Reducing subspaces of $C_{00}$ contractions

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Abstract. Using the Sz.-Nagy–Foias theory of contractions, we obtain
general results about reducibility for a class of completely nonunitary
contractions. These are applied to certain truncated Toeplitz operators,
previously considered by Li–Yang–Lu and Gu. In particular, a negative
answer is given to a conjecture stated by the latter.

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1. Introduction

We will denote by $L^2$ the Lebesgue space $L^2(\mathbb{T}, dm)$, where $dm$ is normalized
Lebesgue measure. The subspace of functions whose negative Fourier
coefficients are zero is denoted by $H^2$; it is identified with the space of analytic functions in the unit disc with square summable Taylor coefficients.
An inner function is an element of $H^2$ whose values have modulus 1 almost everywhere on $\mathbb{T}$.

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If \( \theta \) is an inner function, then the space \( K_\theta = H^2 \ominus \theta H^2 \) is usually called a model space; it has been the focus of much research, in function theory in the unit disc as well as in operator theory (see, for instance, [8, 6]; or [2] for a more recent account). In particular, in the last two decades several papers discuss the so-called truncated Toeplitz operators, introduced in [9], which are compressions to \( K_\theta \) of multiplication operators on \( L^2 \).

Originating with work in [1], the question of reducibility of a certain class of truncated Toeplitz operators has been recently investigated in papers by Yi, Yang, and Lu [4, 5] and Gu [3]. Besides certain remarkable results, they also contain intriguing questions that have not yet found their solution.

The current paper has several purposes. First, we put the problem of reducibility of the truncated Toeplitz operators in a larger context, that of the Sz.-Nagy–Foias theory of completely nonunitary contractions [7], and show that some results in the above quoted papers may be generalized or given more transparent proofs. Secondly, we answer in the negative a conjecture stated in [3] and prove a statement that replaces it.

The plan of the paper is the following. After presenting in the next section the elements of Sz.-Nagy theory that interest us, we obtain in Section 3 some general results about reducibility for completely nonunitary contractions. These results are applied in Section 4 to a certain class of truncated Toeplitz operators. The connection to [5] is achieved in Section 5, while the relation to [3] is the content of the last section.

2. Sz.-Nagy–Foias dilation theory

The general reference for this section is the monograph [7], in particular chapters I, II, and VI.

2.1. Minimal isometric dilation. If \( \mathcal{H} \) is a Hilbert space and \( \mathcal{H}_1 \) is a closed subspace, we will denote by \( P_{\mathcal{H}_1} \) the orthogonal projection onto \( \mathcal{H}_1 \).

A closed subspace \( M \) of \( \mathcal{H} \) is said to be reducing for an operator \( T \) if both \( M \) and \( M^\perp \) are invariant with respect to \( T \). A completely nonunitary contraction \( T \in \mathcal{L}(\mathcal{H}) \) is a linear operator that satisfies \( \|T\| \leq 1 \), and there is no reducing subspace of \( T \) on which it is unitary. The defect of \( T \) is the operator \( D_T = (I - T^*T)^{1/2} \), and the defect space is \( \mathcal{D}_T = D_T \mathcal{H} \).

We write \( T \in C_0 \) if \( T^n \) tends strongly to 0, and \( T \in C_{00} \) if \( T \) and \( T^* \) are in \( C_0 \), that is \( T^n \) and \( T^*n \) both tend strongly to 0. If \( T \in C_{00} \), then it can be shown that \( \dim \mathcal{D}_T = \dim \mathcal{D}_{T^*} \). The subclass of \( C_{00} \) for which this dimension is finite and equal to \( N \) is denoted by \( C_0(N) \). We will mostly be interested by contractions in the class \( C_{00} \).
An isometric dilation of $T$ is an isometric operator $V \in \mathcal{L}(\mathcal{K})$, with $\mathcal{K} \supset \mathcal{H}$, such that $P_H V^n | \mathcal{H} = T^n$ for any $n \in \mathbb{N}$. Note that if $T = P_H V | \mathcal{H}$ and $V H^\perp \subset H^\perp$, then $V$ is a dilation. An isometric dilation $V \in \mathcal{L}(\mathcal{K})$ is called minimal if $\mathcal{K} = \bigvee_{n=0}^\infty V^n \mathcal{H}$. This is uniquely defined, modulo a unitary isomorphism commuting with the dilations; in [7] there is a precise description of its geometric structure. This becomes simpler for contractions in $C_0$; since this is the only case we are interested in, we will describe the minimal isometric dilation in this case.

We will say that a subspace $X \subset \mathcal{K}$ is wandering for $V$ if $V^n X \perp V^m X$ for any $n \neq m$, and in this case we will denote $M_+(X) := \bigoplus_{n=0}^\infty V^n X$. Note that $M_+(X)$ is invariant with respect to $V$.

**Lemma 2.1.** If $T$ is a completely nonunitary contraction and $V$ is its minimal isometric dilation, then $T \in C_0$ if and only if there exist wandering subspaces $L, L^* \subset \mathcal{K}$ for $V$, with $\dim L = \dim D_T$ and $\dim L^* = \dim D_T^*$, such that

\begin{equation}
\mathcal{K} = M_+(L^*) = \mathcal{H} \oplus M_+(L).
\end{equation}

In this case, the operators

\begin{equation}
\phi : D_T x \mapsto (V - T)x, \quad \phi_* : D_{T^*} x \mapsto x - VT^* x
\end{equation}

extend to unitary operators $D_T \to L$ and $D_{T^*} \to L^*$.

**2.2. Analytic vector valued functions.** If $\mathcal{E}$ is a Hilbert space, then $H^2(\mathcal{E})$ is the Hilbert space of $\mathcal{E}$-valued analytic functions in $\mathbb{D}$ with the norms of the Taylor coefficients square summable. As in the scalar case, these functions have strong radial limits almost everywhere on $\mathbb{T}$, and so may be identified with their boundary values, defined on $\mathbb{T}$.

