Numéro d'ordre : ? ?-2008

Université Claude Bernard Lyon 1

Habilitation à Diriger des Recherches

spécialité : **mathématiques pures**

Interactions entre l'analyse complexe et la théorie des opérateurs

---

Emmanuel Fricain

– RAPPORTEURS –

M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?

– JURY –

M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?
M. ? ? ? ? ? ? ? Université ? ? ? ? ?

- Août 2008 -

Sommaire

Liste des travaux présentés

[T1] I. Chalendar, E. Fricain et D. Timotin. Functional models and asymptotically orthonormal sequences. *Ann. Inst. Fourier (Grenoble)*, 53 (2003), no. 5, 1527–1549.

[T2] Chalendar I., E. Fricain et J. Partington. Overcompleteness of sequences of reproducing kernels in model spaces. *Integral Equations Operator Theory*, 56 (2006), no. 1, 45–56.

[T3] E. Fricain. Bases of reproducing kernels in de Branges spaces. *J. Funct. Anal.*, 226 (2005), no. 2, 373–405.

[T4] N. Chevrot, E. Fricain et D. Timotin. On certain Riesz families in vector-valued de Branges-Rovnyak spaces. *soumis.*

[T5] E. Fricain et J. Mashreghi. Boundary behavior of functions in the de Branges-Rovnyak spaces. *Compl. anal. oper. theory*, 2 (2008), 87–97.

[T6] E. Fricain et J. Mashreghi. Integral representations of the n-th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel. *Ann. Inst. Fourier (Grenoble)*, à paraître.

[T7] A. Baranov, E. Fricain et J. Mashreghi. Weighted norm inequalities for de Branges-Rovnyak spaces and their applications. *soumis.*

[T8] E. Fricain et J. Mashreghi. Integral means of the derivatives of Blaschke products. *Glasgow Mathematical Journal*, 50 (2008), no. 2, 233–249.

[T9] E. Fricain et J. Mashreghi. Exceptional sets for the derivatives of Blaschke products. *Proceedings of the St. Petersburg Mathematical Society*, Amer. Math. Soc. Transl., 222 (2008), no. 2, 163-170.

[T10] N. Chevrot, E. Fricain et D. Timotin. The characteristic function of a complex symmetric contraction. *Proc. Amer. Math. Soc.*, 135 (2007), no. 9, 2877–2886.

[T11] I. Chalendar, E. Fricain et D. Timotin. A note on the stability of linear combinations of algebraic operators. *Extracta Mathematicae*, à paraître.

[T12] I. Chalendar, E. Fricain, A. Popov, D. Timotin et V. Troitsky. Finitely strictly singular operators between James spaces. *J. Funct. Anal.*, à paraître.

Chapitre 1

Introduction

Les modèles fonctionnels de Sz.-Nagy–Foias et de Branges–Rovnyak sont devenus des outils incontournables dans de nombreuses questions d'analyse et il nous apparaît donc essentiel de bien comprendre les espaces qui interviennent. Une grande partie des travaux présentés dans cette habilitation [T1–T7] est consacrée à l'étude de ces espaces modèles. Ainsi même si les modèles fonctionnels de Sz.-Nagy–Foias et de Branges–Rovnyak ne font pas ici l'objet d'une étude proprement dite, ils sont utilisés dans au moins deux de nos travaux [T4,T10] et ils sont sous-jacents dans une grande partie des autres [T1,T2,T3,T5,T6,T7]. Ils peuvent donc être vus comme l'origine et la motivation de mes recherches et c'est la raison pour laquelle je vais leur consacrer une assez longue introduction. Une autre partie des travaux présentés ici [T8,T9] traite d'estimations sur les produits de Blaschke qui sont des fonctions méromorphes dans le plan complexe et qui apparaissent naturellement dans la théorie des espaces de Hardy. Une dernière partie des travaux [T10–T12] porte sur diverses questions de théorie des opérateurs.

1.1 La théorie des opérateurs modèles et le shift sur $H^2(E)$: un bref historique

Etant donné X un espace de Banach, on note par $\mathcal{L}(X)$ l'algèbre des opérateurs linéaires et continus de X dans X et pour $T \in \mathcal{L}(X)$, $\sigma(T)$ désigne le *spectre* de T, c'est-à-dire l'ensemble des nombres complexes λ tels que $\lambda Id - T$ n'est pas inversible dans $\mathcal{L}(X)$. La résolvante $R_T(\lambda) := (\lambda Id - T)^{-1}$, fonction définie et analytique en dehors du spectre de T et à valeurs opératorielles, est un des moyens les plus utiles et efficaces pour étudier un opérateur. Elle intervient notamment de façon cruciale dans le calcul fonctionnel et les décompositions spectrales de type Riesz-Dunford. A la fin des années 40, sous l'impulsion de l'école d'analyse fonctionnelle formée à Odessa par M. G. Krein, on a commencé à lier aux opérateurs d'autres fonctions analytiques, d'abord scalaires puis matricielles et enfin opératorielles, l'idée étant que les propriétés de ces fonctions devaient refléter au mieux la structure des opérateurs eux-mêmes. Ces fonctions, appelées *fonctions caractéristiques* de l'opérateur T, sont apparues pour la première fois chez M. Livsitz [138] puis ont été reprises par M. Krein, M. Brodskii, V. Potapov, Yu. Shmulyan, A. Strauss.... La motivation principale se trouvait dans les recherches de Krein sur le prolongement des opérateurs non auto-adjoints sur un espace de Hilbert et il s'agissait d'obtenir des versions continues des modèles triangulaires de type Schur ou Jordan. Finalement dans les années 60, B. Sz.-Nagy et C. Foias d'une part, et L. de Branges et J. Rovnyak d'autre part, ont repris ces idées pour développer deux théories du modèle fonctionnel pour les contractions sur un espace de Hilbert. Finalement en 1977, inspiré par des idées de B. Pavlov, V. Vasyunin [227] a développé une approche dite "sans coordonnées" qui a permis d'unifier toutes ces théories.

D'un point de vue général, un *modèle* pour un opérateur $T : X \longrightarrow X$ est un autre opérateur $M : K \longrightarrow K$ qui, dans un certain sens, est équivalent à T, l'idée étant bien sûr que l'étude de M soit plus simple et permette d'obtenir des informations sur T. En dimension finie, il existe les modèles classiques de Schur ou Jordan. Dans le cas de la dimension infinie, l'un des résultats les plus importants en théorie spectrale est le modèle de J. von-Neumann [231] :

Théorème 1.1.1 (von-Neumann) *Soit N un opérateur normal sur un espace de Hilbert H ($NN^* = N^*N$). Alors il existe une mesure borélienne positive μ sur $\sigma(N)$ et une fonction mesurable $z \longmapsto P(z)$ à valeurs projections orthogonales dans H telles que l'opérateur N est unitairement équivalent à l'opérateur de multiplication*

$$f(z) \longmapsto zf(z), \qquad z \in \sigma(N), \tag{1.1}$$

sur l'espace $\mathfrak{L} := \{f \in L^2(H,\mu) : f(z) \in P(z)H,\ \mu \text{ p.p.}\}$.

Presque toutes les informations intéressantes sur les opérateurs normaux (et en particulier autoadjoints ou unitaires) peuvent s'obtenir par l'étude du modèle défini par (1.1). Si on veut dépasser le cadre de la dimension infinie ou le cadre des opérateurs non-normaux, la situation est beaucoup plus délicate.

