1}{\varphi(q_2)} \sum_{n\geq 2} \psi(n-1,q_1,a_1) z^n,$$

$$F_{a_2,q_2}(z)I_{q_1}(z) = \frac{1}{\varphi(q_1)} \sum_{n\geq 2} \psi(n-1,q_2,a_2) z^n,$$

$$I_{q_1}(z)I_{q_2}(z) = \frac{1}{\varphi(q_1)\varphi(q_2)} \sum_{n\geq 2} (n-1) z^n,$$

where ψ is the Chebyshev function in arithmetic progression (1.5). Hence, the right hand side of (2.6) can be rewritten as

$$\sum_n B(n,q_1,q_2,a_1,a_2) r^n e(n\alpha),$$

with

$$B(n,q_1,q_2,a_1,a_2) = G(n,q_1,q_2,a_1,a_2) - \frac{1}{\varphi(q_2)} \sum_{n\geq 2} \psi(n-1,q_1,a_1) - \frac{1}{\varphi(q_1)} \sum_{n\geq 2} \psi(n-1,q_2,a_2) + \frac{1}{\varphi(q_1)\varphi(q_2)} \sum_{n\geq 2} (n-1) z^n. \quad (2.7)$$

So that

$$E(X, q_1, q_2, a_1, a_2) = \int_0^1 \sum_n \sum_{m \le X} B(n, q_1, q_2, a_1, a_2) r^n e(n\alpha) r^{-m} e(-m\alpha) d\alpha$$
$$= \sum_n \sum_{m \le X} B(n, q_1, q_2, a_1, a_2) r^{n-m} \int_0^1 e((n-m)\alpha) d\alpha$$
$$= \sum_{n \le X} B(n, q_1, q_2, a_1, a_2)$$

since

$$\int_0^1 e(t\alpha) d\alpha = 1 \qquad \text{if } t = 0 \text{ and } 0 \text{ otherwise.}$$

Then we obtain the equation

$$\begin{aligned} S(X, q_1, q_2, a_1, a_2) &= E(X, q_1, q_2, a_1, a_2) + \frac{1}{\varphi(q_2)} \sum_{n \le X} \psi(n-1, q_1, a_1) \\ &+ \frac{1}{\varphi(q_1)} \sum_{n \le X} \psi(n-1, q_2, a_2) - \sum_{n \le X} \frac{n-1}{\varphi(q_1)\varphi(q_2)}. \end{aligned} \tag{2.8}$$

Moreover, we have an explicit formula for the average of $\psi(n, q, a)$ as follows.

Lemma 2.2.8. *Let $X \ge 2$ and $(a, q) = 1$, we have*

$$\sum_{n \le X} \psi(n-1, q, a) = \frac{X^2}{2\varphi(q)} - \frac{1}{\varphi(q)} \sum_{\chi(q)} \overline{\chi}(a) \sum_{\rho_\chi} \frac{X^{\rho_\chi + 1}}{\rho_\chi(\rho_\chi + 1)} + \mathcal{O}(X \log^2(qX)).$$

Proof. We first remark that

$$\sum_{n \le X} \psi(n-1, q, a) = \sum_{\substack{n \le X \\ n \equiv a(q)}} (X - n)\Lambda(n) = \int_1^X \psi(t, q, a) dt. \tag{2.9}$$

Using the orthogonality relation of Dirichlet characters, we have

$$\psi(t, q, a) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \overline{\chi}(a) \psi(t, \chi),$$

where

$$\psi(t, \chi) := \sum_{n \le t} \chi(n)\Lambda(n). \tag{2.10}$$

Let now χ be a primitive character modulo q. Then from Theorem 12.5 and Theorem 12.10 of [35], we obtain for $t, T \ge 2$

$$\psi(t, \chi) = \delta_0(\chi) t - \sum_{|\gamma_\chi| \le T} \frac{t^{\rho_\chi}}{\rho_\chi} + \mathcal{O}\left(\frac{t}{T} \log^2(qtT) + \delta_1(\chi) q^{1/2} \log^2 q \right), \tag{2.11}$$

where $\delta_0(\chi) = 1$ if $\chi = \chi_0$ and $\delta_0(\chi) = 0$ otherwise; $\delta_1(\chi)$ is defined in Lemma 2.1.2.

If χ is imprimitive. Suppose that χ is a character modulo q induced by the primitive character χ^* modulo q^*, $q^* > 1$. Then we have

$$\begin{aligned}\psi(t,\chi) - \psi(t,\chi^*) &\ll \sum_{\substack{n\le t\\(n,q)>1}} \Lambda(n) = \sum_{p|q}\sum_{p^k\le t} \log p\\ &\ll \sum_{p|q}\left[\frac{\log t}{\log p}\right]\log p \ll \log t\sum_{p|q}\log p\\ &\ll \log q \log t.\end{aligned}$$

Hence we rewrite (2.11) for all characters χ modulus q. Let $2 \le t, T \le X$, one has

$$\psi(t,\chi) = \delta_0(\chi)t - \sum_{|\gamma_\chi|\le T}\frac{t^{\rho_\chi}}{\rho_\chi} + \mathcal{O}\left(\frac{X}{T}\log^2(qX) + \delta_1(\chi)q^{1/2}\log^2 q\right). \tag{2.12}$$

Substituting the above formula into (2.10), we obtain

$$\begin{aligned}\psi(t,q,a) =& \frac{t}{\varphi(q)} - \frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{|\gamma_\chi|\le T}\frac{t^{\rho_\chi}}{\rho_\chi}\\ &+\mathcal{O}\left(\frac{1}{\varphi(q)}\sum_{\chi(q)}\left(\frac{X}{T}\log^2(qX) + \delta_1(\chi)q^{1/2}\log^2 q\right)\right).\end{aligned}$$

This leads to the following

$$\psi(t,q,a) = \frac{t}{\varphi(q)} - \frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{|\gamma_\chi|\le T}\frac{t^{\rho_\chi}}{\rho_\chi} + \mathcal{O}\left(\frac{X}{T}\log^2(qX)\right), \tag{2.13}$$

since there is at most one exceptional character mod q.
Substituting (2.13) into (2.9),

$$\begin{aligned}\sum_{n\le X}\psi(n-1,q,a) =& \frac{X^2}{2\varphi(q)} - \frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{|\gamma_\chi|\le T}\frac{X^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)}\\ &+\mathcal{O}(\frac{X^2}{T}\log^2(qX)).\end{aligned}$$

Note that by (2.1.2), one has

$$\sum_{|\gamma_\chi|\ge T}\frac{X^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} \ll X^2\sum_{|\gamma_\chi|\ge T}\frac{1}{|\rho_\chi|^2} \ll \frac{X^2}{T}\log(qT).$$

Therefore Lemma 2.2.8 is proved when we choose $T = X$. □

The proof of Theorem 2.2.6 is complete when we substitute Lemma 2.2.8 into (2.8).

2.2.9 Proof Theorem 2.2.1

From Theorem 2.2.6, to prove Theorem 2.2.1, it is enough to estimate

$$E(X, q_1, q_2, a_1, a_2) \ll_{q_1,q_2} X^{B^*_{q_1}+B^*_{q_2}} \log X \log(q_1 X) \log(q_2 X).$$

By the Cauchy-Schwarz inequality, the integral $E(X, q_1, q_2, a_1, a_2)$ is estimated as

$$|E(X, q_1, q_2, a_1, a_2)|^2 = \left| \int_0^1 (F_{a_1,q_1}(z) - I_{q_1}(z)) (F_{a_2,q_2}(z) - I_{q_2}(z)) \, I\left(X, \frac{1}{z}\right) d\alpha \right|^2$$
$$\ll \left(\int_0^1 |F_{a_1,q_1}(z) - I_{q_1}(z)|^2 \left| I\left(X, \frac{1}{z}\right) \right| d\alpha \right) \left(\int_0^1 |F_{a_2,q_2}(z) - I_{q_2}(z)|^2 \left| I\left(X, \frac{1}{z}\right) \right| d\alpha \right).$$

Noting that $|E(X, q_1, q_2, a_1, a_2)|$ can be estimated as the product of two factors which have the same properties. So we need only consider

$$T_1(X, q, a) := \int_0^1 |F_{a,q}(z) - I_q(z)|^2 \left| I\left(X, \frac{1}{z}\right) \right| d\alpha, \tag{2.14}$$

we remark that

$$|E(X, q_1, q_2, a_1, a_2))| \ll \max_{\substack{q_i,a_i \\ i=1,2}} (T_1(X, q_i, a_i)).$$

We will estimate $T_1(X, q, a)$ by bounding $I\left(X, \frac{1}{z}\right)$ and then the moment of $|F_{a,q}(z) - I(z)|$. We first remark that , for $1 \le y \le X$, by partial summation,

$$I\left(y, \frac{1}{z}\right) = \sum_{n \le y} e^{n/X} e(-n\alpha) = e^{y/X} \sum_{n \le y} e(-n\alpha) - \frac{1}{X} \int_1^y \sum_{n \le t} e(-n\alpha) e^{t/X} dt.$$

Since

$$\sum_{n \le t} e(-n\alpha) \ll \min\left(t, \frac{1}{|\alpha|}\right),$$

we obtain

$$I\left(y, \frac{1}{z}\right) \ll \min\left(y, \frac{1}{|\alpha|}\right). \tag{2.15}$$

We now prove the following lemma.

Lemma 2.2.10. *Let $X \ge 2$ and $|\alpha| \le 1/2$. For any integrable positive function of period 1, we have*

$$\int_{-1/2}^{1/2} f(\alpha) \min\left(X, \frac{1}{|\alpha|}\right) d\alpha \ll \sum_{k=0}^{\mathcal{O}(\log X)} \frac{X}{2^k} \int_0^{2^{k+1}/X} f(\alpha) d\alpha. \tag{2.16}$$

Proof. Because of periodicity, we can restrict ourselves to $[0, 1/2]$. We evaluate the left hand side of (2.16) as

$$\int_0^{1/2} f(\alpha) \min\left(X, \frac{1}{|\alpha|}\right) d\alpha = X \int_0^{1/X} f(X) d\alpha + \int_{1/X}^{1/2} f(\alpha) \frac{d\alpha}{\alpha}.$$

We write $[\frac{1}{X}, \frac{1}{2}]$ as the disjoint union of $[\frac{2^k}{X}, \frac{2^{k+1}}{X}]$, for $0 \le k \le \mathcal{O}(\log X)$. Then the right hand side of the above formula is bounded by

$$\begin{aligned}
&X\int_0^{1/X} f(\alpha)d\alpha + \sum_{k=0}^{\mathcal{O}(\log X)} \int_{2^k/X}^{2^{k+1}/X} f(\alpha)\frac{d\alpha}{\alpha} \\
&\le X\int_0^{1/X} f(\alpha)d\alpha + \sum_{k=0}^{\mathcal{O}(\log X)} \frac{X}{2^k}\int_{2^k/X}^{2^{k+1}/X} f(\alpha)d\alpha \\
&\ll \sum_{k=0}^{\mathcal{O}(\log X)} \frac{X}{2^k}\int_0^{2^{k+1}/X} f(\alpha)d\alpha,
\end{aligned}$$

and this proves the lemma. □

Since $z = z(\alpha) = e^{-1/X}e(n\alpha)$, we replace $f(\alpha) = |F_{a,q}(z) - I(z)|^2$ in Lemma 3.3.4, to obtain

$$T_1(X,q,a) \ll \sum_{k=0}^{\mathcal{O}(\log X)} \frac{X}{2^k}\int_0^{2^{k+1}/X} |F_{a,q}(z) - I(z)|^2\, d\alpha. \tag{2.17}$$

For $1 \le h \le X$, putting

$$\begin{aligned}
W(X,q,a,h) :&= \int_0^{1/2h} |F_{a,q}(z) - I(z)|^2\, d\alpha \\
&= \int_0^{1/2h} \left|\sum_n \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)}\right) e^{-n/X}e(n\alpha)\right|^2 d\alpha,
\end{aligned}$$

where $\delta(n)$ is defined as

$$\delta(n) = 1 \qquad \text{if } n \equiv a (\text{mod } q) \text{ and } 0 \text{ otherwise.} \tag{2.18}$$

Now we recall Gallagher's lemma (see [19], section 4)

$$\int_0^{1/2h} \left|\sum_n a_n e(n\alpha)\right|^2 d\alpha \ll \frac{1}{h^2}\int_{-\infty}^{\infty} \left|\sum_{x<n\le x+h} a_n\right|^2 dx,$$

where $a_n \in \mathbb{C}$ and $\sum_n |a_n| < \infty$. Then for $1 \le h \le X$, we obtain

$$\begin{aligned}
W(X,q,a,h) &\ll \frac{1}{h^2}\int_{-h}^{\infty} \left|\sum_{x<n\le x+h} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)}\right) e^{-n/X}\right|^2 dx \\
&\ll \frac{1}{h^2}\int_0^{h} \left|\sum_{n\le x} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)}\right) e^{-n/X}\right|^2 dx \qquad (2.19)\\
&\quad + \frac{1}{h^2}\int_0^{\infty} \left|\sum_{x<n\le x+h} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)}\right) e^{-n/X}\right|^2 dx.
\end{aligned}$$

Next we evaluate

$$J_{1,a,q}(X,h) := \int_0^h \left| \sum_{n \le x} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)} \right) e^{-n/X} \right|^2 dx$$

$$J_{2,a,q}(X,h) := \int_0^\infty \left| \sum_{x<n\le x+h} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)} \right) e^{-n/X} \right|^2 dx$$

from the two following lemmas.

