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Geodesics for meromorphic connections on Riemann surfaces and dynamics of homogeneous vector fields in several complex variables

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To my grandparents,
Luigia, Mario, Mariuccia and Elio

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Introduction

One of the main open problems in local dynamics of several complex variables is the understanding of the dynamics of holomorphic germs tangent to the identity in a full neighbourhood of the origin.

These are the germs of holomorphic endomorphisms of $\mathbb{C}^n$ fixing the origin and with differential there equal to the identity, and are the immediate multidimensional analogous of the parabolic maps in one complex variable, i.e., of maps of the form

$$f(z) = z + a_{\nu+1} z^{\nu+1} + \dots,$$

with $a_{\nu+1} \neq 0$. In this latter case, the iteration theory of our function is well established. In particular, the Leau-Fatou Flower Theorem gives a precise description of the local dynamics, asserting the presence of petal-shaped basins of attraction for the origin, and proving that the convergence to zero is possible only tangentially to some precise directions, one for each petal.

Moreover, using the description of the dynamics given by the Leau-Fatou Theorem, in 1978 Camacho proved the following Theorem:

Theorem 1 (Camacho, [Cam78]). *Let $f(z) = z + a_{\nu+1} z^{\nu+1} + \dots$, with $a_{\nu+1} \neq 0$, be a germ of holomorphic function tangent to the identity. Then f is locally topologically conjugated to the time-1 map of the homogeneous vector field*

$$Q = z^{\nu+1} \frac{\partial}{\partial z}$$

In particular this means that, from a topological point of view, time-1 maps of vector fields in $\mathbb{C}$ provide a complete list of model for the dynamics.

We remark that such a result was obtained via the dynamics, i.e., with a careful study of the dynamics of the map we wanted to study, the map tangent to the identity, and of the model we wanted to obtain, the time-1 map of the field.

When trying to pass from the one-dimensional complex dynamics to the multi-dimensional one, the maps tangent to the identity are among the first classes of maps that one can try to study, aiming to exploit the (formal) similarity with their well-understood one-dimensional counterpart.

A complete generalization of the Leau-Fatou Theorem in $\mathbb{C}^n$ is not known, yet, but there are a lot of partial results, due to Écalle, Hakim, Abate, Bracci, Tovena and others. One thing is sure: the Flower Theorem cannot be generalized in a trivial way (i.e., with a multi-dimensional flower figure and the associated directions of convergence) because there are examples of new phenomena arising in several complex variables.

So, instead of focusing on a direct generalization of the Flower Theorem, we may try another approach: it is reasonable to hope that Camacho's statement may be generalized to the several complex variables setting, at least for generic maps. If this were the case, understanding the dynamics of time-1 maps of vector fields would give a description of the topological dynamics of generic maps tangent to the identity in several complex variables, thus allowing to solve the topological problem by considering only a, particularly well-behaved, class of the maps we should study. Anyway, time-1 maps of homogeneous vector fields provide an important class of examples to study. In particular, a possible proof of the generalization of Camacho's Theorem to the several complex variables setting may pass through the very understanding of time-1 maps of vector fields.

All these reasons motivate us to the study of the dynamics of time-1 maps of holomorphic homogeneous vector fields. In particular, being interested to discrete orbits of the time-1 maps, we are naturally lead to consider the real integral curves for the field, and to study their asymptotic behaviour.

We immediately recognise that, given our homogeneous vector field Q, we have some particular sets of initial conditions for the integral curves for which the dynamical problem is particularly easy: these are the lines through the origin which are invariant for the field. We call the associated directions in $\mathbb{P}^{n-1}(\mathbb{C})$ the *characteristic directions* for Q, and *characteristic leaves* the associated complex lines through the origin. In particular, we see that an integral curve issuing from a point of a characteristic line will remain in that leaf for any time, and so the dynamics will be one-dimensional.

In particular, we see that if the vector field Q is a multiple of the radial field every direction is characteristic, and so the problem is completely one-dimensional. We call these fields *dicritical*, and *non-dicritical* otherwise. So, we shall be mainly interested in understanding the dynamics of integral curves outside the characteristic leaves of non-dicritical vector fields.

To do so, we need to use a fact from complex geometry: blowing up the origin of $\widehat{\mathbb{C}^n}$, we obtain the exceptional divisor $S \cong \mathbb{P}^{n-1}(\mathbb{C})$. We can consider the normal bundle $N_S^{\otimes\nu}$ of this exceptional divisor and note that there exists a (ν-to-1) holomorphic covering map $\chi_\nu : \mathbb{C}^n \setminus \{0\} \to N_S^{\otimes\nu} \setminus S$, which generalizes the usual biholomorphism betweeen $\widehat{\mathbb{C}^n} \setminus \{0\}$ and $N_S \setminus S$.

Using this map, we can push-forward our vector field Q to a holomorphic field G on $N_S^{\otimes\nu}$. This is possible thanks to the homogeneity of Q. In this way, we see that we have moved our problem of understanding integral curves for a field in $\mathbb{C}^n$ to the one of studying the integral curve for a field on a line bundle.

Our next step is to better understand G. And it is now that the *meromorphic connections*, together with the associated *geodesics*, come into play. The first are, roughly speaking, a way to differentiate meromorphic sections of a holomorphic vector bundle $p : E \to X$ over a Riemann surface X. We shall usually denote them with ∇.

In [ABT04], Abate, Bracci and Tovena showed that it is possible to associate to any non-dicritical homogeneous vector field Q a well-defined and global *canonical section* X_Q of the bundle $(N_S^*)^{\otimes\nu} \otimes TS$. By the canonical identification of the bundles, this corresponds to a morphism $X_Q : (N_S)^{\otimes\nu} \to TS$. In particular, being $N_S^{\otimes\nu}$ a line bundle, this morphism is an isomorphism outside the vanishing locus of X_Q, and it is easy to see that X_Q vanishes exactly on the characteristic directions. So, out of the characteristic directions, we can see the line bundle $N_S^{\otimes\nu}$ inside TS, thus defining a *singular holomorphic foliation* $\mathcal{F}$ on TS, whose leaves are Riemann surfaces.

Following Abate and Tovena, we see that it is possible to introduce a *partial* meromorphic connection ∇ on the bundle $N_S^{\otimes\nu} \to S$ such that G is exactly the geodesic field for ∇. By *partial meromorphic connection* we mean the following: ∇ does not allow to derivate sections of $N_E^{\otimes\nu}$ with respect to every section of TS, but only with respect to tangent vectors which are in the image of $N_E^{\otimes\nu}$ under X_Q. In particular, ∇ is a meromorphic connection when restricted to any leaf of the foliation $\mathcal{F}$ and we call *geodesics* for ∇ the curves which are geodesics for these induced connections, i.e., real curves $\sigma : I \to N_E^{\otimes\nu}$ such that $\nabla_{\sigma'(t)} X^{-1}(\sigma'(t)) \equiv 0$.

So, in order to understand the integral curves for our homogeneous field, we are left to study geodesics for a meromorphic connection on a line bundle. It is actually possible to simplify the problem again. Indeed, we prove that we can pushforward our partial connection ∇ to a partial connection ∇^0 on TS in a natural way: we define

$$\nabla_v^0(s) = X_Q\left(\nabla_{X_Q^{-1}v}\left(X_Q^{-1}s\right)\right)$$

where both v and s are tangent to the leaf. So, also ∇^0 is *partial*, in the sense that it does not allow us to derivate every section of TS, but only vector fields tangent to a leaf with respect to other vector fields tangent to the same leaf. In particular, a curve $\sigma : I \to S$ will be a geodesic for ∇^0 if it is contained in a leaf of the foliation and $\nabla^0_{\sigma'(t)}\sigma'(t) \equiv 0$.

It is clear from the definition of ∇^0 that there is a correspondence between geodesics for ∇^0 and geodesics for ∇ (i.e., integral curves for G), and so also with the integral curves for Q. In particular, we shall prove the following Theorem:

Theorem 2 (Abate, Tovena, [AT11])**.** *Let Q be a non-dicritical homogeneous vector field of degree $\nu + 1 \geq 2$ in $\mathbb{C}^n$ and let $\hat{S}_Q$ be the complement in $\mathbb{C}^n$*

of the characteristic leaves of Q. Then, for $\gamma : I \to \hat{S}_Q$, the following are equivalent:

1. *$\gamma \colon I \to \hat{S}_Q$ is an integral curve for Q in $\mathbb{C}^n$;*
2. *$\chi_\nu \circ \gamma$ is an integral curve for G (i.e., a geodesic for ∇) in $N_S^{\otimes \nu}$;*
3. *$[\gamma]$ is a geodesic for ∇^0 in a leaf of the foliation $\mathcal{F}$ of $\mathbb{P}^{n-1}(\mathbb{C})$.*

Thanks to this result, we see that, in order to study integral curves for our homogeneous vector field, we can study geodesics for meromorphic connections on the tangent bundle of a Riemann surfaces.

So, the study splits in two parts: understanding the foliation in surfaces and studying the asymptotic behaviour of the geodesics for the connection, which are contained in the leaves. In this thesis, we shall be mainly interested in the second one. To do so, we shall need two main ingredients: a characterization of the possible ω-limits for the geodesics on the tangent bundle of a Riemann surface (to understand the behaviour of the direction $[\gamma]$ of an integral curve) and and a way to understand where is $\gamma(t)$ once we know $[\gamma(t)]$ (for example, we would like to know when $\gamma(t)$ tends to the origin or diverges once we know that its direction tends to some $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$).

The first problem is mainly addressed in Chapter 3, with Theorems 3.4.8 (Theorem 4.6 in [AT11]) and 3.4.6. We completely classify all possible ω-limits for a geodesic for a meromorphic connection on the tangent bundle over a compact Riemann surface. This problem, apart from its intrinsic interest, is the first step to the understanding of the asymptotic dynamics of geodesics inside a leaf of the foliation on $\mathbb{P}^{n-1}(\mathbb{C})$ induced by the canonical morphism X_Q.

The second problem is studied in Section 4.2. Here the idea is to use the geodesic field G, for which we can write an explicit formula. If we use local coordinates (z, v) for the bundle $N_S^{\otimes \nu}$, we notice that the (modulus of) v is strictly related to the distance of $\gamma(t)$ to the origin in $\mathbb{C}^n$. So, the approach here is the following: we look for holomorphic local normal forms for the geodesic field near the singularities of the connection and solve the related differential equations. So, once we know the asymptotic behaviour of $[\gamma(t)]$ (and so of $z(t)$), this would gives us information about the asymptotic behaviour of $v(t)$, and so also of $\gamma(t)$. The theory for the singularities of lower order (0 or 1) is well established, thanks to the existence of holomorphic normal forms for the geodesic field. In the irregular case such a holomorphic classification is not known and this prevents to get a precise description in this case. Nonetheless, we are able to prove a partial result about geodesics converging to this kind of singular points.

There is a case in which all these methods are particularly efficient (and this is the reason we pay more attention to the problem of understanding the geodesics inside a leaf than to the one of understanding the foliation):

in $\mathbb{C}^2$, the exceptional divisor is the Riemann sphere, and so the foliation becomes trivial. So, being able to classify the ω-limits of geodesics of the sphere solves both parts of the problem, allowing to obtain a fairly complete description of the dynamics in this situation.

The thesis is divided in six chapters.

In Chapter 1 we collect the background material we shall later need about holomorphic connections on a Riemann surface and foliations on a manifold.

In Chapter 2 we introduce three different foliations on a line bundle on a Riemann surface, respectively of real rank 1, 2 and 3. They will be in some sense included one into the next, in the sense that every leaf of the rank-3 foliation is foliated by the leaves of the rank-2 one, and the same happens for the rank-2 and rank-1 foliations. The rank-1 foliation precisely corresponds to the geodesics, that we shall need in the following sections. Studying the other foliations provides very useful tools to understand the behaviour of this one.

In Chapter 3 we introduce the notion of meromorphic connection on a Riemann surface and study in detail the geodesics with respect to such a connection on the tangent bundle. In particular, we generalize the theory developed by Abate and Tovena for the Riemann sphere to a generic compact Riemann surface, and we classify all the possible ω-limits of geodesics, thus generalizing to this more general setting their previous result about the Riemann sphere. We end this chapter with a detailed study of the geodesics for holomorphic connections on a complex torus, and with an explicit description of the ω-limits in this case.

In Chapter 4 we turn to the original dynamical problem. We study the dynamics of a self-map of a complex manifold, constructing the bridge between the dynamical problem and the geodesics for meromorphic connections. Then, we study in detail the behaviour of the geodesics near a singular point for the connection.

In Chapter 5 we apply all the results obtained so far to the study of the integral curves of holomorphic homogeneous vector fields in $\mathbb{C}^n$, and develop the construction further. In the second part we concentrate on the case of $\mathbb{C}^2$, coming to quite a complete understanding of the situation for a significant class of homogeneous vector fields.

In Chapter 6 we give a holomorphic classification of cubic vector fields in $\mathbb{C}^2$, and, as an application of the theory developed so far, we study in detail the dynamics for the holomorphic representatives, in analogy with what was done by Abate and Tovena for the quadratic ones. This allows to obtain a pretty complete description of the possible phenomena arising in this situations, and to get concrete examples of maps showing a behaviour very different from their counterparts in one complex variable. Moreover, we are able to give concrete examples of singularities with higher irregularity, that could not be obtained in the quadratic case.

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Chapter 1

Preliminary results

In this introductory chapter we introduce the two main objects we shall deal with during all this thesis. These are the *holomorphic connections* on a Riemann surface and the concept of *foliation* of a manifold.

1.1 Holomorphic connections

Throughout this section we shall indicate with S a Riemann surface and with $p : E \to S$ a holomorphic vector bundle on it. $\mathcal{O}_S$ will be the structure sheaf of S, i.e., the sheaf of holomorphic function on S and $\mathcal{E}$ the (sheaf of germs of) holomorphic sections of E.

In classical differential geometry we can define a connection ∇ on the tangent bundle TS of S (and on its powers) as a map $\mathcal{T}_M \to \Omega^1_S \otimes \mathcal{T}_M$, where Ω^1_S is the sheaf of holomorphic 1-forms and $\mathcal{T}_M$ is the sheaf of the sections of the tangent bundle. We shall now introduce a connection on E, which will give us a way to differentiate sections of the general vector bundle E with respect to a given vector field X on S.

Definition 1.1.1. *A holomorphic connection on a holomorphic vector bundle $p : E \to S$ over a Riemann surface is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$ which satisfies the Leibniz rule*

$$\nabla(se) = ds \otimes e + s\nabla e$$

for all $s \in \mathcal{O}_S$ and $e \in \mathcal{E}$.

A horizontal section *of ∇ is a section $e \in \mathcal{E}$ such that $\nabla(e) = 0$.*

In particular, the difference between two holomorphic connections is a tensor.

Let us see what this definition means in local coordinates. First, we take a trivializing atlas for E, which will be of the form $\{(U_\alpha, z_\alpha; e^1_\alpha, \ldots, e^n_\alpha)\}$, with (U_α, z_α) local charts and $e_\alpha^1, \dots, e_\alpha^n$ holomorphic generators for E on U_α. On every U_α, we can find n^2 holomorphic 1-forms $(\eta_\alpha)_j^i$ such that

$$\nabla e_\alpha^i = \sum_j (\eta_\alpha)_j^i \otimes e_\alpha^j,$$

and we shall say that the forms $(\eta_\alpha)_j^i$ represent the connection ∇ on U_α. In fact we see that, for a general section $\overline{s}$ of E, we can locally compute $\nabla\overline{s}$ by representing $\overline{s}_{|U_\alpha}$ as $\sum_j s_\alpha^j e_\alpha^j$ for some holomorphic functions s_α^j on U_α: so

$$\begin{aligned}
\nabla\left(\sum_i s_\alpha^i e_\alpha^i\right) &= \sum_i \left(\nabla s_\alpha^i e_\alpha^i\right) \\
&= \sum_i \left(ds_\alpha^i \otimes e_\alpha^i + s_\alpha^i \nabla e_\alpha^i\right) \\
&= \sum_i \left(ds_\alpha^i \otimes e_\alpha^i + s_\alpha^i \sum_j \left((\eta_\alpha)_j^i \otimes e_\alpha^j\right)\right) \\
&= \sum_i \left(ds_\alpha^i + \sum_j \left(s_\alpha^j (\eta_\alpha)_j^i\right)\right) \otimes e_\alpha^i
\end{aligned}$$

In particular, with the above computation we have proved the following result.

Proposition 1.1.2. *Let E be a trivial line bundle, i.e. $E \cong S \times \mathbb{C}$, and fix a global generator e for E. Let ∇ be a holomorphic connection on E. Then there exists a global $\eta \in \Omega_S^1$ such that*

$$\nabla(se) = (ds + \eta) \otimes e$$

for all $s \in \mathcal{O}_S$.

In the sequel, we shall mainly interested in the case in which E is a line bundle, i.e., a holomorphic vector bundle of rank 1.

We see that in this case we have an atlas $\{(U_\alpha, z_\alpha, e_\alpha)\}$ trivializing E, and in particular we only need a single holomorphic 1-form η_α on every U_α to represent ∇, given by

$$\nabla e_\alpha = \eta_\alpha \otimes e_\alpha$$

So, for a section s, which is locally $s_\alpha e_\alpha$ for some $s_\alpha \in \mathcal{O}(U_\alpha)$, we have

$$\nabla(s_\alpha e_\alpha) = (ds_\alpha + \eta_\alpha) \otimes e_\alpha$$

In the next Chapter we are going to study in more detail holomorphic connections on a line bundle and we shall need some little more background, that we are going to recall now.

Consider our line bundle $p\colon E \to S$ and the trivializing cover $\{(U_\alpha, z_\alpha, e_\alpha)\}$. We know that, on every non empty intersection $U_\alpha \cap U_\beta$, we have a (non-vanishing) holomorphic function $\xi_{\alpha\beta}$ satisfying

$$e_\beta = \xi_{\alpha\beta} e_\alpha. \tag{1.1}$$

In particular, if v_α represents the coordinate in E with respect to e_α, we have that $\xi_{\alpha\beta}$ gives the change of this coordinates on the intersection, by the rule

$$v_\alpha = \xi_{\alpha\beta} v_\beta.$$

It is clear that this set of functions $\xi_{\alpha\beta}$ must satisfy the rule

$$\xi_{\alpha\beta}\xi_{\beta\gamma}\xi_{\gamma\alpha} = 1$$

on every non-empty triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ and, obviously, we have $\xi_{\alpha\alpha} = 1$. This means that the set $\{\xi_{\alpha\beta}\}$ defines an element, that we shall call ξ, in $H^1(S, \mathcal{O}^*)$. We say that ξ *represents* the line bundle E

We have the following useful change rule between two 1-forms representing a connection ∇ on two overlapping charts.

Lemma 1.1.3. *Let $p : E \to S$ be a line bundle over a Riemann surface S, trivialized by the cover $\{(U_\alpha, z_\alpha, e_\alpha)\}$, and let ∇ be a holomorphic connection on E, represented by the 1-form η_α on U_α. Let $\xi_{\alpha\beta}$ be given by* (1.1). *Then, on the intersection of two overlapping charts U_α and U_β we have*

$$\eta_\beta = \eta_\alpha + \frac{\partial \xi_{\alpha\beta}}{\xi_{\alpha\beta}} \tag{1.2}$$

Proof. From (1.1), we get

$$\nabla(e_\beta) = \eta_\beta \otimes e_\beta = \eta_\beta \xi_{\alpha\beta} \otimes e_\alpha \tag{1.3}$$

and

$$\nabla(e_\beta) = \nabla(\xi_{\alpha\beta} e_\alpha) = d\xi_{\alpha\beta} \otimes e_\alpha + \xi_{\alpha\beta}\eta_\alpha \otimes e_\alpha \tag{1.4}$$

Comparing (1.3) and (1.4) we get

$$(\xi_{\alpha\beta}\eta_\beta) \otimes e_\alpha = (\xi_{\alpha\beta}\eta_\alpha + d\xi_{\alpha\beta}) \otimes e_\alpha$$

and the assertion follows dividing by $\xi_{\alpha\beta}$ (it is never vanishing) and noting that $d\xi_{\alpha\beta} = \partial\xi_{\alpha\beta}$ because $\xi_{\alpha\beta}$ is holomorphic. □

Viceversa, we see that a set $\{\eta_\alpha\}$ of forms that satisfy (1.2) actually represent a holomorphic connection.

With the above Lemma we can find a condition on ξ that can ensure the existence of a holomorphic connection on E.

Proposition 1.1.4. *Let $p : E \to S$ be a line bundle over a Riemann surface and $\xi \in H^1(E, \mathcal{O}^*)$ the cohomology class representing E. Then there exists a holomorphic connection on E if and only if ξ vanishes in cohomology under the map $\partial \log : H^1(E, \mathcal{O}^*) \to H^1(S, \Omega^1_S)$, if and only if ξ can be seen as an element of $H^1(S, \mathbb{C}^*)$.*

Proof. If there exists a holomorphic connection, locally represented by η_α, equation (1.2) precisely says that $\partial \log(\xi_{\alpha\beta}) = \frac{\partial \xi_{\alpha\beta}}{\xi_{\alpha\beta}}$ is a coboundary. Conversely, if $\partial \log(\xi_{\alpha\beta}) = 0$ in $H^1(S, \Omega^1_S)$, i.e., it is the coboundary of some $\eta \in H^0(S, \Omega^1_S)$, we can use these η_α as representatives for a holomorphic connection on E. This proves the first equivalence. The second follows from the short exact sequence of sheaves

$$0 \to \mathbb{C}^* \to \mathcal{O}^* \to \Omega^1_S \to 0,$$

in which the third arrow represents the maps $\partial(\log(\cdot))$. We see that the vanishing of ξ under $\partial(\log(\cdot))$ is exactly equivalent to ξ being the image of a class $\hat{\xi} \in H^1(S, \mathbb{C}^*)$ under the map induced by the inclusion $\mathbb{C}^* \hookrightarrow \mathcal{O}^*$. □

It is actually possible to explicitly find the representative $\hat{\xi} \in H^1(S, \mathbb{C}^*)$ of ξ, if we have a connection ∇ on E. In fact, we consider the 1-forms η_α representing ∇ on U_α and (possibly shrinking the open sets of the cover) we find local primitives for η_α, i.e., holomorphic functions $K_\alpha \in \mathcal{O}(K_\alpha)$ such that $\partial K_\alpha = \eta_\alpha$ on U_α. Integrating (1.2) we get

$$K_\beta = K_\alpha + \log(\xi_{\alpha\beta}) + c_{\alpha\beta}$$

for some constant function $c_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Taking an exponential we get

$$\exp(K_\beta) = \exp(K_\alpha)\xi_{\alpha\beta}e^{c_{\alpha\beta}}$$

and this means that we can consider the cocycle

$$\hat{\xi}_{\alpha\beta} = \frac{1}{e^{c_{\alpha\beta}}} = \frac{\exp(K_\alpha)}{\exp(K_\beta)}\xi_{\alpha\beta}. \tag{1.5}$$

It is indeed immediate to verify that $\xi_{\alpha\beta}$ satisfies the (multiplicative) cocycle relations, so that it gives an element of $H^1(S, \mathbb{C}^*)$ (it is non-zero, being an exponential).

Moreover, we see that it actually represents ξ. In fact, we must prove that $\left[\frac{\hat{\xi}}{\xi}\right] = 0$ in $H^1(S, \mathcal{O}^*)$. By definition, this means that there must exist functions g_α on U_α such that, on every intersection $U_\alpha \cap U_\beta$, we have $\frac{g_\alpha}{g_\beta} = \frac{\hat{\xi}_{\alpha\beta}}{\xi_{\alpha\beta}}$. But this is indeed true. In fact, it suffices to consider $g_\alpha = \exp(K_\alpha)$ and we are done.

We end this section with a proposition and a definition, that we shall need in the sequel. The following Proposition gives a useful characterization of horizontal sections.

Proposition 1.1.5. *Let $p : E \to S$ be a line bundle over a Riemann surface, trivialized by a cover $\{U_\alpha, z_\alpha, e_\alpha\}$, and ∇ a holomorphic connection on E, locally represented by η_α. Then,*

$$\nabla\left(\exp(-K_\alpha)e_\alpha\right) \equiv 0,$$

which means that $\exp(-K_\alpha)e_\alpha$ is a horizontal section on U_α. In particular, it means that given a point $z_0 \in U_\alpha$ and an element $v_0(e_\alpha)_{|z_0} \in E_{z_0}$, locally the solution of $\nabla(s) \equiv 0$ with $s(0) = (z_0, v_0)$ satisfies

$$\exp(K_\alpha(z))v \equiv \exp(K_\alpha(z_0))v_0.$$

Proof. It follows from a standard computation. In fact,

$$\begin{aligned}\nabla\left(\exp(-K_\alpha)e_\alpha\right) &= d\exp(-K_\alpha) \otimes e_\alpha + \exp(-K_\alpha)\eta_\alpha \otimes e_\alpha \\ &= -\exp(-K_\alpha)\eta_a \otimes e_\alpha + +\exp(-K_\alpha)\eta_\alpha \otimes e_\alpha \equiv 0.\end{aligned}$$

□

Definition 1.1.6. *The homomorphism $\rho : \pi_1(S) \to \mathbb{C}^*$ corresponding to the class $\hat{\xi}$ under the canonical isomorphism $H^1(S, \mathbb{C}^*) \cong Hom(H_1(S, \mathbb{Z}), \mathbb{C}^*) = Hom(\pi_1(S), \mathbb{C}^*)$ is the* monodromy representation *of the holomorphic connection ∇.*

In particular, if $\pi : \widetilde{S} \to S$ is the universal cover of S, we can identify $\pi_1(S)$ with $\mathrm{Aut}(\pi)$. In this case, we shall indicate with $\rho(\pi) \subseteq \mathbb{C}^*$ the image of $\mathrm{Aut}(\pi) \sim \pi_1(S)$ under ρ and with $|\rho|(\pi)$ the image of $\mathrm{Aut}(\pi)$ under $|\rho|$.

We say that a connection ∇ has real periods *if the image of ρ is contained in S^1, i.e., if the class ξ is actually contained in $H^1(S, S^1)$.In particular, ∇ has real periods if and only if $|\rho|(\pi) = 1$.*

In the next chapter we are going to give a geometrical interpretation of this last definition.

1.2 Foliations

In this section we shall give the main definitions concerning the theory of foliations on a manifold. Since we shall also need real foliations, we shall give all the definitions and properties in the real setting, but it will be clear how to generalize them to the complex one.

Giving a (regular k-dimensional) foliation on a (real) manifold S means, speaking very roughly, subdividing it in a continuum of k-dimensional submanifolds, that will be called the *leaves* of the foliation. This is easily seen to be equivalent to specifying, for every $p \in S$, a k-plane in the fiber of TS over p, which varies smoothly with p. This second construction is known as a *distribution* of k-planes. In the particular case of $k = 1$, the name *line* *field* is also common. We see that a line field is a generalization of a (non singular) vector field, because we only consider the direction of a line in TS_p instead of a direction and, say, a modulus, that would give a specific point in the fiber. In particular, we see that every non singular vector field defines a line field on the manifold S where it is defined, and so a foliation of S.

If we just consider the field without the assumption of being non singular, we come up with an object that is called *singular 1-dimensional foliation*.

We see that every singular foliation induced by a field may be viewed as a regular one on the complement of the singular points of the field.

After giving an idea of what a foliation on a manifold is, we shall now give a proper definition of it.

We shall define a (regular) foliation by saying what it is locally. In particular, the key definition is the following

Definition 1.2.1. *Consider the $n-$dimensional cube $I^n = [0,1]^n = \{(x,y) \in I^k \times I^{n-k}\}$ (or polydisc Δ^n in the complex setting). The* standard foliation *of* dimension*, or* rank*, k (and* codimension *$n-k$) on I^n is the representation of I^n as the disjoint union of k-cubes, called* standard plaques*,*

$$I^n = \bigsqcup_{y \in I^{n-k}} L_y,$$

with $L_y = I^k \times \{y\} \subset I^n$.

Definition 1.2.2. *A* (regular) foliation *of rank k of an n-dimensional manifold S is a partition $S = \bigsqcup L_\alpha$ of the manifold into the disjoint union of connected immersed submanifolds L_α, called leaves, which is locally diffeomorphic to the standard foliation, i.e., such that for every point $p \in S$ there exists a neighbourhood U of p and a C^∞ diffeomorphism $\varphi : U \to I^n$ such that the image of any connected component of $L_\alpha \cap U$ is a standard plaque for the standard foliation in I^n for every α, and conversely.*

The one above is only one possible definition of a foliation. In fact, there are several different definitions or equivalent characterizations. We are going to present here two of them, that we shall need later in the exposition. They are in some sense dual to each other: the idea is that we can think of a k-foliation as an *integrable distribution* of k-plane. By this we mean that, for every point, we assign a k-plane in the fiber of the tangent bundle at that point, such that all these k-planes change smoothly in T_pS as the point p moves. A possible way to let this idea become precise is to ask for a k-dimensional subbundle F of TS. The integrability condition, which is known to be equivalent to $[F,F]_p \subset F_p$ for every $p \in S$, ensures that we can think to the k-plane in each point as tangent to a k-submanifold (that, not surprisingly, will be the leaf of the foliation passing through p). In particular, what we are asking for is that any leaf is an integral leaf for this distribution of planes, a generalization of being an integral curve for a smooth vector field.

In fact, one of the equivalent definition we present is precisely that in any point we want the tangent plane to the leaf to be generated by k fixed non-singular vector fields, independent at every point. The dual characterization is obtained by giving $n-k$ global non-singular smooth forms, again independent at every point, and ask that their common k-dimensional kernel is precisely the k-plane tangent to the leaf.

In the next Theorem we collect these equivalent definition of a foliation (see [IY08] and [War83]).

Theorem 1.2.3. *Giving a (regular) foliation of rank k on a smooth manifold S is equivalent to any of the following:*

- *a collection of $(U_\alpha; F_1^\alpha, \dots F_k^\alpha)$, where $\{U_\alpha\}$ is an open covering of S and, for every α, $(F_1^\alpha, \dots, F_k^\alpha)$ is a $k-$uple of smooth vector fields, linearly independent at every point $p \in U_\alpha$, and hence non vanishing in any point, such that*
  - *if F^α indicates the span of $(F_1^\alpha, \dots, F_k^\alpha)$, we have $[F^\alpha, F^\alpha]_p \subset F_p^\alpha$ for every p, i.e., for every pair $\left(F_i^\alpha, F_j^\alpha\right)$ there exist $c_{ij}^{\alpha,l} \in C^\infty(U_\alpha)$ such that*
  $$[F_i^\alpha, F_j^\alpha] = \sum_{l=1}^{k} c_{ij}^{\alpha,l} F_l^\alpha;$$
  - *if $U_\alpha \cap U_\beta \neq \emptyset$, then there exist invertible $g_i^{\alpha\beta,j} \in C^\infty(U_\alpha \cap U_\beta)$ such that*
  $$F_i^\alpha = \sum_{j=1}^{k} g_i^{\alpha\beta,j} F_j^\beta;$$
- *a collection of $\left(U_\alpha; \omega_1^\alpha, \dots \omega_{n-k}^\alpha\right)$, where $\{U_\alpha\}$ is an open covering of S and, for every α, $(\omega_1^\alpha, \dots, \omega_{n-k}^\alpha)$ is an $n-k$-uple of smooth 1-forms, linearly independent at every point (i.e., such that $\omega_1^\alpha \wedge \dots \wedge \omega_{n-k}^\alpha$ is never vanishing), and hence non vanishing in any point, and such that*
  - *the ideal spanned by $\omega_1^\alpha, \dots, \omega_{n-k}^\alpha$ is closed in the exterior algebra of forms, i.e., if for every ω_i^α there exist $\eta_j^\alpha \in \Omega_S^1$ and $c^{\alpha,ij} \in C^\infty(S)$ such that*
  $$d\omega_i^\alpha = \sum_{j=1}^{n-k} c^{\alpha,ij} \omega_j^\alpha \wedge \eta_j^\alpha;$$
  - *if $U_\alpha \cap U_\beta \neq \emptyset$, then there exist invertible $h^{\alpha\beta,ij} \in C^\infty(S)$ such that*
  $$\omega_i^\alpha = \sum_{j=1}^{k} h^{\alpha\beta,ij} \omega_j^\beta.$$

We remark again that all we have said can be repeated also in the complex setting, with holomorphic vectors, forms and transition functions c, g and h.

We now define singular foliations. By a *singular k-dimensional foliation* on a manifold S we mean a regular foliation on the complement of a one-codimensional set. More precisely:

Definition 1.2.4. *A* singular foliation *of* rank k *of an n-dimensional manifold S is a pair $\mathcal{F} = (\mathcal{F}', \Sigma)$, where Σ (the* singular set*) is a union of one-codimensional submanifolds (or proper analytic sets in the complex setting) of S and $\mathcal{F}'$ is a regular k-foliation on $S \setminus \Sigma$. We will say that the foliation is* saturated *if it cannot be extended to a foliation of a larger open set.*

As for regular foliations, we have a number of different characterizations for singular foliations. We collect some of them in the following Theorem.

Theorem 1.2.5. *Giving a singular foliation of rank k on a smooth manifold S is equivalent to any of the following:*

- *a collection of $(U_\alpha; F_1^\alpha, \dots F_k^\alpha)$, where $\{U_\alpha\}$ is an open covering of S and, for every α, $(F_1^\alpha, \dots, F_k^\alpha)$ is a $k-$uple of smooth vector fields, such that*
  - *if F^α indicates the span of $(F_1^\alpha, \dots, F_k^\alpha)$, we have $[F^\alpha, F^\alpha]_p \subset F_p^\alpha$ for every p, i.e., for every pair $\left(F_i^\alpha, F_j^\alpha\right)$ there exist $c_{ij}^{\alpha,l} \in C^\infty(U_\alpha)$ such that*
    $$[F_i^\alpha, F_j^\alpha] = \sum_{l=1}^{k} c_{ij}^{\alpha,l} F_l^\alpha;$$
  - *if $U_\alpha \cap U_\beta \neq \emptyset$, we have $F^\alpha = F^\beta$ for every point in $U_\alpha \cap U_\beta$;*
- *a collection of $\left(U_\alpha; \omega_1^\alpha, \dots \omega_{n-k}^\alpha\right)$, where $\{U_\alpha\}$ is an open covering of S and, for every α, $(\omega_1^\alpha, \dots, \omega_{n-k}^\alpha)$ is an $n-k$-uple of smooth 1-forms such that*
  - *the ideal ω^α spanned by $\omega_1^\alpha, \dots, \omega_{n-k}^\alpha$ is closed in the exterior algebra of forms, i.e., if for every ω_i^α there exist $\eta_j^\alpha \in \Omega_S^1$ and $c^{\alpha,ij} \in C^\infty(S)$ such that*
    $$d\omega_i^\alpha = \sum_{j=1}^{n-k} c^{\alpha,ij} \omega_j^\alpha \wedge \eta_j^\alpha;$$
  - *if $U_\alpha \cap U_\beta \neq \emptyset$, we have $\omega^\alpha \cap \omega^\beta$ for every point in if $U_\alpha \cap U_\beta$.*

Chapter 2

Three foliations on a line bundle

In this chapter, following [AT11] we shall introduce three foliations, respectively of rank $1, 2$ and 3, on the total space of a line bundle $p : E \to S$ over a Riemann surface S, which can be seen as a manifold of real dimension 4. We shall see that each foliation foliates the larger one, i.e., that the leaves of the rank 1 foliation are contained in those of the rank 2 foliation, and the same happens for the rank 2 with respect to the rank 3. These foliations will turn out to be extremely useful for studying our dynamical problems.

2.1 Definitions and first properties

Roughly speaking, the idea for defining the larger foliation will be to put local metrics on the total space of the bundle $p : E \to S$ and to consider the foliation locally given by the points having the same norm, so obtaining a foliation of rank 3. One of the problems will be to study if two of these local leaves can be part of a unique global leaf.

To define this foliation, we begin by studying the following problem: from the classical differential geometry we know how to associate a connection to a metric on a (real) manifold, which turns out to be related to the ideas of parallel transport and geodesics with respect to the given metric, and also a foliation given by the level sets of this global metric.

Here we would like to do the converse: we have a connection and we would like to use it to define a metric, and so a foliation. We shall see that in general we cannot do so, in the sense that there does not always exist a global metric that we would call *compatible* with our connection (see Definition 2.1.3). What we can find is only a family of local metrics. The crucial point will be the fact that this family is actually a *conformal family of local metrics*, i.e., a local family of metrics differing only by multiplication of positive real functions. So, the leaves for the foliation of any two metrics in this family will be the same, thus allowing to define a global foliation. Not being induced by a global metric, the geometry of this foliation will be much richer than the usual one induced by a global metric.

In order to undertake this program, we recall the concept of metric on a general bundle, different from TS, and then we study the possible relations between this metric and the original connection.

So, we give the following definition.

Definition 2.1.1. *Let S be a complex manifold and $p : E \to S$ a holomorphic vector bundle. A* Hermitian metric *g on E is the assignment of a Hermitian inner product $g_p(\cdot,\cdot)$ (also written as $\langle\cdot,\cdot\rangle_p$) on each fiber E_p of the bundle, such that for any open set $U \in S$ and any R, T smooth sections of E, the function*

$$\langle R, T\rangle : U \to \mathbb{C}$$

$$x \mapsto \langle R(x), T(x)\rangle_x$$

is smooth. We shall call a bundle with an Hermitian metric an Hermitian bundle*.*

We remark that the previous definition is well posed also in the case of a complex bundle over a differentiable manifold. We have not stated it in this case because we shall not need this greater generality in the sequel.

Remark 2.1.2. *We shall use the convention for which the Hermitian product is $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second.*

The next definition specifies what we mean when we ask for a compatibility between a metric g and a connection ∇.

Definition 2.1.3. *Let $p : E \to S$ a complex Hermitian bundle, with metric g. We say that a connection ∇ on E is* adapted*, or* compatible*, with g, and we shall write $\nabla g \equiv 0$, if*

$$d\langle R, T\rangle = \langle \nabla R, T\rangle + \langle R, \overline{\nabla} T\rangle$$

for every smooth sections R, T, that means

$$X(\langle R, T\rangle) = \langle \nabla_X R, T\rangle + \langle R, \nabla_{\overline{X}} T\rangle$$

for every smooth vector field X on S.

It is known that, given an Hermitian metric g on a complex bundle, it is possible to associate a connection which is compatible with g, and it is unique if we ask for some additional conditions. This is the so-called Chern connection.

What we want to do now is studying the opposite question: given a connection on a complex bundle, when is it possible to find an Hermitian metric compatible with it?

To answer this question, we shall restrict to the case we shall be most interested in later: so, until the end of this section, $p : E \to S$ will be a line bundle over S. As a first thing, we can give a precise characterization of what the condition of compatibility means in local coordinates. Given a trivializing atlas $\{U_\alpha, z_\alpha, e_\alpha\}$ for $p : E \to S$, denote by η_α the 1-form representing ∇ on U_α. We can also consider the real function $n_\alpha : U_\alpha \to \mathbb{R}^+$, given by

$$n_\alpha(p) = g_p(e_\alpha, e_\alpha).$$

It is straightforward to see that n_α is a smooth function and, conversely, we see that the function n_α uniquely characterizes the metric on U_α. The following Lemma gives a characterization of the compatibility in terms of n_α.

Lemma 2.1.4. *The metric g is adapted to ∇ if and only if*

$$\partial n_\alpha = n_\alpha \eta_\alpha \tag{2.1}$$

Proof. By definition, the fact that g is adapted to ∇ means that $X\left(g(R,T)\right) = g\left(\nabla_X R, T)\right) + g\left(R, \nabla_{\overline{X}} T\right)$ for every sections R and T of E and any vector field X on S. Being E a line bundle, we locally have, in the trivializing chart $(U_\alpha, z_\alpha, e_\alpha)$, that $R = r_\alpha e_\alpha$ and $T = t_\alpha e_\alpha$, so that we have

$$X\left(g(R,T)\right) = X(r_\alpha \overline{t_\alpha} n_\alpha).$$

So, the condition on g becomes

$$\begin{aligned} dr_\alpha(X)\overline{t_\alpha} n_\alpha + & r_\alpha d\overline{t_\alpha}(X) n_\alpha + r_\alpha \overline{t_\alpha} dn_\alpha(X) = \\ & = (dr_\alpha + r_\alpha \eta_\alpha)(X) n_\alpha \overline{t_\alpha} + r_\alpha n_\alpha \overline{(dt_\alpha + t_\alpha \eta_\alpha)(\overline{X})}, \end{aligned}$$

that, after some cancellations, gives (recalling that $d = \partial + \overline{\partial}$)

$$\partial n_\alpha + \overline{\partial} n_\alpha = n_\alpha \eta_\alpha + n_\alpha \overline{\eta_\alpha}. \tag{2.2}$$

By reasons of type, it is equivalent to $\partial n_\alpha = n_\alpha \eta_\alpha$ (and $(\overline{\partial} n_\alpha = n_\alpha \overline{\eta_\alpha})$). $\square$

Remark 2.1.5. *It is not difficult to find an analogous condition for a vector bundle of rank greater than 1. In this case, we locally have the matrix $A_\alpha = \left(\eta_\alpha^{ij}\right)$ representing the connection, and we get an analogous of the n_α defining a matrix $G = (g_{ij})$, where $g_{ij} = \langle e_\alpha^i, e_\alpha^j \rangle$. The condition in this case is*

$$dG = A^t G + G\overline{A}.$$

We only remark that this condition actually reduces to the (2.1) *in the case of a line bundle (in fact, it is exactly equal to* (2.2) *in that case).*

An important thing about the equation (2.1) is that it can actually be solved, as proved in the next proposition.

Proposition 2.1.6. *Let E be a complex line bundle on a Riemann surface S and let $\nabla : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$ be a holomorphic connection on E. Let $(U_\alpha, z_\alpha, e_\alpha)$ be a local trivializing chart for E and let $\eta \in \Omega^1_S(U_\alpha)$ such that $\nabla e_\alpha = \eta_\alpha \otimes e_\alpha$. Assume we have a holomorphic primitive K_α of η_α on U_α. Then*

$$n_\alpha = \exp(2 \operatorname{Re} K_\alpha) = \exp(K_\alpha + \overline{K_\alpha})$$

is a positive solution for $\partial n_\alpha = n_\alpha \eta_\alpha$. Conversely, if n_α is a positive solution of (2.1), then for any $z_0 \in U_\alpha$ and simply connected neighbourhood $U \subset U_\alpha$ of z_0 there exists a holomorphic primitive $K_\alpha \in \mathcal{O}(U)$ of η_α over U such that $n_\alpha = \exp(2 \operatorname{Re} K_\alpha)$ in U. Furthermore, K_α is unique up to a purely imaginary additive constant. Finally, two (local) solutions of (2.1) differ (locally) by a positive multiplicative constant.

Proof. The fact that $\exp(2 \operatorname{Re} K_\alpha)$ is a solution of (2.1) is a straightforward calculation: in fact we have

$$\partial(\exp(K_\alpha + \overline{K_\alpha})) = \partial(K_\alpha + \overline{K_\alpha}) \exp(K_\alpha + \overline{K_\alpha}) = \eta_\alpha \exp(K_\alpha + \overline{K_\alpha}),$$

because $\partial K_\alpha = \eta_\alpha$ and $\partial \overline{K_\alpha} = 0$, and this proves the statement.

Then, if n_α is a positive solution of (2.1), locally we can define $\operatorname{Re} K_\alpha$ as $\frac{\log n_\alpha}{2}$, which is harmonic because

$$\overline{\partial}\partial \log n_\alpha = \overline{\partial} \frac{\partial n_\alpha}{n_\alpha} = \overline{\partial} \eta_\alpha = 0,$$

and $\operatorname{Im} K_\alpha$ in such a way that K_α becomes holomorphic. Clearly, we have $n_\alpha = \exp(2 \operatorname{Re} K_\alpha)$. Let's see that $\partial K_\alpha = \eta_\alpha$. Being K_α holomorphic, we know that $\partial K_\alpha = 2\partial \operatorname{Re} K_\alpha$, which is equal to $\partial(\log n_\alpha) = \frac{\partial n_\alpha}{n_\alpha}$, and this is η_α because n_α solves (2.1).

Suppose now that we have K_α and $\widetilde{K}_\alpha$ satisfying the statement. We want to prove that their difference is a purely imaginary complex number. First, we know that it is a holomorphic function, as both K_α and $\widetilde{K}_\alpha$ are. Then, from $\partial \overline{K_\alpha} = \partial \overline{\widetilde{K}_\alpha} (= \eta_\alpha)$ we get that the difference is also antiholomorphic, so that it must be a constant c. But if $\operatorname{Re} c \neq 0$ we would have $n_\alpha = \exp(2 \operatorname{Re} K_\alpha) \neq \exp(2 \operatorname{Re} \widetilde{K}_\alpha) = n_\alpha$, which is impossible.

Finally, let n_α and $\widetilde{n}_\alpha$ be two local solutions of (2.1). We have $\partial \log n_\alpha = \eta_\alpha = \partial \log \widetilde{n}_\alpha$. Given a (local) holomophic primitive K_α for η_α we obtain

$$n_\alpha c_1 = \exp(2 \operatorname{Re} K_\alpha) = \widetilde{n}_\alpha c_2,$$

so that $\widetilde{n}_\alpha = c n_\alpha$ for some constant c, that must be real and positive because both n_α and $\widetilde{n}_\alpha$ are. □

So we see that locally the question is always answered in a positive way, and moreover we can also give an explicit formula for the metric. What we are going to do now is to study the global problem, i.e., to state and prove a condition for the existence of a global compatible metric. This in particular gives a geometrical interpretation of Definition 1.1.6.

Proposition 2.1.7. *Let $p : E \to S$ be a complex line bundle on a Riemann surface S, and ∇ a holomorphic connection on it. Then there exists a Hermitian metric compatible with ∇ if and only if ∇ has real periods.*

Proof. By (1.5) we have

$$\operatorname{Re}(K_\alpha - K_\beta) + \log(|\xi_{\alpha\beta}|) = \log|\hat{\xi}_{\alpha\beta}|, \tag{2.3}$$

where K_α and K_β are primitives of η_α and η_β on U_α and U_β.

Suppose we have a metric g compatible with ∇. In particular, it defines functions $n_\alpha = g(e_\alpha, e_\alpha)$ on every U_α. By the change rule (1.1), we must have

$$n_\beta = |\xi_{\alpha\beta}|^2 n_\alpha \tag{2.4}$$

on every non empty intersection. By Proposition 2.1.6, we know that every such n_α must be of the form $n_\alpha = \exp(2\operatorname{Re} K_\alpha)$. Taking a logarithm and dividing, (2.4) gives

$$\operatorname{Re}(K_\beta - K_\alpha) = \log|\xi_{\alpha\beta}|$$

and so, comparing with (2.3), we get $\log|\hat{\xi}_{\alpha\beta}| = 0$, that means that $|\hat{\xi}_{\alpha\beta}| = 1$, i.e., $\hat{\xi}_{\alpha\beta} \in S^1$.

Conversely, if ∇ has real periods, what we have is that there exist constants $c_\alpha \in \mathbb{C}^*$ such that

$$\hat{\xi}_{\alpha\beta} = \frac{c_\beta}{c_\alpha}\widetilde{\xi}_{\alpha\beta} \tag{2.5}$$

with $\widetilde{\xi}_{\alpha\beta} \in S^1$ and what we want to prove is that there exist functions n_α satisfying (2.4), i.e., functions $\widetilde{K}_\alpha$, primitives of η_α, such that every $n_\alpha = \exp(2\operatorname{Re}\widetilde{K}_\alpha)$ satisfies (2.4). Substituting (2.5) in (2.3) we get

$$\operatorname{Re}(K_\beta - K_\alpha) - \log|\xi_{\alpha\beta}| = \log|c_\alpha| - \log|c_\beta|$$

and we see that we can use $\widetilde{K}_\alpha = K_\alpha + \log|c_\alpha|$. $\square$

We are aiming at defining some foliations on E, which are some global objects, starting from local definitions on the U_α's. So, we shall need to know the change rules for the coordinates of the bundle E (and for its tangent bundle TE, when we shall look for a description in terms of vector fields). This is the content of the next Lemma.

Lemma 2.1.8. *Let $p\colon E \to S$ be a line bundle on a Riemann surface S, locally trivialized by a cover $\{(U_\alpha, z_\alpha, e_\alpha)\}$, where e_α is a generator for E on U_α. We denote by (z_α, v_α) the induced local coordinates for E and consider the induced local frame $\left(\partial_\alpha := \frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial v_\alpha}\right)$ and the local coframe $(p^*(dz_\alpha), dv_\alpha)$. Then, in the intersection between two charts U_α and U_β we have*

$$\begin{pmatrix} p^*(dz_\alpha) \\ dv_\alpha \end{pmatrix} = \begin{pmatrix} \frac{\partial z_\alpha}{\partial z_\beta} \circ p & 0 \\ v_\beta \left(\frac{\xi_{\alpha\beta}}{\partial z_\beta} \circ p\right) & (\xi_{\alpha\beta} \circ p) \end{pmatrix} \begin{pmatrix} p^*(dz_\beta) \\ dv_\beta \end{pmatrix} \tag{2.6}$$

and

$$\begin{pmatrix} \partial_\alpha \\ \partial/\partial v_\alpha \end{pmatrix} = \begin{pmatrix} \frac{\partial z_\beta}{\partial z_\alpha} & -v_\beta\left(\left(\frac{\partial z_\beta}{\partial z_\alpha}\frac{\partial \xi_{\alpha\beta}}{\partial z_\beta}\frac{1}{\xi_{\alpha\beta}}\right)\circ p\right) \\ 0 & (\xi_{\beta\alpha}\circ p) \end{pmatrix} \begin{pmatrix} \partial_\beta \\ \partial/\partial v_\beta \end{pmatrix}. \tag{2.7}$$

Proof. We find the first matrix, the second being the transposed of the inverse of the first.

It is clear that $p^*(dz_\beta) = \left(\frac{\partial z_\beta}{\partial z_\alpha}\circ p\right) p^* dz_\beta$, so we only have to express dv_α in terms of $p^*(dz_\beta)$ and dv_β. To do this, recall that $e_\beta = \xi_{\alpha\beta} e_\alpha$, so that

$$v_a = (\xi_{\alpha\beta}\circ p)\, v_\beta. \tag{2.8}$$

So,

$$dv_\alpha = d\left((\xi_{\alpha\beta}\circ p)\, v_\beta\right) = v_\beta d\left(\xi_{\alpha\beta}\circ p\right) + (\xi_{\alpha\beta}\circ p)\, dv_\beta,$$

which gives

$$dv_\alpha = v_\beta\left(\frac{\partial \xi_{\alpha\beta}}{\partial z_\beta}\circ p\right) p^*(dz_\beta) + (\xi_{\alpha\beta}\circ p)\, dv_\beta,$$

where we used the fact that $d\left(\xi_{\alpha\beta}\circ p\right) = \left(\frac{\partial \xi_{\alpha\beta}}{\partial z_\beta}\circ p\right) p^*(dz_\beta)$, which follows from that fact that $\xi_{\alpha\beta}$ is holomorphic. □

We can now start defining our foliations. The idea for the higher-rank one, that we shall call *metric foliation* is the following: suppose we have a global metric adapted to ∇. Then, for every $c \in \mathbb{R}$ we can consider the points v in every fiber of E with $g(v, v) = c$. For a fixed $c \neq 0$, we see that in every fiber the points satisfying this condition form a set diffeomorphic to S^1 and it is not difficult to see that, for these c's, the global set $\{v \in E\colon g(v, v) = c\}$ is diffeomorphic to $S^1 \times S$. Hence, we have a (non-singular) real foliation of $E \setminus S$ of rank 3, with cilinder-like leaves and we also see that we can add S (the 0-level of the metric) to the foliation, thus obtaining a singular foliation of E.

If we do not have a global metric compatible with ∇, we can try to define the foliation locally on sets U_α where we do have a global adapted metric. If we prove that the leaves are the same for every U_α, we can glue them in the intersections of the U_α's and again we find that we have a foliation of $E \setminus S$. Note that, if this works, the foliation will not depend on the particular adapted metric chosen, because, by the last part of Proposition 2.1.6, we know that two metrics local g_α and g'_α compatible with ∇ are represented by functions n_α and n'_α which locally differ by a positive multiplicative constant (and so globally the leaves are the same because locally the level sets are the same).

In the next Proposition we prove that the local foliations actually glue to a global one.

Proposition 2.1.9. *Let $p : E \to S$ be a line bundle over a Riemann surface S and let ∇ be a holomorphic connection on S. Let (U_α) be a trivializing cover (with connected intersections) for E. Then the level sets for any local or global metric adapted to ∇ on U_α glue together to give a rank 3 real foliation on E, singular on S.*

Proof. Let U_α and U_β be two open sets of the cover with non empty intersection. The level sets for the metric are the level sets, on U_α, for the function $g_\alpha(v) = n_\alpha(p(v))|v_\alpha|^2$, defined for $v \in E$, $v = (p(v), v_\alpha e_\alpha)$. We want to prove that, on $U_\alpha \cap U_\beta$, g_α and g_β differ multiplicatively by a real constant. But this is true, because

$$n_\beta(p(v)) = \exp\left(2\operatorname{Re}(K_\alpha - K_\beta)\right) n_\alpha(p(v)) = |\hat{\xi}_{\alpha\beta}|^{-2}|\xi_{\alpha\beta}(p(v))|^2 n_\alpha(p(v))$$

and

$$|v_\beta|^2 = |\xi_{\alpha\beta}(p(v))|^{-2}|v_\alpha|^2,$$

so that

$$n_\beta(p(v))|v_\beta|^2 = |\hat{\xi}_{\alpha\beta}|^{-2} n_\alpha(p(v))|v_\alpha|^2$$

and $|\hat{\xi}_{\alpha\beta}|$ is a constant. □

So, we can give a proper definition for this rank 3 foliation.

Definition 2.1.10. *Let $p : E \to S$ be a line bundle over a Riemann surface S and let ∇ be a holomorphic connection on S. Let (U_α) be a trivializing cover (with connected intersections) for E. Then, the sets which are locally the level sets of any local or global metric compatible with ∇ form a real rank 3 singular foliation on E, regular on $E \setminus S$, called the* metric foliation.

The next result gives a characterization of the metric foliation in terms of real 1-forms defined on the open sets U_α.

Proposition 2.1.11. *The metric foliation is generated on U_α by the 1-form*

$$\varpi_\alpha = \operatorname{Re}\left(|v_\alpha|^2 p^*\eta_\alpha + \overline{v_\alpha} dv_\alpha\right)$$

where, as usual, η_α is the holomorphic 1-form representing ∇ on U_α.

We note that we can write ϖ_α as

$$\varpi_\alpha = |v_\alpha|^2 \operatorname{Re}(\omega_\alpha)$$

with $\omega_\alpha = p^*\eta_\alpha + \frac{1}{v_\alpha} dv_\alpha$. We shall also study the form ω_α later.