Dans la théorie des opérateurs linéaires, le shift sur ℓ^2 (c'est-à-dire l'opérateur de décalage à droite) s'est révélé très vite être un exemple simple mais fondamental. Notamment il est apparu que cet opérateur devait jouer un rôle crucial dans l'étude des modèles pour les opérateurs non-normaux et en particulier les contractions. Rappelons que d'un point de vue abstrait, un opérateur S linéaire et continue sur un espace de Hilbert complexe séparable $\mathcal{H}$ est appelé un *shift* si S est une isométrie et $\|S^{*n}x\| \to 0$, $n \to +\infty$ ($x \in \mathcal{H}$). L'exemple (canonique) le plus simple est le suivant : considérons E un espace de Hilbert et $\ell^2(E)$ l'espace des suites $x = (x_n)_{n\geq 0}$, $x_n \in E$, telles que

$$\|x\|_2^2 := \sum_{n=0}^{\infty} \|x_n\|_E^2 < +\infty.$$

Alors l'opérateur

$$S : (x_0, x_1, x_2, \dots) \longmapsto (0, x_0, x_1, x_2, \dots)$$

est un shift sur $\ell^2(E)$ et son adjoint est

$$S^* : (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots).$$

Un autre exemple (celui qui va intervenir dans les modèles de Sz.-Nagy-Foias et de Branges–Rovnyak) est la transcription analytique de ce shift. Si $\mathbb{D}$ est le disque unité ouvert du plan complexe et E est un espace de Hilbert, on désigne par $H^2(E)$ l'espace de Hardy du disque unité à valeurs vectorielles dans E (voir la sous-section 1.2.1 pour la définition) et l'opérateur

$$S : f(z) \longmapsto zf(z)$$

est un shift sur $H^2(E)$. Son adjoint est donné par

$$S^* : f(z) \longmapsto \frac{f(z) - f(0)}{z}.$$

Dans le cas où $E = \mathbb{C}$, on note plus simplement $\ell^2(E) = \ell^2$ et $H^2(E) = H^2$.

Il est facile de voir que les shifts sur $\ell^2(E)$ et sur $H^2(E)$ sont unitairement équivalents. La *multiplicité* d'un shift S est par définition la dimension de $\ker S^*$, le noyau de S^*. Ainsi, pour tout espace de Hilbert E, la multiplicité du shift sur $\ell^2(E)$ (ainsi que sur $H^2(E)$) est égale à la dimension de E. De plus, deux shifts sont unitairement équivalents si et seulement si ils ont la même multiplicité.

En 1949, dans le cadre d'une étude sur la dynamique des opérateurs linéaires sur un espace de Hilbert, A. Beurling [37] s'est intéressé à la description des sous-espaces invariants du shift de multiplicité 1 sur ℓ^2. L'idée essentielle de Beurling a été de considérer la représentation analytique du shift sur H^2 pour pouvoir utiliser les outils analytiques et la théorie de Nevanlinna. Ainsi, il a démontré que tous les sous-espaces invariants du shift sur H^2 sont de la forme ΘH^2, avec Θ une fonction intérieure (voir la sous-section 1.2.1 pour la définition des fonctions intérieures). Ce résultat a ensuite été étendu dans de nombreuses directions ; citons en particulier [128] pour le cas d'un shift de multiplicité finie, [104, 114, 113] pour le cas d'un shift de multiplicité quelconque et [7] pour un contexte plus général. Dans le cas d'un shift de multiplicité infinie, la fonction intérieure scalaire qui apparait dans le théorème de Beurling est alors remplacée par une fonction à valeurs opératorielles. Plus précisément, on peut formuler le résultat général suivant :

Théorème 1.1.2 (Beurling-Lax-Halmos) *Soient E un espace de Hilbert, S le shift sur $H^2(E)$ et M un sous-espace vectoriel fermé de $H^2(E)$, invariant par S. Alors il existe un espace de Hilbert F et une fonction intérieure Θ, définie sur $\mathbb{D}$ et à valeurs dans l'ensemble des opérateurs linéaires et bornés de F dans E, tels que $M = \Theta H^2(F)$.*

Le premier pas dans l'étude spectrale des opérateurs non-normaux est peut-être le résultat dû à J. von Neumann [231] et H. Wold [232] sur la décomposition des isométries. Plus précisément, ce théorème de structure dit que si V est une isométrie sur un espace de Hilbert H, alors il existe une unique décomposition $H = H_u \oplus H_s$, où $H_u, H_s \in \mathrm{Lat}(V)$ et $S := V|H_s$ est un shift sur H_s et $U := V|H_u$ est un unitaire sur H_u. L'isométrie V est dite *pure* si $H_u = \{0\}$, autrement dit si V est un shift.

Par la suite, G. Rota [187, 188] a montré que si T est une contraction stricte sur un espace de Hilbert E (c'est-à-dire $\|T\| < 1$) et si S désigne le shift sur $H^2(E)$ alors il existe un sous-espace invariant N de S^* tel que T est similaire à $S^*|N$. En utilisant le théorème 1.1.2, on obtient alors que la restriction de l'adjoint du shift aux espaces

$$K_\Theta := H^2(E) \ominus \Theta H^2(F)$$

représente un modèle pour les contractions strictes. Influencés par les idées de Krein, Livsitz, Rota et Beurling, B. Sz.-Nagy et C. Foias ont développé dans les années 60 un modèle fonctionnel pour toutes les contractions sur un espace de Hilbert. Dans cette théorie, le shift sur $H^2(E)$ intervient aussi, mais si on veut construire un modèle universel pour toutes les contractions, nous allons voir qu'il faut compliquer un peu ce modèle (en fait l'espace modèle). De plus, l'autre différence essentielle avec le modèle de Rota est que, dans la théorie de Sz.-Nagy–Foias, on donne une représentation explicite de la fonction analytique qui appara^dans le modèle (il s'agit de la fonction caractéristique que nous avons évoquée au début de cette introduction). Le fait d'avoir une représentation explicite va ainsi permettre de déduire de nombreux résultats sur la contraction T en fonction des propriétés de sa fonction caractéristique. L'utilisation de cette fonction caractéristique va permettre d'introduire une structure analytique très profonde qui va ouvrir la voie à l'utilisation dans ce contexte de toutes les techniques fines d'analyse complexe. Ceci va se révéler vraiment fructueux et contribuer au succès de cette théorie et de ses applications. Parallèlement à la théorie de Sz.-Nagy-Foias, L. de Branges et J. Rovnyak ont développé une autre théorie avec la différence essentielle que les sous-espaces invariants sur lesquels ils restreignent l'adjoint du shift ne sont plus nécessairement fermés pour la norme L^2. Cette différence présente un certain nombre d'avantage et une plus grande souplesse mais elle introduit aussi (comme nous le verrons) d'autres difficultés.

Le point de départ dans la construction du modèle fonctionnel de Sz.-Nagy–Foias est l'utilisation d'une dilatation unitaire minimale U pour la contraction T. Signalons que cette notion de dilatation unitaire minimale leur a également permis de construire un calcul fonctionnel H^∞ pour les contractions qui s'est révélé fort utile. En utilisant le théorème spectral 1.1.1, on réalise alors l'action de la dilatation U et le modèle dépend alors de deux plongements isométriques de certains espaces L^2 dans l'espace de la dilatation unitaire minimale U. Dans [227] et [146], V. Vasyunin a développé un modèle universel, *le modèle fonctionnel sans* *coordonnées*, qui englobe les modèles de Sz.-Nagy–Foias et de Branges–Rovnyak. L'idée est de ne fixer ni la représentation concrête de la dilatation unitaire, ni la dilatation elle-même, mais plutôt de travailler avec un plongement fonctionnel abstrait qui contient toute l'information sur la contraction T.

1.2 Le modèle fonctionnel sans coordonnées : la construction

Avant de décrire la construction du modèle fonctionnel sans coordonnées, rappelons quelques notions sur les fonctions analytiques à valeurs opératorielles et les contractions.