Lemma 2.2.11. *Let $X \ge 2$ and $1 \le h \le X$,*

$$J_{1,a,q}(X,h) \ll H_{a,q}(h) + \frac{h}{\varphi^2(q)},$$

where

$$H_{a,q}(x) := \int_0^x \left(\psi(t,q,a) - \frac{t}{\varphi(q)} \right)^2 dt.$$

Proof. We have

$$\begin{aligned} J_{1,a,q}(X,h) &= \int_0^h \left| \sum_{n\le x} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)} \right) e^{-n/X} \right|^2 dx \\ &\le \int_0^h \left| \sum_{n\le x} \left(\Lambda(n)\delta(n) - \frac{1}{\varphi(q)} \right) \right|^2 dx + \int_0^h \left(\psi(x,q,a) - \frac{[x]}{\varphi(q)} \right)^2 dx \\ &\ll \int_0^h \left(\psi(x,q,a) - \frac{x}{\varphi(q)} \right)^2 dx + \frac{1}{\varphi(q)^2} \int_0^h (x-[x])^2 dx \\ &\le H_{a,q}(h) + \frac{h}{\varphi^2(q)}. \end{aligned}$$

□

Lemma 2.2.12. *Let $X \ge 2$ and $1 \le h \le X$,*

$$J_{2,a,q}(X,h) \ll \frac{h^2}{X^2} \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} \left(H_{a,q}(jX) + \frac{jX}{\varphi^2(q)} \right) + \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} \left(K_{a,q}(jX,h) + \frac{jX}{\varphi^2(q)} \right),$$

where

$$K_{a,q}(x,h) := \int_0^x \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)} \right)^2 dt$$

and $H_{a,q}$ function is defined in the last previous.

Proof. We denote

$$F(n) := \Lambda(n)\delta(n) - \frac{1}{\varphi(q)}.$$

For $y \ge 2$, using partial summation, we have

$$\sum_{n\le y} F(n) e^{-n/X} = e^{-y/X} \sum_{n \le y} F(n) + \frac{1}{y} \int_0^y \sum_{n\le t} F(n) e^{-t/X} dt. \tag{2.20}$$

We replace y in (2.20) by $x+h$ and x, we obtain

$$\begin{aligned}\sum_{x<n\le x+h} F(n)e^{-n/X} &= \left(\sum_{n\le x+h} - \sum_{n\le x}\right) F(n)e^{-n/X}\\ &= e^{-(x+h)/X} \sum_{n\le x+h} F(n) + \frac{1}{X}\int_0^{x+h} \sum_{n\le t} F(n)e^{-t/X} dt\\ &\quad - e^{-x/X}\sum_{n\le x} F(n) - \frac{1}{X}\int_0^{x}\sum_{n\le t} F(n)e^{-t/X} dt \\ &= e^{-x/X} \sum_{x<n\le x+h} F(n) + e^{-x/X} \sum_{n\le x+h} F(n)(e^{-h/X}-1)\\ &\quad + \frac{1}{X}\int_x^{x+h}\sum_{n\le t} F(n)e^{-t/X} dt.\end{aligned} \tag{2.21}$$

Since $e^{-t}-1\le t$ for all non-negative t, then $e^{-h/X}-1\le h/X$. Now squaring out and integrating (2.21), we obtain

$$\begin{aligned}J_{2,a,q}(X,h) &= \int_0^\infty \left|\sum_{x<n\le x+h} F(x)e^{-n/X}\right|^2 dx\\ &\ll \int_0^\infty \left(\sum_{x<n\le x+h} F(n)\right)^2 e^{-2x/X} dx\\ &\quad + \frac{h^2}{X^2}\int_0^\infty \left(\sum_{n\le x+h} F(n)\right)^2 e^{-2x/X} dx\\ &\quad + \frac{1}{X^2}\int_0^\infty \left(\int_x^{x+h}\sum_{n\le t} F(n)e^{-t/X} dt\right)^2 dx.\end{aligned} \tag{2.22}$$

We now proceed to estimate (2.22) in several steps.
Step 1: The first term on the right hand side of (2.22) is

$$\int_0^\infty \left(\sum_{x<n\le x+h} F(n)\right)^2 e^{-2x/X} dx = \sum_{j=0}^\infty \int_{jX}^{(j+1)X} \left(\sum_{x<n\le x+h} F(n)\right)^2 e^{-2x/X} dx$$
$$\le \sum_{j=0}^\infty e^{-2j}\int_{jX}^{(j+1)X} \left(\sum_{x<n\le x+h} F(n)\right)^2 dx \le \sum_{j=0}^\infty \frac{1}{2^j}\int_0^{(j+1)X} \left(\sum_{x<n\le x+h} F(n)\right)^2 dx,$$

since $e^{-2j}\le 2^{-j}$ for all j non-negative integers. Moreover, for $y\ge 2$,

$$\int_0^y \left(\sum_{x<n\le x+h} F(n)\right)^2 dx = \int_0^y \left(\psi(x+h,q,a)-\psi(x,q,a) - \frac{[x+h]}{\varphi(q)} + \frac{[x]}{\varphi(q)}\right)^2 dt$$

$$\ll \int_0^y \left(\psi(x+h,q,a)-\psi(x,q,a) - \frac{h}{\varphi(q)}\right)^2 dt + \int_0^y \left(\frac{[x+h]}{\varphi(q)} - \frac{[x]}{\varphi(q)} - \frac{h}{\varphi(q)}\right)^2 dt$$
$$\ll K_{a,q}(y,h) + \frac{y}{\varphi^2(q)}.$$

Hence we have obtained that

$$\int_0^\infty \left(\sum_{x<n\le x+h} F(n)\right)^2 e^{-2x/X}dx \ll \sum_{j=1}^\infty \frac{1}{2^{j-1}}\left(K_{a,q}(jX,h)+\frac{jX}{\varphi^2(q)}\right). \tag{2.23}$$

Step 2: Next we consider the last term of (2.22). By the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^\infty \left(\int_x^{x+h}\sum_{n\le t} F(n)e^{-t/X}dt\right)^2 dx &\ll h\int_0^\infty\int_x^{x+h}\left(\sum_{n\le t}F(n)\right)^2 e^{-2t/X}dtdx\\ &= h\int_0^\infty\left(\sum_{n\le t}F(n)\right)^2 e^{-2t/X}\left(\int_{t-h}^t dx\right)dt\\ &= h^2\int_0^\infty\left(\sum_{n\le t}F(n)\right)^2 e^{-2t/X}dt.\end{aligned}$$

Moreover, analogously to Step 1, we have

$$\begin{aligned}\int_0^\infty\left(\sum_{n\le t}F(n)\right)^2 e^{-2t/X}dt &\le \sum_{j=0}^\infty\frac{1}{2^j}\int_0^{(j+1)X}\left(\sum_{n\le t}F(n)\right)^2 dt\\ &\ll \sum_{j=1}^\infty\frac{1}{2^{j-1}}\left(H_{a,q}(jX)+\frac{jX}{\varphi^2(q)}\right).\end{aligned}$$

Hence the last term of (2.22) is bounded by

$$\ll \frac{h^2}{X^2}\sum_{j=1}^\infty\frac{1}{2^{j-1}}\left(H_{a,q}(jX)+\frac{jX}{\varphi^2(q)}\right). \tag{2.24}$$

Step 3: Finally, the second term of (2.22) is

$$\begin{aligned}\frac{h^2}{X^2}\int_0^\infty\left(\sum_{n\le x+h}F(n)\right)^2 e^{-2x/X}dx &= e^{2h/X}\frac{h^2}{X^2}\int_0^\infty\left(\sum_{n\le t}F(n)\right)^2 e^{-2t/X}dt\\ &\ll \frac{h^2}{X^2}\sum_{j=1}^\infty\frac{1}{2^{j-1}}\left(H_{a,q}(jX)+\frac{jX}{\varphi^2(q)}\right).\end{aligned} \tag{2.25}$$

Combining (2.23), (2.24), (2.25) and (2.21), the proof is complete. □

Hence, it suffices to estimate the two functions $H_{a,q}(x)$ and $K_{a,q}(x,h)$.

Lemma 2.2.13. *With the notation of Theorem 2.2.1, we have the estimate*

$$H_{a,q}(x) \ll_q x^{2B_q^*+1}.$$

Proof. Using (2.13) for $2\le T\le x$, we can rewrite

$$\begin{aligned}H_{a,q}(x) &= \frac{1}{\varphi^2(q)}\int_0^x\left(\sum_{\chi(q)}\overline{\chi}(a)\sum_{|\gamma_\chi|\le T}\frac{t^{\rho_\chi}}{\rho_\chi}\right)^2 dt + \mathcal{O}\left(\int_0^x \frac{x^2}{T^2}\log^4(qt)dt\right)\\ &\ll \frac{1}{\varphi^2(q)}\sum_{\chi(q)}|\overline{\chi}(a)|^2\int_0^x\left|\sum_{|\gamma_\chi|\le T}\frac{t^{\rho_\chi}}{\rho_\chi}\right|^2 dt + \frac{x^3}{T^2}\log^4(qx).\end{aligned} \tag{2.26}$$

We first consider

$$\int_0^x \left| \sum_{|\gamma_\chi|\le T} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt \ll \int_0^x \left| \sum_{|\gamma_\chi|\le 1} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt + \int_0^x \left| \sum_{1<|\gamma_\chi|\le T} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt.$$

By the well known bound of the Siegel zero, we have

$$\int_0^x \left| \sum_{|\gamma_\chi|\le 1} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt \ll_q x^{2B_q^*+1}.$$

For the last sum, we get

$$\begin{aligned}\int_0^x \left| \sum_{1<|\gamma_\chi|\le T} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt &\ll \sum_{1<|\gamma_\chi|\le T} \sum_{1<|\gamma'_\chi|\le T} \frac{x^{\beta_\chi+\beta'_\chi+1}}{|\gamma_\chi||\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)} \\ &\ll x^{2B_q^*+1} \sum_{1<|\gamma_\chi|\le T} \sum_{1<|\gamma'_\chi|\le T} \frac{1}{|\gamma_\chi||\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)}.\end{aligned}$$

By using Lemma 2.1.3, that is

$$\sum_{\gamma'_\chi} \frac{1}{|\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)} \ll \frac{\log^2(q|\gamma_\chi|)}{|\gamma_\chi|},$$

we then obtain

$$\begin{aligned}\sum_{1<|\gamma_\chi|\le T} \sum_{\gamma'_\chi} \frac{1}{|\gamma_\chi||\gamma'_\chi|(1+|\gamma_\chi-\gamma'_\chi|)} &\ll \sum_{1<|\gamma_\chi|\le T} \frac{\log^2(q|\gamma_\chi|)}{|\gamma_\chi|^2} \\ &\ll \sum_{n=1}^{T} \frac{\log^3(qn)}{n^2} \ll_q 1.\end{aligned}$$

Hence,

$$\int_0^x \left| \sum_{|\gamma_\chi|\le T} \frac{t^{\rho_\chi}}{\rho_\chi} \right|^2 dt \ll_q x^{2B_q^*+1}. \tag{2.27}$$

By substituting (2.27) into (2.26), we obtain

$$H_{a,q}(x) \ll_q x^{2B_q^*+1} + \frac{x^3}{T^2}\log^4(qx).$$

Now, we choose $T = x$. Therefore, since $B_q^* \ge \frac{1}{2}$ we obtain

$$H_{a,q}(x) \ll_q x^{2B_q^*+1} + x\log^4(qx) \ll_q x^{2B_q^*+1}.$$

□

Lemma 2.2.14. *For $1 \le h \le x$ and $x \gg_q 1$, we have*

$$K_{a,q}(x,h) \ll_q hx^{2B_q^*} \log^2(qx).$$

Proof. We consider two cases:

Case 1: Let GRH be true.
Prachar [41] showed that for $x \ge 2, 1 \le q \le x, 1 \le h \le x$, we have

$$\sum_{a(q)}^{*} K_{a,q}(x,h) \ll hx \log^2(qx),$$

where $\sum_{a(q)}^*$ denotes a sum over a set of reduced residues modulo q. Thus this lemma is true under the GRH.

Case 2: Let GRH be false, which means that $B_q > 1/2$.
For $0 < \theta \le 1$, we consider

$$\begin{aligned} K(x,\theta) &= \int_x^{2x} \left(\psi(t+t\theta,q,a) - \psi(t,q,a) - \frac{t\theta}{\varphi(q)} \right)^2 dt \\ &\ll \int_1^2 \int_{xv/2}^{2xv} \left(\psi(t+t\theta,q,a) - \psi(t,q,a) - \frac{t\theta}{\varphi(q)} \right)^2 dt dv \end{aligned}$$

in which we used the idea of Saffari-Vaughan [43].
Using (2.13), we have

$$\begin{aligned} &\int_{xv/2}^{2xv} \left(\psi(t+t\theta,q,a) - \psi(t,q,a) - \frac{t\theta}{\varphi(q)} \right)^2 dt \\ &\ll_q \int_{xv/2}^{2xv} \left| \sum_{\chi(q)} \overline{\chi}(a) \sum_{\rho_\chi} \frac{t^{\rho_\chi}}{\rho_\chi} [(1+\theta)^{\rho_\chi} - 1] \right|^2 dt \\ &\qquad + \int_{xv/2}^{2xv} \left(\log^2(qx(1+\theta)) - \log^2(qx) \right)^2 dt \\ &=: K_1(v) + K_2(v). \end{aligned} \tag{2.28}$$

Trivially, $K_2(v) \ll xv \log^4(qx)$, then

$$\int_1^2 K_2(v) dv \ll x \log^4(qx). \tag{2.29}$$

By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} K_1(v) &\ll_q \sum_{\chi(q)} |\overline{\chi}(a)|^2 \int_{xv/2}^{2xv} \left| \sum_{\rho_\chi} \frac{t^{\rho_\chi}}{\rho_\chi} [(1+\theta)^{\rho_\chi} - 1] \right|^2 dt \\ &= \sum_{\chi(q)} |\overline{\chi}(a)|^2 \int_{xv/2}^{2xv} \sum_{\rho_\chi} \sum_{\rho'_\chi} \frac{[(1+\theta)^{\rho_\chi} - 1]\,[(1+\theta)^{\overline{\rho'_\chi}} - 1]}{\rho_\chi \overline{\rho'_\chi}} t^{\rho_\chi} t^{\overline{\rho'_\chi}} dt \\ &= \sum_{\chi(q)} |\overline{\chi}(a)|^2 \sum_{\rho_\chi} \sum_{\rho'_\chi} \frac{[(1+\theta)^{\rho_\chi} - 1]\,[(1+\theta)^{\overline{\rho'_\chi}} - 1]}{\rho_\chi \overline{\rho'_\chi}} \\ &\qquad\qquad \times \frac{2^{1+\rho_\chi+\overline{\rho'_\chi}} - 2^{-1-\rho_\chi-\overline{\rho'_\chi}}}{\rho_\chi + \overline{\rho'_\chi} + 1} (xv)^{\rho_\chi+\overline{\rho'_\chi}+1}. \end{aligned}$$