Proof. On U_α, we have that the leaves are the level sets of

$$|v_\alpha|^2 n_\alpha.$$

The foliation will be generated by the differential of this function, which is

$$\begin{aligned} d\left(|v_\alpha|^2 n_\alpha\right) =& d\left(|v_\alpha|^2\right) n_\alpha + |v_\alpha|^2 dn_\alpha = 2\operatorname{Re}\left(\overline{v_\alpha} dv_\alpha\right) n_a + 2|v_\alpha|^2 \operatorname{Re}(n_\alpha p^* \eta_\alpha) \\ =& 2n_\alpha \operatorname{Re}\left(\overline{v_\alpha} dv_\alpha + |v_\alpha|^2 p^* \eta_\alpha\right) = 2n_\alpha \varpi_\alpha. \end{aligned}$$

and we are done, because n_α is a positive real function, if we prove the closure in the exterior algebra, i.e., that there exists a form τ_α on U_α such that $d\varpi_\alpha = \tau_\alpha \wedge \varpi_\alpha$. To do this, we note that $d\eta_\alpha = 0$ by reasons of type, so that we have

$$d\omega_\alpha = \operatorname{Re}\left(\left(v_\alpha d\overline{v}_\alpha + \overline{v}_\alpha dv_\alpha\right) p^* \eta_\alpha + d\overline{v}_\alpha \wedge dv_\alpha\right)$$

and we see that it suffices to consider the real 1-form

$$\tau_\alpha = \frac{dv_\alpha}{v_\alpha} + \frac{d\overline{v}_\alpha}{\overline{v}_\alpha} = \frac{2}{|v_\alpha|^2} \operatorname{Re}(\overline{v}_\alpha dv_\alpha).$$

□

On a non empty-intersection $U_\alpha \cap U_\beta$ we have $n_\alpha |v_\alpha|^2 = n_\beta |v_\beta|^2$, so that

$$n_\alpha \varpi_\alpha = \frac{d(|v_\alpha|^2 n_\alpha)}{2} = \frac{d(|v_\beta|^2 n_\beta)}{2} \frac{n_\alpha}{n_\beta} \frac{|v_\alpha|^2}{|v_\beta|^2} = n_\beta \varpi_\beta \frac{n_\alpha}{n_\beta} \frac{|v_\alpha|^2}{|v_\beta|^2} = \varpi_\beta n_\alpha \frac{|v_\alpha|^2}{|v_\beta|^2}.$$

It follows that, on $p^{-1}(U_\alpha \cap U_\beta) \setminus (U_\alpha \cap U_\beta)$,

$$\frac{\varpi_\alpha}{|v_\alpha|^2} = \frac{\varpi_\beta}{|v_\beta|^2},$$

So, we see that the forms $\operatorname{Re}(\omega_\alpha) = \varpi_\alpha / |v_\alpha|^2$ glue to a global form ϖ on $E \setminus S$. The reason for which we used $\overline{\omega_\alpha}$ instead of $\frac{\varpi_\alpha}{|v_\alpha|^2}$ to define the metric foliation is that the first is defined also on S.

It not difficult to see that in fact we have also $\omega_\alpha = \omega_\beta$ on $p^{-1}(U_\alpha \cap U_\beta)$, and so the ω_α's glue to a global form on $E \setminus S$, that we call ω.

This ω is a global complex 1-form on $E \setminus S$, and we see that it gives a rank 1 complex foliation on $E \setminus S$. Being never zero, we also know that the induced foliation will be non-singular. Clearly, from the fact that $\varpi_\alpha = |v_\alpha|^2 \operatorname{Re}(\omega)$, we see that the leaves of this foliation (that, in particular, is also a real rank 2 foliation) are contained in those of the metric foliation.

We give a name to this new foliation and then we give a characterization that will explain the name chosen for it.

Definition 2.1.12. *Let $p : E \to S$ be a line bundle over a Riemann surface S and let ∇ be a holomorphic connection on S. Let (U_α) be a trivializing cover (with connected intersections) for E and ω the complex global form on $E \setminus S$ given on $p^{-1}(U_\alpha) \setminus U_\alpha$ by $\omega_\alpha = p^* \eta_\alpha + \frac{1}{v_\alpha} dv_a$. We call the* horizontal foliation *the complex rank 1 non-singular foliation on $E \setminus S$ induced by ω.*

Remark 2.1.13. *As we saw that the metric foliation can be thought as a non-singular foliation on $E \setminus S$, induced by a global form, or as a singular foliation on all of E, but given by local forms, we can do the same with the horizontal foliation. In fact, in Definition 2.1.12 we used a global form and we got a non-singular foliation on $E \setminus S$. If we consider the forms $v_\alpha \omega_\alpha$, we see that they extend the same foliation to all of S, but they cannot be glued to a single global form.*

Before explaining the geometrical meaning (and the name) of the horizontal foliation, we are going to characterize the two foliations we have constructed so far by means of vector fields (sections of TE).

For the horizontal one it is easy: (locally) we have to look for a single complex vector field, and we immediately see that the local fields

$$H_\alpha = \partial_\alpha - (\eta_\alpha(\partial_\alpha) \circ p) v_\alpha \frac{\partial}{\partial v_\alpha} \tag{2.9}$$

solve the problem, because $\omega_\alpha(H_\alpha) = 0$.

For the metric foliation we need three vector fields, thought as fields over $\mathbb{R}$. Clearly we can use H_α and iH_α. These two fields generate the horizontal foliation as a real rank 2 foliation, and so, speaking very roughly, the third field we are looking for should be, when restricted to any fiber of E, the generator for the foliation of it given by the level sets of the metric. In fact, we see that if we take the vector field $iR := iv_\alpha \frac{\partial}{\partial v_\alpha}$ (somehow orthogonal to the radial field $R = v_\alpha \frac{\partial}{\partial v_\alpha}$) we have $\varpi_\alpha(iR) = 0$ and it is independent from H_α and iH_α, and so together they generate the metric foliation.

Let's now see the geometrical interpretation of the horizontal foliation, also explaining the name used for it. Recall that a section s of E is said to be *horizontal* (with respect to ∇) if $\nabla s = 0$. Then, we have the following result:

Proposition 2.1.14. *A local section s_α of E is an integral curve for H_α if and only if it is a horizontal section.*

Proof. We write our section s_α as (z_α, v_α). $\nabla(s_\alpha) = 0$ means that $dv_\alpha + v_\alpha (\eta_\alpha \circ p) = 0$, which is equivalent to $(dv_\alpha + v_\alpha (\eta_\alpha \circ p)) (\partial_\alpha) = dv_\alpha(\partial_\alpha) + v_\alpha (\eta_\alpha(\partial_\alpha) \circ p) = 0$ because S has complex dimension 1. It means that $\frac{\partial v_\alpha}{\partial z_\alpha} = -v_\alpha (\eta_\alpha(\partial_\alpha) \circ p)$, which is exactly the condition for (z_α, v_α) to be an integral curve for H_α. □

So, we see that the leaves of the horizontal foliation are the horizontal sections of ∇, which means that, for every point $v \in E$, the leaf passing through v is given by the solution of the differential equation $\nabla(s) = 0$ with initial condition v. In particular, on a sufficiently small trivializing open set U_α, given a horizontal leaf L, we have that each connected component of $L \cap p^{-1}(U_\alpha)$ is a holomorphic copy of U_α.

To introduce the third foliation we start considering a special case, i.e., E will be TS. Then, we shall see how to extend this definition to a general line bundle E over S.

Definition 2.1.15. *Let S be a Riemann surface and ∇ a holomorphic connection on TS. A smooth curve $\sigma : I \to S$, where I is an interval of $\mathbb{R}$, is a* geodesic *for ∇ (or a ∇-*geodesic*) if $\nabla_{\sigma'}\sigma' \equiv 0$.*

As a matter of notation, in this section we shall use a tilde to denote objects in this special case ($E = TS$), so that there will not be confusion with the general case we shall study after it, for which we will continue to use the letters without the tildas.

We are going to see that, in this case, we have a strict relation between the fields $\widetilde{H}_\alpha$ we used to characterize the horizontal foliation and the geodesics for ∇. In fact, let us consider the local fields $v_\alpha H_\alpha$ that we used to characterize the singular horizontal foliation on all of E. They differ on the intersections $U_\alpha \cap U_\beta$ by a multiplicative factor $\xi_{\alpha\beta}\frac{\partial z_\beta}{\partial z_\alpha}$, which is in general complex, so that they do not define, when seen as fields over $\mathbb{R}$, a real rank-1 foliation. But suppose we are in the special case with $E = TS$. We see that in this situation we have $\widetilde{\xi}_{\alpha\beta}\frac{\partial z_\beta}{\partial z_\alpha} = 1$, so that these fields not only give a real rank foliation on TS, but also glue to a single global field $\widetilde{G}$ inducing this foliation. Actually, it is possible to see that the integral curves of this field $\widetilde{G}$ are exactly the geodesics for ∇. Summing up, we have the following

Proposition 2.1.16. *Let S be a Riemann surface, $\{(U_\alpha, z_\alpha, \partial_\alpha)\}$ a trivializing cover with connected intersections and ∇ a holomorphic connection on TS. Then, the local fields*

$$\widetilde{v}_\alpha \widetilde{H}_\alpha = \widetilde{v}_\alpha \partial_\alpha - (\eta_\alpha(\partial_\alpha) \circ p)\,(\widetilde{v}_\alpha)^2\,\frac{\partial}{\partial v_\alpha}$$

defined on U_α glue to a global field $\widetilde{G}$ (which is a section of $T(TS)$). Moreover, a curve $\sigma : I \to S$, where $I \subset \mathbb{R}$ is an interval, is an integral curve for $\widetilde{G}$ if and only if it is a ∇-geodesic.

We do not prove this result now, because we are going to show a more general one in a while (see Proposition 2.1.18). In fact, what we would like to do now is to extend this idea to the case in which E is any line bundle over S, looking for a field on E whose integral curves will be suitable generalizations of the geodesics for a connection on TS.

We see that we have two different objects: the line bundle $p : E \to S$, where we have the connection ∇, and the line bundle TS, where we have the vector field tangent to a curve in S.

To relate these two ideas we need a morphism X between E and TS, that would allow us to bring back the tangents to the curves in S to E, where we may use the connection ∇ on E to define the geodesics in this context.

In particular, because of the fact that we shall need only this case, we shall suppose to have an isomorphism $X : E \to S$, so that it is clear that we can speak about preimages of vector fields by X.

The first thing we do is putting the last ideas about these generalized geodesics in a proper definition.

Definition 2.1.17. *Let $p : E \to S$ be a line bundle over a Riemann surface S, ∇ a holomorphic connection on E and X an isomorphism between E and TS. A smooth curve $\sigma : I \to S$, with I an interval in $\mathbb{R}$, is a* geodesic *with respect to ∇ and X if $\nabla_{\sigma'} X^{-1}(\sigma') = 0$.*

In particular, if $E = TS$ (and $X = id$), σ is geodesic if $\nabla_{\sigma'}(\sigma') = 0$.

Then, we look for an analogous of Proposition 2.1.16, in particular for a vector field for this situation. The fact we used in the special case of TS was that, on any non-empty intersection $U_\alpha \cap U_\beta$, the change between $\widetilde{G}_\alpha = v_\alpha H_\alpha$ and $\widetilde{G}_\beta = v_\beta H_\beta$ was $\widetilde{\xi}_{\alpha\beta} \frac{\partial z_\beta}{\partial z_\alpha} = 1$.

In the next proposition we show that it is possible to find suitable local fields such that the change is 1 also in this case, so that we come up with a global field G, and prove that the integral curves for this field are actually the geodesics with respect to ∇ and X.

To fix a notation that we shall use in the statement and in the proof, consider an open set U_α of the cover and the local generator e_α. We can look at his image by X, which will be of the form

$$X(e_\alpha) = X_\alpha \partial_\alpha \tag{2.10}$$

for some holomorphic function $X_\alpha \in \mathcal{O}^*(U_\alpha)$.

Proposition 2.1.18. *Let $p : E \to S$ be a line bundle over a Riemann surface, $X : E \to TS$ an isomorphism, $\{(U_\alpha, z_\alpha\}$ an open cover of S with connected intersections such that $\{(U_\alpha, z_\alpha, e_\alpha)\}$ is a trivializing cover for E and $\{(U_\alpha, z_\alpha, \partial_\alpha)\}$ is a trivializing cover for TS, and ∇ a holomorphic connection on E. Then the local fields*

$$(X_\alpha \circ p)\, v_\alpha H_\alpha = (X_\alpha \circ p)\, v_\alpha \partial_\alpha - ((X_\alpha \eta_\alpha(\partial_\alpha)) \circ p)\, (v_\alpha)^2 \frac{\partial}{\partial v_\alpha} \tag{2.11}$$

defined on U_α glue to a global field G, section of TE. Moreover, a curve $\sigma : I \to S$, where $I \subset \mathbb{R}$ is an interval, is a geodesic with respect to X and ∇, i.e., $\nabla_{\sigma'} X^{-1}(\sigma') = 0$, if and only if $X^{-1}(\sigma')$ is an integral curve for G.

Proof. First we prove that in any non-empty intersection $U_\alpha \cap U_\beta$ we have $X_\alpha v_\alpha H_\alpha = X_\beta v_\beta H_\beta$. By (2.8) and the easy to prove fact that

$$H_\alpha = \frac{\partial z_\beta}{\partial z_\alpha} H_\beta$$

we only need to prove that

$$X_\beta = \xi_{\alpha\beta} \frac{\partial z_\beta}{\partial z_\alpha} X_\alpha.$$

But this follows from

$$X_\beta \frac{\partial}{\partial z_\beta} = X(e_\beta) = \xi_{\alpha\beta} X(e_\alpha) = \xi_{\alpha\beta} X_\alpha \frac{\partial}{\partial z_\alpha} = \xi_{\alpha\beta} \frac{\partial z_\beta}{\partial z_\alpha} X_\alpha \frac{\partial}{\partial z_\beta}$$

and so we get that G is a global field.

For the second statement, we work locally and we derive two identities equivalent to the fact of being a geodesic and of having the preimage of the tangent field to be an integral curve, and then prove that the two identities found are also equivalent.

We start characterizing the fact of being a geodesic. Let $z_\alpha(t)$ be the coordinate for a curve $\sigma : I \to S$ (in U_α). The condition for being a geodesic is $\nabla_{\sigma'} X^{-1}(\sigma') \equiv 0$, which is, noting that $X^{-1}(\sigma') = X^{-1}(z'_\alpha \partial_\alpha) = \frac{z'_\alpha}{X_\alpha} e_\alpha$,

$$\left(d\left(\frac{z'_\alpha}{X_\alpha}\right) \otimes e_\alpha + \frac{z'_\alpha}{X_\alpha} (\eta_\alpha \circ p) \otimes e_\alpha \right) (z'_\alpha \partial_\alpha) = 0,$$

which is equivalent to

$$\left(d\left(\frac{z'_\alpha}{X_\alpha}\right)\right) (z'_\alpha \partial_\alpha) + \frac{z'_\alpha}{X_\alpha} (\eta_\alpha \circ p) (z'_\alpha \partial_\alpha) = 0,$$

which is easily seen to be equivalent to

$$\left(\frac{z'_\alpha}{X_\alpha}\right)' + \eta_\alpha (\partial_\alpha) \frac{(z'_\alpha)^2}{X_\alpha} = 0 \tag{2.12}$$

after noting that $(f \circ z_\alpha)' = z'_\alpha df(\partial_\alpha)$ for any $f \in \mathcal{O}(U_\alpha)$.

On the other hand, a curve $t \to (z_\alpha(t), v_\alpha(t))$ in $p^{-1}(U_\alpha)$ is an integral curve for G if and only if

$$\begin{cases} z'_\alpha = X_\alpha(z_\alpha) v_\alpha \\ v'_\alpha = -\eta_\alpha(\partial_\alpha) X_\alpha(z_\alpha) (v_\alpha)^2 \end{cases} \tag{2.13}$$

Recalling that $X^{-1}(\sigma')$ is locally given by $\left(z_\alpha, \frac{z'_\alpha}{X_\alpha}\right)$, we have that this is an integral curve for G if and only if

$$\begin{cases} z'_\alpha = X_\alpha(z_\alpha) \frac{z'_\alpha}{X_\alpha} \\ \left(\frac{z'_\alpha}{X_\alpha}\right)' = -\eta_\alpha(\partial_\alpha) X_\alpha(z_\alpha) \left(\frac{z'_\alpha}{X_\alpha}\right)^2 \end{cases} \tag{2.14}$$

Being (2.12) and (2.14) clearly equivalent, we get the assertion. $\square$

Remark 2.1.19. *In particular, equation* (2.12) *gives a characterization of being a geodesic with respect to* ∇ *and* X *very similar to the familiar one of the classical differential geometry.*

Remark 2.1.20. *Obviously, we have* $X_\alpha = 1$ *when* $E = TS$ *and we use* $e_\alpha = \partial_\alpha$*, so that* $\widetilde{G} = G$ *in that case and Proposition 2.1.18 reduces to Proposition 2.1.16.*

Having proved Proposition 2.1.18, we can now define the last foliation we need on a line bundle $p : E \to S$.

Definition 2.1.21. *Let* $p : E \to S$ *be a line bundle over a Riemann surface* S*,* $X : E \to TS$ *an isomorphism and* ∇ *a holomorphic connection on* E*. The global holomorphic field* G*, locally given by* (2.11) *on* U_α*, where* $\{(U_\alpha, z_\alpha, e_\alpha)\}$ *is a trivializing cover for* E*, is called the* geodesic field *associated to* ∇ *and* X*. The induced real rank 1 foliation on* $E \setminus S$ *is the* geodesic foliation *associated to* ∇ *and* X*.*

Remark 2.1.22. *In the sequel we shall need to explicitely integrate the geodesic field in some situations. Here we describe a procedure that can often help.*

We use the fact that the geodesics are contained in the horizontal leaves. We locally parametrize these leaves (with a complex variable) and then use this parametrization to get information about the geodesics, too.

For the first task, we look for a holomorphic map $\varphi : V \to E$*, where* V *is an open set in* $\mathbb{C}$*, such that*

$$\varphi' = H_\alpha \circ \varphi$$

We see that, if we write $\varphi(z) = (z_\alpha(\zeta), v_\alpha(\zeta))$*, we must request*

$$\begin{cases} z'_\alpha \equiv 1 \\ v'_\alpha = \eta_\alpha(\partial_\alpha) v_\alpha. \end{cases}$$

It is easy to see that the solution is of the form

$$\begin{cases} z_\alpha(\zeta) = \zeta + c_0 \\ v_\alpha(\zeta) = v_\alpha(\zeta) = c_1 \exp\left(-K_\alpha(\zeta + c_0)\right) \end{cases} \tag{2.15}$$

for some constants $c_0, c_1 \in \mathbb{C}$ *and* K_α *is a holomorphic primitive of* η_α *on* $V + c_0$ *(see Proposition 1.1.5).*

Next, we consider our geodesic, of the form $(z(t), v(t))$*. From* (2.15) *and* (2.13)*, we get*

$$\frac{\exp(K_\alpha(z_\alpha(t)))}{X_\alpha(z_\alpha(t))} z'_\alpha(t) = c_1.$$

Suppose now that we can find a primitive F_α *of* $\exp(K_\alpha)/X_\alpha$*. In this case we have* $F_\alpha(z_\alpha(t)) = c_1 t + c_2$*. Because of the fact that* $F'_\alpha \neq 0$*, we come to*

$$z_\alpha(t) = F_\alpha^{-1}\left(c_1 t + c_2\right),$$

which gives the solution for $z_\alpha(t)$*. It is then easy to solve also for* $v_\alpha(t)$*, obtaining*

$$v_\alpha(t) = \frac{z'_\alpha(t)}{X_\alpha(z_\alpha(t))} = \frac{c_1}{\exp(K_\alpha(z_\alpha(t)))}$$

.

So, we have defined three foliations on a line bundle $p : E \to S$ over a Riemann Surface. In the next two sections we shall study in more detail these foliations in the case of $E = TS$.

2.2 The foliations on the tangent bundle

The aim of this section is to study the first two foliations we have introduced for a line bundle $p : E \to S$ over a Riemann surface, i.e., the metric and the horizontal one, in the special case of $E = TS$.

The setting will be the following: S will be a general Riemann surface, and $\pi : \widetilde{S} \to S$ will be the universal cover, which in particular is a simply connected Riemann surface, i.e., can be thought of as the unit disk $\mathbb{D}$, the plane $\mathbb{C}$ or the sphere $\mathbb{P}^1(\mathbb{C})$. ∇ will be a holomorphic connection on TS and $\{U_\alpha, z_\alpha, \partial_\alpha\}$ a trivializing cover for TS. η_α will be the holomorphic 1-form representing ∇ on U_α.

We see that the connection ∇ on TS induces a connection, that we shall call $\widetilde{\nabla}$, on $T\widetilde{S}$, obtained by setting $\widetilde{\nabla} := \pi^*\nabla$:

$$d\pi\left(\widetilde{\nabla}_{\widetilde{v}}\left(\widetilde{e}\right)\right) = \nabla_{d\pi(\widetilde{v})} d\pi\left(\widetilde{e}\right). \tag{2.16}$$

We shall prove in Chapter 3 (Corollary 3.1.13) that on $T\mathbb{P}^1(\mathbb{C})$ there cannot exist holomorphic connections, so that S (and hence also $\widetilde{S}$) cannot be $\mathbb{P}^1(\mathbb{C})$. So the covering $\widetilde{S}$ can be only $\mathbb{D}$ or $\mathbb{C}$. In any case, $T\widetilde{S}$ is a trivial bundle and so can be trivialized by an atlas consisting of a single chart. We shall call w the coordinate on $\widetilde{S}$, and $\frac{\partial}{\partial w}$ will be the global generator of $T\widetilde{S}$. Because of the fact that $T\widetilde{S}$ is trivialized by a single chart, we can represent ∇ with a single global holomorphic 1-form $\widetilde{\eta}$.

The first thing we would like to do is to relate $\widetilde{\eta}$ (on $\pi^{-1}(U_\alpha)$) with η_α. To do so, as the last piece of notation, we define a function $\pi'_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{C}^*$ by

$$d\pi_w\left(\frac{\partial}{\partial w}\right) = \pi'_\alpha(w)\partial_{\alpha|\pi(w)}.$$

The following Lemma gives the relation between $\widetilde{\eta}$ and η_α, expressed in terms of the function π'_α

Lemma 2.2.1. *Under the hyphoteses above, we have:*

$$\widetilde{\eta} = \pi^* \eta_\alpha + \frac{1}{\pi'_\alpha} d\pi'_\alpha. \tag{2.17}$$

Proof. We prove that

$$\widetilde{\eta}\left(\frac{\partial}{\partial w}\right) = \pi^* \eta_\alpha \left(\frac{\partial}{\partial w}\right) + \frac{1}{\pi'_\alpha} d\pi'_\alpha \left(\frac{\partial}{\partial w}\right).$$

By the definition (2.16) of $\widetilde{\nabla}$, we have

$$\nabla_{\pi'_\alpha \frac{\partial}{\partial z}_{|\pi(w)}} \left(\pi'_\alpha \frac{\partial}{\partial z}_{|\pi(w)}\right) = d\pi \widetilde{\nabla}_{\frac{\partial}{\partial w}} \left(\frac{\partial}{\partial w}\right).$$

The left hand side is equal to

$$\pi'_\alpha \left(d\pi'_\alpha \left(\frac{\partial}{\partial z}_{|\pi(w)}\right) \otimes \frac{\partial}{\partial z}_{|\pi(w)} + \pi'_\alpha \eta_\alpha \left(\frac{\partial}{\partial z}_{|\pi(w)}\right) \otimes \frac{\partial}{\partial z}_{|\pi(w)} \right) \tag{2.18}$$

while the right hand side is equal to

$$d\pi \left(\widetilde{\eta}\left(\frac{\partial}{\partial w}\right) \otimes \frac{\partial}{\partial w} \right) = \left(\pi'_\alpha\right)^2 \eta_\alpha \otimes \frac{\partial}{\partial z}_{|\pi(w)}. \tag{2.19}$$

Equating (2.18) and (2.19) and dividing by $(\pi'_\alpha)^2$, which is everywhere different from zero, we get the assertion. □

We can now start studying the two foliations on TS. The following result permits to relate the foliations on TS to those on $T\widetilde{S}$.

Proposition 2.2.2. *Let S be a Riemann surface and ∇ a holomorphic connection on TS. Let $\pi : \widetilde{S} \to S$ be the universal covering of S and $\widetilde{\nabla}$ the holomorphic connection on $T\widetilde{S}$ induced by ∇. Then $d\pi$ sends the leaves of the metric foliation of $T\widetilde{S}$ onto the leaves of the metric foliation of TS, and the same is true for the horizontal foliation.*

Proof. We prove that the form $\widetilde{\varpi}$ representing the metric foliation on $\widetilde{p} : T\widetilde{S} \to \widetilde{S}$ is (a positive multiple of) the form $(d\pi)^* \varpi$, where ϖ (see Proposition 2.1.11) represents the analogous foliation on $p : TS \to S$, and that the same is true for $\widetilde{\omega}$ and $(d\pi)^* (\omega)$, the forms generating the horizontal foliations.

We start with the horizontal foliation, i.e., with $\widetilde{\omega}$ and $(d\pi)^* (\omega)$. Recall that

$$\omega = p^* \eta_\alpha + \frac{1}{v_\alpha} dv_\alpha$$

and

$$\widetilde{\omega} = \widetilde{p}^* \widetilde{\eta} + \frac{1}{\widetilde{v}} d\widetilde{v},$$

where $\widetilde{v}$ is, in analogy with v_α, the coordinate in the fiber of $T\widetilde{S}$. By definition of π'_α, the map $d\pi : T\widetilde{S} \to TS$ has the form

$$d\pi(w, \widetilde{v}) = \big(\pi(w), \pi'_\alpha(w)\widetilde{v}\big)$$

which means that $v_\alpha \circ d\pi = (\pi'_\alpha \circ \widetilde{p})\, \widetilde{v}$, and so we have

$$\begin{aligned}
(d\pi)^*\omega &= (d\pi)^* \left(p^*\eta_\alpha + \frac{1}{v_\alpha} dv_\alpha\right)\\
&= \widetilde{p}^*\pi^*\eta_\alpha + \frac{1}{v_\alpha \circ d\pi} d\left(v_\alpha \circ d\pi\right)\\
&= \widetilde{p}^*\pi^*\eta_\alpha + d\log\big(\left(\pi'_\alpha \circ \widetilde{p}\right)\widetilde{v}\big)\\
&= \widetilde{p}^*\pi^*\eta_\alpha + d\log\left(\pi'_\alpha \circ \widetilde{p}\right) + d\log\left(\widetilde{v}\right)\\
&= \widetilde{p}^*\widetilde{\eta} + \frac{1}{\widetilde{v}} d\widetilde{v}\\
&= \widetilde{\omega},
\end{aligned}$$

which gives the thesis for the horizontal foliation. Regarding the metric foliations, since $\varpi_\alpha = |v_\alpha|^2 \operatorname{Re}\omega$ and $\widetilde{\varpi} = |\widetilde{v}|^2 \operatorname{Re}\widetilde{\omega}$, we have

$$\left(d\pi\right)^* \varpi_\alpha = |v_\alpha \circ d\pi|^2 \operatorname{Re}\left(\omega \circ d\pi\right) = |\pi'_\alpha \circ \widetilde{p}|^2 |\widetilde{v}|^2 \operatorname{Re}\widetilde{\omega} = |\pi'_\alpha \circ \widetilde{p}|^2 \widetilde{\varpi},$$

where we used the fact that $\left(d\pi\right)^* \omega = \widetilde{\omega}$, proved in the first part of this proof. The assertion follows. □

Proposition 2.2.2 allows us to split our study of the metric and horizontal foliations in two parts: understanding their behaviour in the case of simply connected surfaces admitting holomorphic connections (i.e., $\mathbb{D}$ or $\mathbb{C}$), and then recover information for a generic surface by passing to from the universal cover.

The first task is easy to do: if $\widetilde{S}$ is $\mathbb{D}$ or $\mathbb{C}$, we know that the tangent bundle $T\widetilde{S}$ is isomorphic to the trivial bundle $\widetilde{S} \times \mathbb{C}$. In fact, given an element $v\frac{\partial}{\partial z}_{|z} \in T\widetilde{S}$, we can see it as a pair (z, v) and it is easy to see that this actually gives an isomorphism of bundles.

Let us study the metric foliation. By Definition 2.1.10 (and Proposition 2.1.9), the leaves of this foliation are (locally) the level set of any metric adapted to $\widetilde{\nabla}$. In particular, in this case we have a global metric of this kind, represented by the function

$$n(z) = \exp\left(2 \operatorname{Re}\widetilde{K}(z)\right),$$

where $\widetilde{K}$ is a global primitive of $\widetilde{\eta}$. So, the leaves of the metric foliation are the level set of the function

$$\exp\left(2 \operatorname{Re}\widetilde{K}(z)\right) |v|^2$$

and, using the isomorphism between $T\widetilde{S}$ and $\widetilde{S} \times \mathbb{C}$, we see that any such leaf (except the zero section, corresponding to the 0 set of the norm function) is diffeomorphic to a "cilinder" $\widetilde{S} \times S^1$, by

$$\left(e^{i\theta}, z\right) \mapsto \left(z, \exp\left(-\operatorname{Re}\widetilde{K}(z)\right) \exp\left(\operatorname{Re}\widetilde{K}(z_0)\right) |v_0| e^{2\pi i\theta}\right).$$

Similarly, from Proposition 1.1.5 we see that the leaves of the horizontal foliation are given by

$$\exp\left(\widetilde{K}(z)\right) v = c \tag{2.20}$$

for some constant $c \in \mathbb{C}$ (again, the zero constant gives the zero section). We immediately recognize that the leaves of the horizontal foliation are biholomorphic to $\widetilde{S}$, by the map

$$z \mapsto \left(z, \exp\left(-\widetilde{K}(z)\right) \exp\left(\widetilde{K}(z_0)\right) v_0\right).$$

In particular, we see that each metric leaf is foliated by horizontal ones and that the intersections of these foliations with every fiber of $T\widetilde{S}$ are a circumference (for the metric foliation) or a point (for the horizontal one).

In the remaining part of this section we consider a generic Riemann surface S and $\pi : \widetilde{S} \to S$ will be its universal cover. So, we are in the situation described by the following diagram:

$$\begin{tikzcd} T\widetilde{S} \arrow[r, "\widetilde{p}"] \arrow[d, "d\pi"'] & \widetilde{S} \arrow[d, "\pi"] \\ TS \arrow[r, "p"] & S \end{tikzcd}$$

It is clear that, apart from the foliations on $\widetilde{S}$, to study the foliations on S we need one more ingredient: to understand the action of the automorphism group. In fact, consider a point $z_0 \in S$ and a non-trivial loop γ in S with base point z_0. Start following the leaf of the horizontal foliation (with the metric one is analogous) from a point $v_0 \in T_{z_0}S$ along the path γ. For every point $z \in \gamma$, we find the corresponding point in T_zS given by following the leaf along γ. It may happen (and in general it does) that, when we come back to z_0, we reach a point in $T_{z_0}S$ which is not v_0. Then, we can follow this γ another time, or also consider a different loop, and in general we shall continue to find different return points in the fiber $T_{z_0}S$. In particular, we see that this time the intersection of a leaf with a fiber may be different from a single point.

So, being $\exp\left(\widetilde{K}\right)$ the quantity we have to study to understand the metric and horizontal foliations, we need to find some relation betweeen

$\exp\left(\widetilde{K}\right)$ and $\exp\left(\widetilde{K}\circ\gamma\right)$, where with γ we shall continue to denote both a loop in S and an element of $\mathrm{Aut}(\pi)$. Before doing this, we need to introduce a notation regarding γ. In the same way we defined the function π'_α related to the action of $d\pi_\alpha$, we introduce the function γ' on $\widetilde{S}$, defined by

$$d\gamma_w\left(\frac{\partial}{\partial w}\right) = \gamma'(w)\left(\frac{\partial}{\partial w}\right)_{|\gamma(w)}.$$

We immediately see that, by definition of π'_α, we have

$$\left(\pi'_\alpha\circ\gamma\right)\gamma' = \pi'_\alpha$$

and

$$d\pi_{\widetilde{z}_0} = \gamma'(\widetilde{z}_0)d\pi_{\gamma(\widetilde{z}_0)}.$$

The following Lemma gives a relation between $\gamma^*\widetilde{\eta}$ and $\widetilde{\eta}$. Its proof is essentially the same as the one of Lemma 2.2.1.

Lemma 2.2.3. *Let S be a Riemann surface, ∇ a holomorphic connection on TS, $\pi:\widetilde{S}\to S$ the universal covering of S, $\widetilde{\nabla}$ the holomorphic connection on $T\widetilde{S}$ induced by ∇ and $\widetilde{\eta}$ a global form representing $\widetilde{\nabla}$ and $\gamma\in\mathrm{Aut}(\pi)$. Then,*

$$\widetilde{\eta} = \gamma^*\widetilde{\eta} + \frac{1}{\gamma'}d\gamma'. \tag{2.21}$$

Lemma 2.2.4. *Let S be a Riemann surface, ∇ a holomorphic connection on TS, $\pi:\widetilde{S}\to S$ the universal covering of S, $\widetilde{\nabla}$ the holomorphic connection on $T\widetilde{S}$ induced by ∇ and $\gamma\in\mathrm{Aut}(\pi)$. Let $\widetilde{\eta}$ a global form representing $\widetilde{\nabla}$ and $\widetilde{K}$ a global primitive of $\widetilde{\eta}$. Then*

$$\exp\left(\widetilde{K}\circ\gamma\right) = \frac{\rho(\gamma)}{\gamma'}\exp\left(\widetilde{K}\right)$$

Proof. First, we can integrate (2.21) and take an exponential, to get

$$\exp\left(\widetilde{K}\circ\gamma\right) = \frac{\delta(\gamma)}{\gamma'}\exp\left(\widetilde{K}\right) \tag{2.22}$$

for some $\delta(\gamma)\in\mathbb{C}^*$. Then, we prove that $\delta(\gamma)=\rho(\gamma)$, the monodromy representation. Let $\{(U_\alpha,z_\alpha)\}$ be an open cover of S, made by simply connected open sets, such that all the non-empty intersections are connected. We use it and $\mathrm{Aut}(\pi)$ to construct a suitable open cover for $\widetilde{S}$. To do so, fix for avery α a connected component of $\pi^{-1}(U_\alpha)$ and call it $\widetilde{U}_{\alpha,id}$. Then, denote with $\widetilde{U}_{\alpha,\gamma}$ the component $\gamma(\widetilde{U}_{\alpha,id})$. It is easy to see that $\{\widetilde{U}_{\alpha,\gamma}, z_\alpha\circ\pi, \partial/\partial w\}$ is actually an open cover of $\widetilde{S}$, trivializing $T\widetilde{S}$. Moreover, being $\widetilde{S}$ simply connected, the cocycle representing $T\widetilde{S}$ is trivial.

We choose a holomorphic primitive $\widetilde{K}_\alpha$ of η_α on every U_α. As from (2.21) we derived (2.22), we see that from (2.17) we get

$$\exp(\widetilde{K})_{|\widetilde{U}_{\alpha,\gamma}} = c_{\alpha,\gamma}\exp(K_\alpha \circ \pi)\pi'_{\alpha \widetilde{U}_{\alpha,\gamma}}$$

for some constants $c_{\alpha,\gamma} \in \mathbb{C}^*$.

Now, consider a non-empty intersection $U_\alpha \cap U_\beta$. For every $\gamma \in \mathrm{Aut}(\pi)$ there is a unique $\gamma_1 \in \mathrm{Aut}(\pi)$ such that $\widetilde{U}_{\alpha,\gamma} \cap \widetilde{U}_{\beta,\gamma_1} \neq \emptyset$ and so, in this intersection, we have

$$\begin{aligned} 1 = \frac{\exp(\widetilde{K})}{\exp(\widetilde{K})} &= \frac{c_{\alpha,\gamma}\exp(K_\alpha \circ \pi)\pi'_\alpha}{c_{\beta,\gamma_1}\exp(K_\beta \circ \pi)\pi'_\beta} \\ &= \frac{c_{\alpha,\gamma}\exp(K_\alpha \circ \pi)}{c_{\beta,\gamma_1}\exp(K_\beta \circ \pi)}\left(\frac{\partial z_\alpha}{\partial z_\beta} \circ \pi\right) = \frac{c_{\alpha,\gamma}}{c_{\beta,\gamma_1}}\hat{\psi}_{\alpha\beta}, \end{aligned} \tag{2.23}$$

where $\{\hat{\psi}_{\alpha\beta}\}$ is the locally constant cocycle representing the monodromy of the connection ∇.

Consider now two elements $\gamma_0, \gamma \in \mathrm{Aut}(\pi)$. We have

$$\begin{aligned} \exp(\widetilde{K} \circ \gamma)_{\widetilde{U}_{\alpha,\gamma_0}} &= \exp(\widetilde{K})_{\widetilde{U}_{\alpha,\gamma\gamma_0}} \circ \gamma = c_{\alpha,\gamma\gamma_0}\exp(K_\alpha \circ \pi \circ \gamma)(\pi'_\alpha \circ \gamma) \\ &= \frac{c_{\alpha,\gamma\gamma_0}}{\gamma'}\exp(K_\alpha \circ \pi)\pi'_\alpha = \frac{c_{\alpha,\gamma\gamma_0}}{c_{\alpha,\gamma}}\frac{1}{\gamma'}\exp(\widetilde{K})_{|\widetilde{U}_{\alpha,\gamma_0}}. \end{aligned}$$

So we have $\delta(\gamma) = \frac{c_{\alpha,\gamma\gamma_0}}{c_{\alpha\gamma_0}}$ and we are done, because of the canonical isomorphism between Čech and singular cohomology. □

We are now ready to prove the following Theorem, that describes the geometry of the metric foliation, relating this to the monodromy representation, defined in Definition 1.1.6.

Theorem 2.2.5. *Let ∇ be a holomorphic connection on the tangent bundle $p : TS \to S$ of a Riemann surface S. Let L be a leaf of the metric foliation induced by ∇ on TS and let $v_0 \in T_{z_0}S \cap L$. Then*

$$L \cap T_{z_0}S = |\rho|(\pi)\left(S^1 \cdot v_0\right) \tag{2.24}$$

and

$$\overline{L} \cap T_{z_0}S = \overline{|\rho|(\pi)} \cdot \left(S^1 v_0\right). \tag{2.25}$$

It follows that

1. *if ∇ has real periods then all leaves of the metric foliation are closed in TS;*

2. *if ∇ has not real periods then all leaves of the metric foliation accumulate all points of the zero section of TS.*

Proof. By definition of metric foliation, we have $S^1 \cdot v_0 \subset L \cap T_{z_0}S$. On the universal cover $\widetilde{S}$ of S we consider a lift of L, i.e., a leaf $\widetilde{L}$ for the induced $\widetilde{\nabla}$ such that $L = d\pi\left(\widetilde{L}\right)$. We fix a point $\widetilde{z}_0 \in \pi^{-1}(z_0)$ and $\widetilde{v}_0 \in T_{\widetilde{z}_0}\widetilde{S}$ such that $d\pi(\widetilde{v}_0) = v_0$.

We are going to prove that, if $\gamma \in \pi_1(S, z_0)$,

$$d\pi_{\gamma(\widetilde{z}_0)}\left(\widetilde{L}_{\gamma(\widetilde{z}_0)}\right) = \left|\frac{1}{\rho(\gamma)}\right| d\pi_{\widetilde{z}_0}\left(\widetilde{L}_{\widetilde{z}_0}\right) \tag{2.26}$$

where we see γ as an element of $\mathrm{Aut}(\pi)$ and $\widetilde{L}_{\widetilde{z}}$ denotes $L \cap T_{\widetilde{z}}\widetilde{S}$. From (2.26) and from the fact that $d\pi_{\widetilde{z}_0}\left(\widetilde{L}_{\widetilde{z}_0}\right) = S^1 \cdot v_0$ we can derive that

$$L \cap T_{z_0}(S) = \bigcup_{\gamma \in \mathrm{Aut}(\pi)} d\pi_{\gamma(\widetilde{z}_0)}\left(\widetilde{L}_{\widetilde{z}_0}\right) = |\rho|(\pi) \cdot \left(S^1 \cdot v_0\right),$$

which is (2.24), and (2.25) clearly follows.

So, we have to prove (2.26). We have $(\widetilde{z}_0, \widetilde{v}_0) \in \widetilde{L}$. Consider another point $(\widetilde{z}, \widetilde{v}) \in T\widetilde{S}$. We have

$$(\widetilde{z}, \widetilde{v}) \in \widetilde{L} \Leftrightarrow \exp\left(\mathrm{Re}\,\widetilde{K}(\widetilde{z})\right)|\widetilde{v}| = \exp\left(\mathrm{Re}\,\widetilde{K}(\widetilde{z}_0)\right)|\widetilde{v}_0|,$$

where, as usual, $\widetilde{K}$ denotes a primitive of $\widetilde{\eta}$. Using Lemma 2.2.4 we deduce that

$$(\gamma(\widetilde{z}_0), \widetilde{v}) \in \widetilde{S} \in \widetilde{L} \Leftrightarrow |\widetilde{v}| = \frac{\gamma'(\widetilde{z}_0)}{|\rho(\gamma)|}|\widetilde{v}_0|. \tag{2.27}$$

From this last equivalence we get

$$\widetilde{L}_{\gamma(\widetilde{z}_0)} = \frac{\gamma'(\widetilde{z}_0)}{|\rho(\gamma)|} \cdot \widetilde{L}_{\widetilde{z}_0}$$

and, using $d\pi_{\gamma(\widetilde{z}_0)} = \frac{1}{\gamma'} d\pi_{\widetilde{z}_0}$, we get (2.26).

We now prove the second part of the Theorem. Statement 2 is easy: if ∇ has not real periods, by definition $|\rho|(\pi)$ is not contained in S^1, and so $0 \in \overline{|\rho|(\pi)}$.

Instead, assume that ∇ has real periods. We want to prove that any leaf $L \subset TS \setminus S$ is closed in TS.

To do so, we consider a succession $(z_k, v_k) \in L$, with $(z_k, v_k) \to (z_0, v_0)$ and we prove that $(z_0, v_0) \in L$. We lift L to a leaf $\widetilde{L}$ as before, so that $L = d\pi(\widetilde{L})$. We take preimages $(\widetilde{z}_k, \widetilde{v}_k) \subset \widetilde{L}$ of (z_k, v_k), i.e., such that $\pi(\widetilde{z}_k) = z_k$ and $d\pi_{\widetilde{z}_k}(\widetilde{v}_k)$.

Now we fix a point $\hat{z}_0 \in \pi^{-1}(z_0)$. Our goal is to find a $\hat{v}_0 \in T\widetilde{S}_{\hat{z}_0}$ such that $(\hat{z}_0, \hat{v}_0) \in \widetilde{L}$ and $d\pi_{\hat{z}_0}(\hat{v}_0) = v_0$. This would imply that $(z_0, v_0) \in L$.

We know that $z_k \to z_0$. So, it means that there exist a sequence $\{\gamma_k\} \subset \mathrm{Aut}(\pi)$ such that the succession $\hat{z}_k = \gamma_k^{-1}(\widetilde{z}_k) \to \hat{z}_0$. We consider the elements

$\hat{v}_k = \widetilde{v}_k/\gamma_k'(\hat{z}_k)$. First, we notice that $d\pi_{\hat{z}_k}(\hat{v}_k) = d\pi_{\widetilde{z}_k}(\widetilde{v}_k) = v_k$. Then, by the characterization (2.27) of living in a leaf, we have that $(\hat{z}_k, \hat{v}_k) \in \widetilde{L}$.

This means that the product $\exp\left(\operatorname{Re}\widetilde{K}(\hat{z}_k)\right)|\hat{v}_k|$ is a (non-zero) constant and the fact that $\exp\left(\operatorname{Re}\widetilde{K}(\hat{z}_k)\right)$ tends to $\exp\left(\operatorname{Re}\widetilde{K}(\hat{z}_0)\right)$ as $z_k \to z_0$ says that the sequence $\{\hat{v}_k\}$ is bounded. So, there exists a subsequence of the $\hat{v}_k$'s that converges to a certain $\hat{v}_0 \in \mathbb{C}$. By continuity, we have that $(\hat{z}_0, \hat{v}_k) \in \widetilde{L}$. From the fact that $d\pi\left(\widetilde{L}\right) = L$ we know that $(z_0, d\pi_{\hat{z}_0}(\hat{v}_0)) \in L$. So, if we prove that $d\pi_{\hat{z}_0}(\hat{v}_0) = v_0$ we are done. But this follows from

$$d\pi_{\hat{z}_0}(\hat{v}_0) = \lim_k d\pi_{\hat{z}_k}(\hat{v}_k) = \lim_k v_k = v_0.$$

□

The description of the horizontal foliation is similar. The proof of the next Theorem is essentially the same of the previous one, with all the arguments repeated without taking the modulus of the elements.

Theorem 2.2.6. *Let ∇ be a holomorphic connection on the tangent bundle $p : TS \to S$ of a Riemann surface S. Let L be a leaf of the horizontal foliation induced by ∇ on TS and let $v_0 \in T_{z_0}S \cap L$. Then $p(L) = S$ and*

$$L \cap T_{z_0}S = \rho(\pi) \cdot v_0$$

and

$$\overline{L} \cap T_{z_0}S = \overline{\rho(\pi)} \cdot v_0.$$

It follows that

1. *if ∇ has real periods then all leaves of the horizontal foliation are closed in TS or every leaf is dense in the leaf of the metric foliation containing it;*

2. *if ∇ has not real periods then all leaves of the horizontal foliation accumulate all points of the zero section of TS.*

In the last part of this section we are going to study the dynamics of the geodesics for a holomorphic connection on the tangent bundle of a Riemann surface. We shall first study the problem for simply connected Riemann surfaces (i.e., $\mathbb{D}$ or $\mathbb{C}$, because there are not holomorphic connections on $\mathbb{P}^1(\mathbb{C})$) and then we shall get information about the geodesics on a general Riemann surface looking at what happens on the universal cover, as we did for the metric and horizontal foliations.

So, let us suppose we are given a simply connected Riemann surface $\widetilde{S}$ and a holomorphic connection $\widetilde{\nabla}$ on $T\widetilde{S}$. We start our study defining a function $J : \widetilde{S} \to \mathbb{C}$, which will prove to be extremely useful for the study of the geodesics.

Definition 2.2.7. *Let $\widetilde{\eta}$ be the global 1-form representing $\widetilde{\nabla}$ and let $\widetilde{K}$: $\widetilde{S} \to \mathbb{C}$ be a holomorphic primitive. We define a function $J : \widetilde{S} \to \mathbb{C}$ as a holomorphic primitive of* $\exp(\widetilde{K})$.

We immediately remark that J actually exists, because $\widetilde{S}$ is simply connected, and that J is locally invertible, because $J' = \exp(\widetilde{K})$. In the following Proposition we study the main properties of J.

Proposition 2.2.8. *J is a local isometry from $\widetilde{S}$ to $\mathbb{C}$, if we consider on $\widetilde{S}$ the metric corresponding to $\widetilde{K}$ (see Proposition 2.1.6). Moreover, a smooth curve $\sigma \colon I \to \widetilde{S}$ is a geodesic for $\widetilde{S}$ if and only if there are two constants c_0 and $w_0 \in \mathbb{C}$ such that $J(\sigma(t)) = c_0 t + w_0$. In particular, the geodesic with $\sigma(0) = z_0$ and $\sigma'(0) = v_0 \in \mathbb{C}^*$ is given by*

$$\sigma(t) = J^{-1}\left(c_0 t + J(z_0)\right), \tag{2.28}$$

where $c_0 = \exp\left(\widetilde{K}(z_0)\right) v_0$ and J^{-1} is the analytic continuation of the local inverse of J near $J(z_0)$ such that $J^{-1}\left(J(z_0)\right) = z_0$.

Finally, a curve $\sigma \colon [0, \varepsilon) \to \widetilde{S}$ is a geodesic for $\widetilde{\nabla}$ if and only if

$$\sigma'(t) = \exp\left(-\widetilde{K}(\sigma(t))\right) \exp\left(\widetilde{K}(\sigma(0))\right) \sigma'(0), \tag{2.29}$$

if and only if

$$J(\sigma(t)) = \exp\left(\widetilde{K}(\sigma(0))\right) \sigma'(0) t + J(\sigma(0)). \tag{2.30}$$

Proof. Being $\widetilde{S}$ simply connected, we have an isomorphism between $T\widetilde{S}$ and $\widetilde{S} \times \mathbb{C}$, given by $v \frac{\partial}{\partial z}_{|z} \to (z, v)$. We also know by Proposition 2.1.6 that the metric $\widetilde{g}$ corresponding to $\widetilde{K}$ is characterized by $n(z) = \exp\left(2 \operatorname{Re} \widetilde{K}(z)\right)$. So the lenght of an element $v \frac{\partial}{\partial z} \in T_z \widetilde{S}$ is given by

$$\widetilde{g}_z(v) = \exp\left(\operatorname{Re} \widetilde{K}(z)\right) |v|.$$

By definition of J, this is equal to $|J'(z)v|$, which precisely means that J is a local isometry, and also (2.28) follows. In particular, this says that J locally sends geodesic segments to euclidean segments.

To prove the equivalence of being geodesic with (2.29) we recall that every geodesic lives in a leaf of the horizontal foliation and this means, by (2.20), that

$$\sigma' \exp\left(\widetilde{K}(\sigma)\right) = c$$

for some constant $c \in \mathbb{C}$. In particular, we have

$$\sigma'(t) \exp\left(\widetilde{K}(\sigma(t))\right) = \sigma'(0) \exp\left(\widetilde{K}(\sigma(0))\right)$$

which is clearly equivalent to (2.29). Conversely, we see that every σ satisfying (2.29) satisfies also (2.12), and this gives the equivalence. Finally, equivalence with (2.30) follows from that with (2.29). □

We end this section by proving a statement, analogous to Proposition 2.2.2, about the relation between ∇-geodesics and geodesics for the connection $\widetilde{\nabla}$ induced on the cover by ∇, and studying the action of $\mathrm{Aut}(\pi)$ on the geodesics for $\widetilde{\nabla}$.

Proposition 2.2.9. *Let S be a Riemann surface and ∇ a holomorphic connection on TS. Let $\pi : \widetilde{S} \to S$ be the universal covering of S and $\widetilde{\nabla}$ the holomorphic connection on $T\widetilde{S}$ induced by ∇. A curve $\widetilde{\sigma}\colon I \to \widetilde{S}$ is a geodesic for $\widetilde{\nabla}$ if and only if $\sigma = \pi \circ \widetilde{\sigma}$ is a geodesic for ∇.*

Moreover, every $\gamma \in \mathrm{Aut}(\pi)$ sends $\widetilde{\nabla}$-geodesics in $\widetilde{\nabla}$-geodesics.

Proof. By definition of π we have

$$\nabla_{\sigma'}\sigma' = \nabla_{d\pi(\widetilde{\sigma}')} d\pi(\widetilde{\sigma}') = d\pi\left(\widetilde{\nabla}_{\widetilde{\sigma}'}\widetilde{\sigma}'\right).$$

So, we have

$$\nabla_{\sigma'}\sigma' \equiv 0 \Leftrightarrow \widetilde{\nabla}_{\widetilde{\sigma}'}\widetilde{\sigma}' \equiv 0$$

which gives the first assertion.

For the second, we suppose that σ is a $\widetilde{\nabla}$-geodesic and prove that also $\gamma \circ \sigma$ is. By (2.29), we need to show that

$$(\gamma \circ \sigma)'(t) = \exp\left(-\widetilde{K}(\gamma \circ \sigma(t))\right) \exp\left(\widetilde{K}(\gamma \circ \sigma(0))\right) (\gamma \circ \sigma)'(0). \tag{2.31}$$

But, from Lemma 2.2.4, we know that

$$\begin{aligned} &\exp\left(-\widetilde{K}(\gamma \circ \sigma(t))\right) \exp\left(\widetilde{K}(\gamma \circ \sigma(0))\right) (\gamma \circ \sigma)'(0) \\ =&\frac{\gamma'(\sigma(t))}{\rho(\gamma)} \exp\left(-\widetilde{K}(\sigma(t))\right) \frac{\rho(\gamma)}{\gamma'(\sigma(0))} \exp\left(\widetilde{K}(\sigma(0))\right) \gamma'(\sigma(t))\sigma'(0) \\ =& \left(\gamma'(\sigma(t))\right) [\exp\left(-\widetilde{K}(\sigma(t))\right) \exp\left(\widetilde{K}(\sigma(0))\right) \sigma'(0)] \end{aligned}$$

and this is equal to $\left(\gamma'(\sigma(t))\right)\sigma'(t) = \left(\gamma \circ \sigma\right)'(t)$ by (2.29). $\square$

Chapter 3

Poincaré-Bendixson theorems for meromorphic connections

In this chapter we shall introduce the notion of *meromorphic connection* on a Riemann surface, as an extension of the idea of holomorphic connection. In particular, this will give a way to differentiate meromorphic sections of a holomorphic bundle insted of only holomorphic ones. We shall introduce geodesics for these connections and in particular study their behaviour in the case of connections on the tangent bundle. The main result will be a characterization of the possible ω-limits for a geodesic, which generalizes the analogous statement for the Riemann sphere in [AT11] to the setting of a generic compact Riemann surface.

3.1 Meromorphic connections: generalities

Before giving the definition of a meromorphic connection, we shall review some basic definitions and facts regarding meromorphic sections of a holomorphic bundle over a Riemann surface. As always, $p : E \to S$ will be our holomorphic bundle over a Riemann surface S. We shall denote by $\mathcal{M}$ the sheaf of meromorphic functions on S.

Definition 3.1.1. *A* meromorphic section *of a holomorphic vector bundle $p : E \to S$ over a Riemann surface S defined by a holomorphic cocycle $\xi = \{\xi_{\alpha\beta}\}$ is a meromorphic vector cochain $\{x_\alpha\}$, with $x_\alpha \in \mathcal{M}(U_\alpha) \otimes_{\mathbb{C}} \mathbb{C}^n$ such that, on every non-empty intersection $U_\alpha \cap U_\beta$, we have*

$$x_\alpha = \xi_{\alpha\beta} x_\beta.$$

We shall denote by $\mathcal{M}_E$ the sheaf of meromorphic sections of $p : E \to S$.

What we are going to do now is to find an invariant for all the meromorphic sections of a given line bundle. In this way, we shall be able to associate to any holomorphic line bundle over a Riemann surface this invariant, that will be called the *degree* of the bundle. To define this invariant, we need to recall the following basic definition.

Definition 3.1.2. *The* order *at* 0 *of a meromorphic function* $\varphi(z)$ *defined on an open set* $U \subseteq \mathbb{C}$*, with* $0 \in U$ *is the order of its Laurent series at* 0*, i.e., the only integer* ν *such that* $z^{-\nu}\varphi(\nu)$ *is holomorphic and non-vanishing at* 0*.*

The order at 0 *of a meromorphic vector function* $\overline{\varphi} = (\varphi_1, \ldots, \varphi_n)$ *defined on an open set* $U subseteq \mathbb{C}$*, with* $0 \in U$ *is the minimum of the orders of the components* φ_i*.*

The definitions clearly extend to a generic point different from 0. We remark that for meromorphic vector functions too the order ν at 0 is the only integer such that $z^{-\nu}\overline{\varphi}(z)$ is holomorphic and non-vanishing at 0.

We are now ready to define the *order* of a meromorphic section $s \in \mathcal{M}_E(S)$.

Definition 3.1.3. *The* order ord_z *of a meromorphic section* s *of a holomorphic bundle* $p : E \to S$ *at a point* $z \in S$ *is the order at* 0 *of the corresponding meromorphic vector function in any chart* U_α *trivializing the cover (and containing* z*).*

This definition is well posed because, being the bundle holomorphic, the transition maps are holomorphic and so the order does not depend on the chosen chart. We also remark that the order of a section is non-zero only for a discrete set of points, and hence for a finite set if S is compact. So, in this case it is meaningful to consider the sum of all the orders at the points of S, thus obtaining the following definition of *total order* for a meromorphic section.

Definition 3.1.4. *Let* $p : E \to S$ *be a holomorphic vector bundle over a compact Riemann surface* S*. The* total order *of a meromorphic section* s *of this bundle is the sum of all the orders of* s *at the points of* S*.*

The next Proposition shows that this total order is in fact independent on the section if $p : E \to S$ is a line bundle.