1.2.1 Préliminaires sur les fonctions analytiques à valeurs opératorielles et les contractions

Si E est un espace de Hilbert complexe séparable et si le cercle unité $\mathbb{T}$ du plan complexe est muni de la mesure de Lebesgue normalisée m, alors $L^2(E)$ désigne l'espace de Hilbert des fonctions f définies sur $\mathbb{T}$ et à valeurs dans E, mesurables et telles que

$$\|f\|_2^2 := \int_{\mathbb{T}} \|f(z)\|_E^2 \, dm(z) < +\infty.$$

L'espace de Hardy correspondant $H^2(E)$ est défini comme l'espace des fonctions analytiques dans $\mathbb{D}$ à valeurs dans E, $f(z) = \sum_{n\geq 0} a_n z^n$, $a_n \in E$, telles que

$$\|f\|_2^2 := \sum_{n\geq 0} \|a_n\|_E^2 < +\infty.$$

Il est bien connu que $H^2(E)$ peut s'identifier au sous-espace fermé de $L^2(E)$ des fonctions dont les coefficients de Fourier négatifs sont nuls. Le symbole P_+ (respectivement P_-) désigne alors la projection orthogonale de $L^2(E)$ sur $H^2(E)$ (respectivement sur $H^2_-(E) := L^2(E) \ominus H^2(E)$).

Si E, E_* sont deux espaces de Hilbert séparables, on note par $\mathcal{L}(E, E_*)$ l'espace des opérateurs linéaires et bornés de E dans E_*. Alors $L^\infty(E \to E_*)$ est l'espace de Banach des fonctions f définies sur $\mathbb{T}$, à valeurs dans $\mathcal{L}(E, E_*)$, qui sont faiblement mesurables et telles que

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \|f(\zeta)\|_{\mathcal{L}(E,E_*)} < +\infty.$$

L'espace de Banach $H^\infty(E \to E_*)$ est défini comme l'espace des fonctions analytiques et bornées dans $\mathbb{D}$, à valeurs dans $\mathcal{L}(E, E_*)$, muni de la norme

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} \|f(z)\|_{\mathcal{L}(E.E_*)}.$$

Si f est dans $H^\infty(E \to E_*)$, alors la limite

$$f(e^{i\theta}) := \lim_{r\to 1} f(re^{i\theta})$$

existe pour la topologie forte opérateur pour presque tout $e^{i\theta}$ dans $\mathbb{T}$. Comme précédemment, on peut identifier $H^\infty(E \to E_*)$ avec le sous-espace de $L^\infty(E \to E_*)$ des fonctions dont les coefficients de Fourier négatifs sont nuls.

Précisons maintenant quelques éléments de terminologie concernant les fonctions analytiques contractives. Si Θ est dans la boule unité de $H^\infty(E \to E_*)$, on dit que Θ est *pure* si $\|\Theta(0)e\| < \|e\|$, pour tout $e \in E$, $e \neq 0$; on dit que Θ est *intérieure* si $\Theta(\zeta)$ est une isométrie pour presque tout $\zeta \in \mathbb{T}$ (dans le cas scalaire, cela signifie que Θ est de module 1 presque partout sur $\mathbb{T}$). On dit que Θ est $*$-*intérieure* si $\Theta(\zeta)$ est une co-isométrie pour presque tout $\zeta \in \mathbb{T}$.

Soient Θ_1 dans la boule unité de $H^\infty(E, E_*)$ et Θ_2 dans la boule unité de $H^\infty(E', E'_*)$. On dit que Θ_1 et Θ_2 *coïncident* (au sens de Sz.-Nagy–Foias) s'il existe deux opérateurs unitaires $U : E \longrightarrow E'$, $U_* : E_* \longrightarrow E'_*$ tels que pour tout $z \in \mathbb{D}$, on ait $\Theta_1(z) = U_*^*\Theta_2(z)$.

Nous précisons maintenant quelques notions importantes pour les contractions. Tout d'abord, il existe un lemme analogue à la décomposition de von Neuman–Wold pour les contractions, du à Sz.-Nagy–Foias et Langer indépendamment (voir [155]) : si T une contraction sur un espace de Hilbert H, alors il existe une unique décomposition $H = H_0 \oplus H_u$ telle que $TH_0 \subset H_0$, $TH_u \subset H_u$ et $T_u := T|H_u$ est unitaire, tandis que $T_0 := T|H_0$ est complètement non unitaire, c'est-à-dire que T_0 n'est unitaire sur aucun de ses sous-espaces invariants. On dit que T est complètement non unitaire si $H_u = \{0\}$.

L'opérateur $D_T := (Id - T^*T)^{1/2}$ est appelé l'*opérateur de défaut* de la contraction T. Les *espaces de défaut* de T sont $\mathcal{D}_T = \mathrm{clos}(D_T H)$, $\mathcal{D}_{T^*} = \mathrm{clos}(D_{T^*}H)$, et les *indices de défaut* de T sont $\partial_T = \dim \mathcal{D}_T$, $\partial_{T^*} = \dim D_{T^*}$. Puisque $D_T = D_{T_0} \oplus 0$, $D_{T^*} = D_{T_0^*} \oplus 0$, on a $\mathcal{D}_T = \mathcal{D}_{T_0}$ et $\mathcal{D}_{T^*} = \mathcal{D}_{T_0^*}$.

On dit qu'un opérateur unitaire $U \in \mathcal{L}(\mathcal{H})$ est une *dilatation unitaire* d'une contraction $T \in \mathcal{L}(H)$ si $H \subset \mathcal{H}$ et si

$$T^n = P_H U^n | H, \qquad (n \geq 1).$$

Elle est dite *minimale* si $\mathrm{Span}(U^n H : n \in \mathbb{Z}) = \mathcal{H}$. Un résultat fondamentale de B. Sz.-Nagy [219] affirme que toute contraction possède une dilatation unitaire minimale.

On écrit que $T \in C_{0\cdot}$ si $T^n \to 0$ pour la topologie forte, et $T \in C_{\cdot 0}$ si $T^* \in C_{0\cdot}$. La *fonction caractéristique* Θ_T de la contraction T est la fonction à valeurs opératorielles $\Theta_T(\lambda) : \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$ définie pour $\lambda \in \mathbb{D}$ par

$$\Theta_T(\lambda) := \left\{-T + \lambda D_{T^*}(Id - \lambda T^*)^{-1} D_T\right\} |\mathcal{D}_T. \tag{1.2}$$

On vérifie alors que Θ_T est une fonction de la boule unité de $H^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$, qui est pure. De plus, on a $\Theta_T = \Theta_{T_0}$. Maintenant si T est une contraction complètement non-unitaire, alors T est $C_{\cdot 0}$ si et seulement si Θ_T est intérieure et T est $C_{0\cdot}$ si et seulement si Θ_T est $*$-intérieure.

Une classe de contractions fondamentale dans la suite est formée des opérateurs de Toeplitz dont le symbole est une fonction opératorielle contractive. Plus précisément, si $\varphi \in L^\infty(E \to E_*)$, T_φ désigne l'opérateur de Toeplitz de $H^2(E)$ dans $H^2(E_*)$ défini par

$$T_\varphi(f) := P_+(\varphi f), \qquad (f \in H^2(E)).$$

Alors $T_\varphi \in \mathcal{L}(H^2(E), H^2(E_*))$, $\|T_\varphi\| = \|\varphi\|_\infty$ et $T_\varphi^* = T_{\varphi^*}$, où $\varphi^* \in L^\infty(E_* \to E)$ est défini par $\varphi^*(\zeta) := \varphi(\zeta)^*$, $\zeta \in \mathbb{T}$. On utilisera aussi l'opérateur de Hankel associé $H_\varphi : H^2(E) \longrightarrow H^2_-(E_*)$ défini par

$$H_\varphi(f) := P_-(\varphi f).$$

On a alors

$$\varphi f = T_\varphi f + H_\varphi f, \qquad \|\varphi f\|^2 = \|T_\varphi f\| + \|H_\varphi f\|,$$

et le théorème de Nehari affirme que

$$\|H_\varphi\| = \text{dist } (\varphi, H^\infty(E \to E_*)).$$

1.2.2 La construction du modèle

Pour cette section, on suit la présentation de [152] (voir aussi [155]). Comme on l'a déjà mentionné, l'idée du modèle fonctionnel sans coordonnées est de ne fixer ni la représentation de la dilatation unitaire minimale de T, ni la dilatation elle même. Au contraire, nous allons travailler avec deux plongements fonctionnels abstraits. Cette universalité apporte une plus grande liberté au modèle et explique en partie son intérêt.