Thus,

$$\int_1^2 K_1(v)dv \ll_q \sum_{\chi(q)} |\overline{\chi}(a)|^2 \sum_{\rho_\chi} \sum_{\rho'_\chi} \frac{[(1+\theta)^{\rho_\chi}-1][(1+\theta)^{\overline{\rho'_\chi}}-1]}{\rho_\chi \overline{\rho'_\chi}} \frac{2^{1+\rho_\chi+\overline{\rho'_\chi}}-2^{-1-\rho_\chi-\overline{\rho'_\chi}}}{\rho_\chi+\overline{\rho'_\chi}+1}$$
$$\times \frac{2^{2+\rho_\chi+\overline{\rho'_\chi}}-1}{\rho_\chi+\overline{\rho'_\chi}+2} x^{\rho_\chi+\overline{\rho'_\chi}+1}.$$

By the trivial inequality $|z_1 z_2| \ll |z_1|^2+|z_2|^2$, we hence obtain

$$\begin{aligned}\int_1^2 K_1(v)dv &\ll_q \sum_{\chi(q)} |\overline{\chi}(a)|^2 \sum_{\rho_\chi} \sum_{\rho'_\chi} \left| \frac{[(1+\theta)^{\rho_\chi}-1][(1+\theta)^{\overline{\rho'_\chi}}-1]}{\rho_\chi \overline{\rho'_\chi}} \right| \frac{x^{\rho_\chi+\overline{\rho'_\chi}+1}}{\left|\rho_\chi+\overline{\rho'_\chi}+1\right|^2} \\ &\ll x^{2B_q^*+1} \sum_{\chi(q)} |\overline{\chi}(a)|^2 \sum_{\rho_\chi} \left| \frac{(1+\theta)^{\rho_\chi}-1}{\rho_\chi} \right|^2 \sum_{\rho'_\chi} \frac{1}{\left|\rho_\chi+\overline{\rho'_\chi}+1\right|^2} \\ &\ll x^{2B_q^*+1} \sum_{\chi(q)} |\overline{\chi}(a)|^2 \sum_{\gamma_\chi} \min\{\theta^2, (\gamma_\chi)^{-2}\} \sum_{\gamma'_\chi} \frac{1}{(1+|\gamma_\chi-\gamma'_\chi|)^2}.\end{aligned}$$

By Lemma 2.1.3, we have

$$\sum_{\gamma'_\chi} \frac{1}{(1+|\gamma_\chi-\gamma'_\chi|)^2} \ll \log(q|\gamma_\chi|).$$

Then we have the estimate

$$\begin{aligned}\int_1^2 K_1(v)dv &\ll_q x^{2B_q^*+1} \sum_{\chi(q)} |\overline{\chi}(a)|^2 \left(\sum_{|\gamma_\chi|>\theta^{-1}} \frac{\log(q|\gamma_\chi|)}{\gamma_\chi^2} + \sum_{|\gamma_\chi|\le\theta^{-1}} \theta^2 \log(q|\gamma_\chi|) \right) \\ &\ll_q x^{2B_q^*+1} \theta \log^2\left(\frac{1}{\theta}\right). \end{aligned} \tag{2.30}$$

From (2.28), (2.29) and (2.30), we obtain

$$K(x,\theta) \ll_q x^{2B_q^*+1} \theta \log^2\left(\frac{1}{\theta}\right) + x\log^4(qx).$$

Similar to [43, (6.21)], we have

$$\begin{aligned}&\int_x^{2x} \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)} \right)^2 dt \\ &\ll \frac{x}{h} \int_{h/3x}^{3h/x} \left(\int_x^{3x} \left(\psi(t+t\theta,q,a) - \psi(t,q,a) - \frac{t\theta}{\varphi(q)} \right)^2 dt \right) d\theta \\ &\ll_q hx^{2B_q^*} \log^2\left(\frac{x}{h}\right) + x\log^4(qx).\end{aligned}$$

Since $B_q^* > 1/2$, there exists $\epsilon(q) > 0$ such that $2B_q^* > 1+\epsilon(q)$. Thus,

$$x\log^4(qx) \ll x^{2B_q^*-\epsilon(q)} \log^4(qx) \ll x^{2B_q^*} \log^2(qx) \text{ for } x \gg_q 1.$$

Therefore

$$\int_x^{2x} \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)}\right)^2 dt \ll_q hx^{2B_q^*} \log^2(qx). \tag{2.31}$$

Now we split $K_{a,q}(x,h)$ into two parts:

$$\begin{aligned} K_{a,q}(x,h) = &\int_0^h \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)}\right)^2 dt \\ &+ \int_h^x \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)}\right)^2 dt. \end{aligned}$$

The first one is bounded by

$$\begin{aligned} &\ll \int_0^h \left(\psi(t+h,q,a) - \frac{t+h}{\varphi(q)}\right)^2 dt + \int_0^h \left(\psi(t,q,a) - \frac{t}{\varphi(q)}\right)^2 dt \\ &\ll_q h^{2B_q^*+1}. \end{aligned}$$

We recover $[h,x]$ by the disjoint union of $[\frac{x}{2^{k+1}}, \frac{x}{2^k}]$, for $0 \le k \le \mathcal{O}(\log \frac{x}{h})$. Using (2.31), we rewrite the second part as

$$\begin{aligned} &\ll \sum_{k\ge 0} \int_{x/2^{k+1}}^{x/2^k} \left(\psi(t+h,q,a) - \psi(t,q,a) - \frac{h}{\varphi(q)}\right)^2 dt \\ &\ll_q h \sum_{k\ge 0} \left(\frac{x}{2^k}\right)^{2B_q^*} \log^2(qx) \\ &\ll_q hx^{2B_q^*} \log^2(qx). \end{aligned}$$

Combining the two estimates above, the proof is complete. □

Now, using Lemmas 2.2.11, 2.2.12, 2.2.13, 2.2.14 and (2.19), we complete the proof of Theorem 2.2.1.

$$\begin{aligned} W(X,q,a,h) \ll &\frac{1}{h^2}\left(h^{2B_q^*+1} + \frac{h}{\varphi^2(q)}\right) \\ &+ \frac{1}{X^2}\sum_{j=1}^{\infty} \frac{1}{2^{j-1}}\left((jX)^{2B_q^*+1} + \frac{jX}{\varphi^2(q)}\right) \\ &+ \frac{1}{h^2}\sum_{j=1}^{\infty} \frac{1}{2^{j-1}}\left(h(jX)^{2B_q^*}\log^2(qjX) + \frac{jX}{\varphi^2(q)}\right) \\ \ll_q &h^{2B_q^*-1} + X^{2B_q^*-1}\sum_{j\ge 1}\frac{j^{2B_q^*+1}}{2^{j-1}} + \frac{X^{2B_q^*}}{h}\sum_{j\ge 1}\frac{j^{2B_q^*}}{2^{j-1}}\log^2(qX) \\ \ll_q &\frac{X^{2B_q^*}}{h}\log^2(qX). \end{aligned}$$

Substituting the above inequality into (2.17), we get

$$T_1(X,q,a) \ll_q X^{2B_q^*} \log X \log^2(qX).$$

We conclude

$$\begin{aligned} E(X, q_1, q_2, a_1, a_2) &\ll_{q_1,q_2} T_1(X, q_1, a_1)^{1/2} T_1(X, q_2, a_2)^{1/2} \\ &\ll_{q_1,q_2} X^{B^*_{q_1}+B^*_{q_2}} \log X \log(q_1 X) \log(q_2 X). \end{aligned}$$

Now the proof of Theorem 2.2.1 is complete. □

2.3 Omega-result for the average order

Similar to the classical case, Theorem 1.3.3, we prove an omega-result in the arithmetic progressions showing that the asymptotic formula proved in (1.13) is essentially the best possible.

Theorem 2.3.1. *We have*

$$\begin{aligned} S(X, q_1, q_2, a_1, a_2) = &\frac{X^2}{2\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)} H(X, q_1, a_1) \\ &+ \frac{1}{\varphi(q_1)} H(X, q_2, a_2) + \Omega(X \log\log X), \end{aligned} \tag{2.32}$$

where the implicit constant in the Ω*-symbol depends on* q_1, q_2*.*

Proof. By Theorem 2.2.1, we obtain the main terms in this formula. We will show that

$$G(n, q_1, q_2, a_1, a_2) = \Omega(n \log\log n), \tag{2.33}$$

and then Theorem 2.3.1 will follows. In fact, assuming the error term of Theorem 2.3.1 is $o(X \log\log X)$, one has

$$\begin{aligned} &S(n+1, q_1, q_2, a_1, a_2) - S(n, q_1, q_2, a_1, a_2) \\ &= \frac{(n+1)^2 - n^2}{2\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)} \left(H(n+1, q_1, a_1) - H(n, q_1, a_1)\right) \\ &\qquad + \frac{1}{\varphi(q_1)} \left(H(n+1, q_2, a_2) - H(n, q_2, a_2)\right) + o(n \log\log n) \\ &= o(n \log\log n). \end{aligned}$$

So (2.33) would be false. Hence the error term of Theorem 2.3.1 is $\Omega(X \log\log X)$.

We now show how to prove (2.33). We recall the following result.

Lemma 2.3.2. *Let* q' *be a positive integer,* $q = \prod_{\substack{p \le x \\ p \ne p_1, p \nmid q'}} p$*, where* p_1 *is some prime divisor of the exceptional modulus up to* q *if there exists a Siegel's zero, then*

$$\left| x - \sum_{x \le n \le 2x} \Lambda(n)\chi_0(n) \right| + \sum_{\substack{\chi(q) \\ \chi \ne \chi_0}} \left| \sum_{x \le n \le 2x} \Lambda(n)\chi(n) \right| \le \frac{x}{2}.$$

This lemma is implied by [16, Theorem 7] (see also in [7, Lemma 4]). It follows that, for $(a,q)=1$,

$$\begin{aligned}
\psi(2x,q,a) &= \sum_{\substack{n\le 2x\\ n\equiv a(q)}}\Lambda(n) \ge \sum_{\substack{x\le n\le 2x\\ n\equiv a(q)}}\Lambda(n)\\
&= \frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{x\le n\le 2x}\chi(n)\Lambda(n)\\
&= \frac{1}{\varphi(q)}\sum_{x\le n\le 2x}\Lambda(n)\chi_0(n)+\frac{1}{\varphi(q)}\sum_{\substack{\chi(q)\\ \chi\ne\chi_0}}\overline{\chi}(a)\sum_{x\le n\le 2x}\chi(n)\Lambda(n)\\
&= \frac{x}{\varphi(q)}-\frac{1}{\varphi(q)}\left(x-\sum_{x\le n\le 2x}\Lambda(n)\chi_0(n)-\sum_{\substack{\chi(q)\\ \chi\ne\chi_0}}\overline{\chi}(a)\sum_{x\le n\le 2x}\Lambda(n)\chi(n)\right)\\
&\ge \frac{x}{\varphi(q)}-\frac{1}{\varphi(q)}\left(\left|x-\sum_{x\le n\le 2x}\Lambda(n)\chi_0(n)\right|+\sum_{\substack{\chi(q)\\ \chi\ne\chi_0}}|\overline{\chi}(a)|\left|\sum_{x\le n\le 2x}\Lambda(n)\chi(n)\right|\right)\\
&\ge \frac{x}{\varphi(q)}-\frac{x}{2\varphi(q)}=\frac{x}{2\varphi(q)}.
\end{aligned} \tag{2.34}$$

Let now

$$Q=\prod_{\substack{p\le x\\ p\ne p_1,p\nmid q_1q_2}} p.$$

Hence,

$$\sum_{\substack{n\le 4x\\ Q|n}} G(n,q_1,q_2,a_1,a_2)\ge \sum_{\substack{b=1\\ (b,Q)=1}}^{Q}\sum_{\substack{m\le 2x\\ m\equiv a_1(q_1)\\ m\equiv b(Q)}}\Lambda(m)\sum_{\substack{l\le 2x\\ l\equiv a_2(q_2)\\ l\equiv Q-b(Q)}}\Lambda(l). \tag{2.35}$$

We now use the Chinese remainder Theorem to evaluate the right hand side of (2.35). Since Q and q_1 are coprime, the solutions of the system $m\equiv a_1(q_1)$ and $m\equiv b(Q)$ are given by

$$m\equiv a_1m_1Q+bm_2q_1(q_1Q):=A(q_1Q),$$

where m_1,m_2 are integers satisfying $m_1Q+m_2q_1=1$.

Analogously to the system $l\equiv a_2(q_2)$ and $l\equiv Q-b(Q)$, we have

$$l\equiv: B(q_2Q).$$

Thus, using Lemma 2.3.2, the right hand side of (2.35) is

$$\begin{aligned}
\sum_{\substack{b=1\\ (b,Q)=1}}^{Q}\sum_{\substack{m\le 2x\\ m\equiv A(q_1Q)}}\Lambda(m)\sum_{\substack{l\le 2x\\ l\equiv B(q_2Q)}}\Lambda(l) &= \sum_{\substack{b=1\\ (b,Q)=1}}^{Q}\psi(2x,q_1Q,A)\psi(2x,q_2Q,B)\\
&\ge \sum_{\substack{b=1\\ (b,Q)=1}}^{Q}\frac{x}{2\varphi(q_1Q)}\frac{x}{2\varphi(q_2Q)}=\frac{x^2}{4\varphi(q_1)\varphi(q_2)\varphi(Q)},
\end{aligned}$$

since $(q_1, Q) = 1$, $(q_2, Q) = 1$.