Denote by $T_z^\mathcal{E}$ multiplication by $z$ acting on $H^2(\mathcal{E})$; it is an isometric operator. If $\omega : \mathcal{E} \to \mathcal{E}'$ is unitary, then the notation $\tilde{\omega}$ will indicate the unique unitary extension $\tilde{\omega} : H^2(\mathcal{E}) \to H^2(\mathcal{E}')$ such that $\tilde{\omega} T_z^\mathcal{E} = T_z^{\mathcal{E}'} \tilde{\omega}$.

Suppose $X \subset \mathcal{K}$ is wandering for the isometry $V \in \mathcal{L}(\mathcal{K})$. Then the map $\tilde{\mathcal{F}}_X$, defined by

\begin{equation}
\tilde{\mathcal{F}}_X \left( \sum_{n=0}^\infty V^n x_n \right) = \sum_{n=0}^\infty \lambda^n x_n,
\end{equation}

is unitary from $M_+(X)$ to $H^2(X)$.

Another class of functions that we have to consider take as values operators between two Hilbert spaces $\mathcal{E}, \mathcal{E}'$. More precisely, we will be interested in contractive analytic functions; that is, functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}')$, which
satisfy $\|\Theta(z)\| \leq 1$ for all $z \in \mathbb{D}$. As in the scalar case, $\Theta$ has boundary values $\Theta(e^{it})$ almost everywhere on $\mathbb{T}$.

A contractive analytic function is called pure if $\|\Theta(0)x\| < \|x\|$ for any $x \in E, x \neq 0$. Any contractive analytic function admits a decomposition in a direct sum $\Theta = \Theta_p \oplus \Theta_u$, where $\Theta_p$ is pure and $\Theta_u$ is a constant unitary operator; then $\Theta_p$ is called the pure part of $\Theta$. A contractive analytic function will be called bi-inner if $\Theta(e^{it})$ is almost everywhere unitary. (We prefer this shorter word rather than call them inner and *-inner).

The appropriate equivalence relation for contractive analytic functions is that of coincidence: two analytic functions $\Theta : \mathbb{D} \to \mathcal{L}(E,E^*)$, $\Theta' : \mathbb{D} \to \mathcal{L}(E',E'^*)$ are said to coincide if there exist unitary operators $\omega : E \to E'$, $\omega^* : E'^* \to E^*$, such that $\Theta'(\lambda)\omega = \omega^* \Theta(\lambda)$ for all $\lambda \in \mathbb{D}$.

2.3. **Functional model and characteristic function.** The model theory of Sz.-Nagy and Foias associates to any completely nonunitary contraction $T$ a pure contractive analytic function $\Theta_T(z)$, with values in $\mathcal{L}(D_T,D_T^*)$, defined by the formula

\[(2.4) \quad \Theta_T(z) = -T + zD_T^*(I - zT^*)^{-1}D_T|D_T.\]

A functional model space and an associated model operator are constructed by means of $\Theta_T$, and one can prove that $T$ is unitarily equivalent to this model operator.

As we will be interested only in $C_{00}$ contractions, we will describe the model only in this case, in which it takes a significantly simpler form. The reason is that $T \in C_{00}$ is equivalent to $\Theta_T$ bi-inner. The functional model associated to a bi-inner contractive analytic function $\Theta : \mathbb{D} \to \mathcal{L}(E,E^*)$ is defined as follows: the **model space** is

\[(2.5) \quad \mathcal{H}_\Theta = H^2(E^*) \ominus \Theta H^2(E),\]

while the **model operator** $S_\Theta$ is the compression to $\mathcal{H}_\Theta$ of $T^E_T$. If $\Theta$ is pure, then $T^E_T$ is precisely a minimal unitary dilation of $S_\Theta$.

Note that (2.5) shows that $S_\Theta$ satisfies the assumptions of Lemma 2.1 with $L = \Theta E, L_\ast = E_\ast$. In particular,

\[(2.6) \quad \dim D_{S_\Theta} = \dim E, \quad \dim D_{S_\Theta^*} = \dim E_\ast.\]

Suppose $\Theta : \mathbb{D} \to \mathcal{L}(E,E_\ast)$ and $\Theta' : \mathbb{D} \to \mathcal{L}(E',E'_\ast)$ coincide, by means of the operators $\omega : E \to E'$, $\omega_\ast : E'_\ast \to E_\ast$. Then the unitary $\tilde{\omega}_\ast : H^2(E_\ast) \to H^2(E'_\ast)$ satisfies $\tilde{\omega}(\mathcal{H}_\Theta) = \mathcal{H}_{\Theta'}$ and

$$\tilde{\omega}_\ast S_\Theta = S_{\Theta'} \tilde{\omega}_\ast.
Returning now to the contraction $T$ and its characteristic function, the next lemma is a particular case of one of the basic results in [7, Chapter VI].

**Lemma 2.2.** If $T \in C_{00}$, then the formula (2.4) defines a bi-inner pure analytic function with values in $\mathcal{L}(D_T, D_{T^*})$, and $T$ is unitarily equivalent to $S_{\Theta_T}$. $S_{\Theta_T}$ is called the functional model of $T$.

There is a relation between the functional model and the geometrical structure of a minimal unitary dilation given by (2.1), as shown by the next result.

**Lemma 2.3.** Suppose $T \in C_{00}$, $V \in \mathcal{L}(\mathcal{K})$ is a minimal isometric dilation of $T$, and $L, L_*$ are wandering subspaces for $V$ satisfying (2.1). Extend $\phi, \phi_*$ in (2.2) to unitary operators $\tilde{\phi} : H^2(D_T) \to H^2(L)$, $\tilde{\phi}_* : H^2(D_{T^*}) \to H^2(L_*)$, and define $\Omega = \tilde{S}_{L_*}^* \tilde{\Phi}_* \tilde{\Phi} \tilde{S}_L$ is the inclusion of $M_+(L)$ into $M_+(L_*)$.