On appelle *plongement fonctionnel abstrait* une application linéaire

$$\Pi = (\pi_*, \pi) : \begin{pmatrix} L^2(E_*) \\ L^2(E) \end{pmatrix} \longrightarrow \mathcal{H}$$

satisfaisant les propriétés suivantes :

(i) π et π_* sont des isométries ;

(ii) $\pi_*^*\pi = \Theta$, où Θ est une fonction analytique contractive et pure[1] de $H^\infty(E \to E_*)$;

(iii) l'image de Π est dense dans $\mathcal{H}$.

[1]Dans la définition, on inclut la pureté de $\pi_*^*\pi$ car on est intéressé ici par le modèle ; cela n'est pas essentiel et n'est pas le cas par exemple dans la définition utilisé dans [T4]

On montre alors que l'opérateur $U : \mathcal{H} \longrightarrow \mathcal{H}$ défini par $U\Pi = \Pi z$ est un opérateur unitaire. On définit l'*espace modèle* par

$$K_\Theta = \mathcal{H} \ominus \left(\pi H^2(E) \oplus \pi_* H^2_-(E_*)\right), \tag{1.3}$$

et l'*opérateur modèle* $M_\Theta \in \mathcal{L}(K_\Theta)$ par la formule

$$M_\Theta = P_\Theta U | K_\Theta, \tag{1.4}$$

où P_Θ est la projection orthogonale sur K_Θ. On a alors le théorème fondamental suivant :

Théorème 1.2.1 (voir [152], Theorem 1.17) *L'opérateur M_Θ est une contraction complètement non unitaire dont la fonction caractéristique est Θ. De plus, l'opérateur U est une dilatation unitaire minimale de M_Θ.*

Réciproquement, si T est une contraction complètement non unitaire, alors on considère un plongement fonctionnel abstrait

$$\Pi = (\pi_*, \pi) : \begin{pmatrix} L^2(E_*) \\ L^2(E) \end{pmatrix} \longrightarrow \mathcal{H}$$

telle que $\dim E = \partial_T$, $\dim E_* = \partial_{T^*}$ et $\pi_*^* \pi = \Theta$ coïncide avec Θ_T (on peut montrer qu'un tel plongement fonctionnel existe). Alors, on a le résultat suivant :

Théorème 1.2.2 (voir [152], Theorem 1.18) *Soit T une contraction complètement non unitaire et Θ une fonction analytique contractive qui coïncide avec Θ_T, la fonction caractéristique de T. Alors T est unitairement équivalent à M_Θ.*

Le quintuplé $\{\mathcal{H}, \pi, \pi_*, K_\Theta, M_\Theta\}$ s'appelle le *modèle fonctionnel sans coordonnées* de T.

Pour terminer cette brève description du modèle fonctionnel sans coordonnées, nous introduisons également deux autres plongements isométriques qui interviennent naturellement dans l'étude du spectre absolument continu de la contraction T. En particulier, ils donnent des représentations spectrales pour la restriction de la dilatation unitaire minimale U de T aux parties résiduelles et $*$-résiduelles de $\mathcal{H}$, c'est à dire à

$$\mathcal{R} = \mathcal{H} \ominus \pi_* L^2(E_*) \quad \text{et} \quad \mathcal{R}_* = \mathcal{H} \ominus \pi L^2(E).$$

On vérifie que

$$\mathcal{R} = \operatorname{clos}(\pi - \pi_*\Theta) L^2(E) \quad \text{et} \quad \mathcal{R}_* = \operatorname{clos}(\pi_* - \pi\Theta^*) L^2(E_*).$$

Puisque $(\pi - \pi_*\Theta)^*(\pi - \pi_*\Theta) = \Delta^2$ et $(\pi_* - \pi\Theta^*)^*(\pi_* - \pi\Theta^*) = \Delta_*^2$, où $\Delta = (Id - \Theta^*\Theta)^{1/2}$ et $\Delta_* = (Id - \Theta\Theta^*)^{1/2}$, les décompositions polaires nous donnent deux isométries partielles $\tau : L^2(E) \longrightarrow \mathcal{H}$, $\tau_* : L^2(E_*) \longrightarrow \mathcal{H}$ telles que

$$\tau\Delta = \pi - \pi_*\Theta \quad \text{et} \quad \tau_*\Delta_* = \pi_* - \pi\Theta^*,$$

et dont les noyaux sont respectivement $L^2(E) \ominus \mathrm{clos}(\Delta L^2(E))$, $L^2(E_*) \ominus \mathrm{clos}(\Delta_* L^2(E_*))$. Si on regarde τ comme défini seulement sur $\mathrm{clos}(\Delta L^2(E))$, elle fournit une équivalence unitaire entre l'opérateur de multiplications par z sur $\mathrm{clos}(\Delta L^2(E))$ et $\mathcal{U}|\mathcal{R}$. De même, si τ_* est définit seulement sur $\mathrm{clos}(\Delta_* L^2(E_*))$, elle fournit une équivalence unitaire entre l'opérateur de multiplications par z sur $\mathrm{clos}(\Delta_* L^2(E_*))$ et $\mathcal{U}|\mathcal{R}_*$.

On vérifie alors facilement que

$$\tau^*\pi = \Delta, \quad \tau^*\pi_* = 0, \quad \tau_*^*\pi = 0, \quad \tau_*^*\pi_* = \Delta_* \tag{1.5}$$

et

$$Id = \pi\pi^* + \tau_*\tau_*^* = \pi_*\pi_*^* + \tau\tau^*. \tag{1.6}$$

En particulier, il suit de (1.5) et (1.6) les décompositions orthogonales suivantes :

$$\mathcal{H} = \pi(L^2(E)) \oplus \tau_*(L^2(E_*)) = \pi_*(L^2(E_*)) \oplus \tau(L^2(E)).$$

Ainsi, il existe deux décompositions très utiles de l'espace modèle K_Θ, à savoir

$$K_\Theta = \mathbb{H}' \oplus \mathbb{H}'' = \mathbb{H}'_* \oplus \mathbb{H}''_*, \tag{1.7}$$

où

(1.8)
$$\mathbb{H}'' = K_\Theta \cap \tau(L^2(E)), \quad \mathbb{H}' = K_\Theta \ominus \mathbb{H}'', \quad \mathbb{H}''_* = K_\Theta \cap \tau_*(L^2(E_*)), \quad \mathbb{H}'_* = K_\Theta \ominus \mathbb{H}''_*.$$

1.3 La transcription du modèle fonctionnel

1.3.1 La transcription de Sz.-Nagy–Foias

Etant donnée $\Theta \in H^\infty(E \to E_*)$ une fonction contractive (pure), on note $\Delta = (Id - \Theta^*\Theta)^{1/2}$ (c'est-à-dire l'opérateur de défaut de la contraction Θ). On définit alors

$$\mathcal{H} = \begin{pmatrix} L^2(E_*) \\ \mathrm{clos}\,\Delta L^2(E) \end{pmatrix}, \tag{1.9}$$

et

$$\pi = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} \quad \text{et} \quad \pi_* = \begin{pmatrix} Id \\ 0 \end{pmatrix}. \tag{1.10}$$

Alors l'espace modèle K_Θ obtenu est

$$K_\Theta = \begin{pmatrix} H^2(E_*) \\ \mathrm{clos}\,\Delta L^2(E) \end{pmatrix} \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(E), \tag{1.11}$$

et l'opérateur modèle est

$$M_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = P_\Theta \begin{pmatrix} zf \\ zg \end{pmatrix} = \begin{pmatrix} zf - \Theta[z(\Theta^* f + \Delta g)]\widehat{\ }(0) \\ zg - \Delta[z(\Theta^* f + \Delta g)]\widehat{\ }(0) \end{pmatrix}, \quad \begin{pmatrix} f \\ g \end{pmatrix} \in K_\Theta. \tag{1.12}$$