Hence, we obtain

$$\begin{aligned}\max_{n\le 4x} G(n,q_1,q_2,a_1,a_2) &\ge \frac{Q}{4x}\sum_{\substack{n\le 4x\\ Q|n}} G(n,q_1,q_2,a_1,a_2)\\ &\ge \frac{xQ}{16\varphi(q_1)\varphi(q_2)\varphi(Q)} = \frac{x}{16\varphi(q_1)\varphi(q_2)}\prod_{\substack{p\le x\\ p\ne p_1, p\nmid q_1q_2}}(1-p^{-1})^{-1}\\ &= (1-p_1^{-1})\frac{\prod_{p|q_1q_2}(1-p^{-1})}{16\varphi(q_1)\varphi(q_2)}x\prod_{p\le x}(1-p^{-1})^{-1}\\ &\gg_{q_1,q_2} x\log\log x,\end{aligned}$$

where in the last inequality, we used Proposition 1.1.5.

□

2.4 Cesàro weighted

The Cesàro weighted average for Goldbach numbers was introduced in [30] by Languasco and Zaccagnini. They proved, for $X \ge 2$, and $k > 1$ a real number, that

$$\begin{aligned}\sum_{n\le X} G(n)\frac{\left(1-\frac{n}{X}\right)^k}{\Gamma(k+1)} &= \frac{X^2}{\Gamma(k+3)} - 2\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(\rho+k+2)}X^{\rho+1}\\ &+\sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+k+1)}X^{\rho_1+\rho_2} + \mathcal{O}_k(X^{1/2}).\end{aligned}$$

In the study of these problems, the parameter k plays a crucial role and we would like to keep it as small as possible. The work of Languasco-Zaccagnini treats $k > 1$. Goldston-Yang [19] extended the above for $k = 1$ under the Riemann Hypothesis. Further, Cantarini considered this problem without assuming any hypotheses [30] and Brüdern-Kaczorowski-Perelli [10] gave a general formula for $k > 0$, still for $q = 1$, in which the zero-sum converges conditionally.

In arithmetic progressions, Cantarini-Gambini-Zaccagnini [11] studied the Cesàro weighted version for $q_1 = q_2$ and $k > 1$. We would like to extend this problem to the case $q_1 \ne q_2$ and $k = 1$ in the following theorem.

Theorem 2.4.1. *When X tends to infinity, we have*

$$\begin{aligned}\sum_{n\le X}\left(1-\frac{n}{X}\right)G(n,q_1,q_2,a_1,a_2) &= \frac{X^2}{6\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)}L(X,q_1,a_1)\\ &+\frac{1}{\varphi(q_1)}L(X,q_2,a_2) + \mathcal{O}(X^{B^*_{q_1}+B^*_{q_2}}\log X\log(q_1X)\log(q_2X)),\end{aligned}$$

where the implicit constant depends on q_1, q_2 and for $(a,q) = 1$,

$$L(X,q,a) = -\frac{1}{\varphi(q)}\sum_{\chi(q)}\overline{\chi}(a)\sum_{\rho_\chi}\frac{X^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)(\rho_\chi+2)}.$$

Remark 2.4.2. *When $q_1 = q_2 = 1$ and assuming the RH, the error term in Theorem 2.4.1 is $\mathcal{O}(X\log^3 X)$. It is weaker than the result of Goldston-Yang [19] because some functions which we used in our argument can be estimated in a more efficient way when $q = 1$ and under the RH.*

Proof. First, let

$$\mathcal{K}(X, z) := \sum_{n\le X}\left(1 - \frac{n}{X}\right) z^n. \tag{2.36}$$

Then we have

$$\begin{aligned}
&\int_0^1 (F_{a_1,q_1}(z) - I_{q_1}(z))\,(F_{a_2,q_2}(z) - I_{q_2}(z))\,\mathcal{K}\left(X, \frac{1}{z}\right) d\alpha \\
&= \int_0^1 \sum_n B(n, q_1, q_2, a_1, a_2) r^n e(n\alpha) \sum_{m\le X}\left(1 - \frac{m}{X}\right) r^{-m} e(-m\alpha) d\alpha \\
&= \sum_{n\le X}\left(1 - \frac{n}{X}\right) B(n, q_1, q_2, a_1, a_2).
\end{aligned}$$

From (2.7) and Lemma 2.2.8, we have

$$\begin{aligned}
&\sum_{n\le X}\left(1 - \frac{n}{X}\right) B(n, q_1, q_2, a_1, a_2) \\
&= \sum_{n\le X}\left(1 - \frac{n}{X}\right) G(n, q_1, q_2, a_1, a_2) - \frac{1}{\varphi(q_1)X}\int_1^X \sum_{n\le u}\psi(n-1, q_2, a_2) du \\
&- \frac{1}{\varphi(q_2)X}\int_1^X \sum_{n\le u}\psi(n-1, q_1, a_1) du + \frac{1}{\varphi(q_1)\varphi(q_2)X}\int_1^X \sum_{n\le u}(n-1) du \\
&= \sum_{n\le X}\left(1 - \frac{n}{X}\right) G(n, q_1, q_2, a_1, a_2) \\
&- \frac{1}{\varphi(q_1)\varphi(q_2)X}\int_1^X \left(\frac{u^2}{2} - \sum_{\chi(q_1)}\overline{\chi}(a_1)\sum_{\rho_\chi}\frac{u^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} + \mathcal{O}(u\log^2(q_1 u))\right) \\
&- \left(\sum_{\chi(q_2)}\overline{\chi}(a_2)\sum_{\rho_\chi}\frac{u^{\rho_\chi+1}}{\rho_\chi(\rho_\chi+1)} + \mathcal{O}(u\log^2(q_2 u))\right) du.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
\sum_{n\le X}\left(1 - \frac{n}{X}\right) G(n, q_1, q_2, a_1, a_2) &= \frac{X^2}{6\varphi(q_1)\varphi(q_2)} + \frac{1}{\varphi(q_2)} L(X, q_1, a_1) + \frac{1}{\varphi(q_1)} L(X, q_2, a_2) \\
&+ \mathcal{O}(X(\log^2(q_1 X) + \log^2(q_2 X)) + \mathcal{E}(X, q_1, q_2, a_1, a_2),
\end{aligned}$$

where

$$\mathcal{E}(X, q_1, q_2, a_1, a_2) = \int_0^1 (F_{a_1,q_1}(z) - I_{q_1}(z))\,(F_{a_2,q_2}(z) - I_{q_2}(z))\,\mathcal{K}\left(X, \frac{1}{z}\right) d\alpha.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\,\mathcal{E}(X,q_1,q_2,a_1,a_2)\,|^2 \ll & \int_0^1 |F_{a_1,q_1}(z)-I_{q_1}(z)|^2 \left|\mathcal{K}\left(X,\frac{1}{z}\right)\right| d\alpha \\ & \times \int_0^1 |F_{a_2,q_2}(z)-I_{q_2}(z)|^2 \left|\mathcal{K}\left(X,\frac{1}{z}\right)\right| d\alpha.\end{aligned}$$

We need only to estimate

$$T_2(X,q,a) := \int_0^1 |F_{a,q}(z)-I(z)|^2 \left|\mathcal{K}\left(X,\frac{1}{z}\right)\right| d\alpha.$$

We consider

$$\begin{aligned}\mathcal{K}\left(X,\frac{1}{z}\right) &= \sum_{n\le x}\left(1-\frac{n}{X}\right)e^{n/X}e(-n\alpha) = \frac{1}{X}\int_1^X \sum_{n\le t} e^{n/X}e(-n\alpha)dt \\ &= \frac{1}{X}\int_1^X I\left(t,\frac{1}{z}\right)dt \\ &\ll \min\left(X,\frac{1}{|\alpha|}\right),\end{aligned}$$

since (2.15). Using Lemma 3.3.4 and similarly to the proof of Theorem 2.2.1, we also obtain

$$T_2(X,q,a) \ll_q X^{2B_q^*}\log X\log^2(qX).$$

We conclude

$$\mathcal{E}(X,q_1,q_2,a_1,a_2) \ll_{q_1,q_2} X^{B_{q_1}^*+B_{q_2}^*}\log X\log(q_1X)\log(q_2X).$$

Hence the proof of Theorem 2.4.1 is now complete. □

Chapter 3

Goldbach Representations of an integer as a Sum of many primes

Let $k \geq 2$ be an integer, we define the *weighted* Goldbach function for k primes

$$G_k(n) := \sum_{n_1+n_2+...+n_k=n} \Lambda(n_1)...\Lambda(n_k),$$

and its average order

$$S_k(X) := \sum_{n \leq X} G_k(n).$$

Then we obtain results about $S_k(X)$ as follows.

3.1 An asymptotic result for $S_k(X)$ under the RH

In this section we prove

Theorem 3.1.1. *Let $k \geq 2$, $X \geq k$ and assume the RH holds. Then we have*

$$S_k(X) = \frac{X^k}{k!} + H_k(X) + \mathcal{O}_k(X^{k-1}\log^3 X),$$

with

$$H_k(X) = -k \sum_{\rho} \frac{X^{\rho+k-1}}{\rho(\rho+1)...(\rho+k-1)},$$

where the sum is taken over the non-trivial zeros in the critical line of the Riemann zeta function.

This improves the existing result for $k \geq 4$, where the error term is bounded by $X^{k-1}\log^k X$ ([28], Theorem 1.3) and agrees with the best known upper bound of the error term of $S(X)$ for $k = 2$, i.e., $X \log^3 X$, first obtained in [28] and later in [19].

For our proof we follow the idea of Goldston and Yang [19] for $k = 2$ and a trivial bound (see Lemma 3.1.4).

To prove Theorem 3.1.1, we now use induction on k. We know that this theorem is true when $k = 2$ and suppose that it holds up to $k - 1$, then we prove it for k. We use the notation of [19]. Consider the generating function

$$S_0(\alpha, x) = \sum_{n \leq x} \Lambda_0(n)e(n\alpha), \qquad e(\alpha) = e^{2\pi i\alpha},$$

where $\Lambda_0(n) = \Lambda(n) - 1$. Then for $k \geq 2$, we have

$$\begin{aligned} S_0(\alpha, x)^k &= \sum_{n_1,\ldots,n_k \leq x} \Lambda_0(n_1)\ldots\Lambda_0(n_k)e((n_1 + \cdots + n_k)\alpha) \\ &= \sum_{n \leq kx} B_k(n, x)e(n\alpha), \end{aligned}$$

where

$$B_k(n, x) = \sum_{\substack{n_1,\ldots,n_k \leq x \\ n_1+\cdots+n_k=n}} \Lambda_0(n_1)\ldots\Lambda_0(n_k).$$

When $k \leq n \leq x$, we can express $G_k(n)$ through $B_k(n, x)$ as

$$\begin{aligned} B_k(n, x) &= \sum_{n_1+\cdots+n_k=n} \Lambda_0(n_1)\ldots\Lambda_0(n_k) \\ &= \sum_{n_1+\cdots+n_k=n} (\Lambda(n_1) - 1)\ldots(\Lambda(n_k) - 1) \\ &= G_k(n) - k\sum_{n_1=1}^{n-k+1} G_{k-1}(n - n_1) + \sum_{i=2}^{k-2}(-1)^i\binom{k}{i}\sum_{n_1+\cdots+n_k=n} \Lambda(n_i)\ldots\Lambda(n_k) \\ &+ (-1)^{k-1}k\sum_{n_1+\cdots+n_k=n} \Lambda(n_k) + (-1)^k\sum_{n_1+\cdots+n_k=n} 1. \end{aligned} \tag{3.1}$$

Let $y \geq 2$, we define

$$T(y, \alpha) := \sum_{n \leq y} e(n\alpha). \tag{3.2}$$

We then have the estimate $T(y, \alpha) \ll \min\left(y, \frac{1}{|\alpha|}\right)$. For $x \geq X$, we have

$$\int_0^1 S_0(\alpha, x)^k T(X, -\alpha)d\alpha = \sum_{n \leq X} B_k(n, x).$$

Substituting (3.1) into the above, we obtain

$$\begin{aligned} S_k(X) &= \int_0^1 S_0(\alpha, x)^k T(X, -\alpha)d\alpha + \sum_{i=1}^{k-2}(-1)^{i+1}\binom{k}{i}\sum_{n \leq X}\sum_{n_1+\cdots+n_k=n} \Lambda(n_{i+1})\ldots\Lambda(n_k) \\ &+ (-1)^k k\sum_{n \leq X}\sum_{n_1+\cdots+n_k=n} \Lambda(n_k) + (-1)^{k+1}\sum_{n \leq X}\sum_{n_1+\cdots+n_k=n} 1 \\ &=: N_0 + \sum_{i=1}^{k-2}(-1)^{i+1}\binom{k}{i}N_i + (-1)^k kN_{k-1} + (-1)^{k+1}N_k. \end{aligned} \tag{3.3}$$

Now, we estimate N_i, for $0 \le i \le k$.

Part 1: Main term of N_k.

We have

$$N_k = \sum_{n \le X} \sum_{n_1 + \cdots + n_k = n} 1 = \sum_{k \le n \le X} \binom{n-1}{k-1} = \frac{X^k}{k!} + \mathcal{O}(X^{k-1}). \tag{3.4}$$

Part 2: Main term of N_{k-1}.

$$\begin{aligned} N_{k-1} &= \sum_{n \le X} \sum_{n_1 + \cdots + n_k = n} \Lambda(n_k) = \sum_{n \le X} \sum_{n_k=1}^{X-(k-1)} \sum_{n_1 + \cdots + n_{k-1} = n - n_k} \Lambda(n_k) \\ &= \sum_{n \le X} \sum_{n_k=1}^{X-(k-1)} \binom{n - n_k - 1}{k-2} \Lambda(n_k) = \sum_{n_k=1}^{X-(k-1)} \Lambda(n_k) \left(\frac{(X-n_k)^{k-1}}{(k-1)!} + \mathcal{O}(X^{k-2}) \right) \\ &= \psi_{k-1}(X-k+1) + \mathcal{O}(X^{k-1}), \end{aligned}$$

where for j being a non-negative integer, we denote

$$\psi_j(x) := \frac{1}{j!} \sum_{n \le x} (x-n)^j \Lambda(n). \tag{3.5}$$

We note that for $j \ge 1$, one has

$$\psi_j(x) = \int_0^x \psi_{j-1}(t) dt. \tag{3.6}$$

Moreover, for $j = 1$ we have an explicit formula (see [35, (13.7)])

$$\begin{aligned} \psi_1(t) &= \frac{t^2}{2} - \sum_\rho \frac{t^{\rho+1}}{\rho(\rho+1)} - \frac{\zeta'}{\zeta}(0) t + \frac{\zeta'}{\zeta}(-1) + \mathcal{O}(t^{-1/2}) \\ &= \frac{t^2}{2} - \sum_\rho \frac{t^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}(t). \end{aligned} \tag{3.7}$$

Then by (3.6) and the induction method, it is easy to prove that

$$\psi_j(t) = \frac{t^{j+1}}{(j+1)!} - \sum_\rho \frac{t^{\rho+j}}{\rho(\rho+1)\dots(\rho+j)} + \mathcal{O}(t^j). \tag{3.8}$$

To be able to exchange $\sum_\rho$ and the integral is important to remark that $\sum_\rho$ into (3.7) absolutely converges.