Proposition 3.1.5. *Let* $p : E \to S$ *be a line bundle over a compact Riemann surface* S*. Then, every non trivial meromorphic section of* E *has the same total order.*

Proof. Consider two sections s and s'. Because of the fact that the fibers are 1-dimensional, we have $s = \varphi s'$ for some global meromorphic function $\varphi \in \mathcal{M}(S)$. So, we have

$$\sum_z \operatorname{ord}_z s = \sum_z \operatorname{ord}_z s' + \sum_z \operatorname{ord}_z \varphi.$$

But it is known that, for every meromorphic function φ on a compact Riemann surface, we have $\sum_z \operatorname{ord}_z \varphi = 0$, and the assertion follows. $\square$

So, the total order does not depend on the section, and this allows us to give the following definition.

Definition 3.1.6. *The* degree *of a line bundle $p : E \to S$ over a Riemann surface is the total order of a (and hence any) meromorphic section of E. We shall denote the degree of the bundle with* $\deg E$.

We remark that the degree of E^* is $\deg E^* = \deg E$.

The following Theorem gives a useful characterization for the degree of the bundle we will be most interested in, the tangent bundle.

Theorem 3.1.7. *The degree of the tangent bundle $p : TS \to S$ of a compact Riemann surface is equal to the Euler characteristic of the surface*

$$\deg p = \chi_S.$$

Proof. It follows from a double application of the Riemann-Roch formula. Let us first recall it in this situation: given a compact Riemann surface S and a line bundle L over S we have

$$h^0(S, L) - h^0(S, L^* \otimes (TS)^*) = \deg L + 1 - g, \tag{3.1}$$

where $h^0(X, E)$ is the dimension over $\mathbb{C}$ of the vector space of the holomorphic sections of the line bundle E over S and g is the genus of S.

We first apply (3.1) with L equal to the trivial bundle, for which we have $h^0(S, L) = 1$ (it is the dimension of the space of holomorphic functions on S, which are only the constants because S is compact) and $\deg L = 0$. It means that

$$1 - h^0\left(S, (TS)^*\right) = 1 - g,$$

i.e., we get

$$h^0\left(S, (TS)^*\right) = g. \tag{3.2}$$

Then, applying (3.1) with $L = (TS)^*$ we get

$$g - 1 = \deg\,(TS)^* + 1 - g$$

which gives $\deg\,(TS) = -\deg\,(TS)^* = 2 - 2g$ $\square$

We now come to the definition of a meromorphic connection for a line bundle $p : E \to S$. As we did for the holomorphic connections, we give the definition for a holomorphic vector bundle of any rank, but we immediately restrict to the case of a vector bundle. In particular, in the sequel *meromorphic connection* will always stand for *meromorphic connection on a holomorphic line bundle.*

Definition 3.1.8. *A* meromorphic connection ∇ *on a holomorphic vector bundle* $p : E \to S$ *is a* $\mathbb{C}$*-linear map* $\nabla : \mathcal{M}_E \to \mathcal{M}_S^1 \otimes_{\mathcal{M}(S)} \mathcal{M}_E$ *(where* $\mathcal{M}_S^1 \otimes_{\mathcal{M}(S)} \mathcal{M}_E$ *is the* $\mathcal{M}(S)$*-module of meromorphic fiber-valued 1-forms on* S*) which satisfies the Leibniz rule*

$$\nabla(se) = ds \otimes e + s\nabla e$$

for all $s \in \mathcal{M}_S$ *and* $e \in \mathcal{M}_E$.

Let us consider a meromorphic connection on a line bundle. It is easy to see that, given a trivializing cover $\{U_\alpha, z_\alpha, e_\alpha\}$ for the bundle, we have a meromorphic 1-form η_α on every U_α representing the connection, i.e.,

$$\nabla(s_\alpha) = \eta_\alpha \otimes s_\alpha$$

for every local meromorphic section s_α on U_α, exactly as it happens for holomorphic connections. Another thing that continues to hold is the relation (1.2) between the forms η_α and η_β representing ∇ on two overlapping open sets U_α and U_β,

$$\eta_\beta = \eta_\alpha + \frac{\partial \xi_{\alpha\beta}}{\xi_{\alpha\beta}}. \tag{3.3}$$

Moreover, we see that any collection $\{\eta_\alpha\}$ associated to a trivializing cover and satisfying (3.3) gives a meromorphic connection.

The following definitions will be very useful in the sequel.

Definition 3.1.9. *The* residue *of a meromorphic connection at a point* $z \in S$ *is the residue of any 1-form* η_α *representing it on an open set* U_α *of the cover (when* $z \in U_\alpha$*).*

Being $\xi_{\alpha\beta}$ holomorphic, we see that (3.3) implies that the residue of a meromorphic connection at a point $p \in S$ is well defined, i.e., does not depend on the particular open set U_α and form η_α used to represent it.

In particular, if all the forms η_α's are holomorphic, we see that we re-obtain a holomorphic connection. So, we can really see meromorphic connections as a generalization of holomorphic ones. In particular, we see that every meromorphic connection ∇ on a bundle $p : E \to S$ is a holomorphic connection for the bundle $p : E|_{S^0} \to S^0$, where S^0 is the surface S without the poles of the forms η_α's (which will be called the *poles* of the connection).

We continue to call *horizontal* a section s such that $\nabla s = 0$. We can also define the *geodesics* for a meromorphic connection in the following way: given a curve $\sigma : I \to S^0$ and $X : E \to TS$ a morphism which is an isomorphism on $E|_{S^0}$, σ is said to be a geodesic with respect to the connection ∇ and X if it is a geodesic for the holomorphic connection induced on $E|_{S^0}$.

The last thing we are going to study in this section is the sum of the residues of a meromorphic connection on our bundle $p : E \to S$. We prove that it is equal to $-\deg p$ and so, in particular, for the tangent bundle it will be $-\chi_S$. This means that we cannot have holomorphic connections on Riemann surfaces which are not a complex torus, thus proving the result we used at the beginning of Section 2.2.

The proof will essentially consist of two parts: we shall prove that the sum of the residues is independent on the particular connection and then we shall show a connection for which the equality holds.

We begin with the following Lemma, that in particular will give us the connection for the second part of the proof.

Lemma 3.1.10. *Let s be a meromorphic section of a line bundle $p : E \to S$ over a Riemann surface. Then there exists a unique meromorphic connection on E such that $\nabla s = 0$.*

Proof. By definition, s is defined by a meromorphic cochain $\{s_\alpha\}$ with respect to a cover $\{U_\alpha\}$. By definition, we have

$$s_\alpha = \xi_{\alpha\beta} s_\beta, \tag{3.4}$$

where ξ is the cocycle of the bundle $p : E \to S$. Consider the meromorphic connection locally represented, on U_α, by the 1-form $-\frac{ds_\alpha}{s_\alpha}$. We only have to prove the compatibility identity (1.2), which in this case means

$$-\frac{ds_\beta}{s_\beta} = -\frac{ds_\alpha}{s_\alpha} + \frac{d\xi_{\alpha\beta}}{\xi_{\alpha\beta}}. \tag{3.5}$$

Differentiating (3.4) we get

$$ds_\alpha = d\xi_{\alpha\beta} s_\beta + \xi_{\alpha\beta} ds_\beta$$

and so, dividing by s_α, we get

$$\frac{ds_\alpha}{s_\alpha} = \frac{s_\beta}{s_\alpha} d\xi_{\alpha\beta} + \frac{\xi_{\alpha\beta}}{s_\alpha} ds_\beta,$$

which gives (3.5) because of (3.4).

So, we have a meromorphic connection ∇. The next step is to prove that s actually is horizontal with respect to ∇. But this follows directly from the definition of ∇. In fact, locally,

$$\nabla s_\alpha = ds_\alpha \otimes e_\alpha + s_\alpha \left(-\frac{ds_\alpha}{s_\alpha}\right) \otimes e_\alpha = 0$$

and so s is horizontal.

Finally, suppose we have two connections ∇ and ∇', represented by η_α and η'_α on U_α, such that $\nabla s = \nabla' s = 0$. It means that, on U_α,

$$\begin{cases} ds_\alpha \otimes e_\alpha + s_\alpha \eta_\alpha \otimes e_\alpha = 0 \\ ds_\alpha \otimes e_\alpha + s_\alpha \eta'_\alpha \otimes e_\alpha = 0. \end{cases}$$

So, we would have

$$s_\alpha\left(\eta_\alpha - \eta'_\alpha\right) = 0$$

on U_α. It means that $\eta_\alpha - \eta'_\alpha = 0$ outside the zeroes of s_α, which are a discrete subset of U_α. By the Riemann extension Theorem, we have $\eta_\alpha - \eta'_\alpha = 0$ also on the zeroes of s_α and this means that $\nabla = \nabla'$. □

Theorem 3.1.11. *The sum of the residues of any meromorphic connection over a holomorphic line bundle $p : E \to S$ over a compact Riemann surface is the same for every meromorphic connection. Moreover,*

$$\sum_{p \in S} \mathrm{Res}_p \nabla = -\deg p.$$

Proof. By Proposition 1.1.2 (that is easily seen to hold also for meromorphic connections), the difference between two connections is a tensor, i.e., the (tensor) multiplication by a global meromorphic 1-form η. So, we have

$$\sum_{z \in S} \mathrm{Res}_z(\nabla) - \sum_{z \in S} \mathrm{Res}_z(\nabla) = \sum_{z \in S} \mathrm{Res}_z \eta.$$

Since the sum of the residues of every global meromorphic 1-form on a compact Riemann surface is zero, we obtain the result.

To prove that the sum of the residues is equal to $-\deg(p)$ we consider any meromorphic section s of $p : E \to S$ and the (unique) meromorphic connection on E for which s is horizontal. This connection is locally represented by the form $-\frac{ds_\alpha}{s_\alpha}$, which has residue at any point z equal to the opposite of order of s at z. So, the sum of the residues for this connection, and hence for all, because of the first part, is $-\deg p$. □

Corollary 3.1.12. *Let S be a compact Riemann surface and ∇ a meromorphic connection on $p : TS \to S$, represented by a cochain η of 1-forms η_α. Then*

$$\sum_{z \in S} \mathrm{Res}_z \nabla = -\chi_S.$$

Corollary 3.1.13. *There does not exist any holomorphic connection on the tangent bundle of any compact Riemann surface which is not a torus, in particular on $T\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.*

Proof. A holomorphic connection would have the sum of the residues equal to zero, which, by Corollary 3.1.12, is possible only with $\chi_S = 0$. □

Viceversa, given n points $z_1, \ldots, z_n$ on $\mathbb{P}^1(\mathbb{C})$ and n complex numbers $a_1, \ldots, n_n$ whose sum is -2, we see that we can construct a cochain of meromorphic forms such that the residue at z_i is equal to a_i. In fact, we can clearly suppose that $z_1 = \infty$. We construct a meromorphic function φ in

$\mathbb{C}$ with poles at z_i, and $\mathrm{Res}_{z_i} = a_i$. The residue at ∞ will be forced to be $-2 - \sum_{i=2}^{n} a_i = a_1$. So, considering the connection represented by $\varphi\, dz$, the following Theorem follows.

Theorem 3.1.14. *Let $z_1, \ldots, z_n$ be distinct points in $\mathbb{P}^1(\mathbb{C})$ and $a_1, \ldots, a_n$ be complex numbers such that $\sum a_i = -2$. Then, there exists a meromorphic connection on $p : T\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, holomorphic on $\mathbb{P}^1(\mathbb{C}) \setminus \{z_i\}$, which has residue a_i at z_i.*

3.2 Meromorphic connections on the tangent bundle

In this section, we study in more detail meromorphic connections on the tangent bundle of a compact Riemann surface In particular, we extend the results for $\mathbb{P}^1(\mathbb{C})$ in Section 4 of [AT11] to the case of a generic compact Riemann surface S. We shall use these results in Section 3.3 to obtain, with Theorem 3.4.6, a generalization of Theorem 4.6 of [AT11] (see Theorem 3.4.8) to this more general setting.

To do so, we start introducing the following definitions/notations. Unless otherwise noted, in all this section ∇ will be a meromorphic connection on the tangent bundle of a compact Riemann surface.

Definition 3.2.1. *Let S be a compact Riemann surface. Let ∇ be a meromorphic connection on S and let S^0 be the complement of the poles.*

- *A (n-)*geodesic cycle *is the union of n geodesic segments $\sigma_i \colon [0,1] \to S^0$, disjoint except for the conditions $\sigma_i(0) = \sigma_{i-1}(1)$ and $\sigma_1(0) = \sigma_n(1)$. The points $\sigma_i(0)$ will be called the* vertices *of the geodesic cycle;*
- *A (m-)*multicurve *is a union of m disjoint geodesic cycles. A multicurve will be said to be* disconnecting *if it disconnects S,* non-disconnecting *otherwise;*
- *A* part *is the closure of an open set of S whose boundary is a multicurve.*

We remark that a part may be all of S, when the associated multicurve is non-disconnecting. Moreover, we see that every disconnecting multicurve is the boundary of a part $P \subsetneq S$.

We would like to define a notion of *external angle* at a vertex $v_0 = \sigma_i(0)$ of a geodesic cycle, i.e., the angle between the two tangent vectors $\sigma'_{i-1}(1)$ and $\sigma'_i(0)$. To do this, we consider a local metric associated to ∇ near v_0, found applying Proposition 2.1.6, and we see that we can define our notion of angle using this metric. Furthermore, Proposition 2.1.6 says that every two local metrics we consider must differ by multiplication of a positive real function. It means that the notion of angles does not depend on the chosen metric, and so we can speak of angles, and in particular of external angles, with respect of our meromorphic connection ∇.

We see that the angle is not the only "metric object" that we can associate to a meromorphic connection. In fact, if considering Proposition 2.1.6 again, we can prove the following property of the local metrics adapted to ∇.

Lemma 3.2.2. *Every local metric adapted to a holomorphic connection ∇ is* flat*, i.e., the associated Gauss curvature is everywhere vanishing.*

Proof. Each associated local metric is of the form hg_0, where h is a positive real function, given by $h = \exp(2\operatorname{Re} K) = |\exp(K)|^2$, and g_0 stands for the Euclidean metric. It is known that, in such a situation, the Gauss curvature is given by $-\frac{1}{h}\Delta \log h$, which here is 0 because we have to evaluate a laplacian of the modulus square of a holomorphic function. □

We remark that this property is true only locally: it does not mean that we have a flat *global* metric - actually, it is not even granted that we have a global metric.

It is possible to see that there is also another interesting quantity that we can calculate, even if we do not have a global metric. This is the integral, on a curve, of the *geodesic curvature*. In particular, if we consider a pole p for our connection and a small (clockwise) circle τ around it, not containing other poles, we see that

$$\int_\tau k_g = -2\pi(1 + \operatorname{Re}\operatorname{Res}_p(\nabla)), \tag{3.6}$$

where k_g is the geodesic curvature of τ (see for example [AT11], Theorem 4.1 for the proof).

With all these ingredients at our disposal, it becomes natural to try and apply a Gauss-Bonnet Theorem to study the relation between the residues of the connection ∇ in a part P of S, the external angles at the vertices of the multicurve bounding P and the topology of S.

Theorem 3.2.3 (cp. Theorem 4.1 in [AT11])**.** *Let ∇ be a meromorphic connection on a compact Riemann surface S, with poles $\{p_1, \dots, p_r\}$ and set $S^0 := S \setminus \{p_1, \dots, p_r\}$. Let P be a part of S such that the boundary of P is a disconnecting multicurve γ made by n geodesic cycles. Take as orientation of γ the standard one given by the fact of being the boundary of a part P. Let $z_1, \dots, z_s$ denote the vertices of γ and ε_j the external angle at z_j. Suppose that P contains the poles $\{p_1, \dots, p_g\}$ and denote with H_P the rank of $\pi_1(P)$. Then*

$$\sum_{j=1}^{s} \varepsilon_j = 2\pi \left(2 - n - 2H_P + \sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) \right).$$

Proof. Let τ_g be a small counterclockwise circle bounding a disc near p_j, and let k_g^j the geodesic curvature of τ^j.

With an application of the Gauss-Bonnet Theorem as in [AT11] to P without the small circles containing the poles in P we find that

$$\sum_{j=1}^{g} \int_{\tau_j} k_g^j + \sum_{j=1}^{s} \varepsilon_j = 2\pi\chi_P = 2\pi(2 - n - g - H_P). \tag{3.7}$$

But from (3.6) we get

$$\sum_{j=1}^{g} \int_{\tau_j} k_g^j = -2\pi g - 2\pi \sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla). \tag{3.8}$$

Comparing (3.7) and (3.8) we get the thesis. □

Remark 3.2.4. *The hypothesis that the multicurve disconnects cannot in general be removed. In fact, consider a single geodesic cycle that do not disconnect S. Suppose to cut the surface along this cycle and apply the above argument in this situation. We see that the two geodesic cycle that now form the boundary of the area where we are going to apply our version of the Gauss-Bonnet Theorem (that is, all the area of S) have the same external angles (in absolute value), but, due to the induced orientations, with opposite sign. So, if we apply the arguments of the previous proof we simply get*

$$\sum_{p_j \in S} \operatorname{Res}_{p_j}(\nabla) = -2 + H_S = -\chi_S,$$

which does not give us any further information.

From Theorem 3.2.3 we immediately derive the following Corollary.

Corollary 3.2.5. *If α and β are multicurves such that $\alpha = \beta$ in $H_1(S^0)$, we have*

$$\sum_{j=1}^{s} \varepsilon_j^A = \sum_{j=1}^{s} \varepsilon_j^B$$

Proof. It follows from (3.7) and the fact that the region without poles bounded by the two multicurves has Euler characteristic equal to zero. □

In the next two Corollaries we highlight what happens when the disconnecting multicurve is made up by a single geodesic or by a single geodesic cycle composed by two geodesics.

Corollary 3.2.6 (Case of one disconnecting geodesic - cp. Corollary 4.2 in [AT11])**.** *Let ∇ be a meromorphic connection on a compact Riemann surface S, with poles $\{p_1, \dots, p_r\}$ and set $S^0 := S \setminus \{p_1, \dots, p_r\}$. Let σ be a* *disconnecting geodesic* $1-$*cycle. Let* P *be one of the two parts in which* S *is disconnected by* σ *and* $\varepsilon \in (-\pi, \pi)$ *the unique external angle of* σ*. Then*

$$\varepsilon = 2\pi \left(1 - 2H_P + \sum_P \operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\right).$$

So

$$\sum_{p_j \in P} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) \in (-3/2 + 2H_P, -1/2 + 2H_P).$$

Corollary 3.2.7 (Case of two geodesics such that their union disconnects X - cp. Corollary 4.3 in [AT11])**.** *Let* ∇ *be a meromorphic connection on a compact Riemann surface* S*, with poles* $\{p_1, \ldots, p_r\}$ *and set* $S^0 := S \setminus \{p_1, \ldots, p_r\}$*. Let* γ *be a disconnecting geodesic* $2-$*cycle. Let* P *be one of the two parts in which* S *is disconnected by* γ *and* ε_0 *and* ε_1 *be the two external angles of* γ*. Then*

$$\varepsilon_0 + \varepsilon_1 = 2\pi \left(1 - 2H_P + \sum_P \operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\right)$$

and hence

$$\sum_{p_j \in P} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) \in (-2 + 2H_P, 2H_P).$$

Remark 3.2.8. *In particular, we remark that in a part of* S *bounded by a disconnecting* $2-$*cycle there must necessarily be a pole.*

In the following we shall need to consider *closed* geodesics and *periodic* geodesics for a meromorphic connection. We define them in the following Definition.

Definition 3.2.9. *A geodesic* $\sigma : [0, l] \to S$ *is* closed *if* $\sigma(l) = \sigma(0)$ *and* $\sigma'(l)$ *is a positive multiple of* $\sigma'(0)$*. It is* periodic *if* $\sigma(l) = \sigma(0)$ *and* $\sigma'(l) = \sigma'(0)$*.*

As we shall momentarily see, closed geodesics are not necessarily periodic. This is due to the fact that, starting with a connection, we constructed the local metrics from it, contrarily to the case of Riemannian geometry, where we start with a global metric and we construct a connection from it.

By Corollary 3.2.6 we immediately see that a disconnecting geodesic is closed if if and only if, for every component P of $S \setminus \sigma$, we have

$$\sum_P \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1 + 2H_P. \tag{3.9}$$

We do not study in detail periodic geodesics for a meromorphic connection on a compact Riemann surface, because we shall not need them in the sequel.

We only state a result of [AT11] which characterizes closed and periodic geodesic in the case of the Riemann sphere, in order to give the idea of the difference between these two concepts.

Proposition 3.2.10 (Corollary 4.5 in [AT11])**.** *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0 = \infty, p_1, \dots, p_r\}$, and set $S^0 = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, p_1, \dots, p_r\} \subset \mathbb{C}$. Let $\sigma : [0, l] \to S$ be a geodesic with $\sigma(0) = \sigma(l)$ and no other self-intersections; in particular, σ is an oriented Jordan curve. Let $\{p_1, \dots, p_g\}$ be the poles of ∇ contained in the interior of σ. Then σ is a closed geodesic if and only if*

$$\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$$

and it is a periodic geodesic if and only if

$$\sum_{j=1}^{g} \operatorname{Res}_{p_j}(\nabla) = -1.$$

If σ is closed, at every turn the tangent vector is multiplied by

$$\exp\left(2\pi \sum_{j=1}^{g} \operatorname{Im}\operatorname{Res}_{p_j}(\nabla)\right),$$

and so σ can be extended to an infinite lenght geodesic $\sigma : J \to S$, where J is a half-line (possibly $J = \mathbb{R}$) such that:

1. *if $\sum_{j=1}^{g} \operatorname{Im}\operatorname{Res}_{p_j}(\nabla) < 0$ then $\sigma'(t) \to 0$ as $t \to +\infty$ and $|\sigma'(t)| \to +\infty$ as t tends to the other end of J;*
2. *if $\sum_{j=1}^{g} \operatorname{Im}\operatorname{Res}_{p_j}(\nabla) > 0$ then $\sigma'(t) \to 0$ as $t \to -\infty$ and $|\sigma'(t)| \to +\infty$ as t tends to the other end of J.*

3.3 Minimal sets for fields on compact surfaces

In this section we shall study the following problem: we have a compact real surface M and a, possibly singular, foliation on it. We want to study the leaves of such a foliation and, in particular, its so-called *minimal invariant sets*, which can be seen as the smallest (under inclusion) closed subsets of M containing the leaf of each of its points.

For the sake of clarity, we shall start with a simpler problem: we are going to define and study the problem in the case of vector fields. Then, we shall generalize to line fields.

The first thing we do is giving the following Definition.

Definition 3.3.1. *Let X be a smooth vector field on a real compact surface M. A subset $\Omega \subseteq M$ is* invariant *for X if, for any $p \in \Omega$, the integral curve for X issuing from p is contained in Ω.*

Ω is minimal *for X if it is closed, invariant for X, non-empty and contains no proper closed invariant subsets.*

The following Theorem, proved by Arthur J. Schwartz in [Sch63], characterizes the possible minimal sets for a smooth vector field.

Theorem 3.3.2 (Schwartz [Sch63])**.** *Let M be a compact connected two-dimensional smooth real manifold. Let X be a smooth vector field on M and Ω be a minimal set for X. Then Ω must be one of the following:*

1. *a fixed point for X;*
2. *a single, closed orbit homeomorphic to S^1;*
3. *all of M, and in this case we have that M is the two-dimensional torus.*

It is actually possible to generalize Theorem 3.3.2 to the case of several vector fields. Let $\mathcal{X} = \{X_i\}$ be a family of smooth vector fields on M. A set $\Omega \subset M$ is *invariant* for $\mathcal{X}$ if it is so for every $X_i \in \mathcal{X}$. Again, we can consider the minimal invariant sets for $\mathcal{X}$, defined as a closed, non-empty invariant sets Ω which do not contain any Ω' satisfying the same properties. The next Theorem gives a characterization of these sets.

Theorem 3.3.3 (Hounie [Hou81])**.** *Let M be a compact connected two-dimensional smooth real manifold. Let $\mathcal{X}$ be a family of smooth vector fields on M, and let Ω be a minimal set for $\mathcal{X}$. Then Ω must be one of the following:*

1. *a point which is a common zero for all the fields in Ω;*
2. *a single, closed $\mathcal{X}$-orbit, i.e., an orbit for every element of $\mathcal{X}$, homeomorphic to S^1;*
3. *all of M.*

Consider now a line field Λ on M and the associated foliation. We say that $\Omega \subset M$ is invariant for Λ if it is a union of leaves and singular points. Again, Ω is *minimal* if it is closed, non-empty, invariant and does not properly contain any Ω' with the same properties. The next Theorem gives a characterization of the possible minimal sets.

Theorem 3.3.4 (Hounie [Hou81])**.** *Let M be a compact connected two-dimensional smooth real manifold and let Λ be a smooth line field with singularities on S. Then a Λ-minimal set Ω must be one of the following*

1. *a singularity of Λ;*
2. *a closed integral curve of Λ, homeomorphic to S^1;*
3. *all of S, and in this case Λ is equivalent to an irrational line field on the torus (i.e., there exists a homeomorphism $\varphi : M \to T$ that transforms the given foliation to the one induced by an irrational line field).*

We shall use this Theorem in the following section, to study the possible ω-limits of geodesics for a meromorphic connection.

3.4 ω-limits sets of geodesics

Once we have defined a meromorphic connection on a Riemann surface and a notion of geodesic for it, we can ask which is the asymptotic behaviour of these geodesics, e.g., which is the shape of their ω-limits. In this section, we give an answer in the case of a compact Riemann surface.

The main idea will be to see a geodesic σ as part of a leaf of a suitable foliation $\mathcal{F}$ on S, and then to apply Theorem 3.3.4 to get some information about the minimal sets for $\mathcal{F}$ contained in the ω-limit of σ. Then, we shall use these information to recover the shape of the ω-limit itself.

The following Lemma provides a smooth line field Λ such that σ is (part of) an integral curve of Λ.

Lemma 3.4.1. *Let S be a compact Riemann surface and ∇ a meromorphic connection on S, with poles $p_1, \ldots, p_r \in X$. Let $S^0 = S \setminus \{p_1, \ldots, p_r\}$. Let $\sigma\colon (\varepsilon_-, \varepsilon_+) \to S^0$ be a geodesic for ∇ without selfintersections, maximal in both forward and backward time. Then there exists a smooth line field Λ on S, which has σ as integral curve and, in a neigbourhood of σ, is singular exactly on the poles of ∇. In particular, on the ω-limit of σ, the line field Λ is singular exactly on the poles of ∇.*

In particular, this precisely means that we can see σ as (the support of) an integral curve for a line field on S and apply Theorem 3.3.4 to study the minimal sets contained in its $\omega-$limit.

Proof. If $A \subset S^0$ is open and small enough, we can find a metric g on A compatible with ∇ and an isometry J between A endowed with g and an open set in $\mathbb{C}$ endowed with the euclidean metric; g is unique up to a positive multiple.

We consider an open cover $\mathcal{A}$ of S^0 made up by open sets A_i with the following properties:

1. $\mathcal{A}$ is locally finite;
2. each A_i is endowed with a metric g_i compatible with ∇, and with an isometry $J_i : A_i \to B_i \subseteq \mathbb{C}$, with B_i convex;
3. if σ intersects A_i, we fix a bound on the possible angles between pairs of lines in $\mathbb{C} \supset B_i$ which contain the image of a part of σ. Say that the range of the possible angles between these lines must be less than $\frac{\pi}{4}$. It is possible to do so thanks to the (continuous) dependence of the solution of the geodesic equation from the initial conditions and the fact that the isometry is continuous, too.

We shall start building a line field on every open set A_i of $\mathcal{A}$. Then we shall show how to use them to find a global line field on S^0, and finally we shall extend it to all of S. To do so, let us fix an open set $A \in \mathcal{A}$, together with its image B. We shall now costruct a smooth flow of curves in B, that will correspond to a smooth flow of curves, and so to a line distribution, in A.

If σ does not cross A, we put on B any smooth vector field which is never zero and consider its associated (regular) foliation.

If σ crosses A, we consider the segments $\sigma_n \subset B$ which are the images of the connected components of $\sigma(I) \cap A$ (recall that J sends geodesics segments in A to Euclidean segments in B). In particular, we recall that, by our assumption on A, the angle between σ_i and σ_j is bounded by $\frac{\pi}{8}$ for every pair (i, j). We fix a line l in $\mathbb{C}$ and use it to define the *inclination* θ_n of every σ_n as the angle between l and the line containing σ_n.

We see that the σ_n's disconnect B in some components. We describe now how to costruct the flow in all these components.

1. Suppose we have a σ_0 that disconnects B in two parts, at least one of which that does not contain any σ_i. In such a zone, we define our flow by means of lines parallel to σ_0 and take the associated line field.

2. In a component B_{01} of B bounded by two segments σ_0 and σ_1 and where we do not have other σ_i's, we define the vector field in the following way: we take two points $y_0 \in \sigma_0$ and $y_1 \in \sigma_1$. By convexity of B, the segment joining them is contained in B. We parametrize this segment as $\tau\colon [0, 1] \to B, \tau(t) = (1-t)y_0 + ty_1$ and, for every $t \in [0, 1]$, we consider the line l_t passing through $\tau(t)$ with inclination $\theta_0 + \varphi(t)(\theta_1 - \theta_0)$, where $\varphi(t)$ is a smooth, non decreasing function, $\varphi(t)\colon [0, 1] \to [0, 1]$, which is 0 in a neighbourhood of 0 and 1 in a neighbourhood of 1. We immediately see that the intersections $\widetilde{l_r} = l_r \cap B$ form a smooth flow on B_{01}, and that this flow is smooth also at the boundary of B_{01} (i.e. near σ_0 and σ_1).

3. Being the σ_n disjoint, the only missing case is when we have some σ_n's accumulating to a line σ_0. We add the limit line to the foliation, so that now all B will be divided in zones where we can apply the arguments of cases 2 or 1. Because of the smooth dependence of the geodesics on the initial conditions, the new line will not cause any problem of smoothness.

Clearly the foliations we built in the different parts of B glue to a global one and so we see that we have costructed a smooth line field in B (and so also in A). Note that the inclinations of these lines are bounded by the inclinations of the σ_n, with the same bound if case 3 did not happen, or a bit more (say, less than $\pi/3$), if we needed to add limit lines. So, the angles betweeen all these lines are less than $\frac{\pi}{3}$.

The next step will be to glue the local line fields we have built on the A_i to a global field on S^0. This means that we must specify, for every point

$p \in S^0$, a direction $\lambda(p)$ in TS^0_p, such that the correspondence $p \to \lambda(p)$ is smooth. To do so, we consider a partition of unity ρ_i subordinated to the cover $\mathcal{A}$. If p belongs to a unique A_i, we use as $\lambda(p)$ the one given by the local costruction above. Otherwise, if p belongs to a finite number of A_i's (recall that the cover is locally finite) we do the following. Suppose that, without loss of generality, $p \in A_1 \cap \cdots \cap A_n$. We have n lines in TS^0_p, given by the local constructions on the B_i's. We use the partition of unity to do a convex combination of (the inclinations, measured with respect to any of the B_i's involved, of) these lines, thus obtaining a line in TS^0_p. We remark that, by the arguments before Theorem 3.2.3, the notion of angle is well defined and does not depend on the particular local metric, and so neither on the open set, chosen to define it.

We have thus obtained a smooth line field on S^0 having S^0 as integral curve. We check that it is non-singular near σ. Clearly, it suffices to check that the foliation is non-singular on the intersections $A_1 \cap \cdots \cap A_m$ which intersect σ. For this, we use the hypothesis on the boundedness of the differences of angles among the lines in B_i. We extend this line field to all of S, adding the poles as singular points. The resulting field satisfies the request of the Lemma, and so we are done. □

The following Theorem describes the possible ω-limits of a ∇-geodesic. Before introducing it, we give a couple of definitions that we shall need in the statement and in the proof.

Definition 3.4.2. *A* saddle connection *for a meromorphic connection on the tangent bundle of a Riemann surface with poles $p_1, \dots, p_r$ is a maximal geodesic $\sigma\colon (-\varepsilon_-, \varepsilon_+) \to S^0$ such that $\sigma(t)$ tends to a pole for both $t \to \varepsilon_-$ and ε_+.*

A graph of saddle connections *is a connected graph in S whose vertices are poles and whose arcs are saddle connections. A* spike *is a saddle connection of a graph which does not belong to any cycle of the graph.*

A boundary graph of saddle connections *(or* boundary graph*) is a graph of saddle connections which is also the boundary of a connected open set of S.*

In the sequel we shall need a notion of *disconnecting graph* more meaningful, in our context, than the purely topological one. In fact, consider a graph that does not disconnect S and suppose to be able to add to it a small disconnecting cycle near one of its poles. The new graph disconnects S, and we may take the small disconnecting cycle as small as we like. So, we look for a notion of *disconnecting graph* which should be invariant under these small modifications of the graph.

We actually reserve the term *disconnecting graph* for the usual topological situation, and we are going to call this property *essentially disconnecting* *graph*. To be able to define it, we need to state what we shall call *desingularization of a graph G to a curve γ.*

Consider a boundary graph G and call A the open set of S whose boundary bA is G. For every vertex p_j of G, take a small open ball B_j centered at that pole. Moreover, consider a small open neigbourhood C_i of every spike of G. We see that the union of the following three sets is a curve in S^0, that we call a *desingularization of* G:

- $G \setminus \left(\bigcup_j B_j \cup \bigcup_i C_i\right)$;
- $\left(\bigcup_j bB_j \cap A\right) \setminus \bigcup_i C_i$;
- $(\bigcup_i bC_i \cap A) \setminus \bigcup_j B_j$.

The rationale behind this definition is the following: we take the graph and the boundary of the neighbourhoods of the spikes outside the small balls at the vertices and we connect them with small arcs (which are the boundaries of the small balls). We see that, in particular, we can (uniformly) approximate the graph G with desingularizing curves with respect to any global metric on the compact Riemann surface S.

Remark 3.4.3. *If the graph bounds two open sets of S, there may happen that, using one of them, the resulting desingularizating curve is not connected. In this case, we consider as desingularization the one which is connected.*

We are now ready to give the following definitions.

Definition 3.4.4. *A cycle of saddle connection which is the boundary of a connected open set in S* essentially disconnects *S if every sufficiently well-approximating desingularization of G disconnects S.*

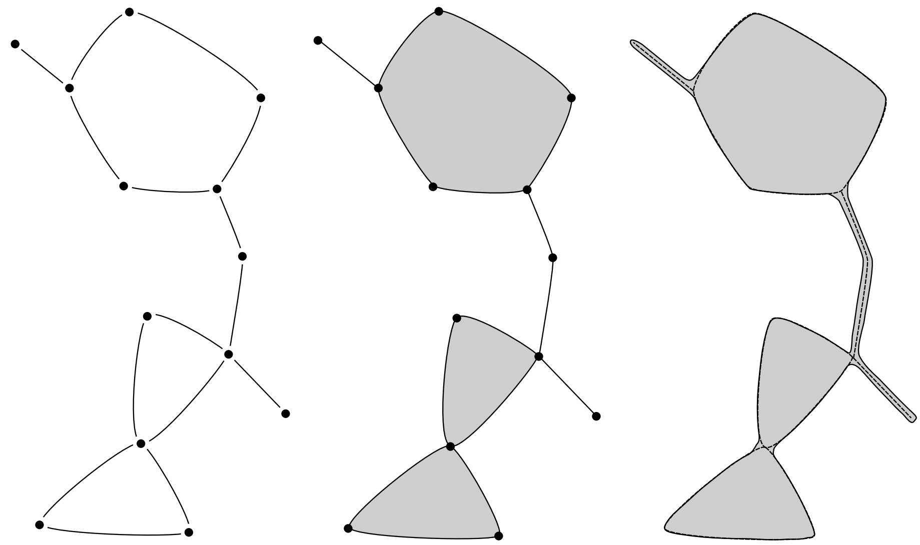
It is clear that, for sufficiently well-approximating desingularization, the fact that one of them is disconnecting implies that all of them are. Because of this we may give the previous definition without specifying which particular desingularization we are using.

It is also clear that our starting example, i.e., the union of a non disconnecting cycle and a small disconnecting one, is not essentially disconnecting.

What we are going to do now is to relate this definition with the property of having disconnecting cycles.

Lemma 3.4.5. *Let G be a graph of saddle connections in S, such that every cycle of G disconnects S. Then G is essentially disconnecting.*

Proof. Because of the fact that every cycle in G disconnects S, we can paint in black, for every cycle $\mathcal{C}$, the part of $S \setminus \mathcal{C}$ not contained in A, the open set of S whose boundary is G. In this way, S will now have every component of $S \setminus G$, except A, painted in black and A, say, white. Now we desingularize



(a) the graph (b) the black region (c) the curve ω

Figure 3.1: Desingularization of the graph

G to a curve γ in the following way, clearly equivalent to the construction above: near a pole connecting two (or more) cycles, we paint in black a little ball and, for every spike, we substitute it with a little black strip following its path, so that now we get a unique black region (see figure 3.1). We call γ the boundary of the black region we have constructed in this way. Clearly γ is (homotopic to) a desingularization of G and, because of the fact that it divides a black region and a white one, by definition γ disconnects S. It follows that G is essentially disconnecting. $\square$

So, we know that if every cycle is disconnecting then the graph is essentially disconnecting, while we can construct examples with an arbitrarily high number of disconnecting cycle (and at least one non disconnecting) which are not essentially disconnecting.

Now we state and prove the main result of this section.

Theorem 3.4.6. *Let S be a compact Riemann surface and ∇ a meromorphic connection on TS, with poles $p_1, \ldots, p_r \in A$. Let $S^0 = S \setminus \{p_1, \ldots, p_r\}$. Let $\sigma\colon [0, \varepsilon_0) \to S^0$ be a maximal geodesic for ∇. Then either*

1. *$\sigma(t)$ tends to a pole of ∇ as $t \to \varepsilon_0$; or*

2. *σ is closed; or*

3. *the ω-limit set of σ is the support of a closed geodesic; or*

4. *the ω-limit set of σ in S is a boundary graph of saddle connections; or*

5. σ intersects itself infinitely many times; or

6. the ω-limit of σ is all of S, and in this case S is a torus.

Moreover, if we are in one of the cases 2, 3 or 4 and if the ω-limit disconnects (for case 2 and 3), or essentially disconnects (for case 4) then σ lives is only one of the resulting components of the complement of the ω-limit. For this component (that is a part P of S) we have that

$$\sum_{p_i \in P} \operatorname{Re}\operatorname{Res}_{p_i}(\nabla) = -1 + 2H_P \ = -\chi_P,$$

i.e., the sum of the residues of the poles contained in this part is equal to $-1 + 2H_P$, where H_P is the rank of $\pi_1(P)$.

Proof. Suppose σ is not closed, nor with infinitely many self-intersections. Then up to changing the starting point of σ we can assume that σ does not self-intersect. Call p_0 the new starting point and W the ω-limit set.

We apply the construction in Lemma 3.4.1, considering p_0 as a virtual pole. In this way, σ becomes maximal in both forward and backward time and the construction can be carried out as done before. In particular, we build a line field on S, singular, in a neighbourhood of σ, exactly on the poles of ∇ and on p_0, such that, in any point contained in the support of σ we have that the line field is generated by the tangent vector of σ.

Being W closed, invariant and non-empty, applying Zorn's Lemma we see that it must contain at least one minimal set for the line field Λ. By Theorem 3.3.4 we know that this minimal set can be all of S (which gives case 6), homeomorphic to S^1 (which gives case 3, see the last Corollary in [Sch63]) or otherwise it must be a singular point for the foliation, that is a pole for ∇ (let it p_0). If W reduces to this pole, we have case 1. Otherwise, we want to prove that we are in the situation described by case 4.

Being W path-connected, there must exist curves in $S^0 \cap W$ that connect z_0 and the pole (or two poles). So, we obtain that there is a (topological) graph, that we call $\widetilde{W}$, inside W.

$\widetilde{W}$ may disconnect S in some components (possibly only one) and σ must lie in one of these components. Moreover, all the arcs of the graph must be in the boundary of this component, in order to be accumulated by σ.

We prove that in the open component with σ, that we call R, there cannot be any point of W. In fact, take a point $z \in S^0 \cap \widetilde{W} \subset W$. We know that σ must accumulate it, and this means that we have segments of σ arbitrarily near z. Start from one of these segments following σ. It will go somewhere in the component R, without self-intersecting, and it will eventually return near z. Now, going on, it is trapped between the graph and the previous part of itself. It has to go to W, so that it cannot return near z, which means that we cannot have points of W in R (see also the last Corollary of [Sch63]).

So, we have found that W must consist of the graph $\widetilde{W}$ and, possibly, of other points in the components of $X \backslash \widetilde{W}$ other from R. But this last possibility is obviously impossible, because σ does not cross the graph (it would mean that we have two leaves of the foliation of Lemma 3.4.1 intersecting in a regular point). So we have found that $W = \widetilde{W}$, which means that W must be only the graph. By the local form of the geodesic, we see that the arcs connecting the poles in $\widetilde{W}$ must be geodesics, too, and this gives 4.

So, we are left to proving the second part of the statement. The first part of the proof already gives that σ lives in only one component and that this must be a part of S. So, we have to prove the formula for the sum of the residues. This is clear for the cases 2 and 3, by Theorem 3.2.3. Let us prove it for case 4, assuming, as given, that W essentially disconnects S.

Consider a generic point $z_0 \in W \cap S^0$. Locally, near z_0, we have the local isometry J with (an open of) $\mathbb{C}$. It means that we can find a geodesic $\tau : [0, \varepsilon) \to S^0$, issuing from z_0, that intersects σ infinitely many times. Let $\sigma(t_n)$ be a succession of points of intersection between σ and τ, with t_n increasing and $\sigma(t_n) \to z_0$. Call σ_n the part of σ between $\sigma(t_n)$ and $\sigma(t_{n+1})$, τ_n the same for τ and denote $\widetilde{\sigma}_n = \sigma_n \cup \tau_n$.

Due to the fact that W is the ω-limit of σ, we have a subsuccession $\widetilde{\sigma}_{n_k}$ of $\widetilde{\sigma}_n$ such that every element is homotopy equivalent to a disconnecting desingularization ω of W. So, in particular, all the elements of $\{\widetilde{\sigma}_{n_k}\}$ disconnect S. We can clearly suppose that $n_{k+1} > n_k + 1$, so that all the $\widetilde{\sigma}_{n_k}$ are disjoint.

We define a notion of "in" and "out" with respect to W in the following way: the outside O is the open part in which lives σ (i.e., the white component) and the inside I is interior part of the union of the other components (i.e., the original black part).

We can do the same with the $\widetilde{\sigma}_{n_k}$, obtaining a succession I_{n_k} of parts of S, where we choose I_{n_k} to be the component of $S \setminus \widetilde{\sigma}_{n_k}$ which contains I.

Then, the sum of the two external angles of $\widetilde{\sigma}_{n_k}$ goes to zero. In fact, up to a subsuccession, we can suppose the the direction of σ at the vertex of I_{n_k} closest to z_0 along τ converges to a direction v_-. The same is true for the directions at the other vertices, that, up to a subsuccession, converge to v_+. Because of the local geometry, we have $v_- = v_+$, so that the sum of the two external angles of $\widetilde{\sigma}_{n_k}$ goes to zero.

But, by finiteness of the number of the poles, we also have that, starting form some $\overline{k}$, every I_{n_k} contains the same poles (which are exactly the poles not contained in the outside of W) and so the sum of their residues is definitely constant. It means that, starting from this $\overline{k}$, the intersections become parallel and we can look for a condition on the sum of the residues "in" and "out" of W by considering the same sum with respect to a I_{n_k}, with k large enough.

So, we get

$$0 = \sum \varepsilon_j = 2\pi \left(1 - 2H_I + \sum_{p_j \in W} + \sum_{p_j \in I} \right) = -2\pi \left(1 - 2H_O + \sum_{p_j \in O} \right),$$

where H_I and H_O denote the ranks of $\pi_1(I)$ and $\pi_1(O)$.

In particular, $\sum^O = -1 + 2H_O$, which is the general analogous of the formula in [AT11] (see Theorem 3.4.8 later). We remark that these last computations hold also in the case in which W does not contain poles, is homeomorphic to S^1 and disconnect S (which corresponds to case 3). □

Remark 3.4.7. *There are examples of all the cases of Theorem 3.4.6 (see Section 3.5, Chapter 6 and [AT11]) except for case 4.*

As a corollary of Theorem 3.4.6, we derive the following result, which was obtained by Abate and Tovena in [AT11]. This Theorem will be very useful in the sequel, in particolar in Chapter 5, to study the behaviour of integral curves for holomorphic homogeneous vector fields in $\mathbb{C}^2$.

Theorem 3.4.8 (Theorem 4.6 in [AT11])**.** *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $p_1, \ldots, p_r \in \mathbb{P}^1(\mathbb{C})$. Let $S^0 = \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, \ldots, p_r\}$. Let $\sigma\colon [0, \varepsilon_0) \to S^0$ be a maximal geodesic for ∇. Then either:*

1. *$\sigma(t)$ tends to a pole of ∇ as $t \to \varepsilon_0$; or*
2. *σ is closed and then surrounds poles $p_1, \ldots, p_g$, with $\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$; or*
3. *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is the support of a closed geodesic surrounding poles $p_1, \ldots, p_g$ with $\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$; or*
4. *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is a a graph of saddle connections; or*
5. *σ intersects itself infinitely many times, and in this case every simple loop of σ surrounds a set of poles whose sum of residues has real part belonging to $(-3/2, -1) \cup (-1, -1/2)$.*

Moreover, in case 4, if the ω-limit disconnects S (i.e., there is at least one cycle in the graph), we have that

$$\sum_P \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$$

where the sum is done on the part P fo the complement of the ω-limit that contains σ.

In particular, a recurrent geodesic either intersects itself infinitely many times or is closed.

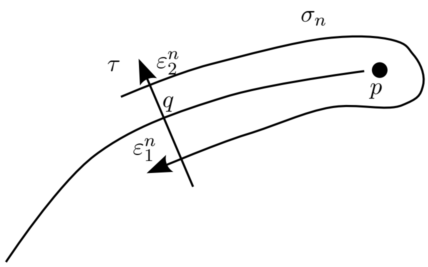


Figure 3.2: A vertex of a spike

Here, by a *simple loop of* σ we mean the restriction of σ to an interval $[t_0, t_1]$ which is a simple loop in S. We notice that, in $\mathbb{P}^1(\mathbb{C})$, every simple loop disconnects $\mathbb{P}^1(\mathbb{C})$ in two parts and so the condition holds in both parts (recall that in this case the sum of the residues of ∇ is -2).

Proof. For the first part we just apply Theorem 3.4.6 and note that H_P will be always zero. The estimate for the simple loops in point 5 follows from Corollary 3.2.6. The last statement of the recurrence follows from the first part. □

Remark 3.4.9. *Let p be a pole belonging to a graph of saddle-connection which is an ω-limit set for a geodesic. Suppose that p is the vertex of only one arc of the graph, i.e., it is a vertex of a spike. Take a transversal to the arc at a point q near p, and consider the passages σ_n of σ near the pole. Let ε_1^n and ε_2^n be the two external angles between σ_n and τ, as in Figure 3.2. Definitively, we have that $\varepsilon_1^n + \varepsilon_2^n = \pi$, because both the parts of σ_n locally at the left and at the right of the arc become parallel (because they tend to the arc). So, we have*

$$\pi = 2\pi\,(1 + \operatorname{Re}\operatorname{Res}_p(\nabla))$$

which gives

$$\operatorname{Re}\operatorname{Res}_p(\nabla) = -\frac{1}{2}.$$

So, in particular, if all the poles have the real part of the residue different from $-1/2$, the graph cannot have spikes.

We see that the arcs in the ω-limit graph not belonging to any cycle are essentially due to the fact that we were able to see the geodesic as a leaf of a foliation instead that as an integral curve of a vector field. Indeed, suppose that we can actually find a vector field on S having σ as integral curve and which, in a neighbourhood of the geodesic, vanishes exactly at the poles of the connections. Then, we see that such an arc in the graph would be accumulated by integral curves for a vector fields, tending to it in two different directions. This is impossible, and so we see that the graph could not have any spike. So, we have the following

Theorem 3.4.10. *Let S be a compact Riemann surface and ∇ a meromorphic connection on TS, with poles $p_1, \ldots, p_r \in A$. Let $S^0 = S \setminus \{p_1, \ldots, p_r\}$. Let $\sigma\colon [0, \varepsilon_0) \to S^0$ be a maximal geodesic for ∇ and suppose we can find a smooth vector field X on S such that for any $t \in [0, \varepsilon_0)$ we have $X(\sigma(t)) = \rho(\sigma(t))\sigma'(t)$, where $\rho(t)$ is a positive real function, and moreover X, in a neighbourhood of the support of σ, vanishes exactly on the poles of ∇. Then either*

1. *$\sigma(t)$ tends to a pole of ∇ as $t \to \varepsilon_0$; or*
2. *σ is closed; or*
3. *the ω-limit set of σ is the support of a closed geodesic; or*
4. *the ω-limit set of σ in S is a graph of saddle connections, consisting only of cycles; or*
5. *σ intersects itself infinitely many times; or*
6. *the ω-limit of σ is all of S, and in this case S is a torus.*

We remark that we actually need that the field we construct is zero only on the poles to avoid the presence of arc in the graph not belonging to any cycle (see below).

It is possible, adapting the proof of Lemma 3.4.1, to look for conditions that ensure the extension of σ' to a vector field, instead that only to a line field. In particular, we see that:

- if in an open set A of the cover, isometric via J to an open $B \subseteq \mathbb{C}$, we have only finitely many zones bounded by passages σ_n's with opposite direction, we can build the field in the same way we built the line field: we only have to take in account the directions of the lines corresponding to the geodesics. To do so, instead of doing a convex combination of the inclinations, we do a convex combination of the tangent vectors. We shall obtain, for any zone bounded by two passages with opposite directions, a central line where the field is zero. Because of the finiteness of the number of the lines, this central line consists of points which are not accumulated by the geodesics, and so we do not have to care whether the resulting field is zero.

- if we have an open set A of the cover, isometric via J to an open $B \in \mathbb{C}$, such that there we have two sequence $\{\sigma_n\}$ and $\{\sigma_m\}$ converging to a limit line $\widetilde{l}$ and such that the all the σ_n's have the same direction and all the σ_m's have the opposite one, we have to define our field as zero on the limit line. In this case, the geodesic locally corresponing to $\widetilde{l}$ is part of the ω-limit and is accumulated form both sides by the geodesic. So, we have an arc in the graph which is not part of a cycle.

3.5 Geodesics on the torus

In this section we study in detail the geodesics for holomorphic connections on the torus. We remark that, by Theorem 3.1.12, we cannot have holomorphic connections on Riemann surfaces different from the torus. Moreover, Theorem 3.4.6 tells us that this is the only case in which all the surface S can be the ω-limit set (and a minimal set for the line field of Lemma 3.4.1). Thus, this study will in particular completely characterize the last possibility in Theorem 3.4.6.

So, our goals are: to characterize holomorphic connections on the torus, and to study the geodesics for them.

We recall that we can see a complex torus as a quotient of $\mathbb{C}$, by the action of a rank-2 lattice, generated over $\mathbb{R}$ by two elements $\lambda_1, \lambda_2 \in \mathbb{C}$. In particular, we can suppose, without loss of generality, that one of the two generators is 1 and the other is some $\lambda \in \mathbb{C}$, with $\operatorname{Im} \lambda > 0$. We shall denote by T_λ the torus associated to λ.

The next results characterize holomorphic connections on a torus.

Lemma 3.5.1. *Every holomorphic connection on a torus is the projection of a connection on the cover $\mathbb{C}$ represented by a constant global form $a\,dz$, with $a \in \mathbb{C}$.*

Proof. We know that the tangent bundle of a torus is trivial. So, the condition for a set of $\{\eta_\alpha\}$ to represent a meromorphic connection becomes, in a trivialization with the identity as trivializing map, that we must have a global η, which is holomorphic, because of the absence of the poles. We can build on $\mathbb{C}$, the cover of the torus, a form $\widetilde{\eta}$, the form associated to the connection induced on $\mathbb{C}$ by the one on the torus. This form $\widetilde{\eta}$ is global and holomorphic. We have to prove that this form is of the form adz for a certain $a \in \mathbb{C}$. But in fact, a holomorphic 1-form on $\mathbb{C}$ is of the form $f(z)dz$ for some holomorphic function f. But we must have $\widetilde{\eta}(z) = \widetilde{\eta}(z+1) = \widetilde{\eta}(z+\lambda)$, which gives $f(z) = f(z+1) = f(z+\lambda)$. But this means that we have an induced holomorphic map on the torus, which must be constant. So $f(z) = a$, for some constant $a \in \mathbb{C}$, and we are done. $\square$

The next step will be to study the geodesics for a holomorphic connection ∇ on a torus. To do this, we shall study geodesics for the associated connection $\widetilde{\nabla}$ on $\mathbb{C}$ represented by some adz and then project them to the torus.

Let $\widetilde{\sigma}$ be a geodesic for $\widetilde{\nabla}$ issuing from a point $z_0 \in \mathbb{C}$ (that may be 0 without loss of generality) with tangent vector v_0.

First, we consider the case with $a = 0$. In this case, the local isometry J is given by $J(z) = cz$, with $c \in \mathbb{C}^*$. So, applying the equation (2.30) in Proposition 2.2.8 we obtain

$$c\widetilde{\sigma}(t) = cv_0 t$$

which means that $\widetilde{\sigma}(t) = v_0 t$ and the geodesics are the euclidean ones.

Let us see what happens if $a \neq 0$. In this case, the local isometry J is given by $J(z) = \frac{1}{a}\exp(az)$. We apply again equation (2.30) in Proposition 2.2.8 to get

$$\frac{1}{a}\exp(a\widetilde{\sigma}(t)) = \exp(az_0)v_0 t + \frac{1}{a}\exp(az_0),$$

which we can solve to obtain

$$\widetilde{\sigma}(t) = \frac{1}{a}\log(1 + av_0 t), \tag{3.10}$$

where log is the branch of the logarithm with $\log 1 = 0$ defined along the half-line $t \mapsto 1 + av_0 t$

So, we see that in general the geodesics for $\widetilde{\nabla}$ are not the euclidean ones. This could be seen also by the form of J, which is not a multiple of the identity. This causes the metric to be distorted with respect to the one of $\mathbb{C}$ and the geodesics to bend. In particular, also the geodesics on the torus will not in general be induced by lines on the covering $\mathbb{C}$.

But we can see from (3.10) that we can also have (non-trivial, i.e., with $a \neq 0$) cases in which a geodesic is in fact the projection of a line, i.e., when av_0 is real. This is precised in the following Proposition.

Proposition 3.5.2. *For any given torus T_λ (i.e., for every parameter λ) and every closed curve σ on it induced by a straight line $\widetilde{\sigma}$ on $\mathbb{C}$, there exists a holomorphic connection ∇ on T_λ (non trivial, i.e., represented by a form $a\,dz$ with $a \neq 0$) such that (the support of) σ is (the support of) a geodesic for ∇.*

Proof. We look for an $a \in \mathbb{C}$ such that the connection represented by the form adz satisfies the statement. Without loss of generality we can suppose that $\widetilde{\sigma}(0) = 0$. Moreover, we have $v_0 = re^{i\theta}$ for some $r \in \mathbb{R}^*$. So we look for an a such that av_0 is real, which gives an a of the form

$$a = \frac{\overline{r}}{e^{i\theta}},$$

for any $\overline{r} \in \mathbb{R}^*$. In fact, we see that in this way we have

$$\widetilde{\sigma}(t) = \frac{e^{i\theta}}{\overline{r}}\log\left(1 + r\overline{r}t\right). \tag{3.11}$$

The argument of log is real, so it gives a straight line, with inclination equal to that of v_0, as desired. $\square$

In particular, Proposition 3.5.2 applies to the case of closed curves on the torus. However, all the geodesics found in this way cannot be periodic. We prove this in the next Proposition, where we also study in detail closed geodesics. We remark here that all geodesics for a holomorphic connection on T_λ are defined for every t.