Cette transcription s'appelle le modèle fonctionnel de Sz.-Nagy–Foias. De plus, on a

$$M_\Theta^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} P_+ \bar{z} f \\ \bar{z} g \end{pmatrix}. \tag{1.13}$$

Si T est une contraction complètement non unitaire, alors T est unitairement équivalent à l'opérateur modèle M_{Θ_T}. Remarquons que si de plus $T \in C_{\cdot 0}$ (ce qui équivaut à Θ_T intérieure), alors le modèle se simplifie et on a

$$K_{\Theta_T} = H^2(E_*) \ominus \Theta_T H^2(E) \tag{1.14}$$

et T est unitairement équivalent à $M_{\Theta_T} = P_{\Theta_T} S | K_{\Theta_T}$. De façon duale, si $T \in C_{0 \cdot}$ (ce qui équivaut à Θ_{T^*} intérieure), alors on obtient que T est unitairement équivalent à $S^* | K_{\Theta_{T^*}}$ et on retrouve alors le théorème de Rota.

Comme nous l'avons déjà mentionné, le modèle fonctionnel de Sz.-Nagy–Foias a connu un vif succès. Notamment on peut retraduire de nombreux propriétés spectrales de la contraction T en fonction de propriétés fonctionnelles sur Θ_T ; on peut alors utiliser des outils fonctionnels pour résoudre ces questions sur la fonction caractéristique et ensuite récupérer des propriétés sur la contraction T. Ce modèle est donc devenu un outil incontournable en théorie des opérateurs et il existe une immense littérature consacré à ce sujet (en particulier, on pourra consulter [220, 52, 125, 155, 152, 157, 153]). On peut notamment citer l'utilisation de ce modèle dans le cadre du problème du sous-espace invariant. Ainsi, Sz.-Nagy–Foias [220] ont obtenu des résultats de classification du lattice des sous-espaces invariants et des décompositions spectrales pour certaines classes de contractions ; l'idée est qu'une factorisation d'un certain type pour la fonction caractéristique Θ_T donne des renseignements sur la structure des sous-espaces invariants de la contraction T. Je voudrais également rappeler que Sarason [196] a utilisé ce modèle pour développer une nouvelle approche à la théorie de l'interpolation qui s'est révélé très fructueuse.

D'un point de vue plus fonctionnel, les espaces modèles en eux même ont également été l'objet de nombreuses recherches, particulièrement quand l'espace modèle se réduit à $H^2(E_*) \ominus \Theta H^2(E)$ (ce qui correspond au cas où Θ est intérieure). Dans cette direction, de nombreux mathématiciens ont tenté de mieux comprendre la géométrie et les propriétés fonctionnelles de ces espaces. La littérature est aussi immense et je voudrais citer les travaux d'Ahern-Clark [1, 2, 3, 51], Aleksandrov [7]–[14], Baranov [29]–[33], Cima [49, 50], Cohn [55]–[58], Dyakonov [73]–[81], Havin–Mashreghi–Nazarov [107]–[108], Makarov-Poltoratski [170, 145], Nikolski [159, 116], Treil [223, 224, 225, 230], Nazarov-Volberg [150]. Je reviendrais dans les chapitres 2 et 3 sur certains de ces travaux.

1.3.2 La transcription de de Branges–Rovnyak

Etant donnée $\Theta \in H^\infty(E \to E_*)$ une fonction contractive (pure), on note W_Θ l'opérateur autoadjoint positif défini par

$$W_\Theta = \begin{pmatrix} Id & \Theta \\ \Theta^* & Id \end{pmatrix},$$

et $W_\Theta^{[-1]}$ désigne l'opérateur égal à 0 sur $\ker(W_\Theta)$ et à l'inverse de W_Θ sur $\mathrm{Im}(W_\Theta)$. On pose alors $\mathcal{H} = L^2(E_* \oplus E, W_\Theta^{[-1]})$, formé des fonctions $f : \mathbb{T} \longrightarrow E_* \oplus E$ mesurables et telles que

$$\int_{\mathbb{T}} \langle W_\Theta^{[-1]}(\zeta) f(\zeta), f(\zeta) \rangle \, dm(\zeta) < +\infty,$$

et

$$\pi = \begin{pmatrix} \Theta \\ Id \end{pmatrix} \quad \text{et} \quad \pi_* = \begin{pmatrix} Id \\ \Theta^* \end{pmatrix}.$$

Alors l'espace modèle $\mathcal{K}_\Theta$ obtenu est

$$\mathcal{K}_\Theta = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H^2(E), g \in H^2_-(E_*), g - \Theta^* f \in \Delta L^2(E) \right\}, \tag{1.15}$$

et l'opérateur modèle est

$$\mathcal{M}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} zf - \Theta[zg]\hat{\ }(0) \\ zg - [zg]\hat{\ }(0) \end{pmatrix}. \tag{1.16}$$

Cette transcription s'appelle le modèle fonctionnel de de Branges–Rovnyak. De plus, on a

$$\mathcal{M}_\Theta^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} P_+ \bar{z} f \\ \bar{z} g - \Theta^* \bar{z} f(0) \end{pmatrix}.$$

En fait, cela ne correspond pas tout à fait au modèle original de de Branges–Rovnyak. Pour expliciter ce modèle original et l'identifier avec la transcription précédente, nous devons introduire les espaces $\mathcal{H}(\Theta)$ et $\mathcal{D}(\Theta)$ de de Branges–Rovnyak. Dans la théorie du modèle fonctionnel de Sz.-Nagy–Foias, un rôle essentiel est joué par le théorème de Beurling-Lax-Halmos (théorème 1.1.2) concernant la description de tous les sous-espaces fermés de $H^2(E_*)$ qui sont S-invariants. De façon similaire, dans la théorie de de Branges–Rovnyak, un rôle essentiel est joué par la description de tous les sous-espaces contractivement inclus dans $H^2(E_*)$ qui sont S-invariants. De Branges–Rovnyak (voir [156, 25, 65]) ont caractérisé ces sous-espaces en montrant qu'ils sont tous de la forme

$$\mathcal{E} = \mathcal{M}(\Theta) := \{\Theta f : f \in H^2(E)\},$$

où Θ est une fonction dans la boule unité de $H^\infty(E \to E_*)$ et $\mathcal{M}(\Theta)$ est équipé de la norme

$$\|\Theta f\|_{\mathcal{M}(\Theta)} = \|f\|_2, \qquad f \in H^2(E).$$

Alors $\mathcal{H}(\Theta)$ est défini comme l'espace complémentaire (au sens de de Branges–Rovnyak) de l'espace $\mathcal{M}(\Theta)$ dans $H^2(E_*)$; plus précisément $\mathcal{H}(\Theta)$ est formé des fonctions f de $H^2(E_*)$ telles que

$$\|f\|^2_{\mathcal{H}(\Theta)} := \sup_{g \in \mathcal{M}(\Theta)} \left(\|f+g\|_2^2 - \|g\|^2_{\mathcal{M}(\Theta)}\right) < +\infty. \tag{1.17}$$

Une autre façon plus opératorielle de définir $\mathcal{H}(\Theta)$ est de poser

$$\mathcal{H}(\Theta) = (Id - T_\Theta T_\Theta^*)^{1/2} H^2(E_*), \tag{1.18}$$

muni du produit scalaire

$$\langle (Id - T_\Theta T_\Theta^*)^{1/2}) f, (Id - T_\Theta T_\Theta^*)^{1/2}) g\rangle_{\mathcal{H}(\Theta)} = \langle f, g\rangle_2, \tag{1.19}$$

pour $f, g \in H^2(E_*) \ominus \ker(Id - T_\Theta T_\Theta^*)$. On vérifie alors que les deux définition coïncident et que $\mathcal{H}(\Theta)$ est un espace de Hilbert inclus contractivement dans $H^2(E_*)$. Notons de plus que S^* laisse invariant $\mathcal{H}(\Theta)$ et il agit sur $\mathcal{H}(\Theta)$ comme une contraction que nous noterons $X_\Theta := S^*|\mathcal{H}(\Theta)$. Remarquons que dans le cas particulier où Θ est intérieure, alors $Id - T_\Theta T_\Theta^*$ est une projection orthogonale et $\mathcal{H}(\Theta) = K_\Theta$.