Moreover, by definition of $\psi_j(x)$, we obtain

Lemma 3.1.2. *Let j be a non-negative integer, then for any positive integer n, we have*

$$\psi_j(x+n) = \psi_j(x) + \mathcal{O}(x^j).$$

Proof. By the definition of $\psi_j(x)$ and (3.8), we have

$$\begin{aligned}\psi_j(x+1) &= \frac{1}{j!}\sum_{m\le x+1}\Lambda(m)(x+1-m)^j\\ &= \frac{1}{j!}\sum_{m\le x}\Lambda(m)[(x-m)^j+\sum_{a=1}^{j}\binom{j}{a}(x-m)^{j-a}]\\ &= \psi_j(x)+\sum_{a=1}^{j}\frac{1}{a!}\psi_{j-a}(x)=\psi_j(x)+\mathcal{O}(x^j).\end{aligned}$$

Hence Lemma 3.1.2 is proved. □

From (3.8) and Lemma 3.1.2, we obtain

$$N_{k-1}=\frac{X^k}{k!}-\sum_{\rho}\frac{X^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)}+\mathcal{O}(X^{k-1}). \tag{3.9}$$

Part 3: Estimate N_0.

Analogously to the idea of Goldston and Yang [19], we define an expected value function for $S_0(\alpha,x)$ as

$$E_X(S_0(\alpha)):=\frac{1}{X}\int_X^{2X}S_0(\alpha,x)dx.$$

We need the following lemmas.

Lemma 3.1.3 ([19], Lemma 7)**.** *Assuming the RH, we have for $1\le h\le X$,*

$$\int_{-1/2h}^{1/2h}E_X(|S_0(\alpha)|^2)d\alpha\ll\frac{X\log^2 X}{h}.$$

Lemma 3.1.4. *Let $x\in[X,2X]$, we have the estimate*

$$S_0(\alpha,x)\ll X.$$

Proof. We have

$$\begin{aligned}S_0(\alpha,x)&\ll\sum_{n\le x}|\Lambda(n)-1|\,|e(n\alpha)|\le\sum_{n\le x}|\Lambda(n)-1|\\ &\le\sum_{n\le x}\Lambda(n)+\sum_{n\le x}1=\psi(x)+[x]\\ &\ll X,\end{aligned}$$

with $x\le 2X$. □

Consider $\alpha\in[-1/2,1/2]$, since $T(X,-\alpha)\ll\min\left(X,\frac{1}{|\alpha|}\right)$, we have

$$\begin{aligned}N_0&\ll\int_{-1/2}^{1/2}|S_0(\alpha,x)|^kT(X,-\alpha)d\alpha=\int_{-1/2}^{1/2}E_X(|S_0(\alpha)|^k)T(X,-\alpha)d\alpha\\ &\ll X\int_{-1/X}^{1/X}E_X(|S_0(\alpha)|^k)d\alpha+\int_{1/X}^{1/2}\frac{E_X(|S_0(\alpha)|^k)}{\alpha}d\alpha.\end{aligned}\tag{3.10}$$

We estimate the first term of (3.10)

$$\begin{aligned}
&X\int_{-1/X}^{1/X} E_X(|S_0(\alpha)|^k)d\alpha = X\int_{-1/X}^{1/X}\frac{1}{X}\int_X^{2X}|S_0(\alpha,x)|^k\,dxd\alpha\\
&\le X\int_{-1/X}^{1/X}\frac{1}{X}\max_{x\in[X,2X]}|S_0(\alpha,x)|^{k-2}\int_X^{2X}|S_0(\alpha,x)|^2\,dxd\alpha\\
&\le X\max_{\substack{x\in[X,2X]\\|\alpha|\le 1/X}}|S_0(\alpha,x)|^{k-2}\int_{-1/X}^{1/X}E_X(|S_0(\alpha)|^2)d\alpha\\
&\ll X^{k-1}\log^2 X,
\end{aligned}$$

where for the last inequality, we use Lemma 3.1.3 and Lemma 3.1.4.

For the second term of (3.10), writing $[1/X,1/2]$ as the disjoint union of $[2^j/X,2^{j+1}/X]$ for $0\le j\le \mathcal{O}(\log X)$, then

$$\begin{aligned}
\int_{1/X}^{1/2}\frac{E_X(|S_0(\alpha)|^k)}{\alpha}d\alpha &\ll \sum_{j=0}^{\mathcal{O}(\log X)}\frac{X}{2^j}\int_{2^j/X}^{2^{j+1}/X}E_X(|S_0(\alpha)|^k)d\alpha\\
&\ll \sum_{j=0}^{\mathcal{O}(\log X)}\frac{X}{2^j}\int_{2^j/X}^{2^{j+1}/X}\frac{1}{X}\max_{x\in[X,2X]}|S_0(\alpha,x)|^{k-2}\int_X^{2X}|S_0(\alpha,x)|^2\,dxd\alpha\\
&\le \sum_{j=0}^{\mathcal{O}(\log X)}\frac{X}{2^j}\max_{\substack{x\in[X,2X]\\ \alpha\in[2^j/X,2^{j+1}/N]}}|S_0(\alpha,x)|^{k-2}\int_{2^j/X}^{2^{j+1}/X}E_X(|S_0(\alpha)|^2)d\alpha\\
&\ll \sum_{j=0}^{\mathcal{O}(\log X)}\frac{X}{2^j}X^{k-2}X\frac{2^{j+1}}{X}\log^2 X \ll X^{k-1}\log^3 X.
\end{aligned}$$

Then we obtain

$$N_0 \ll X^{k-1}\log^3 X. \tag{3.11}$$

Part 4: Main term of N_i, $1\le i\le k-2$.

$$\begin{aligned}
N_i &= \sum_{n\le X}\sum_{n_1+\dots+n_k=n}\Lambda(n_{i+1})\dots\Lambda(n_k)\\
&= \sum_{n\le X}\sum_{n_1+\dots+n_i=i}^{n-(k-i)}\left(\sum_{n_{i+1}+\dots+n_k=n-(n_1+\dots+n_i)}\Lambda(n_{i+1})\dots\Lambda(n_k)\right)\\
&= \sum_{n\le X}\left[\binom{i-1}{i-1}G_{k-i}(n-i)+\dots+\binom{n-(k-i+1)}{i-1}G_{k-i}(k-i)\right]\\
&= \binom{i-1}{i-1}G_{k-i}(X-i)+\left[\binom{i-1}{i-1}+\binom{i}{i-1}\right]G_{k-i}(X-1-i)+\dots\\
&+\left[\binom{i-1}{i-1}+\binom{i}{i-1}+\dots+\binom{i+X-k-1}{i-1}\right]G_{k-i}(k-i).
\end{aligned}$$

Using the formula for m being a non-negative integer, we have

$$\binom{i-1}{i-1}+\binom{i}{i-1}+\dots+\binom{i+m}{i-1}=\binom{i+m+1}{i}. \tag{3.12}$$

Hence we obtain that

$$\begin{aligned}N_i &= \binom{i}{i}G_{k-i}(X-i) + \binom{i+1}{i}G_{k-i}(X-i) + \cdots + \binom{i+X-k}{i}G_{k-i}(k-i)\\ &= \sum_{n\le X}\binom{X-n}{i}G_{k-i}(n)\\ &= \sum_{n\le X}\frac{(X-n)^i}{i!}G_{k-i}(n) + \mathcal{O}\left(\sum_{n\le X}(X-n)^{i-1}G_{k-i}(n)\right)\\ &= T_i(X,k-i) + \mathcal{O}(X^{k-1}),\end{aligned}$$

where for $j\ge 0$,

$$T_j(X,k-i) := \frac{1}{j!}\sum_{n\le X}(X-n)^j G_{k-i}(n).$$

Then we have a property for this function, that is

$$T_{j+1}(X,k-i) = \int_0^X T_j(t,k-i)dt.$$

Moreover, by the induction hypothesis, for $1\le i\le k-2$, we have

$$\sum_{n\le x}G_{k-i}(n) = \frac{x^{k-i}}{(k-i)!} + H_{k-i}(x) + \mathcal{O}_k(x^{k-i-1}\log^3 x).$$

So we calculate

$$\begin{aligned}T_j(X,k-i) = \frac{X^{k-i+j}}{(k-i+j)!} - (k-i)\sum_\rho \frac{X^{\rho+k-i+j-1}}{\rho(\rho+1)\dots(\rho+k-i+j-1)}\\ + \mathcal{O}_k(X^{k-i+j-1}\log^3 X).\end{aligned} \tag{3.13}$$

Replacing $j=i$ in (3.13), we obtain

$$N_i = \frac{X^k}{k!} - (k-i)\sum_\rho \frac{X^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)} + \mathcal{O}_k(X^{k-1}\log^3 X). \tag{3.14}$$

Combining (3.3), (3.4), (3.9), (3.11) and (3.14), we obtain

$$\begin{aligned}S_k(X) &= \frac{X^k}{k!}\left[\sum_{i=1}^k(-1)^{i+1}\binom{k}{i}\right] - \sum_\rho \frac{X^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)}\left[\sum_{i=1}^k(-1)^{i+1}\binom{k}{i}(k-i)\right]\\ &\quad + \mathcal{O}(X^{k-1}\log^3 X)\\ &= \frac{X^k}{k!} - k\sum_\rho \frac{X^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)} + \mathcal{O}_k(X^{k-1}\log^3 X),\end{aligned}$$

where in the last equation, we used

$$\sum_{i=0}^k(-1)^i\binom{k}{i} = (1-1)^k = 0.$$

and

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i) = k(1-1)^{k-1} = 0.$$

Our proof of Theorem 3.1.1 is now complete. □

We now return to the original error term of $\log^k X$. In 2012, Languasco and Zaccagnini [28] stated an asymptotic formula of $S_k(X)$ for $k \geq 3$ as follows.

Theorem 3.1.5. *Let $k \geq 3$ be an integer and assuming the RH, then for $X \geq 3$*

$$S_k(X) = \frac{X^k}{k!} + H_k(X) + \mathcal{O}(X^{k-1}\log^k X).$$

Following their proof for the case $k = 2$, we fill in the details for case $k \geq 3$ using the original Hardy and Littlewood approach to the circle method.

In this section, we will prove asymptotic formulas of Theorem 3.1.5. Furthermore, we will check if Theorem 3.1.1 can be recovered from the original proof of [29], a question posed by the referee.

To prove Theorem 3.1.5, we need the two following results. Firstly,

Theorem 3.1.6. *Let $k \geq 3$ be an integer and assuming the RH. Then for $2 \leq y \leq X$, we have*

$$\left|\sum_{n\leq y}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right]e^{-n/X}\right| \ll X^{k-1}(\log X)^k,$$

where the function $\psi_j(x)$ is defined in (3.5).

The reason why we are able to derive Theorem 3.1.5 from Theorem 3.1.6 is the same for $k = 2$: via partial summation, the exponential weight in Theorem 3.1.6 just varies in the range $[e^{-1/N}, e^{-1}]$, and so the orders of magnitude of the functions involved are not changed.

Moreover, we can prove the following lemma.

Lemma 3.1.7. *Let $k \geq 3$ be an integer and $y \geq 2$, we have*

$$\sum_{n\leq y}\left(\psi_{k-2}(n) - \frac{n^{k-1}}{(k-1)!}\right) = -\sum_{\rho}\frac{y^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)} + \mathcal{O}(y^{k-1}).$$

Proof. From (3.8) with $j = k-2$, we have

$$\begin{aligned}\sum_{n\leq y}\left(\psi_{k-2}(n) - \frac{n^{k-1}}{(k-1)!}\right) &= \int_1^y\left(\psi_{k-2}(t) - \frac{t^{k-1}}{(k-1)!}\right)dt + \mathcal{O}(y^{k-1})\\ &= -\int_1^y\left(\sum_{\rho}\frac{t^{\rho+k-2}}{\rho(\rho+1)\dots(\rho+k-2)} + \mathcal{O}(t^{k-2})\right)dt\\ &\quad + \mathcal{O}(y^{k-1})\\ &= -\sum_{\rho}\frac{y^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)} + \mathcal{O}(y^{k-1}).\end{aligned}$$

Then the proof is complete. □

We follow the notation of [28]. Let $X \geq 2$, then the weighted exponential sums are

$$\widetilde{S}(\alpha) := \sum_{n\geq 1} \Lambda(n) e^{-n/X} e(n\alpha) = \sum_{n\geq 1} \Lambda(n) e^{-nz},$$
$$V(\alpha) := \sum_{n\geq 1} e^{-n/X} e(n\alpha) = \sum_{n\geq 1} e^{-nz},$$
$$\widetilde{R}(\alpha) = \widetilde{S}(\alpha) - \frac{1}{z},$$

where $z = 1/X - 2\pi i\alpha$, $\alpha \in [-1/2, 1/2]$.

Remark that these are some exponential sums originally used by Hardy-Littlewood and rewritten with the congruence condition in Chapter 2 of this thesis.

Then we have the following lemmas.