Proposition 3.5.3. *Let ∇ be a holomorphic connection on the torus T_λ and $\widetilde{\nabla}$ the corresponding connection on the covering $\mathbb{C}$, represented by a global form $\widetilde{\eta} = adz$. Let $\sigma\colon [0,\infty) \to T_\lambda$ be a (non-constant) closed geodesic for ∇. Then:*

- *if $a \neq 0$, σ is the projection of a line of the form $\overline{a}t + b$ on the covering $\mathbb{C}$ and it cannot be periodic;*
- *if $a = 0$, σ is periodic.*

Proof. If $a = 0$, the geodesics are the euclidean ones and so, once σ is closed, it must also be periodic.

So, let us study the problem with $a \neq 0$. In order for σ to be closed and non-trivial, we must have $\widetilde{\sigma}(\overline{t}) = \frac{1}{a}\log(1 + av_0\overline{t}) = n + m\lambda$ for a certain $\overline{t} \in \mathbb{R}$ and $n, m \in \mathbb{Z}$, not both zero ($\widetilde{\sigma}$ is the lift of σ, as usual). Moreover, we want the tangent vector in $\overline{t}$ to be parallel to $\widetilde{\sigma}'(0) = \sigma'(0)$. The derivative $\widetilde{\sigma}'$ is

$$\widetilde{\sigma}'(t) = \frac{1}{a}\frac{1}{1 + av_0t}av_0 = \frac{v_0}{1 + av_0t},$$

so we want $1 + av_0\overline{t} = e^{a(n+m\lambda)}$ to be real. Summing up, we obtain that the conditions are

1. $\exists n, m \in \mathbb{Z}$ such that $\overline{t} = \frac{e^{a(n+m\lambda)}-1}{av_0}$ is real;
2. $e^{a(n+m\lambda)}$ real.

Together, they give av_0 real, which means that v_0 is a real multiple of $\overline{a}$. Being $\widetilde{\sigma} = \frac{1}{a}\log(1 + av_0t)$, we obtain that a geodesic must be the projection of a line of the given form in order to be closed.

Now, when a closed geodesic (that, as we have seen, implies that $\widetilde{\sigma}$ is a line) can be periodic? We want $\sigma'(\overline{t}) = \sigma'(0) = v_0$, which gives $e^{a(n+m\lambda)} = 1$, that is $a(n + m\lambda) = 0$, which is impossible. □

It is easy to see that a geodesic which is the projection of a line, if not closed, is dense, so we have that, for such a geodesic, σ is closed or the ω-limit is all of T.

In the last Theorem of this section we study the geodesics which are not the projection of a line, and give a complete description of the possible ω-limit sets for holomorphic connections on a torus.

Theorem 3.5.4. *Let ∇ be a holomorphic connection on the complex torus T_λ, represented by the global 1-form $a\,dz$. Let $\sigma : [0, +\infty) \to T$ be a maximal (non constant) geodesic for ∇. Then, if $a \neq 0$ the lift $\widetilde{\sigma}$ of σ in the covering $\mathbb{C}$ tends, for $t \to \infty$, to a line l of the form $l(t) = \overline{a}t + b$. So, the ω-limit of σ is (the closure of) the projection of the line l on T_λ. In particular:*

1. *if $\overline{a} \in (\mathbb{Q} \oplus \lambda\mathbb{Q}) \setminus \{0\}$, then:*
   - *σ is closed, non periodic, and is the projection of a line of the form $\overline{a}t + b$ on the covering $\mathbb{C}$, for a certain $b \in \mathbb{C}$; or*
   - *σ is not closed, but its ω-limit is a closed geodesic which is the projection of a line of the form $\overline{a}t + b$ on the covering $\mathbb{C}$, for a certain $b \in \mathbb{C}$.*
2. *if $\overline{a} \neq 0$ and $\overline{a} \notin (\mathbb{Q} \oplus \lambda\mathbb{Q}) \setminus \{0\}$, the ω-limit of σ is all of T_λ.*

In particular, if $a \neq 0$ the ω-limit of σ does not depend on the initial tangent vector v_0.

If $a = 0$, σ is the projection of a line of the form $v_0 t + b$, where $v_0 = \sigma'(0)$. So,

- *if $v_0 \in \mathbb{Q} \oplus \lambda\mathbb{Q}$, then σ is closed and periodic;*
- *if $v_0 \notin \mathbb{Q} \oplus \lambda\mathbb{Q}$, σ is not closed and its ω-limit is all of T_λ.*

Proof. If $a = 0$, we know that the geodesic are the euclidean ones, and the statement follows. So, let us study the case with $a \neq 0$.

We start considering the equation of the lift $\widetilde{\sigma}$,

$$\widetilde{\sigma}(t) = \frac{1}{a} \log(1 + av_0 t) = \frac{\overline{a} \log(1 + av_0 t)}{|a|^2}$$

and its real and imaginary parts

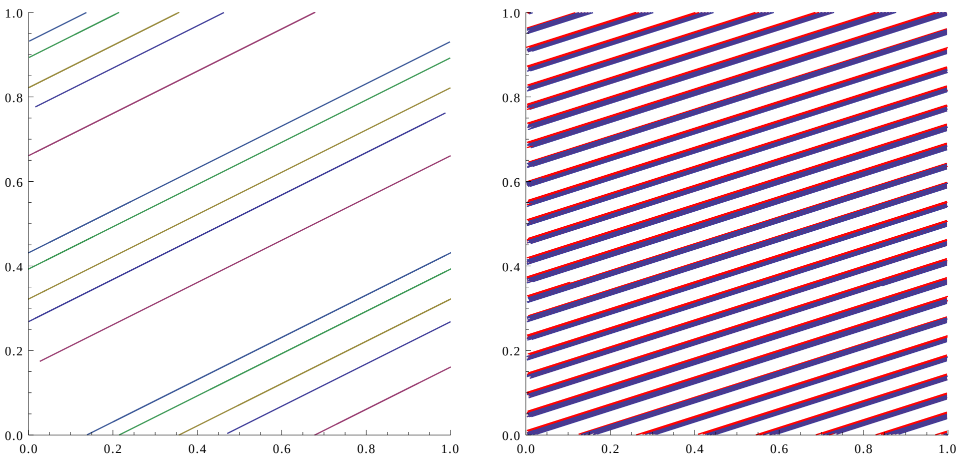
$$\operatorname{Re} \widetilde{\sigma}(t) = \frac{\operatorname{Re}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t)) - \operatorname{Im}(\overline{a}) \operatorname{Im}(\log(1 + av_0 t))}{|a|^2}$$

and

$$\operatorname{Im} \widetilde{\sigma}(t) = \frac{\operatorname{Re}(\overline{a}) \operatorname{Im}(\log(1 + av_0 t)) + \operatorname{Im}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t))}{|a|^2}.$$

For $t \to +\infty$, $\operatorname{Im}(\log(1 + av_0 t))$ is bounded, while $\operatorname{Re}(\log(1 + av_0 t))$ goes to infinity. This means that

$$\begin{aligned}
\lim_{t \to +\infty} \frac{\operatorname{Im} \widetilde{\sigma}(t)}{\operatorname{Re} \widetilde{\sigma}(t)} &= \lim_{t \to +\infty} \frac{\operatorname{Re}(\overline{a}) \operatorname{Im}(\log(1 + av_0 t)) + \operatorname{Im}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t))}{\operatorname{Re}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t)) - \operatorname{Im}(\overline{a}) \operatorname{Im}(\log(1 + av_0 t))} \\
&= \lim_{t \to +\infty} \frac{\operatorname{Im}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t))}{\operatorname{Re}(\overline{a}) \operatorname{Re}(\log(1 + av_0 t))} \\
&= \frac{\operatorname{Im} \overline{a}}{\operatorname{Re} \overline{a}}
\end{aligned}$$



(a) $\overline{a} \in \mathbb{Q} \oplus \lambda\mathbb{Q}$ (b) $\overline{a} \notin \mathbb{Q} \oplus \lambda\mathbb{Q}$

Figure 3.3: Geodesics on the torus

So, $\widetilde{\sigma}(t)$ tends to a line of the form $\overline{a}t + b$ for a certain $b \in \mathbb{C}$. It means that that the ω-limit of σ must be (the closure of) the projection of this line. If $\overline{a}$ is not generated over $\mathbb{Q}$ by 1 and λ, this projection corresponds to a curve dense in the torus T_λ, and so the ω-limit of all the torus T_λ. Conversely, if $\overline{a}$ is generated by 1 and λ, we have that the projection of $\overline{a}t + b$ is a closed curve, and the statement follows. □

In Figure 3.3 we have drawn examples of geodesics in the two differents situations in which $\overline{a} \in \mathbb{Q} \oplus \lambda\mathbb{Q}$ and $\overline{a} \notin \mathbb{Q} \oplus \lambda\mathbb{Q}$, both with $\lambda = i$. In Figure 3.3a we have drawn the geodesics with $a = 2 - i$ and $v(0) = 1 + ki$, with $k = 0, \ldots, 4$, from $t = 500$ to $t = 1000000$ (to highlight the ω-limits). We see that in fact the ω-limits are lines with angular coefficient $1/2$, independently from v. In Figure 3.3b we have done the same with $a = (\pi - i)/200$ and $v(0) = 500$ (in blue) and $v(0) = 500(1 + i)$ (in red), for t from 500 to 4000. Again, we see that the ω-limit depends only on a and not on $v(0)$.

Chapter 4

Holomorphic endomorphisms of complex manifolds

In this Chapter we start talking about complex dynamics. In particular, we shall study holomorphic endomorphisms of a complex manifold fixing a hypersurface. We shall see how to associate to every such holomorphic endomorphism a foliation of the hypersurface and a meromorphic connection on each leaf. Then, we shall study the geodesics for these connections, with particular care to the situation in which the geodesic tends to a singular point for the connection.

The construction will be developed in this general setting. In the next Chapter we shall see how to apply it to the study of the dynamics of holomorphic homogeneous vector fields, showing the connection between the results of this Chapter and the Poincaré-Bendixson theory for geodesics developed in the previous one.

4.1 The main construction

In all this chapter S will be a connected hypersurface in a complex manifold M, with $\dim(M) = n$. A chart for M will be said to be adapted to S if in that chart $S = \{z^1 = 0\}$. We will indicate with $\operatorname{End}(M, S)$ the set of holomorphic endomorphisms of M which fix S pointwise and consider an element $f \in \operatorname{End}(M, S)$, different from the identity.

The following definition was introduced in [ABT04] and will be on primary importance for all this section, and the first step to define a notion of *tangency* between a hypersurface and an endomorphism fixing it.

Definition 4.1.1. *The* f-order of vanishing *at* p *of* $h \in \mathcal{O}_{M,p}$ *is*

$$\nu_f(h, p) = \max\{\mu \in \mathbb{N} \colon h \circ f - h \in \mathcal{I}_{S,p}^{\mu}\},$$

where $\mathcal{I}_S$ *is the ideal sheaf of* S*.*

The order of contact $\nu_f(p)$ *of* f *with* S *in* p *is*

$$\nu_f(p) = \min\{\nu_f(h,p)\colon h \in \mathcal{O}_{M,p}\}.$$

Roughly speaking, the order of contact is a measure of how much f is similar to the identity near S, in a neighbourhood of p. For example, it says how many terms of $h \circ f - h$ we have to calculate, at worst, to be sure that f is not the identity map.

The following Lemmas give a better characterization for $\nu_f(p)$ and, in particular, prove that it does not depend on the point p, allowing us to speak about the order of contact of f with S regardless to the point we use to calculate it. In particular, it says that the number of terms we will calculate at worst to understand that f is not the identity does not depend on p and so, in a certain sense, there are not parts of S where f is more or less tangent to the hypersurface. So, let's start with the following Lemma.

Lemma 4.1.2. *Let* $\left(z^1, \dots, z^n\right)$ *be local coordinates at* $p \in S$*. Given* $h \in \mathcal{O}_{M,p}$*, we have*

$$h \circ f - h = \sum_{j=1}^{n} \left(f^j - z^j\right) \frac{\partial h}{\partial z^j} \mod \mathcal{I}_{S,p}^{2\nu_f(p)}. \tag{4.1}$$

It follows that

$$\nu_f(p) = \min_{j=1,\dots,n} \{\nu_f(z^j, p).\}$$

Proof. We start developing h at p:

$$\begin{aligned} h \circ f(z) - h(z) =& \sum_{j=1}^{n} \left(f^j - p^j\right) \frac{\partial h}{\partial z^j} + \sum_{|I| \geq 2} \frac{\left(f^j - p^j\right)^I}{I!} \frac{\partial^I h}{\partial z^I} \\ &- \sum_{j=1}^{n} \left(z^j - p^j\right) \frac{\partial h}{\partial z^j} - \sum_{|I| \geq 2} \frac{\left(z^j - p^j\right)^I}{I!} \frac{\partial^I h}{\partial z^I} \\ =& \sum_{j=1}^{n} \left(f^j - z^j\right) \frac{\partial h}{\partial z^j} + \sum_{|I| \geq 2} \frac{\left(f^j - z^j\right)^I}{I!} \frac{\partial^I h}{\partial z^I}, \end{aligned}$$

where we used the usual multi-index notation, with $I = (i_1, \dots, i_k) \in \mathbb{N}^k$ and $l^I = \left(l^1\right)^{i_1}, \dots, \left(l^k\right)^{i_k}$. So, we are done if we prove that $\sum_{|I| \geq 2} \frac{\left(f^j - z^j\right)^I}{I!} \in \mathcal{I}_{S,p}^{2\nu_f(p)}$. But we notice that, from the definition of $\nu_f(p)$, we get $(z^j \circ f - z^j) = f^j - z^j \in \mathcal{I}_{S,p}^{\nu_f(p)}$ and we finish recalling that $|I| \geq 2$.

For the second statement, we see that we have $\nu_f(p) \leq \min_{j=1,\dots,n}\{\nu_f(z^j, p)\}$ by definition, while the reverse inequality follows from the first part of the proof. $\square$

Lemma 4.1.3. *For every $h \in \mathcal{O}_{M,p}$, the function $p \to \nu_f(h,p)$ is locally constant. So, it follows in particular that the function $p \to \nu_f(p)$ is constant.*

Proof. Let $\{l^1,\dots,l^k\}$ be a set of generators for $\mathcal{I}_{S,p}$. By definition of $\nu_{h,p}$ there exist germs $g_I \in \mathcal{O}_{M,p}$ such that

$$h \circ f - h = \sum_{|I|=\nu_f(h,p)} l^I g_I. \tag{4.2}$$

By the coherence of the involved sheaves, (4.2) must hold also in a neighbourhood, and so we have $\nu_f(h,p) \leq \nu_f(h,q)$ for q in a neighbourhood of p. Conversely, by definition of $\nu_f(h,p)$ we have that there is at least one $g_{I_0} \notin \mathcal{I}_{S,p}$. But this implies that also $g_{I_0} \notin \mathcal{I}_{S,q}$ for q sufficiently near to p, and this proves the converse inequality.

The second statement follows from the first and the connectedness of S. □

The above results allow us to restate Definition 4.1.1.

Definition 4.1.4. *The* order of contact ν_f *of f with S is*

$$\nu_f = \min\{\nu_f(h,p) \colon h \in \mathcal{O}_{M,p}\}$$

for any point $p \in S$.

We want now to compare this order of contact ν_f to a related quantity, i.e., the minimum of the f-order of vanishing taken only on $\mathcal{I}_{S,p}$ insted of all $\mathcal{O}_{M,p}$, that we could call a sort of "tangential order". So, we give the following Definition.

Definition 4.1.5. *f is* tangential *to S in p if*

$$\min\{\nu_f(h,p) \colon h \in \mathcal{I}_{S,p}\} > \nu_f.$$

We will now prove that also the notion of being tangential does not depend on the chosen point, so that we can speak of f tangential to S regardless to the point. We start with an analougous of Lemma 4.1.2.

Lemma 4.1.6. *Let $\{l^1,\dots,l^k\}$ a set of generators of $\mathcal{I}_{S,p}$. Then, for every $h \in \mathcal{I}_{S,p}$, we have*

$$\nu_f(h,p) \geq \min\{\nu_f(l^1,p),\dots,\nu_f(l^k,p),\nu_f+1\}.$$

In particular, f is tangential at p if and only if

$$\min\{\nu_f(l^1,p),\dots,\nu_f(l^k,p)\} > \nu_f.$$

Proof. The second statement clearly follows from the first and the definition of being tangential. For the first, we write h as $g_1 l^1 + \cdots + g_k l^k$, for some $g_j \in \mathcal{O}_{M,p}$, and develop $h \circ f - h$ as

$$\begin{aligned} h \circ f - h &= \sum_{j=1}^{k} \left[(g_j \circ f) \left(l^j \circ f \right) - g_j l^j \right] \\ &= \sum_{j=1}^{k} \left[(g_j \circ f) \left(l^j \circ f - l^j \right) + (g_j \circ f - g_j) \, l^j \right]. \end{aligned}$$

The first part of the sum gives the terms $\nu_f(l^j, p)$, while in the second we recognise the terms $g_j \circ f - g_j$, which give a term ν_f, each multiplied by l^j, which, being in $\mathcal{I}_{S,p}$, adds 1 to the given order ν_f. □

Remark 4.1.7. *With $M = \mathbb{C}^n$ the last statement is much simpler, because we need only one generator for $\mathcal{I}_{S,p}$. With a generic M, $\mathcal{I}_{S,p}$ may actually require more than one generator.*

The following Corollary follows again from Lemma 4.1.3, that is from the coherence of $\mathcal{I}_{S,p}$.

Corollary 4.1.8. *If f is tangential to S in $p_0 \in S$, then it is tangential to S in every $p \in S$.*

Let us try to understand better what the notions of order of vanishing and of tangential map mean with a concrete example. We consider $M = \mathbb{C}^n$ and the hypersurface S will be $\{z^1 = 0\}$. We work locally, taking 0 as the point near which we work. We start asking which are the self-maps of $\mathbb{C}^n$ which leave $\{z^1 = 0\}$ fixed. We see that, in order to do so, f must be of the form

$$f \begin{pmatrix} z^1 \\ \vdots \\ z^j \\ \vdots \\ z^n \end{pmatrix} = \begin{pmatrix} z^1 + (z^1)^{a_1} \widetilde{f}^1(z^1, \ldots, z^n) \\ \vdots \\ z^j + (z^1)^{a_j} \widetilde{f}^j(z^1, \ldots, z^n) \\ \vdots \\ z^n + (z^1)^{a_n} \widetilde{f}^n(z^1, \ldots, z^n) \end{pmatrix}$$

for some holomorphic functions $\widetilde{f}^j$, not divisible by z^1. We immediately see that $z^j \circ f - z^j = (z^1)^{a_j} \widetilde{f}^j(z^1, \ldots, z^n)$, which means that $\nu_f = \min_{j=1,\ldots,n}\{a_j\}$. Let us see when f is tangential. Locally, $\mathcal{I}_S$ is generated by z^1, so that we see that f is tangential if and only if $a_1 > \min_{j=2,\ldots,n}\{a_j\}$.

Let us continue a bit on this line. We can rewrite f in a form that permits to put in evidence the order ν_f. In fact, we see that there exist functions g_j

such that

$$f\begin{pmatrix} z^1 \\ \vdots \\ z^j \\ \vdots \\ z^n \end{pmatrix} = \begin{pmatrix} z^1 + (z^1)^{\nu_f} g^1(z^1, \dots, z^n) \\ \vdots \\ z^j + (z^1)^{\nu_f} g^j(z^1, \dots, z^n) \\ \vdots \\ z^n + (z^1)^{\nu_f} g^n(z^1, \dots, z^n) \end{pmatrix}$$

with at least one of the g^j's different from zero when restricted to $\{z^1 = 0\}$. In particular, we see that

$$f^j - z^j = (z^1)^{\nu_f} g^j \tag{4.3}$$

and the condition to be tangential is easily seen to be equivalent to $g^1 = 0$ on $S = \{z^1 = 0\}$.

What we have found is true in general: given an endomorphism f fixing an hypersurface S of M, we can find (locally) the functions g^j's such that (4.3) holds and again conclude that f is tangential if and only if locally $g^1 = 0$ when restricted to S.

We want now to study in more detail the functions g^j's just introduced. Obviously, they depend on the chart we are working with. The key fact is that we can construct a global section of a suitable bundle over S, whose main ingredients are precisely the functions g_j's, that will turn out to be extremely useful to study the dynamics of f. This has been done, even in more generality, in [ABT04], considering the more general problem in which S is a submanifold of M of any codimension. We will state and prove it only in the situation we are interested in, i.e., the one in which S is an hypersurface of M. The general proof uses the same ideas and is only more complicated due to the several indices needed.

Proposition 4.1.9. *Consider the (local) sections*

$$\chi_{f,U} = \sum_{j=1}^{n} g^j \frac{\partial}{\partial z^j} \otimes \left(dz^1\right)^{\otimes \nu_f}$$

of the bundle $TM_{|S} \otimes (N_S^*)^{\otimes \nu_f}$. They define a global section of this bundle over S.

Proof. Take two overlapping charts (U, z) and $\left(\hat{U}, \hat{z}\right)$, with $S|_U = \{z^1 = 0\}$ and $S|_{\hat{U}} = \{\hat{z}^1 = 0\}$ and consider the resulting sets of functions $\{g^j\}$ and $\{\hat{g}^j\}$. We want to prove that, on $U \cap \hat{U}$, we have

$$\sum_{j=1}^{n} g^j \frac{\partial}{\partial z^j} \otimes \left(dz^1\right)^{\otimes \nu_f} = \sum_{j=1}^{n} \hat{g}^j \frac{\partial}{\partial \hat{z}^j} \otimes \left(d\hat{z}^1\right)^{\otimes \nu_f} \mod \mathcal{I}_S,$$

i.e., that they are equal when restricted to S.

First, we see that, modulo $\mathcal{I}_S$, $d\hat{z}^1 = \frac{\partial \hat{z}^1}{\partial z^1} dz^1$ ($\frac{\partial \hat{z}^1}{\partial z^j} = 0$ on S for $j \neq 1$) and $\frac{\partial}{\partial \hat{z}^j} = \sum_k \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial}{\partial z^k}$. So, we get

$$\sum_{j=1}^n \hat{g}^j \frac{\partial}{\partial \hat{z}^j} \otimes \left(d\hat{z^1}\right)^{\otimes \nu_f} =$$

$$= \sum_{j=1}^n \hat{g}^j \left(\sum_k \frac{\partial z^k}{\partial \hat{z}^j} \frac{\partial}{\partial z^k}\right) \left(\frac{\partial \hat{z}^1}{\partial z^1}\right)^{\nu_f} \otimes \left(dz^1\right)^{\otimes \nu_f} \quad \bmod \mathcal{I}_S$$

$$= \sum_k \left(\left(\frac{\partial \hat{z}^1}{\partial z^1}\right)^{\nu_f} \sum_j \frac{\partial z^k}{\partial \hat{z}^j} \hat{g}^j\right) \frac{\partial}{\partial z^k} \otimes \left(dz^1\right)^{\otimes \nu_f} \quad \bmod \mathcal{I}_S.$$

So, we need only to prove that $g^k = \left(\frac{\partial \hat{z}^1}{\partial z^1}\right)^{\nu_f} \sum_j \frac{\partial z^k}{\partial \hat{z}^j} \hat{g}^j \mod \mathcal{I}_S$. To do this, consider the function z^k and apply (4.1) in the chart with the coordinate $\hat{z}$. We find that, modulo $\mathcal{I}_S^{2\nu_f}$,

$$z^k \circ f - z^k = \sum_{j=1}^n \left(\hat{z}^j \circ f - \hat{z}^j\right) \frac{\partial z^k}{\partial \hat{z}^j} = \sum_j \left(\hat{z}^1\right)^{\nu_f} \hat{g}^j \frac{\partial z^k}{\partial \hat{z}^j}.$$

On the other hand, $z^k \circ f - z^k$ is also equal to $\left(z^1\right)^{\nu_f} g^k$, and equating the two results we get the desired identity, bacause $\hat{z}' = z' \frac{\partial \hat{z}'}{\partial z'} \mod \mathcal{I}_S^2$. $\square$

Recalling that we have a natural isomorphism between $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ and $\mathrm{Hom}\left(N_S^{\otimes \nu_f}, TM|_S\right)$, we can interpret the section χ_f just built as a section of this latter bundle over S, and so as a morphism X_f between $N_S^{\otimes \nu_f}$ and $TM|_S$. So, we get the following Definition.

Definition 4.1.10. *Given $f \in \mathrm{End}(M, s)$, $f \neq id_M$, χ_f is the* canonical section *and the associated morphism X_f is the* canonical morphism.

We have the following Lemma, which gives a characterization of tangential maps in terms of the canonical morphism.

Lemma 4.1.11. *$f \in \mathrm{End}(M, S)$ is tangential to S if and only if the image of the canonical morphism is contained in TS.*

Proof. We know that, given the usual writing $f^j - z^j = (z^1)^{\nu_f} g^j$, the map f is tangential if and only if $g^1|_S = 0$. But this precisely means that the image of the canonical morphism is contained in TS. $\square$

We remark that $N_S^{\otimes \nu_f}$ is a line bundle, and so it means that X_f, outside its zeroes, is an isomorphism with the image. We stress this thing in the following way.

Definition 4.1.12. *Let $f \in \mathrm{End}(M, s)$ be tangential to S. A point $p \in S$ is said to be singular for f if $X_f(p) = 0$. We will use the notations* $\mathrm{Sing}(f)$ *for the set of singular points and S^0 for $S \setminus \mathrm{Sing}(f)$.*

As we just remarked, X_f is an isomorphism with its image on S^0.

Let us sum up what we have found till now. We have our hypersurface S, two vector bundles on it, $N_S^{\otimes \nu_f}$ and $TM|_S$, and we defined a morphism between them, X_f:

$$\begin{tikzcd} N_S^{\otimes \nu_f} \arrow[rr, "X_f"] \arrow[dr] & & TM_{|S} \arrow[dl] \\ & S & \end{tikzcd}$$

In particular, if f is tangential, we can replace $TM|_S$ with TS by Lemma 4.1.11, and so we have a diagram:

$$\begin{tikzcd} N_S^{\otimes \nu_f} \arrow[rr, "X_f"] \arrow[dr] & & TS \arrow[dl] \\ & S & \end{tikzcd}$$

Restricting to S_0, we know that X_f become injective:

$$\begin{tikzcd} N_S^{\otimes \nu_f} \arrow[rr, hook, "X_f"] \arrow[dr] & & TS \arrow[dl] \\ & S & \end{tikzcd}$$

In particular, via X_f, we can think of $N_{S^0}^{\otimes \nu_f}$ as a 1-dimensional subbundle of TS^0. So, using X_f we can try to differentiate sections of bundles over S^0 with respect to sections of $N_{S^0}^{\otimes \nu_f}$, introducing what we hope will be a a connection, setting

$$\nabla_u(\overline{s}) := \nabla_{X_f(u)}(\overline{s}).$$

It turns out that this definition does not work, but that it is possible to obtain a sort of connection (a *partial connection*) anyway, with a slightly more involved definition. This was done in [ABT04] and it is what we are going to do now.

First, we define in a proper way what we would like a partial connection to be, and then we shall give an expression for the partial connection in this case. Finally, we shall prove that our proposal is in effect a partial connection.

We give the definition of partial connection in a general setting, then we shall specialize to our situation.

Definition 4.1.13. *Let S^0 be a complex manifold and F and E two holomorphic vector bundles over S^0. Suppose we have a morphism $X : F \to TS^0$. A* partial holomorphic X-connection*, or* holomorphic action *of F on E, is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that*

$$\nabla_u(gs) = X(u)(g)s + g\nabla_u(S) \tag{4.4}$$

for all $s \in \mathcal{E}, u \in \mathcal{F}$ and $g \in \mathcal{O}_{S^o}$, where as usual $\mathcal{E}$ and $\mathcal{F}$ are the sheaves of holomorphic sections of E and F.

If X is injective we can identify F with its image in TS^0. In this case we shall call ∇ a partial holomorphic connection along $X(F) \subset TS^0$.

Finally, if both E and F extend to a larger manifold S, with S^0 dense in S, and again X is injective on S^0, we will call ∇ a partial meromorphic connection along X *on E.*

Clearly, in our situation we have $F = N_{S^0}^{\otimes \nu_f}$ and we are in the last case, with S^0 dense in S and $X = X_f$ injective on S^0. The next Theorem says that we can actually define a partial meromorphic connection along X_f on $E = N_S$.

Theorem 4.1.14. *Let S be a hypersurface in a complex manifold M and let $f \in \mathrm{End}(M, S)$, $f \neq id_M$ tangential to S. Then we can define a partial meromorphic connection ∇ along the canonical morphism X_f on N_S by setting*

$$\nabla_u(s) = \pi\left([X_f(\widetilde{u}), \widetilde{s}]_{|S}\right), \tag{4.5}$$

where $s \in \mathcal{N}_S, u \in \mathcal{N}_S^{\otimes \nu_f}, \pi : \mathcal{T}_{M,S} \to \mathcal{N}_S$ is the canonical projection, $\overline{s} \in \mathcal{T}_{M,S}$ is any element such that $\pi\left(\overline{s}_{|S}\right) = s$ and $\overline{u} \in \mathcal{T}_{M,S}^{\otimes \nu_f}$ is any element such that $\pi\left(\widetilde{u}_{|S}\right) = u$.

Proof. We have to prove that the definition does not depend on the chosen chart and on the extensions $\widetilde{s}$ and $\widetilde{u}$. Then we shall prove that it actually defines a partial meromorphic connection.

First, we note that the section χ_f is (locally) defined also in a neighbourhood of S, with value 0 if one of the terms in the tensor product is different from dz^1, which amounts to say that also the morphism is locally defined and zero on elements of the basis with at least one term tangent to S. We still call this extension X_f

We refer to [ABT04], Sections 4 and 5, for the proof of the independence from the chosen chart. Here we prove that the definition does not depend on the particular extensions $\widetilde{s}$ and $\widetilde{u}$. Take extensions $\widetilde{s}, \widetilde{s}', \widetilde{u}, \widetilde{u}'$. We immediately see that $(\widetilde{s} - \widetilde{s}')_{|S}$ is tangent to S and that $(\widetilde{u} - \widetilde{u}')_{|S}$ is made up by terms of the form $u_1 \otimes \cdots \otimes u_{\nu_f}$, with at least one u_i tangent to S. So, $X_f\left(\widetilde{u} - \widetilde{u}'\right)_{|S} = 0$.

We want to prove that

$$\left[X_f(\widetilde{u}'), \widetilde{s}'\right]_{|S} - [X_f(\widetilde{u}), \widetilde{s}]_{|S}$$

is tangent to S, so that it would vanish under π. But we can write

$$\begin{aligned}\left[X_f(\widetilde{u}'), \widetilde{s}'\right]_{|S} =& [X_f(\widetilde{u}), \widetilde{s}]_{|S} + \left[X_f(\widetilde{u}), \widetilde{s}' - \widetilde{s}\right]_{|S} \\ &+ \left[X_f(\widetilde{u}' - \widetilde{u}), \widetilde{s}\right]_{|S} + \left[X_f(\widetilde{u}' - \widetilde{u}), \widetilde{s}' - \widetilde{s}\right]_{|S}.\end{aligned}$$

So, we want to prove that the sum of the last three terms is tangent to S. But the last two are zero because we saw that $X_f\left(\widetilde{u} - \widetilde{u}'\right)_{|S} = 0$ and the first of the three is the commutator of two terms, both tangent to S.

Now, we prove that the definition actually gives a partial meromorphic connection. To do this, we need to prove the equality (4.4) ($\mathbb{C}$-linearity is clear). First, we would like to prove that

$$\nabla_{gu}(s) = g\nabla_u(s),$$

with $g \in \mathcal{O}_S$. So, let us extend g to some $g \in \mathcal{O}_M$. By the definition of ∇ we have

$$\begin{aligned}\nabla_{gu}(s) =& \pi\left([\widetilde{g}X_f(\widetilde{u}), \widetilde{s}]\,|_S\right) \\ =& \pi\left(g[X_f(\widetilde{u}), \widetilde{s}]|_S - \widetilde{s}(\widetilde{g})|_S X_f(\widetilde{u})\right) \\ =& g\,\pi\left([X_f(\widetilde{u}, \widetilde{s})]|_S\right) \qquad \text{(because the second term is tangential)} \\ =& g\,\nabla_u(s).\end{aligned}$$

Finally, we verify the Leibniz rule:

$$\begin{aligned}\nabla_u(gs) =& \pi\left([X_f(\widetilde{u}), \widetilde{g}\,\widetilde{s}]_{|S}\right) \\ =& \pi(g[X_f(\widetilde{u}), \widetilde{s}]|_S + X_f(\widetilde{u})(\widetilde{g})\widetilde{s}|S) \\ =& g\nabla_u(s) + \pi(X_f(u)g|_S s)\end{aligned}$$

because, since $X_f(u) \in \mathcal{I}_S$, its action on $\widetilde{g}$ depends only on $\widetilde{g}|_S = g$. This completes the proof. $\square$

So, we have constructed a partial meromorphic connection on N_S. Our next step will be to use it to define a partial meromorphic connection on

$N_S^{\otimes \nu_f}$. To do so, and also to understand better what is happening, we want to find a concrete formula for the connection found so far.

Let us denote by $\partial_1 := \pi\left(\frac{\partial}{\partial z^1}\right)$ a local generator of N_{S_0}, so that $\partial_1 \otimes \cdots \otimes \partial_1 = \partial_1^{\otimes \nu_f}$ is a local generator for $N_{S_0}^{\otimes \nu_f}$. The following Lemma gives the desired formula.

Lemma 4.1.15. *Let ∇ the partial meromorphic connection along the canonical morphism X_f on N_S defined by* (4.5)*. Then, locally we have*

$$\nabla_{\partial_1{}^{\otimes \nu_f}} \partial_1 = -\frac{\partial g^1}{\partial z^1}|_{U \cap S^0} \partial_1 .$$

Proof. We can use $\widetilde{s} = \frac{\partial}{\partial z^1}$ as (local) extension of ∂_1 and $\widetilde{u} = \left(\frac{\partial}{\partial z^1}\right)^{\otimes \nu_f}$ as (local) extension of $\partial_1^{\otimes \nu_f}$. So, we have

$$\begin{aligned} \nabla_{\partial_1{}^{\otimes \nu_f}} \partial_1 =& \pi\left(\left[X_f\left(\frac{\partial}{\partial z^1}^{\otimes \nu_f}\right), \frac{\partial}{\partial z^1}\right]_{|S}\right) = \pi\left(\left[\sum_{j=1}^n g^j \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^1}\right]_{|S}\right) \\ =& \pi\left(\left(-\sum_{j=1}^n \frac{\partial g^j}{\partial z^1}\frac{\partial}{\partial z^1}\right)_{|S^0}\right) = -\frac{\partial g^1}{\partial z^1}_{|S^0} \partial_1 . \end{aligned}$$

□

Given a line bundle, in this case N_S, and a connection on it, it is possible to induce a connection on some tensor power of the bundle. In our situation, we can induce a partial meromorphic connection on $N_S^{\otimes \nu_f}$ by setting

$$\nabla(s_1 \otimes \cdots \otimes s_{\nu_f}) = \sum_{j=1}^{\nu_f} s_1 \otimes \cdots \otimes \nabla s_j \otimes \cdots \otimes s_{\nu_f} .$$

It is easy to verify that this actually is a partial meromorphic connection on $N_S^{\otimes \nu_f}$, and also to generalize in a obvious way the formula of Lemma 4.1.15, obtaining

$$\nabla_{\partial_1{}^{\otimes \nu_f}}\left(\partial_1{}^{\otimes \nu_f}\right) = -\nu_f \frac{\partial g^1}{\partial z^1}|_{U \cap S^0} (\partial_1)^{\otimes \nu_f} . \tag{4.6}$$

Having at disposal a connection, it is natural to define a concept of geodesic related to it, as we did in Chapter 2. This is what we are going to do. Before doing that, we will rephrase a bit the results we have just talked about.

To do so, consider the canonical morphism $X_f : N_S^{\otimes \nu_f} \to TS$ associated to a tangential map f. We see that the image of X_f is given, on S^0, by a rank 1 subbundle of TS^0, which means that we have a distribution of lines in TS, i.e., we obtain a (complex 1-dimensional) foliation $\mathcal{F}_f$ on S, regular precisely on S^0. On S^0, we can look for a generator of this foliation, i.e., for a section v_0 of TS which in every point of S^0 generates the associated line of the distribution (and is zero on the singular points). It is easy to see that we can simply consider

$$v_0 = X_f \left(\partial_1^{\otimes \nu_f} \right) = \sum_{j=2}^{n} g^j_{|U \cap S} \frac{\partial}{\partial z^j}. \tag{4.7}$$

Our goal now is to define a partial meromorphic connection $\nabla^0 : \mathcal{F}_f \to \mathcal{F}_f^* \otimes \mathcal{F}_f$ along the identity on $\mathcal{F}_f$, holomorphic on S^0, to be thought in the following way: given a leaf F_f of the induced foliation, we want to differentiate fields tangent to the leaf (i.e., elements of $\mathcal{F}_f$) with respect to other fields tangent to F_f, (the reason fo $\mathcal{F}_f^*$). To do that, we have at our disposal the connection ∇, which permits to differentiate sections of $N_S^{\otimes \nu_f}$ with respect to other sections of $N_S^{\otimes \nu_f}$, and the canonical morphism $X_f : N_S^{\otimes \nu_f} \to TS$.

It is then natural to put

$$\nabla^0_v s = X_f \left(\nabla_{X_f^{-1}(v)} X_f^{-1}(s) \right) \tag{4.8}$$

for s and v tangent to the leaf, using X_f to bring them back to $N_S^{\otimes \nu_f}$.

Putting together all the leaves, we see that we have a (partial) connection on S which permits to differentiate fields tangent to any leaf with respect to another field tangent to the leaf, which, by costruction, restricts to a standard holomorphic connection on each leaf of $\mathcal{F}_f$.

Remark 4.1.16. *In particular, we see that we have constructed a foliation of the hypersurface S in Riemann surfaces and, on each of these leaves F, we have two line bundles, $(N_S)|_F$ and the tangent TF to the leaf itself, with a morphism X_f between them. Moreover, we have a standard holomorphic connection on the first bundle and an induced connection on the second. We see that we precisely are in the setting we studied in Chapter 2.*

In Chapter 2, we defined the geodesics with respect to the connection on the first bundle and the morphism between the two (see Definition 2.1.17). Here, it is then natural to introduce the following Definition.

Definition 4.1.17. *A ∇^0-geodesic is a real curve $\sigma : I \to S^0$ such that $\sigma'(t) \in (\mathcal{F}_t)_{\sigma(t)}$ for all $t \in I$ and $\nabla^0_{\sigma'}\sigma' = 0$.*

We remark that this definition is consistent with the connection ∇^0 we have, in the sense that to verify the property of being geodesic we only need to be able to differentiate a vector field, σ', tangent to a leaf, with respect to the same σ', and so with respect to a field again tangent to the leaf.

There is one situation in which the similarity with Chapter 2 is even stronger: if M has complex dimension 2, S is a Riemann surface itself and so the foliation consists of only one leaf. So, in this situation, the partial connection is in fact a standard connection on all of S. In the sequel, this will be in fact the case we will be mostly interested in.

In the remaining parto of this section, we are going to study the local form of the geodesics and to introduce a global field, whose integral curves will be exactly the geodesics, as we did in Chapter 2.

We start with the first task and, as it happened in Chapter 2, we will derive an equation for the ∇^0-geodesics which will have the form of the common geodesic equation in differential geometry. We have just remarked that, for a geodesic σ, the tangent field σ' is tangent to the leaf, and it means that it is multiple of the generator v_0 introduced before. We define σ_0' to be this multiple, i.e., we have

$$\sigma' = \sigma_0' v_0. \tag{4.9}$$

Moreover, by definition of v_0, it means that

$$\left(\sigma^j\right)' = \sigma_0' \left(g^j \circ \sigma\right)$$

for $j = 1, \ldots, n$ and, in particular, that $\sigma^1 \equiv 0$. We are then ready to state and prove the following Lemma.

Lemma 4.1.18. *$\sigma \colon I \to S^0$ is a ∇^0-geodesic if and only if $X_f^{-1}(\sigma') = \sigma_0' \partial_1^{\otimes \nu_f}$, with*

$$\left(\sigma_0'\right)' - \nu_f \left(\frac{\partial g^1}{\partial z^1} \circ \sigma\right) \left(\sigma_0'\right)^2 = 0. \tag{4.10}$$

Proof. The fact that S^0 is not of complex dimension 1 causes not much trouble to the proof of equation (2.12). In fact, the condition of being ∇^0-geodesic here is

$$\nabla^0_{\sigma'} \sigma' \equiv 0 \Leftrightarrow \left(\nabla_{X_f^{-1}(\sigma')} X_f^{-1}(\sigma')\right) \equiv 0$$

that, by (4.9) and (4.7), becomes

$$\nabla_{\left(\sigma_0' \partial_1^{\otimes \nu_f}\right)} \left(\sigma_0' \partial_1^{\otimes \nu_f}\right) \equiv 0.$$

By definition of ∇, we have

$$(\sigma_0')[X_f(\partial_1^{\otimes \nu_f})(\sigma_0') + \sigma_0' \nabla_{\partial_1^{\otimes \nu_f}} \partial_1^{\otimes \nu_f}] \equiv 0,$$

which is, by (4.7) and (4.6),

$$\begin{aligned} 0 \equiv &(\sigma_0')\left(v_0(\sigma_0') - \sigma_0'\nu_f\frac{\partial g^1}{\partial z^1}\right)\\ = &(\sigma')(\sigma_0') - \nu_f(\sigma_0')^2\frac{\partial g^1}{\partial z^1}, \end{aligned}$$

which is the assertion. □

Our next goal is to find a (holomorphic) vector field on the total space of a bundle on S, whose integral curves are precisely the geodesics for the connection ∇. The problem in using TS is that we only have geodesics contained in the leaves of the foliation (and so, for example, it is not true that for any point and "speed", i.e., tangent vector at that point, it is possible to find a geodesic having these as starting conditions). We may solve this problem by constructing a field for every leaf, but then we would have to glue them together (and also to take into account the topology of the leaves). So, it turns out that the best thing to do is not to use TS, or part of it, as the bundle, but to define a field on $N_S^{\otimes\nu_f}$ instead, and ask to have the preimages under X_f of the geodesics as the integral curves of the field.

So, we define locally a field G on the total space of $p : N_S^{\otimes\nu_f} \to S$, by

$$G_{|p^{-1}(U)} = \left(\sum_{j=2}^{n} g^j_{|U\cap S} v\frac{\partial}{\partial z^j}\right) + \nu_f\frac{\partial g^1}{\partial z^1}|_{U\cap S}\,(v)^2\,\frac{\partial}{\partial v} \tag{4.11}$$

where, as in Chapter 2, v stands for the coordinate of the fiber of the line bundle, in this case $N_S^{\otimes\nu_f}$.

The last Proposition of this section ensures that the local fields $G_{p^{-1}(U)}$ actually glue to a global field G, which safisfied the required property about its integral curves.

Proposition 4.1.19. *Let $f \in \mathrm{End}(M,S)$ be tangential. Then the field G defined with* (4.11) *is a global holomorphic vector field on the total space of $p : N_S^{\otimes\nu_f} \to S$ and for a curve $\sigma : I \to S^0$ the following are equivalent:*

1. *σ is ∇^0-geodesic;*

2. *the image of σ is contained in a leaf of $\mathcal{F}_f$ and $X_f^{-1}(\sigma')$ is an integral curve of G.*

Proof. If M has complex dimension two, and so S is a Riemann surface, the proof is the same as in Proposition 2.1.18. In fact, in this case we have

$$\frac{\partial}{\partial z^2} = \frac{1}{g^2}v_0$$

and the 1-form η representing ∇^0 is

$$\eta = -\left(\nu_f \frac{1}{g^2}\frac{\partial g^1}{\partial z^1}\right) dz^2 \tag{4.12}$$

So, the calculations are the same as in Proposition 2.1.18 with $X_\alpha = g^2$, and so with $X_\alpha \eta_\alpha(\partial_\alpha) = g^2 \eta\left(\frac{\partial}{\partial z^2}\right) = -\nu_f \frac{\partial g^1}{\partial z^1}$. The general case follows from similar computations and the second statement follows from the geodesic equation (4.10), as in the proof of Proposition 2.1.18. □

4.2 Local study of singularities

In this section we are going to study the geodesic flow for the connection ∇ near the singularities of the morphism X.

We know that a geodesic must be contained in one leaf of the foliation given by the canonical morphism. So, the reason for this study is clear: to understand the dynamics of the geodesics for the connection ∇^0, we need to know two things: one is the dynamics of the foliation itself of the hypersurface S, and the other is the behaviour of the geodesics inside the leaf they live in. In this section, we are going to study this second problem. Because of the characterization of the possible ω-limits (in the compact case) of Theorem 3.4.6, it is natural to try and study the singularities of the morphism and the geodesics tending to these.

We remark that, in the case in which the hypersurface is a Riemann surface, i.e., the ambient manifold M is of complex dimension 2, the first problem, to understand the dynamics of the foliation, becomes trivial and so this study will give a complete picture of what is going on, at least in this situation.

So, in all this section we shall restrict ourself to consider bundles over a Riemann surface. Moreover, because of the fact that the arguments will be essentially the same, and in fact the exposition will become more clear, we shall work in the following (slightly) more general setting: S will be a Riemann surface, E a line bundle over it and X a morphism $X\colon E \to TS$, singular on a set $\mathrm{Sing}(X) \subset S$ and ∇ a meromorphic connection on E, holomorphic on $S^0 := S \setminus \mathrm{Sing}(X)$. We shall make the following two assumptions: $\mathrm{Sing}(X)$ will be a discrete set and the geodesic field of ∇, defined on $E|_{S^0}$, will extend holomorphically to all of E. We see that, in the case we are interested in, these assumptions are actually satisfied. ∇^0 will be the connection induced on TS by ∇ via X, represented by the form η^0.

Our aim is to study the geodesic flow near the singular points of X. To do this, we consider a singularity p_0 and consider a trivializing chart $(U_\alpha, z_\alpha, e_\alpha)$ at p_0, such that, in this chart, p_0 becomes the point 0.

The first thing we do is relating the form η to η^0, by means of the function X_α defined as in (2.10) (in particular, we remark that from $X(p_0) = 0$ we get $X_\alpha(0) = 0$). To avoid confusions between the two connections, from now on shall denote with ρ the residues for ∇ and with Res the residues of ∇^0.

Lemma 4.2.1. *The two forms η and η^0 representing ∇ and ∇^0 are related by*

$$\eta^0_\alpha = \eta_\alpha - \frac{1}{X_\alpha} dX_\alpha \tag{4.13}$$

and therefore

$$\mathrm{Res}_{p_0}(\nabla^0) = \rho_{p_0}(\nabla) - \mathrm{ord}_{p_0}(X_\alpha). \tag{4.14}$$

Proof. Because of the fact that S has complex dimension 1, we are done if we prove that

$$\eta^0 \left(\partial_\alpha\right) = \eta \left(\partial_\alpha\right) - \frac{1}{X_\alpha} dX_\alpha \left(\partial_\alpha\right).$$

By definition of η^0 we have

$$\nabla^0_{\partial_\alpha} \left(\partial_\alpha\right) = \eta^0 \left(\partial_\alpha\right) \otimes \partial_\alpha.$$

On the other side, by definition of ∇^0, we have

$$\begin{aligned}
\nabla^0_{\partial_\alpha} \left(\partial_\alpha\right) =& X \left(\nabla_{X^{-1}(\partial_\alpha)} \left(X^{-1}\left(\partial_\alpha\right)\right)\right) = X \left(\nabla_{X^{-1}(\partial_\alpha)} \left(\frac{1}{X_\alpha} e_\alpha\right)\right) \\
&= X \left(X \circ X^{-1}\left(\partial_\alpha\right) \left(\frac{1}{X_\alpha}\right) \otimes e_\alpha + \frac{1}{X_\alpha} \nabla_{X^{-1}(\partial_\alpha)} e_\alpha\right) \\
&= X \left(-\frac{1}{(X_\alpha)^2} dX_\alpha(\partial_\alpha) \otimes e_\alpha + \frac{1}{X_\alpha} \eta_\alpha \otimes e_\alpha\right) \\
&= -\frac{1}{X_\alpha} dX_\alpha(\partial_\alpha) \otimes \partial_\alpha + \eta_\alpha \otimes \partial_\alpha
\end{aligned}$$

This proves (4.13), and (4.14) follows from (4.13). □

The assumption that G extends holomorphically to all of E means that the product $X_\alpha \eta_\alpha(\partial_\alpha)$ is a holomorphic function, for every chart U_α. We call Y_α this function, so that we can rewrite η_α as $\frac{Y_\alpha}{X_\alpha} dz_\alpha$ and

$$G_\alpha = X_\alpha v_\alpha \partial_\alpha - Y_\alpha \left(v_\alpha\right)^2 \frac{\partial}{\partial v_\alpha}. \tag{4.15}$$

We will also need to consider the order of the two functions X_α and Y_α, so that we define numbers $\mu_{X,\alpha}, \mu_{Y,\alpha}$ and functions h^X_α, h^Y_α non-vanishing at 0 (i.e., at p_0) such that $X_\alpha = (z_\alpha)^{\mu_{X,\alpha}} h^X_\alpha$ and $Y_\alpha = (z_\alpha)^{\mu_{Y,\alpha}} h^Y_\alpha$. By (4.14), we immediately see that $\mu_{X,\alpha} = \mu_X$ for every α, i.e., it does not depend on the chosen chart. With these notations, we can use (4.13) to obtain the following new expression for η^0.

Lemma 4.2.2. *The 1-form η^0 representing the connection ∇^0 is locally given by*

$$\eta^0 = \left(\frac{h_\alpha^Y}{z^{\mu_X - \mu_{Y,\alpha}} h_\alpha^X} - \frac{\mu_X}{z_\alpha} - \frac{\left(h_\alpha^X\right)'}{h_\alpha^X} \right) dz_\alpha .$$

Proof. It is an easy computation: from

$$\eta_\alpha^0 = \eta_\alpha - \frac{1}{X_\alpha} dX_\alpha = \frac{Y_\alpha}{X_\alpha} dz_\alpha - \frac{1}{X_\alpha} dX_\alpha$$

we get, substituting the expressions for X_α and Y_α,

$$\eta_\alpha^0 = \left(\frac{z_\alpha^{\mu_{Y,\alpha}} h_\alpha^Y}{z_\alpha^{\mu_X} h_\alpha^X} - \frac{\mu_X z_\alpha^{\mu_X - 1} h_\alpha^X + z_\alpha^{\mu_X} \left(h_\alpha^X\right)'}{z_\alpha^{\mu_X} h_\alpha^X} \right) dz_\alpha$$

and the assertion follows. □

In the remaining part of this section we are going to study possible normal forms for the geodesic field near a singularity. In fact, finding a holomorphic normal form is of huge help in understanding the local dynamics of the geodesic, that is the problem we are studying. We will see that the normal form will strongly depend on a number, called the *irregularity* of the singularity, defined as $m = \mu_X - \mu_{Y,\alpha}$. So, we are lead to the following definition.

Definition 4.2.3. *Let p_0 a singularity for the morphism $X \colon E \to TS$ and μ_X and $\mu_{Y,\alpha}$ the orders at 0 of the functions X_α and Y_α in (4.15), where U_α is a chart centered in p_0. Then:*

- *if $\mu_X \leq \mu_{Y,\alpha}$, p_0 is an* apparent singularity*;*
- *if $\mu_X = \mu_{Y,\alpha} + 1$, p_0 is a* Fuchsian singularity*;*
- *if $\mu_X > \mu_{Y,\alpha} + 1$, p_0 is an* irregular singularity*, of* irregularity $m = \mu_X - \mu_{Y,\alpha}$*;*
- *if $\mu_{Y,\alpha} \geq 1$, p_0 is a* degenerate singularity*.*

The next Lemma in particular says that these definition are all well posed.

Lemma 4.2.4. *Under a general change of coordinates for the bundle,*

$$(z_\beta, v_\beta) = \varphi(z_\alpha, v_\alpha) = (\psi(z_\alpha), \xi(z_\alpha) v_\alpha) \tag{4.16}$$

with ψ and ξ holomorphic and such that $\psi(0) = 0$ (i.e., if 0 was a singular point, it remains so), $\psi'(0) \neq 0$ and $\xi \neq 0$ the functions X_α and Y_α change according to the rules

$$X_\beta \circ \psi = \frac{\psi' X_\alpha}{\xi}$$

and

$$Y_\beta \circ \psi = \frac{1}{\xi} Y_\alpha - \frac{\xi'}{\xi^2} X_\alpha .$$

Proof. We note that we are in the situation described in Lemma 2.1.8, with $\psi' = \frac{\partial z_\beta}{\partial z_\alpha}$ and $\xi = \xi_{\beta\alpha} = \frac{1}{\xi_{\alpha\beta}}$ So, Lemma 2.1.8 and the formula

$$v_\alpha = \xi_{\alpha\beta} v_\beta = \frac{1}{\xi} v_\beta$$

give

$$\begin{aligned} G =& X_\alpha v_\alpha \partial_\alpha - Y_\alpha (v_\alpha)^2 \frac{\partial}{\partial v_\alpha} \\ =& X_\alpha \frac{v_\beta}{\xi} \left(\frac{\partial z_\beta}{\partial z_\alpha} \partial_\beta - v_\beta \frac{\partial z_\beta}{\partial z_\alpha} \frac{\partial \xi_{\alpha\beta}}{\partial z_\beta} \frac{1}{\xi_{\alpha\beta}} \frac{\partial}{\partial v_\beta} \right) - Y_\alpha \frac{v_\beta^2}{\xi^2} \xi \frac{\partial}{\partial v_\beta} \\ =& \left(X_\alpha \frac{1}{\xi} \psi' \right) v_\beta \frac{\partial}{\partial z_\beta} - \left(\frac{Y_\alpha}{\xi} + \frac{\partial (1/\xi)}{\partial z_\alpha} X_\alpha \right) (v_\beta)^2 \frac{\partial}{\partial v_\beta}, \end{aligned}$$

so that we have

$$X_\beta = \frac{\psi'}{\xi} X_\alpha$$

and

$$Y_\beta = \frac{1}{\xi} Y_\alpha + \frac{\partial (1/\xi)}{\partial z_\alpha} X_\alpha = \frac{1}{\xi} Y_\alpha - \frac{1}{\xi^2} \frac{\partial \xi}{\partial z_\alpha} X_\alpha = \frac{1}{\xi} Y_\alpha - \frac{1}{\xi^2} \xi' X_\alpha .$$

□

Corollary 4.2.5. *Definition 4.2.3 is well posed. In particular, if U_α and U_β are two charts both centered at a singular point p_0:*

1. *if $\mu_X \leq \mu_{Y,\alpha}$, then also $\mu_X \leq \mu_{Y,\beta}$;*
2. *if $\mu_X = \mu_{Y,\alpha} + 1$, then also $\mu_X = \mu_{Y,\beta} + 1$;*
3. *if $\mu_X > \mu_{Y,\alpha} + 1$, then $\mu_{Y,\beta} = \mu_{Y,\alpha}$ and in particular $\mu_X > \mu_{Y,\beta} + 1$;*
4. *if $\mu_{Y,\alpha} \geq 1$, then $\mu_{Y,\beta} \geq 1$.*

Proof. They all follow from the formulas in Lemma 4.2.4. □

In the remaining part of this chapter we shall look for normal forms for the geodesic field near a singularity of X and use them to study the dynamics of the geodesics. Before starting with this program, we prove a Lemma that gives a first simplification of the geodesic field.

