STABILITY AND BIFURCATIONS IN PROJECTIVE HOLOMORPHIC DYNAMICS

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ABSTRACT. We survey the classical theory of stability for rational maps and give a new proof based on potential and ergodic theoretic tools. We then exploit this approach to present a generalization of this theory which is valid in any dimension. This text is partially based on the lectures given by the first author at IMPAN during the Simons Semester *Dynamical Systems* held in Banach Center.

In these lectures we will discuss stability and bifurcation phenomena for holomorphic families of rational maps and, more generally, of endomorphisms of $\mathbb{P}^k$. More precisely, we will describe the classical results independently obtained by Lyubich and Mañé, Sad and Sullivan about the stability of rational maps and then present a new approach leading to their generalization in any dimension.

The four first lectures all deal with the case of $\mathbb{P}^1$. In the first lecture we briefly introduce the framework and present the tools which will be used. The second lecture is devoted to the classical proof of Lyubich-Mañé-Sad-Sullivan theorem and emphasizes the aspects which fail in higher dimension. In the third lecture we present an alternative proof of the Lyubich-Mañé-Sad-Sullivan theorem which is based on pluripotential and ergodic theories. This approach is the one we will follow in the higher dimensional setting and we discuss which are the difficulties related to that generalization. We also introduce in that lecture the notion of bifurcation current by means of Lyapunov exponents. The fourth lecture is a digression on some possibilities offered by the bifurcation currents techniques.

The remaining lectures deal with a generalization of Lyubich-Mañé-Sad-Sullivan theorem to $\mathbb{P}^k$ which has been obtained by C. Dupont and the authors of these notes. This generalization is based on new methods and ideas which, at least partially, have already been explained in the special case of $\mathbb{P}^1$ in the third lecture. In the fifth lecture, we first introduce some tools and, in particular, some concepts to deal with holomorphic motions of Julia sets, we then describe the main result and present a general strategy for its proof. In the two next lectures, admitting a fundamental lemma concerning Misiurewicz parameters, we prove how to obtain stability (holomorphic motions) of Julia sets from stability of repelling cycles. The eighth lecture is devoted to the proof of the fundamental lemma. In the last two lectures we cover some related aspects of our result and present a list of open questions.

As we have tried to stress the ideas and concepts rather than the technical aspects, our presentation is mostly informal and our style is that of Notes. In the subsection 3.2

however, we gather some material for the first time and thus give precise and complete proofs.

For background on dynamics in several complex variables, we refer to the lecture notes [DS10] by T.C. Dinh and N. Sibony. For background on bifurcation currents, we refer to the survey [Duj14] by R. Dujardin or the lecture notes [Ber13] by the first author. In particular, an extensive bibliography may be found in these three texts.

1. Holomorphic families

1.1. Definitions and examples.

Definition 1.1. *A* holomorphic family *of endomorphisms of* $\mathbb{P}^k$ *is a holomorphic map* $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ *of the form* $f(\lambda, z) = (\lambda, f_\lambda(z))$ *such that:*

  (1) M *is a complex manifold;*
  (2) *the algebraic degree* $\deg_a f_\lambda$ *of* f_λ *is constant:* $\deg_a f_\lambda \equiv d \geq 2$.

We shall say that d *is the degree of the holomorphic family* f.

Let us recall a few basic facts about holomorphic endomorphisms of projective spaces.

  (1) Each f_λ can be lifted to $\mathbb{C}^{k+1}$ through the canonical projection $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$:

$$
\begin{array}{ccc}
\mathbb{C}^{k+1} \setminus \{0\} & \xrightarrow{\ F_\lambda\ } & \mathbb{C}^{k+1} \setminus \{0\} \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{P}^k & \xrightarrow{\ \ \ f_\lambda\ \ \ } & \mathbb{P}^k
\end{array}
$$

  Here F_λ is a non-degenerate d-homogeneous polynomial map, i.e., $F_\lambda^{-1}\{0\} = \{0\}$ and $F_\lambda(tz) = t^d F_\lambda(z), \forall t \in \mathbb{C}, \forall z \in \mathbb{C}^{k+1}$ where $d = \deg_a f_\lambda$.
  (2) The map $f_\lambda : \mathbb{P}^k \to \mathbb{P}^k$ is a finite ramified covering whose topological degree $\deg_{\text{top}} f_\lambda$ is equal to d^k.
  (3) For every $\lambda_0 \in M$, the family f can be lifted to a family

$$
\begin{aligned}
F: \Omega \times \mathbb{C}^{k+1} & \to & \Omega \times \mathbb{C}^{k+1} \\
(\lambda, z) & \mapsto & (\lambda, F_\lambda(z))
\end{aligned}
$$

  for every sufficiently small ball $\Omega \subset M$ centered at λ_0.

The following are **basic examples** of holomorphic families.

  (1) The space $\mathcal{H}_d(\mathbb{P}^k)$ of all holomorphic endomorphisms of algebraic degree d of $\mathbb{P}^k$. Actually $\mathcal{H}_d(\mathbb{P}^k) \approx \mathbb{P}^{N_{d,k}} \setminus Z$ where Z is an hypersurface in $\mathbb{P}^{N_{d,k}}$ and $N_{d,k} = \frac{(k+1)(d+k)!}{d!k!}$.
  (2) M is any complex submanifold of $\mathcal{H}_d(\mathbb{P}^k)$. This is particularly interesting when M is dynamically defined.
  (3) The polynomial quadratic family; $k = 1, M = \mathbb{C}$ and $f_\lambda(z) = z^2 + \lambda$.

(4) The degree d polynomial family; $k = 1, M = \mathbb{C}^{d-1}$ and f_λ is defined by the condition: f'_λ is unitary, the critical set $\mathsf{C}_{f_\lambda} = \{0, \lambda_1, \ldots, \lambda_{d-2}\}$ and $f_\lambda(0) = \lambda_{d-1}$.

1.2. **The equilibrium measure of f_λ.**

A reference for this subsection and the next one is given by the lecture notes [DS10]. Let ω denote the Fubini-Study form of $\mathbb{P}^k$. For each λ, one has $d^{-1}f_\lambda^*\omega = \omega + dd^c v_\lambda$, where v_λ is a smooth function on $\mathbb{P}^k$. One may for instance take $v_\lambda(z) = d^{-1}\log\frac{\|F_\lambda(\tilde{z})\|}{\|\tilde{z}\|}$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{C}^{k+1}$ and $\pi(\tilde{z}) = z$.

By induction one gets $d^{-n}\left(f_\lambda^n\right)^*\omega = \omega + dd^c g_n(\lambda, z)$, where the function g_n is given by $g_n(\lambda, z) := v_\lambda + \cdots + d^{-(n-1)}v_\lambda \circ f_\lambda^{n-1}$.

It is clear that g_n is locally uniformly converging to some function $g(\lambda, z)$. One thus gets $\lim_n d^{-n}\left(f_\lambda^n\right)^*\omega = \omega + dd^c_z g(\lambda, z)$. One then sets

$$T_\lambda := \omega + dd^c_z g(\lambda, z).$$

This is a positive closed $(1,1)$-current on $\mathbb{P}^k$, it is called the *Green current* of f_λ. The function $g(\lambda, z)$ is called the *Green function* of f_λ. By construction we have

$$f_\lambda^* T_\lambda = dT_\lambda.$$

Let us stress that $g_n \to g$ locally uniformly and that the g_n's are smooth. It turns out that g is Hölder continuous, a fact which was first proved by M. Kosek [Kos97].

The *equilibrium measure* μ_λ of f_λ is defined by

$$\mu_\lambda := (T_\lambda)^k = T_\lambda \wedge \cdots \wedge T_\lambda.$$

By construction μ_λ is a probability measure on $\mathbb{P}^k$ such that

$$f_\lambda^* \mu_\lambda = d^k \mu_\lambda \ \text{ and } \ (f_\lambda)_* \mu_\lambda = \mu_\lambda.$$

Two crucial facts about the measure μ_λ are that it is ergodic and does not give mass to proper analytic subsets of $\mathbb{P}^k$.

1.3. **Two fundamental equidistribution theorems.**

Theorem 1.2. Repelling periodic points equidistribute μ_λ: $d^{-kn}\sum_{z\in\mathcal{R}_n(\lambda)}\delta_z \to \mu_\lambda$, where $\mathcal{R}_n(\lambda) := \{n\text{-periodic repelling points in } J_\lambda\}$.

Theorem 1.3. Iterated preimages equidistribute μ_λ: *there exists a proper algebraic subset $\mathcal{E}$ of $\mathbb{P}^k$ which is contained in the postcritical set of f_λ and such that $d^{-kn}\sum_{f_\lambda^n(z)=a}\delta_z \to \mu_\lambda$ for all $a \in \mathbb{P}^k \setminus \mathcal{E}$.*

Note that the exceptional set $\mathcal{E}$ is well understood when $k = 1$. For $k \geq 2$, one knows that $\mathcal{E}$ is the largest proper algebraic subset of $\mathbb{P}^k$ which is totally invariant by f_λ. We refer to the papers of Briend-Duval [BD99, BD09] and Dinh-Sibony [DS03] for the proofs and to the papers of Brolin [Bro65], Freire-Lopes-Mañé [FLM83] and Lyubich [Ly83b] for the one-dimensional case.

1.4. **Goals.** In these lectures, we aim to

- study the stability of the ergodic dynamical systems $(J_\lambda, f_\lambda, \mu_\lambda)$;
- understand how J_λ depends on λ.

We will first review the classical results in dimension $k = 1$ and then discuss their generalization to higher dimension.

2. When $k = 1$: Lyubich-Mañé-Sad-Sullivan Theorem

2.1. **The statement.** Here we make the mild assumption that $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ is a holomorphic family (of degree d) such that

$$\mathsf{C}_f = \mathrm{Crit} f = \{(\lambda, c_j(\lambda)) \colon \lambda \in M, 1 \le j \le 2d - 2\},$$

where the maps $c_j : M \to \mathbb{P}^1$ are holomorphic.

The results obtained by Lyubich [Ly83a] and Mañé, Sad and Sullivan [MSS83] in the early 80's are essentially summarized by the next theorem and its corollary.

Theorem 2.1. (Lyubich, Mañé-Sad-Sullivan) *For every $\lambda_0 \in M$ the following assertions are equivalent:*

(1) *J_λ moves holomorphically on some neighbourhood of λ_0;*
(1') *J_λ moves continuously on some neighbourhood of λ_0;*
(2) *the repelling cycles of f_λ move holomorphically on some neighbourhood of λ_0;*
(2') *f_λ has no unpersistent neutral cycle near λ_0;*
(3) *$(f_\lambda^n(c_j(\lambda)))$ is normal at λ_0 for all $1 \le j \le 2d - 2$ (stability of the critical orbits).*

In view of that, one defines

$$\text{Stability locus of } f := \mathrm{Stab}(f) := \{\lambda_0 \colon (1) \dots (3) \text{ occur}\}$$
$$\text{Bifurcation locus of } f := \mathrm{Bif}(f) := M \setminus \mathrm{Stab}(f).$$

The above theorem has the following fundamental consequence.

Corollary 2.2. $\mathrm{Stab}(f)$ *is dense in* M.

Example: the Mandelbrot set. *Consider the quadratic polynomial family which is given by $p : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{C} \times \mathbb{P}^1$, where $p(\lambda, z) = (\lambda, z^2 + \lambda)$. The* Mandelbrot set M_2 *is defined as:*

$$M_2 := \{\lambda \in \mathbb{C} \colon (p_\lambda^n(0))_n \text{ is bounded in } \mathbb{C}\}.$$

It follows from the above theorem that

$$\lambda_0 \in \mathrm{Bif}(p) \Leftrightarrow (p_\lambda^n(0))_n \text{ is } \underline{not} \text{ normal at } \lambda_0 \Leftrightarrow \lambda_0 \in bM_2.$$

We now explain the concepts involved in the above Theorem.

2.2. Holomorphic motions.

Definition 2.3. *A holomorphic motion of $K \subset \mathbb{P}^1$ over a complex manifold M and centered at $\lambda_0 \in M$ is a map $h : M \times K \to \mathbb{P}^1$ such that*

   i) $h(\lambda_0, \cdot) = Id_K$;
   ii) $h(\lambda, \cdot)$ *is one-to-one, for every $\lambda \in M$;*
   iii) $h(\cdot, z)$ *is holomorphic on M, for every $z \in K$.*

Remark 2.4. *The λ-lemma says that such a motion can always be extended to $M \times \overline{K}$ and is continuous on $M \times \overline{K}$, moreover this extension is unique. We shall later implicitely give the proof of this simple and basic fact.*

It is useful to think about holomorphic motions in terms of laminations.
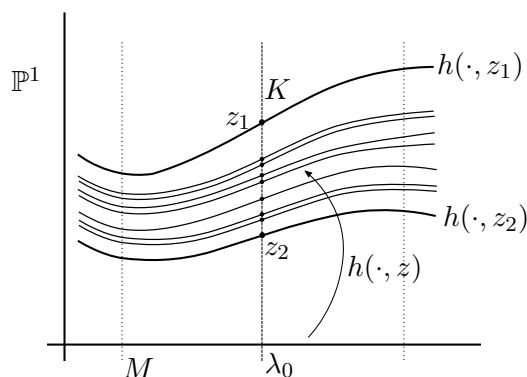
Definition 2.5. *Let $\mathcal{R}_n(\lambda) := \{n\text{-periodic repelling points of } f_\lambda\}$. We say that the repelling cycles of f_λ move holomorphically over $\Omega \subset M$ if for every $n \geq 1$ there exists a set of holomorphic maps $\rho_{j,n} : \Omega \to \mathbb{P}^1$ such that $\mathcal{R}_n(\lambda) = \{\rho_{j,n}(\lambda) \colon 1 \leq j \leq N_d(n)\}$, for every $\lambda \in \Omega$. Here $N_d(n) := Card(\mathcal{R}_n(\lambda))$.*

Remark 2.6. *This is equivalent to say that there exists a holomorphic motion of $\mathcal{R}_n(\lambda_0)$ over Ω and centered at λ_0 for every $n \geq 1$. The important fact here is that Ω is the same for <u>all</u> n. Indeed, if n is fixed and Ω is a sufficiently small neighbourhood of λ_0, such a motion of $\mathcal{R}_n(\lambda_0)$ always exists by the implicit function theorem.*

Definition 2.7. *One says that J_λ moves holomorphically over M if there exists a holomorphic motion h of J_λ which is centered at $\lambda_0 \in M$ and such that $h_\lambda := h(\lambda, \cdot)$ conjugates the dynamics:*

$$
\begin{array}{ccc}
J_{\lambda_0} & \xrightarrow{\;h_\lambda\;} & J_\lambda \\
{\scriptstyle f_{\lambda_0}}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\
J_{\lambda_0} & \xrightarrow{\;h_\lambda\;} & J_\lambda
\end{array}
$$

2.3. **Continuity in the Hausdorff topology.**

$$\mathrm{Comp}^*(\mathbb{P}^1) := \{\text{non-empty compact subsets of } \mathbb{P}^1\}.$$

For $K \in \mathrm{Comp}^*(\mathbb{P}^1)$ we set $K_\varepsilon := \varepsilon$-neighbourhood of K. For a map $E : M \to \mathrm{Comp}^*(\mathbb{P}^1)$ we have the following definitions (by $\approx$ we mean: sufficently close):

E is upper semicontinuous (usc) at $\lambda_0 \Leftrightarrow \forall \varepsilon > 0 : E(\lambda) \subset (E(\lambda_0))_\varepsilon$ for $\lambda \approx \lambda_0$
E is lower semicontinuous (lsc) at $\lambda_0 \Leftrightarrow \forall \varepsilon > 0 : E(\lambda_0) \subset (E(\lambda))_\varepsilon$ for $\lambda \approx \lambda_0$
E is continuous at $\lambda_0 \Leftrightarrow E$ is ucs and lsc at λ_0.