Lemma 3.1.8 ([28], Lemma 2.4)**.** *We have*

$$V(\alpha) = \frac{1}{z} + \mathcal{O}(1).$$

To proceed further we need the following results:

Lemma 3.1.9 ([28], Lemma on page 797)**.** *Assume the RH holds. For $\alpha \in [-1/2, 1/2]$, we have*

$$|\widetilde{R}(\alpha)| \ll (X^{1/2} + X|\alpha|^{1/2}) \log X.$$

Lemma 3.1.10 ([27])**.** *Assume that the RH holds. Let X be a sufficiently large integer. For $0 \leq \xi \leq 1/2$, we have*

$$\int_{-\xi}^{\xi} |\widetilde{R}(\alpha)|^2 d\alpha \ll X\xi \log^2 X.$$

Lemma 3.1.11 ([29], Lemma 8)**.** *Let $k \geq 2$ be an integer, then for $1 \leq l \leq X$ we have*

$$\int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^k} d\alpha = e^{-n/N} \frac{n^{k-1}}{(k-1)!} + \mathcal{O}(1).$$

Proof. With z as above, by the residue theorem we obtain

$$\begin{aligned}\int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^k} d\alpha &= \frac{e^{-n/X}}{2\pi i} \int_{1/X-\pi i}^{1/X+\pi i} \frac{e(-nz)}{z^k} d\alpha \\ &= e^{-n/X} \frac{n^{k-1}}{(k-1)!} + \frac{e^{-n/X}}{2\pi i} \int_{\Gamma} \frac{e^{-n/X}}{(k-1)!} \\ &= e^{-n/X} \frac{n^{k-1}}{(k-1)!} + \mathcal{O}(1),\end{aligned}$$

where we denote by Γ the left half of the circle $|z - 1/X| = \pi$. □

PROOF OF THEOREM 3.1.6

From the definitions of $\widetilde{S}(\alpha)$ and $G_k(n)$, we calculate

$$\widetilde{S}(\alpha)^k = \left(\sum_{m\geq 1}\Lambda(m)e^{-m/X}e(m\alpha)\right)^k = \sum_{n\geq 1}G_k(n)e^{-n/X}e(n\alpha).$$

Then we obtain

$$G_k(n)e^{-n/X} = \int_{-1/2}^{1/2}\widetilde{S}(\alpha)^k e(-n\alpha)d\alpha.$$

So that

$$\sum_{n\leq y}G_k(n)e^{-n/X} = \sum_{n\leq y}\int_{-1/2}^{1/2}\widetilde{S}(\alpha)^k e(-n\alpha)d\alpha = \int_{-1/2}^{1/2}\widetilde{S}(\alpha)^k T(y,-\alpha)d\alpha, \tag{3.15}$$

where $T(y,\alpha)$ is defined in (3.2). Since $\widetilde{S}(\alpha) = \widetilde{R}(\alpha) + \frac{1}{z}$, we rewrite (3.15) as

$$\begin{aligned}\sum_{n\leq y}G_k(n)e^{-n/X} &= \int_{-1/2}^{1/2}\left(\widetilde{R}(\alpha)+\frac{1}{z}\right)^k T(y,-\alpha)d\alpha \\ &= \int_{-1/2}^{1/2}\frac{T(y,-\alpha)}{z^k}d\alpha + k\int_{-1/2}^{1/2}\frac{\widetilde{R}(\alpha)}{z^{k-1}}T(y,-\alpha)d\alpha \\ &\quad + \sum_{i=2}^{k-1}\binom{k}{i}\int_{-1/2}^{1/2}\frac{\widetilde{R}(\alpha)^i}{z^{k-i}}T(y,-\alpha)d\alpha + \int_{-1/2}^{1/2}\widetilde{R}(\alpha)^k T(y,-\alpha)d\alpha \\ &= I_0 + kI_1 + \sum_{i=2}^{k-1}\binom{k}{i}I_i + I_k.\end{aligned} \tag{3.16}$$

To prove Theorem 3.1.6 we will estimate I_i for $i = 0, 1, \ldots, k$.
Estimate of I_0. Using Lemma 3.1.11, we obtain

$$I_0 = \sum_{n\leq y}\int_{-1/2}^{1/2}\frac{e(-n\alpha)}{z^k}d\alpha = \sum_{n\leq y}e^{-n/X}\frac{n^{k-1}}{(k-1)!} + \mathcal{O}(y). \tag{3.17}$$

Estimate of I_1. By Lemma 3.1.8,

$$\frac{1}{z^{k-1}} = V(\alpha)^{k-1} + \mathcal{O}_k(V(\alpha)^{k-2}).$$

Then I_1 become

$$\begin{aligned}I_1 &= \int_{-1/2}^{1/2}\frac{\widetilde{R}(\alpha)}{z^{k-1}}T(y,-\alpha)d\alpha \\ &= \int_{-1/2}^{1/2}\widetilde{R}(\alpha)V(\alpha)^{k-1}T(y,-\alpha)d\alpha + \mathcal{O}_k\left(\int_{-1/2}^{1/2}\widetilde{R}(\alpha)V(\alpha)^{k-2}T(y,-\alpha)d\alpha\right).\end{aligned}$$

Since $\widetilde{R}(\alpha) = \widetilde{S}(\alpha) - V(\alpha) + \mathcal{O}(1)$, we obtain

$$\begin{aligned}
\int_{-1/2}^{1/2} \widetilde{R}(\alpha)V(\alpha)^{k-1}T(y,-\alpha)d\alpha &= \int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))V(\alpha)^{k-1}T(y,-\alpha)d\alpha \\
&\quad + \mathcal{O}_k\left(\int_{-1/2}^{1/2} |V(\alpha)|^{k-1}|T(y,-\alpha)|d\alpha\right) \\
&= \int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))V(\alpha)^{k-1}T(y,-\alpha)d\alpha \\
&\quad + \mathcal{O}_k(X^{k-1}).
\end{aligned} \tag{3.18}$$

Remark that

$$\widetilde{S}(\alpha) - V(\alpha) = \sum_{n=1}^{\infty}(\Lambda(n) - 1)e^{-n/X}e(n\alpha),$$

and for any s positive integer.

$$V(\alpha)^s = \sum_{n=s}^{\infty} \sum_{n_1+\ldots n_s=n} e^{-n/X}e(n\alpha) = \sum_{n=s}^{\infty}\binom{n-1}{s-1}e^{-n/X}e(n\alpha). \tag{3.19}$$

Then we obtain

$$\begin{aligned}
\int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))V(\alpha)^{k-1}T(y,-\alpha)d\alpha =& \sum_{n\le y}\int_{-1/2}^{1/2}\left(\sum_{m_1=1}^{\infty}(\Lambda(m_1)-1)e^{-m_1/X}e(m_1\alpha)\right) \\
& \times\left(\sum_{m_2=k-1}^{\infty}\binom{m_2-1}{k-2}e^{-m_2/X}e(m_2\alpha)\right)e(-n\alpha)d\alpha \\
=& \sum_{n\le y}\sum_{m_1=1}^{\infty}(\Lambda(m_1)-1)\sum_{m_2=k-1}^{\infty}\binom{m_2-1}{k-2} \\
& \times e^{-(m_1+m_2)/X}\int_{-1/2}^{1/2} e(m_1+m_2-n)d\alpha \\
=& \sum_{n\le y} e^{-n/X}\sum_{m_1=1}^{n-k+1}(\Lambda(m_1)-1)\binom{n-m_1-1}{k-2},
\end{aligned}$$

where for the last equality, we use

$$\int_{-1/2}^{1/2} e(m_1+m_2-n)d\alpha = 1 \text{ if } m_1+m_2 = n \text{ and } 0 \text{ otherwise.}$$

In the following we will need two properties of binomials, i.e.,

$$\begin{aligned}
\binom{n}{r} &= \frac{n!}{r!(n-r)!} = \frac{n^r}{r!} + \mathcal{O}_r(n^{r-1}), \\
\sum_{n=a}^{b}\binom{n}{a} &= \binom{b+1}{a+1}.
\end{aligned}$$

Using the above forme and (3.8), we obtain

$$\begin{aligned}\sum_{m_1=1}^{n-k+1} \Lambda(m_1)\binom{n-m_1-1}{k-2} &= \sum_{m_1=1}^{n-k+1} \Lambda(m_1)\frac{(n-m_1-k+1)^{k-2}}{(k-2)!} \\ &\quad + \mathcal{O}_k\left(\sum_{m_1=1}^{n-k+1} \Lambda(m_1)(n-m_1-k+1)^{k-3}\right) \\ &= \psi_{k-2}(n-k+1) + \mathcal{O}_k(n^{k-2}).\end{aligned}$$

By Lemma 3.1.2, we can infer

$$\sum_{m_1=1}^{n-k+1} \Lambda(m_1)\binom{n-m_1-1}{k-2} = \psi_{k-2}(n) + \mathcal{O}_k(n^{k-2}). \tag{3.20}$$

Moreover, we also have

$$\begin{aligned}\sum_{m_1=1}^{n-k+1} \binom{n-m_1-1}{k-2} &= \sum_{X=k-2}^{n-2} \binom{N}{k-2} = \binom{n-1}{k-1} \\ &= \frac{n^k}{(k-1)!} + \mathcal{O}(n^{k-2}).\end{aligned}$$

Therefore, combining all the previous equations, we obtain

$$\int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))V(\alpha)^{k-1}T(y,-\alpha)d\alpha = \sum_{n\le y} e^{-n/X}\left(\psi_{k-2}(n) - \frac{n^{k-1}}{(k-1)!}\right) + \mathcal{O}_k(y^{k-1}).$$

Hence

$$I_1 = \sum_{n\le y} e^{-n/X}\left(\psi_{k-2}(n) - \frac{n^{k-1}}{(k-1)!}\right) + \mathcal{O}_k(X^{k-1}). \tag{3.21}$$

Estimate of I_i for $2 \le i \le k-1$. Analogously to I_1, using Lemma 3.1.8, we have

$$\begin{aligned}I_i &= \int_{-1/2}^{1/2} \frac{\widetilde{R}(\alpha)^i}{z^{k-i}} T(y,-\alpha)d\alpha \\ &= \int_{-1/2}^{1/2} \widetilde{R}(\alpha)^i V(\alpha)^{k-i}T(y,-\alpha)d\alpha + \mathcal{O}_k\left(\int_{-1/2}^{1/2} \widetilde{R}(\alpha)^i V(\alpha)^{k-i-1}T(y,-\alpha)d\alpha\right).\end{aligned}$$

The first term of I_i can be estimated by

$$\int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))^i V(\alpha)^{k-i}T(y,-\alpha)d\alpha + \mathcal{O}_k\left(\int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))^i V(\alpha)^{k-i-1}T(y,-\alpha)d\alpha\right).$$

Our work now is to bound

$$I_{i,M} = \int_{-1/2}^{1/2} (\widetilde{S}(\alpha) - V(\alpha))^i V(\alpha)^{k-i}T(y,-\alpha)d\alpha. \tag{3.22}$$

From (3.19) and

$$(\widetilde{S}(\alpha) - V(\alpha))^i = \sum_{n\geq 1} \sum_{n_1+\cdots+n_k=n} (\Lambda(n_1) - 1) \dots (\Lambda(n_i) - 1), \tag{3.23}$$

we have

$$I_{i,M} = \sum_{n=1}^{y} e^{-n/N} \binom{n-i-1}{k-i-1} \sum_{m=i}^{n} \sum_{n_1+\cdots+n_i=m} (\Lambda(n_1) - 1) \dots (\Lambda(n_i) - 1). \tag{3.24}$$

Moreover, we also obtain

$$M_i = \sum_{m=i}^{n} \sum_{n_1+\cdots+n_i=m} (\Lambda(n_1) - 1) \dots (\Lambda(n_i) - 1) \ll n^{i-1} \log^{i+1} n.$$

This implies $I_{i,M} \ll_k y^{k-1} \log^k y$. Hence

$$I_i \ll_k X^{k-1} \log^k X. \tag{3.25}$$

Estimate of I_k. Since $T(y, \alpha) \ll \min\left(y, \frac{1}{|\alpha|}\right)$,we have

$$\begin{aligned} I_k &\ll \int_{-1/2}^{1/2} \left|\widetilde{R}(\alpha)\right|^k |T(y, -\alpha)| d\alpha \\ &\ll y \int_{-1/y}^{1/y} \left|\widetilde{R}(\alpha)\right|^k d\alpha + \int_{1/y}^{1/2} \frac{\left|\widetilde{R}(\alpha)\right|^k}{\alpha} d\alpha. \end{aligned}$$

Using Lemma 3.1.10 and Lemma 3.1.9, we obtain

$$\begin{aligned} y \int_{-1/y}^{1/y} \left|\widetilde{R}(\alpha)\right|^k d\alpha &\ll y \left|\widetilde{R}(\alpha)\right|^{k-2} \int_{-1/y}^{1/y} \left|\widetilde{R}(\alpha)\right|^2 d\alpha \\ &\ll y \left(X^{1/2} + \frac{X}{y^{1/2}}\right)^{k-2} (\log X)^{k-2} \frac{X}{y} \log^2 X \ll X^{k-1} \log^k X. \end{aligned}$$

Moreover $[1/y, 1/2]$ can be written as the disjoint union of $[2^j/y, 2^{j+1}/y]$ for $0 \leq j \leq J$, for J satisfies $2^J/y \leq 1/2 \leq 2^{J+1}/y$. Then

$$\begin{aligned} I_{k,1} := \int_{1/y}^{1/2} \frac{\left|\widetilde{R}(\alpha)\right|^k}{\alpha} d\alpha &\ll \sum_{j=0}^{J} \frac{y}{2^j} \int_{2^j/y}^{2^{j+1}/y} |\widetilde{R}(\alpha)|^k d\alpha \\ &\ll \sum_{j=0}^{J} \frac{y}{2^j} \left(X^{1/2} + X\left(\frac{2^{j+1}}{y}\right)^{1/2}\right)^{k-2} (\log X)^{k-2} X \frac{2^{j+1}}{y} \log^2 X \\ &\ll \sum_{j=0}^{J} X^{k/2} \log^k X + \sum_{j=0}^{J} \left(\frac{2^{j+1}}{y}\right)^{(k-2)/2} X^{k-1} \log^k X \\ &=: I_{k,2} + I_{k,3}. \end{aligned}$$