Lemma 4.2.6. *By a change of type* (4.16)*, we can suppose that $h_\alpha^X = 1$.*

Proof. It suffices to consider a change with $\psi = id$ and $\xi = h_\alpha^X$. In particular we see that with this change of coordinates Y_α becomes

$$Y_\beta = \frac{1}{h_\alpha^X}\left(z_\alpha^{\mu_{Y,\alpha}} h_\alpha^Y - z_\alpha^{\mu_X}\left(h_\alpha^X\right)'\right)$$

so that we obtain that

- if $\mu_{Y,\alpha} < \mu_X$, then $\mu_{Y,\beta} = \mu_{Y,\alpha}$ and $h_\beta^Y = \frac{1}{h_\alpha^X}\left(h_\alpha^Y - z_\beta^{\mu_X - \mu_{Y,\alpha}}\left(h_\alpha^X\right)'\right)$;
- if $\mu_{Y,\alpha} \geq \mu_X$, then $\mu_{Y,\beta} = \mu_X + \mathrm{ord}_{p_0}\left(\left(h_\alpha^X\right)' - z_\alpha^{\mu_{Y,\alpha}-\mu_X} h_\alpha^Y\right) \geq \mu_X$ and $h_\beta^Y = \frac{1}{z^{\mu_{Y,\beta}-\mu_X} h_\alpha^X}\left(z_\alpha^{\mu_{Y,\alpha}-\mu_X} h_\alpha^Y - \left(h_\alpha^X\right)'\right)$.

□

4.2.1 Apparent singularities

We start our study of the singularities of X with the apparent case. This is characterized by the fact that, at a singularity p_0 of this kind, we have $\mu_{Y,\alpha} \geq \mu_X$. In particular, this means that p_0 is not a pole for the 1-form η, and so also for the meromorphic connection ∇, while the 1-form η^0 is such that, locally, $\eta^0(\frac{\partial}{\partial z})$ has a pole of order 1 and residue $-\mu_X$ (see Lemma 4.2.2).

The next Theorem gives the holomorphic classification of geodesic fields at an apparent singularity.

Theorem 4.2.7. *Let $p : E \to S$ be a line bundle over a Riemann surface S. Suppose we have a morphism $X : E \to TS$ which is an isomorphism over $S^0 = S \setminus \mathrm{Sing}(X)$ and a meromorphic connection ∇ on E which is holomorphic on S^0 and such that the geodesic field G extends holomorphically from E_{S^0} to all of E.*

Let $p_0 \in \mathrm{Sing}(X)$ be an apparent singularity of order μ. Then, there exists a chart (U, z, e) centered at p_0 such that there G is given by

$$G = \begin{cases} zv\frac{\partial}{\partial z} & \text{if } \mu = 1 \\ z^\mu(1 + az^{\mu-1}v)\frac{\partial}{\partial z} \text{ with } a \in \{0,1\}, & \text{if } \mu > 1. \end{cases} \tag{4.17}$$

If $\mu > 1$, then $a \in \{0, 1\}$ and is a holomorphic and formal invariant.

Proof. By Lemma 4.2.6 we can suppose that G as the form

$$G = z_\alpha^{\mu_X} v_\alpha \partial_\alpha - z_\alpha^{\mu_{Y,\alpha}} h_\alpha^Y (v_\alpha)^2 \frac{\partial}{\partial v_\alpha}$$

in some chart U_α. Now, we prove that there exists a change of coordinates to a new z_β such that the new Y_β becomes 0. In fact, with the notations of Lemma 4.2.4, the request is that

$$0 = Y_\beta \circ \psi = \frac{1}{\xi} Y_\alpha - \frac{\xi'}{\xi^2} X_\alpha. \tag{4.18}$$

We look for a holomorphic solution ξ for (4.18), different from zero in a neighbourhood of 0, i.e., of p_0. To do so, we fix $\xi(0) = 1$ and notice that the Cauchy problem

$$\begin{cases} \xi' = z_\alpha^{\mu_{Y,\alpha}-\mu_X} h_\alpha^Y \xi \\ \xi(0) = 0 \end{cases} \tag{4.19}$$

is equivalent to equation (4.18), so that we obtain a holomorphic solution ξ. Taking the change of coordinates characterized by $\psi = id$ and ξ this solution, we obtain that the new field becomes

$$G = z_\beta^{\mu_X} h_\beta^X v_\beta \partial_\beta,$$

with in particular $h_\beta^X(0) = h_\alpha^X(0) = 1$. So, we found a (local) field without the component in $\frac{\partial}{\partial v_\beta}$, so that the problem can be simplified by means of the classification of 1-dimensional vector fields tangent to $\mathbb{C}$. In fact, Theorem 5.25 of [IY08] says that any 1-dimensional analytic vector field of the form

$$F(z) = z^\mu(1 + \dots)\frac{\partial}{\partial z}$$

is analitically conjugated to its polynomial formal normal form

$$F_0(z) = z^\mu(1 + az^{\mu-1}),$$

where $a \in \mathbb{C}$ is a formal and holomorphic invariant (see Theorem 4.24 of [IY08] for this normal form). This solves the problem if $\mu = 1$ while, if $\mu > 1$, we are left with a geodesic filed of the form

$$G = z^\mu(1 + az^{\mu-1})\frac{\partial}{\partial z}$$

for some $a \in \mathbb{C}$. We are going to prove that, if $a \neq 0$, we can let a become 1 with a last change of coordinate of the form

$$(z, v) \mapsto (\gamma z, \delta v).$$

In fact, under this change of coordinates, we see that the field G takes the form

$$G = z^\mu(\delta\gamma^{\mu-1} + a\delta\gamma^{2\mu-2}z^{\mu-1})v\frac{\partial}{\partial z}.$$

Using $\delta = a$ and γ such that $\delta\gamma^{\mu-1} = 1$ we get

$$G = z^\mu(1 + z^{\mu-1})v\frac{\partial}{\partial z},$$

as desired. □

Definition 4.2.8. *The formal and holomorphic invariant a is called the* apparent index *of the singularity p_0.*

The next step will be to exploit the local form given by Theorem 4.2.7 to study the dynamics of the geodesics for our connection ∇ on the line bundle E near an apparent singularity. To do this, we first study the solution of (4.17) and then apply this to our dynamical problem.

Lemma 4.2.9. *Consider the system of differential equations on* $\mathbb{C}^2$

$$\begin{cases} z' = z^\mu \left(1 + az^{\mu-1}\right) v \\ v' = 0 \end{cases} \tag{4.20}$$

with μ an integer ≥ 1, $a \in \mathbb{C}$, and initial conditions $z(0) = z_0$ and $v(0) = \overline{v}_0$. Then

1. *if $\mu = 1$ the solution is defined for every forward time and*
   - *if* $\operatorname{Re} \overline{v}_0 < 0$*, then $z(t) \to 0$ as $t \to \infty$;*
   - *if* $\operatorname{Re} \overline{v}_0 > 0$*, then $|z(t)| \to \infty$ as $t \to \infty$;*
   - *if* $\operatorname{Re} \overline{v}_0 = 0$*, then $|z(t)|$ is constant and the solution is periodic;*
2. *if $\mu > 1$ and $a = 0$ then, for every initial condition z_0 for z,*
   - *if* $\operatorname{Re}(\overline{v}_0 z_0^{\mu-1}) > 0$ *and* $\operatorname{Im}(\overline{v}_0 z_0^{\mu-1}) = 0$*, then the solution diverges in finite forward time;*
   - *if* $\operatorname{Re}(\overline{v}_0 z_0^{\mu-1}) > 0$ *and* $\operatorname{Im}(\overline{v}_0 z_0^{\mu-1}) \neq 0$*, then the solution is $z(t)$ defined for every forward time, bounded, and eventually goes to zero, but the maximum of $|z(t)|$ on the solution can be arbitrarily high. In particular,* $\max |z(t)| \to \infty$ *as* $\operatorname{Im}(\overline{v}_0 z_0^{\mu-1}) \to 0$ *);*
   - *if* $\operatorname{Re}(\overline{v}_0 z_0^{\mu-1}) \leq 0$ *then the solution is defined for every forward time and $z(t) \to$ as $t \to \infty$, with $|z(t)| \leq |z_0|$ for every t;*
3. *if $\mu > 1$ and $a \neq 0$ then the system admits fixed points with $z \neq 0$. Moreover, for every z_0 there exists a v_0 such that, if the initial condition $\overline{v}_0$ is ζv_0,*
   - *if* $\operatorname{Re}(\zeta/a) > 0$ *then the solution is defined for every forward time and $z(t) \to 0$ as $t \to +\infty$;*
   - *if* $\operatorname{Re}(\zeta/a) < 0$*, then the solution $z(t)$ either goes to one of the fixed points of the system or is periodic, surrounding zero.*

Proof. We divide the proof in the three cases of the statement

1. We have $\mu = 1$, so that the system (4.20) reduces to

   $$z' = z\overline{v}_0,$$

   which can be easily solved with

   $$z(t) = z_0 \exp(\overline{v}_0 t).$$

We see that the behaviour of the solution depends only on $\operatorname{Re}\overline{v}_0$ and we obtain the assertion.

2. We have $\mu > 1$ and $a = 0$. Now the first equation of (4.20) becomes

$$z' = z^\mu \overline{v}_0$$

which is solved by (a suitable branch of)

$$z(t) = z_0 \left(1 - \frac{\overline{v}_0 z_0^{\mu-1}}{\mu-1} t\right)^{-\frac{1}{\mu-1}}.$$

We see that we have the three possibilities of the statement:

- if $\overline{v}_0 z_0^{\mu-1} \in \mathbb{R}^+$, then there exists $t \in \mathbb{R}^+$ such that the solution diverges at time t;
- if $\operatorname{Re}(\overline{v}_0 z_0^{\mu-1}) > 0$ and $\operatorname{Im}(\overline{v}_0 z_0^{\mu-1}) \neq 0$, the solution is defined for every time. Moreover, the term in parenthesis

$$1 - \frac{\overline{v}_0 z_0^{\mu-1}}{\mu-1} t \tag{4.21}$$

represents a line, and so its modulus is bounded from below. It is easy to see that in fact the least possible modulus of this quantity is

$$\left|\frac{\operatorname{Im}(\overline{v}_0 z_0^{\mu-1})}{\overline{v}_0 z_0^{\mu-1}}\right|.$$

It follows that, for every t,

$$|z(t)| \leq |z_0| \left(\frac{|\overline{v}_0 z_0^{\mu-1}|}{|\operatorname{Im}(\overline{v}_0 z_0^{\mu-1})|}\right)^{\frac{1}{\mu-1}}.$$

In particular, it follows that every solution is bounded and goes to zero, but the maximum of $|z(t)|$ goes to infinity as the quantity $|\operatorname{Im}(\overline{v}_0 z_0^{\mu-1})|$ goes to zero;

- if $\operatorname{Re}(\overline{v}_0 z_0^{\mu-1}) \leq 0$, then the term (4.21) diverges as $t \to \infty$ and so $z(t) \to 0$. Moreover, since

$$\left|1 - \frac{\overline{v}_0 z_0^{\mu-1}}{\mu-1} t\right|^2 \geq 1 - 2\operatorname{Re}\frac{\overline{v}_0 z_0^{\mu-1}}{\mu-1} + \left|\frac{\overline{v}_0 z_0^{\mu-1}}{\mu-1}\right|^2 \geq 1,$$

we have $|z(t)| \leq |z_0|$, as desired;

3. Here $\mu > 1$ and $a \neq 0$. We immediately observe that the fixed points are given by $z = 0$ and the solutions of $az_0^{\mu-1} = -1$. Then, we see that a solution of the system (4.20) must satisfy $v(t) \equiv \overline{v}_0$ and

$$\left(1 + \frac{1}{az(t)^{\mu-1}}\right) \exp\left(-\left(1 + \frac{1}{az(t)^{\mu-1}}\right)\right) \exp\left(-\frac{(\mu-1)\,\overline{v}_0}{a} t\right) = c_0, \tag{4.22}$$

where

$$c_0 = \left(1 + \frac{1}{az(0)^{\mu-1}}\right) \exp\left(-\left(1 + \frac{1}{az(0)^{\mu-1}}\right)\right)$$

To prove this, it suffices to show that the derivative in t of the left hand side is zero. We can rewrite (4.22) as

$$w(t) \exp(-w(t)) \exp\left(-\frac{(\mu-1)\,\overline{v}_0}{a} t\right) = c_0, \tag{4.23}$$

where

$$w(t) = 1 + \frac{1}{az(t)^{\mu-1}}. \tag{4.24}$$

The derivative of the left hand side of (4.23) is easily seen to be equal to

$$\left(e^{-w}\right)\left(w' - ww' - w\frac{(\mu-1)\overline{v}_0}{a}\right) \exp\left(-\frac{(\mu-1)\overline{v}_0}{a} t\right).$$

Substituting $w' = \frac{1-\mu}{az^{\mu}} z'$ and equation (4.24) we get that this derivative is actually zero.

We rewrite it as

$$w(t) \exp(-w(t)) = c_0 \exp\left(\frac{(\mu-1)\,\overline{v}_0}{a} t\right), \tag{4.25}$$

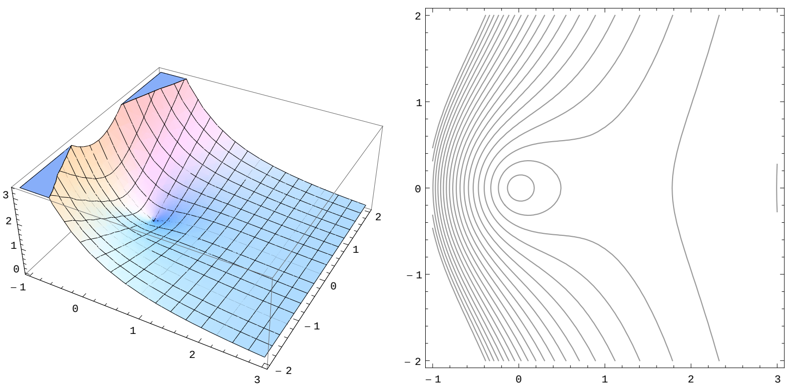
to divide the parts with and without w (i.e., z). Then, we notice that the important parameter will be (the real part of) $\frac{\overline{v}_0}{a}$, so that we get the following cases:

- if $\mathrm{Re}\left(\frac{\overline{v}_0}{a}\right) > 0$, then the modulus of the right hand side of (4.25) diverges as $t \to +\infty$. So, we need that also

  $$|w(t) \exp(-w(t))| = |w(t)| \exp\left(-\mathrm{Re}\, w(t)\right) \to \infty.$$

  In particular, we must have $|w(t)| \to \infty$, which means that $z(t) \to 0$;
- if $\mathrm{Re}\left(\frac{\overline{v}_0}{a}\right) = 0$, the right hand side of (4.25) has constant module c_0 and argument going to infinity. Thus, we have to study the level sets of the function $g(w) = |we^{-w}|$.

  In Figure 4.1 we have drawn a plot of this function, with variable $w = x + iy$ and the level sets of it. We see that there exists a



(a) Graph of g (b) Levels of g

Figure 4.1: $g(w) = |we^{-w}|$

value c such that if $|c_0| \geq c$ the c_0-level of g is connected (and unbounded), while if $|c_0| < c$ the level sets has two components, one bounded and one unbounded. So, we have the following two cases:

- if $|c_0| \geq c$, we have $\arg w(t)$ bounded and $|\operatorname{Im} w(t)| \to +\infty$. But this means that also $|w(t)| \to +\infty$, which gives $z(t) \to 0$;
- if $|c_0| < c$ and $w(0)$ is in the unbounded component of $\{g = c_0\}$ the description is like in the previous case and we have $\sigma_{v_0}(t) \to p_0$. On the other hand, if $w(0)$ belongs to the bounded component, we know that the $\arg w(t) \to \infty$, which means that $w(t)$ describes this level set infinitely many times. But this means that $w(t)$ is closed (and thus periodic, because it satisfies a first order ODE);

• if $\operatorname{Re}\left(\frac{\overline{v}_0}{a}\right) < 0$, then the modulus of the right hand side of (4.25) goes to zero as $t \to +\infty$. So, we have two possibilities:

- $|w(t)| \to 0$, and so $z(t)$ tends to one of the fixed points; or
- $\operatorname{Re} w(t) \to +\infty$, so that we have also $|w(t)| \to \infty$, and so $z(t) \to 0$.

□

Theorem 4.2.10. *Let $p : E \to S$ be a line bundle over a Riemann surface S. Suppose we have a morphism $X : E \to TS$ which is an isomorphism over $S^0 = S \setminus \operatorname{Sing}(X)$ and a meromorphic connection ∇ on E which is holomorphic on S^0 and such that the geodesic field G extends holomorphically from E_{S^0} to all of E. Let $p_0 \in \operatorname{Sing}(X)$ be an apparent singularity of order μ and apparent index a if $\mu > 1$. Let $\sigma : [0, \varepsilon) \to S^0$ be a geodesic for ∇ such that $\sigma(t) \to p_0$ as $t \to \varepsilon$.*

Then, for $t \to \varepsilon$, $\sigma'(t) \to O_{p_0}$ *and* $X^{-1}(\sigma'(t))$ *tends to a non-zero element of* E_{p_0}.

Moreover, there is a neighbourhood $U \subset S$ *of the singular point* p_0 *such that, if* $z_0 \in U \setminus \mathrm{Sing}(X)$ *and* $\sigma_{v_0} : [0, \varepsilon) \to S^0$ *is the maximal geodesic with initial conditions* $\sigma_{v_0}(0) = z_0$ *and* $\sigma'_{v_0}(0) = X(v_0)$ *(where* $v_0 = \overline{v}_0 e$ *and* e *is the generator of* E_{z_0} *in a local chart), then*

1. *if* $\mu = 1$*:*
   - *if* $\mathrm{Re}\,\overline{v}_0 < 0$*, then* $\sigma_{v_0}(t) \to p_0$ *as* $t \to +\infty$*;*
   - *if* $\mathrm{Re}\,\overline{v}_0 > 0$*, then* $\sigma_{v_0}(t)$ *escapes the neighbourhood* U *as* $t \to +\varepsilon$*;*
   - *if* $\mathrm{Re}\,\overline{v}_0 = 0$*, then* $\sigma_{v_0}(t)$ *is a periodic geodesic surrounding* p_0*;*
2. *if* $\mu > 1$ *and* $a = 0$ *there exists a non-zero direction* $\widetilde{v}_0 \in E_{z_0}$ *and a neighbourhood* $\widetilde{V}$ *of* $\mathbb{R}^+\widetilde{v}_0$ *such that:*
   - *if* $\overline{v}_0 \in \widetilde{V}$*, then* $\sigma_{v_0}(t)$ *escapes the neighbourhood* U *as* $t \to +\varepsilon$*;*
   - *if* $\overline{v}_0 \notin \widetilde{V}$*, then* $\sigma_{v_0}(t) \to p_0$ *as* $t \to +\infty$*;*
3. *if* $\mu > 1$ *and* $a = 1$ *then there is a non-zero direction* $v_0 \in E_{z_0}$ *such that, if the initial direction of* σ_v *is* $v = \zeta v_0$*,*
   - *if* $\mathrm{Re}(\zeta) > 0$*, then* $\sigma_v(t) \to p_0$ *as* $t \to +\infty$*;*
   - *if* $\mathrm{Re}(\zeta) < 0$*, then either* $\sigma_v(t) \to p_0$*, or* σ_v *escapes the neighbourhood* U*, or it is a periodic geodesic surrounding* p_0*;*
   - *if* $\mathrm{Re}(\zeta) = 0$*, then either* $\sigma_v(t) \to p_0$ *as* $t \to +\varepsilon$ *or* σ_v *escapes the neighbourhood* U*.*

Proof. First, by Theorem 4.2.7 we can suppose that the field G has the form (4.17) in a suitable chart centered at the singular point p_0 and, by Proposition 4.1.19 we know that a curve $\sigma(t) = (z(t))$ is a ∇-geodesic if and only if $X^{-1}(\sigma') = (z(t), v(t))$ is an integral curve for G. It means that $(z(t), v(t))$ must be a solution of (4.20) and the request that $\sigma(t) \to p_0$ means that $z(t) \to 0$. From (4.20) we see that $v(t) \equiv v(0) \neq 0$, and so $X^{-1}(\sigma'(t))$ tends to a non-zero element of E_{p_0}, Moreover, from $z(t) \to 0$, we have $z'(t) \to 0$, and so $\sigma'(t) \to O_{p_0}$. So we have proved the first part of the Theorem.

Statements 1-3 follow from Lemma 4.2.9. We only remark that the $\widetilde{v}_0$ of case 2 is equal to $z_0^{1-\mu}$ and that in case 3 we take U sufficiently small so that $z_0^{\mu-1} \neq -1$ (otherwise we would have $z(t) \equiv z_0$, and this implies that we cannot be in the domain of a chart where the field is of the form (4.17)). So, all the solutions of (4.20) that go to a fixed point different to zero now correspond to geodesics escaping U. □

4.2.2 Fuchsian and irregular singularities

In this section we are going to provide a formal (i.e., by means of conjugation by non-necessarily convergent power series) classification of (the geodesic fields near) Fuchsian and irregular singularities, and also a holomophic one in the Fuchsian case. This will allow us to obtain a description of the dynamics of the geodesic flow near a Fuchsian singularity analogous to the one given in Theorem 4.2.10 for apparent ones. Then, we shall prove other results concerning the dynamics of a geodesic near a Fuchsian or irregular singularity.

We start with the formal classification.

Theorem 4.2.11. *Let $p : E \to S$ be a line bundle over a Riemann surface S. Suppose we have a morphism $X : E \to TS$ which is an isomorphism over $S^0 = S \setminus \mathrm{Sing}(X)$ and a meromorphic connection ∇ on E which is holomorphic on S^0 and such that the geodesic field G extends holomorphically from E_{S^0} to all of E.*

Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian or irregular singularity such that, in a chart $(U_\alpha, z_\alpha, e_\alpha)$ centered at p_0 we have

$$G = z_\alpha^{\mu_X} \left(a_0 + a_1 z_\alpha + \dots\right) v_\alpha \partial_\alpha - z_\alpha^{\mu_Y} \left(b_0 + b_1 z_\alpha + \dots\right) (v_\alpha)^2 \frac{\partial}{\partial v_\alpha},$$

with $\mu_X > \mu_Y$ and $a_0, b_0 \neq 0$. Call $\rho = b_0/a_0 \neq 0$. If p_0 is Fuchsian, then $\rho = \mathrm{Res}_{p_0}(\nabla)$. Then:

1. *if p_0 is Fuchsian, then*

   (a) *if $\mu_Y - \rho \notin \mathbb{N}^*$, then G is formally conjugated to*

   $$z^{\mu_X - 1} \left(zv\partial - \rho v^2 \frac{\partial}{\partial v} \right); \tag{4.26}$$

   (b) *if $\mu_Y - \rho = n \in \mathbb{N}^*$, then G is formally conjugated to*

   $$z^{\mu_X - 1} \left(zv\partial - \rho v^2 \left(1 + a z^n\right) \frac{\partial}{\partial v} \right); \tag{4.27}$$

   with $a \in \{0, 1\}$ which is a formal invariant;

2. *if p_0 is irregular of irregularity $m = \mu_X - \mu_Y > 1$, then G is formally conjugated to*

   $$z^{\mu_X - m} \left(z^m v\partial - v^2 \left(1 + \widetilde{\rho} z^{m-1}\right) \frac{\partial}{\partial v} \right), \tag{4.28}$$

   where $\widetilde{\rho} = \mathrm{Res}_{p_0}(\nabla)$ (and so is a formal invariant).

Proof. Proving that the given espressions are normal forms means that we are looking for new charts in which the geodesic fields become of the given form. Namely, if (z_α, v_α) are the initial coordinates, we are looking for a change of type (4.16), composed with a liner map,

$$(z_\beta, v_\beta) = \varphi\,(z_\alpha, v_\alpha) = \left(\frac{\psi(z_\alpha)}{\gamma}, \frac{\xi(z_\alpha)v_\alpha}{\delta}\right)$$

with ψ and ξ non necessarily convergent power series. In particular, we write X_α and Y_α as

$$\begin{cases} X_\alpha = z_\alpha^{\mu_X} \sum_{j=0}^{+\infty} a_j z_\alpha^j \\ Y_\alpha = z_\alpha^{\mu_{Y,\alpha}} \sum_{j=0}^{+\infty} b_j z_\alpha^j \end{cases}$$

and look for coordinates in which we would have

$$\begin{cases} X_\beta = z_\beta^{\mu_X} \\ Y_\beta = \rho z_\beta^{\mu_X - 1} \end{cases}$$

for case 1a,

$$\begin{cases} X_\beta = z_\beta^{\mu_X} \\ Y_\beta = \rho\left(1 + z_\beta^n\right) z_\beta^{\mu_X - 1} \end{cases}$$

for case 1b, and

$$\begin{cases} X_\beta = z_\beta^{\mu_X} \\ Y_\beta = \rho\left(1 + z_\beta^{m-1}\right) z_\beta^{\mu_X - m} \end{cases}$$

for case 2. We shall do this in two steps: first, we shall use only a formal change of type (4.16). Then, we shall study the action of a linear change.

For the first task, following the usual (formal) Poincaré-Dulac method, we shall study the effect of a change of the form

$$\varphi_n\,(z_\alpha, v_\alpha) = \left(z_\alpha + c_1 z_\alpha^{n+1}, v_\alpha(1 + c_2 z_\alpha^n)\right),$$

with $c_1, c_2 \in \mathbb{C}$ and not both zero. proving that, with it, we can modify the n-esim coefficients of X_α and Y_α to the desired quantities without altering the coefficients with index $< n$. Then, we shall consider the infinite composition

$$\varphi = \cdots \circ \varphi_n \circ \varphi_1 \circ \varphi_0,$$

that will be a formal power series giving the formal conjugation to the desired form.

First, we study the action of φ_n on X_α and Y_α, up to the term of order n. By Lemma 4.2.4 we have

$$X_\beta = z_\beta \left(\sum_{j=0}^{n-1} a_j z_\beta^j + \left(a_n + a_0\left((n+1-\mu_X)c_1 - c_2\right)\right) z_\beta^n + o(z_\beta^n)\right)$$

and

$$Y_\beta = \begin{cases} z_\beta^{\mu_Y} \left(\sum_{j=0}^{n-1} b_j z_\beta^j \right) + & \text{if } m = 1 \\ + z_\beta^{\mu_Y} \left(\left(b_n - (\mu_Y c_1 + c_2) b_0 - n c_2 a_0 \right) z_\beta^n + o(z_\beta^n) \right) & \text{(Fuchsian case)} \\ & \\ z_\beta^{\mu_Y} \left(\sum_{j=0}^{n-1} b_j z_\beta^j \right) + & \text{if } m > 1 \\ + z_\beta^{\mu_Y} \left(\left(b_n - (\mu_Y c_1 + c_2) b_0 \right) z_\beta^n + o(z_\beta^n) \right) & \text{(irregular case)} \end{cases}$$

So, we see that the terms of order less than n are actually left unmodified by φ_n.

The next step is to find a couple (c_1, c_2) such that the terms of order n become as in the assertion. We see that the condition are

$$\begin{cases} a_0(\mu_X - n - 1)c_1 + a_0 c_2 = a_n \\ \mu_Y b_0 c_1 + (n a_0 + b_0) c_2 = b_n \end{cases} \tag{4.29}$$

if $m = 1$, and

$$\begin{cases} a_0(\mu_X - n - 1)c_1 + a_0 c_2 = a_n \\ \mu_Y b_0 c_1 + b_0 c_2 = b_n \end{cases} \tag{4.30}$$

if $m > 1$. We want to solve both these systems for the unknown c_1 and c_2. In particular, this can surely be done if their determinant is non-zero. The determinant of (4.29) is

$$n a_0^2 \left(\mu_Y - \rho - n \right),$$

so that, if $n \neq \mu_Y - \rho$, the first system is solvable. In particular, this gives the assertion for the case 1a. Moreover, even if $n = \mu_Y - \rho$, the system (4.29) becomes

$$\begin{cases} a_0(\rho c_1 + c_2) = a_n \\ a_0 \mu_Y (\rho c_1 + c_2) = b_n \end{cases} \tag{4.31}$$

and we see that we can anyway solve the first equation of (4.31), thus getting a normal form expressed by

$$z^{\mu_X - 1} \left(z v \partial - \rho v^2 \left(1 + a z^n \right) \frac{\partial}{\partial v} \right), \tag{4.32}$$

for some $a \in \mathbb{C}$.

Let us now consider the irregular case and the associated system (4.30). Now the determinant is

$$a_0 b_0 (m - n - 1)$$

and so, if $n \neq m - 1$, we can find the values c_1 and c_2 that satisfy (4.30). Otherwise, system (4.30) becomes

$$\begin{cases} a_0(\mu_Y c_1 + c_2) = a_n \\ b_0(\mu_Y c_1 + c_2) = b_n \end{cases} \tag{4.33}$$

and we see that we can solve the first equation also in this case, thus obtaining

$$z^{\mu_X - m}\left(z^m v\partial - \rho v^2\left(1 + az^{m-1}\right)\frac{\partial}{\partial v}\right), \tag{4.34}$$

as (partial) normal form.

Now we consider the linear change

$$(z_\alpha, v_\alpha) = (\gamma z_\beta, \delta v_\beta). \tag{4.35}$$

Our goal is to reduce (4.32) to (4.27) and (4.34) to (4.28), thus proving the assertion.

In both case, the geodesic field has the form

$$G_\alpha = z_\alpha^{\mu - m}(z_\alpha^m v_\alpha \frac{\partial}{\partial z_\alpha} - \rho v_\alpha^2(1 + az_\alpha^q)\frac{\partial}{\partial v_\alpha}),$$

with $m = 1$ and $q = \mu_Y - \rho$ in case 1b and $q = m - 1$ in case 2. Under the linear change (4.35) G_α becomes

$$\begin{aligned} G_\beta &= \gamma^{\mu - m} z_\beta^{\mu - m}\left(\gamma^m z_\beta^m \delta v_\beta \frac{1}{\gamma}\frac{\partial}{\partial z_\beta} - \rho\delta^2 v_\beta^2\left(1 + a\gamma^q z_\beta^q\right)\frac{1}{\delta}\frac{\partial}{\partial v_\beta}\right) \\ &= z_\beta^{\mu - m}\left(\delta\gamma^{\mu - 1} z_\beta^m v_\beta \frac{\partial}{\partial z_\beta} - \rho\delta\gamma^{\mu - m} v_\beta^2\left(1 + a\gamma^q z_\beta^q\right)\frac{\partial}{\partial v_\beta}\right) \end{aligned} \tag{4.36}$$

Now, in the Fuchsian resonant case, (4.36) is

$$G_\beta = z_\beta^{\mu - 1}\left(\delta\gamma^{\mu - 1} z_\beta v_\beta \frac{\partial}{\partial z_\beta} - \rho\delta\gamma^{\mu - 1} v_\beta^2\left(1 + a\gamma^n z_\beta^n\right)\frac{\partial}{\partial v_\beta}\right)$$

and we see that there exists a pair (γ, δ) solving

$$\begin{cases} a\gamma^n = 1 \\ \delta\gamma^{\mu - 1} = 1. \end{cases}$$

With two such γ and δ, the geodesic field becomes

$$G = z^{\mu - 1}\left(zv\frac{\partial}{\partial z} - \rho v^2(1 + z^n)\frac{\partial}{\partial v}\right),$$

as desired.

In the irregular case, (4.36) becomes

$$G_\beta = z_\beta^{\mu - m}\left(\delta\gamma^{\mu - 1} z_\beta^m v_\beta \frac{\partial}{\partial z_\beta} - \rho\delta\gamma^{\mu - m} v_\beta^2\left(1 + a\gamma^{m-1} z_\beta^{m-1}\right)\frac{\partial}{\partial v_\beta}\right)$$

and we see that with (γ, δ) satisfying

$$\begin{cases} \delta\gamma^{\mu - 1} = 1 \\ \rho\delta\gamma^{\mu - m} = 1, \end{cases}$$

which means

$$\begin{cases} \delta = \gamma^{1-\mu} \\ \rho\gamma^{1-m} = 1, \end{cases}$$

we get

$$G_\beta = z_\beta^{\mu-m}\left(z_\beta^m v_\beta \frac{\partial}{\partial z_\beta} - v_\beta^2\left(1 + \widetilde{\rho} z_\beta^{m-1}\right)\frac{\partial}{\partial v_\beta}\right),$$

and we are done. □

Remark 4.2.12. *If we try to do a linear change of coordinates also in case 1a, we see that we cannot simplify the form* (4.26) *any further. In fact, the residue* ρ *must remain unmodified under the change of coordinates.*

Doing the computations, after the substitutions we would get

$$G_\beta = z_\beta^{\mu-1}\left(\delta\gamma^{\mu-1} z_\beta v_\beta \frac{\partial}{\partial z_\beta} - \rho\delta\gamma^{\mu-1} v_\beta^2 \frac{\partial}{\partial v_\beta}\right)$$

and we see that requiring $\delta\gamma^{\mu-1} = 1$ *brings this expression again in the form*

$$G_\beta = z_\beta^{\mu-1}\left(z_\beta v_\beta \frac{\partial}{\partial z_\beta} - \rho v_\beta^2 \frac{\partial}{\partial v_\beta}\right).$$

Remark 4.2.13. *In both the cases 1b and 2 of Theorem 4.2.11, when the determinant of the corresponding system was zero, we decided to solve the first equation, thus obtaining the best possible form,* $z_\beta^{\mu_X}$*, for* X_β*, but having to hold an extra term in* Y_β*. Clearly, it is possible to do the opposite choice, i.e., solve the second equations of systems* (4.31) *and* (4.33)*. Doing so, we see that we can obtain*

$$G = z^{\mu_X-1}\left(z\left(1 + a'z^n\right)v\partial - \rho v^2\frac{\partial}{\partial v}\right)$$

as another normal form for case 1b and

$$z^{\mu_X-m}\left(z^m\left(1 + a'z^{m-1}\right)v\partial - \rho v^2\frac{\partial}{\partial v}\right)$$

for case (2)*.*

Definition 4.2.14. *The formal invariant* $a \in \mathbb{C}$ *is called resonant index.*

It is actually possible to prove that, in the Fuchsian case, the formal normal form of Theorem 4.2.11 is in fact also a *holomorphic* normal form, in the sense that the changes of coordinates needed to obtain the forms (4.26) and (4.27) are holomorphic. We do not prove this here, but we derive from this holomorphic classification the following Theorem, the analogous of Theorem 4.2.10 for Fuchsian singularities with vanishing resonant index. For a proof of the fact that this normal form is a holomorphic one, refer to [AT11], pages 2665-2669. We remark that a holomorphic classification in the case of irregular singularities is not known, yet.

Theorem 4.2.15. *Let $p : E \to S$ be a line bundle over a Riemann surface S. Suppose we have a morphism $X : E \to TS$ which is an isomorphism over $S^0 = S \setminus \mathrm{Sing}(X)$ and a meromorphic connection ∇ on E which is holomorphic on S^0 and such that the geodesic field G extends holomorphically from E_{S^0} to all of E. Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian singularity of order $\mu_X > 1$ and let $\rho = \mathrm{Res}_{p_0}(\nabla)$. Suppose the resonant index vanishes if $\mu_X - 1 - \rho \in \mathbb{N}^*$. Then, there exists a neighbourhood $U \subset S$ of p_0 such that, for every $z_0 \in U \cap S^0$,*

1. *if $\mathrm{Re}\,\rho < \mu_Y$, then we have at least one geodesic issuing from z_0 that escapes U, and every geodesic issuing from z_0 that does not escape goes to p_0. Moreover, for any of these geodesics σ going to p_0 inside U we have:*
   - *(a) if $\mu_Y \,\mathrm{Re}\,\rho < |\rho|^2$, then $X^{-1}(\sigma'(t)) \to 0$ as $\sigma(t) \to p_0$;*
   - *(b) if $\mu_Y \,\mathrm{Re}\,\rho > |\rho|^2$, then $\left|X^{-1}(\sigma'(t))\right| \to +\infty$ as $\sigma(t) \to p_0$;*
   - *(c) if $\mu_Y \,\mathrm{Re}\,\rho = |\rho|^2$, then $X^{-1}(\sigma'(t))$ accumulates a circumference in E_{p_0};*
2. *if $\mathrm{Re}\,\rho > \mu_Y$, then all geodesics issuing from z_0 but one escape U; furthermore, the exceptional geodesic σ_0 tends to p_0 in finite time with $X^{-1}(\sigma_0'(t)) \to +\infty$ as $\sigma_0(t) \to p_0$;*
3. *if $\mathrm{Re}\,\rho = \mu_Y$, but $\rho \neq \mu_Y$, then the geodesics not escaping U are either closed, with $X^{-1}(\sigma')$ either tending to 0 or diverging, or accumulate the support of a closed geodesic in U, with $X^{-1}(\sigma') \to 0$;*
4. *if $\rho = \mu_Y$ (necessarily > 0), then for every $z_0 \in U \cap S^0$ there is a non-zero direction $v_0 \in E_{z_0}$ such that, if $v = \zeta v_p \in E_p$ and σ_v is the geodesic issuing from z_0 tangent to $X(v)$:*
   - *(a) if $\mathrm{Re}\,\zeta < 0$, then σ_v converges to p_0 staying in U but with $|X^{-1}(\sigma_v'(t))| \to +\infty$;*
   - *(b) if $\mathrm{Re}\,\zeta > 0$, then σ_v escapes U;*
   - *(c) if $\mathrm{Re}\,\zeta = 0$, then σ_v is periodic and surrounds p_0.*

Proof. We consider as neighbourhood U a local chart centered in the singularity p_0 found with (the holomorphic counterpart of) Theorem 4.2.11. In particular, (4.26) gives that a curve σ is a geodesic if and only if $X^{-1}(\sigma'(t))$ satisfies

$$\begin{cases} z'(t) = z(t)^{\mu_Y+1} v \\ v'(t) = -\rho z(t)^{\mu_Y} v^2, \end{cases} \tag{4.37}$$

where ρ is the residue of ∇ at p_0

We start studying the case(s) in which $\rho \neq \mu_Y$. It is possible to integrate the system (4.37) obtaining as solution

$$\begin{cases} z(t) = z_0(1+ct)^{1/(\rho-\mu_Y)} = z_0 \exp\left(\frac{1}{\rho-\mu_Y}\log(1+ct)\right) \\ v(t) = v_0(1+ct)^{-\rho/(\rho-\mu_Y)} = v_0 \exp\left(\frac{-\rho}{\rho-\mu_Y}\log(1+ct)\right), \end{cases} \tag{4.38}$$

where $c = (\rho - \mu_Y)\, z_0^{\mu_Y} v_0$ and log denotes the branch of the logarithm with $\log 1 = 0$.

Consider the modulus of the solutions,

$$\begin{cases} |z(t)| = |z_0| \exp\left(\operatorname{Re}\left(\frac{1}{\rho-\mu_Y}\right)\log|1+ct| - \operatorname{Im}\left(\frac{1}{\rho-\mu_Y}\right)\arg(1+ct)\right) \\ |v(t)| = |v_0| \exp\left(\operatorname{Re}\left(\frac{-\rho}{\rho-\mu_Y}\right)\log|1+ct| - \operatorname{Im}\left(\frac{-\rho}{\rho-\mu_Y}\right)\arg(1+ct)\right). \end{cases} \tag{4.39}$$

and suppose that $\operatorname{Re}\rho < \mu_Y$, which means that $\operatorname{Re}(\rho-\mu_Y)^{-1} < 0$. We see that $\arg(1+ct)$ is bounded, so that the term that will decide the asymptotic behaviour of $z(t)$ is $\log|1+ct|$ (with its coefficient). In fact, we recognize two possibilities:

- if $c = ((\rho-\mu_Y)z_0^{\mu_Y}v_0) \notin \mathbb{R}^-$, $\log|1+ct|$ is defined (and remains bounded from below) for every t, so that we get $\sigma(t) \to p_0$ for $t \to +\infty$. For what concerns $v(t)$, we have to look at the coefficient $\operatorname{Re}\frac{-\rho}{\rho-\mu_Y}$ in front of $\log|1-ct|$. We have

$$\operatorname{Re}\frac{-\rho}{\rho-\mu_Y} = \frac{-|\rho|^2 + \mu_Y \operatorname{Re}\rho}{|\rho-\mu_Y|^2}$$

  and cases 1a, 1b and 1c follow.

- if $c = ((\rho-\mu_Y)z_0^{\mu_Y}v_0) \in \mathbb{R}^-$, there exists a $t \in \mathbb{R}^+$ such that $ct = -1$, so that $\log|1+ct|$ diverges to $-\infty$ and $|z(t)|$ diverges, which means that the geodesic leaves U.

If $\operatorname{Re}\rho > \mu_Y$ it is easy to see that the situation for $z(t)$ will be the opposite of the previous one. Moreover, the condition $\operatorname{Re}\rho > \mu_Y$ implies that $|\rho|^2 > \mu_Y \operatorname{Re}\rho$, so that $|v(t)| \to +\infty$ in the only case in which $\sigma(t) \to p_0$, and so part 2 is proved.

Let us now consider the case with $\operatorname{Re}\rho = \mu_Y$ and $\rho \neq \mu_Y$. Now we have that the real part of $\frac{1}{\rho-\mu_Y}$, which played a major role in the two previous cases, is zero. In fact, we can write $\rho = \mu_Y - \frac{i}{\gamma}$, for some $\gamma \in \mathbb{R}^*$, so that we get $\frac{1}{\rho-\mu_Y} = i\gamma$. Substituting these into (4.38) we obtain

$$\begin{cases} z(t) = & z_0 \exp\left(-\gamma\arg(1+ct) + i\gamma\log|1+ct|\right) \\ v(t) = & v_0 \exp\left(-\log|1+ct| + \mu_Y\gamma\arg(1+ct)\right) \\ & \cdot\exp\left(-i\left(\arg(1+ct) + \mu_Y\gamma\log|1+ct|\right)\right) \end{cases}$$

where now c can be expressed as $-i\gamma z_0^{\mu_Y} v_0$. We have the following cases:

- if $c \in \mathbb{R}^-$, then $1 + ct$ will be real for every t for which the solution will be defined (i.e., before $-c^{-1}$). So $\arg(1 + ct)$ is constant and $z(t)$ describes a closed geodesic, with $v(t)$ diverging with $t \to -c^{-1}$;

- if $c \in \mathbb{R}^+$, the solution will be defined for all t (until it escapes U). Again, we have that $\arg(1 + ct)$ is constant, so that $\sigma(t)$ is a closed geodesic. Now, $v(t) \to 0$ as $t \to +\infty$;

- if $c \notin \mathbb{R}$ (c cannot be zero) the geodesic is defined for all times (unless it escapes) and $|z(t)|$ converges, so that $\sigma(t)$ tends to the support of a closed geodesic and the expression for $v(t)$ gives that $v(t) \to 0$, as wished.

We are left to prove case 4. Here the solution of (4.37) is different. Namely, we have

$$\begin{cases} z(t) = z_0 \exp\left(z_0^{\mu_Y} v_0 t\right) \\ v(t) = v_0 \exp\left(-\mu_Y z_0^{\mu_Y} v_0 t\right). \end{cases} \tag{4.40}$$

We see that we have the following cases, giving the different possibilities of case 4:

- if $\operatorname{Re}\left(z_0^{\mu_Y} v_0\right) < 0$, then $z(t) \to 0$, so that $\sigma(t) \to p_0$ for $t \to +\infty$, and $|v(t)| \to +\infty$;

- if $\operatorname{Re}\left(z_0^{\mu_Y} v_0\right) > 0$ the solution $z(t)$ of (4.40) goes to infinity, which means that the geodesic escapes U;

- if $\operatorname{Re}\left(z_0^{\mu_Y} v_0\right) = 0$ the geodesic is periodic.

□

Remark 4.2.16. *We may try to generalize Theorem 4.2.15 in the resonant case. In this situation, we have $\mu_Y - \rho \in \mathbb{N}^*$, so that the only possibility is that $\operatorname{Re}\rho < \mu_Y$ as in case 1. Numerical studies (see [AT11]) seem to suggest that the assertion of case 1 should hold in this case, too, but this has not been proved, yet.*

Remark 4.2.17. *Consider a Fuchsian singularity such that $\operatorname{Re}\rho < \mu_Y$. We see that it cannot appear as a vertex of a graph of saddle connection accumulated by some geodesic. In fact, this would require the existence of infinitely many geodesics, arbitrarily close to the singularity, and escaping in both forward and backward time. But the case 1 of Theorem 4.2.15 excludes this possibility. The same is true for apparent singularities, by Theorem 4.2.10. So, we see that the vertices of a graph of saddle connections accumulated by a geodesic must be irregular or Fuchsian with $\operatorname{Re}\rho \geq \mu_Y$.*

The next result we present concerns geodesics which tend to a Fuchsian singularity p_0 with real residue ρ_{p_0}. We shall need it in the next chapter. To study this problem we could think to work in the following way: suppose we have an explicit formula for a metric adapted to ∇ in a pointed neighbourhood of p_0. In that case we could then easily study the metric foliation (and the geodesics, which live inside those leaves) and in particular its behaviour near the singularity.

The main ingredient we used to construct a solution of (2.1) in Proposition 2.1.6 was a local primitive of the 1-form η representing ∇. It is known that we can find such a primitive on a simply connected open set, or also on other domains, if and only if the condition

$$\int_\gamma \eta = 0 \tag{4.41}$$

is satisfied for every closed loop in the open set. In this case we are exactly in the opposite situation: being p_0 Fuchsian, the residue ρ_{p_0} must be different from 0, and so for every simple loop surrounding p_0 the integral (4.41) is not zero. But we see that we actually do not need a primitive of η to solve (2.1), but only its exponential. So, we can argue in the following way. We locally write $\eta = kdz$ for some meromorphic function k, with

$$k(z) = k^*(z) + \frac{\rho_{p_0}}{z} \tag{4.42}$$

with $\operatorname{Res}_{p_0} k^* = 0$. Then, we formally integrate (4.42) to get

$$K(z) = K^*(z) + \rho_{p_0} \log z.$$

This is not a globally well-defined primitive, but it is indeed a locally defined (multivalued) primitive. If we consider the exponential required by Proposition 2.1.6, we get

$$\exp(2 \operatorname{Re} K(z)) = \exp(2 \operatorname{Re} K^*(z))|z|^{\operatorname{Re} \rho_{p_0}} \exp(-(\operatorname{Im} \rho_{\rho_{p_0}}) \arg(z)).$$

So, we see that if $\operatorname{Im} \rho_{p_0} = 0$, i.e., if ρ_{p_0} is real, the last expression reduces to

$$\exp(2 \operatorname{Re} K(z)) = \exp(2 \operatorname{Re} K^*(z))|z|^{\rho_{p_0}}. \tag{4.43}$$

So, in this case we have a local metric in a pointed neighbourhood of p_0 and so we have an explicit expression for the metric foliation there. This allows us to prove the following Proposition.

Proposition 4.2.18. *Let E be a line bundle on a Riemann surface S, X a morphism $X : E \to TS$ which is an isomorphism on $S^0 = S \setminus \operatorname{Sing}(X)$ and ∇ a meromorphic connection on E, holomorphic on $E_{|S^0}$, such that the geodesic field extends holomorphically form $E_{|S^0}$ to the whole of E. Let $p_0 \in \operatorname{Sing}(X)$ be a Fuchsian singularity, with residue $\rho_{p_0} \in \mathbb{R}^*$. Then:*

1. *if $\rho_{p_0} < 0$, then all leaves of the metric foliation over p tend to the zero section as $p \to p_0$;*

2. *if $\rho_{p_0} > 0$, then all leaves of the metric foliation over $p \in S^0$ tend to infinity as $p \to p_0$.*

In particular, if $\sigma : [0, \varepsilon) \to S^0$ is a geodesic with $\sigma(t) \to p_0$ as $t \to \varepsilon$, then

1. *if $\rho_{p_0} < 0$, then $X^{-1}(\sigma'(t))$ tends to 0_{p_0} as $t \to \varepsilon$;*

2. *if $\rho_{p_0} > 0$ then $|X^{-1}(\sigma'(t))| \to \infty$ as $t \to \varepsilon$.*

Proof. Consider a small chart $(U_\alpha, z_\alpha, e_\alpha)$ centered at p_0. From (4.43), we have that, near p_0, a local metric g_α on $U_\alpha \setminus \{p_0\}$ adapted to ∇ is given by

$$g_\alpha(z_\alpha, v_\alpha) = \exp(2 \operatorname{Re} K^*_\alpha(z_\alpha))|z_\alpha|^{2\rho_{p_0}}|v_\alpha|^2$$

where the notations are as in the argument before this Proposition. Notice that, since p_0 is Fuchsian, then k^*_α is holomorphic at p_0. Being the leaves of the metric foliation on $U_\alpha \setminus \{p_0\}$ the level sets of g_α, the first part of the Proposition is proved. The second follows from the first, recalling that every geodesic leaf is contained in a metric one. □

We end this section with some results about irregular singularities The main difficulty we encounter when trying to study this kind of singularities is that we do not have a holomorphic normal form for the geodesic field near the singular point. Thus, we shall not be able to give a detailed description of this case, analogous to those given in Theorems 4.2.10 and 4.2.15 for apparent and Fuchsian singularities.

Anyway, we see that we can say something about geodesics tending to the singularities also in this case. Here we prove a result concerning geodesics converging to the origin staying inside some particular sectors.

From Lemma 4.2.6 we know that there exists a local chart (U, z) centered at the singular point p_0 such that, in this chart, the geodesic field takes the form

$$G = z^{\mu_X} v \frac{\partial}{\partial z} - z^{\mu_X - m} h_Y v^2 \frac{\partial}{\partial v} \tag{4.44}$$

where m is the irregularity of the pole and h_Y can be taken of the form

$$h_Y = (1 + \rho_{p_0} z^{m-1} + \widetilde{g}(z)),$$

with $\widetilde{g}(z) = z^m g(z)$, with g holomorphic. In fact to obtain this form we need only a finite number of steps of the Poincaré-Dulac method presented in the proof of 4.2.11, and so the conjugation is in fact holomorphic. We keep g of order m, but we see that we may consider it of any arbitrarily high order.

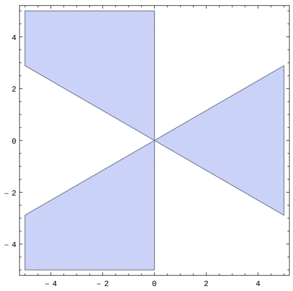


Figure 4.2: Re $\left(\frac{1}{z^3}\right) > 0$

We know that the integral curves for G, i.e., the geodesics, are included in the leaves of the horizontal foliation. This, by Proposition 1.1.5, are locally the level sets of the function

$$\exp(K(z))v$$

where K is, as usual, a local primitive of η. Here, η is $\frac{h_Y}{z^m}dz$ and it means that, if $(z(t), v(t))$ is a geodesic, it must satisfy

$$\exp\left(-\frac{m-1}{z^{m-1}} + \rho_{p_0}\log z + G(z)\right)v = c,$$

where $G(z)$ is a primitive of g (and we can assume $G(0) = 0$), for some constant $c \in \mathbb{C}^*$.

Suppose now that $z(t) \to 0$. We want to study the behaviour of $v(t)$, and in particular to look for conditions for $v(t) \to 0$ or $|v(t)| \to \infty$.

It is clear that, for $z(t) \to 0$, we have $G(z) \to 0$, so that we can avoid considering this term. So, we are left to study the expression

$$\exp\left(-\frac{m-1}{z^{m-1}} + \rho_{p_0}\log z\right) = \exp\left(-\frac{m-1}{z^{m-1}}\right)z^{\rho_{p_0}}.$$

If $\arg z(t)$ is bounded, the dominant term is the first. So, in particular, we have the following:

- if, for every $t > \bar{t}$, we have Re $\left(\frac{1}{z^{m-1}}\right) > 0$, then $\exp\left(-\frac{m-1}{z^{m-1}}\right) \to 0$, and so $v(t) \to \infty$;

- if, for every $t > \bar{t}$, we have Re $\left(\frac{1}{z^{m-1}}\right) < 0$, then $\exp\left(-\frac{m-1}{z^{m-1}}\right) \to 0$, and so $v(t) \to 0$;

In Figure 4.2 we have drawn the part of the plane where Re $\left(\frac{1}{z^{m-1}}\right) > 0$ with $m = 4$.We see that we have $2(m-1)$ sectors, each one of the same angle, such that if $z(t)$ tends to the origin staying in one of these sectors, then we have $v(t) \to 0$ or $v(t) \to \infty$, in an alternate way.

We study now what happens if, for infinitely many $t_n \to \infty$, we have $\mathrm{Re}\left(\frac{1}{z(t_n)^{m-1}}\right) = 0$, i.e., if $z(t_n)$ belongs to one of the lines separating the open sectors. We see that, for these t_n's, we have that $\left|\exp\left(-\frac{m-1}{z^{m-1}}\right)\right|$ is constant and different from 0. So, the relevant term becomes $z^{\rho_{p_0}}$ (where we remark that ρ_{p_0} is the residue of ∇at p_0). We see that now we have the following possibilities:

- if $\mathrm{Re}(\rho_{p_0}) > 0$, we have that $|z(t)|^{\rho_{p_0}} \to 0$, and so $|v(t)| \to \infty$;
- if $\mathrm{Re}(\rho_{p_0}) < 0$, we have that $|z(t)|^{\rho_{p_0}} \to \infty$, and so $|v(t)| \to 0$;
- if $\mathrm{Re}(\rho_{p_0}) = 0$, then we have that $|v(t)|$ is constant

We summarize some of these considerations in the following Proposition, for later reference.

Proposition 4.2.19. *Let E be a line bundle on a Riemann surface S, X a morphism $X : E \to TS$ which is an isomorphism on $S^0 = S \setminus \mathrm{Sing}(X)$ and ∇ a meromorphic connection on E, holomorphic on $E_{|S^0}$. Let p_0 be an irregular singularity of irregularity m, such that in a suitable chart centered at p_0 the geodesic field takes the form (4.44). Then, we have $2(m-1)$ radial sectors, each one with opening $\pi/(m-1)$, such that, if $z(t)$ tends to p_0 staying in one of these sectors, then, in an alternate way, $v(t)$ goes to zero or diverges.*

Remark 4.2.20. *By numerical studies (see Chapter 6, in particular Section 6.7) it seems that in fact this should be the general situation, i.e., for every irregular singularity of irregularity m there is a suitable chart centered at it such that, in this chart, we can find $2(m-1)$ sectors at the origin such that, for every geodesic $(z(t), v(t))$, if $z(t) \to 0$, then this happens staying inside one of the sectors (and $v(t)$ goes to zero or diverges, accordingly). Moreover, it seems that there actually exist directions of convergence: for every geodesic $(z(t), v(t))$, then $z(t)$ can converge to the origin only tangentially to some precise directions.*

Chapter 5

Dynamics in $\mathbb{C}^n$

In this chapter we shall see, following [AT11], the relation between the dynamics of a germ of endomorphism tangent to the identity in $\mathbb{C}^n$ and the Poincaré-Bendixson-type theorems we discussed in Chapter 3. After developing the theory in any dimension, we shall concentrate on the case $n = 2$, for which we shall be able to give quite a complete description of the dynamics.

5.1 The construction in this case

In this section we are going to study in more detail the construction of the previous chapter in the situation in which $M = \widehat{\mathbb{C}^n}$, the blow-up of $\mathbb{C}^n$ at the origin, and $S = \mathbb{P}^{n-1}(\mathbb{C})$ is the exceptional divisor. Then, we shall see how all this applies to the study of germs of endomorphisms tangent to the identity.

First of all, let us fix all the notations we shall use in this and in the following sections. We begin by fixing coordinates to work with and compute the changes between them.

