DEGENERATION OF QUADRATIC POLYNOMIAL ENDOMORPHISMS TO A HÉNON MAP

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ABSTRACT. For an algebraic family (f_t) of regular quadratic polynomial endomorphisms of $\mathbb{C}^2$ parametrized by $\mathbb{D}^*$ and degenerating to a Hénon map at $t = 0$, we study the continuous (and indeed harmonic) extendibility across $t = 0$ of a potential of the bifurcation current on $\mathbb{D}^*$ with the explicit computation of the non-archimedean Lyapunov exponent associated to (f_t). The individual Lyapunov exponents of f_t are also investigated near $t = 0$. Using (f_t), we also see that any Hénon map is accumulated by the bifurcation locus in the space of quadratic holomorphic endomorphisms of $\mathbb{P}^2$.

1. INTRODUCTION

Our aim is to study an algebraic family of regular quadratic polynomial endomorphisms of $\mathbb{C}^2$ parametrized by a punctured open disk and degenerating to a Hénon map at the puncture, paying a particular attention to the asymptotic behaviour of the individual Lyapunov exponents and their sums. For one dimensional meromorphic families of rational functions, such degenerations towards rational functions of lower (topological) degrees have been studied in [DeM16, FG17, DO17]. Favre recently introduced a general framework to study such degenerations for holomorphic endomorphisms of $\mathbb{P}^k$ in [Fav16] and we here provide the first concrete study in dimension k higher than 1. We also study the geometry of the *bifurcation locus*, in the sense of [BBD18], in the space of quadratic holomorphic endomorphisms of $\mathbb{P}^2$. The geometry of the bifurcation locus near the line at infinity of the moduli space $\mathcal{M}_2 \cong \mathbb{C}^2$ of the quadratic rational functions on $\mathbb{P}^1$ and that near the hyperplane at infinity of the natural parameter space $\cong \mathbb{C}^3$ of quadratic polynomial skew products on $\mathbb{C}^2$ have been studied in [BG15] and [AB18], respectively.

Let us be more specific and precisely state our results. In the rest of this article, we fix

$$c \in \mathbb{C}^* \quad \text{and} \quad p(w) = w^2 + c_1 w + c_2 \in \mathbb{C}[w].$$

For each

$$(g, h) \in \mathbb{C}[z, w] \times \mathbb{C}[z, w] \quad \text{such that } \deg g = 2,\ g_{zz} \in \mathbb{C}^*, \text{ and } \deg h \leq 2, \tag{1.1}$$

we focus on the algebraic family

$$f_t(z, w) = f_t(z, w; g, h) = \begin{pmatrix} w \\ cz + p(w) \end{pmatrix} + t \begin{pmatrix} g(z, w) \\ h(z, w) \end{pmatrix}, \quad t \in \mathbb{D}, \tag{1.2}$$

of quadratic polynomial endomorphisms of $\mathbb{C}^2$ parametrized by $\mathbb{D}$; we also set the constants $g_{zz}/2! =: G_g = G \in \mathbb{C}^*$ and $h_{zz}/2! =: H_h = H \in \mathbb{C}$, respectively, and set

$$\tilde{g}(z, w) := g(z, w) - Gz^2.$$

For $t = 0$, the map $f_0(z, w) = (w, cz + p(w))$ is a Hénon map (see e.g. [FM89]) and is independent of (g, h). For every $0 < |t| \ll 1$, f_t satisfies the condition

$$\liminf_{\|(z,w)\| \to \infty} \frac{\|f_t(z, w)\|}{\|(z, w)\|^2} > 0 \tag{1.3}$$

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(here $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^2$), or equivalently, f_t is a *regular* quadratic polynomial endomorphism of $\mathbb{C}^2$, that is, it extends to a holomorphic endomorphism of $\mathbb{P}^2$ (see e.g. [BJ00]). In particular, if $0<|t|\ll 1$, then f_t admits the (second) Julia set J_{f_t}, which is contained in $\mathbb{C}^2$ and coincides with the support of the unique maximal entropy measure μ_{f_t} of (the holomorphic extension to $\mathbb{P}^2$ of) f_t, and the sum $L(f_t)$ of the two *individual* Lyapunov exponents $\chi_1(f_t)\ge\chi_2(f_t)$(indeed $\ge\log\sqrt{2}$ [BD99]) of f_t with respect to μ_{f_t} is given by

$$L(f_t)=\int_{\mathbb{C}^2}\log|\det Df_t|\mu_{f_t}\in\mathbb{R}. \tag{1.4}$$

We also call $L(f_t)$ *the Lyapunov exponent* of f_t with respect to μ_{f_t}. Here and below, we fix the trivialization of the tangent bundle $T\mathbb{C}^2$ of $\mathbb{C}^2$ induced by the orthonormal frame (∂_z,∂_w) of $T\mathbb{C}^2$, and identify the derivative df_t of f_t with the $M(2,\mathbb{C})$-valued function $(z,w)\mapsto\det(Df_t)_{(z,w)}$, by convention.

We regard the set of all (g,h) as in (1.1) as $(\mathbb{C}^*\times\mathbb{C}^2)\times\mathbb{C}^3$, parametrizing it by the coefficients of g,h to mention the local uniformity of the estimates.

1.1. **Degeneration of the Lyapunov exponent.** Our first interest is in the asymptotic behaviour of $L(f_t)$ as $t\to 0$, where $f_t=f_t(z,w;g,h)$, for each (g,h) as in (1.1). Such a behaviour has been studied for meromorphic families of rational functions on $\mathbb{P}^1$ by DeMarco [DeM16]. In our situation, it follows from Favre's generalization [Fav16] of DeMarco's estimate that there is a non-negative constant α such that

$$L(f_t)=\alpha\log|t|^{-1}+o\big(\log|t|^{-1}\big)\quad\text{as } t\to 0$$

and that the constant α is characterized as the *non-archimedean* Lyapunov exponent associated to the meromorphic family $(f_t)_{t\in\mathbb{D}^*}$, regarded as a single rational function defined over a field of formal Laurent series at $t=0$. The function $t\mapsto L(f_t)-\alpha\log|t|^{-1}$ is continuous and subharmonic on $0<|t|\ll 1$ (see e.g. [DS10] for more details), and is a potential of the *bifurcation current* (indeed *measure*) on $0<|t|\ll 1$ associated to the family (f_t) in the sense of [BBD18].

In the following, it is convenient to say that the pair (g,h) (or the associated $(G,H)=(G_g,H_h)$) is *non-exceptional* if

$$\left|\frac{H}{G}\right|\neq|c|. \tag{1.5}$$

Our first principal result answers affirmatively Favre's general question [Fav16, Problem 1] in our context by establishing the continuous (and indeed harmonic) extendibility of the potential $t\mapsto L(f_t)-\alpha\log|t|^{-1}$ across $t=0$, with the concrete value $\alpha=1/2$, for any non-exceptional (g,h).

