Modulation instability of the nonlinear Schrödinger equation

Numerical simulations

Master 1 Physique fondamentale
SCOL - 2020/2021

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Objectives

- Run numerical simulations of the propagation of wave packets according to the nonlinear Schrödinger equation;
- Explore different scenarios by varying the initial condition and the parameters.

Abstract

You will study numerically the nonlinear Schrödinger equation and especially the phenomenon of modulation instability (MI). A script written in Python is provided that enables fast simulations and interactive display of the results.

1 / The equation

You are provided with a Python code that integrates the following equation using the split-step method [1]:

$$i\frac{\partial \Psi}{\partial Z} - \frac{\beta_2}{2}\frac{\partial^2 \Psi}{\partial T^2} + \gamma|\Psi|^2\Psi = 0 \tag{1}$$

where $\Psi(Z,T)$ is the envelope of the electromagnetic field, Z is the propagation distance, T is the time defined in the reference frame travelling at the group velocity of the carrier wave. β_2 and γ are the group velocity dispersion (GVD) coefficient and the nonlinear coefficient at the carrier frequency respectively.

By now you should be familiar with this equation, at least in the following adimensionnal form:

$$i\frac{\partial \Psi}{\partial z} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial t^2} + |\Psi|^2\Psi = 0 \tag{2}$$

As you can see, by some change of variables adimensionnal distance and time z and t are introduced. The nonlinear coefficient becomes unity and most importantly, the GVD coefficient is assumed to be negative which corresponds to the so-called *focusing regime* [2].

In our simulations we will consider this adimensional equation to focus on the phenomenology but feel free to play with realistic parameters later!

[1] See `https://en.wikipedia.org/wiki/Split-step_method` for a bit of explanation.

[2] In fiber optics, this corresponds to the anomalous dispersion regime.

2 / Before propagation simulations: the MI gain

In this section we take a brief look at some analytical result obtained with Eq. (2). Studying the linear stability analysis of this equation with respect to small harmonic perturbation of its steady-state solution, one finds that the latter is *modulationally unstable* for frequency components of the the perturbation below a cut-off value $\Omega_c = 2$. The gain $g(\Omega)$ experienced by these frequency components is given by

$$g(\Omega) = |\Omega|\sqrt{\Omega_c^2 - \Omega^2} \tag{3}$$

We recal that Ω is the pulsation difference between the carrier wave and the harmonic perturbation. It is obviously related to a frequency detuning $f = \Omega/(2\pi)$.

- Using the Python script `MI_gain.py`. Plot the spectrum of the modulation instability gain as defined by Eq. (3);
- Verify that the following values are in accordance with the equation: (i) the cut-off pulsation, (ii) the maximum gain pulsation, (iii) the value of maximum gain;
- What is the gain for $\Omega = 0$? What does it mean?

3 / First simulation: Propagation of a gaussian pulse

The rest of this course is dedicated to numerical simulations of wave propagation according to Eq. (2). To do this, you have several Python codes ready to be used. The main code is `main_NLSE_MI.py` that allows you to define all the relevant parameters of the simulations including the initial wavefield, the propagation of which you want to simulate. Running this file will perform the propagation simulation. `plotlib.py` contains a list of commands that allows you to plot the results of the simulation in various forms. Running each cell will plot a different figure. You don't need to modify any of the scripts in the subscript folder, at least for now.

- In `main_NLSE_MI.py`, modify the initial field (U) such that it corresponds to a real gaussian of amplitude 1 and width 1. Perform the simulation for a propagation distance of 20. What do you observe?
- Perfom the same simulation for an amplitude of 2. What has changed?

4 / Modulation instability of a continuous wave

Once you master the previous simulations we move on to the actual topic of this course.

- Modify the initial field such that it now consists of a real gaussian pulse of amplitude 1 and width 5 (Simulation parameters might need to be changed). Observe the strong difference of spatio-temporal dynamics;
- Modify the initial field to a continuous wave perturbed by a small modulation whose frequency corresponds to the maximum MI gain. Observe and comment. Is there anything that seems odd?
- After correcting the previous issue, increase/decrease the amplitude of the perturbation and observe;
- Increase the modulation frequency so that it doesn't fall in the gain bandwidth anymore;
- Reduce the frequency of the perturbation below half of the cut-off. What happens?
- Start now from a purely continuous wavefield but set the `noise` variable to 1. This adds white noise to the initial field. Observe the so-called *spontaneous modulation instability* phenomenon that often arise in experiments!

5 / Dam break of a square wave packet

- Set the initial wavefield to a super-gaussian pulse of width 60. Observe what happens on the edges of this "box" during propagation.