So, we are given $\pi : M \to \mathbb{C}^n$, the blow-up of the origin, and $S = \pi^{-1}(0)$ the exceptional divisor. We will use $w = \left(w^1, \dots, w^n\right)$ as coordinates for $\mathbb{C}^n$, to be able to use the letters z's for the charts we are going to introduce. We set $H_j \subset \mathbb{C}^n$ to be the set of points such that $w^j \neq 0$ and note that we can use the open sets $U_j := \pi^{-1}(H_j) \cup (S \setminus L_j)$ (where L_j is the hyperplane in S corresponding to $\{w^j = 0\}$) as a cover for M. We define coordinates $z_j = \left(z_j^1, \dots, z_j^n\right)$ on U_j, defined by

$$\left(z_j^1, \dots, z_j^n\right) = \left(\frac{w^1}{w^j}, \dots, \frac{w^{j-1}}{w^j}, w^j, \frac{w^{j+1}}{w^j}, \dots, \frac{w^n}{w^j}\right),$$

so that we also get that the projection π on the open U_j is given by

$$\pi(z_j(p)) = \left(z_j^1 z_j^j, \dots, z_j^j, \dots, z_j^n z_j^j\right). \tag{5.1}$$

In particular, we see that the chart (U_j, z_j) is centered at the point $[0, \dots, 1, \dots, 0] \in \mathbb{P}^{n-1}(\mathbb{C})$, where the 1 is in the j-th position.

The following Lemma gives the coordinate changes betweeen z_i and z_j on $U_i \cap U_j$, as well as the change rules for dz_j^h and $\frac{\partial}{\partial z_j^h}$. We omit the proof, which consists enterely of standard computations.

Lemma 5.1.1. *The coordinate change in $U_i \cap U_j = \{z_i^j, z_j^i \neq 0\}$ are*

$$z_j^h = \begin{cases} z_i^i z_i^j & \textit{if } (h = j) \\ 1/z_i^j & \textit{if } (h = i) \\ z_i^h / z_i^j & \textit{if } (h \neq i, j) \end{cases}$$

The change rules for differentials are

$$dz_j^h = \begin{cases} z_i^i dz_i^j + z_i^j dz_i^i & \textit{if } (h = j) \\ -dz_i^j/(z_i^j)^2 & \textit{if } (h = i) \\ \frac{1}{z_i^j} d_i^h - \frac{z_j^h}{(z_i^j)^2} dz_i^j & \textit{if } (h \neq i, j) \end{cases}$$

The change rules for tangent vectors are

$$\frac{\partial}{\partial z_j^h} = \begin{cases} \frac{1}{z_i^j} \frac{\partial}{\partial z_i^i} & \textit{if } (h = j) \\ z_i^j \left(2 z_i^i \frac{\partial}{\partial z_i^i} - \sum_{k=1}^n z_i^k \frac{\partial}{\partial z_i^k} \right) & \textit{if } (h = i) \\ \frac{1}{z_i^j} d_i^h - \frac{z_j^h}{(z_i^j)^2} dz_i^j & \textit{if } (h \neq i, j) \end{cases}$$

The last coordinates we need to introduce are the ones for the bundle $p : N_S^{\otimes \nu} \to S$. We notice that we can use $\{U_j \cap S\} = \{\{z_j^j = 0\}\}$ as trivializing cover and use coordinates (ζ_j, v_j) on $p^{-1}(U_j \cap S)$, where ζ_j denotes the point of S and is the standard coordinate induced by the z_j coordinate on U_j,

$$\zeta_j = \left(\zeta_j^1, \dots \zeta_j^{n-1} \right) = \left(z_j^1, \dots, \widehat{z_j^j}, \dots, z_j^n \right) \in \mathbb{C}^{n-1}.$$

We use v_j (corresponding to the generator $\partial_j^{\otimes \nu_f}$) as coordinate for the fiber, so that we get (ζ_j, v_j) as local coordinates for the bundle.

Now, let us take a map f tangent to the identity in $\mathbb{C}^n$ and denote by $\hat{f} \in \mathrm{End}(M, S)$ the blow-up of this map at the origin. We have $\pi \circ f = \hat{f} \circ \pi$ and, in particular, $\hat{f}$ will be the identity on S.

We can write (the components of) f as a sum of its homogeneous terms, i.e.,

$$f^j(w) = w^j + \sum_{h \geq 2} Q_h^j(w)$$

and, if possible, start the sum with an higher index, if for every j we have $Q^j_h = 0$, for every $h \leq \nu$ for some ν. So, we obtain

$$f^j(w) = w^j + \sum_{h \geq \nu+1} Q^j_h(w)$$

with Q^j_h an homogeneous polynomial of degree h. The integer $\nu + 1$ is called the *order* of the map f and is the least integer $k \geq 2$ such that not all Q^j_k's are zero.

In the previous sections we have constructed a connection ∇^0 on the tangent bundle of a hypersurface pointwise fixed by a holomorphic endomorphism, depending on the endomorphism itself. What we are going to do now is to prove that, in the case we are considering, i.e., if $f \in \mathrm{End}(\mathbb{C}^n, 0)$, the connection related to $\hat{f}$ actually depends only on the homogeneous polynomials $\{Q^1_{\nu+1}, \dots, Q^n_{\nu+1}\}$ when $\hat{f}$ turns out to be tangential to S.

We start giving a precise characterization of when $\hat{f}$ actually is tangential to S. To do so, we associate to our map f the vector field

$$Q_f = \sum_{j=1}^n Q^j_{\nu+1} \frac{\partial}{\partial w^j}$$

in $\mathbb{C}^n$, which is homogeneous of degree $\nu + 1$.

Definition 5.1.2. *We say that a homogeneous vector field is* dicritical *if it is a multiple of the radial vector field*

$$\sum_{j=1}^n w^j \frac{\partial}{\partial w^j}.$$

We say that a map f tangent to the identity is dicritical *if its associated field Q_f is dicritical.*

This condition is easily seen to be equivalent, for a field $Q = \sum_{j=1}^n Q^j \frac{\partial}{\partial w^j}$, to

$$w^h Q^k = w^k Q^h$$

for all $h, k = 1, \dots, n$.

This condition on the associated field turns out to be the only obstruction for the $\hat{f}$ to be tangential to S, in the sense of Definition 4.1.5. In fact, we are going to prove that $\hat{f}$ is tangential to S if and only if f is non-dicritical.

To do so, we need an explicit expression for the blow-up $\hat{f}$. This is done in the next Lemma in the chart U_1, and for other charts it may be found in the same way.

Lemma 5.1.3. *In the chart (U_1, z_1) we have*

$$\hat{f}_1^j(z_1) = \begin{cases} z_1^1 + \left(z_1^1\right)^{\nu+1} \sum_{h\geq\nu+1} \left(z_1^1\right)^{h-\nu-1} Q_h^1\left(1,\zeta_1\right) & (\text{if } j = 1) \\ z_1^j + \left(z_1^1\right)^{\nu} \frac{\sum_{h\geq\nu+1}\left(z_1^1\right)^{h-\nu-1}\left[Q_h^j(1,\zeta_1)-z_1^j Q_h^1(1,\zeta_1)\right]}{1+\left(z_1^1\right)^{\nu-1}Q_h^1(1,\zeta_1)} & (\text{if } j \neq 1), \end{cases} \tag{5.2}$$

where we set $\hat{f}_1$ for $z_1 \circ \hat{f}$.

Proof. First, let us write $w = \left(w^1, w'\right)$, where $w' = \left(w^2, \ldots, w^n\right)$ and, analogously, $z_1 = \left(z_1^1, z_1'\right)$, $f = (f^1, f')$ and $\hat{f} = (\hat{f}^1, \hat{f}')$, and recall that $\pi\colon (z_1^1, z_1') \mapsto (z_1^1 z_1' z_1^1)$, while its local inverse is $(w^1, w') \mapsto (w^1, w'/w^1)$.

From the relation $f \circ \pi = \pi \circ \hat{f}$ we get

$$\begin{aligned}\left(f^1\left(z_1^1, z_1^1 z_1'\right), f'\left(z_1^1, z_1^1 z_1'\right)\right) &= f\left(z_1^1, z_1^1 z_1'\right) \\ &= \left(\hat{f}^1\left(z_1^1, z_1'\right), \hat{f}^1\left(z_1^1, z_1'\right)\hat{f}'\left(z_1^1, z_1'\right)\right),\end{aligned}$$

so that we have

$$\begin{cases} f^1\left(z_1^1, z_1^1 z_1'\right) = \hat{f}^1\left(z_1^1, z_1'\right) \\ f'\left(z_1^1, z_1^1 z_1'\right) = \hat{f}^1\left(z_1^1, z_1'\right)\hat{f}'\left(z_1^1, z_1'\right). \end{cases} \tag{5.3}$$

It follows that

$$\begin{aligned}\hat{f}^1(z_1^1, z_1') = f^1\left(z_1^1, z_1^1 z_1'\right) &= z_1^1 + \sum_{h\geq\nu+1} Q_h^1\left(z_1^1, z_1^1 z_1'\right) \\ &= z_1^1 + \sum_{h\geq\nu+1} \left(z_1^1\right)^h Q_h^1\left(1, z_1'\right),\end{aligned}$$

where in the last step we used the homogeneity of Q_h^1, and this proves the formula for $j = 1$. For $j \neq 1$, we use the second equation in (5.3) to get

$$\begin{aligned}\hat{f}'\left(z_1^1, z_1'\right) &= \frac{f'\left(z_1^1, z_1^1 z_1'\right)}{\hat{f}^1\left(z_1^1, z_1'\right)} = \frac{f'\left(z_1^1, z_1^1 z_1'\right)}{f^1\left(z_1^1, z_1^1 z_1'\right)} \\ &= \frac{z_1^1 z_1' + \sum_{h\geq\nu+1}\left(z_1^1\right)^h Q_h'\left(1, z_1'\right)}{z_1^1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^h Q_h^1\left(1, z_1'\right)} \\ &= \frac{z_1' + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h'\left(1, z_1'\right)}{1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h^1\left(1, z_1'\right)} \\ &= z_1' + \frac{z_1' + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h'\left(1, z_1'\right)}{1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h^1\left(1, z_1'\right)} \\ &\quad - \frac{z_1'\left(1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h^1\left(1, z_1'\right)\right)}{1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h^1\left(1, z_1'\right)} \\ &= z_1' + \frac{z_1' + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1}\left(Q_h'\left(1, z_1'\right) - z_1' Q_h^1\left(1, z_1'\right)\right)}{1 + \sum_{h\geq\nu+1}\left(z_1^1\right)^{h-1} Q_h^1\left(1, z_1'\right)}\end{aligned}$$

and we are done. $\square$

Corollary 5.1.4. *$\hat{f}$ is tangential to S if and only if f is non-dicritical.*

Proof. If f is dicritical, all the $Q^j_{\nu+1}$ are not zero, so that in particular there exist a point $p \in S$ such that $\nu_{\hat{f}}\left(z^1_1, p\right) = \nu + 1$. Moreover, all the differences in the first term of the sum in the numerator of the second formula in (5.2) are zero, so that also $\nu_{\hat{f}}\left(z^j_1, p\right) \geq \nu + 1$ for $j \neq 1$. It means, by definition, that $\hat{f}$ is not tangential.

Conversely, if f is not dicritical, there will be a j with

$$Q^j_{\nu+1}\left(1, \zeta_1\right) - z^j_1 Q^1_{\nu+1}\left(1, \zeta_1\right) \neq 0,$$

which means that $\nu_{\hat{f}}\left(z^j_1, p\right)$ will be equal to ν for some p, and, being $\nu_{\hat{f}}\left(z^1_1, p\right) \geq \nu + 1$, it means that $\hat{f}$ is tangential to S. $\square$

The next Lemma allows to see that the associated geodesic field G on $N_S^{\otimes \nu}$ depends only on $\{Q^1_{\nu+1}, \ldots, Q^n_{\nu+1}\}$ when $\hat{f}$ is tangential, i.e., when f is non dicritical.

Lemma 5.1.5. *In the chart U_1, we have*

- $\frac{\partial g^1_1}{\partial z^1_1}|_{U_1 \cap S} = Q^1_{\nu+1}\left(1, \zeta_1\right)$;
- $g^j_{1|U_1 \cap S} = Q^p_{\nu+1}\left(1, \zeta_1\right) - \zeta_1^{j-1} Q^1_{\nu+1}\left(1, \zeta_1\right)$ *for* $j = 2, \ldots, n$;
- $G_{|p^{-1}(U_1 \cap S)} =$

$$\left(\sum_{j=2}^{n-1}\left[Q^j_{\nu+1}\left(1, \zeta_1\right) - \zeta_1^{j-1} Q^1_{\nu+1}\left(1, \zeta_1\right)\right] v_1 \frac{\partial}{\partial \zeta_1^h}\right) + \nu Q^1_{\nu+1}\left(1, \zeta_1\right) v_1^2 \frac{\partial}{\partial v_1}.$$

In particular, the morphism X_f, the connections ∇ and ∇^0 and the associated geodesic field G depend only on $\{Q^1_{\nu+1}, \ldots, Q^n_{\nu+1}\}$.

Proof. The first equality follows immediately from the first of (5.2). For the second, we use the second of (5.2) to get, for $j \neq 1$,

$$g^j = Q^j_{\nu+1}\left(1, \zeta_1\right) - z^j_1 Q^1_h\left(1, \zeta_1\right),$$

which gives the desired formula recalling that, in the chart U_1, we have $\zeta^j_1 = z_1^{j-1}$. The last equality follows from the first two and the definition of G. $\square$

So, we see that we can associate to a non-dicritical $(\nu+1)$-homogeneous field $\sum_{j=1}^n Q^j \frac{\partial}{\partial w^j}$ the morphism $X_Q : N_S^{\otimes \nu} \to TS$ and the connection ∇, which will be the common morphism and connection for all the endomorphisms of the form

$$f(z) = z + Q(z) + O\left(\|z\|\right)^{\nu+2}.$$

The next thing we are going to do is to understand better the meaning of the singular points, i.e., the zeroes, of X_Q. To do so, we introduce a definition and then prove that the objects we have defined are precisely the zeroes of X_Q, thus giving a new characterization of $\mathrm{Sing}(\hat{f})$.

Definition 5.1.6. *A* characteristic direction *of a homogeneous vector field $Q = \sum_{j=1}^n Q^j \frac{\partial}{\partial w^j}$ is a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ such that the line $L_v = \mathbb{C}v$ is Q-invariant. L_v will be called* characteristic leaf *and we shall say that $[v]$ is degenerate if $Q_{|L_v} \equiv 0$, non-degenerate otherwise.*

We have the following Proposition, giving an equivalence between characteristic directions and singular points:

Proposition 5.1.7. *The singular points of X_Q are exactly the characteristic directions of Q.*

Proof. We shall use the second formula of Lemma 5.1.5. In the chart U_1, we know that $[v]$ is characteristic if and only if there exixts $\lambda \in \mathbb{C}$ such that

$$\begin{pmatrix} Q^1_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) \\ \vdots \\ Q^j_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) \\ \vdots \\ Q^n_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \vdots \\ z_1^j \\ \vdots \\ z_1^n \end{pmatrix},$$

where $[1, \dots, z_1^j, \dots z_1^n]$ is a representative for $[v]$. This is clearly equivalent to

$$\begin{cases} Q^1_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = \lambda \\ & \vdots \\ Q^j_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = \lambda z_1^j \\ & \vdots \\ Q^n_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = \lambda z_1^n \end{cases}$$

which is

$$\begin{cases} Q^1_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = \lambda \\ & \vdots \\ Q^j_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = z_1^j Q^1_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) \\ & \vdots \\ Q^n_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right) & = z_1^n Q^1_{\nu+1}\left(1, z_1^2, \dots, z_1^n\right). \end{cases}$$

Thanks to Lemma 5.1.5, and recalling that $\zeta_1 = \left(z_1^2, \dots, z_1^n\right)$, we see that this is equivalent to $g^j = 0$, for $j = 2, \dots, n$, which precisely means that $[v]$ is a zero for X_Q. $\square$

What we have done so far has been to introduce G using something like a pull-back by X_Q, in the sense that we prescribed some properties on the images on the integral curves of G. What we want to do now is to give another description of G, that will turn out to be the pushforward of the field Q by a map $\chi_\nu : \mathbb{C}^n \setminus 0 \to N_S^{\otimes \nu}$. In this way, we shall be able to relate the integral curves for G, which we can study by means of the connection ∇^0 on the foliation of TS, and the integral curves of Q, i.e., the integral curves for the field associated to our starting endomorphism f. In particular, if we consider f_Q, the time-1 map of Q, we see that studying the integral curves for Q permits to undersand the behaviour of the discrete orbits of f_Q. So, all we have done gives us a way to approach the original problem of studying the orbits of a time-1 map: we take its field Q and, to study its integral curves, we can use the integral curves of G, which we can study looking at their images in the foliation on TS. Here, we can use Poincaré-Bendixson-type Theorems to understand the topological behaviour of the geodesics for ∇^0, which are contained in a leaf of the foliation of TS, thus getting information about the asymptotic behavior of the integral curves for the initial endomorphism.

So, let us introduce this map χ_ν:

Definition 5.1.8. *In the chart $H_j = \{w^j \neq 0\}$ we define $\chi_\nu : \mathbb{C}^n \setminus \{0\} \to N_S^{\otimes \nu} \setminus S$ as $\chi_\nu(w) = (\zeta_j(w), v_j(w))$, where*

$$\begin{cases} \zeta_j(w) = \left(\frac{w^1}{w^j}, \dots, \widehat{\frac{w^j}{w^j}}, \dots, \frac{w^n}{w^j} \right) \\ v_j(w) = (w_j)^\nu \end{cases} \tag{5.4}$$

The next proposition shows that χ_ν is actually a well defined map between $\mathbb{C}^n \setminus \{0\}$ and $N_S^{\otimes \nu} \setminus S$.

Proposition 5.1.9. *The map χ_ν defined in* (5.4) *actually gives a ν-to-1 holomorphic covering map between $\mathbb{C}^n \setminus \{0\}$ and $N_S^{\otimes \nu} \setminus S$.*

Proof. Using Lemma 5.1.1 it is easy to see that the changes of coordinates for $N_S^{\otimes \nu}$ are

$$\zeta_j^h = \begin{cases} \zeta_i^h / \zeta_i^j & \text{if } 1 \leq h \leq j-1, i \leq h \leq n-1 \\ \zeta_i^{h+1} / \zeta_i^j & \text{if } j \leq h \leq i-2 \\ 1/\zeta_i^j & \text{if } h = i-1 \end{cases}$$

if $j < i$ and similar ones if $j > i$, and

$$v_j = \left(\zeta_i^j\right) v_i.$$

It easily follows that the map χ_ν is well defined, and it is then clear that it is a holomorphic ν-to-1-covering. □

We are now ready to prove the announced result that $G = d\chi_\nu Q$.

Theorem 5.1.10. *Let Q be a non-dicritical homogeneous vector field of degree $\nu + 1 \geq 2$ in $\mathbb{C}^n$ and G the associated geodesic field on the total space of $p : N_S^{\otimes \nu} \to S$. Then*

$$d\chi_\nu(Q) = G.$$

Proof. First, we remark that the fact that Q is ν-homogeneous ensures that we can define its pushforward by χ_ν.

To evaluate $d\chi_\nu(Q)$ we need to know the (local) pushforward of the tangent vectors $\frac{\partial}{\partial w^h}$. From the definition of χ_ν, we find that

$$d\chi_\nu\left(\frac{\partial}{\partial w^h}\right) = \begin{cases} \frac{1}{w^j}\frac{\partial}{\partial \varsigma_j^h} & \text{if } h < j \\ \frac{\left(\sum_{k=1}^{j-1} w^k \frac{\partial}{\partial \varsigma_j^k} + \sum_{k=j}^{n-1} w^{k+1} \frac{\partial}{\partial \varsigma_j^k}\right)}{(w^j)^2} + \nu\left(w^j\right)^{\nu-1}\frac{\partial}{\partial v_j} & \text{if } h = j \\ \frac{1}{w^j}\frac{\partial}{\partial \varsigma_j^{h-1}} & \text{if } h > j, \end{cases}$$

and the assertion follows. □

As a corollary, we get the relation between the integral curves for the homogeneous vector field Q and integral curves for G. We define $\hat{S}_Q := \pi^{-1}(S^0) \setminus \{0\} = \{w \in \mathbb{C}^n \setminus \{0\} \colon [w] \in S^0\}$, where S^0 is the complement in S of the characteristic directions.

Corollary 5.1.11. *Let Q be a non-dicritical homogeneous vector field of degree $\nu + 1 \geq 2$ in $\mathbb{C}^n$ and G the associated geodesic field on the total space of $p : N_S^{\otimes \nu} \to S$. Then a real curve $\gamma : I \to \hat{S}_Q$ is an integral curve for Q if and only if $\chi_\nu \circ \gamma$ is an integral curve for G. Moreover:*

- *if $\gamma \colon I \to \hat{S}_Q$ is an integral curve for Q, then $[\gamma] \colon I \to \mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^0-geodesic;*

- *if $\sigma \colon I \to \mathbb{P}^{n-1}$ is a ∇^0-geodesic then there exist ν integral curves $\gamma_1, \ldots, \gamma_\nu : I \to \hat{S}_Q$ for Q, differing only by the multiplication by a ν-th root of unity, such that $[\gamma_j] = \sigma$ for every $j = 1, \ldots, \nu$.*

Proof. It follows immediately from Theorem 5.1.10 and Proposition 4.1.19. □

5.2 Behaviour on characteristic directions

In this section we describe the dynamics of the field Q inside a characteristic leaf. By definition, we have that each integral curve for Q starting on a characteristic leaf remains there, and so the problem is 1-dimensional. In particular, this also solves the problem for dicritical maps, for which all directions are characteristic (see Section 6.2.2). We remark that a qualitative description is given by the Leau-Fatou Flower Theorem. What follows actually is the study of the topological model given by Camacho ([Cam78]).

Proposition 5.2.1. *Let Q be a holomorphic homogeneous vector field in $\mathbb{C}^n$, of degree $\nu + 1 \geq 2$. Let $[v]$ be a characteristic direction, with λ_0 such that $Q^j(v) = \lambda_0 v^j$. Then,*

- *if the direction $[v] \in \mathbb{P}^{n-1}$ is degenerate, then the dynamics of Q on L_v is trivial, i.e., every point is fixed by the flow;*
- *if the direction $[v] \in \mathbb{P}^{n-1}$ is not degenerate, then the integral curve for Q with $\zeta_0 v$ as starting point is*

$$\gamma_{\zeta_0 v}(t) = \frac{\zeta_0 v}{(1 - \nu\lambda_0\zeta_0^\nu t)^{1/\nu}}. \tag{5.5}$$

  In particular, we do not have recurrent integral curves and

  – *if $\lambda_0\zeta_0^\nu \notin \mathbb{R}^+$ then $\lim_{t\to+\infty} \gamma_{\zeta_0 v}(t) = 0$;*

  – *if $\lambda_0\zeta_0^\nu \in \mathbb{R}^+$ then $\lim_{t\to+\left(\lambda_0\zeta_0^\nu\nu\right)^{-1}} \|\gamma_{\zeta_0 v}(t)\| = +\infty$;*

Proof. If the direction is degenerate, we have that the field is zero on the leaf and so everything is fixed.

If the direction is non-degenerate, we recast the problem in $\mathbb{C}$ and then we bring it back on the leaf. To do so, we take a parametrization of the leaf, i.e., a function

$$\varphi : \mathbb{C} \to L_v$$

$$\zeta \mapsto \zeta v.$$

We pull-back the field $Q_{|L_v}$ to $\mathbb{C}$ with $d\varphi^{-1}$, obtaining

$$d\varphi^{-1}(Q_{|L_v}) = \lambda_0\zeta^{\nu+1}\frac{\partial}{\partial\zeta}$$

and we integrate this field on $\mathbb{C}$, obtaining

$$\zeta(t) = \frac{\zeta_0}{(1 - \lambda_0\nu\zeta_0^\nu t)^{1/\nu}}$$

as integral curve issuing from ζ_0, as we may verify directly. So, we bring our integral curve on $\mathbb{C}$ back to $L_v \subset \mathbb{C}^n$, obtaining

$$\gamma_{\zeta_0 v}(t) = \frac{\zeta_0 v}{(1 - \nu\lambda_0\zeta_0^\nu t)^{1/\nu}},$$

as wanted. From this formula, it is easy to verify the last two assertions: if $\lambda_0\zeta_0^\nu \in \mathbb{R}^+$ there is a real time for which the denominator goes to zero, and so $\|\gamma(t)\|$ diverges. If not, the denominator is never zero and so the curve is defined for any (positive) t and the denominator diverges as $t \to +\infty$. □

5.3 Dynamics in $\mathbb{C}^2$

In this section we shall apply all the results proved so far to the study of the dynamics of homogeneous vector fields in $\mathbb{C}^2$. We see that this dimension is very particular for what concerns the method introduced before. In fact, when studying the problem in $\mathbb{C}^n$, we come to study geodesics for a connection on a foliation in Riemann surfaces of $\mathbb{P}^{n-1}$, and so the problem splits in two parts: study the leaves of the foliation and the geodesics inside them. The behaviour and topology of these foliation may in general be quite complicated, not allowing to direcly get too many information from the use of Poincaré-Bendixson-type theorems. But in dimension 2 we see that the problem of understanding the foliation is very easy: $\mathbb{P}^1$ is already the unique leaf, allowing us to use Theorem 3.4.8 directly. This is what we are going to do now. The next chapter will be devoted to the study of concrete examples, i.e., the dynamics of homogeneous vector fields of degree 3 in $\mathbb{C}^2$, as an application of this method.

We start by reviewing the setting and the results obtained so far in this situation. This will help to clarify ideas and concepts in a more “visualizable” setting, and also to have a concise and complete treatment of this special case.

So, let M be a 2-dimensional complex manifold (in our case, $\hat{\mathbb{C}}^2$) and S a hypersurface, i.e., a 1-dimensional complex submanifold of M, and $f \in \mathrm{End}(M, S)$ different from the identity and tangential to S. In a chart (U_α, z_α) adapted to S, i.e., where $S = \{z_\alpha^1 = 0\}$ we can write f as

$$f\begin{pmatrix} z_\alpha^1 \\ z_\alpha^2 \end{pmatrix} = \begin{pmatrix} (z_\alpha^1)^{\nu_f} g_\alpha^1 \\ (z_\alpha^1)^{\nu_f} g_\alpha^2 \end{pmatrix},$$

with $g^1_{\alpha|S} = 0$. So, the canonical morphism becomes

$$X_{f,\alpha} = g_\alpha^2 \frac{\partial}{\partial z_\alpha^2} \otimes (dz_\alpha^1)^{\otimes \nu_f}$$

and the partial meromorphic connection ∇ along X_f on N_S introduced in Chapter 4 by (4.5), the induced connection on $N_S^{\otimes \nu_f}$ and the ∇^0 on TS

are all standard meromorphic connections. In particular, we see that ∇^0 is represented, on (U_α, z_α), by the 1-form

$$\eta^0_\alpha = \left(\frac{\partial g^2_\alpha / \partial z^2_\alpha}{g^2_\alpha} - \nu_f \frac{1}{g^2_\alpha} \frac{\partial g^1_\alpha}{\partial z^1_\alpha} \right) dz^2_\alpha.$$

So, we see that we have all the ingredients to consider on $N_{S^0}^{\otimes \nu_f}$, a line bundle over the Riemann surface S^0, the three foliations that we studied in Chapter 2:

- the metric foliation on $N_{S^0}^{\otimes \nu_f} \setminus S^0$, a real non-singular foliation of rank 3, that we can extend to a singular foliation to all of $N_{S^0}^{\otimes \nu_f}$ by adding the zero section as exceptional leaf;
- the horizontal foliation on $N_{S^0}^{\otimes \nu_f}$, a complex non-singular foliation of rank 1;
- the geodesic foliation on $N_{S^0}^{\otimes \nu_f}$, a real foliation of rank 1, singular only on the zero section, where we use the fact that X_f is an isomorphism between $N_{S^0}^{\otimes \nu_f}$ and TS^0.

We can also consider the geodesic field G on $N_{S^0}^{\otimes \nu_f}$, defined by (4.11). In the proof of Proposition 4.1.19 we already noticed that in this special case, i.e., when M has dimension 2, it is the same field we studied in Chapter 2. Here, it becomes of the form

$$G_{p^{-1}(U_\alpha)} = g^2_{\alpha | U_\alpha \cap S} v_\alpha \frac{\partial}{\partial z^2_\alpha} + \nu_f \left.\frac{\partial g^1_\alpha}{\partial z^1_\alpha}\right|_{U_\alpha \cap S} v^2_\alpha \frac{\partial}{\partial v_\alpha},$$

which is also

$$G_{p^{-1}(U_\alpha)} = X_\alpha v_\alpha H_\alpha,$$

with H_α as defined in (2.9) (obviously, here $\partial_\alpha = \frac{\partial}{\partial z^2}$) and X_α given by

$$X_\alpha = g^2_\alpha|_{U_\alpha \cap S^0}.$$

Let us now consider our starting problem: we have a holomorphic homogeneous vector field in $\mathbb{C}^2$ and we want to study its dynamics. So, let us consider $\mathbb{C}^2$ with the usual coordinates (w^1, w^2) and a vector field of the form

$$Q(w^1, w^2) = Q^1(w^1, w^2) \frac{\partial}{\partial w^1} + Q^2(w^1, w^2) \frac{\partial}{\partial w^2}$$

with Q^1 and Q^2 homogeneous of the same total degree $\nu + 1$. Blowing-up the origin, we get the exceptional divisor $S \cong \mathbb{P}^1(\mathbb{C})$. By what we said in Section 5.1 we can cover $M = \hat{\mathbb{C}}^2$ with two charts, that we can call (U_0, z_0)

and (U_∞, z_∞), centered respectively at the points $0 = [1,0] \in \mathbb{P}^1(\mathbb{C})$ and $\infty = [0,1] \in \mathbb{P}^1(\mathbb{C})$. By (5.1) we get

$$\begin{cases} w^1 = z_0^1 \\ w^2 = z_0^1 z_0^2 \end{cases}$$

and

$$\begin{cases} w^1 = z_\infty^1 z_\infty^2 \\ w^2 = z_\infty^2. \end{cases}$$

Consider now the bundle $N_S^{\otimes \nu}$. We trivialize it with the two charts (U_0, ζ_0, v_0) and $(U_\infty, \zeta_\infty, v_\infty)$, with

$$\zeta_\infty = \frac{1}{\zeta_0} \text{ and } v_\infty = \zeta_0^\nu v_0$$

Using these coordinates, it is easy to rewrite all the quantities we need. The result is the following.

$$\begin{array}{c} \text{Chart } U_0 \text{ at } [1,0]\text{, coordinate } \zeta_0 \\ \hline \frac{g_0^1}{\partial z_0^1}_{|U_0 \cap S} = Q_1(1,\zeta_0) \\ X_0 = g^2_{0\,U_0\cap S} = Q_2(1,\zeta_0) - \zeta_0 Q_1(1,\zeta_0) \\ \eta_0 = -\frac{\nu Q_1(1,\zeta_0)}{Q_2(1,\zeta_0)-\zeta_0 Q_1(1,\zeta_0)} d\zeta_0 \\ \omega_{|p^{-1}(U_0\cap S^0)} = -\frac{\nu Q_1(1,\zeta_0)}{Q_2(1,\zeta_0)-\zeta_0 Q_1(1,\zeta_0)} d\zeta_0 + \frac{1}{v_0} dv_0 \\ H_0 = \frac{\partial}{\partial \zeta_0} + \frac{\nu Q_1(1,\zeta_0)}{Q_2(1,\zeta_0)-\zeta_0 Q_1(1,\zeta_0)} v_0 \frac{\partial}{\partial v_0} \\ G_0 = (Q_2(1,\zeta_0) - \zeta_0 Q_1(1,\zeta_0))\, v_0 \frac{\partial}{\partial \zeta_0} + \nu Q_1(1,\zeta_0)(v_0)^2 \frac{\partial}{\partial v_0} \end{array}$$

$$\begin{array}{c} \text{Chart } U_\infty \text{ at } [0,1]\text{, coordinate } \zeta_\infty \\ \hline \frac{g_\infty^1}{\partial z_\infty^2}_{|U_\infty \cap S} = Q_1(\zeta_\infty,1) \\ X_\infty = g^2_\infty = Q^1(\zeta_\infty,1) - \zeta_\infty Q_2(\zeta_\infty,1) \\ \eta_\infty = -\frac{Q_1(\zeta_\infty,1)}{Q^1(\zeta_\infty,1)-\zeta_\infty Q_2(\zeta_\infty,1)} d\zeta_\infty \\ \omega_{p^{-1}(U_\infty\cap S^0)} = -\frac{Q_1(\zeta_\infty,1)}{Q^1(\zeta_\infty,1)-\zeta_\infty Q_2(\zeta_\infty,1)} d\zeta_\infty + \frac{1}{v_\infty} dv_\infty \\ \frac{\partial}{\partial \zeta_\infty} + \frac{Q_1(\zeta_\infty,1)}{Q^1(\zeta_\infty,1)-\zeta_\infty Q_2(\zeta_\infty,1)} v_\infty \frac{\partial}{\partial v_\infty} \\ G_\infty = \left(Q^1(\zeta_\infty,1) - \zeta_\infty Q_2(\zeta_\infty,1)\right) v_\infty \frac{\partial}{\partial \zeta_\infty} + \nu Q_2(\zeta_\infty,1)(v_\infty)^2 \frac{\partial}{\partial v_\infty} \end{array}$$

Remark 5.3.1. *In all this approach, the important quantities are the* order *of a singularity, i.e., the order of vanishing of the morphism X, and the* residue *of the connection at the singular points.*

Other quantities had been introduced before to study this problem. In particular, we recall the index $\iota_{[v]}$, *that we may here define as*

$$\iota_{[v]} = -\frac{1}{\nu} \operatorname{Res}_{[v]}(\nabla)$$

and the director $\delta_{[v]}(Q)$ *of a non-degenerate characteristic direction* $[v]$*, defined as*

$$\delta_{[v]}(Q) = \frac{1}{Q^1(1, v_0)} \frac{\partial(Q^2(1,\zeta) - \zeta Q^1(1,\zeta))}{\partial \zeta}(v_0)$$

(see [Éca81a, Éca81b, Éca85] and [Hak98]). We shall not concentrate on these quantities and on their relation with our problem.

The following Theorems are mainly restatements of the general Theorems 4.2.10 and 4.2.15 and of Proposition 4.2.18.

Theorem 5.3.2 (Corollary 7.3 in [AT11])**.** *Let Q be a homogeneous holomorphic vector field in $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be an apparent singularity of X_Q of order $\mu \geq 1$ and apparent index $a \in \mathbb{C}$ if $\mu > 1$. Then:*

1. *if the direction $[\gamma(t)] \in \mathbb{P}^1(\mathbb{C})$ of an integral curve $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{0\}$ of Q tends to $[v_0]$ as $t \to \varepsilon$, then $\gamma(t)$ tends to a non-zero point of the characteristic leaf $L_{v_0} \subset \mathbb{C}^2$;*

2. *no integral curve of Q tends to the origin tangent to $[v_0]$;*

3. *there is an open set of initial conditions whose integral curves tend to a non-zero point of L_{v_0};*

4. *if $\mu = 1$, then Q admits periodic orbits of arbitrarily long periods accumulating at the origin.*

Proof. We can have two different situations, due to the fact that the field Q is identically zero on the leaf of an apparent singularity. So, an integral curve can be a unique point of L_{v_0} or it cannot intersect L_{v_0}. In the first case the assertions are clearly true, so we are going to consider integral curves of the second kind. Moreover, being the characteristic direction Q-invariant, we can suppose that our integral curve lives in $\hat{S}_Q$. So, we have an integral curve $\gamma(t)$, and, in order to apply Theorem 4.2.10, we consider its projection $\sigma = [\gamma(t)] = p \circ \chi_\nu \circ \gamma$, converging to $[v_0]$. We clearly have $X_Q^{-1}(\sigma') = \chi_\nu \circ \gamma$, and Theorem 4.2.10 implies that this object tends to a non-zero element of $(N_{\mathbb{P}^1(\mathbb{C})}^{\otimes \nu})_{v_0} = \chi_\nu(L_{v_0})$. We only need to show that this implies that $\gamma(t)$ tends to a unique point of L_{v_0} (among the ν possible points), but this follows from the fact that the limit is connected. In particular, 2 immediately follows, and 3 follows from the cases of Theorem 4.2.10. We are left with proving 4: this follows from the corresponding cases of Theorem 4.2.10, where we proved the existence of periodic orbits through any point in a suitable neighbourhood. From the proof of that Theorem, we see that the period of these orbits is inversely proportional to the modulus of v_0, and so the assertion follows. We remark that we cannot have periodic orbits with $\mu > 1$. In fact, by (4.14), we have $\mathrm{Res}_{[v_0]}(\nabla^0) = \rho_{[v_0]}(\nabla) - \mu = -\mu$ and so, in order for the projection of the integral curve to be closed, we need $\mu = 1$. □

Proposition 5.3.3 (Corollary 7.5 in [AT11]). *Let Q be a homogeneous holomorphic vector field in $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian singularity of X_Q with real residue $\rho \in \mathbb{R}^*$. Let $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{0\}$ be an integral curve of Q such that $[\gamma(t)]$ tends to $[v_0]$ as $t \to \varepsilon$. Then:*

- *if $\rho < 0$, then $\gamma(t)$ tends to the origin as $t \to \varepsilon$;*
- *if $\rho > 0$, then $\gamma(t)$ diverges as $t \to \varepsilon$.*

In particular, if $\rho > 0$, no integral curve can tend to the origin tangent to $[v_0]$ (outside the characteristic leaf L_{v_0}).

Theorem 5.3.4 (Corollary 8.5 in [AT11]). *Let Q be a homogeneous vector field in $\mathbb{C}^2$ of degree $\nu + 1 \geq 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian singularity of X_Q of order $\mu_X \geq 1$, residue $\rho \in \mathbb{C}^*$ and resonant index $a \in \mathbb{C}$ if $\mu_X - 1 - \rho \in \mathbb{N}^*$. Put $\mu_Y = \mu_X - 1$. Then:*

1. *if the direction $[\gamma(t)] \in \mathbb{P}^1(\mathbb{C})$ of an integral curve $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{0\}$ of Q tends to $[v_0]$ as $t \to \varepsilon$ and γ is not contained in the characteristic leaf L_{v_0}, then:*
   - *(a) if $\mathrm{Re}\,\rho < \mu_Y$ and $\mu_Y \mathrm{Re}\,\rho < |\rho|^2$, then $\gamma(t)$ tends to the origin;*
   - *(b) if $\rho = \mu_Y > 0$, or $\mathrm{Re}\,\rho > \mu_Y$, or $\mathrm{Re}\,\rho < \mu_Y$ and $\mu_Y \mathrm{Re}\,\rho > |\rho|^2$, then $\|\gamma(t)\| \to +\infty$;*
   - *(c) if $\mathrm{Re}\,\rho < \mu_Y$ and $\mu_Y \mathrm{Re}\,\rho = |\rho|^2$ then $\gamma(t)$ accumulates a circumference in L_{v_0}.*

Furthermore there is a neighbourhood $U \subset \mathbb{P}^1(\mathbb{C})$ of $[v_0]$ such that an integral curve γ issuing from a point $z_0 \in \mathbb{C}^2 \setminus L_{v_0}$ with $[z_0] \in U \setminus \{[v_0]\}$ can have one of the following behaviours, where $\hat{U} = \{z \in \mathbb{C}^2 \setminus \{0\} : [z] \in U\}$:

2. *if $\mathrm{Re}\,\rho > \mu_Y$, then*
   - *(a) either $\gamma(t)$ escapes $\hat{U}$, and this happens for a Zariski dense open set of initial conditions in $\hat{U}$; or*
   - *(b) $[\gamma(t)] \to [v_0]$ but $\|\gamma(t)\| \to \infty$;*
3. *if $\mathrm{Re}\,\rho = \mu_Y$ but $\rho \neq \mu_Y$, then*
   - *(a) either $\gamma(t)$ escapes $\hat{U}$; or*
   - *(b) $\gamma(t) \to 0$ without being tangent to any direction, and $[\gamma(t)]$ is a closed curve or accumulates a closed curve in $\mathbb{P}^1(\mathbb{C})$ surrounding $[v_0]$; or*

(c) *$\|\gamma(t)\| \to \infty$ without being tangent to any direction, and $[\gamma(t)]$ is a closed curve in $\mathbb{P}^1(\mathbb{C})$ surrounding v_0;*

in particular, no integral curve ouside L_{v_0} can converge to the origin tangent to $[v_0]$;

4. *if $\rho = \mu_Y > 0$, then*

   (a) *either $\gamma(t)$ escapes $\hat{U}$, and this happens for an open set $\hat{U}_1 \subset \hat{U}$ of initial conditions; or*

   (b) *$[\gamma(t)] \to [v_0]$ with $\|\gamma(t)\| \to \infty$, and this happens for an open set $\hat{U}_2 \subset \hat{U}$ of initial conditions such that $\hat{U}_1 \cup \hat{U}_2$ is dense in $\hat{U}$; or*

   (c) *γ is a periodic integral curve with $[\gamma]$ surrounding $[v_0]$;*

   in particular, no integral curve outside L_{v_0} converge to the origin tangent to $[v_0]$, but we have periodic integral curves of arbitrarily long period accumulating the origin;

5. *if $\operatorname{Re}\rho < \mu_Y$ and $a = 0$, then $[\gamma(t)] \to [v_0]$ for an open dense set $\hat{U}_0$ of initial conditions in $\hat{U}$, and $\gamma(t)$ escapes $\hat{U}$ for $z \in \hat{U} \setminus \hat{U}_0$; more precisely:*

   (a) *if $\mu_Y \operatorname{Re}\rho < |\rho|^2$ then $\gamma(t) \to 0$ tangent to $[v_0]$ for all $z \in \hat{U}_0$;*

   (b) *if $\mu_Y \operatorname{Re}\rho > |\rho|^2$ then $||\gamma(t)|| \to \infty$ tangent to $[v_0]$ for all $z \in \hat{U}_0$;*

   (c) *if $\mu_Y \operatorname{Re}\rho = |\rho|^2$ then $\gamma(t)$ accumulates a circumference in L_{v_0}*

Proof. The necessary condition for having a resonant index is $\mu_Y - \rho \in \mathbb{N}^*$, which in particular implies $\mu_Y > \rho \in \mathbb{Z}$. So, in cases 2-4 the resonant index is zero and so cases 2-5 (and also part 1 when the resonant index is zero) follow from Theorem 4.2.15. So, we need only to prove part 1 when the resonant index is not zero (and so $\mu_Y > \rho \in \mathbb{Z}$). But in this case:

- we have $\mu_Y \operatorname{Re}\rho = \mu_Y\rho < |\rho|^2$ if and only if $\rho < 0$;
- the condition for case 1c is never satisfied.

So, the assertion follows applying Proposition 4.2.18 in this situation.

We remark that, by Theorem 4.2.15, in the cases not covered by case 1 we cannot actually have geodesics going to the singular point. □

For a similar study in the case of irregular singularities we would need an analogous of Theorems 4.2.10 and 4.2.15 in this situation. We recall that those two Theorems were proved by means of the holomorphic normal form of the geodesic field near the singularity, which is not known yet for irregular ones. But by Proposition 4.2.19 we get the following result.

Proposition 5.3.5. *Let Q be a homogeneous holomorphic vector field in $\mathbb{C}^2$ of degree $\nu+1 \geq 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be an irregular singularity of X_Q, of irregularity $m \geq 2$ and residue ρ. Let $\gamma : [0,\varepsilon) \to \mathbb{C}^2 \setminus \{0\}$ be an integral curve of Q such that $[\gamma(t)]$ tends to $[v_0]$ as $t \to \varepsilon$. Then, in a suitable chart of $\mathbb{P}^1(\mathbb{C})$ centered at $[v_0]$, we have $2(m-1)$ sectors of equal opening centered at $[v_0]$ such that is $[\gamma(t)]$ definitively stay in one of these sectors, then, alternately, $v(t)$ goes to zero or diverges.*

We also state here a result by Liz Vivas (see [Viv11]) giving the existence of basins of attraction for the origin for *degenerate* irregular direction.

Theorem 5.3.6. *Let f be a germ of holomorphic endomorphism of $\mathbb{C}^2$ tangent to the identity and $[v]$ a degenerate and irregular characteristic direction for f. Then there exists an open basin V attracted to the origin along $[v]$.*

So we see that, given a holomorphic homogeneous vector field in $\mathbb{C}^2$, we can study its integral curves through their projections on $\mathbb{P}^1(\mathbb{C})$. To study these, we have the Poincaré-Bendixson Theorem 3.4.8 and the local results of this chapter. Using all these tools, we see that we can give a fairly complete description of the dynamics for a large class of vector fields. For example, we see that in the generic situation we have all the singularities that become Fuchsian of order 1. Moreover, we see that, without losing the property of being generic, we can also ask that for no subset of the singularities the real part of the sum of the induced residues is equal to -1, thus excluding the possibilities 2, 3 and 4 of Theorem 3.4.8. A sample Theorem it is possible to prove in this situation is the following, which solves the problem in the generic case.

Theorem 5.3.7 (Corollary 8.6 in [AT11])**.** *Let Q be a non-dicritical homogeneous holomorphic vector field in $\mathbb{C}^2$ of degree $\nu+1 \geq 2$, such that all the characteristic directions are Fuchsian of order 1. Assume that for no set of $g \geq 1$ characteristic directions the real part of the sum of the residues is equal to $g-1$ (i.e. the real part of the sum of the induced residues is equal to -1). Let $\gamma : [0,\varepsilon) \to \mathbb{C}^2$ be a maximal integral curve of Q. Then we have the following two cases.*

1. *If $\gamma(0)$ belongs to a characteristic leaf L_{v_0}, then the image of γ is contained in L_{v_0}. Moreover, either $\gamma(t) \to 0$ (and this happens for a Zariski dense set of initial conditions in L_{v_0}), or $\|\gamma(t)\| \to +\infty$, and in this case the integral curve is a straight line.*

2. *If $\gamma(0)$ does not belong to any characteristic leaf, then either:*

   (a) γ converges to the origin tangentially to a characteristic direction $[v_0]$ whose residue has negative real part;

(b) $\|\gamma(t)\| \to +\infty$ *tangentially to a characteristic direction* $[v_0]$ *whose residue has positive real part; or*

(c) $[\gamma] : [0, \varepsilon) \to \mathbb{P}^1(\mathbb{C})$ *intersects itself infinitely many times.*

Furthermore, if 2c never occurs, then 2a holds for a Zariski dense open set of initial conditions.

Chapter 6

Cubic vector fields in $\mathbb{C}^2$

In this last chapter we are going to study in detail the dynamics of homogeneous cubic vector fields in $\mathbb{C}^2$. The quadratic case has been studied by Abate and Tovena in [AT11], starting from the holomorphic (and linear) classification of these maps made by Abate in [Aba05]. Here, we want to give a holomorphic classification of 2-dimensional cubic vector fields and then try to apply the results of the previous sections to study the dynamics of the time-1 maps of these fields, as done in [AT11] for the quadratic maps.

6.1 Preliminary remarks

So, let us suppose we have a cubic vector field in $\mathbb{C}^2$, of the form

$$\begin{aligned} Q(z,w) = \left(az^3 + bz^2w + czw^2 + dw^3\right)\frac{\partial}{\partial z} \\ + \left(Az^3 + Bz^2w + Czw^2 + Dw^3\right)\frac{\partial}{\partial w}. \end{aligned}$$

In order to study a holomorphic classification of these fields, we need to know how they change under a holomorphic change of coordinates of $\mathbb{C}^2$. So, let χ be a biholomorphism of $\mathbb{C}^2$. It is easy to see that, in the coordinates induced by the change χ, the field Q becomes

$$\widetilde{Q} = d\chi^{-1}\left(Q \circ \chi^{-1}\right),$$

which, in coordinates, means

$$\widetilde{Q}^j = \frac{\partial\left(\chi^{-1}\right)^j}{\partial w^i}\left(Q \circ \chi^{-1}\right)^i$$

We immediately see that the resulting $\widetilde{Q}$ is still a vector field of order at least 3 and, in particular, that $\widetilde{Q}_3$, the homogeneous part of $\widetilde{Q}$ of degree 3, actually depends only on Q and the linear part of χ. So, we have the following result.

Proposition 6.1.1. *Let Q and $\widetilde{Q}$ be two holomorphic cubic vector fields in $\mathbb{C}^n$. Then Q and $\widetilde{Q}$ are holomorphically conjugated if and only if they are linearly conjugated.*

In particular, we remark the following analogous for cubic maps of Corollary 1.3 in [Aba05].

Corollary 6.1.2. *Let $f, g \in \mathrm{End}(\mathbb{C}^n, 0)$ be two cubic maps fixing the origin and tangent to the identity. Then f and g are holomorphically conjugated if and only if they are linearly conjugated.*

We shall divide our analysis of cubic vector fields on the basis of the number of the characteristic directions. In fact, we know that a cubic field in $\mathbb{C}^2$ can be dicritical or must have 4 characteristic directions (see, e.g., [AT03]), counted with multiplicity. For any possible number of characteristic directions, we shall first give a holomorphic classification of the fields and then study the dynamics of the representatives found. As a matter of notation, for simplicity and for the sake of clarity, in this chapter we shall denote by z and w the two coordinates of $\mathbb{C}^2$.

6.2 Dicritical case

6.2.1 Classification

Proposition 6.2.1. *Let Q be a dicritical holomorphic cubic vector field in $\mathbb{C}^2$. Then Q is linearly (and hence holomorphically) conjugated to the field*

$$Q(z, w) = z^2 w \frac{\partial}{\partial z} + z w^2 \frac{\partial}{\partial w}.$$

Proof. There exists a function λ, not identically zero, such that, for every $(z, w) \in \mathbb{C}^2$, we have

$$\begin{cases} az^3 + bz^2 w + czw^2 + dw^3 = \lambda(z, w) z \\ Az^3 + Bz^2 w + Czw^2 + Dw^3 = \lambda(z, w) w \end{cases}$$

It means that we must have

$$d = A = 0,\ a = B,\ b = C,\ c = D$$

and the field Q is of the form $Q_{abc} = \left(az^3 + bz^2 w + czw^2\right) \frac{\partial}{\partial z} + \left(az^2 w + bzw^2 + cw^3\right) \frac{\partial}{\partial w}$. We want to see when two such fields are conjugated by linear change of coordinates, i.e., by a

$$\chi(z, w) = (nz + mw, hz + kw) = \begin{pmatrix} n & m \\ h & k \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix},$$

with $nk - mh \neq 0$. It is easy to check that the field Q_{abc} becomes, under conjugation by this χ, the field $Q_{a'b'c'}$, where

$$\begin{cases} a' = an^2 + bhn + ch^2, \\ b' = 2amn + b(kn + hm) + 2chk, \\ c' = am^2 + bkm + ck^2. \end{cases}$$

Let's see what happens under some particular conjugations:

1. $\chi(z, w) = (nz, kw)$. Then we see that the change becomes

   $$\begin{cases} a' = an^2 \\ b' = bkn \\ c' = ck^2. \end{cases}$$

   From this we see that we can reduce the problem to the study of the fields Q_{010}, Q_{100}, Q_{001}, Q_{011}, Q_{110}, Q_{1b1}, with $b \in \mathbb{C}$;

2. $\chi(z, w) = (w, z)$. The change becomes

   $$\begin{cases} a' = c \\ b' = b \\ c' = a, \end{cases}$$

   so that we can reduce our representatives to Q_{010}, Q_{001}, Q_{011}, Q_{1b1}, with $b \in \mathbb{C}$;

3. $\chi(z, w) = (z - w, w)$. We get

   $$\begin{cases} a' = a \\ b' = b - 2a \\ c' = a - b + c. \end{cases}$$

   Thus, we have $d\chi^{-1}(Q_{010} \circ \chi) = Q_{0,1,-1} \sim Q_{011}$;

4. for Q_{001} we see that the general change becomes

   $$\begin{cases} a' = h^2 \\ b' = 2hk \\ c' = k^2. \end{cases}$$

   So, with $h = k = 1$, it becomes one field of the family Q_{1b1}. Thus, we are left with the field Q_{010} and the family Q_{1b1}.

5. Finally, we prove that any field of the family Q_{1b1} is conjugated to Q_{010}. Consider Q_{010}. Then the change is

$$\begin{cases} a' = hn \\ b' = kn + hm \\ c' = km \end{cases}$$

and the system is solvable if we put $a' = c' = 1$.

Summing up, we have obtained that every field Q_{abc} is conjugated to Q_{010}, and this gives the assertion. □

6.2.2 Dynamics

We want to study the dynamics of the dicritical vector field $z^2w\frac{\partial}{\partial z} + zw^2\frac{\partial}{\partial w}$ (the dynamics for case 0 is pretty trivial...). This is easy to describe, thanks to Proposition 5.2.1. We see that the lines $L_{(0,1)}$ and $L_{(1,0)}$ are pointwise fixed. To study the other lines through the origin, let $[v_0] \in \mathbb{P}^1(\mathbb{C})$, $[v_0] \neq 0, \infty$; choose a representative v_0 and let λ_0 such that $Q^j = \lambda_0 v^j$ the associated eigenvalue. Let ζ be a parametrization of L_{v_0}. By (5.5), we see that the half lines $l^+ = \{\lambda\zeta_0^2 \in \mathbb{R}^+\}$ and $l^- = \{\lambda\zeta_0^2 \in \mathbb{R}^-\}$ are totally invariant for the flow. So,

- the integral curve issuing from a point in l^- goes to the origin in forward time and to infinity in (finite) backward time;
- the integral curve issuing from a point in l^+ goes to infinity in (finite) forward time and to the origin in backward time;
- the integral curve issuing from any other point go to the origin both in forward and backward time, without intersecting l^+ and l^-.

6.3 One characteristic direction

6.3.1 Classification

Proposition 6.3.1. *Let Q be a holomorphic homogeneous vector field in $\mathbb{C}^n$ with one characteristic direction. Then, Q is linearly (and hence holomorphically) conjugated to one of the following:*

- $1_{000} : Q(z,w) = -z^3\frac{\partial}{\partial w}$;
- $1_{100} : Q(z,w) = -z^3\frac{\partial}{\partial z} - \left(z^3 + z^2w\right)\frac{\partial}{\partial w}$;
- $1_{010} : Q(z,w) = -z^2w\frac{\partial}{\partial z} - \left(z^3 + zw^2\right)\frac{\partial}{\partial w}$;
- $1_{a01} : f(z) = -\left(az^3 + zw^2\right)\frac{\partial}{\partial z} - \left(z^3 + az^2w + w^3\right)\frac{\partial}{\partial w}$.

Proof. Up to a linear change of coordinates, we can suppose that the characteristic direction is $[0;1]$. Using this condition, we find that $d=0$. Now we want to exclude other characteristic directions, i.e., we want that, for any v, $[1,v]$ is not a characteristic direction. So, we ask for the system

$$\begin{cases} a+bv+cv^2=\lambda \\ A+Bv+Cv^2+Dv^3=\lambda v \end{cases}$$

to have no solutions. It means that

$$A\neq 0, a=B, b=C, c=D,$$

so that the fields become of the form

$$Q_{abcA}=\left(az^3+bz^2w+czw^2\right)\frac{\partial}{\partial z}+\left(Az^3+az^2w+bzw^2+cw^3\right)\frac{\partial}{\partial w}.$$

The possible linear changes of coordinates which hold $[0,1]$ fixed are of the form $\chi(z,w)=(hz,kz+lw)$, with $hl\neq 0$. After some calculations, we get

$$d\chi^{-1}\left(Q_{abcA}\circ\chi\right)=Q_{a'b'c'A'}$$

where

$$\begin{cases} a'=ah^2+bhk+ck^2 \\ b'=l(bh+2ck) \\ c'=cl^2 \\ A'=Ah^3/l. \end{cases}$$

We remark that $A\neq 0$, so that $A'\neq 0$. Moreover, the fact that the third parameter (c) is 0 is a holomorphic invariant. So, we have the following cases:

- $a=b=c=0$: we have $a'=b'=c'=0$ and $A'=Ah^3/l$, so that every germ in this family is equivalent to $Q_{000,-1}$, i.e., $Q(z,w)=-z^3\frac{\partial}{\partial w}$;
- $b=c=0, a\neq 0$: we obtain $b'=c'=0, a'=ah^2, A'=Ah^3/l$ and we get that every germ in this family is equivalent to $g_{-1,00,-1}$, i.e., $f(z,w)=-z^3\frac{\partial}{\partial z}-(z^3+z^2w)\frac{\partial}{\partial w}$;
- $c=0, b\neq 0$: we get $a'=ah^2+bhk, b'=lhb, c=0, A'=Ah^3/l$. We can solve and obtain that every germ is equivalent to $Q_{0,-1,0,-1}$;
- $c\neq 0$: we obtain that every germ in this family is equivalent to one of the family $Q_{-a,0,-1,-1}$.