(i) The map $\tilde{S}_L^* \Phi T^* e_i$ is a bi-inner pure analytic function with values in $\mathcal{L}(M_+(L), M_+(L_*))$.

(ii) We have

\begin{align}
\Omega H_{\Theta_T} &= H, \quad \Omega(D_{T^*}) = L_*, \quad \Omega \Theta_T(D_T) = L, \\
\Omega T_z^{D_{T^*}} &= V \Omega, \quad \Omega S_{\Theta_T} = T \Omega.
\end{align}

(iii) If $\Theta = \phi_\Theta^* \Theta^* T^*$ is written $\Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n$ (with $\Theta_n : L \to L_*$), then

\begin{align}
\Theta_n = P_{L_*} (V^*)^n J,
\end{align}

where $J$ denotes the embedding of $L$ into $M_+(L_*)$.

### 3. Reducibility

In the sequel of the paper we will be interested by reducibility of certain contractions. Fortunately, this can be easily characterized through characteristic functions.

**Lemma 3.1.** Suppose $T \in C_{00}$ has characteristic function $\Theta_T : D_T \to D_{T^*}$. Then the following are equivalent.

(i) $T = T_1 \oplus T_2$.

(ii) There exist nontrivial orthogonal decompositions $D_T = E^1 \oplus E^2$, $D_{T^*} = E_1^* \oplus E_2^*$ which diagonalize $\Theta_T(\lambda)$ for all $\lambda \in \mathbb{D}$; that is,

\begin{align}
\Theta_T(\lambda) = \begin{pmatrix}
\Theta_1(\lambda) & 0 \\
0 & \Theta_2(\lambda)
\end{pmatrix}.
\end{align}

In this case $\dim D_{T_1} = \dim E^i = \dim D_{T_1^*} = \dim E_i^*$, and $\Theta_{T_1}$ coincides with $\Theta_i$. 

Theorem 3.1. Suppose $T \in \mathcal{C}_0$. Then $T$ is reducible if and only if there exist nontrivial subspaces $E \subset \mathcal{D}_T$, $E_* \subset \mathcal{D}_T^*$, such that $\Theta_T(e^{it})E = E_*$ for almost all $t$.

Proof. If nontrivial subspaces as assumed exist, then, since $\Theta$ almost all $t$ follows from (2.6). Suppose that $\Theta$ are invariant with respect to $S_{\Theta}$. Since this last operator is unitarily equivalent to $T$, $T$ is also reducible. Moreover, $S_{\Theta}|_{\mathcal{H}_{\Theta}}$ is unitarily equivalent to $S_{\Theta_1}$, and the equality of the dimensions follows from (2.6).

Corollary 3.2. Suppose $T \in \mathcal{C}_0$. Then $T$ is reducible if and only if there exist nontrivial subspaces $E \subset \mathcal{D}_T$, $E_* \subset \mathcal{D}_T^*$, such that $\Theta_T(e^{it})E = E_*$ for almost all $t$.

Proof. If nontrivial subspaces as assumed exist, then, since $\Theta_T(e^{it})$ is unitary almost everywhere, we also have $\Theta_T(e^{it})E_1^* = E_1^*$ for almost all $t$. The decompositions $\mathcal{D}_T = E \oplus E_1^*$, $\mathcal{D}_T^* = E_* \oplus E_1^*$ satisfy then (3.1).

The following is a geometrical reformulation of Corollary 3.2 in terms of the spaces $L, L_*$ appearing in an arbitrary minimal isometric dilation of $T$.

Corollary 3.3. Suppose $T \in \mathcal{C}_0$ and $V \in \mathcal{L}(\mathcal{K})$ is a minimal dilation of $T$, such that (2.1) is valid for $L, L_*$ wandering subspaces for $V$. Let $d$ be a finite positive integer or $\infty$. Then:

(i) If $T$ has a nontrivial reducing subspace such that the restriction has $d$-dimensional defects, then there exist nontrivial subspaces $L^1 \subset L$, $L_*^1 \subset L_*$, both of dimension $d$, such that

$$L^1 \subset M_+(L^1_*)$$

(ii) The converse also holds if $d < \infty$.

Proof. (i). Suppose $T$ has a reducing subspace with defect of dimension $d$. We apply Lemma 3.1, which gives decomposition (3.1), where $\Theta_i(\lambda) : E^i \to E^i_*$, and $E^1 = \dim E^1_* = d$. So $\Theta_1 H^2(E^1) \subset H^2(E^1_*)$; in particular, if we look at $E^1$ as the constant functions in $H^2(E^1)$, we have

$$\Theta_1 E^1 \subset H^2(E^1_*)$$

Denote then $L^1 = \phi E^1$ and $L_*^1 = \phi_* E^1_*$ ($\phi, \phi_*$ in (2.2)). We consider the unitary operator $\Omega$ from Lemma 2.3. Formulas (2.7) yield also $\Omega E^1_* = L_*^1$, $\Omega \Theta_1(E^1) = L^1$, and $\Omega(H^2(E^1_*)) = M_+(L^1_*)$. Therefore (3.3) implies (3.2).

(ii) Conversely, suppose we have the required spaces satisfying (3.2); therefore $M_+(L^1) \subset M_+(L^1_*)$. Define $\Theta'(\lambda) = \phi_* \Theta_1(\lambda) \phi^* : L \to L_*$. By using $\mathcal{F}_{L_*}$, we obtain $\Theta'(\varepsilon^1)L_1^1 \subset L^1_*$
almost everywhere. Since \( \dim L^1 = \dim L^1_* = d < \infty \), we have in fact 
\( \Theta'(e^{it})L^1 = L^1_* \) almost everywhere. As in the proof of Corollary 3.2, it follows that 
\( \Theta'(e^{it})L^1_{\perp} \subset L^1_{\perp} \) almost everywhere, whence we may obtain a 
decomposition similar to (3.1). This implies the reducibility of \( \Theta' \), and thus 
the reducibility of \( \Theta_T \) and of \( T \). \( \square \)

In particular, we obtain a nice result if we consider reducing subspaces 
with defects of dimension 1.