2.4. **Proof of Lyubich-Mañé-Sad-Sullivan Theorem.** The structure of the proof is as follows:

$$(1) \Longrightarrow (3) \Longrightarrow (2) \Longrightarrow (1)$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$(1') \qquad\qquad (2')$$

We focus on the proof of $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ and the proof of the Corollary 2.2, stressing the arguments which cannot be adapted to $\mathbb{P}^k$, with $k \geq 2$.

$(1) \Rightarrow (3)$ Assume that $(f_\lambda^n(c(\lambda)))_n$ is <u>not</u> normal at λ_0 but J_λ moves holomorphically near λ_0. Then one has the following picture:
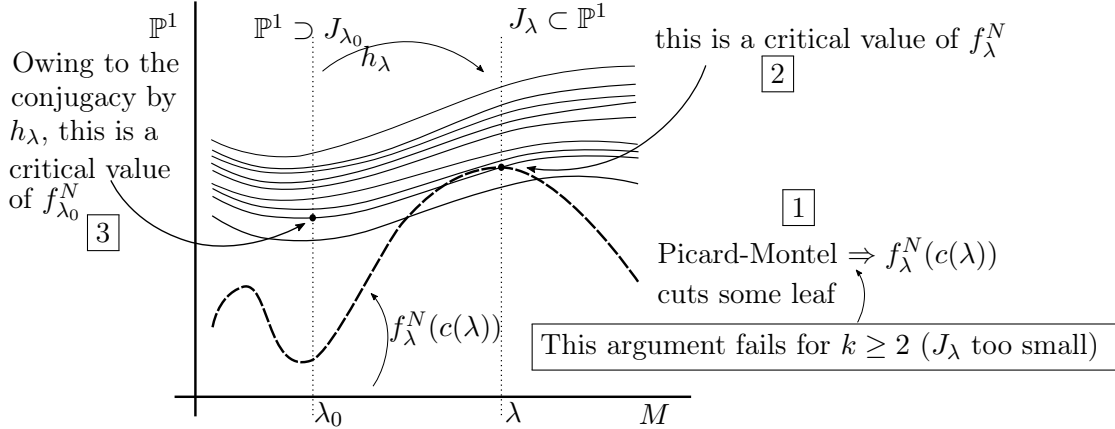

The picture shows that the set of critical values of $f_{\lambda_0}^N$ contains an open piece of J_{λ_0}: this is impossible.

$(2) \Rightarrow (1)$ By assumption we have the following picture for every n.
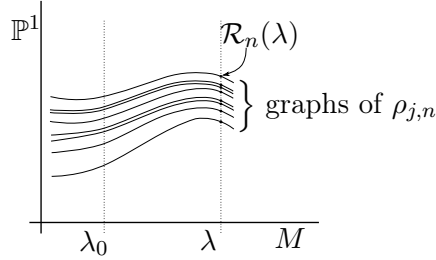
One easily sees (using lifts to $\mathbb{C}^2 \setminus \{0\}$) that the family $\{\rho_{j,n} \colon 1 \leq j \leq N_d(n), n \geq 1\}$ is normal. Since $\cup_n \mathcal{R}_n(\lambda)$ is dense in J_λ, the idea is to take limits to get the expected lamination. The only problem is that of uniqueness of limits. The picture below explains why two distinct limits cannot meet.

If the graphs of two limits γ_1, γ_2 meet, then for $\rho_{j,n}$ close to γ_1 and ρ_{j_n} close to γ_1...
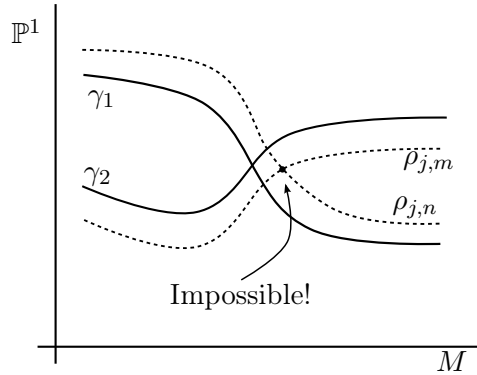
... the graphs of $\rho_{j,n}$ and $\rho_{j,m}$ must meet by Hurwitz lemma!

Fails when $k \geq 2$

$(3) \Rightarrow (2)$ The idea is as follows. Assume that a repelling cycle of f_{λ_0} does not move holomorphically over M. Then one finds a holomorphic map $\rho : U_0 \to \mathbb{P}^1$ such that $f_\lambda^{n_0}(\rho(\lambda)) = \rho(\lambda)$, $\lambda_0 \in U_0$ and $\rho(\lambda_0)$ is repelling for $f_{\lambda_0}^{n_0}$ but $\rho(\lambda_1)$ is attracting for $f_{\lambda_1}^{n_0}$. The orbit of a critical point $c_i(\lambda)$ is contained in the basin of attraction of $\rho(\lambda_1)$. In this situation, $(f_\lambda^n(c_i(\lambda)))_n$ cannot be normal on U_0 as the picture below shows.

Also fails for $k \geq 2$
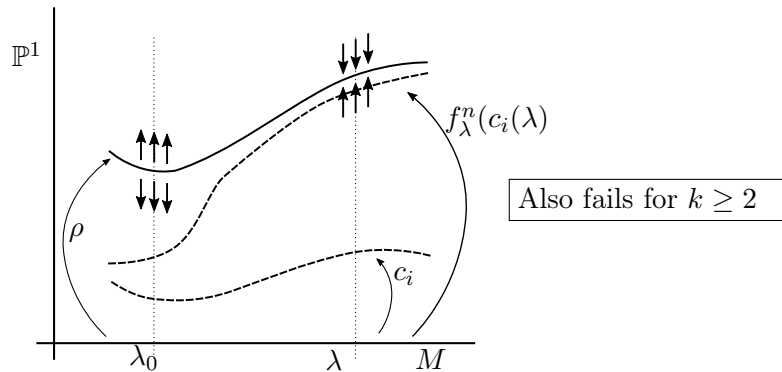
2.5. **Proof of Corollary 2.2.** Let $\lambda_0 \in \mathrm{Bif}(f)$. After a small perturbation $(\lambda_0 \to \lambda_1)$ one gets f_{λ_1}, which has an attracting cycle of period ≥ 3. Let $B(\lambda)$ be the basin of that cycle (exists for $\lambda \approx \lambda_1$). If for some $1 \leq i \leq 2d - 2$ the family $(f_\lambda^n(c_i(\lambda)))_n$ were <u>not</u> normal at λ_1, then, using Picard-Montel Theorem, one sees that after a small perturbation $(\lambda_1 \to \lambda_2)$ one has $\left(f_{\lambda_2}^{n_0}(c_i(\lambda_2)) \right) \in B(\lambda_2)$ and thus $(f_\lambda^n(c_i(\lambda)))_n$ is normal at λ_2.

One repeats this argument a finite $(\leq 2d - 2)$ number of times to get $\lambda_k \in \mathrm{Stab}(f)$ which is arbitrarily close to λ_0.

The fact that this argument cannot be extended to $\mathbb{P}^k$, $k \geq 2$, is essentially due to the non-finiteness of C_{f_λ}.

3. When $k = 1$: a potential theoretic approach to bifurcations

3.1. **The main idea explained for polynomials families.** We consider here a holomorphic polynomial family

$$
\begin{aligned}
p \colon M \times \mathbb{C} &\to & M \times \mathbb{C} \\
(\lambda, z) &\mapsto & (\lambda, p_\lambda(z))
\end{aligned}
$$

where p_λ is a unitary polynomial of degree $d \geq 2$. As in the previous lecture, we assume that the critical set $\mathsf{C}_p = \{(\lambda, c_j(\lambda)) \colon 1 \leq j \leq d - 1\}$, where $c_j \colon M \to \mathbb{C}$ are holomorphic.

The *Green function* G of p on $M \times \mathbb{C}$ is defined by

$$
G(\lambda, z) := \lim_n d^{-n} \log^+ |p_\lambda^n(z)| .
$$

As the convergence is locally uniform, the function G is *psh* and continuous. Moreover, one easily sees that

- $G(\lambda, z) = 0 \Leftrightarrow (p_\lambda^n(z))$ is bounded
- $J_\lambda = b\{G(\lambda, z) = 0\}$
- $\mu_\lambda = dd_z^c G(\lambda, z)$.

The Lyapunov exponent $L(\lambda)$ of p_λ with respect to its equilibrium measure μ_λ is defined by

$$
L(\lambda) := \int \log \left| p_\lambda'(z) \right| \mu_\lambda(z).
$$

Note that $\log |p_\lambda'| \in L^1(\mu_\lambda)$, since μ_λ has a continuous potential.

Proposition 3.1. $L(\lambda)$ *is the exponential rate of growth of* $(p_\lambda^n)'(z)$ *for* μ_λ*-almost every point* z.

Proof. Apply Birkhoff Theorem (μ_λ is ergodic). For μ_λ-almost every z,

$$
\int \log \left| p_\lambda'(z) \right| \mu_\lambda(z) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \log \left| p_\lambda'(p_\lambda^k(z)) \right| = \lim_n \frac{1}{n} \log \left| (p_\lambda^n)'(z) \right| .
$$

$\square$

The following formula [Prz85] relates the Lyapunov exponent with the Green function.

Theorem 3.2. ***Przytycki Formula.*** $L(\lambda) = \log d + \sum_{j=1}^{d-1} G(\lambda, c_j(\lambda))$.

Proof. Integrate by parts.

$$L(\lambda) = \int \log |p'_\lambda| \, \mu_\lambda = \int \log \left| d \prod_j (z - c_j(\lambda)) \right| \mu_\lambda$$

$$= \log d + \sum_j \int \log |z - c_j(\lambda)| \, dd^c_z G(\lambda, z) = \log d + \sum_j \left\langle \delta_{c_j(\lambda)}, G(\lambda, \cdot) \right\rangle.$$

$\square$

Corollary 3.3. *The function $L(\lambda)$ is psh, continuous and bigger than $\log d$.*

Applying dd^c_λ to Przytycki Formula one gets the following fundamental formula:

(1)
$$\boxed{dd^c_\lambda L(\lambda) = \sum_{j=1}^{d-1} dd^c_\lambda G(\lambda, c_j(\lambda))}$$

| deals with holomorphic | detects the instability |
| motions of repelling cycles | of the orbit of $c_j(\lambda)$ |

Formula (1) offers a possibility to avoid Picard-Montel arguments in the proof of Lyubich-Mañé-Sad-Sullivan Theorem and, more generally, provides another way to look at stability questions in dimension one. Although this will be discussed in details in the next subsection, the proposition below roughly indicates how this is possible for polynomial families.

Proposition 3.4. *The following hold:*
  (1) $G(\lambda, c_i(\lambda))$ *is pluriharmonic on a ball $B \subset M \Leftrightarrow (p^n_\lambda(c_i(\lambda)))_n$ is normal on B;*
  (2) *if the repelling cycles of p move holomorphically over $U \subset M$, then L is pluriharmonic on U.*

Proof.   (1) Set $u_n(\lambda) := p^n_\lambda(c_i(\lambda))$, and recall that $d^{-n} \log^+ |u_n(\lambda)| \to G(\lambda, c_i(\lambda))$ locally uniformly.

  • $(u_n)_n$ normal on $B \xRightarrow[B]{\text{shrink}} \begin{cases} \text{if } u_n \xrightarrow{\text{unif}} \infty \Rightarrow & G(\lambda, c_i(\lambda)) \text{ is a limit of} \\ & \text{pluriharmonic functions on B} \\ \text{if } u_{n_k} \xrightarrow{\text{unif}} h \text{ holomorphic} \Rightarrow G(\lambda, c_i(\lambda) = 0 \text{ on } B \end{cases}$

  • Assume $G(\lambda, c_i(\lambda))$ is pluriharmonic on B.
       <u>First case</u>: $G(\lambda, c_i(\lambda_0)) = 0$, $\lambda_0 \in B$. Then $G(\lambda, c_i(\lambda)) = 0$ on B (maximum principle) and thus $(u_n(\lambda))_n$ is uniformly bounded on B.
       <u>Second case</u>: $G(\lambda, c_i(\lambda)) > 0$ on B. Then $p^n_\lambda(c_i(\lambda)) \to \infty$ locally uniformly on B.
  (2) This is a direct consequence of the approximation formula which will be discussed in the next lecture.

$\square$

3.2. **A new approach to the Lyubich-Mañé-Sad-Sullivan theory.** We aim here to give a proof of Theorem 2.1 which exploits the ideas described in the former subsection. As it turns out, this approach can be partially adapted to the higher dimensional setting and we shall also explain which are the main technical difficulties related to this achievement.

We first recall our notations and introduce some new concepts. Let us consider a holomorphic family

$$f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$$

$$(\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

of degree $d \geq 2$ and whose critical set C_f is given by the graphs of $2d - 2$ holomorphic maps $c_j : M \to \mathbb{P}^1$ as follows:

$$\mathsf{C}_f = \operatorname{Crit} f = \{(\lambda, c_j(\lambda)) \colon \lambda \in M, 1 \leq j \leq 2d - 2\}.$$

Definition 3.5. *A parameter λ_0 is said* Misiurewicz *if there exist integers $p_0, n_0 \geq 1$ and a holomorphic map σ defined on some neighbourhood of λ_0 such that $\sigma(\lambda) \in \mathcal{R}_{n_0}(\lambda)$ and $\sigma(\lambda_0) = f^{p_0} \circ c_j(\lambda_0)$ but $\sigma \neq f^{p_0} \circ c_j$ for some $1 \leq j \leq 2d - 2$.*

We recall that $\mathcal{R}_n(\lambda) = \{n\text{-periodic repelling points of } f_\lambda\}$ and that the cardinal of $\mathcal{R}_n(\lambda)$ is equivalent to d^n for every λ.

We endow $\mathcal{O}(M, \mathbb{P}^1) := \{\gamma : M \to \mathbb{P}^1 : \gamma \text{ holomorphic}\}$ with the topology of local uniform convergence; this is a metric space. The space of interest here is the (possibly empty) subspace

$$\mathcal{J} := \{\gamma \in \mathcal{O}(M, \mathbb{P}^1) \colon \gamma(\lambda) \in J_\lambda, \forall \lambda \in M\}.$$

We have two natural maps. The first one is a self-map on $\mathcal{J}$

$$\mathcal{F} : \mathcal{J} \to \mathcal{J}$$

$$\gamma \mapsto \mathcal{F}(\gamma)$$

defined by $\mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$ and the second one is a projection

$$p_\lambda : \mathcal{J} \to \mathbb{P}^1$$

$$\gamma \mapsto \gamma(\lambda).$$

Definition 3.6. *An equilibrium web for f is a compactly supported probability measure $\mathcal{M}$ on $\mathcal{J}$ such that $\mathcal{F}_* \mathcal{M} = \mathcal{M}$ and $(p_\lambda)_* \mathcal{M} = \mu_\lambda$ for all $\lambda \in M$.*

The existence of an equilibrium web is a weak form of holomorphic motion of Julia sets, it rather means that the measures μ_λ move holomorphically over M. As we shall see, the two equidistribution theorems described in Subsection 1.3 combined with the Banach-Alaoglu compactness theorem will allow us to construct such equilibrium webs.

For each $\lambda \in M$, the Lyapunov exponent of the system $(J_\lambda, f_\lambda, \mu_\lambda)$ is

$$L(\lambda) := \int \log |f'_\lambda| \, \mu_\lambda.$$

One may easily check that it does not depend on the choice of the metric $|\cdot|$ on $\mathbb{P}^1$. We have to generalize Przytycki formula to rational maps. This has been done by DeMarco [dM03] and a different proof, more in the spirit of integration by part, has been given by Bassanelli and the first author [BB07] (see also [Ber13]). Applying dd^c_λ, this formula yields:

Theorem 3.7. dd^cL-**Formula.** $dd^c_\lambda L(\lambda) = (\pi_M)_* \left(\left(\omega + dd^c_{\lambda,z} g(\lambda, z) \right) \wedge [\mathsf{C}_f] \right).$

Recall that ω is the Fubini-Study form on $\mathbb{P}^1$, $g(\lambda, z)$ the Green function of f and $[\mathsf{C}_f]$ the current of integration on the critical set C_f of f in $M \times \mathbb{P}^1$.