Theorem 1. *(i) Pick (g_0,h_0) as in (1.1). Then for every $\beta>0$ small enough, we have*

$$\log\frac{(1-\beta)\cdot 4\big||H/G|-|c|\big|^{1/2}}{|G|^{1/2}}\le L(f_t)-\frac{1}{2}\log|t|^{-1}\le\log\frac{(1+\beta)\cdot 4\big(|H/G|+|c|\big)^{1/2}}{|G|^{1/2}} \tag{1.6}$$

for every $t\in\mathbb{D}^*$ close enough to 0 and every (g,h) close enough to (g_0,h_0), recalling that $f_t=f_t(z,w;g,h)$ and $(G,H)=(G_g,H_g)$. In particular, for every (g,h), the non-archimedean Lyapunov exponent α associated to the meromorphic family $(f_t)_{t\in\mathbb{D}^*}$ equals $1/2$.

(ii) Pick a non-exceptional (g,h). Then for the algebraic family $(f_t)_{t\in\mathbb{D}^*}$ in (1.2) associated to this (g,h), the continuous and subharmonic function $t\mapsto L(f_t)-(1/2)\log|t|^{-1}$ is harmonic on $0<|t|\ll 1$ and extends harmonically across $t=0$, satisfying

$$\lim_{t\to 0}\Big(L(f_t)-\frac{1}{2}\log|t|^{-1}\Big)=\log\frac{4\max\big\{|c|,|H/G|\big\}^{1/2}}{|G|^{1/2}}. \tag{1.7}$$

Notice that a similar continuous extendability result has been obtained for meromorphic families of polynomials in one variable by Favre–Gauthier [FG17] and that examples of discontinuity at $t=0$ have been obtained in [DO17] for meromorphic families of rational functions on $\mathbb{P}^1$.

1.2. **Accumulation of the bifurcation locus to the Hénon locus.** For a holomorphic family of rational functions on $\mathbb{P}^1$, the theory of J-stability/bifurcation, originating from the seminal papers by Mañé-Sad-Sullivan [MSS83], Lyubich [Lyu83], DeMarco [DeM01, DeM03], is now classical. A generalization of this theory to holomorphic families of holomorphic endomorphisms of $\mathbb{P}^k$ parametrized by a complex manifold M was recently developed in [BBD18, Bia16]. There the *bifurcation locus* in the parameter space M is *defined* as the support of the *bifurcation current* on M; the bifurcation current on M is the dd^c of the Lyapunov exponent function $\lambda \mapsto L(f_\lambda) := \int_{\mathbb{P}^k} \log|\det Df_\lambda|\mu_{f_\lambda}$ on M, where μ_{f_λ} is the unique maximal entropy measure of f_λ on $\mathbb{P}^k$. In dimension $k=2$, if in addition M is simply connected, then the bifurcation locus coincides with, e.g., the complement in M of the locus where all the repelling cycles of f_λ in the second Julia sets J_{f_λ} of f_λ move holomorphically. We refer to [BBD18] for more details, and to [Bia16] for an analogous (slightly weaker) characterization valid in any dimension k.

Let us now focus on the family

$$f_{t,G,H}(z,w) := \begin{pmatrix} w \\ cz + p(w) \end{pmatrix} + t\begin{pmatrix} Gz^2 \\ Hz^2 \end{pmatrix}, \quad (t,G,H) \in \mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}, \tag{1.8}$$

of quadratic polynomial endomorphisms of $\mathbb{C}^2$, which are regular for every $t \in \mathbb{D}^*$ since the leading homogeneous term $(Gz^2, w^2 + Hz^2)$ of $f_{t,G,H}$ maps only $(0,0)$ to $(0,0)$.

The following is our second principal result.

Theorem 2. *The bifurcation locus in the parameter space $\mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}$ of the family $(f_{t,G,H})$ accumulates to $\{t=0\}$ in $\mathbb{D} \times \mathbb{C}^* \times \mathbb{C}$ tangentially to the locus $|H/G| = |c|$ in $\mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}$.*

The set $\mathrm{Hol}_2(\mathbb{P}^2)$ of all quadratic holomorphic endomorphisms of $\mathbb{P}^2$ is a Zariski open subset in $\mathbb{P}^{N_2}$, where $N_2 = 3 \cdot 4!/(2!2!) - 1 = 17$, in the coefficients parametrization, and in turn is regarded as the parameter space of the holomorphic family of (all) quadratic holomorphic endomorphisms of $\mathbb{P}^2$. We also note that all Hénon maps live in $\mathbb{P}^{N_2} \setminus \mathrm{Hol}_2(\mathbb{P}^2)$ in the coefficients parametrization.

The following immediate consequence of Theorem 2 is also one of our principal results, and answers affirmatively a question by Johan Taflin.

Corollary. *The Hénon locus in $\mathbb{P}^{N_2} \setminus \mathrm{Hol}_2(\mathbb{P}^2)$ is accumulated by the bifurcation locus of $\mathrm{Hol}_2(\mathbb{P}^2)$.*

The proof of Theorem 2 (so that of Corollary) is based on Theorem 1 and is purely analytical. In former studies, the presence of bifurcations has been established by means of more geometric arguments; see e.g. [BT17, Duj17, BB16, Taf17, AB18, Bie18].

1.3. **Individual Lyapunov exponents.** It follows from a result by Pham [Pha05] that, for any holomorphic family $(f_t)_{t\in M}$ parametrized by a complex manifold M of holomorphic endomorphisms of $\mathbb{P}^k$ of a given degree $d > 1$, if we denote by $\chi_1(f_t) \geq \cdots \geq \chi_k(f_t)$ all the individual Lyapunov exponents of f_t for each $t \in M$, then for every $j \in \{1,\ldots,k\}$, the function $t \mapsto \sum_{\ell=1}^{j} \chi_\ell(f_t)$ on M is plurisubharmonic.

Let us focus on our family (f_t) as in (1.2). Recall that the function $t \mapsto L(f_t) \equiv \chi_1(f_t) + \chi_2(f_t)$ is continuous and subharmonic on $0 < |t| \ll 1$, so by the above result by Pham, the function $t \mapsto \chi_1(f_t)$ is subharmonic and the function $t \mapsto \chi_2(f_t)$ is lower semicontinuous, in general. We conclude this introduction with the following precision of Theorem 1(ii).

Theorem 3. *Pick a non-exceptional (g,h). Then for the algebraic family $(f_t)_{t\in\mathbb{D}^*}$ in (1.2) associated to this (g,h), the functions $t \mapsto \chi_1(f_t) - \frac{1}{2}\log|t|^{-1}$ and $t \mapsto \chi_2(f_t)$ are harmonic on $0 < |t| \ll 1$ and extend harmonically across $t = 0$, satisfying*

$$\lim_{t\to 0}\Big(\chi_1(f_t) - \frac{1}{2}\log|t|^{-1}\Big) = \log\frac{2\max\big\{|c|, |H/G|\big\}^{1/2}}{|G|^{1/2}} \quad \textit{and} \quad \lim_{t\to 0}\chi_2(f_t) = \log 2.$$

The proof of the harmonicity of $\chi_1(f_t), \chi_2(f_t)$ is based on the full strength of Berteloot–Dupont–Molino's approximations, that is, approximations of not only $\chi_1(f_t) + \chi_2(f_t)(= L(f_t))$ but also $\chi_1(f_t)$ [BDM08, Theorem 1.5].

1.4. **Organization of the article.** In Section 2, we establish a key estimate, which is a development of an estimate appearing in [Duj17]. In Section 3, we show Theorem 1. In Section 4, we show Theorem 2 (and Corollary) using Theorem 1(i). We also include a comparison of our analysis here with the study of bifurcations of quadratic polynomial skew products in [AB18]. In Section 5, we show Theorem 3, which is a precision of Theorem 1(ii). To make such a precision, we also recall some standard facts from ergodic theory as well as Berteloot–Dupont–Molino's approximations.