**Corollary 3.4.** An operator \( T \in C_{00} \) has a reducing subspace with defects 
of dimension 1 if and only if there exists \( y \in L, \ y_* \in L_* \), \( y, y' \neq 0 \), and a 
scalar inner function \( u \), such that \( y = u(V)y_* \). In this case the characteristic 
function of the reduced operator is precisely \( u \).

**Proof.** By Corollary 3.3 applied to \( d = 1 \), the existence of a reducing sub-
space with defects of dimension 1 is equivalent to the existence of elements 
of norm 1 \( y \in L, \ y_* \in L_* \), such that \( y \in M_+(y_*) \). The Fourier representation 
\( \mathcal{F}_y \), maps \( M_+(y_*) \) onto \( H^2 \); more precisely, from (2.3) it follows that 
\( \mathcal{F}_y(f(V)y_*) = f \). In particular, \( y \) is a wandering vector for \( V \), which implies 
that \( u := \mathcal{F}_y, y \) is an inner function.

If we denote by \( \mathcal{H}_1 \) the reducing subspace of dimension 1 obtained, then 
have \( \mathcal{H}_1 = M^+(y_*) \subset M^+(y) \). Through the Fourier representation \( \mathcal{F}_y \), 
this becomes \( H^2 \subset uH^2 \). By comparing with the general formula for the 
functional model, we see that the characteristic function of the reduced 
operator is \( u \). \( \square \)

**Remark 3.5.** Part of the results in this section may be extended to more 
general contractions. Thus Lemma 3.1 is true for a general completely 
nonunitary contraction; we have then to use in the proof the more compli-
cated general form of the functional model associated to \( T \). Appropriately 
modified versions of Corollaries 3.2 and 3.3 can also be stated. However, 
since the statements are less neat, we have preferred to restrict ourselves to 
the case \( T \in C_{00} \), which will be used in the applications in the sequel of the 
paper.

4. A class of contractions

In the rest of the paper we will work in the Hardy space \( H^2 \), applying 
the above results to a particular class of contractions. By \( T_\varphi \) we will denote 
the usual Toeplitz operator on \( H^2 \), that is, the compression of the operator 
of multiplication by \( \varphi \) on \( H^2 \). Recall here that, for a scalar inner function 
\( K_\theta = H^2 \subset \theta H^2 = \mathcal{H}_\theta \) (see (2.5) with \( \mathcal{E} = \mathcal{E}_* = \mathbb{C} \)).
Let then $\theta, B$ be two scalar inner functions that satisfy the basic assumption

$$\ker T_{\theta B} = \{0\}. \tag{4.1}$$

Note that $f \in \ker T_{\theta B}$ if and only if $\theta f \in \ker T_B = K_B$, whence (4.1) is equivalent to $\theta H^2 \cap K_B = \{0\}.$

We will consider the operator $A_{\theta B}^0 \in \mathcal{L}(K_\theta)$, defined by

$$A_{\theta B}^0 = P_{K_\theta} T_B|K_\theta. \tag{4.2}$$

The operator $A_{\theta B}^0$ is usually called the truncated Toeplitz operator on $K_\theta$ with symbol $B$. It is known [9] that truncated Toeplitz operators are complex symmetric; that is, there exist a complex conjugation $C_\theta$ on $K_\theta$ such that

$$(A_{\theta B}^0)^* = C_\theta A_{\theta B}^0 C_\theta. \tag{4.3}$$

The next theorem identifies concretely a minimal isometric dilation of $A_{\theta B}^0$; it is a generalization of [5, Lemma 3.1].

**Theorem 4.1.** Let $B$ and $\theta$ two inner functions satisfying (4.1). The operator $T_B \in \mathcal{L}(H^2)$ is a minimal isometric dilation of $A_{\theta B}^0$. For this minimal isometric dilation we have

$$L = \theta K_B, \quad L^* = K_B. \tag{4.4}$$

**Proof.** $T_B$ is an isometry on $H^2$, and $T_B(K_\theta^\perp) = T_B(\theta H^2) \subset \theta H^2 = K_\theta^\perp$. Thus it follows from (4.2) that $T_B$ is a dilation of $A_{\theta B}^0$. We show now by induction according to $n$ that

$$K_\theta + BK_\theta + \cdots + B^n K_\theta = K_{B^n \theta}. \tag{4.5}$$

Equality (4.5) is obviously true for $n = 0$. Suppose that it is true up to $n - 1$. We are left then to prove that

$$K_{B^{n-1} \theta} + B^n K_\theta = K_{B^n \theta}. \tag{4.6}$$

It is immediate from the definitions that the left hand side is contained in the right hand side. On the other hand, we have

$$K_{B^n \theta} = K_{B^{n-1} \theta} \oplus B^{n-1} \theta K_B = B^n K_\theta \oplus K_{B^n}. \quad \text{If } f \in K_{B^n \theta} \text{ is orthogonal to } K_{B^{n-1} \theta} \text{ as well as to } B^n K_\theta, \text{ it follows that } f \in (\theta B^{n-1} K_B) \cap K_{B^n}. \quad \text{So } f = \theta B^{n-1} g \text{ with } g \in K_B; \text{ and also } f \perp B^n H^2, \text{ which means } \theta g \perp B H^2, \text{ or } \theta g \in K_B. \text{ It follows that } 0 = T_{\theta B}(\theta g) = T_{\theta B}g. \text{ By (4.1), this implies } g = 0, \text{ whence } f = 0.
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Since $\left(\bigvee_{n} K_{B^n}\theta\right)^\perp = \bigcap_{n} B^n\theta H^2 = \{0\}$,

it follows that

(4.7) $H^2 = \bigvee_{n=0}^{\infty} T^n_BK_\theta$.