We shall also use an approximation formula for L which basically says that the Lyapunov exponent of f_λ (with respect to μ_λ) is the asymptotic limit of the averages of Lyapunov exponents of repelling n-cycles. This approximation formula was first proved for endomorphisms of $\mathbb{P}^k$ by C. Dupont, L. Molino and the first author in [BDM08], a simpler proof adapted to the case of $\mathbb{P}^1$ is written in [Ber13], Y. Okuyama [Oku12] has given another proof.

Theorem 3.8. **Approximation Formula.** $L(\lambda) = \lim_n d^{-n} \sum_{z \in \mathcal{R}_n(\lambda)} \frac{1}{n} \log |(f^n_\lambda)'(z)|.$

We are now ready to state our version of Theorem 2.1. The parameter space M will be assumed to be simply connected. Note that the equivalence $(1) \Leftrightarrow (3)$ is a new result, it says that either all repelling cycles move holomorphically or they asymptotically all bifurcate (see [Ber17]).

Theorem 3.9. *For $\lambda_0 \in M$ and $n \in \mathbb{N}$ we denote by $N_d(n)$ the number of n-repelling periodic points of f_{λ_0} and by $\tau_0(n)N_d(n)$ the number of those which move holomorphically over M. Then the following assertions are equivalent.*

(1) $\limsup_n \tau_0(n) > 0$
(2) f *admits an equilibrium web* $\mathcal{M}$
(3) *all repelling cycles of f_λ move holomorphically over M (i.e. $\tau_0(n) = 1, \forall n$)*
(4) J_λ *move holomorphically over M*
(5) $dd^cL(\lambda) = 0$ *on M*
(6) $(f^n_\lambda(c_j(\lambda)))_n$ *is a normal family for every $1 \le j \le 2d - 2$*
(7) *there are no Misiurewicz parameters in M.*

The next diagram describes the proof of the above Theorem. Continuous arrows indicate that the corresponding parts of the proof easily adapt in higher dimension, while for the implications marked with a dashed arrow one would need major modifications of the argument. Parenthesis indicate that the corresponding property will need to be modified when dealing with dimension $k \ge 2$.
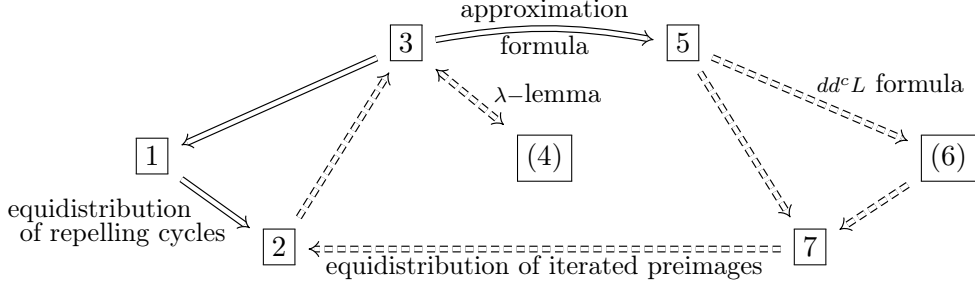
Proof. $(1) \Rightarrow (2)$. By assumption, for every $n \in \mathbb{N}^*$ there exists a collection of holomorphic maps $\rho_{j,n} : M \to \mathbb{P}^1$ where $1 \leq j \leq \tau_0(n)N_d(n) =: N'_d(n)$ and such that one has $\{\rho_{j,n}(\lambda) : 1 \leq j \leq N'_d(n)\} \subset \mathcal{R}_n(\lambda)$ for all $\lambda \in M$.

We may thus define a sequence $(\mathcal{M}_n)_n$ of discrete measures on $\mathcal{J}$ by setting

$$\mathcal{M}_n := d^{-n} \sum_{j=1}^{N'_d(n)} \delta_{\rho_{j,n}}.$$

Since $\limsup_n |\mathcal{M}_n| = \limsup_n \tau_0(n)N_d(n) =: \tau > 0$, Banach-Alaoglu Theorem gives a subsequence $(\mathcal{M}_{n_q})_q$ such that $\mathcal{M}_{n_q} \to \mathcal{M}$ and $|\mathcal{M}| = \tau$. Since by construction $\mathcal{F}_*(\mathcal{M}_{n_q}) = \mathcal{M}_{n_q}$ we get $\mathcal{F}_*\mathcal{M} = \mathcal{M}$.

Let us now set $\sigma_\lambda := p_{\lambda*}\mathcal{M}$ for every $\lambda \in M$. One has $|\sigma_\lambda| = |\mathcal{M}| = \tau$ and it follows from $p_\lambda \circ \mathcal{F} = f_\lambda \circ p_\lambda$ that $f_{\lambda*}\sigma_\lambda = \sigma_\lambda$. Moreover

$$\sigma_\lambda = p_{\lambda*}\mathcal{M} = \lim_q p_{\lambda*}\mathcal{M}_{n_q} = \lim_q d^{-n_q} \sum_{j=1}^{N'_d(n_q)} \rho_{j,n_q}(\lambda) \leq \lim_q d^{-n_q} \sum_{z \in \mathcal{R}_{n_q}(\lambda)} \delta_z = \mu_\lambda$$

where the last equality follows from the Equidistribution Theorem 1.2. Writing now $\mu_\lambda = \tau\left(\frac{\sigma_\lambda}{\tau}\right) + (1-\tau)\left(\frac{\mu_\lambda - \sigma_\lambda}{1-\tau}\right)$ one deduces from the ergodicity of μ_λ that $\sigma_\lambda = \tau\mu_\lambda$ for every $\lambda \in \Omega_0$. Then $\frac{1}{\tau}\mathcal{M}$ is an equilibrium web for f.

$(2) \Rightarrow (3)$. Let us first observe that for every $(\lambda_0, z_0) \in M \times J_{\lambda_0}$ there exists $\gamma \in \operatorname{supp}\mathcal{M}$ such that $z_0 = \gamma(\lambda_0)$. Indeed, since $J_{\lambda_0} = \operatorname{supp}\mu_{\lambda_0} = \operatorname{supp}(p_{\lambda_0*}\mathcal{M})$ there exist a sequence $(\gamma_k)_k$ in $\operatorname{supp}\mathcal{M}$ such that $\gamma_k(\lambda_0) \to z_0$. Then, as $\mathcal{M}$ is compactly supported, we can take for γ any limit of $(\gamma_k)_k$.

Now assume that z_0 is a repelling n-periodic point for f_{λ_0}. Then, by the implicit function theorem, there exists a neighbourhood V_{λ_0} of λ_0 and a holomorphic map $w : V_{\lambda_0} \to \mathbb{P}^1$ such that $w(\lambda_0) = z_0$ and $w(\lambda)$ is n-periodic for f_λ. Let us show that w coincides on V_{λ_0} with the above map γ. All we shall actually need is that $\gamma(\lambda_0) = w(\lambda_0)$ and that the family $\left(f_\lambda^{pn}(\gamma(\lambda))\right)_p = (\mathcal{F}^{pn}(\gamma))_p$ is normal. Our argument is local and so we can work on $\mathbb{C}$. Since z_0 is repelling, we can shrink V_{λ_0} and find $A > 1$, $r > 0$ such that

$$(2) \qquad |w(\lambda) - f_\lambda^n(z)| = |f_\lambda^n(w(\lambda)) - f_\lambda^n(z)| \geq A|w(\lambda) - z|$$

when $\lambda \in V_{\lambda_0}$ and $|w(\lambda) - z| < r$. On the other hand, as $f_\lambda^p(\gamma(\lambda)) = (\mathcal{F}^p \cdot \gamma)(\lambda)$ and $\mathcal{M}$ is compactly supported and $\mathcal{F}$-invariant, one sees that $\left(f_\lambda^{pn}(\gamma(\lambda))\right)_p$ is a normal family. We can therefore shrink again V_{λ_0} so that $|w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))| < r$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. Combining this with (2) we obtain $r > |w(\lambda) - f_\lambda^{pn}(\gamma(\lambda))| \geq A^p|w(\lambda) - \gamma(\lambda)|$ for every $p \geq 1$ and $\lambda \in V_{\lambda_0}$. This implies $w(\lambda) = \gamma(\lambda)$ on V_{λ_0} since $A > 1$.

By analytic continuation we have $f_\lambda^n(\gamma(\lambda)) = \gamma(\lambda)$ for every $\lambda \in M$ and it thus remains to see that all n-periodic points $\gamma(\lambda)$ are repelling. Let $m(\lambda)$ denote the multiplier of the n-cycle to which $\gamma(\lambda)$ belongs, the function m is holomorphic on M. From $\gamma \in \operatorname{supp}\mathcal{M}$ we immediately deduce that $\gamma(\lambda) \in J_\lambda$ and thus $|m(\lambda)| \geq 1$ for every $\lambda \in M$, by the maximum modulus principle applied to $1/m$ we must therefore have $|\mu(\lambda)| > 1$ for every $\lambda \in M$.

$(3) \Rightarrow (1)$. This is obvious.

$(3) \Leftrightarrow (4)$. To deduce the existence of a holomorphic motion of Julia sets from that of repelling cycles one uses the classical λ-lemma which has been discussed in subsection 2.2. A holomorphic motion of Julia sets conjugates the dynamics (see Definition 2.7) and thus induces a holomorphic motion of all repelling cycles.

$(3) \Rightarrow (5)$. By assumption, $\mathcal{R}_n(\lambda) = \{\rho_{j,n}(\lambda) : 1 \leq j \leq N_d(n)\}$ for every $n \in \mathbb{N}^*$ and every $\lambda \in M$, the maps $\rho_{j,n} : M \to \mathbb{P}^1$ being holomorphic. Thus, by the Approximation Formula (Theorem 3.8), we have $L(\lambda) = \lim_n d^{-n} \sum_{1 \leq j \leq N_d(n)} \frac{1}{n} \log |(f_\lambda^n)'(\rho_{j,n}(\lambda))|$ where the convergence is pointwise. The sequence on the right-hand side is a locally bounded sequence of pluriharmonic function on M. Then the convergence actually occurs in L^1_{loc} and L is pluriharmonic on M (i.e. $dd^c L = 0$).

$(5) \Rightarrow (6)$. For simplicity we assume that M is one dimensional. Since the problem is local we can replace M by any small disc D of $\mathbb{C}$. Let us set $u_{n,j}(\lambda) := f_\lambda^n \circ c_j(\lambda)$ and $\varphi_{j,n}(\lambda) := (\lambda, u_{n,j}(\lambda))$. We have to show that the families $(u_{n,j}(\lambda))_n$ are normal on D. Since $|f_*^n[\mathsf{C}_f]|_{D \times \mathbb{P}^1} = \sum_{j=1}^{2d-2} \int_D \varphi_{j,n}^* \left(\omega + dd^c|\lambda|^2\right) = \sum_{j=1}^{2d-2} \int_D dd^c|\lambda|^2 + u_{n,j}^*\omega$ one sees that it suffices to show that $|f_*^n[\mathsf{C}_f]|_{D \times \mathbb{P}^1}$ is uniformly bounded. A different proof will be given in the next subsection.

We will actually establish the estimate $|f_*^n[\mathsf{C}_f]|_{D \times \mathbb{P}^1} = d^n \int_D dd^c L + O(1)$ and then conclude by the $dd^c L$-Formula 3.7. We first note that the functorial relation $f_\lambda^{n*}T_\lambda = d^n T_\lambda$ seen in subsection 1.2 can be rewritten as $f^{n*}(\omega + dd_{\lambda,z}^c g) = d^n(\omega + dd_{\lambda,z}^c g)$ which implies

that $f^{n*}\omega = d^n(\omega + dd^c_{\lambda,z}g) - dd^c_{\lambda,z}(g \circ f^n)$. Then

$$
\begin{aligned}
|f^n_*[\mathsf{C}_f]|_{D\times\mathbb{P}^1} &= \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge f^{n*}(\omega + dd^c|\lambda|^2) \\
&= d^n \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge (\omega + dd^c_{\lambda,z}g) + \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge \left(dd^c|\lambda|^2 - dd^c_{\lambda,z}(g \circ f^n)\right) \\
&= d^n \int_D dd^c L + \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge dd^c|\lambda|^2 - \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge dd^c_{\lambda,z}(g \circ f^n) \\
&= \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge dd^c|\lambda|^2 - \int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge dd^c_{\lambda,z}(g \circ f^n)
\end{aligned}
$$

where the third equality comes from the $dd^c L$-Formula and the last one from our assumption that $dd^c L$ vanishes on D.

To conclude it remains to check that $\int_{D\times\mathbb{P}^1} [\mathsf{C}_f] \wedge dd^c_{\lambda,z}(g \circ f^n)$ is uniformly bounded. This follows from the fact that the function g is bounded and can be seen using integration by parts.

$(6) \Rightarrow (7)$. Assume that λ_0 is a Misiurewicz parameter. Then, by definition, there exist a critical map c_j and a holomorphic map σ defined on some neighbourhood V_0 of λ_0 such that $\sigma(\lambda)$ is n_0-periodic and repelling for every $\lambda \in V_0$ and $f^{p_0}_{\lambda_0}(c_j(\lambda_0) = \sigma(\lambda_0)$ but $f^{p_0} \circ c_j \neq \sigma$. In that situation, the family $(f^{n+p_0} \circ c_j)_n$ cannot be normal at λ_0. Indeed, we have shown in $(2) \Rightarrow (3)$ that $f^{p_0} \circ c_j$ would coincide with σ if $(f^{n+p_0} \circ c_j)_n$ were normal.

$(7) \Rightarrow (2)$. We will show that for every parameter $\lambda_0 \in M$ there exists a small ball $B_0 \subset M$ centered at λ_0 and an equilibrium web for the restriction of f to $B_0 \times \mathbb{P}^1$. This will imply that the repelling cycles of f move holomorphically on B_0 (recall that we already know that (2) and (3) are equivalent). The holomorphic motion of repelling cycles over M then follows since M is simply connected. Again, as (2) and (3) are equivalent, we get the existence of an equilibrium web over M .

Let z_0 be a n_0-periodic repelling point for f_{λ_0}. By the implicit function theorem, there exist a ball B_0 centered at λ_0 and a holomorphic map $\sigma : B_0 \to \mathbb{P}^1$ such that $\sigma(\lambda_0) = z_0$ and $\sigma(\lambda)$ is n_0-periodic for f_λ. By assumption, there are no Misiurewicz parameters and thus the graph Γ_σ of σ cannot meet the post-critical set $\cup_{p\geq 1} f^p(\mathsf{C}_f)$ unless one of the maps c_j satisfies a relation of the form $f^{p_0+n_0} \circ c_j = f^{p_0} \circ c_j$ on M which means that $c_j(\lambda)$ is pre-periodic for every $\lambda \in M$. As C_f is the union of the $(2d-2)$ graphs of the maps c_j, we are sure to find a periodic repelling point z_0 for which $\Gamma_\sigma \cap (\cup_{p\geq 1} f^p(\mathsf{C}_f)) = \emptyset$.