2. A key lemma

In this section we show a lemma concerning the position of the (second) Julia set J_{f_t} of f_t, which is needed in the sequel and inspired by [Duj17, Lemma 5.2]. From now on, set $B(r) := \{z \in \mathbb{C} : |z| < r\}$ for every $r > 0$ and $A(r,s) := \{z \in \mathbb{C} : r < |z| < s\}$ for any $r, s \in \mathbb{R}$ satisfying $0 < r < s$; as convention, we also set $A(0,s) := B(s)$ for every $s > 0$.

Lemma 2.1. *Pick $(g,h) = (g_0,h_0)$ as in (1.1). Then there is $\beta \in (0,1/2)$ so small that for every (g,h) close enough to (g_0,h_0) and every $t \in \mathbb{D}^*$ close enough to 0, recalling that $f_t = f_t(z,w;g,h)$ and $(G,H) = (G_g,H_g)$ and setting*

$$U_t = U_t(\beta; g,h) := A\left(\frac{1-\beta}{|Gt|}, \frac{1+\beta}{|Gt|}\right) \times B\left(\frac{\big(|H/G| + |c|\big)^{1/2}(1+2\beta)}{|Gt|^{1/2}}\right) \quad \text{and}$$

$$V_t = V_t(\beta; g,h) := A\left(\frac{1-\beta}{|Gt|}, \frac{1+\beta}{|Gt|}\right) \times A\left(\frac{\big||H/G| - |c|\big|^{1/2}(1-2\beta)}{|Gt|^{1/2}}, \frac{\big(|H/G| + |c|\big)^{1/2}(1+2\beta)}{|Gt|^{1/2}}\right),$$

we have $f_t^{-1}(U_t) \Subset V_t$, and in particular, $J_{f_t} \subset V_t$.

Proof. Pick (g_0,h_0). Let us see the former assertion. Suppose to the contrary that there exist

- a sequence (β_n) in $(0,1)$ tending to 0 as $n \to \infty$,
- a sequence $((g_n,h_n))$ tending to (g_0,h_0) as $n \to \infty$,
- a sequence (t_n) in $\mathbb{D}^*$ tending to 0 as $n \to \infty$, and
- a sequence $((z_n,w_n))$ in $\mathbb{C}^2$

such that for every $n \in \mathbb{N}$, we have $(z_n,w_n) \in \mathbb{C}^2 \setminus V_{t_n}(\beta_n; g_n,h_n)$ and

$$(u_n,v_n) := f_{t_n}(z_n,w_n;g_n,h_n) \in \overline{U_{t_n}(\beta_n;g_n,h_n)}.$$

Claim. *As $n \to \infty$, $w_n + t_n\tilde{g}_n(z_n,w_n) = o\big(t_nz_n^2\big)$.*

Proof. Otherwise, taking a subsequence if necessary, there exists $C > 0$ such that for every $n \in \mathbb{N}$,

$$|w_n + t_n\tilde{g}_n(z_n,w_n)| \geq C|t_nz_n^2|. \tag{2.1}$$

Let us first see that, taking a further subsequence if necessary, there exists $C' > 0$ such that for every $n \in \mathbb{N}$,

$$\max\{|w_n|, |t_nz_nw_n|, |t_nw_n^2|\} \geq C' \max\{|t_nz_n^2|, |t_n|^{-1}\}; \tag{2.1$'$}$$

indeed, taking a subsequence if necessary, there are exactly two possibilities;

(a) if $|t_nz_n^2| \geq |t_n|^{-1}$ for every $n \in \mathbb{N}$, then also by (2.1), we have $|w_n + t_n\tilde{g}_n(z_n,w_n)| \geq C\max\{|t_nz_n^2|, |t_n|^{-1}\}$, so if $n \gg 1$, then $\max\{1, |g_{zw}|, |g_{ww}|/2\}\max\{|w_n|, |t_nz_nw_n|, |t_nw_n^2|\} \geq C\max\{|t_nz_n^2|, |t_n|^{-1}\}$ (recall $\tilde{g}_n(z,w) := g_n(z,w) - G_nz^2$ and $g_{zw}, g_{ww} \in \mathbb{C}$), which yields (2.1$'$) in this case.

(b) If $|t_nz_n^2| \leq |t_n|^{-1}$ for every $n \in \mathbb{N}$ but, to the contrary, $\max\{|w_n|, |t_nz_nw_n|, |t_nw_n^2|\} = o(t_n^{-1})$ as $n \to \infty$, then $|z_n| \leq |t_n|^{-1}$ for every $n \in \mathbb{N}$, and then under the assumption (2.1), recalling $\tilde{g}_n(z,w) := g_n(z,w) - G_nz^2$, we also have $t_nz_n^2 = o(t_n^{-1})$ as $n \to \infty$, that is, $z_n = o(t_n^{-1})$ as $n \to \infty$. Then we must have $u_n = w_n + t_ng(z_n,w_n) = o(t_n^{-1})$ as $n \to \infty$, which contradicts $(u_n,v_n) \in \overline{U_{t_n}(\beta_n;g_n,h_n)}$ for every $n \in \mathbb{N}$. Hence (2.1$'$) also holds in this case.

Once (2.1′) is at our disposal, we can deduce a contradiction as follows. Taking a subsequence if necessary, there are exactly three possibilities.

(1) If $|w_n| \geq \max\{|t_n z_n w_n|, |t_n w_n^2|\}$ for every $n \in \mathbb{N}$, then by (2.1′), we have $|w_n| \geq C'|t_n|^{-1}$ and, moreover, $|z_n|^2 \leq |t_n^{-1} w_n|/C' \leq (|w_n|/C')^2$, that is, $|z_n| \leq |w_n|/C'$ for every $n \in \mathbb{N}$. Then $v_n := cz_n + p(w_n) + t_n h(z_n, w_n) = (1 + o(1))w_n^2$ as $n \to \infty$, so by $(u_n, v_n) \in \overline{U_{t_n}(\beta_n; g_n, h_n)}$, we have $w_n = O(|t_n|^{-1/4})$ as $n \to \infty$. Then we must have $0 < C' \leq |t_n^{-1} w_n| = O(|t_n|^{3/4}) \to 0$ as $n \to \infty$, which is impossible.

(2) If $|t_n z_n w_n| \geq \max\{|w_n|, |t_n w_n^2|\}$ for every $n \in \mathbb{N}$, then for every $n \in \mathbb{N}$, we have $|z_n| \geq |t_n|^{-1}$ and, by (2.1′), also have $|z_n| \leq |w_n|/C'$, so that $|t_n z_n^2| \leq |t_n w_n^2|/(C')^2$ and $|t_n z_n w_n| \leq |t_n w_n^2|/C'$. Then $v_n := cz_n + p(w_n) + t_n h(z_n, w_n) = (1 + o(1))w_n^2$ as $n \to \infty$, so by $(u_n, v_n) \in \overline{U_{t_n}(\beta_n; g_n, h_n)}$, we have $w_n = O(|t_n|^{-1/4})$ as $n \to \infty$. Then we must have $1 \leq |t_n z_n| \leq |t_n w_n|/C' = O(|t_n|^{3/4}) \to 0$ as $n \to \infty$, which is impossible.