Therefore $T_B$ is a minimal isometric dilation of $A_\theta^B$.

Then

(4.8) $H^2 = \bigoplus_{n=0}^{\infty} B^nK_B = \bigoplus_{n=0}^{\infty} T^n_BK_B = M_+(K_B),$

whence $L_* = K_B$.

On the other hand, we have

(4.9) $K_{B\theta} = K_\theta \oplus \theta K_B = K_B \oplus BK_\theta$.

Therefore

(4.10) $H^2 = \bigoplus_{n=0}^{\infty} B^nK_B = K_\theta \oplus \theta H^2 = K_\theta \oplus \bigoplus_{n=0}^{\infty} T^n_B\theta K_B = K_\theta \oplus M_+(\theta K_B),$

whence $L = \theta K_B$. \qed

Corollary 4.2. With the above assumptions, $A_\theta^B$ is in $C_{00}$.

Proof. In view of equation (4.10), it follows from Lemma 2.1 that $A_\theta^B$ is in $C_0$. On the other hand, it follows from (4.3) that

$$((A_\theta^B)^n)_{n=0}^{\infty} = C_\theta(A_\theta^B)^nC_\theta,$$

whence $A_\theta^B$ is also in $C_0$. \qed

Using the identification of a minimal unitary dilation in Theorem 4.1 we may compute the characteristic function of $A_\theta^B$. The next theorem generalizes [3, Theorem 2.4] (see Section 6 below).

Theorem 4.3. Let $B$ and $\theta$ two inner functions satisfying (4.1). The characteristic function of $A_\theta^B$ is $\Phi : \mathbb{D} \to \mathcal{L}(K_B)$ defined by

(4.11) $\Phi(\lambda) = A^B_{1-\lambda}$.

Proof. We have identified in Theorem 4.1 $L, L_*$ with $\theta K_B, K_B$ respectively. We intend to apply Lemma 2.3 (iii). Since we want to consider the characteristic function of $A$ as an analytic function with values in $\mathcal{L}(K_B)$, the embedding $J$ is precisely multiplication by $\theta$. Then, if $\Phi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Phi_n$, (2.8)
yields
\[ \Phi_n f = P_{K_B} T^n \theta f \]
for \( f \in K_B \). Thus \( \Phi_n = A_{\theta K_B}^n \). Therefore
\[ \Phi(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_{\theta K_B}^n = A_{\theta}^{\sum_{n=0}^{\infty} \lambda^n K_B} = A_{\frac{\theta}{1-\lambda \overline{\theta}}} \].

We may also obtain a more precise form of Corollary 3.4.

**Corollary 4.4.** Let \( B \) and \( \theta \) two inner functions satisfying (4.1). Then the following assertions are equivalent:

(i) The operator \( A_{\theta B}^0 \) has a reducing subspace such that the restriction has one-dimensional defects.

(ii) There exist \( u \) inner and \( h_1, h_2 \in K_B \), \( h_1, h_2 \neq 0 \), such that
\[ \theta = \frac{h_2}{h_1} (u \circ B) \] (4.12)

(iii) There exist \( u, v_1, v_2 \) inner, with
\[ \ker T_{v_1 B} \cap \ker T_{v_2 B} \neq \{0\} \]
such that
\[ \theta = \frac{v_2}{v_1} (u \circ B) \] (4.13)

**Proof.** The equivalence of (i) and (ii) follows by applying in this case Corollary 3.4. We have \( L^*_K = K_B \), \( L = \theta K_B \), and so the existence of the required reducing subspace is equivalent to the existence of \( h_1, h_2 \in K_B \), \( h_1, h_2 \neq 0 \) and an inner function \( u \), such that \( \theta h_1 = u(V) h_2 \). Since \( V = T_B \), \( u(V) \) is multiplication by \( u \circ B \), and we have
\[ \theta h_1 = h_2 (u \circ B) \] (4.14)

If (ii) is true, then we must have \( h_i = v_i g \) for some inner functions \( v_1, v_2 \) and \( g \) outer, so (4.13) is true. Note that, if \( v \) is an analytic and bounded function in \( \mathbb{D} \), then
\[ vh \in K_B \iff h \in \ker T_{v B} \] (4.15)

So \( v_1 g, v_2 g \in K_B \) is equivalent to \( g \in \ker T_{v_1 B} \cap \ker T_{v_2 B} \).

The implication (iii) \( \implies \) (ii) follows easily by reversing the steps. \( \square \)

Note that the function \( u \) in (ii) and (iii) of the previous corollary is non constant because otherwise \( \theta h_1 \in K_B \), and thus \( h_1 \in \ker T_{\theta B} \) which contradicts hypothesis (4.1).
5. A particular case

Let us consider now the particular case when $B$ is a finite Blaschke product. Denote $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. If $B$ has roots (counting with multiplicities) $w_1, \ldots, w_k$, it is known that

\[ K_B = \left\{ \frac{p(z)}{\prod_{i=1}^k (1 - \bar{w}_iz)} : p \text{ polynomial of degree } \leq k - 1 \right\}. \]

In this case condition (4.1) has a simple equivalent form.

Lemma 5.1. If $B$ is a finite Blaschke product, then (4.1) is satisfied if and only if

\[ \dim K_B \leq \dim K_\theta. \]

Proof. Indeed, first assume that (5.2) is satisfied, and let $f \in \ker T_\theta B$. Then $\theta f \in \ker T_\theta = K_B$, whence $f = T_\theta(\theta f) \in T_\theta K_B \subset K_B$. If $f \neq 0$, then $\theta = \frac{\theta f}{f}$ is a quotient of two polynomials of degree at most $\deg B - 1$, which contradicts assumption (5.2).