Now, as $f^n : \left(B_0 \times \mathbb{P}^1\right) \setminus f^{-n}\left(\cup_{1\leq p\leq n} f^p(\mathsf{C}_f)\right) \to \left(B_0 \times \mathbb{P}^1\right) \setminus \left(\cup_{1\leq p\leq n} f^p(\mathsf{C}_f)\right)$ is a covering of degree d^n, there exist d^n holomorphic graphs $\Gamma_{\sigma_{j,n}}$ such that $f^n\left(\Gamma_{\sigma_{j,n}}\right) = \Gamma_\sigma$. Note that by the invariance of Julia sets all the $\sigma_{j,n}$ are in $\mathcal{J}$ and that $\mathcal{F}^n \cdot \sigma_{j,n} = \sigma$ by

construction. Moreover it is a classical fact that the family $(\sigma_{j,n})_{j,n}$ is normal. Let us set

$$\mathcal{M}_n := \frac{1}{d^n} \sum_{j=1}^{d^n} \delta_{\sigma_{j,n}}.$$

Then $\mathcal{F}_* \mathcal{M}_{n+1} = \mathcal{M}_n$ and, by the Equidistribution Theorem 1.3, we have

$$\lim_n p_{\lambda*} \mathcal{M}_n = \lim_n \frac{1}{d^n} \sum_{j=1}^{d^n} \delta_{\sigma_{j,n}(\lambda)} = \lim_n \frac{1}{d^n} \sum_{f_\lambda^n(z)=\sigma(\lambda)} \delta_z = \mu_\lambda$$

for every $\lambda \in B_0$. It follows that every limit $\mathcal{M}$ of $(\mathcal{M}_n)_n$ is a compactly supported probability measure on $\mathcal{J}$ such that $p_{\lambda*}\mathcal{M} = \mu_\lambda$ for every $\lambda \in B_0$. If $\mathcal{M}$ were $\mathcal{F}$ invariant then it would be an equilibrium web for $f|_{B_0 \times \mathbb{P}^k}$. In order to get equilibrium webs, it suffices to replace $\mathcal{M}_n$ by the Cesaro average $\frac{1}{n} \sum_{1 \le k \le n} \mathcal{M}_k$. $\qquad\square$

We will now describe the main difficulties which appear when adapting the above proof to the higher dimensional setting. We follow the same organization and do not discuss the implications which are easy to adapt.

$(2) \Rightarrow (3)$. The proof follows the same lines but a serious difficulty appears for showing that all n-periodic points $\gamma(\lambda)$ are repelling. One has to use some linearization argument and thus have to deal with delicate problems of resonances. This is why we work in dimension $k = 2$ or assume that $M = \mathcal{H}_d(\mathbb{P}^k)$.

$(3) \Rightarrow (4)$. One of the main difficulties appears here. The λ-Lemma fails because of the lack of Hurwitz theorem. We rather show that any equilibrium web $\mathcal{M}$ somehow contains a measurable holomorphic motion of $\mathcal{J}$. This is done by studying the dynamical system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. The sections 6 and 7 are devoted to a description of that proof.

$(7) \Rightarrow (2)$. The proof follows the same lines but it is not at all obvious that one can find a graph Γ_σ which is not contained in $\cup_{p \ge 1} f^p(\mathsf{C}_f)$. We have to use a delicate entropy argument here. This point is not discussed in these notes (see also the Question A in Section 10).

$(5) \Rightarrow (7)$. We have to argue directly here since the normal family arguments fail and we cannot follow the route $(5) \Rightarrow (6) \Rightarrow (7)$. We rather show directly that Misiurewicz parameters belong to the support of $dd^c L$ by using a dynamical rescaling argument. This is discussed in section 8. Note however that the computation given in the above proof of $(5) \Rightarrow (6)$ is valid in any dimension and yields some estimate on the growth of the volume of $f^n(\mathsf{C}_f)$.

3.3. **The bifurcation current.** Let us still consider a holomorphic family

$$f \colon M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$$
$$(\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

whose degree is $d \geq 2$. It follows immediately from the results of the above subsection that the support of $dd^c L(\lambda)$ is precisely the bifurcation locus $\mathrm{Bif}(f)$. This was first proved by DeMarco [dM03] who defined the bifurcation current T_{bif} of the family f by

$$T_{\mathrm{bif}} := dd^c_\lambda L(\lambda).$$

As shown by the dd^c-formula (Theorem 3.7), the current T_{bif} is a $(1,1)$-current on M which is positive and closed.

We will now decompose the bifurcation current T_{bif} as a sum of closed positive currents T_i, each of them detecting the instability of the critical orbits $(f^n_\lambda(c_i(\lambda)))_n$ or, in other words, the activity of the critical points c_i. These currents are called *activity currents*.

For every $1 \leq i \leq 2d - 2$ we denote by $[\mathsf{C}_i]$ the current of integration on the graph of c_i (note that this graph is also an hypersurface in $M \times \mathbb{P}^1$) and define a closed positive $(1,1)$-current T_i by

$$T_i := (\pi_M)_* \left(\left(\omega + dd^c_{\lambda,z} g(\lambda, z) \right) \wedge [\mathsf{C}_i] \right).$$

By the $dd^c L$- formula (Theorem 3.7) we have

$$T_{\mathrm{bif}} = \sum_{i=1}^{2d-2} T_i.$$

Let us show that $(f^n_\lambda(c_i(\lambda))_n$ is normal when $T_i = 0$ (it will then be clear that the converse is also true). This is a local problem so we can replace M by any small ball $\Omega \subset M$ on which the family f can be lifted to a family F of d-homogeneous non-degenerate polynomial maps on $\mathbb{C}^{k+1}$:

$$
\begin{aligned}
F \colon \Omega \times \mathbb{C}^{k+1} &\rightarrow \quad \Omega \times \mathbb{C}^{k+1} \\
(\lambda, z) &\mapsto \quad (\lambda, F_\lambda(z)).
\end{aligned}
$$

We may also assume that the maps c_j can be lifted to maps $\widehat{c}_j : \Omega \to \mathbb{C}^{k+1} \setminus \{0\}$. Let us consider the Green function $G(\lambda, \cdot)$ of F_λ. This is a plurisubharmonic and continuous function on $\Omega \times \mathbb{C}^{k+1}$ which is defined by

$$G(\lambda, z) := \lim_n d^{-n} \ln \|F^n_\lambda(z)\|.$$

We stress that G satisfies the following homogeneity property: $G(\lambda, tz) = \ln|t| + G(\lambda, z)$ for every $t \in \mathbb{C}$ and every $(\lambda, z) \in \Omega \times \mathbb{C}^{k+1}$ (see [Ber13] for details).

A straightforward computation shows that $G_\lambda(\widehat{c}_i(\lambda))$ is a potential for T_i which means that $dd^c G_\lambda(\widehat{c}_i(\lambda)) = T_i$, the vanishing of T_i thus means that the function $G_\lambda(\widehat{c}_i(\lambda))$ is pluriharmonic on Ω. We shall see that this implies the normality of the family $(f^n_\lambda(c_i(\lambda)))_n$ (we proved a similar fact in Proposition 3.4 for polynomial families).

The function $G_\lambda(\widehat{c}_i(\lambda))$ being pluriharmonic, there exists a non-vanishing holomorphic function h_i on Ω such that $G_\lambda(\widehat{c}_i(\lambda)) = \ln|h_i|$ which, owing to the homogeneity property

of G_λ, can be rewritten as $G_\lambda \left(\frac{\widehat{c}_i(\lambda)}{h_i(\lambda)} \right) = 0$. After replacing $\widehat{c}_i$ by $\frac{\widehat{c}_i(\lambda)}{h_i(\lambda)}$ we thus have

$$\{F_\lambda^n(\widehat{c}_i(\lambda)) : \ n \in \mathbb{N}, \lambda \in \Omega\} \subset \cup_{\lambda \in \Omega} G_\lambda^{-1}\{0\}.$$

After shrinking Ω the set $\cup_{\lambda \in \Omega} G_\lambda^{-1}\{0\}$ becomes a relatively compact subset in $\mathbb{C}^2$ and Montel's theorem then tells us that the family $(F_\lambda^n(\widehat{c}_i(\lambda)))_n$ is normal on Ω. It follows that $(f_\lambda^n(c_i(\lambda)))_n$ is normal on Ω too.

4. When $k = 1$: application of the bifurcation currents techniques

We consider here a holomorphic family $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$, $(\lambda, z) \mapsto (\lambda, f_\lambda(z))$ of degree d rational maps on $\mathbb{P}^1$. Our aim is to use bifurcation current techniques to prove that certain parameters in M (for a sufficently "big" parameter space M) can be accumulated either by hyperbolic parameters or by parameters for which f_λ has the maximal number of distinct neutral cycles (i.e. $2d - 2$ by the Fatou-Shishikura inequality). Let us recall that a parameter λ is hyperbolic if and only if f_λ has $2d - 2$ distinct attracting cycles. A parameter for which f_λ has $2d - 2$ distinct neutral cycles will be called a Shishikura parameter. The results discussed in this section have been obtained by Bassanelli and the first author [BB07].

Let us recall that the equilibrium measure of f_λ is denoted by μ_λ. The Lyapunov exponent $L(\lambda)$ of $(J_\lambda, f_\lambda, \mu_\lambda)$ is given by $L(\lambda) = \int_{\mathbb{P}^1} \log |f_\lambda'| \, \mu_\lambda$. It follows from the Mañé-Manning ([Man84],[Mañ88]) formula $\dim_H \ \mu_\lambda = \frac{h_{top}(f_\lambda)}{L(\lambda)}$ and the Misiurewicz-Przytycki inequality $h_{top}(f_\lambda) \geq \log d$ that $L(\lambda) \geq \frac{\log d}{2}$. We shall use here the Approximation formula (see Theorem 3.8).

Let us consider the following subsets of the parameter space M:

$$\mathsf{Per}_n(w) := \{\lambda \in M : f_\lambda \text{ has a } n\text{-cycle of multiplier } w\}.$$

One may show that $\mathsf{Per}_n(w)$ is a (singular) hypersurface in M. More precisely, there exists a collection of functions $p_n(\lambda, w)$ which are unitary polynomials of degree $N_d(n) \sim \frac{d^n}{n}$ in w and whose coefficients are holomorphic function in λ, such that $\mathsf{Per}_n(w) = \{p_n(\cdot, w) = 0\}$ for $w \neq 1$ (the case $w = 1$ is more delicate).

According to the Poincaré-Lelong formula, the integration current $[\mathsf{Per}_n(w)]$ on the hypersurface $\mathsf{Per}_n(w)$ is given by

$$[\mathsf{Per}_n(w)] = dd_\lambda^c \log |p_n(\lambda, w)|.$$

We are interested in comparing the limits of $d^{-n}[\mathsf{Per}_n(w)]$ and the bifurcation current $T_{bif} = dd_\lambda^c L(\lambda)$.

4.1. **Averaging the multipliers.** We aim here to establish the following formula for the bifurcation current.

Theorem 4.1. *For every $r \geq 0$ one has $T_{bif} = \lim_n \frac{d^{-n}}{2\pi} \int_0^{2\pi} \left[\mathsf{Per}_n(re^{i\theta}) \right] d\theta$.*

Proof. Set $L_n^r(\lambda) := \frac{d^{-n}}{2\pi} \int_0^{2\pi} \log \left| p_n(\lambda, re^{i\theta}) \right| d\theta$, for $r \geq 0$. Denoting the roots of $p_n(\lambda, \cdot)$ by $w_{n,j}(\lambda), 1 \leq j \leq N_d(n)$ (taken with mutiplicity) we get

$$L_n^r(\lambda) = \frac{d^{-n}}{2\pi} \int_0^{2\pi} \log \left| \prod_{j=1}^{N_d(n)} \left(re^{i\theta} - w_{n,j}(\lambda) \right) \right| d\theta$$

$$= \frac{d^{-n}}{2\pi} \sum_{j=1}^{N_d(n)} \int_0^{2\pi} \log \left| \left(re^{i\theta} - w_{n,j}(\lambda) \right) \right| d\theta$$

$$= d^{-n} \sum_{j=1}^{N_d(n)} \log \max \left(r, |w_{n,j}(\lambda)| \right) d\theta.$$

By Fatou Theorem, there exists $n(\lambda) \in \mathbb{N}$ such that all n-cycles of f_λ are repelling for $n \geq n(\lambda)$. Thus, for $0 \leq r \leq 1$ and $n \geq n(\lambda)$, we obtain

$$L_n^r(\lambda) = d^{-n} \sum_{j=1}^{N_d(n)} \log |w_{n,j}(\lambda)| = d^{-n} \sum_{\mathcal{R}_n(\lambda)} \frac{1}{n} \log \left| (f_\lambda^n)'(z) \right|$$

and by the approximation formula $\lim_n L_n^r(\lambda) = L(\lambda)$.

When $r > 1$ we have (always for $n \geq n(\lambda)$)

$$L_n^r(\lambda) = d^{-n} \sum_{j=1}^{N_d(n)} \log |w_{j,n}(\lambda)| + d^{-n} \sum_{1 \leq |w_{j,n}(\lambda)| < r} \log \frac{r}{|w_{j,n}(\lambda)|}$$

and therefore $0 \leq L_n^r(\lambda) - L_n^0(\lambda) \leq d^{-n} N_d(n) \log r$. As $d^{-n} N_d(n) \sim \frac{1}{n}$, we get again $\lim_n L_n^r(\lambda) = L(\lambda)$. Since the sequence $(L_n^r(\lambda))_n$ is a sequence of *psh* functions which is locally uniformly bounded from above (easy to check), the convergence actually occurs in $L_{loc}^1(M)$. Taking dd^c we thus have $T_{bif} = dd^c L = \lim_n dd^c L_n^r = \lim_n \frac{d^{-n}}{2\pi} \int_0^{2\pi} \left[\mathsf{Per}_n(re^{i\theta}) \right] d\theta$.
□

4.2. **Perturbation of Lattès examples.** We aim here to use the above formula and simple potential theoretic arguments to show that rigid Lattès examples are accumulated by hyperbolic parameters or by Shishikura parameters. Let us recall that a Lattès map is a rational function f which is induced on the Riemann sphere from a dilation on a complex torus $D : T \to T$ by mean of some elliptic function $p : T \to \mathbb{P}^1$.

$$
\begin{array}{ccc}
T & \xrightarrow{\ D\ } & T \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathbb{P}^1 & \xrightarrow{\ f\ } & \mathbb{P}^1
\end{array}
$$

Such maps are of course very rare. They have been characterized by A. Zdunik [Zdu90] by the minimality of their Lyapunov exponent.

Theorem 4.2. f_λ *is a Lattès example* $\Leftrightarrow L(\lambda) = \frac{\log d}{2}$.

For simplicity, we consider the moduli space of quadratic rational maps which (according to Milnor [Mil93]) can be considered as a family $f : M \times \mathbb{P}^1 \to M \times \mathbb{P}^1$ parametrized by $M = \mathbb{C}^2$.

We shall use the Monge-Ampère measure associated to the *psh* function L. By definition this positive measure is given by $dd^c L \wedge dd^c L$. The fact that this product of two closed positive current is well defined follows from the continuity of L. Indeed, for any closed positive current S, the function L is integrable with respect to the trace measure of S and therefore the current LS is well defined. One then set $dd^c L \wedge S := dd^c(LS)$ (recall that S closed means that $dd^c S = 0$).

The *bifurcation measure* of the family f has been introduced by Bassanelli and the first author who gave its first properties. It is defined by

$$\mu_{\text{bif}} := \frac{1}{2} dd^c L \wedge dd^c L.$$

It is an elementary property that the strict minima of L belong to the support of $dd^c L \wedge dd^c L$. Since degree two Lattès example are rigid, they correspond to isolated parameters in M. The theorem of Zdunik thus tells us that they belong to the support of the bifurcation measure. This is also true for all Lattès example as it has been proved by X. Buff and T. Gauthier [BG13]. To achieve our goal it remains to show that any parameter λ_0 in the support of the bifurcation measure μ_{bif} can be accumulated either by hyperbolic parameters or by Shishikura parameters. This is actually a direct consequence of the following approximation formula which is obtained from Theorem 4.1 by mean of elementary potential theoretic arguments:

$$\mu_{\text{bif}} = \lim_n \frac{2^{-(n+k(n))}}{2(2\pi)^2} \int_{[0,2\pi]^2} [\mathsf{Per}_n(re^{i\theta_1})] \wedge [\mathsf{Per}_{k(n)}(re^{i\theta_2})] \, d\theta_1 d\theta_2.$$

In the above formula, $k(n)$ is a suitable increasing sequence of integers. The choice of $0 < r < 1$ shows that the support of μ_{bif} is accumulated by hyperbolic parameters while the choice $r = 1$ shows that it can be accumulated by Shishikura parameters.