(3) If $|t_n w_n^2| \geq \max\{|w_n|, |t_n z_n w_n|\}$ for every $n \in \mathbb{N}$, then for every $n \in \mathbb{N}$, we have $|w_n| \geq |t_n|^{-1}$, and by (2.1′), also have $|w_n| \geq \sqrt{C'}|z_n|$, so that $|t_n z_n^2| \leq |t_n w_n^2|/C'$ and $|t_n z_n w_n| \leq |t_n w_n^2|/\sqrt{C'}$. Then $v_n := cz_n + p(w_n) + t_n h(z_n, w_n) = (1 + o(1))w_n^2$ as $n \to \infty$, so by $(u_n, v_n) \in \overline{U_{t_n}(\beta_n; g_n, h_n)}$, we have $w_n = O(|t_n|^{-1/4})$ as $n \to \infty$. Then we must have $1 \leq |t_n w_n| = O(|t_n|^{3/4}) \to 0$ as $n \to \infty$, which is impossible. Hence the claim holds. □

For every $n \in \mathbb{N}$ large enough, by the equality $u_n = w_n + G_n t_n z_n^2 + t_n \tilde{g}_n(z_n, w_n)$ and the above Claim, we have $z_n = (1 + o(1))(u_n/(G_n t_n))^{1/2}$ as $n \to \infty$, so that recalling that $(u_n, v_n) \in \overline{U_{t_n}(\beta_n; g_n, h_n)}$ and $\beta_n \searrow 0$, we have

$$z_n \in A\left(\frac{(1 + o(1))(1 - \beta_n)^{1/2}}{|G_n t_n|}, \frac{(1 + o(1))(1 + \beta_n)^{1/2}}{|G_n t_n|}\right) \subset A\left(\frac{1 - \beta_n}{|G_n t_n|}, \frac{1 + \beta_n}{|G_n t_n|}\right).$$

Moreover, since $v_n = cz_n + p(w_n) + t_n h(z_n, w_n)$ and $(u_n, v_n) \in \overline{U_{t_n}(\beta_n; g_n, h_n)}$ for every $n \in \mathbb{N}$, we have

$$w_n^2 + c_1 w_n = v_n - t_n h(z_n, w_n) - cz_n - c_2 = -t_n H_n z_n^2 - cz_n + o(t_n^{-1}) \quad \text{as } n \to \infty.$$

Since $(z_n, w_n) \notin V_{t_n}(\beta_n; g_n, h_n)$ for every $n \in \mathbb{N}$, taking a subsequence if necessary, there are two possibilities;

(i) if $|w_n| \leq \big||H_n/G_n| - |c|\big|^{1/2}(1 - 2\beta_n)/|G_n t_n|^{1/2}$ for every $n \in \mathbb{N}$, then $w_n = O(t_n^{-1/2})$ as $n \to \infty$, and then $w_n^2 = -t_n H_n z_n^2 - cz_n + o(t_n^{-1})$ as $n \to \infty$.

(ii) Alternatively, if $|w_n| \geq (|H_n/G_n| + |c|)^{1/2}(1 - 2\beta_n)/|G_n t_n|^{1/2}$ for every $n \in \mathbb{N}$, then $w_n = o(w_n^2)$ as $n \to \infty$, and then $w_n^2 = (-t_n H_n z_n^2 - cz_n + o(t_n^{-1}))/(1 + o(1))$ as $n \to \infty$.

So in any case,

$$w_n^2 = (1 + o(1))(-t_n H_n z_n^2 - cz_n + o(t_n^{-1})) \quad \text{as } n \to \infty.$$

Hence recalling that $\beta_n \searrow 0$, we also have

$$\frac{\big||H_n/G_n| - |c|\big|(1 - 2\beta_n)^2}{|G_n t_n|} < |w_n|^2 < \frac{\big(|H_n/G_n| + |c|\big)(1 + 2\beta_n)^2}{|G_n t_n|}$$

for every $n \in \mathbb{N}$ large enough. Hence we must have $(z_n, w_n) \in V_{t_n}(\beta_n; g_n, h_n)$. This gives the desired contradiction.

Once the former assertion is at our disposal, the latter assertion follows from, e.g., the fact that $\mu_{f_t} = \lim_{n\to\infty} 2^{-2n}(f_t^n)^*\Omega$ weakly on $\mathbb{C}^2$ for any smooth probability measure Ω compactly supported in $\mathbb{C}^2$ (see for instance [DS10]). □

3. Proof of Theorem 1

Let us see the former assertion (i). Pick $(g, h) = (g_0, h_0)$ as in (1.1). By Lemma 2.1, we can fix $\beta \in (0, 1/2)$ so small that for every (g, h) close enough to (g_0, h_0) and every $t \in \mathbb{D}^*$ close enough to 0, recalling that $f_t = f_t(z,w;g,h)$, we have $J_{f_t} \subset V_t (= V_t(\beta; g,h))$, so that

$$L(f_t) = \int_{V_t} \log|\det Df_t| \mu_{f_t}(z,w).$$

For every $(z,w) \in \mathbb{C}^2$, we compute as

$$\begin{aligned}\det(Df_t)_{(z,w)} &= \det\begin{pmatrix} tg_z & 1+tg_w \\ c+th_z & p'(w)+th_w\end{pmatrix} = \det\begin{pmatrix} 2tGz+t\tilde{g}_z & 1+tg_w \\ c+th_z & (2w+c_1)+th_w\end{pmatrix} \\ &= 4tGzw + 2tGzc_1 + t\tilde{g}_z(p'(w)+th_w) - (1+tg_w)(c+th_z) \\ &= 4tGzw\cdot\left(1+\frac{2tGzc_1+t\tilde{g}_z(p'(w)+th_w)-(1+tg_w)(c+th_z)}{4tGzw}\right).\end{aligned}$$

From the definition of V_t, we have

$$\frac{4(1-\beta)\big||H/G|-|c|\big|^{1/2}(1-2\beta)}{|G|^{1/2}|t|^{1/2}} \le |4tGzw| \le \frac{4(1+\beta)\big(|H/G|+|c|\big)^{1/2}(1+2\beta)}{|G|^{1/2}|t|^{1/2}} \quad \text{on } V_t,$$

and moreover, $\sup_{V_t}|\det(Df_t)_{(z,w)} - 4tGzw| = O(1)$ as $t\to 0$, uniformly on (g,h) near (g_0,h_0). Hence increasing β slightly if necessary, we have

(3.1)

$$\frac{4(1-\beta)\big||H/G|-|c|\big|^{1/2}(1-2\beta)}{|G|^{1/2}|t|^{1/2}} \le |\det(Df_t)_{(z,w)}| \le \frac{4(1+\beta)\big(|H/G|+|c|\big)^{1/2}(1+2\beta)}{|G|^{1/2}|t|^{1/2}} \quad \text{on } V_t$$

for every $t \in \mathbb{D}^*$ close enough to 0 and every (g,h) close enough to (g_0,h_0). In particular, (1.6) holds, increasing β slightly if necessary.