We have the estimates

$$I_{k,2} \le J X^{k/2} \log^k X \ll X^{k/2} \log^{k+1} X$$
$$I_{k,3} = \frac{X^{k-1}\log^k X}{y^{(k-2)/2}} \frac{2^{(J+2)(k-2)/2} - 2^{(k-2)/2}}{2^{(k-2)/2} - 1}$$
$$\ll_k \left(\frac{2^J}{y}\right)^{(k-2)/2} X^{k-1}\log^k X \ll_k X^{k-1}\log^k X.$$

Thus we split into two cases:

Case 1: for $k = 2$.

$$I_{k,3} \ll I_{k,2}.$$

So we obtain

$$I_k \ll I_{k,3} \ll X \log^3 X.$$

Case 2: for $k \ge 3$.

$$I_{k,2} \ll I_{k,3}.$$

Hence

$$I_k \ll I_{k,2} \ll_k X^{k-1}\log^k X.$$

Combining the above and (3.17), (3.21), (3.25), we complete the proof of Theorem 3.1.6. □

PROOF OF THEOREM 3.1.5

As for $k = 2$, we notice that for $k \ge 3$, Theorem 3.1.5 is a consequence of Theorem 3.1.6 since, by partial summation, we have

$$\sum_{n\le X}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right]$$
$$= \sum_{n\le X} e^{n/X}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right] e^{-n/X}$$
$$= e\sum_{n\le X}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right] e^{-n/X}$$
$$- \frac{1}{X}\int_1^X \sum_{n\le y}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right] e^{-n/X} dy + \mathcal{O}(1).$$

By Theorem 3.1.6, for $k \ge 3$ we obtain

$$\sum_{n\le X}\left[G_k(n) - k\psi_{k-2}(n) + \frac{n^{k-1}}{(k-2)!}\right] \ll_k X^{k-1}\log^k X.$$

On the other hand,

$$\sum_{n\leq X}\left[G_k(n)-k\psi_{k-2}(n)+\frac{n^{k-1}}{(k-2)!}\right]=\sum_{n\leq X}G_k(n)-k\sum_{n\leq X}\left(\psi_{k-2}(n)-\frac{n^{k-1}}{(k-1)!}\right)$$
$$-\sum_{n\leq X}\frac{n^{k-1}}{(k-1)!}$$
$$=\sum_{n\leq X}G_k(n)-k\sum_{n\leq X}\left(\psi_{k-2}(n)-\frac{n^{k-1}}{(k-1)!}\right)$$
$$-\frac{X^k}{k!}+\mathcal{O}_k(X^{k-1}).$$

Since Lemma 3.1.7, we obtain

$$\sum_{n\leq X}G_k(n)=\frac{X^k}{k!}-k\sum_{\rho}\frac{X^{\rho+k-1}}{\rho(\rho+1)...(\rho+k-1)}+\mathcal{O}_k(X^{k-1}\log^k X),$$

which completes the proof of Theorem 3.1.5. □

It was pointed out recently by one of the reviewers of the thesis, Alessandro Languasco, that estimating Lemma 3.1.9 better, as follows, would improve the log-power from k to 3.

Lemma 3.1.12. *Assume the RH holds. For $\alpha\in[-1/2,1/2]$*

$$|\widetilde{R}(\alpha)|\ll X.$$

Proof. By the definition of $\widetilde{R}$, we have

$$|\widetilde{R}(\alpha)|=|\sum_{n\geq 1}(\Lambda(n)-1)e^{-n/X}e(n\alpha)|$$
$$\ll\sum_{n\geq 1}|(\Lambda(n)-1)e^{-n/X}|$$
$$\ll\sum_{n\geq 1}(\Lambda(n)+1)e^{-n/X}$$
$$\ll X,$$

by the PNT and by partial summation.

The above would give the estimate $I_k\ll X^{k-1}\log^3 X$. □

Remark 3.1.13. *Lemma 3.1.12 is better than Lemma 3.1.9 only for*

$$\frac{1}{\log^2 X}\leq|\alpha|\leq\frac{1}{2}.$$

In the remaining alpha-range Lemma 3.1.9 is much sharper.

3.2 Omega result for $S_k(X)$

Analogously to the case $k=2$, we expect an omega-result of the average order of $G_k(n)$ as was studied by Bhowmik, Schlage-Puchta [8], who proved the error term is $\Omega(X^{k-1})$, while [6] shows a similar result for the error term to be $\Omega_{\pm}(X^{k-1})$. In this section, we prove the following result.

Theorem 3.2.1. *Let $k \geq 2$, we have*

$$S_k(X) = \frac{X^k}{k!} + H_k(X) + \Omega(X^{k-1} \log\log X),$$

where the implicit constant in the Ω-symbol depends on k.

To do that, we use the idea in [7] for $k = 2$. We show that for n sufficiently large, $G_k(n) = \Omega(n^{k-1} \log\log n)$. Then Theorem 3.2.1 will be proved because if Theorem 3.2.1 is false, that means

$$S_k(n) = \frac{n^k}{k!} + H_k(n) + o(n^{k-1} \log\log n).$$

This implies

$$G_k(n) = S_k(n) - S_k(n-1) = o(n^{k-1} \log\log n).$$

Let $q = \prod_{\substack{p<x \\ p\nmid q_1}} p$ be the product of primes which are less than x and not divisible by q_1, where q_1 is the exceptional modulus up to q if there exists a Siegel's zero. Since (2.34), for $(a, q) = 1$ we have

$$\psi(2x, q, a) = \sum_{\substack{n\leq 2x \\ n\equiv a(q)}} \Lambda(n) \geq \frac{x}{2\varphi(q)}.$$

Let b be an integer coprime to q, then

$$\sum_{\substack{n\leq 4x \\ n\equiv b(q)}} G_2(n) \geq \sum_{(a,q)=1} \psi(2x, q, a)\psi(2x, q, b-a) \geq \frac{x^2}{4\varphi(q)},$$

$$\sum_{\substack{n\leq 6x \\ n\equiv b(q)}} G_3(n) \geq \sum_{(a,q)=1} \psi(2x, q, a) \sum_{\substack{n\leq 4x \\ n\equiv b-a(q)}} G_2(n) \geq \frac{x^3}{8\varphi(q)}, \ldots,$$

$$\sum_{\substack{n\leq 2(k-1)x \\ n\equiv b(q)}} G_{k-1}(n) \geq \sum_{(a,q)=1} \psi(2x, q, a) \sum_{\substack{n\leq 2(k-2)x \\ n\equiv b-a(q)}} G_{k-2}(n) \geq \frac{x^{k-1}}{2^{k-1}\varphi(q)}.$$

Thus,

$$\sum_{\substack{n\leq 2kx \\ q|n}} G_k(n) \geq \sum_{(a,q)=1} \psi(2x, q, a) \sum_{\substack{n\leq 2(k-1)x \\ n\equiv q-a(q)}} G_{k-1}(n) \geq \frac{x^k}{2^k\varphi(q)}.$$

So that

$$\frac{2kx}{q} \max_{n\leq 2kx} G_k(n) \geq \frac{x^k}{2^k\varphi(q)}.$$

Therefore, we obtain

$$\begin{aligned}\max_{n\leq 2kx} G_k(n) &\geq \frac{x^{k-1}}{2^{k+1}} \frac{q}{\varphi(q)} = \frac{x^{k-1}}{2^{k+1}} \prod_{p<x} \left(1 - p^{-1}\right)^{-1} \prod_{p_1|q_1} \left(1 - p_1^{-1}\right) \\ &\gg_k x^{k-1} \log\log x.\end{aligned}$$

□

3.3 The equivalence

For $k = 2$, we know that there is a strong connection between the Riemann Hypothesis and the average order, i.e., the RH is equivalent to

$$S_2(X) = \frac{X^2}{2} + \mathcal{O}(X^{3/2+\epsilon}),$$

for any $\epsilon > 0$, as proved in [5] and [4]. The method of [4] was generalized in [9] to obtain a zero-free region for the Riemann zeta-function.

In this section, we prove that for $k \geq 2$, a good estimate for $S_k(X)$ is equivalent to the Riemann Hypothesis.

Theorem 3.3.1. *Let $k \geq 2$ be a fixed integer and $X \geq k$, the RH is equivalent to*

$$S_k(X) = \frac{X^k}{k!} + \mathcal{O}_k(X^{k-1/2+\epsilon}),$$

for any $\epsilon > 0$.

The following theorem is an important result for proving Theorem 3.3.1.

Theorem 3.3.2. *Let $k \geq 2$. We assume that there exists $0 < \delta < 1$ such that*

$$S_k(X) = \frac{X^k}{k!} + \mathcal{O}_k(X^{k-\delta}).$$

Then for any non-trivial zero ρ of Riemann zeta function, we have $\Re(\rho) < 1 - \frac{\delta}{2(k+1)}$.

3.3.3 Proof of Theorem 3.3.2

In this part, we consider the power series for $|z| < 1$,

$$F(z) = \sum_{n \geq 1} \Lambda(n) z^n. \tag{3.26}$$

We take the k^{th} power of $F(z)$ and obtain

$$F(z)^k = \sum_{n \geq 1} G_k(n) z^n = (1 - z) \sum_{n \geq 1} S_k(n) z^n.$$

By the assumption in Theorem 3.3.2, we have

$$\begin{aligned} \sum_{n \geq 1} S_k(n) z^n &= \sum_{n \geq 1} \left(\frac{n^k}{k!} + \mathcal{O}_k(n^{k-\delta}) \right) z^n \\ &= \frac{1}{k!} \sum_{n \geq 1} n^k z^n + \mathcal{O}_k \left(\sum_{n \geq 1} n^{k-\delta} |z|^n \right). \end{aligned} \tag{3.27}$$

We then evaluate the main term in (3.27) as follows.

Lemma 3.3.4. *For any integer $k \geq 2$, we have*

$$\sum_{n\geq 1} n^k z^n = \frac{k!}{(1-z)^{k+1}} + \mathcal{O}(|1-z|^{-k}).$$

Proof. We note that there exist unique real numbers $a_0, a_1, \dots, a_k$ such that

$$\sum_{j=0}^{k} \binom{n+j}{j} a_j = n^k \tag{3.28}$$

holds for all positive integers n. From (3.28), we can rewrite

$$\sum_{n\geq 1} n^k z^n = \frac{a_k}{(1-z)^{k+1}} + \frac{a_{k-1}}{(1-z)^k} + \cdots + \frac{a_0}{1-z}.$$

We calculate the value of a_k as follow:

$$a_k = k^k - \binom{k}{1}(k-1)^k + \binom{k}{2}(k-2)^k - \cdots + (-1)^{k-1}\binom{k}{k-1}. \tag{3.29}$$

We just need to prove

$$a_k = k!.$$

Let $k \geq 2$, we define the function

$$f_{k,i} = k^i - \binom{k}{1}(k-1)^i + \binom{k}{2}(k-2)^i - \cdots + (-1)^{k-1}\binom{k}{k-1}. \tag{3.30}$$

We note that

$$f_{k,i} = k(f_{k,i-1} + f_{k-1,i-1}) \tag{3.31}$$

and

$$f_{k,i} = 0, \text{ for all } 1 \leq i \leq k-1. \tag{3.32}$$

In fact, we prove (3.32) by induction on i. When $i = 1$,

$$\begin{aligned} f_{k,1} &= k - \binom{k}{1}(k-1) + \binom{k}{2}(k-2) - \cdots + (-1)^{k-1}\binom{k}{k-1} \\ &= \binom{k}{1} - 2\binom{k}{2} + 3\binom{k}{3} - \cdots + (-1)^{k-1}k\binom{k}{k}, \end{aligned}$$

since

$$(k-n)\binom{k}{n} = (n+1)\binom{k}{n+1}, \text{ for all } 0 \leq n \leq k.$$

On the other hand,

$$(1-x)^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^j,$$

and differentiating we obtain

$$k(1-x)^{k-1} = \sum_{j=0}^{k} (-1)^j j \binom{k}{j} x^{j-1}. \tag{3.33}$$

Replacing x in (3.33) by 1, we obtain $f_{k,1} = 0$.

Next, assuming that (3.32) is true for $1 \le i \le k-2$, we now prove that it is true for $i+1$. By (3.31), we have

$$f_{k,i+1} = k(f_{k,i} + f_{k-1,i}) = 0.$$

From (3.31) and (3.32), we obtain

$$f_{k,k} = k f_{k-1,k-1} = \cdots = k! f_{1,1} = k!.$$

Thus

$$a_k = f_{k,k} = k!.$$

□

Using Lemma 3.3.4, we rewrite (3.27) as follows

$$\sum_{n\ge 1} S_k(n) z^n = \frac{1}{(1-z)^{k+1}} + \mathcal{O}_k\left((1-|z|)^{\delta-k-1}\right).$$

Hence, we obtain

$$\begin{aligned} F(z)^k &= (1-z)\sum_{n\ge 1} S_k(n) z^n \\ &= \frac{1}{(1-z)^k} + \mathcal{O}_k\left(|1-z|(1-|z|)^{\delta-k-1}\right) \\ &= \frac{1}{(1-z)^k} + \mathcal{O}_k\left(|1-z| N^{k+1-\delta}\right) \end{aligned} \tag{3.34}$$

on the circle $|z| = R = 1 - \frac{1}{N}$, for a large positive integer N. We remark that the error term is less than the absolute value of the main term if $|1-z| < N^{\frac{\delta}{k+1}-1}$. We call this is a major arc on $|z| = R$, denoted by $\mathfrak{M}$. The rest of the circle is called a minor arc, denoted by $\mathcal{M}$.