What we have proved is actually an equidistribution statement for the bifurcation measure. Full details on the content of this section are given in [Ber13]. Sharp equidistribution results for the bifurcation measure have recently be obtained by T. Gauthier, Y. Okuyama and G. Vigny [GOV17].

5. When $k \geq 1$: extension of Lyubich-Mañé-Sad-Sullivan Theorem

We consider here a holomorphic family

$$f \colon M \times \mathbb{P}^k \to M \times \mathbb{P}^k$$
$$(\lambda, z) \mapsto (\lambda, f_\lambda(z))$$

whose degree is $d \geq 2$. Recall that, for every $\lambda \in M$, we have an ergodic dynamical system $(J_\lambda, f_\lambda, \mu_\lambda)$, where μ_λ is the equilibrium measure of f_λ (see Lecture 1). The sum of the Lyapunov exponents of $(J_\lambda, f_\lambda, \mu_\lambda)$ is given by the following expression:

$$L(\lambda) := \int_{\mathbb{P}^k} \log \left| \det f'_\lambda(z) \right| \mu_\lambda(z).$$
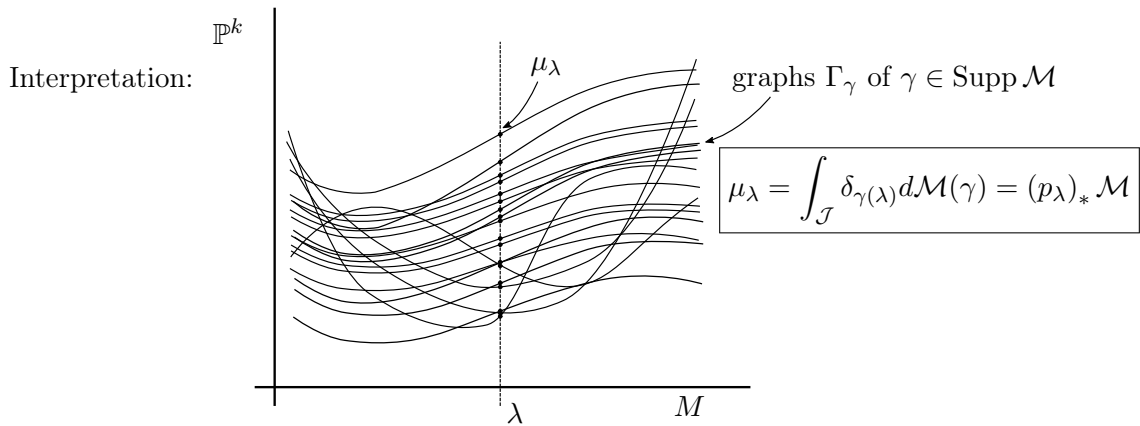
Our aim in this lecture is to describe a recent result which extends Lyubich-Mañé-Sad-Sullivan theorem in higher dimension and is due to C. Dupont and the authors [BBD15].

5.1. **A substitute to the notion of holomorphic motions of Julia sets.** Let us recall a few notations and concepts which have been introduced in subsection 3.2 for families of rational maps. We endow $\mathcal{O}(M, \mathbb{P}^k) := \left\{ \gamma \colon M \to \mathbb{P}^k \colon \gamma \text{ holomorphic} \right\}$ with the topology of local uniform convergence; this is a metric space. The space of interest here is the (possibly empty) subspace

$$\mathcal{J} := \left\{ \gamma \in \mathcal{O}(M, \mathbb{P}^k) \colon \gamma(\lambda) \in J_\lambda, \forall \lambda \in M \right\}.$$

We have two natural maps. The first one is a self-map on $\mathcal{F} \colon \mathcal{J} \to \mathcal{J}$ which is defined by $\mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda))$ and the second one is a projection $p_\lambda \colon \mathcal{J} \to \mathbb{P}^k$ which is defined by $p_\lambda(\gamma) = \gamma(\lambda)$.

Definition 5.1. *An equilibrium web for f is a compactly supported probability measure $\mathcal{M}$ on $\mathcal{J}$ such that $\mathcal{F}_* \mathcal{M} = \mathcal{M}$ and $(p_\lambda)_* \mathcal{M} = \mu_\lambda$ for all $\lambda \in M$.*

Interpretation:

$$\mu_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma) = (p_\lambda)_* \mathcal{M}$$

The existence of such $\mathcal{M}$ somehow means that the equilibrium measures μ_λ's are "holomorphically glued" over M. It is also possible to associate to M a web current in the sense of Dinh: $W_\mathcal{M} := \int_\mathcal{J} [\Gamma_\gamma] \, d\mathcal{M}(\gamma)$. This allows the use of "calculus".

The support of an equilibrium web $\mathcal{M}$ is a quite wild object; we seek for a cleaner one:

Definition 5.2. *An equilibrium lamination for f is an $\mathcal{F}$-invariant subset $\mathcal{L}$ of $\mathcal{J}$ such that:*

- *$\Gamma_\gamma \cap \Gamma_{\gamma'} = \emptyset, \forall \gamma \neq \gamma' \in \mathcal{L}$;*
- *$\mu_\lambda \left(\{ \gamma(\lambda) \colon \gamma \in \mathcal{L} \} \right) = 1, \forall \lambda \in M$;*
- *$\Gamma_\gamma \cap GO(\mathsf{C}_f) = \emptyset, \forall \lambda \in \mathcal{L}$;*
- *$\mathcal{F} : \mathcal{L} \to \mathcal{L}$ is d^k-to-1.*

*Here $GO(\mathsf{C}_f)$ denotes the grand orbit (by f) of the critical set C_f of f.*

Before stating our main result, let us precise a definition. The repelling J-cycles of f_λ are the repelling cycles of f_λ which belong to J_λ. This is not automatic when $k \geq 2$, note however that in the Briend-Duval equidistribution Theorem 1.2 in Lecture 1, the repelling J-cycles of f_λ equidistribute μ_λ.

Theorem 5.3. *Let M be an open and simply connected subset of $\mathcal{H}_d(\mathbb{P}^k)$. Then the following assertions are equivalent:*

(1) *the repelling J-cycles of f_λ move holomorphically over M;*
(2) *$dd^c L \equiv 0$;*
(3) *f admits an equilibrium lamination.*

Although the proof of this result will be given in the next sections, we start with a description of it.

5.2. **Strategy of the proof and tools.** We shall only prove (1)$\Rightarrow$\{(2) and (3)\} and we shall briefly discuss (3) $\Rightarrow$ (1).

Recall that (1) means:
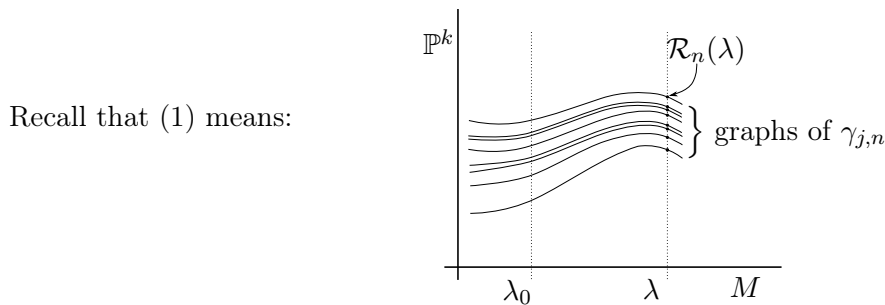
where $\gamma_{j,n} \in \mathcal{J}$ and $\{\gamma_{j,n}(\lambda), j\} = \mathcal{R}_n(\lambda) := \{n - \text{periodic repelling points of } f_\lambda \text{ in } J_\lambda\}$.

• The implication $(1) \Rightarrow (2)$ is obtained by using a generalization of the approximation formula seen in Lecture 4 due to C. Dupont, L. Molino and the first author [BDM08]:

$$L(\lambda) = \lim_n d^{-kn} \sum_{\mathcal{R}_n(\lambda)} \frac{1}{n} \log \left| \det \left(f^n_\lambda \right)' (z) \right|.$$

In our situation this yields $L(\lambda) = \lim_n d^{-kn} \sum_{j=1}^{N_d(n)} \frac{1}{n} \log \left| \det \left(df^n_\lambda \right) (\gamma_{j,n}(\lambda)) \right|$ and shows that L is a pointwise limit of pluriharmonic functions. Since these functions are locally uniformly bounded from above, the convergence occurs in L^1_{loc} and thus L is pluriharmonic. We shall see in Lecture 9 another proof of this fact.

• The implication $(1) \Rightarrow$ *existence of an equilibrium web* $\mathcal{M}$ can be proved by applying Banach-Alaoglu Theorem to $\mathcal{M}_k := d^{-kn} \sum \delta_{\gamma_{j,n}}$ and using the equidistribution theorem for repelling orbits seen in Lecture 1.

• To deduce the *existence of an equilibrium lamination* from the existence of $\mathcal{M}$ is much harder. Its proof exploits the dynamical properties of the system $(\mathcal{J}, \mathcal{F}, \mathcal{M})$. In particular, we need first prove that $\mathcal{M}$ is *acritical* and then replace it with a new web which is both acritical and ergodic.

Definition 5.4. *A web* $\mathcal{M}$ *is said to be acritical if* $\mathcal{M} \left(\{ \gamma \in \Gamma_\gamma \colon \Gamma_\gamma \cap GO(C_f) \neq \emptyset \} \right) = 0.$

The fact that $\mathcal{M}$ is acritical is proved by using the

> **Fundamental Lemma:** $dd^c L \equiv 0 \Rightarrow$ No Misiurewicz parameters in M

Definition 5.5. *A parameter* λ_0 *is said Misiurewicz if there exist integers* $p_0, n_0 \geq 1$ *and a holomorphic map* σ *defined on some neighbourhood of* λ_0 *such that* $\sigma(\lambda) \in \mathcal{R}_{p_0}(\lambda)$ *and* $\Gamma_\sigma \cap W \neq \emptyset$ *but* $\Gamma_\sigma \not\subseteq W$ *for some irreducible component* W *of* $f^{n_0}(C_f)$.
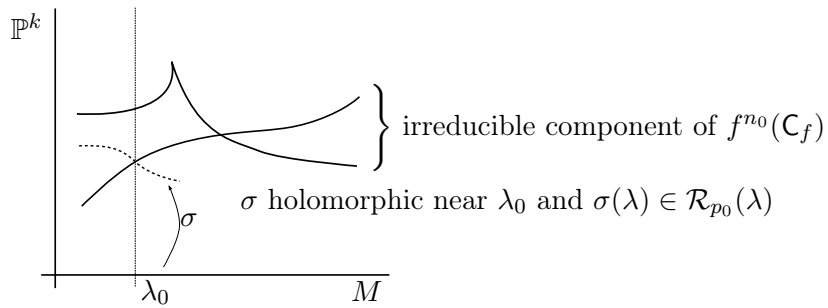
An argument of extremality based on Choquet's decomposition theorem then allows to replace $\mathcal{M}$ by an acritical <u>and</u> ergodic equilibrium web.

• $(3) \Rightarrow (1)$. The proof here follows an idea which works well in dimension one: if a repelling cycle does not move holomorphically then a Siegel disc must appear and this

obviously creates a discontinuity of $\lambda \to J_\lambda$ in the Hausdorff topology, in particular this is not compatible with the holomorphic motion of J_λ.

In $\mathbb{P}^{k \geq 2}$, a Siegel $k-$polydisc would also create a discontinuity but when a repelling cycle does not move holomorphically one simply obtains a Siegel disc, that is a one dimensional object. Is is however possible to show that this is an obstruction to the existence of an holomorphic motion of Julia sets. The question of the discontinuity of $\lambda \to J_\lambda$ in the Hausdorff topology remains open.

Although this strategy seems quite natural, one has to face several technical difficulties to implement it in higher dimension. In particular we must deal with possible persistent resonances in the holomorphic family which is considered. This is exactly why, at least when $k \geq 3$, our result is given for the full family $\mathcal{H}_d(\mathbb{P}^k)$.

To end this description, let us stress that the technical (and potential theoretic) part of the proof is concentrated in the Fundamental Lemma and relies on the following generalization of the Przytycki-De Marco formula (see Lecture 3) due to Bassanelli and the first author:

$$dd^c L = (p_M)_* \left(\left(dd^c_{\lambda,z} g + \omega \right)^k \wedge [\mathsf{C}_f] \right).$$

The remaining lectures will be devoted to the proof of the above results. In Lecture 6 we will construct an acritical and ergodic equilibrium web from holomorphic motions of repelling $\mathcal{J}$-cycles (assuming the Fundamental Lemma). In Lecture 7, we will explain how to extract an equilibrium lamination from an acritical and ergodic equilibrium web. In Lectures 8 and 9 we will respectively prove the Fundamental Lemma and the density of Misiurewicz parameters in Supp $dd^c L$ (see Theorem 9.4).

6. From motion of cycles to acritical equilibrium webs

We assume here that a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ enjoys the property that its repelling J-cycles move holomorphically over M. Our aim is to show that f admits an ergodic acritical equilibrium web.

6.1. **Proof of** (1) $\Rightarrow$ **the existence of an equilibrium web** $\mathcal{M}$. We argue here exactly like in dimension $k = 1$ (see the subsection 3.2). The holomorphic motion of repelling J-cycles means that for every $n \in \mathbb{N}^*$ we have a subset $\{\gamma_{j,n} \colon 1 \leq j \leq N_d(n)\}$ of $\mathcal{J}$ such that $\mathcal{R}_n(\lambda) = \{\gamma_{j,n}(\lambda) \colon 1 \leq j \leq N_d(n)\}$ for all $\lambda \in M$. Recall that $\mathcal{R}_n(\lambda) := \{n\text{-periodic repelling points of } f_\lambda \text{ in } J_\lambda\}$.
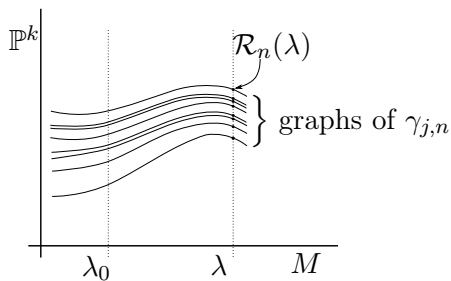
Set $\mathcal{M}_n := d^{-kn} \sum_{j=1}^{N_d(n)} \delta_{\gamma_{j,n}}$. This is a sequence of discrete probability measures on $\mathcal{J}$ (actually $|\mathcal{M}_n| \sim 1$). Moreover, $\cup_n \operatorname{Supp} \mathcal{M}_n$ is relatively compact (a simple Montel normality argument using local lifts of f to $M \times \mathbb{C}^{k+1}$ explains that). Then, by Banach-Alaoglu Theorem we get: $\mathcal{M}_{n_i} \to \mathcal{M}$. $\mathcal{M}$ is clearly a compactly supported measure on $\mathcal{J}$. Moreover $\mathcal{F}_* \mathcal{M} = \lim_i \mathcal{F}_* \mathcal{M}_{n_i} = \lim_i \mathcal{M}_{n_i} = \mathcal{M}$ and $(p_\lambda)_* \mathcal{M} = \lim_i (p_\lambda)_* \mathcal{M}_{n_i} = \lim_i d^{-kn_i} \sum_j \delta_{\gamma_{j,n_i}(\lambda)} = \lim_i d^{-kn_i} \sum_{z \in \mathcal{R}_n(\lambda)} \delta_z = \mu_\lambda$ where the last equality comes from Briend-Duval equidistribution Theorem 1.2. $\mathcal{M}$ is thus an equilibrium web of f.