Maintenant pour pouvoir définir le modèle complet de de Branges–Rovnyak, nous devons introduire l'espace vectoriel suivant

$$\mathcal{D}_0(\Theta) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in \mathcal{H}(\Theta), g \in H^2(E) \text{ et } z^{n+1} f - \Theta P_+ z^{n+1} J g \in \mathcal{H}(\Theta),\ n \geq 0 \right\},$$

où $J : L^2(E) \longrightarrow L^2(E)$ est l'opérateur unitaire défini par $(Jg)(z) = \overline{z} g(\overline{z})$, $g \in L^2(E)$. Pour $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}_0(\Theta)$, $n \geq 0$, on pose

$$u_n := \left\|z^{n+1} f - \Theta P_+ z^{n+1} J g\right\|^2_{\mathcal{H}(\Theta)} + \|P_+ z^{n+1} J g\|_2^2.$$

Alors, en utilisant la définition (1.17), on vérifie que la suite $(u_n)_n$ est croissante et donc elle admet une limite (éventuellement $+\infty$). On définit alors l'espace $\mathcal{D}(\Theta)$ comme l'espace des fonctions $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}_0(\Theta)$ telles que

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2_{\mathcal{D}(\Theta)} := \lim_{n \to +\infty} \left(\left\|z^{n+1} f - \Theta P_+ z^{n+1} J g\right\|^2_{\mathcal{H}(\Theta)} + \|P_+ z^{n+1} J g\|_2^2 \right) < +\infty.$$

L'espace $\mathcal{D}(\Theta)$, muni de $\|\cdot\|_{\mathcal{D}(\Theta)}$, est un espace de Hilbert, contenu contractivement dans $H^2(E_*) \oplus H^2(E)$. De plus, l'opérateur i défini par

$$i\left((Id - T_\Theta T_\Theta^*)h\right) := (Id - T_\Theta T_\Theta^*)h \oplus JP_-\Theta^* h, \qquad h \in H^2(E_*),$$

injecte isométriquement l'espace $\mathcal{H}(\Theta)$ dans l'espace $\mathcal{D}(\Theta)$. Son adjoint est donné par $i^*(f \oplus g) = f$.

Le modèle original de de Branges–Rovnyak est alors

$$\mathrm{BR}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} P_+\bar{z}f \\ zg - \Theta(\bar{z})^* f(0) \end{pmatrix}, \qquad \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(\Theta).$$

Pour passer de ce modèle original à notre transcription initiale, il suffit de considérer l'unitaire

$$\mathcal{J} = \begin{pmatrix} Id & 0 \\ 0 & J \end{pmatrix} : \mathcal{H}(\Theta) \oplus L^2(E) \longrightarrow \mathcal{H}(\Theta) \oplus L^2(E).$$

On a alors

$$\mathcal{K}_\Theta = \mathcal{J}\mathcal{D}(\Theta) \quad \text{et} \quad \mathcal{J}\mathcal{M}_\Theta^*\mathcal{J} = \mathrm{BR}.$$

Signalons que dans le cas $\mathrm{clos}(\Delta H^2(E)) = \mathrm{clos}(\Delta L^2(E))$ (ce qui correspond au cas où la contraction T est telle que T^* ne contient aucune partie isométrique), alors le modèle de de Branges–Rovnyak se simplifie considérablement ; en effet, dans ce cas, l'espace modèle se réduit à $\mathcal{H}(\Theta)$, l'opérateur modèle BR se réduit à X_Θ et on peut montrer que

$$X_\Theta^* f = Sf - \Theta j^* f \qquad (f \in \mathcal{H}(\Theta)), \tag{1.20}$$

où $j : e \longrightarrow S^*\Theta e$ est une contraction de E dans $\mathcal{H}(\Theta)$. Autrement si T est une contraction complètement non unitaire et complètement non co-isométrique, alors T est unitairement équivalent à l'opérateur défini sur $\mathcal{H}(\Theta)$ par la formule (1.20). Ainsi nous voyons que l'un des avantages du modèle de de Branges–Rovnyak par rapport à celui de Sz.-Nagy–Foias est qu'il se simplifie dans un certain sens plus souvent.

Comme les modèles de Sz.-Nagy–Foias et de Branges–Rovnyak sont des transcriptions d'un même modèle, on sait qu'ils sont unitairement équivalents. Indépendamment, Nikolski–Vasyunin [156] et Ball [25] explicitent complètement cette équivalence unitaire, en donnant une formule pour l'opérateur unitaire qui permet de passer d'un modèle à l'autre. Plus précisément, si on note $M_\Theta \in \mathcal{L}(K_\Theta)$ l'opérateur modèle de Sz.-Nagy–Foias et si on pose

$$U(f \oplus g) = f \oplus J(\Theta^* f + \Delta g),$$

alors U est un opérateur unitaire de K_Θ sur $\mathcal{D}(\Theta)$ et on a $\mathrm{BR}^* = UM_\Theta\mathcal{U}^{-1}$. De plus, si π_* est l'isométrie définie par (1.10), alors on a $i^*U = \pi_*^*$ et π_*^* est une isométrie partielle de K_Θ sur $\mathcal{H}(\Theta)$.

Le modèle de de Branges–Rovnyak a été généralisé par plusieurs auteurs dans différentes directions ; notamment dans [26], Ball utilise des idées analogues pour construire un modèle pour des opérateurs qui ne sont pas nécessairement des contractions ; voir aussi [175]. De plus, les espaces de de Branges–Rovnyak interviennent dans l'étude de plusieurs questions centrales en analyse comme l'interpolation des fonctions analytiques dans les classes de Schur et de Carathéodory [82, 17] ou la théorie des systèmes et le problème de diffusion inverse [64, 27, 18, 19]. D'autre part, Sarason a consacré une série de papiers [197, 198, 199, 200, 201, 202, 142, 143, 144] et un livre [203] aux espaces de Branges–Rovnyak $\mathcal{H}(\Theta)$ (dans le cas scalaire). Notamment, dans [142, 143, 144], Sarason étudie les multiplicateurs des espaces de de Branges–Rovnyak, dans [199] il montre comment on peut retrouver très simplement des résultats de Carathéodory en utilisant ces espaces. Enfin dans [201, 202], il montre comment ces espaces $\mathcal{H}(\Theta)$ sont reliés à des questions difficiles et importantes de théorie des fonctions avec en particulier la description des points exposés de la boule unité de H^1. Citons également le travail de de Branges [61] concernant sa preuve de la conjecture de Bieberbach qui utilise de façon cruciale les espaces $\mathcal{H}(\Theta)$. Enfin, plus récemment les espaces de de Branges–Rovnyak sont apparus dans diverses questions de théorie des opérateurs ou d'analyse complexe. Je voudrais citer un travail d'Hartmann-Sarason-Seip [106] dans lequel ces trois auteurs donnent un critère pour la surjectivité d'un opérateur de Toeplitz et la preuve utilise les espaces $\mathcal{H}(\Theta)$. Il existe aussi un papier récent d'Anderson-Rovnyak [20] où des estimations généralisées de type Schwarz-Pick sont données et un article de Jury [119] dans lequel les opérateurs de composition sont étudiés par des méthodes basées sur les espaces $\mathcal{H}(\Theta)$. Malgré tout, les espaces $\mathcal{H}(\Theta)$ restent encore mystérieux et de nombreuses questions restent ouvertes. Une partie des travaux de cette habilitation avait pour objectif de comprendre un peu mieux ces espaces.