We consider the major arc. Taking the complex k^{th} root of (3.34), then $F(z)$ can be written as

$$F(z) = \frac{\omega}{1-z} + \mathcal{O}_k\left(|1-z|^k N^{k+1-\delta}\right), \tag{3.35}$$

where ω is a k^{th} root of unity, i.e., $\omega = \cos\frac{2\pi l}{k} + i \sin\frac{2\pi l}{k}$, for $0 \le l \le k-1$. We prove now that $\omega = 1$. In fact, we suppose that ω is not a real number, i.e.,

$$l \in \mathcal{Z} = \begin{cases} \{1, 2, \ldots, k-1\}, & \text{if } k \text{ is odd} \\ \{1, 2, \ldots, k-1\} \backslash \{\frac{k}{2}\}, & \text{if } k \text{ is even.} \end{cases}$$

On the circle $|z| = 1 - \frac{1}{N}$, we choose a real number

$$z_0 = 1 - \frac{1}{N}.$$

From the definition of $F(z)$, we imply that $F(z_0)$ is also a real number. Furthermore, $F(z_0)$ can be written as

$$\begin{aligned} F(z_0) &= \frac{\omega}{1-z_0} + \mathcal{O}_k\left(|1-z_0|^k N^{k+1-\delta}\right) \\ &= N\omega + \mathcal{O}_k\left(N^{1-\delta}\right). \end{aligned}$$

We note that a necessary condition for $F(z_0)$ to be a real number is that the distance from $N\omega$ to the real line is at most $\mathcal{O}_k\left(N^{1-\delta}\right)$, i.e.,

$$\begin{aligned} \mathcal{O}_k\left(N^{1-\delta}\right) &\geq N\left|\sin\frac{2\pi l}{k}\right|, \qquad l \in \mathcal{Z} \\ &\gg_k N. \end{aligned}$$

This is impossible with $\delta \in (0,1)$. Hence ω is also a real number. We consider two cases for the positive integer k.
Case 1: if k is odd, this implies $\omega = 1$.
Case 2: if k is even, $\omega = \pm 1$. Since $F(z)$ is continuous and its coefficients are non-negative, the sign of ω is $+$, i.e., $\omega = 1$. We conclude

$$F(z) = \frac{1}{1-z} + \mathcal{O}_k\left(|1-z|^k N^{k+1-\delta}\right) \text{ on } \mathfrak{M}. \tag{3.36}$$

Now we introduce the kernel function,

$$K(z) = z^{-N-1}\frac{1-z^N}{1-z},$$

then $K(z) \ll |1-z|^{-1}$. By Cauchy's integral formula, the Chebyshev function can be written as

$$\begin{aligned} \psi(N) &= \frac{1}{2\pi i}\int_{|z|=R} F(z)K(z)dz \\ &= N + \frac{1}{2\pi i}\int_{|z|=R}\left(F(z) - \frac{1}{1-z}\right)K(z)dz. \end{aligned} \tag{3.37}$$

We split the circle $|z| = R$ into the major arc $\mathfrak{M}$ and the minor arc $\mathcal{M}$.

On $\mathfrak{M}$, from (3.36), we obtain

$$\begin{aligned} \int_{\mathfrak{M}}\left(F(z) - \frac{1}{1-z}\right)K(z)dz &\ll_k \int_{\mathfrak{M}}\left(|1-z|^k N^{k+1-\delta}\right)K(z)dz \\ &\ll_k \int_{\mathfrak{M}}\left(|1-z|^{k-1} N^{k+1-\delta}\right)dz \\ &\ll_k \left(N^{\frac{\delta}{k+1}-1}\right)^{k-1} N^{k+1-\delta} N^{\frac{\delta}{k+1}-1} \\ &= N^{1-\frac{\delta}{k+1}}. \end{aligned}$$

On $\mathcal{M}$, by Cauchy-Schwarz inequality, we have

$$\int_{\mathcal{M}}\left(F(z) - \frac{1}{1-z}\right)K(z)dz \ll \left(\int_{\mathcal{M}}|K(z)|^2dz\right)^{1/2}\left(\int_{\mathcal{M}}\left|F(z) - \frac{1}{1-z}\right|^2dz\right)^{1/2}.$$

Moreover, similar to [4] for $k = 2$, we have the estimations

$$\int_{\mathcal{M}} \left| F(z) - \frac{1}{1-z} \right|^2 dz \leq \int_{|z|=R} \left| F(z) - \frac{1}{1-z} \right|^2 dz \ll N \log N$$
$$\int_{\mathcal{M}} |K(z)|^2 dz \ll \int_{\mathcal{M}} \frac{1}{|1-z|^2} dz \ll N^{1-\frac{\delta}{k+1}}.$$

Hence, we have an estimation on the minor arc

$$\int_{\mathcal{M}} \left(F(z) - \frac{1}{1-z} \right) K(z) dz \ll N^{1-\frac{\delta}{2(k+1)}} (\log N)^{1/2}. \tag{3.38}$$

Combining the major arc, minor arc and from (3.37), we obtain

$$\psi(N) - N \ll_k N^{1-\frac{\delta}{2(k+1)}} (\log N)^{1/2}.$$

Therefore, for any non-trivial zero ρ of Riemann zeta function, we have

$$\Re(\rho) < 1 - \frac{\delta}{2(k+1)} < 1.$$

Hence Theorem 3.3.2 is proved. □

We will apply this theorem for proving Theorem 3.3.1, it will be stated in the following section.

3.3.5 Proof of Theorem 3.3.1

Assuming the Riemann Hypothesis, we can easily deduce the asymptotic formula of $S_k(X)$ in Theorem 3.3.1. Now we prove the reverse in the following steps.

Step 1: Granville showed a formula of $S_k(X)$ without using the RH ([20, 21, (1.3)]), that is

$$S_k(X) = \frac{X^k}{k!} + \sum_{\rho} r_k(\rho) \frac{X^{\rho+k-1}}{\rho+k-1} + \mathcal{O}_k(X^{k-2+\frac{4B+2}{3}+o(1)}), \tag{3.39}$$

where B is defined in (1.12) and

$$r_k(\rho) := -\frac{k}{\rho \dots (\rho+k-2)}.$$

Remark 3.3.6. *We know that* $1/2 \leq B \leq 1$.

Step 2: We define the corresponding Dirichlet series of $G_k(n)$

$$f_k(s) = \sum_{n \geq 1} \frac{G_k(n)}{n^s},$$

which converges absolutely and is analytic for $\Re(s) > k$. From (3.39) in step 1, we have

$$\begin{aligned} f_k(s) &= s\int_1^\infty S_k(x)x^{-s-1}dx \\ &= s\int_1^\infty \left(\frac{x^k}{k!} + \sum_\rho r_k(\rho)\frac{x^{\rho+k-1}}{\rho+k-1} + \mathcal{O}_k(x^{k-2+\frac{4B+2}{3}+o(1)})\right)x^{-s-1}dx \\ &= \frac{1}{(s-k)(k-1)!} + \sum_\rho \frac{r_k(\rho)}{s-\rho-k+1} + s\int_1^\infty \mathcal{O}_k(x^{k-2+\frac{4B+2}{3}+o(1)})x^{-s-1}dx \\ &+ \frac{1}{k!} + \sum_\rho \frac{r_k(\rho)}{\rho+k-1}. \end{aligned}$$

From the above, the series $f_k(s)$ can be continued meromorphically to the haft plane $\{\Re(s) > k-2+\frac{4B+2}{3}\}$.

Step 3: We assume that $B < 1$. Then we obtain

$$k-1+B = \inf\{\sigma_0 \geq k-\frac{1}{2} : f_k(s) - \frac{1}{(s-k)(k-1)!} \text{ is analytic on } \Re(s) > \sigma_0\}. \tag{3.40}$$

We then prove (3.40). From step 2, the right-hand side of (3.40) is at most $k-2+\frac{4B+2}{3} \leq k-1+B$, since $B \leq 1$.

For the reverse inequality, the right-hand side of (3.40) is at least $k-\frac{1}{2}$, then (3.40) is true if $B = \frac{1}{2}$. So we can assume $\frac{1}{2} < B < 1$. Hence

$$\max\{k-2+\frac{4B+2}{3}, k-\frac{1}{2}\} < k-1+B.$$

There exists $\epsilon > 0$ such that

$$\max\{k-2+\frac{4B+2}{3}, k-\frac{1}{2}\} < k-1+B-\epsilon.$$

By the definition of B, there exists a non-trivial zero ρ such that

$$B-\epsilon < \Re(\rho).$$

Thus we obtain

$$k-\frac{1}{2} < k-1+B-\epsilon < \Re(\rho+k-1).$$

We consider in the haft plane $\{\Re(s) > k-1+B-\epsilon\}$, $f_k(s)$ has a pole at $\rho+k-1$. So the right-hand side of (3.40) $\geq k-1+B-\epsilon$. Let ϵ tend to 0, the proof of (3.40) is completed.

Step 4: Let

$$E_k(X) = S_k(X) - \frac{X^k}{k!}.$$

Then by the assumption of this part, $E_k(X) \ll_k X^{k-\frac{1}{2}+\epsilon}$.

Moreover,

$$s\int_1^\infty E_k(x)x^{-s-1}dx = f_k(s) + \frac{s}{(s-k)k!}.$$

Then

$$f_k(s) - \frac{1}{(s-k)(k-1)!} = s\int_1^\infty E_k(x)x^{-s-1}dx + \frac{1}{k!}. \tag{3.41}$$

We note that the right-hand side of (3.41) is analytic on $\{\Re(s) > k - \dfrac{1}{2}\}$. Then from Step 3, we obtain

$$k - 1 + B \leq k - \frac{1}{2},$$

i.e.,

$$B \leq \frac{1}{2}.$$

We conclude $B = \dfrac{1}{2}$, that means the Riemann Hypothesis is true.

Step 5: Notice that in Step 3, we assume that $B < 1$. So we need to exclude the case $B = 1$. By assumption,

$$S_k(X) = \frac{X^k}{k!} + \mathcal{O}_k(X^{k-\delta}),$$

with $\delta = \frac{1}{2} - \epsilon$. Applying Theorem 3.3.2, we obtain $\Re(\rho) < 1 - \dfrac{\delta}{2(k+1)} < 1$. This implies $B < 1$. □

Appendix: Further directions of work

In this section, we indicate some further directions of related work .

Firstly, we can study the oscillating term in the Goldbach conjecture. We consider

$$H_k(x) = -k \sum_{\rho} \frac{x^{\rho+k-1}}{\rho(\rho+1)\dots(\rho+k-1)}.$$

For $k = 2$, it was studied by Fujii [15] who proved that under the linear independence conjecture on the first 70 zeros on the critical line of the Riemann zeta function the oscillating term

$$R_2(x) = \Re \sum_{\gamma>0} \frac{x^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}$$

with the sum over ordinates of the non-trivial zeros of $\zeta(s)$, the inequalities

$$R_2(x) > 0.012, \quad R_2(x) < -0.012$$

hold for an unbounded sequence of positive real numbers x. More recently Mossinghoff and Trudgian [36] improved the above bounds to

$$R_2(x) < -0.022978, \quad R_2(x) > 0.021030$$

under the RH and that these oscillations are almost optimal. In the case of several factors, consider

$$R_k(x) = \Re \sum_{\gamma>0} \frac{x^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)\dots(k-1/2+i\gamma)},$$

where the sum is over ordinates of the non-trivial zeros of the Riemann zeta function in the upper half plane. We would like to determine bounds on the oscillations of $R_k(x)$.

- Upper bounds on $|R_k(x)|$ that are valid for all $x > 0$.
- Lower bounds on the oscillations.

We have first that

$$|R_k(x)| \leq \sum_{\gamma>0} \left(\prod_{i=1}^{k} (\gamma^2 + (i - \frac{1}{2}) \right)^{-1/2} := \sum_{\gamma>0} M_k(\gamma).$$

For $k = 2$, we have a simple expansion de $M_2(\gamma)$

$$M_2(\gamma) = \frac{1}{\gamma^2 + \frac{1}{4}} - \frac{1}{\gamma^4} + \frac{2}{\gamma^6} - \frac{61}{16\gamma^8} + \mathcal{O}(\gamma^{-10}).$$

Then we have

$$\begin{aligned} |R_2(x)| &\leq \sum_{\gamma>0} \frac{1}{\gamma^2 + \frac{1}{4}} = \sum_{\rho} \Re(\rho^{-1}) \\ &= 1 + \frac{\gamma^*}{2} - \frac{\log 4\pi}{2} \approx 0.02309, \end{aligned}$$

where γ^* is the Euler–Mascheroni constant.
For small k, Mossinghoff calculated

$$\begin{aligned} M_3(\gamma) &= \frac{1}{\gamma^3} - \frac{35}{8\gamma^5} + \frac{2639}{128\gamma^7} - \frac{107395}{1024\gamma^9} + \mathcal{O}(\gamma^{-11}) \\ M_4(\gamma) &= \frac{1}{\gamma^4} - \frac{21}{2\gamma^6} + \frac{1659}{16\gamma^8} - \frac{16829}{16\gamma^10} + \mathcal{O}(\gamma^{-12}) \\ M_5(\gamma) &= \frac{1}{\gamma^5} - \frac{165}{8\gamma^7} + \frac{46563}{128\gamma^9} - \frac{6462665}{1024\gamma^11} + \mathcal{O}(\gamma^{-13}). \end{aligned}$$

We can use the first known zeros of $\zeta(s)$ to obtain crude bounds. We would like to get better bounds for small k and in general for $k \geq 3$.

A similar question is to quantify the asymptotic results obtained in this thesis. This involves obtaining explicit numerical bounds on the error terms. For the classical case of the Prime Number Theorem, $\psi(x) \sim x$ was recently quantified, in the tradition of Rosser and Schoenfeld (1975), by Trudgian and Platt, as

$$|\psi(x) - x| \leq A \log^B x \exp(-C\sqrt{\log x})$$

for positive constants A, B, C and for $x \geq \exp(1000)$. For the case of arithmetic progressions, investigations of Bennett, Martin, O'Bryant and Rechnitzer (2018) have given sharp numerical bounds for

$$|\psi(x, q, a) - \frac{x}{\varphi(q)}| \leq C\frac{x}{\log x}$$

for $x \geq B$ for explicit constants B, C. In this context Bhowmik, Ernvall-Hytönen and Palojärvi have obtained numerical bounds for the Goldbach functions $S(x)$ and $S(x, q, a, b)$. On the same lines we would like to study the numerical versions of $S_k(x)$ and $S(x, q_1, q_2, a_1, a_2)$ to find cut-off points where the asymptotic results obtained can be replaced by inequalities.

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