□

6.3.2 Dynamics

We study the dynamics for the representatives of the holomorphic classification (cp. [AT11], page 2676). Regardless of the holomorphic class of the field, we have only one characteristic direction, which has order 4, residue $\rho = 2$ and induced residue Res $= -2$. The behaviour in the characteristic line is explained in Proposition 5.2.1, so here we shall study what happens outside that line. Theorem 3.4.8 and Corollary 5.1.11 tell us that the direction $[\gamma]$ of every integral curve γ outside the characteristic line tends to $[0, 1]$, in both forward and backward time. In fact, the presence of a unique singularity with residue -2 prevents the possibility of any other ω-limit for $[\gamma]$.

So, we want to understand if the integral lines go to the origin or diverge as t tends to the supremum of the values of definition for γ (the singularity cannot be apparent, because the residue is non-zero), and also try to see if this value is finite or infinite.

We shall use coordinates (ζ, v) for the chart centered in $[0, 1]$ for all this section.

Case 1_{000} In this case, the associated vector field is

$$Q = -z^3 \frac{\partial}{\partial w}$$

and in particular we have

$$Q^1(\zeta, 1) = 0$$

and

$$Q^2(\zeta, 1) = -\zeta^3,$$

so that we obtain (recall that here $\nu = 2$)

$$G = \zeta^4 v \frac{\partial}{\partial \zeta} - 2\zeta^3 v^2 \frac{\partial}{\partial v}.$$

We see that the unique singularity $[0, 1]$ is degenerate and Fuchsian of order 4, with vanishing resonant index. Here, $\mu_Y = 3 > 2 = \rho$ (the residue), so that Theorem 5.3.4 says that $\|\gamma(t)\|$ goes to infinity in both forward and backward time. We can also write down explicitely the solution issuing form $(z_0, w_0) \in \mathbb{C}^2$:

$$\gamma(t) = (z_0, w_0 - z_0^3 t).$$

We notice that the solution diverges, as we said, and the solution is defined for every $t \in \mathbb{R}$.

Case 1_{100} In this case, the vector field is

$$Q = -z^3 \frac{\partial}{\partial z} - (z^3 + z^2 w) \frac{\partial}{\partial w}$$

and we have

$$Q^1(\zeta,1) = -\zeta^3 \text{ and } Q^2(\zeta,1) = -\zeta^2(1+\zeta).$$

The singularity $\infty = [0,1]$ is again degenerate, and the field G has the form

$$G = \zeta^4 v \frac{\partial}{\partial z} - 2\zeta^2(1+\zeta)v^2 \frac{\partial}{\partial v}, \tag{6.1}$$

so that ∞ is an irregular singularity. As in [AT11] for the analogous case 1_{10}, we can write down the explicit solution and obtain the asymptotic behaviour as follows. So, let us suppose we have $\sigma(t) = (\zeta(t), v(t))$ integral curve for G, corresponding to an integral curve γ for our field in $\mathbb{C}^2$. We know that the horizontal foliation is given by the field

$$H = \frac{\partial}{\partial \zeta} - 2\frac{1+\zeta}{\zeta^2} v \frac{\partial}{\partial v},$$

so that we obtain the condition

$$\exp\left(\frac{-2}{\zeta}\right)\zeta^2 v \equiv c_0$$

from the fact the the geodesic lies in a leaf of the horizontal foliation. Moreover, we see that we also have

$$\zeta' = c_0 \zeta^2 \exp\left(\frac{2}{\zeta}\right)$$

(see Remark 2.1.22). We can solve and find

$$\begin{cases} \zeta(t) = -\dfrac{2}{\log(2c_0 t + 2c_1)} \\[2ex] v(t) = \dfrac{c_0 \log^2(2c_0 t + 2c_1)}{8(c_0 t + c_1)}, \end{cases}$$

where c_0 and c_1 are given by

$$\log(2c_1) = -\frac{2}{\zeta(0)} \text{ and } c_0 = 2c_1 \zeta^2(0) v(0).$$

Now, recall that here $\nu = 2$ and the map $\chi_2 : \mathbb{C}^2 \setminus 0 \to N_{\mathbb{P}^1}^{\otimes 2} \setminus \mathbb{P}^1$ is given by $\chi_2(z,w) = \left(\frac{z}{w}, w^2\right)$, so that $\chi_2^{-1}(\zeta, v) = \left(\zeta v^{1/2}, v^{1/2}\right)$ (two solutions, it is a double cover). From the form of the solution v, we see that we have two cases:

- if $\zeta^2(0)v(0) \in \mathbb{R}^-$, there is a positive t for which the denominator of $v(t)$ vanishes. So, v diverges in finite forward time and goes to 0 in infinite backward time. It means that the integral curve issuing form a point $(z_0, w_0) \in \mathbb{C}^2$, with $z_0^2 \in \mathbb{R}^-$ diverges in finite forward time and goes to the origin in infinite backward time, in both case tangentially to $[0, 1]$;
- if $\zeta^2(0)v(0) \in \mathbb{R}^+$, the description is the opposite of the previous one. There is a negative t for which the denominator of $v(t)$ vanishes. So, v diverges in finite backward time and goes to 0 in infinite forward time. So, the integral curve issuing form a point $(z_0, w_0) \in \mathbb{C}^2$, with $z_0^2 \in \mathbb{R}^-$ diverges in backward time direction in finite time and goes to the origin in infinite forward time, in both case tangentially to $[0, 1]$;
- if $\zeta^2(0)v(0) \notin \mathbb{R}$, then v goes to 0 in infinite time in both forward and backward time, so that if $(z_0, w_0) \in \mathbb{C}^2$, with $z_0^2 \notin \mathbb{R}$, then the issuing integral curve goes to the origin in the same way.

Remark 6.3.2. *The case $z_0 = 0$ means that the point (z_0, w_0) lies in the characteristic direction, and we see from the form of the associated field that this line is left pointwise fixed.*

Case 1_{010} The field here is

$$Q(z, w) = -z^2 w \frac{\partial}{\partial z} - (z^3 + zw^2) \frac{\partial}{\partial w}$$

and the associated geodesic field is

$$G = \zeta^4 v \frac{\partial}{\partial z} - 2\zeta(\zeta^2 + 1)v^2 \frac{\partial}{\partial v}.$$

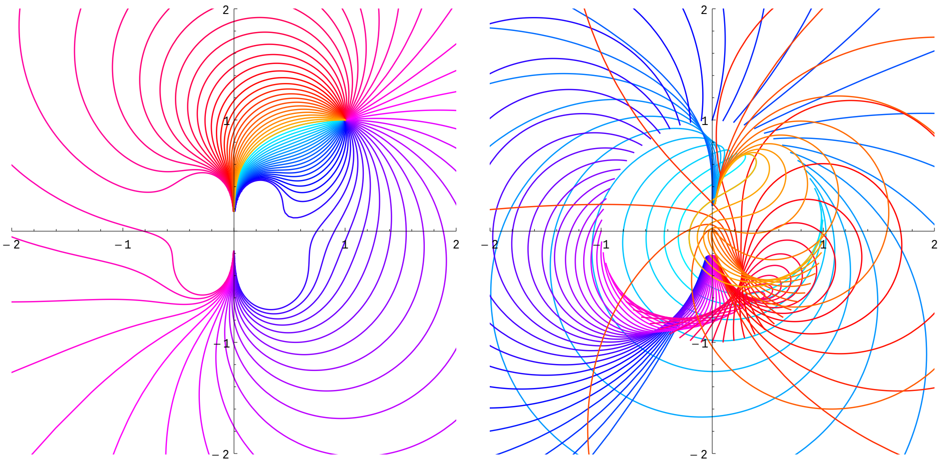
So here the singularity is irregular, with irregularity 3. With the usual arguments we get

$$\begin{cases} \exp\left(-\dfrac{1}{\zeta^2}\right)\zeta^2 v \equiv c_0 \\ \\ \zeta' = c_1 \zeta^2 \exp\left(\dfrac{1}{\zeta^2}\right) \end{cases}$$

which has solution

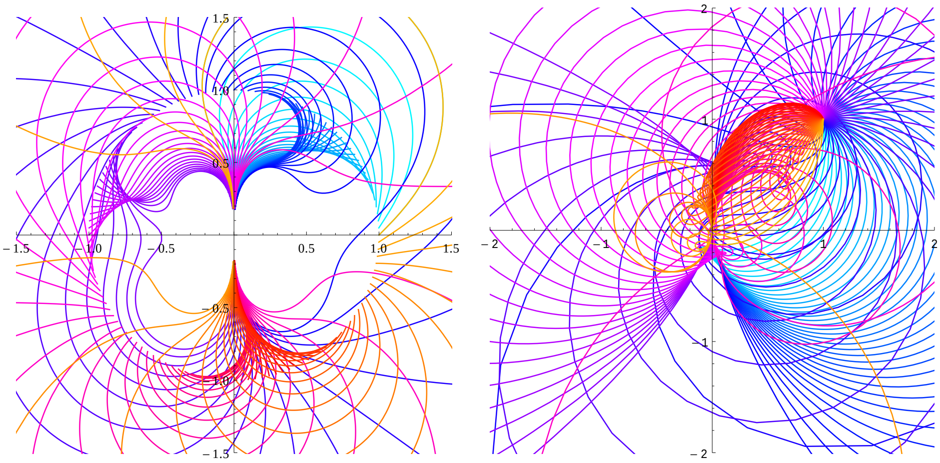
$$\begin{cases} \zeta(t) = -\dfrac{1}{F^{-1}(c_0 t + c_1)} \\ \\ v(t) = -c_0 F^{-1}(c_0 t + c_1) \exp\left(F^{-1}(c_0 t + c_1)^2\right) \end{cases}$$

where F is a primitive of $\exp(-w^2)$. It is possible to study this function to get information in this case.



$\zeta(t)$ $v(t)$

(a) $\zeta(0) = 1 + i, v(0) = \cos(\theta) + i\sin(\theta)$



$\zeta(t)$ $v(t)$

(b) $\zeta(0) = \cos(\theta) + i\sin(\theta), v(0) = 1 + i$

Figure 6.1: Case 1_{010}

Numerical studies suggest the presence of directions of convergence to zero for $\zeta(t)$, as described in Remark 4.2.20. In Figure 6.1a we have drawn $\zeta(t)$ and $v(t)$ for integral curves issuing from $\zeta(0) = 1+i$ with $v(0) = \cos(\theta) + i\sin(\theta)$, with $\theta = 0, \frac{\pi}{32} \dots 2\pi$. Drawings of the same colour are the components of the same curve. In Figure 6.1b we vary $z(0) = \cos(\theta) + i\sin(\theta)$, with $\theta = 0, \frac{\pi}{64} \dots 2\pi$, and hold $v(0) = 1+i$ fixed. We note two directions of convergence, corresponding to the sectors of Proposition 5.3.5 and Remark 4.2.20, and that $v(t) \to 0$ for all these integral curves.

Case 1_{a01} We have

$$Q(z,w) = -(az^3 + zw^2)\frac{\partial}{\partial z} - (z^3 + az^2w + w^3)\frac{\partial}{\partial w}$$

and the geodesic field in this case is

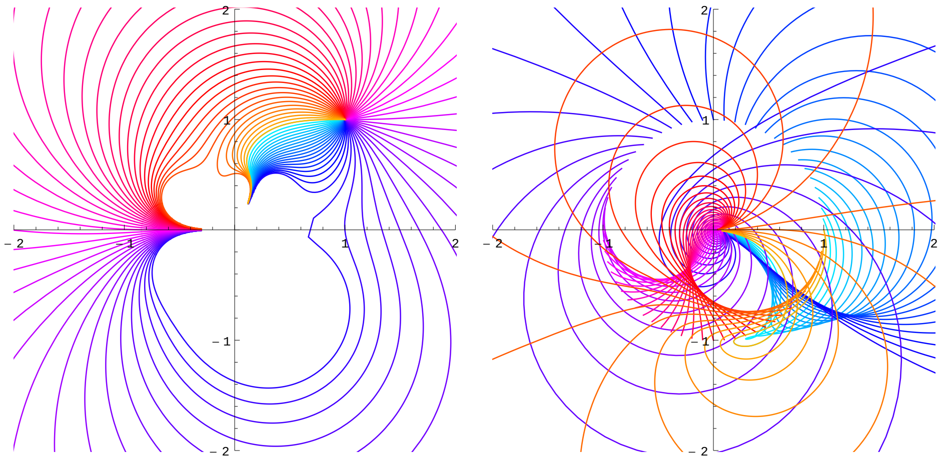
$$G = \zeta^4 v \frac{\partial}{\partial \zeta} - 2(\zeta^3 + a\zeta^2 + 1)v^2\frac{\partial}{\partial v},$$

with $[0,1]$ irregular singularity, of irregularity 4. With considerations analogous to the previous cases, we get

$$\begin{cases} \exp\left(-\dfrac{2a}{\zeta} - \dfrac{2}{3\zeta^3}\right)\zeta^2 v \equiv c_0 \\[2ex] \dfrac{\exp\left(-\frac{2a}{\zeta} - \frac{2}{3\zeta^3}\right)}{\zeta^2}\zeta' = c_1. \end{cases}$$

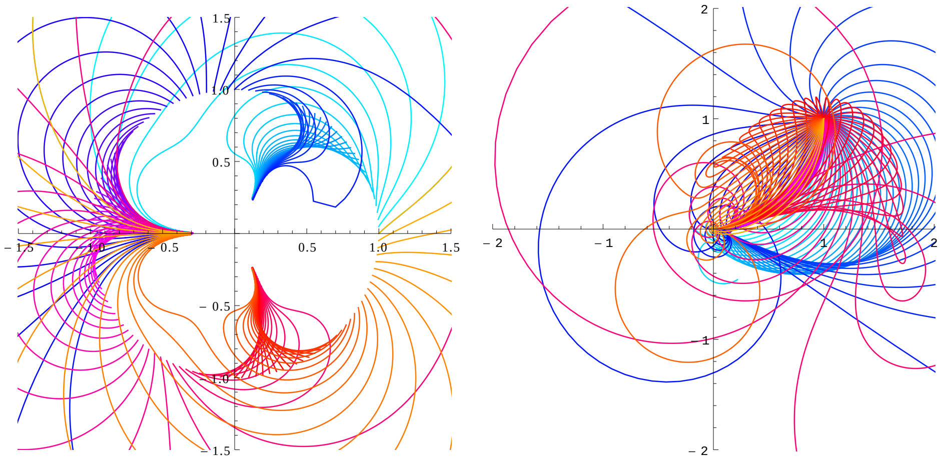
Integrating equation 2 would provide informations on the orbits. Proposition 5.3.5 provides basic information about curves converging to the singularity.

In Figure 6.2 we have studied this case numerically, with $a = 1$ and the same initial conditions we used in the previous case. This time the irregularity is 4, and in fact we observe the presence of 3 directions of attraction for generic projections of curves (we expect exceptional directions of convergence separating curves tending to zero with different directions). This is the only case of all our study in which we have irregularity 4, and thus the only example we have to concretely study the situation with 3 sectors of convergence. We remark that, in order to obtain this, it is necessary to consider vector fields of degree at least 3: in fact, in the study of the quadratic case done in [AT11] the maximal possible irregularity is 3, and so this phenomenon is impossible to be seen.



$\zeta(t)$ $v(t)$

(a) $\zeta(0) = 1 + i, v(0) = \cos(\theta) + i\sin(\theta)$



$\zeta(t)$ $v(t)$

(b) $\zeta(0) = \cos(\theta) + i\sin(\theta), v(0) = 1 + i$

Figure 6.2: Case 1_{a10}

6.4 Two characteristic directions

6.4.1 Classification

Proposition 6.4.1. *Let Q be a holomorphic homogeneous vector field in $\mathbb{C}^2$ with two characteristic directions. Then, if the orders of the two directions are 3 and 1, Q is linearly (and thus holomorphically) conjugated to one of the following:*

- $2_{0001} : Q(z,w) = w^3 \frac{\partial}{\partial w}$;
- $2_{0010} : Q(z,w) = zw^2 dvz$;
- $2_{00c1} : Q(z,w) = czw^2 \frac{\partial}{\partial z} + w^3 \frac{\partial}{\partial w}$, *with* $c \in \mathbb{C}^*$;
- $2_{1010} : Q(z,w) = \left(z^3 + zw^2\right) \frac{\partial}{\partial z} + z^2 w \frac{\partial}{\partial w}$;
- $2_{10c1} : Q(z,w) = \left(z^3 + czw^2\right) \frac{\partial}{\partial z} + \left(z^2 w + w^3\right) \frac{\partial}{\partial w}$, *with* $c \in \mathbb{C} \setminus \{1\}$;
- $2_{0110} : Q(z,w) = \left(z^2 w + zw^2\right) \frac{\partial}{\partial z} + zw^2 \frac{\partial}{\partial w}$;
- $2_{01c1} : Q(z,w) = \left(z^2 w + czw^2\right) \frac{\partial}{\partial z} + \left(zw^2 + w^3\right) \frac{\partial}{\partial w}$, *with* $c \in C \setminus \{1\}$;
- $2_{a110} : Q(z,w) = \left(az^3 + z^2 w + zw^2\right) \frac{\partial}{\partial z} + \left(az^2 w + zw^2\right) \frac{\partial}{\partial w}$, *with* $a \in \mathbb{C}^* \setminus \{1\}$;
- $2_{a1c1} : Q(z,w) = \left(az^3 + z^2 w + czw^2\right) \frac{\partial}{\partial z} + \left(az^2 w + czw^2 + w^3\right) \frac{\partial}{\partial w}$, *with* $a \in \mathbb{C}^*, c \in \mathbb{C} \setminus \{1\}$.

Otherwise, if the orders of the two characteristic directions are both 2, g is holomorphically conjugated to one of the following:

- $2'_{0100} : Q(z,w) = \left(z^2 w\right) \frac{\partial}{\partial z}$;
- $2'_{0b10} : Q(z,w) = \left(bz^2 w\right) \frac{\partial}{\partial z} + \left(zw^2\right) \frac{\partial}{\partial w}$, *with* $b \neq 0, 1$ *(equivalent to* $2'_{0\frac{1}{b}10}$*)*;
- $2'_{1100} : Q(z,w) = \left(z^3 + z^2 w\right) \frac{\partial}{\partial z} + z^2 w \frac{\partial}{\partial w}$;
- $2'_{1b10} : Q(z,w) = \left(z^3 + bz^2 w\right) \frac{\partial}{\partial z} + \left(z^2 w + zw^2\right) \frac{\partial}{\partial w}$, *with* $b \neq 0, 1$;
- $2'_{1bC1} : Q(z,w) = \left(z^3 + bz^2 w + zw^2\right) \frac{\partial}{\partial z} + \left(z^2 w + Czw^2 + w^3\right) \frac{\partial}{\partial w}$, *equivalent to* $2'_{1Cb1}$, *with* $b, C \in \mathbb{C}, b \neq C$.

Proof. Without loss of generality, we can assume that the two characteristic directions are $[0;1]$ and $[1:0]$. We clearly have two different situations, depending on the fact that the two multiplicities are 2,2 or 3,1 (1,3 is clearly equivalent to 3,1 with a swap of the coordinates).

We start imposing that these two directions are characteristic, obtaining $d = A = 0$. The fact that we do not have other characteristic directions means that the system

$$\begin{cases} au^3 + bu^2 + cu = \lambda u \\ Bu^2 + Cu + D = \lambda \end{cases}$$

has no solutions $u \neq 0$. We find the equation

$$(a - B)u^3 + (b - C)u^2 + (c - D)u = 0.$$

Solving, we find that we have three cases:

- $a = B, b = C, c \neq D$. It means that the direction $[0, 1]$ has order 1 and $[1, 0]$ has 3;
- $a = B, b \neq C, c = D$. It means that the direction $[0, 1]$ has order 3 and $[1, 0]$ has 1. It is equivalent to the previous case, as said before;
- $a \neq B, b = C, c = D$. We have that the two orders are 2 and 2.

So, we have two different situations. The first is the one with multiplicities 1 and 3

$$Q^1_{abcD}(z, w) = \left(z + az^3 + bz^2w + czw^2\right)\frac{\partial}{\partial z} + \left(w + az^2w + bzw^2 + Dw^3\right)\frac{\partial}{\partial w}$$

with $c \neq D$; the second one is

$$Q^2_{abCc}(z, w) = \left(z + az^3 + bz^2w + czw^2\right)\frac{\partial}{\partial z}\left(w + az^2w + Czw^2 + cw^3\right)\frac{\partial}{\partial w}$$

with $b \neq C$.

The linear maps that keep the two directions $[0; 1]$ and $[1; 0]$ fixed are of the form $\chi(z, w) = (hz, kw)$. Let us study the two types separetely.

Case 3-1 The action of the conjugation is

$$d\chi^{-1}\left(Q^1_{abcD} \circ \chi\right) = Q^1_{ah^2,bhk,ck^2,Dk^2}.$$

The vanishing of a, b, c, D is a linear invariant. So, we have the following possibilities:

- $a = b = 0$: we see that it is of the form Q^1_{00cD}. If c or D is not zero, we can let it become 1 with the parameter k, so that we see that the family is parametrized by one parameter, $[c; D] \in \mathbb{P}^1(\mathbb{C})$. We divide this case in the three families $2_{0001}, 2_{0010}, 2_{00c1}$, with $c \in \mathbb{C}^* \setminus \{1\}$ as a parameter;

- $b = 0, a \neq 0$: we obtain, using h, that a becomes 1 and the map is equivalent to Q^1_{10cD}, where again we only have to consider the ratio $\lambda = [c; D] \in \mathbb{P}^1(\mathbb{C})$. We divide this case in two families: 2_{1010}, when $D = 0$, and 2_{10c1}, with $c \in \mathbb{C} \setminus \{1\}$ as parameter;

- $a = 0, b \neq 0$: the same as the previous case, with b that goes to 1. Again we obtain two families, that we call 2_{0110} and 2_{01c1}, where $c \in \mathbb{C} \setminus \{1\}$;

- $a, b \neq 0$: we have two parameters, i.e., we have $[c, D]$ as usual and a or b as a second parameter. We get the family 2_{a1cD}, and we divide it into 2_{a110} and 2_{a1c1} (with $c \neq 1$).

Case 2-2 The effect of the conjugation is

$$d\chi^{-1}\left(Q^2_{abCc} \circ \chi\right) = Q^2_{ah^2,bhk,Chk,ck^2}$$

and the classification is similar to the previous one. First we obtain:

1. $a = c = 0$: we obtain Q^2_{0bC0}, which depends only on $[b; C] = \lambda'$;

2. $a = 0, c \neq 0$: they are equivalent to Q^2_{0bC1}, again depending on $\lambda' = [b; C]$;

3. $c = 0, a \neq 0$: we get Q^2_{1bC0};

4. $a, c \neq 0$: again two parameters. We can use b and C and obtain $2'_{1bC1}$, with two complex parameters.

Then, we can use the swap of the coordinates, $\chi(z, w) = (w, z)$, which gives $d\chi^{-1}\left(Q^2_{abCc} \circ \chi\right) = Q^2_{c,C,b,a}$. We obtain that the two cases 2 and 3 are equivalent, with $2'_{1[\lambda_1;\lambda_2]0} \sim 2'_{0,[\lambda_2;\lambda_1]1}$. Moreover, $2'_{1bC1} \sim 2'_{1Cb1}$. Finally, in the first case we can consider the cases Q^2_{0100} and Q^2_{0b10}, with $b \in \mathbb{C}^*$ (and $Q^2_{0b10} \sim Q^2_{0\frac{1}{b}10}$), and we can do the same in case 2, obtaining only Q^2_{1100} and Q^2_{1b10} (here $b \in \mathbb{C}$). In the last case we have the family $2'_{1bC1}$, with $b, C \in \mathbb{C}$, not both zero (we shall see later that we do not need to divide it into more families) and $g^2_{1bC1} \sim g^2_{1Cb1}$. □

6.4.2 Dynamics, case 3-1, $2_{\bullet\bullet\bullet\bullet}$

We start studying the dynamics for the representatives of the form $2_{\bullet\bullet\bullet\bullet}$, i.e., the ones with one characteristic direction of order 3 and the other of order 1. We use the charts (U_0, ζ_0) and (U_∞, ζ_∞) introduced in Section 5.1, centered at $[0, 1]$ and $[1, 0]$ respectively. In the following tables we write down all the relevant objects in this case.

Chart at $[1,0]$, coordinate ζ_0
$Q_1(1,\zeta_0) = a + b\zeta_0 + c\zeta_0^2$
$Q_2(1,\zeta_0) = a\zeta_0 + b\zeta_0^2 + D\zeta_0^3\zeta_0^3$
$X_0 = (D-c)\zeta_0^3$
$\eta_0 = -\frac{2\left(a+b\zeta_0+c\zeta_0^2\right)}{(D-c)\zeta_0^3} d\zeta_0$
$G_0 = (D-c)v_0\zeta_0^3 \frac{\partial}{\partial\zeta_0} + 2\left(a+b\zeta_0+c\zeta_0^2\right) v_0^2 \frac{\partial}{\partial v_0}$
order 3
residue $\rho^0 = \frac{2c}{c-D}$
ind. residue $\mathrm{Res}^0 = \frac{2c}{c-D} - 3 = \frac{c-3D}{D-c}$

Chart at $[0,1]$, coordinate ζ_∞
$Q_1(\zeta_\infty,1) = a\zeta_\infty^3 + b\zeta_\infty^2 + c\zeta_\infty$
$Q_2(\zeta_\infty,1) = a\zeta_\infty^2 + b\zeta_\infty + D$
$X_\infty = (c-D)\zeta_\infty$
$\eta_\infty = -\frac{2\left(D+b\zeta_\infty+a\zeta_\infty^2\right)}{(c-D)\zeta_\infty} d\zeta_\infty$
$G_\infty = (c-D)v_\infty\zeta_\infty \frac{\partial}{\partial\zeta_\infty} + 2\left(D+b\zeta_\infty+a\zeta_\infty^2\right) v_\infty^2 \frac{\partial}{\partial v_\infty}$
order 1
residue $\rho^\infty = \frac{2D}{D-c}$
ind. residue $\mathrm{Res}^\infty = \frac{2D}{D-c} - 1 = \frac{D+c}{D-c}$

From the previous tables it is possible to recover the kind of singularity of $[1,0]$ and $[0,1]$ as the four parameters a, b, c and D vary. The results are collected in the following table.

				$[1,0]$	(3)	$[0,1]$	(1)
0	0	0	1	Apparent		Fuchsian	
0	0	1	0	Fuchsian		Apparent	
0	0	c	1	Fuchsian		Fuchsian	
1	0	1	0	Irregular (3)		Apparent	
1	0	c	1	Irregular (3)		Fuchsian	
0	1	1	0	Irregular (2)		Apparent	
0	1	c	1	Irregular (2)		Fuchsian	
a	1	1	0	Irregular (3)		Apparent	
a	1	c	1	Irregular (3)		Fuchsian	

From this table it is now easy to understand the way we decided to divide some cases of the classification into subcases. The description of the different cases follows.

Case 2_{0010} In this and in the next two cases we do not have irregular singularities and moreover we can integrate directly the field and obtain an explicit formula for the integral curve $\gamma(t)$ issuing from (z_0, w_0). Here, we have

$$Q(z,w) = zw^2 \frac{\partial}{\partial z}$$

and it is easy to see that the solution is

$$\gamma(t) = (z_0 e^{w_0^2 t}, w_0).$$

We see that the value $\operatorname{Re} w_0^2$ dictates the behaviour of the integral curve. In fact, $z(t)$ may go to 0 (and so $[\gamma(t)] \to [0,1]$), diverge (and so $[\gamma(t)] \to [1,0]$) or be periodic (in accord with Theorem 5.3.2).

Case 2_{0001}

Here the field is

$$Q(z,w) = w^3 \frac{\partial}{\partial w}.$$

We see that the first coordinate is constant, while we can solve for the second one and obtain

$$\gamma(t) = \left(z_0, \frac{1}{\sqrt{2}\sqrt{\frac{1}{2w_0^2} - t}} \right),$$

with a suitable determination of the square root. We see that now it is important to consider the parameter w_0^2. In fact, we have:

- if $w_0^2 \in \mathbb{R}^+$, then $w(t)$ diverges as $t \to \frac{1}{2w_0^2}$ (and so the projection goes to the singularity $[0,1]$) and goes to zero as $t \to -\infty$ (and so the projection goes to $[1,0]$);

- if $w_0^2 \in \mathbb{R}^-$, then $w(t)$ diverges as $t \to \frac{1}{2w_0^2}$ (and so the projection goes to the singularity $[0,1]$) and goes to zero as $t \to +\infty$ (and so the projection goes to $[1,0]$);

- if $w_0^2 \notin \mathbb{R}$, then $w(t)$ goes to zero for both $t \to \pm\infty$ and so the projection goes to $[1,0]$.

Case 2_{00c1}, with $c \in \mathbb{C}$ Here the field is

$$Q(z,w) = czw^2 \frac{\partial}{\partial z} + w^3 \frac{\partial}{\partial w}.$$

We see that the component $w(t)$ of $\gamma(t)$ will the same of the case 2_{0001}. So, we can solve also for the first coordinate, obtaining

$$\gamma(t) = \left(z_0 \left(\frac{1}{2w_0^2} \right)^{c/2} \frac{1}{\left(\frac{1}{2w_0^2} - t \right)^{c/2}}, \frac{1}{\sqrt{2}\sqrt{\frac{1}{2w_0^2} - t}} \right),$$

which gives the explicit solution of the problem in this case. Moreover, from the equation of the solution we immediately see that

$$z(t)w(t)^{-c} = z_0 \left(\frac{2}{2w_0^2} \right)^{c/2}$$

and so this product is constant. In particular, we have that the product $|z(t)||w(t)|^{-\operatorname{Re} c}$ is constant. It follows that

- if $\operatorname{Re} c < 0$, if $|w(t)| \to \infty$, then $|z(t)| \to 0$, and viceversa. So, we have the following (see the previous case for the behaviour of $w(t)$):
  - if $w_0^2 \in \mathbb{R}^+$, then $w(t)$ diverges as $t \to \frac{1}{2w_0^2}$ (and so $z(t) \to 0$ and the projection goes to the singularity $[0,1]$) and goes to zero as $t \to -\infty$ (and so $|z(t)| \to \infty$ and the projection goes to $[1,0]$);
  - if $w_0^2 \in \mathbb{R}^-$, then $w(t)$ diverges as $t \to +\infty$ (and so $z(t) \to 0$ and the projection goes to the singularity $[0,1]$) and $w(t)$ goes to zero as $t \to \frac{1}{2w_0^2}$ (and so $|z(t)| \to \infty$ and the projection goes to $[1,0]$);
  - if $w_0^2 \notin \mathbb{R}$, then $w(t)$ goes to zero for both $t \to \pm\infty$ and so $z(t)$ diverges and the projection goes to $[1,0]$;
- if $\operatorname{Re} c = 0$, we have that $|z(t)|$ is constant, so that the asymptotic behaviour is the same as in Case 2_{0001};
- if $0 < \operatorname{Re} c < 1$, we have that $|w(t)|$ and $|z(t)|$ have the same limit, but $|w(t)|$ goes to zero or diverges faster that $|z(t)|$. So, in the cases in which $w(t) \to 0$ we have that the projection goes to $[1,0]$, while if $|w(t)| \to \infty$ we have that $[\gamma] \to [0,1]$;
- if $\operatorname{Re} c = 1$ we have that $\left| \frac{z(t)}{w(t)} \right|$ is constant and equal to $z_0 \sqrt{\left(\frac{2}{2w_0^2} \right)}$. We note that, if we write $c = 1 + i\gamma$ for some $\gamma \in \mathbb{R}$, we have that the two induced residues are $-1 - \frac{2}{\gamma} i$ (for $[1,0]$) and $-1 + \frac{2}{\gamma} i$ (for $[0,1]$). In particular, we see that $\operatorname{Re}\operatorname{Res}^0 = \operatorname{Re}\operatorname{Res}^\infty = -1$ and we may have

closed geodesics on $\mathbb{P}^1(\mathbb{C})$, or geodesics whose ω-limit is a closed one. So, we see that the presence of induced residues equal to -1 makes this case much more complicated than the other ones;

- if $\operatorname{Re} c > 1$, we have that $|z(t)|$ has the same limit as $|w(t)|$, but goes to 0 or ∞ faster that $|w(t)|$. So, we have that, in the cases in which $w(t) \to 0$ we have also $z(t) \to 0$, with $[\gamma] \to [0,1]$, and when $w(t) \to \infty$ we have also $z(t) \to \infty$, with $[\gamma] \to [1,0]$.

Cases 2_{1010}, 2_{0110} and 2_{a110} In all these cases we have $[1,0]$ as irregular singularity of order 3 (and irregularity 2 or 3) and $[0,1]$ as apparent singularity (of order 1). Thus the two residues are 2 and 0. It means that every integral curve must go to a characteristic direction for both $t \to \pm\infty$, or be a periodic integral curve, whose projection on $\mathbb{P}^1(\mathbb{C})$ separates the two singularities (or have projection going to some graph with $[1,0]$ as only vertex). So, an integral curve must have one of the following behaviours:

- it tends to a non-zero point of $L_{[0,1]}$; or
- it diverges tangent to $[1,0]$; or
- it goes to the origin tangent to $[1,0]$; or
- is periodic and its projection on $\mathbb{P}^1(\mathbb{C})$ separates the two singularities;
- has projection going to a graph whose only vertex is $[1,0]$.

Proposition 5.3.5 gives some information about integral curves whose projection goes to the irregular singularity staying in a sector.

Cases 2_{10c1}, 2_{01c1} and 2_{a1c1} In these three cases we have $[1,0]$ as irregular singularity of order 3, irregularity 2 or 3, residue $\rho^0 = \frac{2c}{c-1}$ and induced residue $\operatorname{Res}^0 = \frac{c-3}{1-c}$, while $[0,1]$ is a Fuchsian singularity of order 1, residue $\rho^\infty = \frac{2}{1-c}$ and induced residue $\rho^\infty = \frac{1+c}{1-c}$. In particular, we remark that the induced residues are always different from -1 and this means that we cannot have periodic integral curves, and not even integral curves whose projection is closed.

Depending on the actual value of the parameter c we can have integral curves whose projection on $\mathbb{P}^1(\mathbb{C})$ self-intersects infinitely many times, or tends to a graph. Apart from these possibilities, we see that the projection of every integral curve must go to a singular point. If it goes to the irregular singularity staying in a sector, Proposition 5.3.5says that the integral curve will go to the origin or diverge, depending on the sector. Otherwise, the projection goes to the Fuchsian singularity. Depending on the value of c, Theorem 5.3.4 (and, in the resonant case, Proposition 5.3.3) provides a description of the geodesics tending to the pole.

6.4.3 Dynamics, case 2-2, $2'_{\bullet\bullet\bullet\bullet}$

The following table collects all the relevant objects of this case.

Chart at $[1,0]$, coordinate ζ_0	Chart at $[0,1]$, coordinate ζ_∞
$Q_1(1,\zeta_0) = a + b\zeta_0 + c\zeta_0^2$	$Q_1(\zeta_\infty,1) = a\zeta_0^3 + b\zeta_0^2 + c\zeta_\infty$
$Q_2(1,\zeta_0) = a\zeta_0 + C\zeta_0^2 + c\zeta_0^3$	$Q_2(\zeta_\infty,1) = a\zeta_\infty^2 + C\zeta_\infty + c$
$X_0 = (C-b)\zeta_0^2$	$X_\infty = (b-C)\zeta_\infty^2$
$\eta_0 = -\frac{2\left(a+b\zeta_0+c\zeta_0^2\right)}{(C-b)\zeta_0^2}d\zeta_0$	$\eta_\infty = -\frac{2\left(a\zeta_\infty^2+C\zeta_\infty+c\right)}{(b-C)\zeta_\infty^2}d\zeta_\infty$
$G_0 = (C-b)v_0\zeta_0^2\frac{\partial}{\partial\zeta_0}$	$G_\infty = (b-C)v_\infty\zeta_\infty^2\frac{\partial}{\partial\zeta_\infty}$
$+2\left(a+b\zeta_0+c\zeta_0^2\right)v_0^2\frac{\partial}{\partial v_0}$	$+2\left(a\zeta_\infty^2+C\zeta_\infty+c\right)v_\infty^2\frac{\partial}{\partial v_\infty}$
order 2	order 2
residue $\frac{2b}{b-C}$	residue $\frac{2C}{C-b}$
ind. residue $\mathrm{Res}^0 = \frac{2b}{b-C} - 2 = \frac{2C}{b-C}$	ind. residue $\mathrm{Res}^\infty = \frac{2C}{C-b} - 2 = \frac{2b}{C-b}$

The following one classifies the singularities depending on the value of the four parameters involved.

				$[1,0]$	(2)	$[0,1]$	(2)
0	1	0	0	Fuchsian		Apparent	
0	b	1	0	Fuchsian		Fuchsian	
1	1	0	0	Irregular (2)		Apparent	
1	b	1	0	Irregular (2)		Fuchsian	
1	b	C	1	Irregular (2)		Irregular (2)	

Case $2'_{0100}$ We are studying the field

$$Q(z,w) = z^2 w\frac{\partial}{\partial z}.$$

Here $[1;0]$ is Fuchsian, with residue 2 and induced residue 0 and

$$G_0 = -\zeta_0^2 v_0\frac{\partial}{\partial\zeta_0} + 2v_0^2\zeta_0\frac{\partial}{\partial v_0},$$

while $[0;1]$ is an apparent singularity, with residue 0 and induced residue -2 and

$$G_\infty = -v_\infty\zeta_\infty^2\frac{\partial}{\partial\zeta_\infty}.$$

Because of Theorem 3.4.8, we know that the projection of every maximal geodesic must tend to one of this points, for both $t\to-\infty$ and $t\to+\infty$.

But we can also solve the differential equation directly, obtaining that the integral line issuing from (z_0, w_0) is given by

$$\gamma(t) = \left(-\frac{1}{w_0\left(t - \frac{1}{w_0 z_0}\right)}, w_0 \right).$$

We notice that the behaviour of γ depends on the fact that $w_0 z_0 \in \mathbb{R}$ or not (one between z_0 and w_0 need to be different from zero, because we want to start outside the characteristic lines). We have:

- $w_0 z_0 \in \mathbb{R}^+$: then the first coordinate of $\gamma(t)$ diverges for $t \to \frac{1}{w_0 z_0}$ and goes to zero for $t \to -\infty$;
- $w_0 z_0 \in \mathbb{R}^-$: then the first coordinate diverges for $t \to \frac{1}{w_0 z_0}$ and goes to zero for $t \to +\infty$;
- $w_0 z_0 \notin \mathbb{R}$: then, the first coordinates goes to zero for both $t \to \pm\infty$.

We remark that the behaviour near $[1, 0]$ is as expected from Theorem 5.3.4 and Proposition 5.3.3.

Case $2'_{0b10}$, with $b \in \mathbb{C}^*$ We recall that $g^2_{0b10} \sim g^2_{0\frac{1}{b}10}$.

We have two Fuchsian singularities: $[1, 0]$, with residue $\frac{2b}{b-1}$, induced residue $\frac{2}{1-b}$ and

$$G_0 = -bv_0\zeta_0^2 \frac{\partial}{\partial \zeta_0} + 2b\zeta_0 + v_0^2 \frac{\partial}{\partial v_0}$$

and $[0, 1]$, with residue $\frac{2}{1-b}$, induced residue $\frac{2b}{1-b}$ and

$$G_\infty = bv_\infty\zeta_\infty^2 \frac{\partial}{\partial \zeta_\infty} + 2\zeta_\infty v_\infty^2 \frac{\partial}{\partial v_\infty}.$$

For both of them, we have $\mu_Y = 1$.

We want to study these (induced) residues, varying b. Clearly we can suppose that $\operatorname{Re}\rho^0 \leq \operatorname{Re}\rho^\infty$. We get:

- if $b = -1$ then $\rho^0 = \rho^\infty = 1 = \mu_Y$ and $\operatorname{Res}^0 = \operatorname{Res}^\infty = -1$; almost all integral curves diverge tangentially to a characteristic direction, or are periodic integral curve whose projection on $\mathbb{P}^1(\mathbb{C})$ separates the two singularities and does not intersect any chart in which we can find the Fuchsian normal forms for a singularity. Exceptional curves are periodic, surrounding a (and hence both) characteristic direction. In particular, we see that we cannot have integral curves going to the origin (outside the characteristic leaves) but we have periodic integral curves of arbitrarily long period accumulating the origin;

- if $b \in S^1 \setminus \{-1\}$, then we have $\operatorname{Re}\rho^0 = \operatorname{Re}\rho^\infty = 1 = \mu_Y$, but $\rho^0, \rho^\infty \neq 1 = \mu_Y$, which means that $\operatorname{Re}\operatorname{Res}^0 = \operatorname{Re}\operatorname{Res}^\infty = -1$ but $\operatorname{Res}^0, \operatorname{Res}^\infty \neq -1$. So, we have that every integral curve goes to the origin without being tangent to any direction or diverges, again without being tangent to any direction;

- if b lies between the two circumferences $\Gamma_{-1/2}$ of centre -1 and radius 2 and $\Gamma_{-3/2}$ of centre $1/3$ and radius $2/3$, but $b \notin S^1$, then $\operatorname{Re}\operatorname{Res}^0, \operatorname{Re}\operatorname{Res}^\infty \in (-3/2, -1) \cup (-1, -1/2)$ and so $\operatorname{Re}\rho^0, \operatorname{Re}\rho^\infty \in (1/2, 1) \cup (1, 3/2)$. Being $\operatorname{Re}\rho^0 < \operatorname{Re}\rho^\infty$ and $\rho^0 + \rho^\infty = 2$, we have that $1/2 < \rho^0 < 1 = \mu_Y < \rho^\infty < 3/2$. So, see that may have integral curves whose projection on $\mathbb{P}^1(\mathbb{C})$ intersects itself infinitely many times. Otherwise, almost all integral curves goes to the origin tangent to $[1, 0]$ (we remark here that we are not in the resonant case). Exceptional curves diverge tangent to $[0, 1]$;

- otherwise, $\operatorname{Re}\operatorname{Res}^0, \operatorname{Re}\operatorname{Res}^\infty \notin (-3/2, -1/2)$ and in particular $\operatorname{Re}\rho^0 \leq 1/2$ and $\operatorname{Re}\rho^\infty \geq 3/2$. The picture is as in the previous case, but without the possibility of infinitely self-intersecting geodesics. Almost every integral curve goes to the origin tangent to $[1, 0]$ and exceptional curves diverge tangent to $[0, 1]$.

We are left with the three cases $2'_{1100}$, $2'_{1b10}$ (with $b \in \mathbb{C}$) and $2'_{1bC1}$ (again with $b \in \mathbb{C}$). In all these cases we have at least an irregular singularity and this does not allow to get a very complete description. Depending on the values of the parameters, we see that we can have periodic curves, whose projection surrounds singularities with the sum of the real part of the residues equal to -1, curves whose projection goes to some graph, and also integral curves whose projection self-intersects infinitely many times. Moreover, we may have integral curves whose projection goes to an irregular singularity. A basic description of some of these curves is given in Proposition 5.3.5. Here we list the other possibilities for the integral curves, depending on the particular case (i.e., on the kind of the other singularity):

1. in case $2'_{1100}$ we have an apparent singularity, so an integral curve can go to a non-zero point of the apparent leaf;

2. in case $2'_{1b10}$ we have [0,1] as Fuchsian singularity, of order 2 and residue ρ^∞, and so the description follows from Theorem 5.3.4. We remark that, if this is resonant, then the residue is real and negative and so we can apply Proposition 5.3.3 to study the problem in this case. So, if $\operatorname{Re}\rho^\infty > 1$, then exceptional curves may diverge tangent to $[0, 1]$. If $\operatorname{Re}\rho^\infty = 1$ but $\operatorname{Re}\rho^\infty \neq 1$ we may have integral curve going to the origin or diverging without being tangent to any direction. If $\rho^\infty = 1$, we may have curves diverging tangent to $[0, 1]$, too. Finally, if $\operatorname{Re}\rho^\infty < 1$ and the resonant index is vanishing, we must have $\mu_Y \operatorname{Re}\rho^\infty = \rho^\infty < |\rho|^2$

(and so we may have curves going to the origin tangent to $[0,1]$) or $\operatorname{Re}\rho^\infty > |\rho|^2$ (and in this case ρ is real and a curve whose projection goes to $[0,1]$ diverges tangent to this direction). Finally, if the resonant index is non-vanishing, we remarked that we must have that the residue is real and negative, and so, by Proposition 5.3.3, an integral curve whose projection goes to $[0,1]$ must go to the origin;

3. in case $2'_{1bC1}$ both the two singularities are irregular. At now, this is the worst case to study, because we have two irregular singularities and no singular point of other kind, so that the possibilities listed above are a complete list of the possible behaviours.

6.5 Three characteristic directions

6.5.1 Classification

First, we can suppose that the three directions are $[1,0]$, $[0,1]$ and $[1,1]$. This says that we have the following constraints on the coefficients:

$$A = 0, d = 0, a + b + c = A + B + C =: S.$$

So the field is of the form

$$Q(z,w) = \left(az^3 + bz^2w + (S - a - b)zw^2\right)\frac{\partial}{\partial z} + \left(Bz^2w + Czw^2 + (S - B - C)w^3\right)\frac{\partial}{\partial w}$$

We must impose that $[1,v]$ is not a characteristic direction for $v \neq 0,1$. It means that the system

$$\begin{cases} a + bv + (B + C + D - a - b)v^2 = \lambda \\ Bv + Cv^2 + Dv^3 = \lambda v \end{cases}$$

has not solutions $v \neq 0,1$. So we obtain the equation

$$(B - a)v + (C - b)v^2 + (a + b - B - C)v^3 = 0.$$

We divide by v (the zero solution) and by $v-1$ (for the solution $v=1$). Thus we get

$$w(a + b - B - C) + a - B = 0.$$

We see that we have the following cases:

- if $a = B$, we want $a+b-B-C \neq 0$, i.e., $b \neq C$. $v = 0$ is again solution and this means that the direction $[1;0]$ is the one with multiplicity 2;
- if $a \neq B$, we have two possibilities:

- $a + b - B - C = 0$, that means that the direction $[0; 1]$ has multiplicity 2;
- $a + b - B - C = B - a$, that means that the direction $[1; 1]$ has multiplicity 2.

With a permutation of the characteristic directions, we can study only the case in which the direction $[1; 0]$ has multiplicity 2.

We have thus obtained the field

$$
\begin{aligned}
Q(z, w) = &\left(az^3 + bz^2 w + (C + D - b)zw^2\right) \frac{\partial}{\partial z} \\
&+ \left(az^2 w + Czw^2 + Dw^3\right) \frac{\partial}{\partial w},
\end{aligned}
$$

where $b \neq C$. We have four parameters so far. Let us see what happens if we conjugate our map with one that fixes the characteristic directions, i.e., $\chi(z, w) = (hz, hw)$. We obtain that all the coefficients are multiplied by h^2 with such a conjugation, so that we can suppose that one of the four parameters is 1 (if $\neq 0$).

For now, we stop here the classification, i.e., we do not divide here in subfamilies the maps with three characteristic directions. We shall do in in the next section, using as parameters some relations among a, b, C and D that we shall derive from the dynamics.

We observe in fact that we have another possible χ, the one that exchanges the directions $[0, 1]$ and $[1, 1]$ and keeps $[1, 0]$ fixed. It is the map

$$
\chi_e(z, w) = (-z + w, w) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix},
$$

The effect of a conjugation by such a map is

$$
\begin{cases}
a' = a \\
b' = -2a - b \\
C' = -2a - C \\
D' = a + C + D.
\end{cases}
$$

We shall use this map later.

6.5.2 Dynamics

Here are the tables collecting the objects we are going to use in our study, as well as order, residue and induced residue for every singularity.

Chart at $[1,0]$, coordinate ζ_0	Chart at $[0,1]$, coordinate ζ_∞
$Q_1(1,\zeta_0) = a + b\zeta_0 + (C+D-b)\zeta_0^2$	$Q_1(\zeta_\infty,1) = a\zeta_0^3 + b\zeta_0^2 + (C+D-b)\zeta_\infty$
$Q_2(1,\zeta_0) = a\zeta_0 + C\zeta_0^2 + D\zeta_0^3$	$Q_2(\zeta_\infty,1) = a\zeta_\infty^2 + C\zeta_\infty + D$
$X_0 = (b-C)\zeta_0^2\,(\zeta_0-1)$	$X_\infty = (b-C)\zeta_\infty\,(\zeta_\infty-1)$
$\eta_0 = -\frac{2\left(a+b\zeta_0+(C+D-b)\zeta_0^2\right)}{(b-C)\zeta_0^2(\zeta_0-1)}d\zeta_0$	$\eta_\infty = -\frac{2\left(a\zeta_\infty^2+C\zeta_\infty+D\right)}{(b-C)\zeta_\infty(\zeta_\infty-1)}d\zeta_\infty$
$G_0 = (b-C)\zeta_0^2\,(\zeta_0-1)\,v_0\frac{\partial}{\partial\zeta_0}$	$G_\infty = (b-C)\zeta_\infty\,(\zeta_\infty-1)\,v_\infty\frac{\partial}{\partial\zeta_\infty}$
$+2\left(a+b\zeta_0+(C+D-b)\zeta_0^2\right)v_0^2\frac{\partial}{\partial v_0}$	$+2\left(a\zeta_\infty^2+C\zeta_\infty+D\right)v_\infty^2\frac{\partial}{\partial v_\infty}$

$[1,0]$	$[1,1]$	$[0,1]$
order 2	order 1	order 1
residue $\rho^0 =$ $\frac{2(a+b)}{b-C}$	residue $\rho^1 =$ $\frac{-2(a+C+D)}{b-C}$	residue ρ^∞ $\frac{2D}{b-C}$
induced residue $\mathrm{Res}^0 =$ $\frac{2a+2C}{b-C}$	induced residue $\mathrm{Res}^1 =$ $\frac{-2a-b-C-2D}{b-C}$	induced residue $\mathrm{Res}^\infty =$ $\frac{2D-b+C}{b-C}$

We see that the important parameters that decide the kind of the singularity of a given family are a and then b for $[1,0]$ ("0"), $a+C+D$ for $[1,1]$ ("1") and D for $[0,1]$ ("∞"). In particular, we have that:

- if $a \neq 0$, then 0 is 2-irregular. If $a = 0$ and $b \neq 0$, then 0 is Fuchsian. If $a = b = 0$, then 0 is apparent;
- if $D = 0$, then ∞ is apparent, otherwise ($D \neq 0$) ∞ is Fuchsian;
- if $a + C + D = 0$, then 1 is apparent, otherwise ($a + C + D \neq 0$) is Fuchsian.

Summing up, we have (remember that $b \neq C$):

a	b	C	D	$[1,0]$	$[1,1]$	$[0,1]$
0	0	$= -a-D = 0$	0	-	-	-
0	0	$= -a-D = -D$	$\neq 0$	apparent	apparent	Fuchsian
0	0	$\neq -a-D = 0$	0	apparent	Fuchsian	apparent
0	0	$\neq -a-D = -D$	$\neq 0$	apparent	Fuchsian	Fuchsian
0	$\neq 0$	$= -a-D = 0$	0	Fuchsian	apparent	apparent
0	$\neq 0$	$= -a-D = -D$	$\neq 0$	Fuchsian	apparent	Fuchsian
0	$\neq 0$	$\neq -a-D = 0$	0	Fuchsian	Fuchsian	apparent
0	$\neq 0$	$\neq -a-D = -D$	$\neq 0$	Fuchsian	Fuchsian	Fuchsian
$\neq 0$	any	$= -a-D = 0$	0	2-irregular	apparent	apparent
$\neq 0$	any	$= -a-D$	$\neq 0$	2-irregular	apparent	Fuchsian
$\neq 0$	any	$\neq -a-D = -a$	0	2-irregular	Fuchsian	apparent
$\neq 0$	any	$\neq -a-D$	$\neq 0$	2-irregular	Fuchsian	Fuchsian

We note that the first line gives the identity map, and so we don't need to consider it. Moreover, we see that, in every block of four maps, the second and the third are equivalent, by a conjugation with the map χ_e introduced before. So, will will not study the second case of every block. Finally, we recall that we can always make a coefficient different from 0 become 1. So, we are now ready to study the dynamics for these representatives. As a last remark, we note that the maps corresponding to the lines of this last table (except the first and the second in every block) give the holomorphic classification of the maps in this case.

Case 3_{00C0}, with $C \in \mathbb{C}^*$, i.e., 3_{0010} The field in this case is

$$Q(z,w) = zw^2 \frac{\partial}{\partial z} + zw^2 \frac{\partial}{\partial w}$$

and these are the residues in this case:

	$[1,0]$	$[1,1]$	$[0,1]$
ρ	0	2	0
Res	-2	1	-1

We are going to show, in particular, that in this case no integral curve can go to the origin, but that there exist curves that go to infinity with direction which tends to the Fuchsian characteristic direction $[1,1]$. The first part is quite clear, because, by the form of the field, we immediately see that $Q^1 = Q^2$ and so the difference between the two components of the solution is constant. In particular, to go to zero we should have that the two coordinates of the starting point are equal, but this would mean that it lies on the characteristic leaf of $[1,1]$.

We shall apply the general theory, and also see that we can come to some of the results by means of explicit formulas.

We know that the system has one Fuchsian direction, $[1,1]$, of order 2 and residue $\rho^1 = 2$ (we notice in particular that $\mu_X - 1 - \rho^1 = -2 \notin \mathbb{N}^*$, so that we do not have a resonant index). Using Theorem 5.3.4 we find that we are in case 1b, and so we know that, if an integral curve $\gamma(t)$ is such that $[\gamma(t)] \to [1,1]$, then $\|\gamma(t)\| \to +\infty$. Moreover, we see that case 2 is verified, and it means that we can find a neighbourhood $U \subset \mathbb{P}^1(\mathbb{C})$ of $[1,1]$ such that every integral curve $\gamma(t)$ with projection $[\gamma]$ issuing from a point of $U \setminus \{[1,1]\}$ either escapes from (the preimage under the projection of) this neighbourhood (and this happens for a Zariski dense open set of initial conditions), or we have $[\gamma(t)] \to [1,1]$, but $\|\gamma(t)\| \to \infty$.

Let's now study the other two singularities, $[1,0]$ and $[0,1]$, which are apparent singularities. We know, from Theorem 5.3.2, that if an integral curve $\gamma : [0,\varepsilon) \to \mathbb{C}^2 \setminus \{0\}$ has the projection that goes to one of these singularities as $t \to \varepsilon$, then $\gamma(t)$ tends to a non zero element of the corresponding characteristic leaf and, in particular, that no integral curve can go to the origin tangentially to any of these two characteristic directions. Moreover, we know that this happens for an open set of initial conditions for $[\gamma(0)]$ in a neighbourhood of $[0,1]$ or $[1,0]$.