1.4 Plan du mémoire

Dans le chapitre 2, nous nous intéressons à l'aspect géométrique des espaces de de Branges–Rovnyak $\mathcal{H}(\Theta)$. Plus précisément, nous étudions les suites de noyaux reproduisants de ces espaces et tentons de comprendre quand une suite de noyaux reproduisants forme une base orthonormale, une base asymptotiquement orthonormale, une base de Riesz, une suite minimale ou bien encore une suite surcomplète.

Dans le chapitre 3, nous adoptons un point de vue plus fonctionnel pour étudier les espaces $\mathcal{H}(\Theta)$ dans le cas scalaire. Nous essayons d'analyser le comportement des fonctions de $\mathcal{H}(\Theta)$ lorsqu'on s'approche du cercle unité $\mathbb{T}$ et tentons de relier ce comportement à la fonction Θ elle-même. Notamment, nous montrons comment on peut obtenir des inégalités de type Bernstein et nous appliquons ces inégalités à des problèmes de plongement type Volberg–Treil et à des questions sur la stabilité des bases.

Dans le chapitre 4, nous étudions le comportement au bord des produits de Blaschke. Nous donnons des estimations sur la croissance, à la fois des dérivées n-ièmes d'un produit de Blaschke B, mais aussi de sa dérivée logarithmique, lorsqu'on s'approche du bord ; ces estimations se font en fonction de la vitesse de convergence de la suite des zéros de B.

Dans le dernier chapitre, notre travail porte sur plusieurs questions de théorie des opérateurs. Dans un premier temps, nous étudions une classe de contractions particulières (celle qui sont complexes symétriques) à travers leurs fonctions caractéristiques. Puis, nous nous intéressons aux opérateurs algébriques à travers une question de stabilité spectrale. Enfin, nous terminons par une question liée au problème du sous-espace invariant.

Chapitre 2

Géométrie des espaces de de Branges-Rovnyak

Dans ce chapitre, nous allons présenter les travaux [T1]–[T4] qui portent sur la géométrie des suites de noyaux reproduisants des espaces de de Branges–Rovnyak $\mathcal{H}(b)$.

2.1 Introduction

2.1.1 Les noyaux reproduisants des espaces de de Branges–Rovnyak

On se donne une fonction b dans la boule unité de $H^\infty(E \to E*)$. Rappelons que $\mathcal{H}(b)$ est défini comme l'image de l'opérateur $(Id - T_b T_b^*)^{1/2} H^2(E_*)$, muni du produit scalaire

$$\langle (Id - T_b T_b^*)^{1/2}) f, (Id - T_b T_b^*)^{1/2}) g \rangle_b := \langle f, g \rangle_2, \tag{2.1}$$

pour $f, g \in H^2(E_*) \ominus \ker(Id - T_b T_b^*)$. Si $\lambda \in \mathbb{D}$ et si $e \in E_*$, alors d'après la formule de Cauchy, on a

$$\langle f(\lambda), e \rangle_{E_*} = \langle f, k_{\lambda.e} \rangle_2,$$

pour toute fonction $f \in H^2(E_*)$, avec $k_{\lambda,e}$ le noyau de Cauchy de $H^2(E_*)$ défini par

$$k_{\lambda,e}(z) := \frac{1}{1 - \overline{\lambda} z} e, \qquad (z \in \mathbb{D}).$$

Ainsi en particulier, on obtient que la forme linéaire $f \longmapsto \langle f(\lambda), e \rangle_{E_*}$ est continue sur $H^2(E_*)$. Comme $\mathcal{H}(b)$ est inclus contractivement dans $H^2(E_*)$, cette forme linéaire est aussi continue si on la restreint à $\mathcal{H}(b)$. Le théorème de Riesz implique alors qu'il existe une unique fonction $k_{\lambda,e}^b$ dans $\mathcal{H}(b)$ telle que

$$\langle f, k_{\lambda,e}^b \rangle_b = \langle f(\lambda), e \rangle_{E_*},$$

pour toute fonction $f \in \mathcal{H}(b)$. Cette fonction $k^b_{\lambda,e}$ s'appelle le *noyau reproduisant* de $\mathcal{H}(b)$, associé au point $\lambda \in \mathbb{D}$ et au vecteur $e \in E_*$. En fait, on peut donner une formule explicite pour ce noyau reproduisant. En utilisant la définition du produit scalaire dans $\mathcal{H}(b)$, on remarque d'abord que

$$k^b_{\lambda,e} = (Id - T_b T_b^*) k_{\lambda,e}, \tag{2.2}$$

et comme $T_b^*(k_{\lambda,e}) = b(\lambda)^* k_{\lambda,e}$, on obtient

$$k^b_{\lambda,e}(z) = \frac{Id - b(z)b(\lambda)^*}{1 - \overline{\lambda} z} e. \tag{2.3}$$

Dans le cas scalaire ($\dim E = \dim E_* = 1$), on note k_λ et k^b_λ les noyaux respectifs de H^2 et $\mathcal{H}(b)$.

2.1.2 Les systèmes d'exponentielles et le lien avec les espaces modèles

Si l'on considère une fonction f dans $L^2(0, 2\pi)$, on peut la développer en série de Fourier L^2-convergente

$$f(t) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{int}, \qquad t \in (0, 2\pi),$$

où

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt.$$

Que se passe-t-il si on remplace la base orthonormale $(e^{int})_{n \in \mathbb{Z}}$ de $L^2(0, 2\pi)$ par un système d'exponentielles $(e^{i\mu_n t})_{n \in \mathbb{Z}}$ quelconque ? Cette question trouve son origine dans les travaux de R. E. Paley et N. Wiener [164] et ceux de N. Levinson [135] et a donné naissance à la théorie des séries de Fourier non-harmoniques (voir [234] pour un survey sur le sujet).

Un premier point de vue consiste à regarder la famille $(e^{i\mu_n t})_{n \in \mathbb{Z}}$ comme une petite perturbation du système trigonométrique classique et étudier les perturbations admissibles qui conservent de bonnes propriétés. Ceci a ouvert la voie à toute une direction de recherche. Concernant des résultats de stabilité pour les bases d'exponentielles on pourra consulter [164, 117, 70, 120] et pour des résultats de stabilité pour la propriété de complétude mentionnons [134, 135, 15, 183, 184, 185, 92, 206, 207, 208, 166, 233, 234, 83]. Un autre point de vue qui trouve aussi son origine dans les travaux de Paley–Wiener a été développé par B. Ja. Levin. Dans cette méthode, un rôle fondamental est joué par une certaine fonction entière de type exponentiel fini et qui s'annule aux points μ_n ($n \in \mathbb{Z}$), correspondant aux fréquences du système d'exponentielles étudié. Si cette fonction entière G (appelée fonction génératrice pour la famille d'exponentielles $(e^{i\mu_n t})_{n \in \mathbb{Z}}$) vérifie une certaine propriété de croissance, alors B. Levin [129, 130] et V. Golovin

[97] ont démontré que la famille $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ forme une base inconditionnelle de $L^2(0,a)$, où a est lié aux caractéristiques de la fonction G. Ainsi il existe un lien très étroit entre les propriétés des systèmes d'exponentielles et la théorie des fonctions entières. Enfin un troisième point de vue concernant ce problème des bases d'exponentielles a été introduit par N. Nikolski dans [159] puis dans [116] avec S. Hruščev et B. Pavlov (on pourra voir aussi [153, 165, 115, 161]). Cette nouvelle approche, basée sur le modèle fonctionnel de Sz.-Nagy–Foias, a non seulement permis de retrouver tous les résultats classiques mais elle a aussi donné de nombreux résultats nouveaux qui ont ouvert la voie à d'autres développements. Les publications [89, 90] (issues de ma thèse) et les travaux [T1–T4] (présentés dans ce chapitre) s'inscrivent dans cette direction de recherche.