Suppose now that $\dim K_B > \dim K_\theta$. Then $\theta H^2$ has finite codimension in $H^2$ strictly smaller than $\dim K_B$, whence $\theta H^2 \cap K_B \neq \{0\}$. Applying (4.15) in case $\nu = \theta$, it follows that $\ker T_{\theta B} \neq \{0\}$, contradicting (4.1).

Condition (5.2) is precisely the one considered in [3] and [5]. To discuss this case, we need one more elementary lemma.

Lemma 5.2. Suppose $h_1, h_2$ are two polynomials of degree at most $k - 1$ and

\[ |h_1| = |h_2| \text{ a.e. on } \mathbb{T}. \]

Then,

\[ \frac{h_2}{h_1} = \frac{B_2}{B_1}, \]

where $B_i$ are Blaschke products with $\deg B_1 + \deg B_2 \leq k - 1$.

Proof. First, a general remark. Suppose that $h$ is a polynomial and write $h(z) = z^p g(z)$, with $p \in \mathbb{N} \cup \{0\}$ and $g(0) \neq 0$. Denote the roots (counting with multiplicities) of $g$ by $\alpha_1, \ldots, \alpha_l$. Then, $h^o$, the outer part of $h$, is a polynomial of degree $\deg g$, which has roots $Z_\alpha(h) \cup Z_i(h)$, where $Z_\alpha(h) := \{\alpha_i : |\alpha_i| \geq 1\}$ and $Z_i(h) := \{1/\bar{\alpha}_i : 0 < |\alpha_i| < 1\}$.

We may assume that $h_1, h_2$ have no common roots (otherwise we cancel them). It also follows then that $h_1$ and $h_2$ have no roots on $\mathbb{T}$ (since this would be a common root by (5.3) ). Also, only one of them may have the root 0; suppose it is $h_1$, and write, as above, $h_1(z) = z^p g_1(z)$, with $g_1(0) \neq 0$. 

Assumption (5.3) implies that the outer parts of $g_1$ and $h_2$ coincide. Since $g_1$ and $h_2$ have no common roots, but $g_1 = h_2$, we must have $Z_o(g_1) = Z_i(h_2)$ and $Z_i(g_1) = Z_o(h_2)$. Then we can write $h_2/g_1 = B_2/B_1$, with

$$B_1 = z^p \prod_{\alpha_i \in Z_i^o(h_1)} \phi_{\alpha_i}, \quad B_2 = \prod_{\alpha_i \in Z_i^o(h_2)} \phi_{\alpha_i},$$

where $Z_i^o(p) = \{ \alpha_i : p(\alpha_i) = 0, 0 < |\alpha_i| < 1 \} = \{ 1/\bar{\alpha}_i : \alpha_i \in Z_i(p) \}$. Since we have

$$\deg B_1 + \deg B_2 = p + |Z_i(g_1)| + |Z_i(h_2)| = p + |Z_i(g_1)| + |Z_o(g_1)| \leq k - 1,$$

the lemma is proved. \qed

The next theorem generalizes [5, Theorem 1.4].

**Theorem 5.3.** Suppose $B$ is a finite Blaschke product, while $\theta$ is an inner function with $\deg \theta \geq \deg B$. Then the operator $A^\theta_B$ has a reducing subspace such that the restriction has one-dimensional defects if and only if

$$\theta = \frac{B_2}{B_1}(u \circ B),$$

where $u$ is a non constant inner function, while $B_1, B_2$ are finite Blaschke products with $\deg B_1 + \deg B_2 \leq \deg B - 1$.

**Proof.** We apply to this case Corollary 4.4 (ii). The existence of the required reducing subspace is then equivalent to the existence of $h_1, h_2 \in K_B$ and an inner function $u$, such that

$$\theta h_1 = h_2(u \circ B).$$

By (5.1), it is equivalent to assume in this equality that $h_i$ are polynomials of degree $\leq k - 1$, where $k = \deg B$. Taking absolute values, we obtain, since $\theta$ and $u \circ B$ are inner, that $|h_1| = |h_2|$ on $\mathbb{T}$. We may then apply Lemma 5.2 to obtain the desired formula (5.4).

The converse is immediate, since (5.4) implies (5.5), with the degrees of $h_1$ and $h_2$ at most $k - 1$. If we further write $g_i(z) = \frac{h_i(z)}{\prod_{t=1}^{\ell} (1 - \bar{\omega}_i z)}$, we obtain

$$\theta g_1 = g_2(u \circ B).$$

Since $g_i \in K_B$, this is equivalent, by Corollary 4.4, to the existence of the required reducing subspace. \qed

The condition becomes simpler if $\theta$ is singular.
Theorem 5.4. Let $\theta$ be a singular inner function and let $B$ be a finite Blaschke product. Then the operator $A_{\theta}^B$ has a reducing subspace such that the restriction has one-dimensional defects if and only if
\begin{equation}
\theta = S \circ B,
\end{equation}
for some singular inner function $S$.

Proof. According to Theorem 5.3, it is sufficient to prove that (5.6) and (5.4) are equivalent. The implication (5.6) $\Rightarrow$ (5.4) is clear. Assume now that (5.4) is satisfied, that is we can write
\[B_1 \theta = B_2 (u \circ B),\]
where $B_1$ and $B_2$ are finite Blaschke products with $\deg B_1 + \deg B_2 \leq N - 1$ and $N = \deg B$.

Since $\theta$ is singular, $B_2$ must be a factor of $B_1$ and may be canceled. So we may assume $B_2 = 1$, or $B_1 \theta = u \circ B$, where $\deg B_1 \leq N - 1$.