6.2. **The equilibrium web $\mathcal{M}$ is acritical.** As a first step we will show that $\mathcal{M}$ satifies the following key property

(3) $$\Gamma_\gamma \cap \mathsf{C}_f^+ \neq \emptyset \Rightarrow \Gamma_\gamma \subset \mathsf{C}_f^+, \forall \gamma \in \operatorname{Supp} \mathcal{M}.$$

Recall that $\mathsf{C}_f^+ = \cup_{m \geq 0} f^m(\mathsf{C}_f)$. According to the former subsection and the Fundamental Lemma, there are no Misiurewicz parameter in M. In other words: $\mathcal{M}_{n_k}$ satisfy (3) for all k. By Hurwitz: $\mathcal{M} = \lim_k \mathcal{M}_{n_k}$ also satisfies (3).

As a second step we show that $\mathcal{M}(\mathcal{J}_s) = 0$, where $\mathcal{J}_s := \{\gamma \in \Gamma : \Gamma_\gamma \cap GO(\mathsf{C}_f) \neq \emptyset\}$. First of all,

$$\mathcal{M}\left(\left\{\gamma \in \mathcal{J} : \Gamma_\gamma \cap \mathsf{C}_f^+ \neq \emptyset\right\}\right) \overset{\text{by (3)}}{\leq} \mathcal{M}\left(\left\{\gamma \in \mathcal{J} : \Gamma_\gamma \subset \mathsf{C}_f^+\right\}\right)$$
$$= \mathcal{M}\left(\left\{\gamma \in \mathcal{J} : \gamma(\lambda_0) \in \mathsf{C}_{f_{\lambda_0}}^+\right\}\right) = \mu_{\lambda_0}\left(\mathsf{C}_{f_{\lambda_0}}^+\right) = 0$$

with annotations $\forall \lambda_0 \in M$ and $(p_{\lambda_0})_\star \mathcal{M} = \mu_{\lambda_0}$,

where the last equality is due to the fact that μ_{λ_0} does not give mass to pluripolar sets. Then $\mathcal{M}(\mathcal{J}_s) = 0$ follows from the $\mathcal{F}$-invariance of $\mathcal{M}$.

6.3. **Existence of acritical ergodic webs.** Let $\mathcal{M}_0$ be an acritical equilibrium web for f (constructed in the above step) and let $\mathcal{K} := \operatorname{Supp} \mathcal{M}_0$. Recall that $\mathcal{K}$ is compact in the metric space $\mathcal{J}$. We consider the spaces

$$\mathcal{P}_{web}(\mathcal{K}) := \{\text{equilibrium webs of } f \text{ supported in } \mathcal{K}\}$$
$$\cap$$
$$\mathcal{P}_{inv}(\mathcal{K}) := \{\mathcal{F} - \text{invariant probability measures supported in } \mathcal{K}\}.$$

Note that these are compact metric spaces for the weak$\star$-topology.

The ergodic $\mathcal{F}$-invariant probability measures on $\mathcal{K}$ are precisely the extremal points of $\mathcal{P}_{inv}(\mathcal{K})$. It thus suffices to check that:
   (1) extremality in $\mathcal{P}_{web}(\mathcal{K}) \Rightarrow$ extremality in $\mathcal{P}_{inv}(\mathcal{K})$;
   (2) there exists an acritical web in $\mathcal{P}_{web}(\mathcal{K})$ which is extremal.

(1) follows easily from the fact that $(p_\lambda)_\star \mathcal{M} = \mu_\lambda$ for every $\mathcal{M} \in \mathcal{P}_{web}(\mathcal{K})$ and that the μ_λ's are ergodic.

(2) follows from Choquet Decomposition Theorem applied to $\mathcal{M}_0$:

$$\mathcal{M}_0 = \int_{\mathrm{Ext}(\mathcal{P}_{web})} \mathcal{E} \, d\nu_0(\mathcal{E})$$

where ν_0 is a probability measure for which $\mathrm{Ext}(\mathcal{P}_{web})$ has full measure.

Then $0 = \mathcal{M}_0(\mathcal{J}_s) = \int_{\mathrm{Ext}(\mathcal{P}_{web})} \mathcal{E}(\mathcal{J}_s) d\nu_0(\mathcal{E})$ implies that ν_0-almost all $\mathcal{E}$ in $\mathrm{Ext}(\mathcal{P}_{web})$ are actually acritical.

7. From ergodic acritical equilibrium webs to equilibrium laminations

We assume here that a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ admits an ergodic and acritical web $\mathcal{M}$. Our goal is to explain how the support of $\mathcal{M}$ can be "cleaned" in order to get an equilibrium lamination $\mathcal{L}$ for f. We refer to the lecture 5 for definitions. The manifold M is supposed to be simply connected.

7.1. A dynamical system associated to $\mathcal{M}$.

Let us recall that $\mathcal{M}$ is a compactly supported probability measure on $\mathcal{J}$ such that $\mu_\lambda = \int_{\mathcal{J}} \delta_{\gamma(\lambda)} d\mathcal{M}(\gamma)$ and which is invariant by the map

$$\mathcal{F} : \mathcal{J} \to \mathcal{J} \qquad \mathcal{F}(\gamma)(\lambda) = f_\lambda(\gamma(\lambda)).$$
$$\gamma \mapsto \mathcal{F}(\gamma)$$

Let $\mathcal{K}$ be the support of $\mathcal{M}$ and $\mathcal{X} := \mathcal{K} \setminus \mathcal{J}_s$ where, as before, $\mathcal{J}_s$ is the subset of elements γ in $\mathcal{J}$ whose graphs Γ_γ do not meet the grand orbit $GO(\mathsf{C}_f)$ of the critical set C_f of f. Saying that $\mathcal{M}$ is acritical means that $\mathcal{M}(\mathcal{J}_s) = 0$ and thus that $\mathcal{X}$ has full measure. Using the fact that M is simply connected, one easily sees that $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is onto. To summarize:

$(\mathcal{X}, \mathcal{F}, \mathcal{M})$ is an ergodic dynamical system and the map $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is onto.

We will now transform the dynamical system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ into an invertible one by means of the classical construction of natural extension. The natural extension $\widehat{\mathcal{X}}$ of $\mathcal{X}$ is the set of all possible "histories" for points $\gamma \in \mathcal{X}$:

$$\widehat{\mathcal{X}} := \{\widehat{\gamma} := (\ldots, \gamma_{-j}, \gamma_{-j+1}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_j, \ldots) \; : \; \gamma_j \in \mathcal{X} \text{ and } \mathcal{F}(\gamma_j) = \gamma_{j+1}\}.$$

We have projections $\pi_j : \widehat{\mathcal{X}} \to \mathcal{X}, \widehat{\gamma} \mapsto \gamma_j$ which are onto for all $j \in \mathbb{Z}$.

The shift $\widehat{\mathcal{F}} : \mathcal{X} \to \mathcal{X}$ is clearly invertible and satisfies $\pi_0 \circ \widehat{\mathcal{F}} = \mathcal{F} \circ \pi_0$. It is a classical result that there exists a probability measure $\widehat{\mathcal{M}}$ on $\mathcal{X}$ which is $\widehat{\mathcal{F}}$-invariant and such that $(\pi_j)_* \widehat{\mathcal{M}} = \mathcal{M}$ for all $j \in \mathbb{Z}$. Moreover, $\widehat{\mathcal{M}}$ is ergodic when $\mathcal{M}$ is ergodic. We have thus transfered our problem to the

invertible and ergodic dynamical system $\left(\widehat{\mathcal{X}}, \widehat{\mathcal{F}}, \widehat{\mathcal{M}} \right)$.

In the following, in order to keep the notation as light as possible, we shall continue to use the system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$. The arguments above show that we can actually *think* that our system is invertible.

7.2. **Contraction and conclusion.** The following lemma gives the key estimate that will allow us to recover an equilibrium lamination out of the support of our ergodic and acritical equilibrium web $\mathcal{M}$. As explained in the previous subsection, we can think that our system $(\mathcal{X}, \mathcal{F}, \mathcal{M})$ is invertible.

Lemma 7.1. Contraction Lemma. *There exist a measurable function $\eta : \mathcal{X} \to]0, 1]$ and a positive constant A such that for $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ the following hold:*

(1) *f^{-n} is defined on a tubular neighbourhood $T(\gamma, \eta(\gamma))$ of Γ_γ; and*
(2) *$f^{-n}\left(T(\gamma, \eta(\gamma))\right) \subset T(\mathcal{F}^{-n}\gamma, e^{-nA})$.*
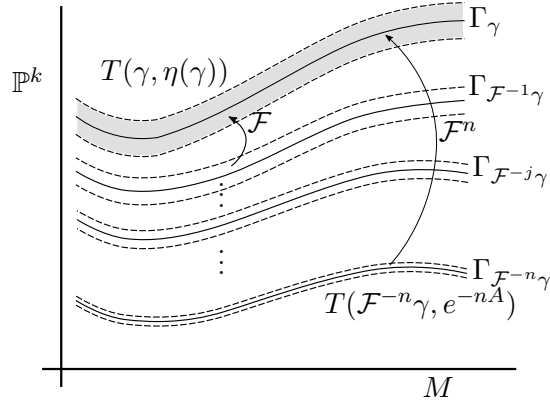
We can now show how to cut out a set of zero measure from the support of $\mathcal{M}$ in order to recover a set of non-intersecting graphs. Let us fix a small ball $B \subset M$. We define the *ramification* $R_B(\gamma)$ over B of an element $\gamma \in \mathcal{X}$ as

$$R_B(\gamma) = \sup_{\gamma' \in \mathrm{Supp}\,\mathcal{M}, \Gamma_\gamma \cap \Gamma_{\gamma'} \neq \emptyset} \sup_B d(\gamma(\lambda), \gamma'(\lambda)).$$

Fix now a $\alpha > 0$ and consider the set of elements $\{\gamma \colon R_B(\gamma) > \alpha\}$. Notice that, by the Contraction Lemma 7.1, we have $R_B(\mathcal{F}^{-n}\gamma) \to 0$ as $n \to +\infty$. Then, Poincaré recurrence Theorem (applied to the inverse system $\left(\mathcal{X}, \mathcal{F}^{-1}, \mathcal{M}\right)$) implies that $\{\gamma \colon R_B(\gamma) > \alpha\}$ has zero $\mathcal{M}$-measure for every $\alpha > 0$. So, $\mathcal{M}\left(\{\gamma \colon R_B(\gamma) > 0\}\right) = 0$ and we can thus consider the full-measure subset $\mathcal{L}^+$ of graphs with zero ramification. It is then not difficult to build the desired lamination $\mathcal{L}$ starting with the set $\mathcal{L}^+$.

We are thus left with proving the Contraction Lemma 7.1. This is the content of the next subsections.

7.3. **Proof of the contraction Lemma 7.1: reduction to some estimate.** Let us start describing the general philosophy of the proof. We exploit a method, developed by Briend-Duval, which proves the analogous statement at a fixed parameter. Namely, given an endomorphism g of $\mathbb{P}^k$ of algebraic degree d, almost every point x (with respect to the equilibrium measure μ) is contained in a ball $B(x, \eta(x))$ where the inverse g^{-n} is defined for every n and satisfies the contraction property $g^{-n}(B(x, \eta(x))) \subset B(g^{-n}(x), e^{-n\chi_1})$. Here χ_1 denotes the smallest Lyapunov exponent of the system $(\mathbb{P}^k, g, \mu)$, which is known [BD99] to be greater that $\frac{\log d}{2}$.

The main steps of the method are as follows.

- An asymptotic estimate of $\left\|dg^{-1}(\cdot)\right\|$ over the inverse orbit $\{g^{-j}(x)\}$ of a point x yields an estimate of the radius of a ball centered at x where g^{-n} is defined, for every n, and of the asymptotic rate of contraction of g^{-n} on this ball.
- The asymptotic estimate of $\left\|dg^{-1}\right\|$ is obtained from

$$\text{for } \mu - \text{a.e. } x \in \mathbb{P}^k: \ \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\|dg^{-1}(g^{-j}(x))\right\| = \int_{\mathbb{P}^k} \log \left\|dg^{-1}(x)\right\| \mu(x)$$

where the equality comes from the ergodicity of μ and Birkhoff Theorem. Since

$$\frac{1}{k} \int \log \left\|dg^{-k}\right\| \to -\chi_1,$$

up to replacing g by a sufficiently high iterate we can assume that the above integral is $\leq -\frac{\chi_1}{2} < 0$.

In our setting, the same method reduces the problem to prove that

$$\text{for } \mathcal{M} - \text{a.e. } \gamma \in \mathcal{X}: \ \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \max_\lambda \left\|df_\lambda^{-1}(\mathcal{F}^{-j}\gamma(\lambda))\right\| < 0.$$

As $\mathcal{M}$ is ergodic, we still know that the limit equals $\int_{\mathcal{X}} \log \max_\lambda \left\|df_\lambda^{-1}(\gamma(\lambda))\right\| \mathcal{M}(\gamma)$. So, we only need to prove that this integral is negative. In order to get this, we show that

(4)
$$\lim_{n\to\infty} \frac{1}{n} \int_{\mathcal{X}} \log \max_\lambda \left\|df_\lambda^{-n}(\gamma(\lambda))\right\| \mathcal{M}(\gamma) < 0$$

and then get the desired estimate after replacing our system by a suitable high iterate f^N.

7.4. **Proof of the contraction Lemma 7.1: the estimate (4).** Let us introduce some notations.

Notation 7.2. *We set*
- $u_n(\gamma, \lambda) = \log \left\|df_\lambda^{-n}(\gamma(\lambda))\right\|$; *and*
- $\widehat{u}_n(\gamma, \lambda) = \log \max_\lambda \left\|df_\lambda^{-n}(\gamma(\lambda))\right\|$.

Notice that the functions $u_n(\gamma, \cdot)$'s are *psh* (this will be useful in the sequel). With these notations, (4) becomes

$$(5) \qquad\qquad \lim_{n\to\infty} \frac{1}{n} \int_{\mathcal{X}} \widehat{u}_n(\gamma)\mathcal{M}(\gamma) < 0.$$

In order to establish (5), we shall need the following two facts:

Fact 7.3. *For $\mathcal{M}$-almost every $\gamma \in \mathcal{X}$ the following holds:*
$$\tfrac{1}{n}u_n(\gamma, \lambda) \to -\chi_1(\lambda), \text{ for almost every } \lambda \in \mathcal{M}.$$

Proof. First of all recall that, by Oseledec Theorem, for every $\lambda \in M$ the set

$$J_{\lambda,1} := \left\{ x \in J_\lambda : \frac{1}{n}\log\left\| df_\lambda^{-n} \right\| \to -\chi_1(\lambda) \right\}$$

has full μ_λ-measure. So, since $(p_\lambda)_* \mathcal{M} = \mu_\lambda$, we have that

$$\forall \lambda \in M : \tfrac{1}{n}u_n(\gamma, \lambda) \to -\chi_1(\lambda) \text{ for } \mathcal{M}\text{-almost every } \gamma \in \mathcal{X}.$$

We can then consider the subset E of the product space $\mathcal{X} \times M$ given by

$$E := \left\{ (\gamma, \lambda) \in \mathcal{X} \times M : \frac{1}{n}u_n(\gamma(\lambda)) \to -\chi_1(\lambda) \right\}.$$

By Tonelli Theorem, we have (denoting by Leb the Lebesgue measure on M):

$$\underset{=}{\overset{(\mathcal{M} \otimes \mathrm{Leb})(E)}{}}$$

$$\int_M \underbrace{\mathcal{M}\left(\{\gamma \in \mathcal{X} : (\gamma, \lambda) \in E\}\right)}_{=1 \forall \lambda \in M} \mathrm{Leb}(\lambda) \qquad \int_{\mathcal{X}} \underbrace{\mathrm{Leb}\left(\{\lambda \in M : (\gamma, \lambda) \in E\}\right)}_{=\mathrm{Leb}(M) \text{ for } \mathcal{M}-\mathrm{a.e.}\gamma \in \mathcal{X}} \mathcal{M}(\gamma)$$

$$\qquad\qquad\qquad\qquad\qquad \Rightarrow$$

$$\square$$

Fact 7.4. *Let $V_0 \Subset W_0 \Subset M$. Then*
   (1) $\frac{u_n}{n}(\gamma, \cdot)$ *are locally uniformly bounded on V_0, for $\mathcal{M}$-a.e. $\gamma \in \mathcal{X}$;*
   (2) $\widehat{u}_n \in L^1(\mathcal{M})$.