The last part of Theorem 5.3.2 says that if the order of the singularity is 1, or if it is greater than 1 but the apparent index is not zero, we have periodic orbits of arbitrarily long periodic accumulating the origin near the singular direction. Now, this must happen for the singularity $[0,1]$, which has order 1 and in fact has induced residue equal to -1, thus allowing the presence of these period orbits. What about the singularity $[1,0]$, of order 2? Here we can work backwards: we know that the induced residue is -2, and this means that we cannot have periodic orbits accumulating $[1,0]$. So, we find that the apparent index of $[1,0]$ must be zero.

We shall now try to find an explicit solution for an integral curve, and we shall see that the computations are in accord with the results from the general theory.

First, as we already noticed, we have $Q^1 = Q^2$. It means that, if we write our solution as $\gamma(t) = (z(t), w(t))$, then $z'(t) = w'(t)$ and so the two components of $\gamma(t)$ will differ, at every time, of the initial difference $c := z(0) - w(0)$. So we can solve for $w(t)$, using $z(t) = c + w(t)$. We get

$$w'(t) = (w(t) + c)w(t)^2.$$

Dropping the t in order to simplify the notations, we get

$$\frac{w'}{(w+c)w^2} = 1.$$

Integrating, we obtain

$$t + c_1 = \frac{1}{c^2}\log\left(1 + \frac{c}{w}\right) - \frac{1}{cw}$$

for some c_1 depending on the initial conditions. Here we see that in effect we can have, for $\|z\|, \|w\| \to \infty$ (which means that we tend to $[1,1]$), or $z = w + c \to 0$, which stands for the $[0,1]$. A careful study of this equation can give information on the convergences, depending on c.

As a last remark, we note that if we consider the geodesic field

$$G_0 = -\zeta_0^2(\zeta_0 - 1)v_0\frac{\partial}{\partial \zeta_0} + 2\zeta_0^2 v_0^2 \frac{\partial}{\partial v_0}$$

we find, in the usual way, that

$$(\zeta_0 - 1)^2\, v_0 = c_0$$

for some c_0 and this again implies that, if $\zeta_0 \to 1$ (which means that the projection of the integral curve tends to [1,1]), we must have $v \to \infty$, and so the $\|\gamma(t)\|$ must diverge, while it does not for $\zeta_0 \to 0$ (which stands for the singularity [0,1]). The same happens for G_∞, and this implies that also for [0,1] the solution does not diverge and goes to a non-zero point of the leaf.

Case 3_{00CD}, with $C, D \in \mathbb{C}^*$ and $C \neq -D$, i.e., 3_{00C1}, $C \neq -1, 0$

	$[1,0]$	$[1,1]$	$[0,1]$
ρ	0	$2+\frac{2}{C}$	$-\frac{2}{C}$
Res	-2	$1+\frac{2}{C}$	$-1-\frac{2}{C}$

The field is

$$(C+1)zw^2\frac{\partial}{\partial z} + (Czw^2 + w^3)\frac{\partial}{\partial w}$$

We know that the two singularities [1,1] and [0,1] are Fuchsian, while [1,0] is apparent. So, we are going to use Theorems 5.3.2 and 5.3.4 to study this case.

We study the situation depending on the value $\mathrm{Re}(-2/C)$:

- if $\mathrm{Re}(-2/C) \leq -1/2$ or $\mathrm{Re}(-2/C) \geq 5/2$ we have one Fuchsian direction with positive real part of the residue and one with negative. So, almost every integral curve goes to the origin tangent to the Fuchsian characteristic direction with negative real part of the residue or to a non-zero point of the leaf of the apparent singularity. Exceptional curves diverge tangent to the Fuchsian direction with positive real part of the residue;

- if $-1/2 < \mathrm{Re}(-2/C) < 0$ or $2 < \mathrm{Re}(-2/C) < 5/2$ the description is as in the previous case, with the extra possibility of integral curves with projection on $\mathbb{P}^1(\mathbb{C})$ with infinite self-intersections;

- we cannot have $-2/C = 0$ or $-2/C = 2$ ($C = -1$ is excluded by hypothesis), and so the residues cannot be -1 (this would have been the only possibility to have graphs as ω-limits of the projections);

- if $\mathrm{Re}(-2/C) = 0$ or $\mathrm{Re}(-2/C) = 2$, it means that one Fuchsian induced residue has vanishing real part, while the other has real part equal to 2. It means that almost every integral curve goes to a point of the apparent leaf, goes to zero without being tangent to any direction, or diverges in the same way. Exceptional curves diverge tangent to the Fuchsian direction with real part of the residue equal to 2;

- if $0 < \mathrm{Re}(-2/C) < 1/2$ or $3/2 < \mathrm{Re}(-2/C) < 2$ we can have integral curves with infinitely self-intersecting projections. Apart from these curves, almost every integral curve goes to a non-zero point of the apparent leaf, and exceptional curves diverge tangent to a Fuchsian direction.

- if $1/2 \leq \mathrm{Re}(-2/C) \geq 3/2$ almost every integral curve goes to a non-zero point of the apparent leaf, and exceptional curves diverge tangent to a Fuchsian direction.

Case $3_{0b00}(b \neq 0) \sim 3_{0100}$

	$[1,0]$	$[1,1]$	$[0,1]$
ρ	2	0	0
Res	0	-1	-1

The field in this case is

$$\left(z^2 w - z w^2\right) \frac{\partial}{\partial z}.$$

We see that the second component of a solution $(z(t), w(t))$ is fixed, so that, also in this case, like in 3_{0010}, there are no integral curves going to the origin (unless $w(0) = 0$, but it would mean that the initial point is in a characteristic leaf).

So, let's study what happens near the singularities: $[1,0]$ is Fuchsian, of order 2, with $\mu_Y = 1$ and $\rho^0 = 2$, so that Theorem 5.3.4 implies that there exists a neighbourhood $U \subset \mathbb{P}^1(\mathbb{C})$ of [1,0] such that every integral curve issuing from a point in $U \setminus \{[1,0]\}$ either has the projection that escapes U (Zariski open set of initial conditions) or diverges tangentially to $[1,0]$. In both cases, it means that, as we said, the origin cannot attract anything.

For the other two singularities, they are apparent of order 1, and Theorem 5.3.2 says that there are periodic orbits of arbitrarily long period accumulating them. This is coherent with the fact the the two induced residues are -1.

The global description is as follows: almost every integral curve is either periodic, with projection on $\mathbb{P}^1(\mathbb{C})$ surrounding an apparent singularity, or goes to a non-zero point of an apparent leaf. Exceptional integral curves diverge tangent to the Fuchsian direction.

Thanks to the fact the the coordinate w is fixed, we can also integrate the field directly, obtaining

$$\begin{cases} z(t) = \dfrac{w_0 z_0}{z_0 + (w_0 - z_0)\exp(w_0^2 t)} \\ \\ w(t) = w_0 \end{cases}$$

Again, from this expression we can recover the main limits of integral curves.

3_{0bC0}**, with** $b, C \in \mathbb{C}^*, b \neq C$**, i.e.** $3_{01C0}, C \neq 0, 1$

	$[1,0]$	$[1,1]$	$[0,1]$
ρ	$\frac{2}{1-C}$	$\frac{-2C}{1-C}$	0
Res	$\frac{2C}{1-C}$	$\frac{-1-C}{1-C}$	-1

The field is

$$\left(z^2w + Czw^2\right)\frac{\partial}{\partial z} + Czw^2\frac{\partial}{\partial w}$$

[1,0] is Fuchsian of order 2, [1,1] is Fuchsian of order 1 and [0,1] is apparent of order 1.

The study of this case in completely analogous to the one done for 3_{00CD}, with the only difference that now μ_Y is 1 for the Fuchsian singularity of order two and the order of the apparent singularity is 1. Again, we see that, varying C, we get examples of the different behavious stated in Theorems 5.3.2 and 5.3.4. Moreover, in this case we might have graphs as ω-limits of projections of integral curves: the graph may have as vertices both the two Fuchsian singularities, or also only $[1,0]$, if $C = -1$.

Case $3_{0bCD}(b, D \neq 0, C \neq -D, C \neq b) \sim 3_{01CD}, C \neq -D, C \neq 1$

	$[1,0]$	$[1,1]$	$[0,1]$
ρ	$\frac{2}{1-C}$	$\frac{-2(C+D)}{1-C}$	$\frac{2D}{1-C}$
Res	$\frac{2C}{1-C}$	$\frac{-1-C-2D}{1-C}$	$\frac{2D-1+C}{1-C}$

We are studying the field

$$Q(z,w) = (z^2w + (C+D-1)zw^2)\frac{\partial}{\partial z} + (Czw^2 + Dw^3)\frac{\partial}{\partial w}$$

Here we have three Fuchsian singularities: $[1,0]$ of order 2 and $[1,1]$ and $[0,1]$ of order 1. We remark that, if $[1,0]$ has non-vanishing resonant index, it must have negative residue (from $\rho = 1 - n$). So, by Proposition 5.3.3, if the direction of an integral curve goes to $[1,0]$, then the integral curve must go to the origin.

Apart from possible integral curves with projection with infinite self-intersections, or going to a graph, or periodic orbits, we have that generic integral curves have one of the following behavious:

- go to the origin tangent to a Fuchsian direction of order 1 and real negative real part of the residue, or;

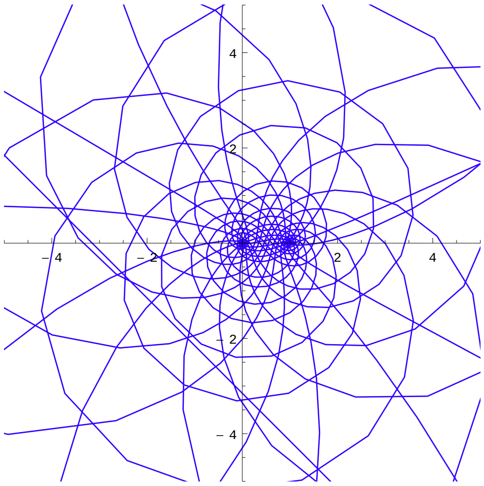


Figure 6.3: $3_{01CD}, \zeta(0) = 10 + i, v(0) = 1$

- go to the origin or diverge without being tangent to any direction (this possibility requires that the $\operatorname{Re}\rho^1 = 0$, $\operatorname{Re}\rho^\infty = 0$ or $\operatorname{Re}\rho^0 = 1$ but $\operatorname{Re}\rho^0 \neq 1$); or
- diverge tangent to $[1, 0]$, the Fuchsian direction of order 2 (if $\rho^0 = 1$); or
- go to the origin tangent to $[1, 0]$.

Exceptional integral curves diverge tangent to a Fuchsian direction of order 1 and residue with positive real part.

We want to study better a particular subcase of this case: suppose that $\operatorname{Re}\rho^0 > 1$ and $\operatorname{Re}\rho^1, \rho^\infty > 0$. By Theorem 5.3.4, the projection of almost every integral curve does not converge to any characteristic. Moreover, under these conditions, we see that we can never exclude the possibility of infinitely self-intersecting projections. In fact, for this we would need

$$\begin{cases} \operatorname{Re}\rho^0 \geq 3/2 \\ \operatorname{Re}\rho^1, \operatorname{Re}\rho^\infty \geq 1/2 \\ \rho^0 + \rho^1 + \rho^\infty = 2, \end{cases}$$

which is clearly impossible. In Figure 6.3 we have drawn an integral curve $\zeta(t)$, with $C = -1/2$ and $D = 1/4$, so that $\rho^0 = 4/3$, $\rho^1 = \rho^\infty = 1/3$ and $\operatorname{Res}^0 = \operatorname{Res}^1 = \operatorname{Res}^\infty = -2/3$ (compare with [AT11], Example 8.3).

So, we are left with the following three cases:

1. Case $3_{ab00}, a, b \neq 0, b \neq -a \sim 3_{1b00}, b \neq -1, 0$;

2. Case $3_{abC0}(a \neq 0, C \neq -a, C \neq b) \sim 3_{1bC0}, C \neq -1, C \neq b$;

3. Case $3_{abCD}, (a, D \neq 0, C \neq -a - D, C \neq b) \sim 3_{1bCD}, D \neq 0, C \neq -1 - D, C \neq b$.

In all of three we have that $[1, 0]$ is an irregular singularity. We have only partial results, due to the lack of a precise description of irregular singularities. The following is a description of the integral curves which are not periodic, whose projection does not self-intersect infinitely many times and does not tend to a graph, and that do not go to the irregular singularity. In this last case, Proposition 5.3.5 provides some basic information about the convergence of some integral curves.

1. In case 3_{ab00} we have two apparent singularities. All the integral curves that do not behave as said above go from a point of an apparent leaf to another point of an apparent leaf.

2. In case 3_{acC0} we have one apparent singularity and one Fuchsian singularity, both of order 1. So, almost all integral curves with a behaviour different from the ones above go to a non-zero point of the apparent leaf or, if the real part of the residue of the Fuchsian singularity is negative, converge to the origin tangent to the Fuchsian direction, or, if the residue of the Fuchsian singularity is purely imaginary, can go to the origin or diverge without being tangent to any direction. Exceptional curves diverge tangent to the Fuchsian direction, and this can happen only if the residue of the Fuchsian direction has positive real part.

3. In case 3_{acCD} we have two Fuchsian singularities, both of order 1. So, apart form the behaviours above, if at least one of the residues of the Fuchsian sungularities have negative or zero real part, almost all integral curves go to the origin tangent to one of the Fuchsian direction whose residue has negative real part, or go to the origin or diverge without being tangent to any direction (and this requires the presence of a Fuchsian direction with purely imaginary residue). In this case, exceptional integral curves diverge tangentially to a Fuchsian direction whose residue has positive real part. If all the Fuchsian residues have positive real part, then the generic behaviour must be among the ones listed above.

6.6 Four characteristic directions

6.6.1 Classification

We can suppose that the four characteristic directions are [0,1], [1,0], [1,1] and [1,y], with $y \neq 0, 1$. So, if we take the generic field

$$\begin{aligned} Q(z,w) &= \left(az^3 + bz^2w + czw^2 + dw^3\right)\frac{\partial}{\partial z} \\ &\quad + \left(Az^3 + Bz^2w + Czw^2 + Dw^3\right)\frac{\partial}{\partial w} \end{aligned}$$

we have that the fact that [0,1] is characteristic implies $d = 0$, [1,0] characteristic implies $A = 0$, [1,1] implies $a + b + c = B + C + D$. We want to impose y as the fourth solution of the equation

$$Bw + Cw^2 + Dw^3 = aw + bw^2 + (B + C + D - a - b)w^3.$$

Because of the zero solution, we obtain

$$B + Cw + Dw^2 = a + bw + (B + C + D - a - b)w^2,$$

which is

$$(w - 1)\left((a + b - B - C)w + (a - B)\right) = 0$$

so we want

$$y = \frac{B - a}{a + b - B - C},$$

which must be different from 0 and 1 (and so $a \neq B$ and $B - a \neq a + b - B - C$), so that the field becomes

$$\begin{aligned} Q(z,w) &= \left(az^3 + bz^2w + \left(D + \frac{a - B}{y}\right)zw^2\right)\frac{\partial}{\partial z} \\ &\quad + \left(Bz^2w + \left(a + b - B + \frac{a - B}{y}\right)zw^2 + Dw^3\right)\frac{\partial}{\partial w}. \end{aligned}$$

So, our parameters are a, b, B, D and a map χ which keeps the characteristic directions fixed must be of the form $\chi(z,w) = (hz, hw), h \neq 0$, as we saw in the revious section. The effect of a conjugation by χ is to multiply all these coefficients by h^2.

6.6.2 Dynamics

We start with the usual tables collecting all the data of this case.

Chart at $[1,0]$, coordinate ζ_0
$Q_1(1,\zeta_0) = a + b\zeta_0 + (D + \frac{a-B}{y})\zeta_0^2$
$Q_2(1,\zeta_0) = B\zeta_0 + (a+b-B+\frac{a-B}{y})\zeta_0^2 + D\zeta_0^3$
$X_0 = \frac{B-a}{y}\zeta_0\,(\zeta_0-1)\,(\zeta_0-y)$
$\eta_0 = -\frac{2\left(a+b\zeta_0+(D+\frac{a-B}{y})\zeta_0^2\right)}{\frac{B-a}{y}\zeta_0(\zeta_0-1)(\zeta_0-y)}d\zeta_0$
$G_0 = \frac{B-a}{y}\zeta_0\,(\zeta_0-1)\,(\zeta_0-y)\,v_0\frac{\partial}{\partial\zeta_0}\ +2\left(a+b\zeta_0+(D+\frac{a-B}{y})\zeta_0^2\right)v_0^2\frac{\partial}{\partial v_0}$

Chart at $[0,1]$, coordinate ζ_∞
$Q_1(\zeta_\infty,1) = a\zeta_\infty^3 + b\zeta_\infty^2 + (D+\frac{a-B}{y})\zeta_\infty$
$Q_2(\zeta_\infty,1) = B\zeta_\infty^2 + (a+b-B+\frac{a-B}{y})\zeta_\infty + D$
$X_\infty = \frac{a-B}{y}\zeta_0\,(\zeta_0-1)\,(\zeta_0-y)$
$\eta_\infty = -\frac{2\left(B\zeta_\infty^2+(a+b-B+\frac{a-B}{y})\zeta_\infty+D\right)}{(a-B)\zeta_\infty(\zeta_\infty-1)\left(\zeta_\infty-\frac{1}{y}\right)}d\zeta_\infty$
$G_\infty = (a-B)\,\zeta_\infty\,(\zeta_\infty-1)\left(\zeta_\infty-\frac{1}{y}\right)v_\infty\frac{\partial}{\partial\zeta_\infty}$
$+2\left(B\zeta_\infty^2+(a+b-B+\frac{a-B}{y})\zeta_\infty+D\right)v_\infty^2\frac{\partial}{\partial v_\infty}$

	residue	induced residue
$[1,0]$	$\rho^0 = \frac{2a}{a-B}$	$\mathrm{Res}^0 = \frac{a+B}{a-B}$
$[1,1]$	$\rho^1 = -\frac{2((a+b+D)y+(a-B))}{(B-a)(1-y)}$	$\mathrm{Res}^1 = \frac{y(B-3a-2b-2D)+(B-a)}{(B-a)(1-y)}$
$[1,y]$	$\rho^y = -2\frac{a+by+(a-B)y+Dy^2}{(B-a)(y-1)}$	$\mathrm{Res}^y = \frac{y^2(-2D)+y(B-2b-a)-3a+B}{(B-a)(y-1)}$
$[0,1]$	$\rho^\infty = \frac{2Dy}{B-a}$	$\mathrm{Res}^\infty = \frac{2Dy+a-B}{B-a}$

We see that:

- if $a=0$, then $[1,0]$ is apparent with $\mathrm{Res}^0=-1$, otherwise it is Fuchsian, with $\mathrm{Res}^0\neq -1$;
- if $(a+b+D)y+(a-B)=0$, then $[1,1]$ is apparent with $\mathrm{Res}^1=-1$, otherwise it is Fuchsian, with $\mathrm{Res}^1\neq-1$;
- if $Dy^2+y(a+b-B)-a=0$, then $[1,y]$ is apparent with $\mathrm{Res}^y=-1$, otherwise it is Fuchsian, with $\mathrm{Res}^y\neq-1$;
- if $D=0$ then $[0,1]$ is apparent with $\mathrm{Res}^\infty=-1$, otherwise it is Fuchsian with $\mathrm{Res}^\infty\neq-1$.

We shall need to consider only one case with every possible number of apparent singularities. Moreover, we remark that we cannot have all the singularities becoming apparent, because this would imply that all the coefficients of the map are zero, and so the map would be the identity.

The last remark we make before starting with the dynamical study is about the presence of graphs of saddle connections as ω-limits of projections of integral curves. First, the vertices of the graph can only be Fuchsian singularities, because of Remark 4.2.17, and moreover they must all have positive real part of the residue.

Remark 6.6.1. *Here we have $\mu_Y = 0$ and order 1 for every Fuchsian singularity, so that the condition $\operatorname{Re}\rho < \mu_Y$ becomes $\operatorname{Re}\rho < 0$ (and $\operatorname{Re}\operatorname{Res} < -1$). So, we see that every graph of saddle connections on $\mathbb{P}^1(\mathbb{C})$ which is an ω-limit for the projection of some integral curve can have as vertices only Fuchsian singularities with $\operatorname{Re}\rho \geq 0$ (and $\operatorname{Re}\rho = 1/2$ in order to be the vertices of a spike). In the following study we shall not concentrate on these cases, as well as on curves with infinitely self-intersecting projection.*

[1,0], [1,1] and $[1,y]$ apparent singularities It means that we have

$$\begin{cases} a = 0 \\ (b+D)y - B = 0 \\ Dy^2 + y(b-B) = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ Dy = B. \end{cases}$$

Moreover, we recall that we can suppose that one of the non-zero coefficients is 1. So, let $D = 1$, so that G_0 becomes

$$G_0 = \zeta_0 (\zeta_0 - 1)(\zeta_0 - y) v_0 \frac{\partial}{\partial \zeta_0}$$

and we see that the three apparent singularities are not in normal form.

Clearly, we have $\rho^0 = \rho^1 = \rho^y = 0$ and $\rho^\infty = 2$. Theorem 5.3.2, case 2, implies that ($\mu_Y = 0$ at the Fuchsian singularity [0,1]) there is an open neighbourhood $U \subset \mathbb{P}^1$ of $[0,1]$ such that, for a Zariski dense open subset of initial conditions in U for $[\gamma(t)]$, we have that $[\gamma(t)]$ escapes U, otherwise $[\gamma(t)] \to [0,1]$ with $\|\gamma(t)\| \to \infty$. So, we see that no integral curve can go to the origin (also because of Proposition 5.3.3). The projection of almost all integral curves are saddle connections between two apparent singularities on $P^1(\mathbb{C})$, which means that almost all integral curve for the field in $\mathbb{C}^2$ go from a non zero element of $L_{[1,a]}$ to another non zero element of $L_{[1,a']}$, where a and a' are $0, 1$ or y. Exceptional curves may diverge tangentially to [0,1] or have a periodic projection around one of the apparent singularities (Theorems 5.3.2 and 5.3.4).

[1,0] and [1,1] apparent singularities We have $a = (b+D)y - B = 0$. As before, we recall that we can assume that one non-zero coefficient is 1. We take $b = 1$, so that we have

$$G_0 = \frac{B}{y} \zeta_0 (\zeta_0 - 1)(\zeta_0 - y) v_0 \frac{\partial}{\partial \zeta_0} - 2\zeta_0 (\zeta_0 - 1)(v_0)^2 \frac{\partial}{\partial v_0}.$$

We have $\rho^0 = \rho^1 = 0$, and so $\rho^y + \rho^\infty = 2$. We see that at least one among $\operatorname{Re}\rho^y$ ad $\operatorname{Re}\rho^\infty$ must be positive. Suppose that $\operatorname{Re}\rho^\infty > 0$ and also that $\operatorname{Re}\rho^y \leq \operatorname{Re}\rho^\infty$. We see that we can have the following cases:

- $\operatorname{Re}\rho^y > 0$ (and so $0 < \operatorname{Re}\rho^y \leq 1$ and $1 \leq \operatorname{Re}\rho^\infty < 2$). In this case, from Theorem 5.3.4 we know that if an integral curve $\gamma(t)$ is such that $[\gamma(t)]$ goes to $[1, y]$ or [0,1], then $\|\gamma(t)\|$ diverges. We study the different possibilities:

    1. $0 < \operatorname{Re}\rho^y < 1/2$ and $3/2 \leq \operatorname{Re}\rho^\infty < 2$. We see that, for example, $-1 < \operatorname{Re}\operatorname{Res}^y < -1/2$, so that there may be integral curves whose projections self-intersects infinitely many times.
    2. $\operatorname{Re}\rho^y = 1/2$ and $\operatorname{Re}\rho^\infty = 3/2$. In this case, we cannot have the infinite selfintersections case.
    3. $1/2 < \operatorname{Re}\rho^y < 1$ and $1 \leq \operatorname{Re}\rho^\infty < 3/2$. Again, we see that the pairs of singularities $\{0, y\}, \{0, \infty\}, \{1, y\}$ and $\{1, \infty\}$ have the sum of the two induced residues in $(-3/2, -1) \cup (-1, -1/2)$ and so we can have integral curves whose projections selfintersects infinitely many times.
    4. $\operatorname{Re}\rho^y = \operatorname{Re}\rho^\infty = 1$. As in case 2, we cannot have infinite self-intersections.

  Moreover, by Remark 6.6.1 we cannot have integral curves whose projection tends to a graph of saddle connections. So, in cases 2 and 4 we get a fairly complete description of the situation: almost all integral curves go from a non-zero point of a characteristic leaf ($L_{[1,0]}$ or $L_{[1,1]}$) to another non-zero point of a characteristic leaf. Exceptional curves diverge tangentially to $[1, y]$ or $[0, 1]$ or have periodic projections surrounding an apparent singularity. In the other two cases, we may also have integral curves with infinitely self-intersecting projections.

- $\operatorname{Re}\rho^y = 0$. This is similar to case 4 of the previous possibility, with the only difference that we may have a graph with $[1, y]$ as only vertex and we cannot have integral curves whose projection goes to $[1, y]$, but we have instead integral curves with closed projection, that may go to the origin or diverge without being tangent to any direction.

- $\operatorname{Re}\rho^y < 0$. Also this case is similar to the first, with the main difference being the fact that if the projection of an integral curve goes to $[1, y]$, then the integral curve goes to the origin instead of diverging. We have the following two possibilities:

    1. if $-1/2 < \operatorname{Re}\rho^y < 0$, we have $-3/2 < \operatorname{Re}\rho^y < -1$, so that we can have infinitely many selfintersections for the projections of the integral curves;

2. if $\operatorname{Re}\rho^y \le -1/2$, we see that no subset of the singularities has the sum of the induced residues in $(-3/2,-1) \cup (-1,-1/2)$.

So, we get the description for case 2: almost all integral curves go from a non-zero point of a characteristic leaf to another non-zero point of a characteristic leaf. Exceptional curves diverge tangentially to $[0,1]$, tend to the origin tangentially to $[1,y]$, or have periodic projections surrounding an apparent singularity. In the other case, we may also have integral curves with infinitely self-intersecting projections.

[1,0] apparent singularity It means that $a = 0$. We have $\rho^0 = 0$ and

$$\rho^1 + \rho^y + \rho^\infty = 2.$$

Again, we see that the real part of at least one of these residues must be positive, and suppose it is $\operatorname{Re}\rho^\infty > 0$. We can also suppose that $\operatorname{Re}\rho^1 \le \operatorname{Re}\rho^y \le \operatorname{Re}\rho^\infty$. So, we have the possibilities listed below (recall that $\mu_Y = 0$ for all the three Fuchsian directions). In each of these cases it is possible to find the values of the residues for which we can have infinitely self-intersecting geodesics, as in the previous paragraph for the case of two apparent singularities. If this does not happen, we can give a complete picture of the dynamics of the associated maps. The description of a geodesics whose projection tends to a singularity does not depend on the presence of these self-intersecting geodesics.

- $\operatorname{Re}\rho^1, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$. No integral curve can go to the origin. If the projection $[\gamma]$ of an integral curve γ goes to $[1,1], [1,y]$ or $[0,1]$, then $\|\gamma(t)\| \to \infty$, while if $[\gamma(t)] \to [1,0]$, then $\gamma(t)$ tends to a non-zero point of $L_{[1,0]}$.

  We may have integral curves whose projection on $\mathbb{P}^1(\mathbb{C})$ self-intersects infinitely many times has as ω-limit a graph with the three Fuchsian singularities as vertices (Remark 6.6.1) or also a graph with two of the three Fuchsian singularities as vertices (and this case requires that the third Fuchsian singularity has residue with real part equal to 1, again by Remark 6.6.1).

  If we do not have these behaviours, we see that the generic behaviour for an integral curve is to connect two points of the apparent leaf. Exceptional curves diverge tangent to a Fuchsian direction or are periodic, surrounding the apparent singularity.

- $\operatorname{Re}\rho^1 < 0, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$. If an integral curve goes to the origin, it must do it tangent to $[1,1]$. Conversely, if the projection $[\gamma]$ of an integral curve γ goes to $[1,1]$ then $\gamma(t) \to 0$. If $[\gamma]$ goes to $[1,y]$ or $[0,1]$, then $\|\gamma(t)\| \to \infty$, while if $[\gamma(t)] \to [1,0]$, then $\gamma(t)$ tends to a non-zero point of $L_{[1,0]}$.

By Remark 6.6.1, we cannot have graphs of saddle connections as ω-limits for projections of integral curves. In fact, all the vertices should have a residue with positive real part, and it means that they would be $[1, y]$ and $[0, 1]$. But we would also need that $\operatorname{Re}\rho^y + \operatorname{Re}\rho^\infty = 1$, which would imply $\operatorname{Re}\rho^1 = 1$, which is false.

So, if we do not have infinitely-self intersecting projections of integral curves, the description is as follows: almost all integral curves go to 0 tangent to $[1, 1]$ or go to a point of $L_{[1,0]}$. Exceptional curves diverge tangent to $[1, y]$ or $[0, 1]$, the Fuchsian directions.

- $\operatorname{Re}\rho^1, \operatorname{Re}\rho^y < 0, \operatorname{Re}\rho^\infty > 0$. An integral curve γ goes to the origin if and only if its projection goes to $[1, 1]$ or $[1, y]$, and this case it converges tangent to the corresponding direction in $\mathbb{P}^1(\mathbb{C})$. If an integral curve γ is such that $[\gamma(t)] \to [1, 0]$, then it tends to a non-zero point of $L_{[1,0]}$. If $[\gamma(t)] \to [0, 1]$, then $\gamma(t)$ diverge tangent to $[0, 1]$.

  By Remark 6.6.1 we know that we cannot have integral curves whose projection accumulates a graph of saddle connections. So, if there are not integral curves with infinitely self-intersecting projections, we have that the generic behaviour is converging to the origin tangent to $[1, 1]$ or $[1, y]$, or to a non-zero point of $L_{[1,0]}$. Exceptional curves diverge tangent to $[0, 1]$.

- $\operatorname{Re}\rho^1 = 0, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$. We see that we have $\rho^1 \neq 0$, because otherwise $[1, 1]$ would not be a Fuchsian direction. All integral curves going to the origin cannot converge tangent to any direction. If the projection $[\gamma]$ of an integral curve γgoes to $[1, 0]$, γ tends to a non-zero value of $L_{[1,0]}$. If $[\gamma] \to [1, 1]$, γ may converge to the origin or diverge, in both cases without being tangent to any direction. If $[\gamma(t)] \to [1, y]$ or $[1, \infty]$, then $\gamma(t)$ diverge, tangent to $[1, y]$ or $[1, \infty]$.

  For what concerns the presence of graphs as ω-limits of projections of integral curves, we see that there may these graphs, with $[1, 1]$ as only vertex or with all the three Fuchsian singularities as vertices.

  If there are not integral curves with such a graph as ω-limit for the projection, or with infinitely self-intersecting projection, almost all geodesics tend to a non-zero value of $L_{[1,0]}$, go to the origin without being tangent to any direction, or diverge, without being tangent to any direction, too. Exceptional curves diverge tangent tangent to $[0, 1]$

- $\operatorname{Re}\rho^1 < 0, \operatorname{Re}\rho^y = 0, \operatorname{Re}\rho^\infty > 0$. If an integral curve γ goes to the origin it may do it tangent to $[1, 1]$ or without being tangent to any direction.

  If the projection $[\gamma]$ of an integral curve goes to $[1, 0]$, then $\gamma(t)$ tends to a non-zero element of $L_{[1,0]}$, as usual. If $[\gamma] \to [1, 1]$, then $\gamma(t)$ tends to the origin tangent to $[1,1]$. If $[\gamma(t)] \to [1,y]$, then γ may go to the origin or diverge, in both cases without being tangent to any direction. If $[\gamma(t)] \to [0,1]$, then $[\gamma(t)]$ diverges tangent to $[0,1]$.

The description of the possible graphs is similar to the previous case: there may be only $[1,y]$ as vertex, or we may have all the three Fuchsian singularities.

If there are not such graphs as ω-limits of projections, nor infinitely self-intersecting projections of integral curves, almost all geodesics tend to the origin tangent to $[1,1]$ or without being tangent to any direction, or tend to an element of $L_{[1,0]}$, or diverge without being tangent to any direction. Exceptional curves diverge tangent to $L_{[0,1]}$.

- $\operatorname{Re}\rho^1, \operatorname{Re}\rho^y = 0, \operatorname{Re}\rho^\infty = 2$. All integral curves going to the origin do it without being tangent to any direction and if an integral curve γ goes to the origin, then its projection goes to $[1,1]$ or to $[1,y]$.

  If $[\gamma(t)] \to [1,1]$, then $\gamma(t)$ goes to the origin or diverge, in any case without being tangent to any direction. The same is true for $[1,y]$. If $[\gamma(t)] \to [1,0]$ then $\gamma(t)$ tends to a non-zero element of $L_{[1,0]}$ and if $[\gamma(t)] \to [0,1]$ then $\gamma(t)$ diverges tangent to $[0,1]$, as usual.

All Fuchsian This is the generic case of all the problem. We have four Fuchsian singularities, with $\rho^0 + \rho^1 + \rho^y + \rho^\infty = 2$ As before, we can suppose that

$$\operatorname{Re}\rho^0 \le \operatorname{Re}\rho^1 \le \operatorname{Re}\rho^y \le \operatorname{Re}\rho^\infty$$

and in particular $\operatorname{Re}\rho^\infty > 0$. We have the following possibilities:

- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0 = 0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1 = 0 \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y = 0 \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0 < 0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0 < 0, \operatorname{Re}\rho^1 = 0, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0 < 0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y = 0, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1 < 0, \operatorname{Re}\rho^y, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1 < 0, \operatorname{Re}\rho^y = 0, \operatorname{Re}\rho^\infty > 0$;
- $\operatorname{Re}\rho^0, \operatorname{Re}\rho^1, \operatorname{Re}\rho^y < 0, \operatorname{Re}\rho^\infty > 0$.

We see that, given an integral curve γ,

1. if $[\gamma(t)]$ tends to a singularity with $\operatorname{Re}\rho < 0$, then $\gamma(t)$ goes to the origin tangent to that direction;

2. if $[\gamma(t)]$ tends to a singularity with $\operatorname{Re}\rho = 0$, then $\gamma(t)$ may go to the origin or diverge, in any case without being tangent to any direction;

3. if $[\gamma(t)]$ tends to a singularity with $\operatorname{Re}\rho > 0$, then $\gamma(t)$ diverges tangent to that direction.

Almost all integral curve have a behaviour of kind 1 or 2 (or have infinitely self-intersecting projection, or the projection has a graph of saddle connections as ω-limit). Exceptional curves diverge tangent to some direction with positive real part of the residue.

6.7 A final remark: a glimpse of higher irregularity and degree

In this last section we shall not prove anything, but only show pictures, trying to motivate our assertions (see Remark 4.2.20) about integral curves near irregular singularities. We want to give some examples of singularities of higher irregularity, necessarily with fields of degree higher than 3, and to show pictures that seem to indicate directions of convergence to 0 for the coordinate $z(t)$.

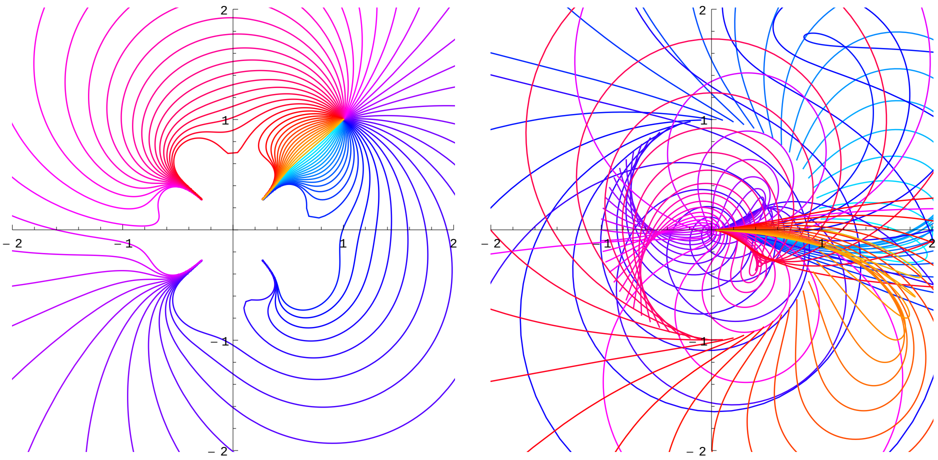
Furthermore, we shall give some pictures showing how these direction vary as the degeneracy of the singularity increases (that means, as the irregularity of the singularity decreases).

We remark that we shall always work with a unique singularity. This means that the induced residue is always -2, and this prevents from getting insights about the general case. Moreover, in all this section, when we speak about *directions of convergence* we mean the directions whose existence is suggested by the numerical experiments.

Our setting will be the following: we suppose to have a homogeneous vector field of degree $\nu + 1$,

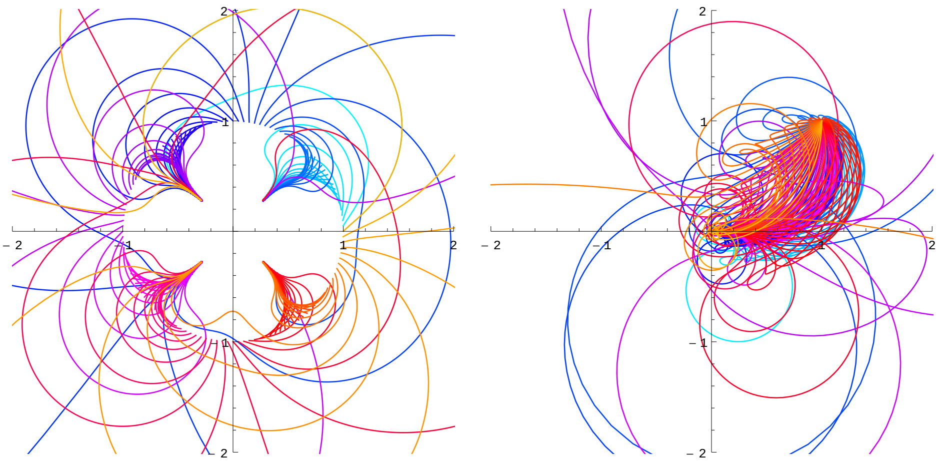
$$\begin{aligned} Q(z,w) =& (a_{\nu+1}z^{\nu+1} + a_\nu z^\nu w + \cdots + a_0 w^{\nu+1})\frac{\partial}{\partial z} \\ &+ (b_{\nu+1}z^{\nu+1} + b_\nu z^\nu w + \cdots + b_0 w^{\nu+1})\frac{\partial}{\partial w}, \end{aligned}$$

with a unique characteristic direction, that, without loss of generality, will be $[0, 1]$. It is easy to see that the same argument used in Section 6.3.1 gives that $a_0 = 0$, $b_{\nu+1} \neq 0$ and $a_i = b_{i-1}$ for $i = 1, \ldots, \nu + 1$. Moreover, we can



$\zeta(t)$ $v(t)$

(a) $\zeta(0) = 1 + i, v(0) = \cos\theta + i\sin\theta$



$\zeta(t)$ $v(t)$

(b) $\zeta(0) = \cos\theta + i\sin\theta, v(0) = 1 + i$

Figure 6.4: A singularity of irregularity 5

suppose that a non-zero coefficient is -1, and we set $b_{\nu+1} = -1$. So, Q must be of the form

$$Q(z,w) =(a_{\nu+1}z^{\nu+1} + a_\nu z^\nu w + \cdots + a_1 z w^\nu)\frac{\partial}{\partial z} + (-z^{\nu+1} + a_{\nu+1}z^\nu w + \cdots + a_1 w^{\nu+1})\frac{\partial}{\partial w}$$

and the geodesic field G, in the chart centered at $[0,1]$, takes the form

$$G = \zeta^{\nu+2} v\frac{\partial}{\partial\zeta} - \nu(\zeta^{\nu+1} - a_{\nu+1}\zeta^\nu - \cdots - a_1)v^2\frac{\partial}{\partial v}.$$

Renaming the coefficients in order to avoid confusion, we see that we have a geodesic field of the form

$$G = \zeta^{\nu+2} v\frac{\partial}{\partial\zeta} - \nu(\zeta^{\nu+1} + a_\nu\zeta^\nu + \cdots + a_0)v^2\frac{\partial}{\partial v} \tag{6.2}$$

for some complex numbers a_i, for $i = 0, \ldots, \nu$. In particular, the singularity has order $\nu + 2$, is non-degenerate if and only if $a_0 \neq 0$ and irregular if at least one of the a_i is non-zero.

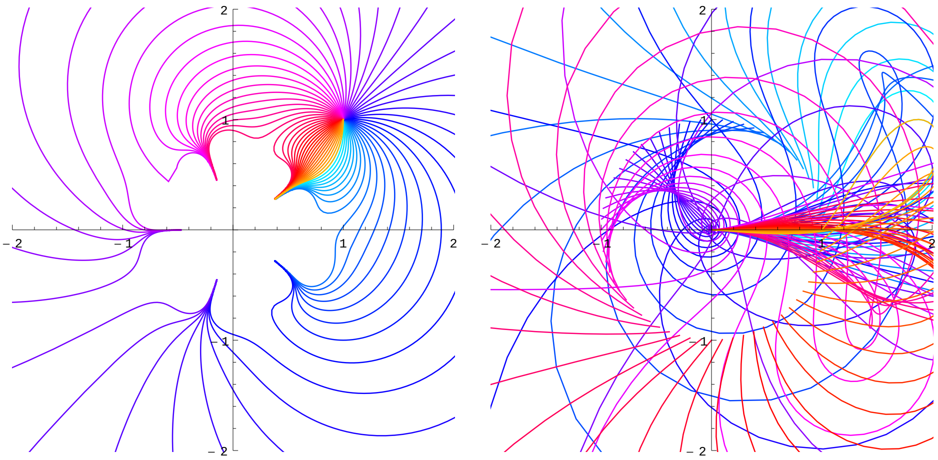
In the following we show a numerical study of the non-degenerate case for $\nu = 3$ and $\nu = 4$ (i.e., for fields of degree 4 and 5). We remark that Thereom 5.3.6 ([Viv11]) ensures the existence of an open basin of attraction for the origin in the case of degenerate singularities.

In Figure 6.4 we have considered the case $\nu = 3$, so that the irregularity of $[0, 1]$ is 5. We have used the parameters $a_0 = 1, a_1 = a_2 = 0$ and $a_3 = 0.3$. In Figure 6.4a we have drawn the coordinates $\zeta(t)$ and $v(t)$ of the integral curves issuing from $(1 + i, \cos\theta + i\sin\theta)$, with $\theta = 0, \frac{1}{32}, \ldots, 2\pi$, while in Figure 6.4b there are the integral curves issuing from $(\cos\theta + i\sin\theta, 1 + i)$, with $\theta = 0, \frac{\pi}{64}, \ldots 2\pi$. In particular, we recognise that the convergence to $\zeta = 0$ seems to happen tangent to four directions. Moreover, the integral curves definitively stay in one of the sectors of Proposition 4.2.19 and in fact we see that $v(t) \to 0$. We remark that there may exist exceptional curves whose first coordinate tends to the origin tangent to another direction, separating the projections on $\mathbb{P}^1(\mathbb{C})$ of the ones going to two different singularities. If the coordinate $z(t)$ of one of these exceptional curves goes to 0 in one of the sectors given by Proposition 4.2.19, we see that, for this curve, we must have $v(t) \to \infty$.

In Figure 6.5 we do the same for the case $\nu = 4$, i.e., for the unique singularity of a homogeneous vector field of degree 5. The geodesic field is

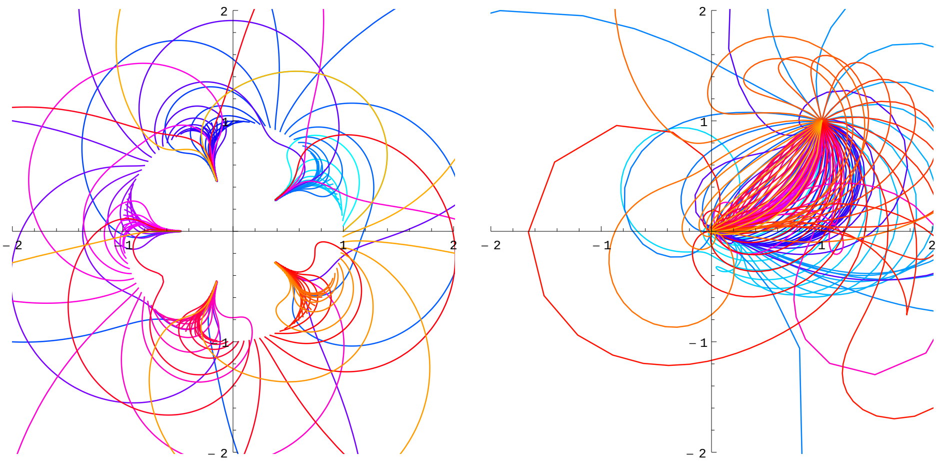
$$G = \zeta^6 v\frac{\partial}{\partial\zeta} - 4(\zeta^5 + 0.1\zeta^4 + 1)v^2\frac{\partial}{\partial v}.$$

The description is similar to the previous one, except for the fact that now there may be five directions of convergence for the projections of the geodesics.



$\zeta(t)$ $v(t)$

(a) $\zeta(0) = 1 + i, v(0) = \cos\theta + i\sin\theta$



$\zeta(t)$ $v(t)$

(b) $\zeta(0) = \cos\theta + i\sin\theta, v(0) = 1 + i$

Figure 6.5: A singularity of irregularity 6

The last thing we want to do is to see how the directions change as we change the parameters. Suppose all the coefficients in (6.2) are non-zero. In particular, we know that, since $a_0 \neq 0$, then the singularity is non-degenerate with irregularity $\nu + 2$ and (we my believe that) there are $\nu + 1$ directions of convergence to zero for $\zeta(t)$. If we let a_0 become zero, the singularity becomes degenerate, with irregularity $\nu + 1$ and the directions become ν. Now we can let a_1 become 0, and see that the irregularity decreases to ν (and the directions to $\nu - 1$).

In Figures 6.6 and 6.7 we precisely do this. We show, respectively, the integral curves issuing from $(1 + i, \cos\theta + i\sin\theta)$ and $(\cos\theta + i\sin\theta, 1 + i)$ for different geodesic fields, of the form

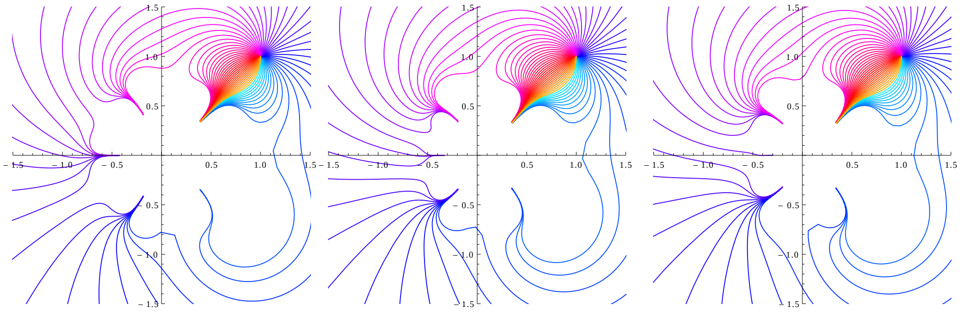
$$G = \zeta^6 v \frac{\partial}{\partial \zeta} - 4h_Y v^2 \frac{\partial}{\partial v},$$

where the h_Y is written under every image.

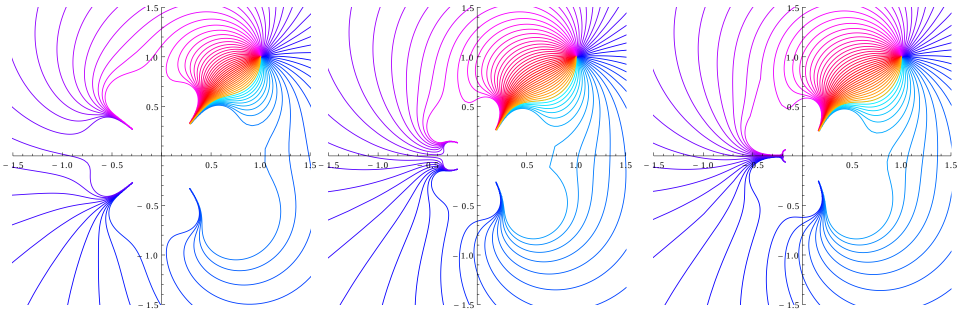
In particular we see that every time that the irregularity decreases, we have two directions that seem to collapse to a single one.

Once again, we remark that the study of the dynamics of the geodesics near irregular singularities is an open problem. There are partial results, as the ones already mentioned, but this case is not fully understood as the apparent and the Fuchsian one.

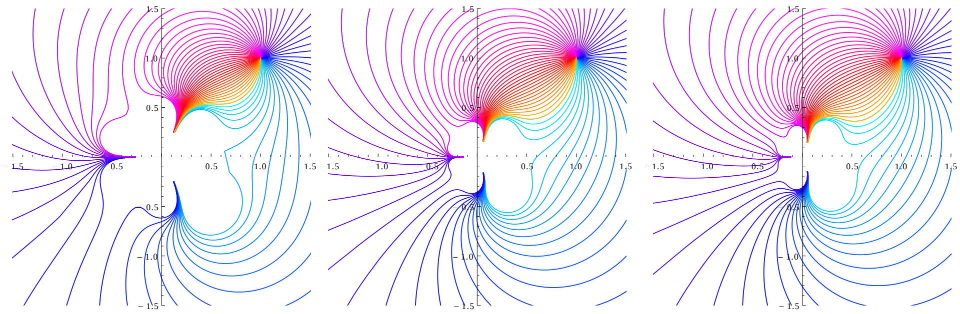
The existence of these directions of convergence is not proved, and pictures like the ones shown here cannot prove anything, but they can anyway give an insight about what is happening.



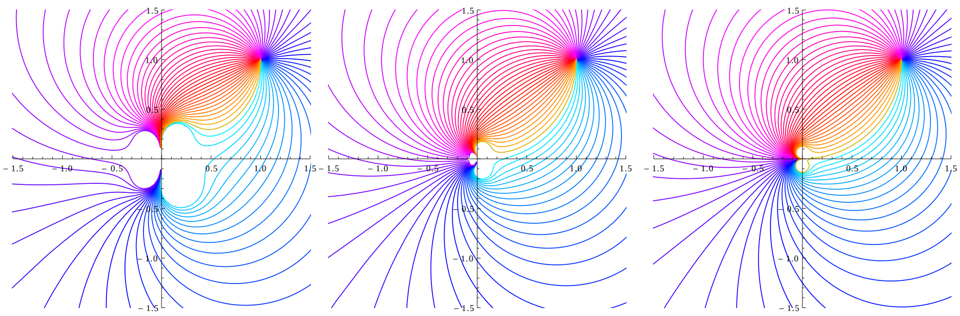
$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 1$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 0.5$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 0.4$



$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + 0.2\zeta$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + 0.1\zeta$

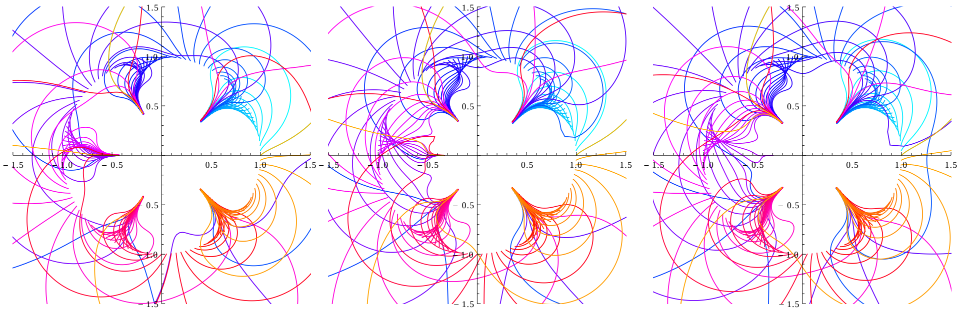


$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.1\zeta^2$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.07\zeta^2$

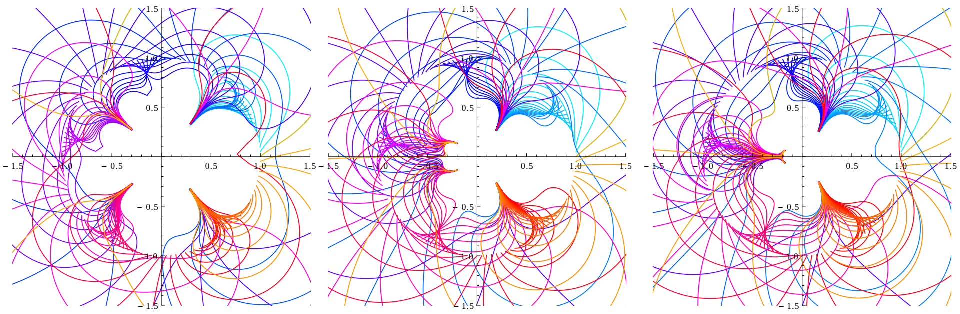


$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3$ $\quad$ $\zeta^5 + 0.1\zeta^4 + 0.02\zeta^3$ $\quad$ $\zeta^5 + 0.1\zeta^4$

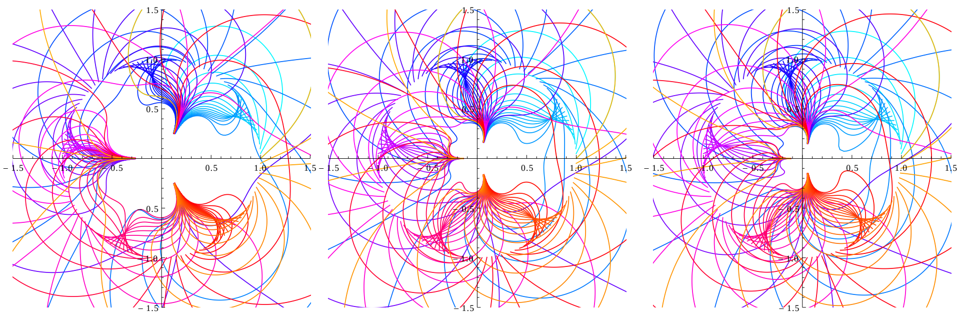
Figure 6.6: $\zeta(0) = 1 + i, v(0) = \cos\theta + i\sin\theta$



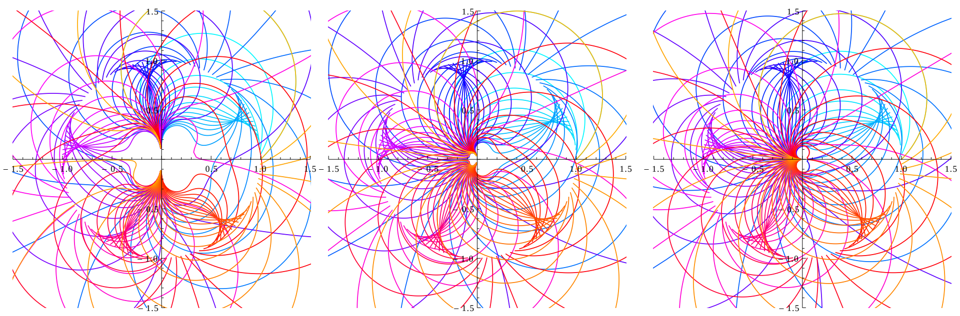
$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 1$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 0.5$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta + 0.4$



$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + \zeta$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + 0.2\zeta$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2 + 0.1\zeta$



$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.5\zeta^2$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.1\zeta^2$ $\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3 + 0.07\zeta^2$



$\zeta^5 + 0.1\zeta^4 + 0.2\zeta^3$ $\zeta^5 + 0.1\zeta^4 + 0.02\zeta^3$ $\zeta^5 + 0.1\zeta^4$

Figure 6.7: $\zeta(0) = \cos\theta + i\sin\theta, v(0) = 1 + i$

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