Write then $u = B_3 S$, where $B_3$ is a Blaschke product and $S$ is the singular part of $u$. Thus we have
\[B_1 \theta = (B_3 \circ B) (S \circ B).
\]
We have $\deg (B_3 \circ B) = \deg B_3 \deg B$; so, if $B_3$ is not constant, then
\[\deg (B_3 \circ B) \geq \deg B = N > \deg B_1.
\]
The contradiction obtained implies that $B_3$ is constant, and so
\[B_1 \theta = S \circ B.
\]
Since the right hand side is singular, it follows that $B_1$ is constant, which proves the theorem. \qed

6. The case $B(z) = z^N$

The case $B(z) = z^N$ is investigated at length in [3]. In particular, the characteristic function of $A_{\theta}^B_{z^N}$ is computed; let us show how Gu’s result follows from Theorem 4.3 above.

We use the canonical basis of $K_B$ formed by $1, z, \ldots, z^{N-1}$. To obtain the matrix of $A_{\theta}^B_{z^N}$, consider first $A_{\theta}^B_{z^N}$. We have
\[
\frac{z^n}{1 - \lambda B} = \sum_{j=0}^{\infty} \lambda^j z^{n-jN} = \sum_{j=0}^{\infty} \lambda^j z^{N(n'-j)+m},
\]
where \( n = Nn' + m \), with \( 0 \leq m \leq N - 1 \). Since \( A_{z^p}^B \) is nonzero only for \(-(N-1) \leq p \leq N - 1\), we have to consider in the above sum only two terms, corresponding to \( j = n' \) and \( j = n' + 1 \). Thus

\[
A_{z^n}^B = A_{\lambda^{n'+m}+\lambda^{n'+1}z_{m-N}}^B.
\]

Its matrix with respect to the canonical basis is

\[
(6.1) \quad A_{\lambda^{n'+m}+\lambda^{n'+1}z_{m-N}}^B = \begin{pmatrix}
\ddots & \ddots & \lambda^{n'+1} & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \\
\lambda^{n'} & \ddots & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \\
\end{pmatrix},
\]

with two nonzero constant diagonals (one in case \( m = 0 \)), corresponding to entries \( a_{ij} \) with \( i - j = m \) or \( i - j = m - N \).

Therefore, if we decompose

\[
\theta(z) = \theta_0(z^N) + z\theta_1(z^N) + \cdots + z^{N-1}\theta_{N-1}(z^N),
\]

then

\[
(6.2) \quad A_{\theta_{z^N}}^B = \begin{pmatrix}
\theta_0(\lambda) & \lambda\theta_{N-1}(\lambda) & \lambda\theta_{N-2}(\lambda) & \cdots & \lambda\theta_1(\lambda) \\
\theta_1(\lambda) & \theta_0(\lambda) & \lambda\theta_{N-1}(\lambda) & \cdots & \lambda\theta_2(\lambda) \\
\theta_2(\lambda) & \theta_1(\lambda) & \theta_0(\lambda) & \cdots & \lambda\theta_3(\lambda) \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
\theta_{N-1}(\lambda) & \theta_{N-2}(\lambda) & \theta_{N-3}(\lambda) & \cdots & \theta_0(\lambda)
\end{pmatrix},
\]

This is precisely the form given by [3, Theorem 2.4].

In the sequel we will solve a conjecture about \( A_{z^N}^B \) left open in [3]. This appears as Conjecture 3.5 therein, and has the following statement.

**Conjecture 6.1.** Suppose \( B(z) = z^N \). Then the following are equivalent:

(i) \( A_{z^N}^B \) has a reducing subspace such that the restriction has one-dimensional defects.

(ii) \( \theta(z) = b(z)u(z^N) \) for some inner function \( u \), while either \( b \equiv 1 \) or

\[
(6.3) \quad b(z) = \prod_{i=1}^{l} \psi_{\alpha_i,J_i},
\]

where \( l \leq N - 1 \), \( J_i \subset \{0, \ldots, N - 1\} \), and \( \psi_{\alpha,J} \) is defined by

\[
(6.4) \quad \psi_{\alpha,J}(z) = \prod_{\alpha\in J} \phi_{\omega^\alpha}(z).
\]
[3, Theorem 3.4] shows that (i) $\implies$ (ii), while (ii) $\implies$ (i) is proved only for $N = 3$ in [3, Section 5].

**Theorem 6.2.** Conjecture 6.1 is false for $N = 4$.

**Proof.** Take two different nonzero values $\alpha, \beta \in \mathbb{D}$, and define
\[
\theta(z) = \frac{(z^2 - \alpha^2)(z^2 - \beta^2)}{(1 - \bar{\alpha}^2 z^2)(1 - \beta^2 z^2)}.
\]
We have then
\[
\theta(z) = \psi_{\alpha, J} \psi_{\beta, J}
\]
with $J = \{0, 2\} \subset \{0, 1, 2, 3\}$, so $\theta$ satisfies condition (ii) of Conjecture 6.1.

On the other hand, if $\theta$ would satisfy condition (i), it would follow by Theorem 5.3 that one should have
\begin{equation}
B_2(z) \theta(z) = B_1(z) u(z^4),
\end{equation}
with $u$ inner and $B_1, B_2$ finite Blaschke products with $\deg B_1 + \deg B_2 \leq 3$.

Obviously $u$ has also to be a finite Blaschke product. Equating the degrees in both sides yields
\[
\deg B_1 + 4 = \deg B_2 + 4 \deg u.
\]
First, $\deg B_1 = 3$ would imply $\deg B_2 = 0$, so $7 = 4 \deg u$: a contradiction. So the degree of the left hand side of (6.5) is between 4 and 6, which implies $\deg u = 1$. Again equating the degrees yields $\deg B_1 = \deg B_2 = 0$ or 1.