Now let us see the convergence assertion in the latter assertion (ii). Pick a non-exceptional (g,h). From the above computation of $\det(Df_t)_{(z,w)}$, we also have

$$L(f_t) - \frac{1}{2}\log|t|^{-1} = \int_{V_t}\log|4tGzw|\mu_{f_t}(z,w) - \frac{1}{2}\log|t|^{-1} + o(1) \quad \text{as } t\to 0,$$

and we compute as

$$\begin{aligned}&\int_{V_t}\log|tzw|\mu_{f_t}(z,w) - \frac{1}{2}\log|t|^{-1} = -\frac{3}{2}\log|t|^{-1} + \int_{V_t}\log|z|\mu_{f_t}(z,w) + \int_{V_t}\log|w|\mu_{f_t}(z,w) \\ &= -\frac{3}{2}\log|t|^{-1} + \int_{V_t}\log|z|\mu_{f_t}(z,w) + \frac{1}{2}\int_{V_t}\log|w|^2\mu_{f_t}(z,w) \\ &= -\frac{3}{2}\log|t|^{-1} + \int_{V_t}\log|z|\mu_{f_t}(z,w) + \frac{1}{2}\int_{V_t}\left(\log|cz+tHz^2| + \log\left|1+\frac{o(t^{-1})}{cz+tHz^2}\right|\right)\mu_{f_t}(z,w) \\ &= \frac{3}{2}\left(\int_{V_t}\log|z|\mu_{f_t}(z,w) - \log|t|^{-1}\right) \\ &\quad + \frac{1}{2}\int_{V_t}\log|c+tHz|\mu_{f_t}(z,w) + \frac{1}{2}\int_{V_t}\log\left|1+\frac{o(t^{-1})}{cz+tHz^2}\right|\mu_{f_t}(z,w),\end{aligned}$$

where setting $(u,v) := f_t(z,w) \in J_{f_t} \subset V_t$, we also used the estimate

$$w^2 = -th(z,w) - cz + v - c_1 w - c_2 = -tHz^2 - cz + o(t^{-1}) \quad \text{as } t \to 0.$$

Since $(1-\beta) \le |tGz| \le (1+\beta)$ on V_t, we have

$$\int_{V_t}\log|z|\mu_{f_t}(z,w) - \log|t|^{-1} = -\log|G| + o(1) \quad \text{as } t\to 0$$

(as $\beta \to 0$). Similarly, under the assumption that (g,h) is non-exceptional, since

$$0 < \frac{\big||c|-|H/G|\big|}{|G|} \le \liminf_{t\to 0}\left|\frac{cz+tHz^2}{t^{-1}}\right| \le \limsup_{t\to 0}\left|\frac{cz+tHz^2}{t^{-1}}\right| \le \frac{|c|+|H/G|}{|G|}$$

(as $\beta \to 0$), we have

$$\int_{V_t} \log \left| 1 + \frac{o(t^{-1})}{cz + tHz^2} \right| \mu_{f_t}(z,w) = o(1) \quad \text{as } t \to 0.$$

It remains to show that $\int_{V_t} \log |c + tHz| \mu_{f_t}(z,w) = \log \max\{|c|, |H/G|\} + o(1)$ as $t \to 0$. Under the (linear) coordinates system change $(z,w) \mapsto (Z,W) = (Gtz, tw)$ on $\mathbb{C}^2$, setting

$$\tilde{f}_t(Z,W) := \begin{pmatrix} Z \\ W \end{pmatrix} \circ f_t \circ \begin{pmatrix} Z \\ W \end{pmatrix}^{-1} = \begin{pmatrix} Z^2 + GW + Gt^2 \tilde{g}\left(\frac{Z}{Gt}, \frac{W}{t}\right) \\ tp\left(\frac{W}{t}\right) + c\frac{Z}{Gt} + th\left(\frac{Z}{Gt}, \frac{W}{t}\right) \end{pmatrix},$$

$$\tilde{\mu}_t := \mu_{\tilde{f}_t} = \begin{pmatrix} Z \\ W \end{pmatrix}_* \mu_{f_t},$$

$$\tilde{U}_t := \begin{pmatrix} Z \\ W \end{pmatrix}(U_t) = A(1-\beta, 1+\beta) \times B\left(\frac{\left(|H/G| + |c|\right)^{1/2}(1+2\beta)}{|G|^{1/2}} |t|^{1/2} \right), \quad \text{and}$$

$$\begin{aligned} \tilde{V}_t :=& \begin{pmatrix} Z \\ W \end{pmatrix}(V_t) \\ =& A(1-\beta, 1+\beta) \times A\left(\frac{\left||H/G| - |c|\right|^{1/2}(1-2\beta)}{|G|^{1/2}} |t|^{1/2}, \frac{\left(|H/G| + |c|\right)^{1/2}(1+2\beta)}{|G|^{1/2}} |t|^{1/2} \right) \end{aligned}$$

and letting $p_1 : (Z,W) \mapsto Z$ be the projection to the first coordinate, we compute as

$$\begin{aligned} \int_{V_t} \log |c + tHz| \mu_{f_t}(z,w) &= \int_{\tilde{V}_t} \begin{pmatrix} Z \\ W \end{pmatrix}_* (\log |c + tHz| \mu_{f_t}(z,w)) \\ &= \int_{\tilde{V}_t} \log \left| c + \frac{H}{G} Z \right| \left(\begin{pmatrix} Z \\ W \end{pmatrix}_* \mu_{f_t} \right)(Z,W) = \int_{\tilde{V}_t} \log \left| c + \frac{H}{G} Z \right| \tilde{\mu}_t(Z,W) \\ &= \int_{A(1-\beta,1+\beta)} \log \left| c + \frac{H}{G} Z \right| ((p_1)_* \tilde{\mu}_t)(Z). \end{aligned}$$

Set $S^1_Z := \{Z \in \mathbb{C}_Z : |Z| = 1\}$. We claim that $\lim_{t \to 0} (p_1)_* \tilde{\mu}_t = m_{S^1_Z}$ weakly on $\mathbb{C}_Z$; let ν be any weak limit point of $\tilde{\mu}_t$ on $\mathbb{C}^2$ as $t \to 0$, which is supported by S^1_Z (as $\beta \to 0$). Set $\tilde{D}_t := \tilde{U}_t \cap (\mathbb{R}_{>0} \times \mathbb{C})$. For every $n \in \mathbb{N}$, if $0 < |t| \ll 1$, then by (3.1), we have $\det Df_t \neq 0$ on V_t (under the assumption that (g,h) is non-exceptional), so that $\tilde{f}^n_t : \tilde{f}^{-n}_t(\tilde{U}_t) \to \tilde{U}_t$ is an unbranched covering of degree 2^{2n}, $\tilde{f}^{-n}_t(\tilde{D}_t)$ consists of 2^{2n} analytic disks, $\tilde{f}^{-n}_t(\tilde{U}_t \setminus \tilde{D}_t)$ consists of 2^{2n} components, and for each component U of $\tilde{f}^{-n}_t(\tilde{U}_t \setminus \tilde{D}_t)$, $\tilde{\mu}_t(U) = 1/2^{2n}$ since $\tilde{\mu}_t(\tilde{U}_t \setminus \tilde{D}_t) = 1$ and $\tilde{f}^*_t \tilde{\mu}_t = 2^2 \tilde{\mu}_t$ on $\mathbb{C}^2$; moreover, as $t \to 0$, $\tilde{f}^{-n}_t(\tilde{D}_t)$ tends to the set of all 2^n-th roots of unity in $\mathbb{C}_Z$ and for each component V of $S^1_Z \setminus \{2^n\text{-th roots of unity}\}$, exactly $2^{2n}/2^n$ components of $\tilde{f}^{-n}_t(\tilde{U}_t \setminus \tilde{D}_t)$ tend to V (as $\beta \to 0$). Hence for every $n \in \mathbb{N}$ and every component V of $S^1_Z \setminus \{2^n\text{-th roots of unity}\}$, we have $\nu(V) = (2^{2n}/2^n) \cdot 1/2^{2n} = 1/2^n$, which implies that $(p_1)_* \nu = m_{S^1_Z}$ on $\mathbb{C}_Z$ (by Caratheodory's theorem). Hence the claim holds.