This fact is crucial for the proof of (5). Its proof is elementary but quite technical and we shall therefore only sketch it. We shall make use of the following elementary fact about holomorphic functions from the unit disk $\mathbb{D}$ to $\mathbb{D}^*$:

Compactness statement: *there exists $0 < \alpha \leq 1$ such that $\sup_{V_0} |\phi| \leq |\phi(\lambda)|^\alpha$ for every $\lambda \in \mathbb{D}(0, 1/2)$ and every holomorphic function $\phi : \mathbb{D} \to \mathbb{D}^*$.*

Idea of proof of Fact 7.4. Notice that $u_n(\gamma, \lambda) = \log\left(\frac{1}{\delta(df_\lambda^n(\gamma(\lambda)))} \right)$, where we denote by $\delta(A)$ the smallest singular value of a matrix A. Moreover, notice that all the matrices that we are considering satisfy $\delta(A) \neq 0$. If the function δ were holomorphic, the above

statement and the compactness of V_0 and $\mathrm{Supp}\,\mathcal{M}$ would imply that for every n the function $u_n(\gamma, n)$ satisfy the estimate

$$\text{(6)} \qquad \delta\left(df_\lambda^n\left(\gamma(\lambda')\right)\right) \leq c^{n(1-\alpha)}\delta\left(df_\lambda^n\left(\gamma(\lambda)\right)\right)^\alpha,$$

holding for any two points $\lambda, \lambda' \in V_0$, where $c = \max_{\gamma,\lambda} \delta(df_\lambda(\gamma(\lambda)))$. The difficulty is overcome as follows. For every matrix A involved, $c_1 |\det A|^k \leq \delta(A)^k \leq |\det A|$ for some constant c_1. So, even if δ is not a holomorphic function, it is bounded above and below by two such functions, for which we can find estimates as above.

Once we have established (6), we can transfer all the estimates we have to do with a variable λ to a single suitable one (given by the Fact 7.3), and thus apply the analogous (known) results in the fixed-parameter setting in order to conclude. $\qquad \square$

We can now explain why the limit in (5) is negative. Since

(1) the sequence $\widehat{u}_n$ is subadditive (i.e., $\widehat{u}_{n+m} \leq \widehat{u}_n + \widehat{u}_m \circ \mathcal{F}^n$),
(2) $\widehat{u}_n \in L^1(\mathcal{M})$, and
(3) $\mathcal{M}$ is ergodic

we can apply the ergodic version of Kingman Subadditive Theorem and thus get the existence of an L such that

(a) $\frac{1}{n} \int_{\mathcal{X}} \widehat{u}_n(\gamma)\mathcal{M}(\gamma) \to L$; and
(b) for $\mathcal{M}$-a.e. $\gamma \in \mathcal{X}$: $\frac{1}{n}\widehat{u}_n(\gamma) \to L$.

In particular, this proves that the limit in (5) exists and that we can compute it on almost every element $\gamma \in \mathcal{X}$. We are thus left with proving that $L < 0$. In order to do so, we consider a $\gamma \in \mathcal{X}$ with the following properties:

(1) $\frac{1}{n}\widehat{u}_n(\gamma) \to L$ (from (b));
(2) $\frac{1}{n}u_n(\gamma, \cdot)$ is uniformly bounded on $V_0 \Subset M$ (from Fact 7.4); and
(3) $\frac{1}{n}u_n(\gamma, \lambda) \to -\chi_1(\lambda)$ for Leb-a.e. $\lambda \in V_0$ (from Fact 7.3).

We then claim that $L \leq -\frac{\log d}{2}$. Suppose to the contrary that $L > -\frac{\log d}{2}$. Pick $U_0 \Subset V_0$. Recall that $\widehat{u}_n(\gamma) = \max_\lambda u_n(\gamma, \lambda)$. Up to extracting a subsequence, one finds points $\lambda_n \in U_0$ and a positive ε such that $\frac{u_n(\gamma, \lambda_n)}{n} \geq -\frac{\log d}{2} + \varepsilon$. Up to extracting another subsequence, we can assume that $\lambda_n \to \lambda_0 \in \overline{U_0}$. Now, there exists an r such that $B(\lambda_n, r) \subset V_0$ for every n, and, since every $u_n(\gamma, \cdot)$ is psh, the submean inequality for each of them at λ_n yields

$$-\frac{\log d}{2} + \varepsilon \leq \frac{u_n(\gamma, \lambda_n)}{n} \leq \frac{1}{|B(\lambda_n, r)|}\int_{B(\lambda_n, r)}\frac{u_n(\gamma, \lambda)}{n}.$$

Since by (3) the sequence $\frac{u_n(\gamma, \lambda)}{n}$ converges to $\chi_1(\lambda)$ for almost every λ and is uniformly bounded (by (2)), the Lebesgue dominated convergence Theorem gives:

$$-\frac{\log d}{2} + \varepsilon \leq \frac{1}{|B(\lambda_0, r)|}\int_{B(\lambda_0, r)} -\chi_1(\lambda).$$

Since $\chi_1(\lambda) \geq \frac{\log d}{2}$, this gives the desired contradiction.

8. Proof of the fundamental lemma

Let $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ be a holomorphic family of degree $d \geq 2$. Let $L(\lambda) := \int_{\mathbb{P}^k} \log |\det f'_\lambda(z)| \, \mu_\lambda(z)$ be the sum of the Lyapunov exponents of the system $(J_\lambda, f_\lambda, \mu_\lambda)$. The Fundamental Lemma, stated in Subsection 5.2, says that Misiurewicz parameters belong to the support of $dd^c L$. Our aim in this lecture is to prove it.

We will follow the proof given in [BBD15]. An alternative and more geometric proof, which is also valid in more general settings, is given in [Bi16a].

Recall that $\mu_\lambda = (dd^c_z g(\lambda, z) + \omega)^k$ and $g = \lim_n g_n$ (locally uniformly on $M \times \mathbb{P}^k$) with g_n smooth (see Lecture 1).

We shall use the formula (see Lecture 5)

$$(7) \qquad dd^c L = (p_M)_* \left(\left(dd^c_{\lambda,z} g + \omega \right)^k \wedge [\mathsf{C}_f] \right).$$

The function L is *psh* and continuous on M.

8.1. **Step 1: Simplifications.** In our computations, we will <u>assume that g is smooth</u>. This is possible since

$$f^* \left(dd^c g + \omega \right) = d \left(dd^c g + \omega \right)$$

$$f^* \left(dd^c g_{n+1} + \omega \right) = d \left(dd^c g_n + \omega \right)$$

where the g_n are smooth and locally uniformly converging to g.

We assume that the parameter space M is a disc centered at the origin 0 in $\mathbb{C}$ and replace $\mathbb{P}^k$ by $\mathbb{C}^k$. The assumption that 0 is a Misiurewicz parameter is then summarized by the following picture.
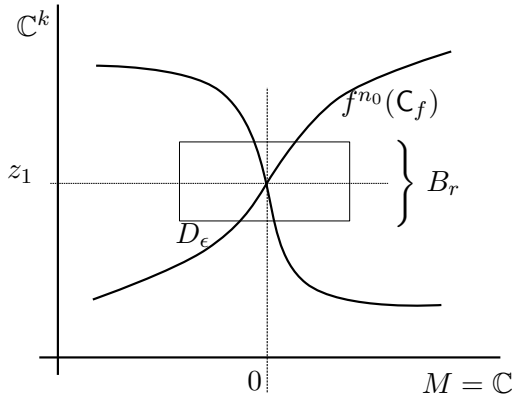
- $f_\lambda(z_1) = z_1, \forall \lambda \in D_\varepsilon$ and $z_1 \in J_\lambda$ and repelling;

- $\exists A > 1$ such that $\|f_\lambda(z) - f_\lambda(z')\| \geq A \|z - z'\|$, $\forall \lambda \in D_\varepsilon, z, z' \in B_r$;

- $(\lambda, z_1) \in f^{n_0}(\mathsf{C}_f) \cap (D_\varepsilon \times B_r) \Rightarrow \lambda = 0.$

8.2. Step 2: Lower bound for $\langle dd^c L, 1_{D_\varepsilon}\rangle$ using (7). Pick $(0, z_0) \in \mathsf{C}_f$ such that $f^{n_0}(z_0) = z_1$. After shrinking, we have that $f^{n_0} : U \to D_\varepsilon \times B_r$ is proper.
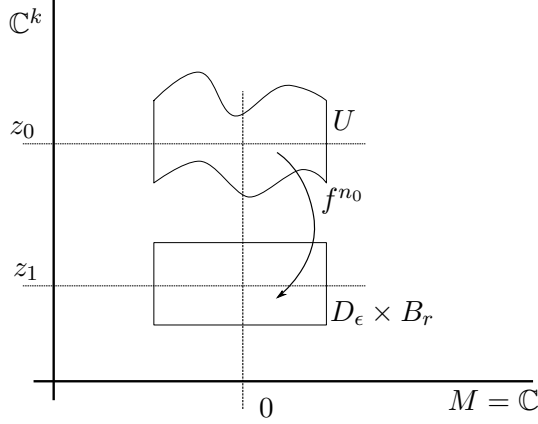
Note that $1_U \leq 1_{D_\varepsilon} \circ p_M$, where
$$p_M : M \times \mathbb{C}^k \to M$$
$$(\lambda, z) \mapsto \lambda.$$

We now compute:

$$\langle dd^c L, 1_{D_\varepsilon}\rangle \overset{\text{by (7)}}{=} \left\langle \overbrace{(dd^c g + \omega)^k \wedge [\mathsf{C}_f]}^{\text{positive measure}}, 1_{D_\varepsilon} \circ p_M \right\rangle$$

$$\geq \underset{1_U \leq 1_{D_\varepsilon} \circ p_m}{\left\langle (dd^c g + \omega)^k \wedge [\mathsf{C}_f], 1_U \right\rangle} = \underset{\text{think } g \text{ smooth}}{\left\langle 1_U [\mathsf{C}_f], (dd^c g + \omega)^k \right\rangle}$$

$$= \left\langle 1_U [\mathsf{C}_f], d^{-n_0 k} (f^{n_0})^* (dd^c g + \omega)^k \right\rangle$$

$$= \left\langle d^{-n_0 k} f_*^{n_0} (1_U [\mathsf{C}_f]), (dd^c g + \omega)^k \right\rangle$$

$$\geq \underset{f^{n_0} : U \to D_\varepsilon \times B_r \text{ proper}}{d^{-n_0 k} \left\langle [f^{n_0}(\mathsf{C}_f)] 1_{D_\varepsilon \times B_r}, (dd^c g + \omega)^k \right\rangle}.$$

Setting $A_0 := 1_{D_\varepsilon \times B_r} [f^{n_0}(\mathsf{C}_f)]$ we have proved that

$$\langle dd^c L, 1_{D_\varepsilon}\rangle \geq d^{-n_0 k} \left\| A_0 \wedge (dd^c g + \omega)^k \right\|.$$

8.3. Step 3: Transfer from the dynamical space $\{0\} \times \mathbb{P}^k$ to the parameter space via a dynamical rescaling. Define inductively A_p by $A_{p+1} := 1_{D_\varepsilon \times B_r} f_*(A_p)$, one may check that

$$A_p \to m \left[\{0\} \times B_r \right] \text{ for some } m \geq 1.$$
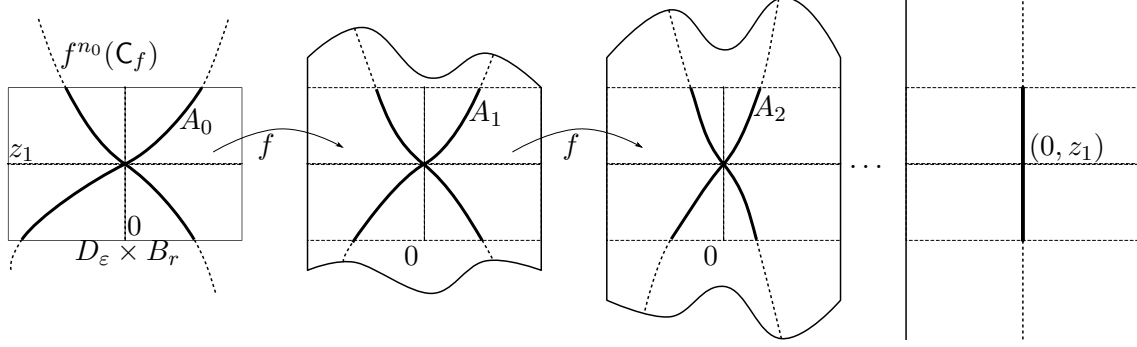
Claim: $\left\| A_p \wedge (dd^c g + \omega)^k \right\| \leq d^{pk} \left\| A_0 \wedge (dd^c g + \omega)^k \right\|$

Conclusion:

$$\langle dd^c L, 1_{D_\varepsilon} \rangle \underset{\boxed{\text{Step 1}}}{\geq} d^{-n_0 k} \left\| A_0 \wedge (dd^c g + \omega)^k \right\| \underset{\boxed{\text{Claim}}}{\geq} d^{-(n_0+p)k} \left\| A_p \wedge (dd^c g + \omega)^k \right\|.$$

This implies that $\langle dd^c L, 1_{D_\varepsilon} \rangle > 0$ since

$$\left\| A_p \wedge (dd^c g + \omega)^k \right\| \to \left\langle m \left[\{0\} \times B_r \right], (dd^c g + \omega)^k \right\rangle = m \mu_{\lambda_0}(B_r) > 0.$$

$$\boxed{\mu_{\lambda_0} = (dd_z^c g(0, z) + \omega)^k} \qquad \boxed{z_1 \in J_{\lambda_0}}$$

Proof of the Claim: (again think g smooth)

$$\begin{aligned}
\left\| A_{p+1} \wedge (dd^c g + \omega)^k \right\| &= \left\langle 1_{D_\varepsilon \times B_r} f_*(A_p), (dd^c g + \omega)^k \right\rangle \\
&= \left\langle A_p, f^* \left(1_{D_\varepsilon \times B_r} (dd^c g + \omega)^k \right) \right\rangle \\
&= \left\langle A_p, (1_{D_\varepsilon \times B_r} \circ f) \, d^k (dd^c g + \omega)^k \right\rangle \\
&\leq d^k \left\langle A_p, (dd^c g + \omega)^k \right\rangle = d^k \left\| A_p \wedge (dd^c g + \omega)^k \right\|.
\end{aligned}$$

$\square$

9. Density of Misiurewicz parameters in the support of $dd^c L$

We consider a family $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ as in the last lecture. Our aim is to prove tha Misiurewicz parameters are dense in $\operatorname{Supp} dd^c L$.

9.1. **A general condition for the vanishing of $dd^c L$.**

Theorem 9.1. *If f admits an equilibrium web $\mathcal{M}$ such that*

$$\forall \gamma \in \operatorname{Supp} \mathcal{M} : \Gamma_\gamma \cap C_f \neq \emptyset \Rightarrow \Gamma_\gamma \subset C_f$$

then $dd^c L \equiv 0$ on M.