Now $u(z^4)$ has either the root 0 of multiplicity 4, or four distinct roots. Both possibilities are incompatible with the fact that the left hand side of (6.5) has either 2 or three roots. We have obtained the desired contradiction, so $\theta$ does not satisfy (i) of Conjecture 6.1. \hfill \Box

In fact, we may replace Conjecture 6.1 with a precise result. We will need the next lemma, also proved in [3, Theorem 3.4].

**Lemma 6.3.** If $\alpha \in \mathbb{D}$, then
\[
\phi_\alpha^N(z^N) = \prod_{i=0}^{N-1} \phi_{\omega^i \alpha}(z).
\]

**Theorem 6.4.** Suppose $B(z) = z^N$. Then the following are equivalent:

(i) $A_B^0$ has a reducing subspace such that the restriction has one-dimensional defects.
(ii) $\theta(z) = b(z)u(z^N)$ for some inner function $u$, while $b$ is either 1 or a finite Blaschke product given by (6.3), where $l \leq N - 1$, $J_i \subset \{0, \ldots, N - 1\}$, $\psi_{\alpha_i}^j$ are defined by (6.4), and, moreover,

$$\sum_{i=1}^l \min\{|J_i|, N - |J_i|\} \leq N - 1.$$  

Proof. (i) $\Rightarrow$ (ii). From Theorem 5.3 we know that $\theta$ is given by (5.4), where $B_1$ and $B_2$ have no common roots. We may denote the roots of $B_1$ (counting multiplicities) by

$$\{\alpha_1^1, \ldots, \alpha_{s_1}^1; \alpha_1^2, \ldots, \alpha_{s_2}^2; \ldots; \alpha_1^p, \ldots, \alpha_{s_p}^p\},$$

where, for each $i = 1, \ldots, p$, the values $\alpha_{s_1}^1, \ldots, \alpha_{s_i}^i$ are all distinct, and

$$(\alpha_{s_1}^1)^N = \cdots = (\alpha_{s_p}^p)^N.$$  

Similarly, we denote the roots of $B_2$ by

$$\{\beta_1^1, \ldots, \beta_{r_1}^1; \beta_1^2, \ldots, \beta_{r_2}^2; \ldots; \beta_1^q, \ldots, \beta_{r_q}^q\},$$

where, for each $i = 1, \ldots, q$, the values $\beta_{r_1}^1, \ldots, \beta_{r_i}^i$ are all distinct, and

$$(\beta_{s_1}^1)^N = \cdots = (\beta_{s_q}^q)^N.$$  

Note that the condition $\deg B_1 + \deg B_2 \leq N - 1$ is transcribed as

$$s_1 + \cdots + s_p + r_1 + \cdots + r_q \leq N - 1.$$  

In particular, $p + q \leq N - 1$.

Now, it is easy to see that, for each $i = 1, \ldots, q$, the Blaschke product

$$\phi_{\beta_1^i} \cdots \phi_{\beta_{r_i}^i}$$

is equal to $\psi_{\beta_1^i, J_i}$ for some $J_i \subset \{0, \ldots, N - 1\}$. So

$$B_2 = \prod_{i=1}^q \psi_{\beta_1^i, J_i}.$$  

The matter is more subtle as concerns $B_1$: it appears at the denominator, which we do not want. We have, similarly,

$$B_1 = \prod_{i=1}^p \psi_{\alpha_1^i, J_i}$$  

for some $J_i' \subset \{0, \ldots, N - 1\}$. 

The factor $\phi_{\alpha_1}(z)$ must be canceled by a factor in $u(z^N)$, so $\alpha_1$ must be a root of $u(z^N)$. But then $u(z^N)$ must also have the roots $\omega^j \alpha_1$ for $j = 1, \ldots, N - 1$, and so

$$u(z^N) = \prod_{j=0}^{N-1} \phi_{\omega^j \alpha_1}(z) u_1(z^N).$$

Since

$$\frac{\phi_{\omega^j \alpha_1}(z)}{\psi_{\alpha_1, J_1''}} = \psi_{\alpha_1, J_1''}$$

with $J_1'' = \{0, \ldots, N - 1\} \setminus J_1', \psi_{\alpha_1, J_1''}$, we have

$$u(z^N) \left/ \psi_{\alpha_1, J_1''} \right. = \psi_{\alpha_1, J_1''} u_1(z^N).$$

We may continue the argument (or use an appropriate induction) to obtain

$$u(z^N) B_1(z) = \prod_{i=1}^p \psi_{\alpha_i, J_i''} u'(z^N)$$

for an inner function $u'$, where $J_i'' = \{0, \ldots, N - 1\} \setminus J_i'$. From (5.4), (6.8), and (6.10) it follows that

$$\theta(z) = \prod_{i=1}^q \psi_{\beta_i, J_i} \prod_{i=1}^p \psi_{\alpha_i, J_i''} u'(z^N).$$

This is exactly the form given by (6.3). Moreover $\min\{|J_i|, N - |J_i|\} \leq r_i$ and $\min\{|J_i''|, N - |J_i''|\} \leq s_i$, so (6.7) implies (6.6).

(ii) $\implies$ (i). Suppose $b(z)$ is given by (6.3), with (6.6) satisfied. Define

$$B_2 = \prod_{\min\{|J_i|, N - |J_i|\} = |J_i|} \psi_{\alpha_i, J_i}$$

and

$$B_1 = \prod_{\min\{|J_i|, N - |J_i|\} = N - |J_i|} \psi_{\alpha_i, N\setminus J_i}.$$ 

Then

$$\theta(z) = \frac{B_2(z)}{B_1(z)} u_1(z^N),$$

where

$$u_1(z) = u(z) \prod_{\min\{|J_i|, N - |J_i|\} = N - |J_i|} \phi_{\alpha_i, N}(z);$$

note that we have used Lemma 6.3. □
References


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