Once this claim is at our disposal, recalling also $J_{f_t} \subset V_t$, we have the desired convergence

$$\lim_{t \to 0} \int_{A(1-\beta,1+\beta)} \log \left| c + \frac{H}{G} Z \right| ((p_1)_* \tilde{\mu}_t)(Z) = \int_{\mathbb{C}_Z} \log \left| c + \frac{H}{G} Z \right| m_{S^1_Z}(Z) = \log \max \left\{ |c|, \left| \frac{H}{G} \right| \right\}$$

since the function $Z \mapsto \log\left|c + (H/G)Z\right|$ is continuous near S^1_Z under the assumption $|H/G| \neq |c|$.

Finally, let us see the harmonicity assertion of $t \mapsto L(f_t) - (1/2)\log|t|^{-1}$ on $0 < |t| \ll 1$ in the latter assertion (ii). If $0 < |t| \ll 1$, then by $f^{-1}_t(U_t) \Subset V_t$ and (3.1) (and the assumption that $|H/G| \neq |c|$), we have (not only $J_{f_t} \subset V_t$ but also) $V_t \cap \bigcup_{n \in \mathbb{N} \cup \{0\}} f^n_t(C_{f_t}) = \emptyset$, where $C_{f_t} := \{p \in \mathbb{C}^2 : \det(Df_t)_p = 0\}$ is the critical set of f_t. In particular, by [BBD18, (F)⇒(B) in Theorem 1.1], the function $t \mapsto L(f_t)$ is harmonic on $0 < |t| \ll 1$. □

4. Proof of Theorem 2

Let us first note that by an argument similar to that in the final paragraph in the proof of Theorem 1, for every $(G_0, H_0) \in \mathbb{C}^* \times \mathbb{C}$ satisfying the assumption $|H_0/G_0| \neq |c|$, there exists $0 < r_0 \ll 1$ such that for every (G, H) close enough to (G_0, H_0) and every $t \in \mathbb{D}^*_{r_0}$, the parameter (t, G, H) is not in the bifurcation locus in the parameter space $\mathbb{D}^* \times \mathbb{C}^* \times \mathbb{C}$ of the family $(f_{t,G,H})$. In particular, for every $G_0 \in \mathbb{C}^*$, every sequence (t_n) in $\mathbb{D}^*$ tending to 0 as $n \to \infty$, and every *bounded* sequence (H_n) in $\mathbb{C}$, if for every $n \in \mathbb{N}$, the function $H \mapsto L(f_{t_n,G_0,H})$ on $\mathbb{C}$ is not harmonic on any open neighborhood of $H = H_n$, then $\lim_{n\to\infty} |H_n/G_0| = |c|$.

Let us next see that for every $G_0 \in \mathbb{C}^*$ and every $R > 2|cG_0|$, if $0 < |t| \ll 1$, then the function $H \mapsto L(f_{t,G_0,H})$ is not harmonic on $\{|H| < R\}$; otherwise, there exist $R > 2|cG_0|$ and a sequence (t_n) in $\mathbb{D}^*$ tending to 0 as $n \to \infty$ such that for every $n \in \mathbb{N}$, the function $H \mapsto L(f_{t_n,G_0,H})$ is harmonic on the open disk $\{|H| < R\}$. Then using the estimate (1.6) of $L(f_{t,g_0,0})$ for $0 < |t| \ll 1$, the mean value theorem for the harmonic functions $H \mapsto L(f_{t_n,G_0,H})$ on $\{|H| < R\}$, and the (lower) estimate (1.6) of $L(f_{t,G_0,H})$ for $0 < |t| \ll 1$, which holds uniformly on the circle $\{|H| = R\}$, we must have

$$\begin{aligned}\log \frac{4|c|^{1/2}}{|G_0|^{1/2}} &= \lim_{n\to\infty}\Big(L(f_{t_n,G_0,0}) - \frac{1}{2}\log|t_n|^{-1}\Big)\\ &= \lim_{n\to\infty}\Big(\int_0^{2\pi} L\Big(f_{t_n,G_0,Re^{i\theta}}\Big)\frac{d\theta}{2\pi} - \frac{1}{2}\log|t_n|^{-1}\Big) \geq \log\frac{4\big|R/|G_0| - |c|\big|^{1/2}}{|G_0|^{1/2}} > \log\frac{4|c|^{1/2}}{|G_0|^{1/2}}\end{aligned}$$

(as $\beta \to 0$), which is impossible. Now the proof of Theorem 2 is complete. □

We conclude this section with a description of similarities between Theorem 2 and one of the main results in [AB18], where the accumulation of the bifurcation locus to the hyperplane at infinity for holomorphic families of quadratic polynomial skew products on $\mathbb{C}^2$ has been completely described.

A comparison with the bifurcation of polynomial skew products. Let us introduce in the family (1.8) (after the coordinate change $(z, w) \mapsto (Gtz, w)$) an extra parameter $\eta \in \{0, 1\}$ as

$$f_{t,G,H,\eta}(z, w) := \begin{pmatrix} z^2 + \eta \cdot tGw \\ w^2 + \frac{H}{tG^2}z^2 + \frac{c}{tG}z \end{pmatrix}, \tag{4.1}$$

so that the family (1.8) (after the above coordinate change) corresponds to the choice $\eta = 1$ of the parameter η. In the case $\eta = 0$, we get a family of regular quadratic *polynomial skew products* on $\mathbb{C}^2$ of the form studied in [AB18]. It is there proved by a method different from here and based on the characterization of stability by means of the boundedness of the critical orbits that the bifurcation locus in the parameter space $\mathbb{C}^3$ of the family $(z, w) \mapsto (z^2, w^2 + Az^2 + Bz + C)$ of quadratic polynomial skew products on $\mathbb{C}^2$ accumulates to the subset

$$\{[A, B, C] \in \mathbb{P}^2_\infty : Az^2 + Bz + C = 0 \text{ for some } z \in S^1\}$$

in the hyperplane at infinity $\mathbb{P}^2_\infty$ of $\mathbb{C}^3$, see [AB18, Theorem C]. Setting $\eta = 1$, $A = \frac{H}{tG^2}$, $B = \frac{c}{tG}$, and $C = 0$, the condition that $Az^2 + Bz + Cz = 0$ for some $z \in \mathbb{C}$ reduces to

$$c + \frac{H}{G}z = 0 \quad \text{for some } z \in S^1.$$

This is equivalent to the *bifurcation condition* $|H/G| = |c|$ that we found in Theorem 2.