Proof. 1) To simplify we consider a lifted family $F : B \times \mathbb{C}^{k+1} \to B \times \mathbb{C}^{k+1}$ where B is a (small) disc in $\mathbb{C}$. Note that in this case $\operatorname{Supp} \mathcal{M} \subset \mathcal{O}(B, \mathbb{C}^{k+1})$. Set $\operatorname{Jac}(\lambda, z) := \det df_\lambda(z)$. We associate to $\mathcal{M}$ the web current $W_\mathcal{M} := \int [\Gamma_\gamma] \, d\mathcal{M}(\gamma)$. It is a result due to Pham [Pha05] that the current $\log |\operatorname{Jac}| W_\mathcal{M}$ is well defined. Moreover, Pham has given a generalized version of the $dd^c L$ formula (7) of the last lecture which yields:

$$0 \leq dd^c L \leq (p_B)_* \left(dd^c \left(\log |\operatorname{Jac}| W_\mathcal{M} \right) \right).$$

It thus suffices to show that $\log |\operatorname{Jac}| W_\mathcal{M}$ is dd^c-closed.

2) A formal computation.

$$\left\langle \underset{\boxed{\text{test form}}}{\log |\operatorname{Jac}| W_\mathcal{M}, dd^c \overset{\downarrow}{\phi}} \right\rangle = \langle W_\mathcal{M}, \log |\operatorname{Jac}| \, dd^c \phi \rangle$$

$$= \int \langle [\Gamma_\gamma], \log |\operatorname{Jac}| \, dd^c \phi] \rangle \, d\mathcal{M}(\gamma)$$

$$= \int d\mathcal{M}(\gamma) \underbrace{\left(\int_\mathbb{C} \log |\operatorname{Jac}(\lambda, \gamma(\lambda))| \, dd^c(\phi \circ \gamma) \right)}_{=0 \text{ if } \Gamma_\gamma \cap \{\operatorname{Jac} = 0\} = \emptyset}$$

$$= 0.$$

3) To make the above computation rigorous, one uses the following estimate:

$$\mathcal{M} \left(\{\gamma : \Gamma_\gamma \cap \{|\operatorname{Jac}| < \varepsilon\} \neq \emptyset\} \right) \lesssim \varepsilon^\tau.$$

Let us set $\mathcal{S}_\varepsilon := \{\gamma \in \operatorname{Supp} \mathcal{M} : \Gamma_\gamma \cap \{|\operatorname{Jac}| < \varepsilon\} \neq \emptyset\}$. Let $\lambda_0 \in B$ fixed.

Claim: $\mathcal{S}_\varepsilon \subset \left\{ \gamma : \gamma(\lambda_0) \in \left(C_{F_{\lambda_0}} \right)_{A \varepsilon^a} \right\}$, for some $A, a > 0$.

The estimate above follows from the Claim:

$$\underset{\boxed{(p_{\lambda_0})_* \mathcal{M} = \mu_{\lambda_0}}}{\mathcal{M}(\mathcal{S}_\varepsilon)} \leq \mu_{\lambda_0} \left(\left(C_{F_{\lambda_0}} \right)_{A \varepsilon^a} \right) \underset{\boxed{\mu_{\lambda_0} \text{ has Hölder potentials}}}{\lesssim \varepsilon^\tau}$$

To prove that claim:

i) $\gamma \in \mathcal{S}_\varepsilon$ and $\Gamma_\gamma \not\subset \{\operatorname{Jac} = 0\} \Rightarrow \gamma \in \mathcal{S}_\varepsilon$ and $\Gamma_\gamma \cap \{\operatorname{Jac} = 0\} = \emptyset$. Then, as $B \ni \lambda \mapsto \operatorname{Jac}(\lambda, \gamma(\lambda)) \in \Delta_R \setminus \{0\}$ contracts the Kobayashi distances we get $|\operatorname{Jac}(\lambda_0, \gamma(\lambda_0))| < \varepsilon^\alpha$, for some $0 < \alpha \leq 1$. By compactness of $\operatorname{Supp} \mathcal{M}$, the estimate is uniform in γ.

ii) The claim then follows from a Łojasiewicz inequality. $\square$

Remark 9.2. *The above theorem also provides another proof of the fact that the holomorphic motion of repelling J-cycles implies the vanishing of $dd^c L$ (implication (1) $\Rightarrow$ (2) in Lecture 5).*

9.2. **Density of Misiurewicz parameters.** We aim here to prove the following Lemma.

Lemma 9.3. *Let $\lambda_0 \in M$ and B be a ball centered at λ_0 with no Misiurewicz parameters inside. Then λ_0 does not belong to the support of $dd^c L$.*

Note that combining this Lemma with the Fundamental Lemma proved in Lecture 8, we get:

Theorem 9.4. $\mathrm{Supp}\, dd^c L = \overline{\{Misiurewicz\ parameters\}}.$

Proof. After shrinking B, we find $\gamma_0 : B \to \mathbb{P}^k$ holomorphic such that $\gamma(\lambda_0) \in J_\lambda$ and $\gamma_0(\lambda)$ is repelling and n_0-periodic for all λ. It is not clear at all that the graph of such a γ_0 is not included in C_f^+. We claim that one can actually find γ_0 in such a way that $\Gamma_{\gamma_0} \not\subseteq \mathsf{C}_f^+$. The proof of this fact is not elementary and will not be discussed here.

Since there are no Misiurewicz parameters in B and $\Gamma_{\gamma_0} \not\subseteq \mathsf{C}_f^+$ we have $\Gamma_{\gamma_0} \cap \mathsf{C}_f^+ = \emptyset$. We may thus lift γ_0 by f^n:

$$
\begin{array}{ccc}
 & & B \times \mathbb{P}^k \\
(\lambda, \gamma_{n,j}(\lambda)) \quad \nearrow & & \downarrow f^n \\
B & \longrightarrow & B \times \mathbb{P}^k \\
\lambda & \longmapsto & (\lambda, \gamma_0(\lambda))
\end{array}
$$

Let us set $\mathcal{M}_n := d^{-kn} \sum_j \delta_{\gamma_{n,j}}$. Using Banach-Alaoglu Theorem and the theorem of equidistribution of iterated preimages we find an equilibrium web $\mathcal{M}$ for f such that

$$\mathcal{M}_{n_i} \to \mathcal{M}.$$

Since by construction $\Gamma_\gamma \cap \mathsf{C}_f = \emptyset$ for all $\gamma \in \mathrm{Supp}\, \mathcal{M}_{n_i}$, Hurwitz Theorem shows that $\Gamma_\gamma \cap \mathsf{C}_f = \emptyset$ or $\Gamma_\gamma \subset \mathsf{C}_f$ for all $\gamma \in \mathrm{Supp}\, \mathcal{M}$. By the above theorem we thus have $dd^c L \equiv 0$ on B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

10. Further results and open questions

The goal of this last section is to present some related results which appeared after the Simons Semester at IMPAN and list a few open questions.

10.1. **Further results.** Let us start by comparing more closely the one dimensional Theorem 2.1 and its generalization, Theorem 5.3. As we said above, condition 3 in Theorem 5.3 can be seen as an analogue of the holomorphic motion of the Julia sets in dimension one (and thus of the first condition in Theorem 2.1). In both Theorems we have a condition on the repelling cycles contained in the Julia sets and, in both cases, the (pluri)harmonicity

of the Lyapunov function is equivalent to the other notions of stability (see Lecture 3 for the dimension one). The following result (see [BBD15, Bi16a]) can be seen as a higher-dimensional analogue of the fact that the stability is equivalent to the normality of the critical orbits (condition 3 in Theorem 2.1), by interpreting the normality as a volume growth condition (this also gives a replacement of the property 6 in Theorem 3.9 valid in any dimension).

Theorem 10.1. *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^k$ of algebraic degree d. Then*

$$dd^c L \neq 0 \Leftrightarrow \limsup_{n \to \infty} \frac{1}{n} \log \|(f^n)_* \, \mathcal{C}_f\| > \log d^{k-1}.$$

Let us mention that this result is at the heart of the alternative geometric approach given in [Bi16a] to the proof of the Fundamental Lemma (see Lecture 8). It is also one of the crucial points in the generalization of the theory of stability explained in these notes to the more general setting of polynomial-like maps of large topological degree, see [Bi16b] for complete details.

Two main questions are thus left open after the analysis above:
(Q1) is the Hausdorff continuity of the Julia sets equivalent to the other notions of stability (as in dimension one, see Theorem 2.1)?
(Q2) is the stability locus dense (as in dimension one, see Corollary 2.2)?

Before adressing the above two questions, we remark that Theorem 5.3 applies only for the study of the family of all endomorphisms of $\mathbb{P}^k$ of a given degree, and not to a generic subfamily (although the equivalence of all the conditions but the one on the motion of repelling cycles holds on every family, see [BBD15]). Nevertheless, in [Bi16a] a slightly weaker condition than the holomorphic motion of all the repelling cycles is stated and proved to be equivalent to all the others for any arbitrary family. The condition and the result are as follows.

Definition 10.2. *Let f be an arbitrary holomorphic family of endomorphisms of $\mathbb{P}^k$ of a given degree $d \geq 2$, with parameter space M. We say that* asymptotically all J-cycles move holomorphically *on M if there exists a subset $\mathcal{P} = \cup_n \mathcal{P}_n \subset \mathcal{J}$ such that*

(1) *every $\gamma \in \mathcal{P}_n$ is n-periodic; and*
(2) *for every $M' \Subset M$, asymptotically every element of $\mathcal{P}$ is repelling, i.e.,*

$$\frac{\text{Card} \{ \text{ repelling cycles in } \mathcal{P}_n \}}{\text{Card} \, \mathcal{P}_n} \to 1.$$

Theorem 10.3. *Let f be an arbitrary holomorphic family of endomorphisms of $\mathbb{P}^k$ of degree $d \geq 2$. Assume that the parameter space is simply connected. Then the following are equivalent:*

(1) *asymptotically all J-cycles move holomorphically;*
(2) *there exists an equilibrium lamination for f;*
(3) *the Lyapunov function is pluriharmonic;*

(4) *there are no Misiurewicz parameters.*

Thus, it makes sense to talk about stability and bifurcation loci for arbitrary families of endomorphisms of $\mathbb{P}^k$. The following Theorem from [BT17] gives an example of a family of endomorphisms of $\mathbb{P}^2$ for which the answer to both the above questions (Q1) and (Q2) is negative.

Theorem 10.4 (Bianchi, Taflin). *The family of endomorphisms of $\mathbb{P}^2$ given by*

$$f_\lambda = \left[-x(x^3 + 2z^3) : y(z^3 - x^3 + \lambda(x^3 + y^3 + z^3)) : z(2x^3 + z^3) \right]$$

with $\lambda \in \mathbb{C}^*$ satisfies the following properties:
  (1) *the Julia set of f_λ depends continuously on λ, for the Hausdorff topology;*
  (2) *the bifurcation locus coincides with $\mathbb{C}^*$.*

The family above is called the *elementary Desboves family* and these maps were previously studied by Bonifant-Dabija [BD02] and Bonifant-Dabija-Milnor [BDM07]. Notice that the above result does not imply the existence of an open set in the bifurcation locus for the family $\mathcal{H}_d(\mathbb{P}^k)$ (a *robust bifurcation*). The existence of such an open set was proved by Dujardin [Duj16], who presents two mechanisms leading to robust bifurcations.

Theorem 10.5 (Dujardin). *The bifurcation locus has non empty interior in the space $\mathcal{H}_d(\mathbb{P}^k)$ for every $k \geq 2$ and $d \geq 2$.*

Let us now focus on the case $k = 2$. The methods by Dujardin give open sets in the bifurcation locus near maps of the form $(z, w) \mapsto (p(z), q(w))$, where p is a bifurcating polynomial in the family of degree d polynomials $\mathcal{P}_d(\mathcal{C})$, and q has a rather specific form. This means that these maps are in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^k)$. He then asks whether any product map of the form $(z, w) \mapsto (p(z), q(w))$ (where p or q is bifurcating in $\mathcal{P}_d(\mathcal{C})$) is contained in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^2)$. This question was positively answered by Taflin [Taf17].

Theorem 10.6 (Taflin). *Let $f(z, w) = (p(z), q(w))$ be a product of two polynomial maps of degree d in $\mathbb{C}$. Then if p or q belongs to the bifurcation locus in $\mathcal{P}_d(\mathcal{C})$, f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.*

The Hausdorff dimension of the bifurcation locus can also be studied for generic families. Let us recall the fundamental result of Shishikura [Shi98] stating that the Hausdorff dimension of the boundary of the Mandelbrot set (which is the bifurcation locus of the quadratic family, see Lecture 2) is equal to 2. From this one can deduce [McM00, Tan98] that the Hausdorff dimension of the bifurcation locus is always maximal for every family of rational maps. For families of endomorphisms of $\mathbb{P}^k$, in [BB16] the authors prove an estimate for the Hausdorff dimension which depends on the values of the Lyapunov exponents. In particular, this dimension is always maximal near (higher-dimensional) isolated Lattès examples.

Theorem 10.7. *Let f be any holomorphic family of endomorphisms of $\mathbb{P}^k$, parametrized by M. Assume that $\lambda_0 \in M$ is such that f_{λ_0} is a Lattès example, and λ_0 is accumulated*

by parameters λ for which f_λ is not a Lattès example. Then the Hausdorff dimension of the bifurcation locus is maximal at λ_0.

This is coherent with a conjecture in [Duj16], which states that Lattès maps should be contained in the closure of the interior of the bifurcation locus of $\mathcal{H}_d(\mathbb{P}^k)$. A step towards the settling of this conjecture in dimension 2 has been recently proved by Biebler.

Theorem 10.8 (Biebler). *For every Lattès map L of degree d on $\mathbb{P}^2$ there exists an integer $n_0 = n(L)$ such that for every $n \geq n_0$ the iterate L^n belongs to the closure of the interior of the bifurcation locus of the family $\mathcal{H}_{d^n}(\mathbb{P}^2)$.*

10.2. **Open questions.**

(A) Is it possible to give an elementary proof of the following fact? If for a family f there are no Misiurewicz parameters in some open subset Ω of the parameter space M then there exists a holomorphic map $\gamma : \Omega \to \mathbb{P}^k$ whose graph avoids the postcritical set $\cup_{p \geq 0} f^p(C_f)$. Note that $\gamma \in \mathcal{J}$ is not necessary in the proof of Thm 9.4.

(B) Is Theorem 5.3 true for any holomorphic family (it is proved for dimension $k = 2$ or when $k \geq 2$ and the parameter space M is an open subset of $\mathcal{H}_d(\mathbb{P}^k)$) ? The difficulty here is with resonances phenomena in the proof of implication $(3) \Rightarrow (1)$.

(C) Is it possible to find natural holomorphic families $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ (with $k \geq 2$) for which the bifurcation locus is not empty but has empty interior?

(D) Are stability and Hausdorff continuity of Julia sets equivalent for skew-products families?

(E) Is it possible to better understand bifurcations within skew-products families (see for instance [AB18])?

(F) What do the exterior powers $dd^c L \wedge dd^c L \wedge \cdots \wedge dd^c L$ detect for holomorphic families $f : M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ with $k \geq 2$?

(G) Is it possible to establish equidistribution results similar to those discussed in subsection 4.1 for higher dimensional holomorphic families (this is related to the approximation formula for the sum of Lyapunov exponents proved in [BDM08])?

(H) When does a true holomorphic motion of Julia sets exist? Do counter-examples exist for stable families?

(I) What can be said about the set of elements of $\mathcal{H}_d(\mathbb{P}^k)$ which belong to some one-dimensional holomorphic family on which the bifurcation locus has empty interior?

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