The topologies of the (second) Julia sets. In [AB18, Theorem D], hyperbolic components in the parameter space $\mathbb{C}^3$ of the family $(z, w) \mapsto (z^2, w^2 + Az^2 + Bz + C)$ of quadratic polynomial skew products on $\mathbb{C}^2$ near the hyperplane at infinity $\mathbb{P}^2_\infty$ are also classified; the proof was based on a monodromy argument, and distinguished hyperbolic components in terms of the topologies of the (second) Julia sets.

In our situation $\eta = 1$ in (4.1), by a similar monodromy argument, it is also possible to observe that for every $G_0 \in \mathbb{C}^*$, the topologies of the (second) Julia set $J_{f_{t,G,H,1}}$ of $f_{t,G,H,1}$ at parameters $|H| \ll 1$ and $|H| \gg 1$ and $t \in \mathbb{D}^*$ close enough to 0 are incompatible, so that the two kinds of parameters (G_0, H, t_0) for $|H| \ll 1$ and $H \gg 1$ and $0 < |t_0| \ll 1$ cannot belong to the same *hyperbolic* component in the parameter space $\mathbb{C}$ of the family $(f_{t_0,G_0,H})_{H\in\mathbb{C}}$ (notice that this is not enough to provide a proof of Theorem 2).

Proposition 4.1. *Let $\mathcal{C}$ be a Cantor set. Then for every $(G_0, H_0) \in \mathbb{C}^* \times \mathbb{C}$, the following hold;*

(i) *when $|H_0| \ll 1$, for every $t \in \mathbb{D}^*$ close enough to 0 and every (G, H) close enough to (G_0, H_0), $J_{f_{t,G,H}}$ is a suspension of $\mathcal{C}$ set over S^1.*

(ii) *when $|H_0| \gg 1$, for every $t \in \mathbb{D}^*$ close enough to 0 and every (G, H) close enough to (G_0, H_0), $J_{f_{t,G,H}}$ is homeomorphic to $S^1 \times \mathcal{C}$.*

The proof is done by an argument similar to that in [AB18, Section 7] for the case of polynomial skew products, and we would thus omit it.

5. Proof of Theorem 3

Pick a non-exceptional (g, h) and let $f_t = f_t(z, w; g, h)$. Set $\mathcal{C}_\eta := \{(x, y) \in \mathbb{C}^2 : |y| > \eta|x| > 0\}$ for each $\eta > 0$. Recall that $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^2$ and let $p_2(z, w) = w$ be the projection to the second coordinate.

Claim. *For every $\delta \in (0, 1)$, there exists $\eta_0 > 0$ so large that for every $t \in \mathbb{D}^*$ close enough to 0 and every $(z, w) \in J_{f_t}$, the subset $\mathcal{C}_{\eta_0}$ is invariant under $(Df_t)_{(z,w)}$, that is, $(Df_t)_{(z,w)}(C_{\eta_0}) \subset C_{\eta_0}$, and for every $(x, y) \in \mathcal{C}_{\eta_0}$,*

$$(1 - \delta) \cdot |2w| \le \frac{\|(Df_t)_{(z,w)}(x, y)\|}{\|(x, y)\|} \le (1 + \delta) \cdot |2w|, \tag{5.1}$$

so in particular that, for every $n \in \mathbb{N}$,

$$\bigl(2(1 - \delta)\bigr)^n \prod_{j=0}^{n-1} \bigl|p_2(f_t^j(z, w))\bigr| \le \frac{\|D(f_t^n)_{(z,w)}(x, y)\|}{\|(x, y)\|} \le \bigl(2(1 + \delta)\bigr)^n \prod_{j=0}^{n-1} \bigl|p_2(f_t^j(z, w))\bigr|. \tag{5.1$'$}$$

Proof. For every $\alpha \in \mathbb{C}$, every $t \in \mathbb{D}^*$, and every $(z, w) \in \mathbb{C}^2$, we compute as

$$(Df_t)_{(z,w)} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 2tGz + t\tilde{g}_z & 1 + tg_w \\ c + th_z & (2w + c_1) + th_w \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 2tGz + t\tilde{g}_z + (1 + tg_w)\alpha \\ c + th_z + (2w + c_1 + th_w)\alpha \end{pmatrix}.$$

By Lemma 2.1, for every $\beta \in (0, 1)$ small enough, there exists $\eta_0 > 0$ so large that for every $t \in \mathbb{D}^*$ close enough to 0, every $(z, w) \in J_{f_t}$, and every $(1, \alpha) \in \mathcal{C}_{\eta_0}$, we have ($J_{f_t} \subset V_t$ and moreover) the estimates

$$\begin{aligned} |2tGz + t\tilde{g}_z + (1 + tg_w)\alpha| &\le (1 + 4^{-1}\beta) \cdot (1 + |g_{zw}/G|)|\alpha| \quad \text{and} \\ (1 - 4^{-1}\beta) \cdot 2|w\alpha| \le |c + th_z + (2w + c_1 + th_w)\alpha| &\le (1 + 4^{-1}\beta) \cdot 2|w\alpha|. \end{aligned}$$

So, in particular

$$(1 - \beta) \cdot \frac{2|w|}{1 + |g_{zw}/G|} \le \frac{|c + th_z + (2w + c_1 + th_w)\alpha|}{|2tGz + t\tilde{g}_z + (1 + tg_w)\alpha|} \quad \text{and}$$

$$(1 - 4^{-1}\beta)|2w\alpha| \le \bigl\|(Df_t)_{(z,w)}(1, \alpha)\bigr\| \le (1 + 4^{-1}\beta)\Bigl(\frac{1 + |g_{zw}/G|}{|2w|} + 1\Bigr)|2w\alpha|.$$

Noting also that $\inf_{(z,w)\in V_t} |w| = O(t^{-1/2})$ as $t \to 0$ (under the assumption that (g, h) is non-exceptional), the invariance $(Df_t)_{(z,w)}(C_{\eta_0}) \subset C_{\eta_0}$ holds and, for every $\delta \in (0, 1)$, we obtain the desired estimate (5.1), decreasing $\beta \in (0, 1)$ and increasing $\eta_0 > 0$ if necessary. $\square$

Let us first see the final convergence assertion on $\chi_1(f_t), \chi_2(f_t)$ as $t \to 0$. Recall two convergence results from ergodic theory. For $0 < |t| \ll 1$, by the Oseledec multiplicative ergodic theorem, for μ_{f_t}-almost every $p \in \mathbb{C}^2$, the limit $\Lambda_p := \lim_{n\to\infty}\big((Df_t^n)_p^*(Df_t^n)_p\big)^{1/(2n)}$ exists in $M(2,\mathbb{C}^2)$ with respect to the operator norm topology and has the two individual Lyapunov exponents $\chi_1(f_t) \ge \chi_2(f_t)$ of f_t as the two eigenvalues. Moreover, there is a canonical filtration $\mathbb{C}^2 \supset E_{2,p} \supset \{0\}$ of $\mathbb{C}^2$ by the invariant subspace $E_{2,p}$ under Λ_p such that

$$\lim_{n\to\infty}\frac{1}{n}\log\|D(f_t^n)_p(v)\| = \begin{cases}\chi_1(f_t) & \text{for every } v \in \mathbb{C}^2 \setminus E_{2,p},\\ \chi_2(f_t) & \text{for every } v \in E_{2,p}\setminus\{0\}.\end{cases}$$

On the other hand, for every $0 < |t| \ll 1$, by the Birkhoff ergodic theorem, for μ_{f_t}-almost every $(z,w) \in \mathbb{C}^2$, we also have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\big|p_2(f_t^j(z,w))\big| = \int_{\mathbb{C}^2}\log|w|\mu_{f_t}(z,w).$$

Once the Claim and the above two convergence results from ergodic theory are at our disposal, for every $\delta \in (0,1)$, there is $\eta_0 > 0$ so large that for every $0 < |t| \ll 1$, we have

$$\left|\chi_1(f_t) - \log 2 - \int_{V_t}\log|w|\mu_{f_t}(z,w)\right| \le \log(1-\delta)^{-1}.$$

On the other hand, in the proof of Theorem 1(ii), we have already seen that

$$\begin{aligned}&\int_{V_t}\log|w|\mu_{f_t}(z,w) - \frac{1}{2}\log|t|^{-1}\\ &= \frac{1}{2}\left(\int_{V_t}\log|z|\mu_{f_t}(z,w) - \log|t|^{-1}\right) + \frac{1}{2}\int_{V_t}\left(\log|c+tHz| + \log\left|1 + \frac{o(t^{-1})}{cz+tHz^2}\right|\right)\mu_{f_t}(z,w)\\ &= -\frac{1}{2}\log|G| + \frac{1}{2}\log\max\left\{|c|, \left|\frac{H}{G}\right|\right\} + o(1) \quad \text{as } t \to 0.\end{aligned}$$

Hence we have the convergence

$$\lim_{t\to 0}\left(\chi_1(f_t) - \frac{1}{2}\log|t|^{-1}\right) = \log\frac{2\max\{|c|, |H/G|\}^{1/2}}{|G|^{1/2}}$$

(as $\delta \to 0$), and in turn have the convergence $\chi_2(f_t) = L(f_t) - \chi_1(f_t) \to \log 2$ as $t \to 0$, also by Theorem 1(ii).

Now let us see the harmonicity assertion on $t \mapsto \chi_1(f_t), \chi_2(f_t)$. Recall that by Berteloot–Dupont–Molino [BDM08, Theorem 1.5], for every $0 < |t| \ll 1$,

$$\begin{cases}\lim_{n\to\infty} 2^{-2n}\sum_{p\in R(f_t^n)\cap J_{f_t}}\frac{1}{n}\log\|D(f_t^n)_p\| = \chi_1(f_t) \quad \text{and}\\ \lim_{n\to\infty} 2^{-2n}\sum_{p\in R(f_t^n)\cap J_{f_t}}\frac{1}{n}\log|\det(D(f_t^n)_p)| = \chi_1(f_t) + \chi_2(f_t),\end{cases}\tag{5.2}$$

where we denote by $R(f_t^n)$ the set of all repelling fixed points of f_t^n in $\mathbb{C}^2$ and by $\|D(f_t^n)_p\|$ the operator norm of the differential $D(f_t^n)_p$ for each $p \in R(f_t^n) \cap J_{f_t}$.

We also claim that there is $r_0 \in (0,1)$ so small that for every $t \in \mathbb{D}_{r_0}^*$, every $n \in \mathbb{N}$, and every $p \in R(f_t^n) \cap J_{f_t}$, the absolute values of the two eigenvalues of the differential $D(f_t^n)_p$ are different and > 1, so in particular letting $\lambda_{1,p,n}(t), \lambda_{2,p,n}(t)$ be the two eigenvalues of $D(f_t^n)_p$ such that $|\lambda_{1,p,n}(t)| > |\lambda_{2,p,n}(t)|$, the above approximations in (5.2) yield

$$\begin{cases}\lim_{n\to\infty} 2^{-2n}\sum_{p\in R(f_t^n)\cap J_{f_t}}\frac{1}{n}\log|\lambda_{1,p,n}(t)| = \chi_1(f_t) \quad \text{and}\\ \lim_{n\to\infty} 2^{-2n}\sum_{p\in R(f_t^n)\cap J_{f_t}}\frac{1}{n}\log|\lambda_{2,p,n}(t)| = \chi_2(f_t);\end{cases}\tag{5.2$'$}$$

indeed, by the above Claim, fixing $\delta \in (0,1)$ and then $\eta_0 > 0$ large enough, for every $0 < |t| \ll 1$ and every $p = (z,w) \in R(f_t^n) \cap J_{f_t}$, the Möbius transformation A induced by $D(f_t^n)_p \in \mathrm{GL}(2,\mathbb{C})$ on the projectivization $\mathbb{P}^1$ of $\mathbb{C}^2 \setminus \{(0,0)\}$ maps the open spherical disk $\mathcal{D}_{\eta_0}$ in $\mathbb{P}^1$ corresponding to the cone $\mathcal{C}_{\eta_0} \cup \{(x,y) : x = 0\}$ minus $\{(0,0)\}$ to a relatively compact subset in $\mathcal{D}_{\eta_0}$. This implies the existence of a fixed point of A in $\mathcal{D}_{\eta_0}$, and in turn that of an eigenvector v_1 of $D(f_t^n)_p$ in $\mathcal{C}_{\eta_0}$. Setting $(z_j, w_j) := f_t^{j-1}(p)$ for each $j \in \{1, \dots, n\}$, the eigenvalue λ_1 of $D(f_t^n)_p$ associated to v_1 satisfies $2(1+\delta)|w_1 \cdots w_n|^{1/n} \geq |\lambda_1|^{1/n} \geq 2(1-\delta)|w_1 \cdots w_n|^{1/n} \to \infty$ as $t \to 0$, and the other eigenvalue λ_2 of $D(f_t^n)_p$ then satisfies

$$\frac{2+o(1)}{1+\delta} = \left(\prod_{j=1}^{n} \frac{|4tGz_jw_j + O(1)|}{2(1+\delta)\cdot|w_j|}\right)^{1/n} \leq$$

$$\leq |\lambda_2|^{1/n} = \left(\frac{|\det D(f_t^n)_p|}{|\lambda_1|}\right)^{1/n} \leq \left(\prod_{j=1}^{n} \frac{|4tGz_jw_j + O(1)|}{2(1-\delta)\cdot|w_j|}\right)^{1/n} = \frac{2+o(1)}{1-\delta} \quad \text{as } t \to 0,$$

and both the divergence of $|\lambda_1|^{1/n}$ and the bounds of $|\lambda_2|^{1/n}$ as $t \to 0$ are uniform on n, p. Hence the claim holds.

Recall now that the function $t \mapsto L(f_t)$ is harmonic on $0 < |t| \ll 1$ (seen in Theorem 1(ii)), so that by [BBD18, (B)⇒(A) in Theorem 1.1], for any simply connected subdomain D in $0 < |t| \ll 1$, the set function $t \mapsto \bigcup_{n\in\mathbb{N}} R(f_t^n) \cap J_{f_t}$ is regarded as a holomorphic motion parametrized by D. Consequently, applying (a variant of) Harnack's theorem (see e.g. [Ran95, Theorem 1.3.10]) to the *convergent* sequences of *positive harmonic* functions on D in (5.2′), both the functions $t \mapsto \chi_1(f_t)$ and $t \mapsto \chi_2(f_t)$ are harmonic on D. □

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