Expliquons maintenant cette approche introduite par Hruščev-Nikolski-Pavlov. On note $\mathbb{D}$ le disque unité ouvert du plan complexe et $\mathbb{C}_+$ le demi-plan supérieur ouvert. Considérons $\mathcal{F}$ la transformée de Fourier sur $L^2(\mathbb{R})$ et U l'opérateur unitaire qui envoie l'espace de Hardy $H^2(\mathbb{D})$ sur l'espace de Hardy $H^2(\mathbb{C}_+)$. Alors un théorème de Paley–Wiener affirme que $\mathcal{F}UH^2(\mathbb{D}) = H^2(\mathbb{C}_+)$ et un calcul élémentaire montre que, si $|\lambda| < 1$, alors

$$\mathcal{F}Uk_\lambda = c(\lambda)e^{-i\overline{\mu}t},$$

où $c(\lambda) \in \mathbb{C}$, $\mu = \omega(\lambda) := i\frac{1+\lambda}{1-\lambda}$. D'autre part, si Θ_a est la fonction intérieure (scalaire) définie par

$$\Theta_a(z) := e^{-a\frac{1+z}{1-z}}, \qquad a > 0, z \in \mathbb{D},$$

alors $\mathcal{F}UK_{\Theta_a} = L^2(0,a)$. De plus, on voit facilement que, si $\lambda \in \mathbb{D}$, $\mu = \omega(\lambda)$, alors

$$\mathcal{F}Uk_\lambda^{\Theta_a} = c(\lambda)e^{-i\overline{\mu}t}\chi_{(0,a)}.$$

On rappelle ici que si Θ est une fonction intérieure (c'est-à-dire une fonction holomorphe et bornée dans $\mathbb{D}$ dont les limites radiales sont de module 1 presque partout sur $\mathbb{T} = \partial\mathbb{D}$), alors K_Θ est l'espace modèle associé à Θ par $K_\Theta = H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D})$ et on a $K_\Theta = \mathcal{H}(\Theta)$ et $Id - T_\Theta T_\Theta^*$ est la projection orthogonale de H^2 sur K_Θ que nous notons aussi P_Θ.

Comme $\mathcal{F}U$ est un opérateur unitaire, on voit que toutes les propriétés géométriques (complétude, minimalité, uniforme minimalité, base...) d'une suite d'exponentielles sur $L^2(0,a)$ sont les mêmes que la suite associée de noyaux reproduisants de K_{Θ_a}. Nikolski a alors considéré le problème plus général suivant : *étant donné Θ une fonction intérieure quelconque (scalaire), peut-on caractériser les suites $(\lambda_n)_{n\geq 1}$ de $\mathbb{D}$ telles que la suite de noyaux reproduisants $(k_{\lambda_n}^\Theta)_{n\geq 1}$ forme une base inconditionnelle de K_Θ ?*

Outre le lien avec les exponentielles et les applications en théorie du contrôle notamment (voir [155, 116, 160, 23] pour les nombreuses applications des systèmes d'exponentielles), l'une des motivations pour s'intéresser à ce problème général vient du modèle fonctionnel. Ainsi la théorie spectrale d'une contraction T dépend beaucoup de la géométrie des noyaux reproduisants de K_{Θ_T}, où Θ_T est la fonction caractéristique de T, comme le souligne le résultat suivant de [116] : soit $T \in \mathcal{L}(H)$ une contraction complètement non unitaire, de class C_{00} et dont les indices de défaut sont égaux à 1 ; si $\Theta_T = SB$ est la décomposition canonique de Θ_T en un facteur intérieur singulier S et en un produit de Blaschke B associé à une suite de zéros $(\lambda_n)_{n\geq 1}$, alors la réunion des vecteurs propres de T et T^* forme une base inconditionnelle de H si et seulement si la suite de noyaux reproduisants $(k^S_{\lambda_n})_{n\geq 1}$ forme une base inconditionnelle de K_S et $\sup_{n\geq 1} |S(\lambda_n)| < 1$.

2.1.3 La méthode initiée par N. Nikolski

Pour résoudre le problème des bases de noyaux reproduisants de l'espace modèle K_Θ, l'idée essentielle de N. Nikolski, inspiré des travaux de S. Hruščëv [115] et B. Pavlov [165], est de regarder la famille $(k^\Theta_{\lambda_n})_{n\geq 1}$ comme une distorsion de la famille $(k_{\lambda_n})_{n\geq 1}$, en se basant sur la formule (2.2). Comme on peut caractériser la propriété de base inconditionnelle de la suite $(k_{\lambda_n})_{n\geq 1}$ par la condition de Carleson, on est alors ramené pour résoudre le problème à étudier l'inversibilité de la projection $P_\Theta|K_B : K_B \longrightarrow K_\Theta$, où B est le produit de Blaschke associé à la suite $(\lambda_n)_{n\geq 1}$ (on sait que dès que $(\lambda_n)_{n\geq 1}$ est une suite de Blaschke, la suite de noyaux reproduisants associée est minimale et complète dans K_B). Cette méthode nécessite de supposer que l'opérateur $P_\Theta : k_{\lambda_n} \longmapsto k^\Theta_{\lambda_n}$ ne déforme pas trop les normes des noyaux reproduisants dans le sens que

$$\sup_{n\geq 1} \frac{\|k_{\lambda_n}\|}{\|P_\Theta k_{\lambda_n}\|} < +\infty.$$

Il est facile de voir que cette condition est équivalente à la condition suivante

$$\sup_{n\geq 1} |\Theta(\lambda_n)| < 1. \tag{2.4}$$

Une grande partie de la théorie s'est donc développée sous cette hypothèse supplémentaire. Signalons que dans le cas particulier des systèmes d'exponentielles, la condition (2.4) signifie que la partie imaginaire des fréquences de notre système est bornée inférieurement, ce qui est souvent le cas dans les applications en théorie du contrôle notamment.

Maintenant pour résoudre le problème de l'inversibilité de la projection $P_\Theta|K_B$, N. Nikolski a utilisé le langage des opérateurs de Hankel et Toeplitz et il a ainsi obtenu le critère suivant (on note h^Θ_λ le noyau reproduisant normalisé de K_Θ associé au point $\lambda \in \mathbb{D}$).

Théorème 2.1.1 (Nikolski) *Soient $(\lambda_n)_{n\geq 1}$ une suite de Blaschke dans le disque unité, B le produit de Blaschke associé et Θ une fonction intérieure satisfaisant* (2.4). *Les assertions suivantes sont équivalentes :*

(i) $(h^\Theta_{\lambda_n})_{n\geq 1}$ *est une suite (resp. une base) de Riesz de* K_Θ ;

(ii) $(\lambda_n)_{n\geq 1}$ *vérifie la condition de Carleson et l'opérateur* $P_\Theta : K_B \longrightarrow K_\Theta$ *est un isomorphisme sur son image (resp. sur* K_Θ*)* ;

(ii) $(\lambda_n)_{n\geq 1}$ *vérifie la condition de Carleson et dist* $(\Theta\overline{B}, H^\infty) < 1$ *(resp. et dist* $(B\overline{\Theta}, H^\infty) < 1$*).*

Dans [116], Hruščëv, Nikolski et Pavlov ont donné d'autres critères notamment en fonction de l'argument de $\Theta\overline{B}$ sur $\mathbb{T}$. Comme on l'a déjà mentionné, ce résultat a ouvert la voie à toute une direction de recherche et les propriétés géométriques des noyaux reproduisants de H^2 et K_Θ ont été beaucoup étudiées notamment par Vasyunin [226], Ivanov [23], Boricheva [42], Treil [224], Clark [51], Baranov [32] et Makarov–Poltoratski [145]. Dans [89], j'ai étudié le problème des bases dans le cas vectoriel et j'ai notamment donné des résultats de stabilité ; dans [90], j'ai considéré le problème de la complétude des noyaux reproduisants de K_Θ et également le problème de la complétude de la biorthogonale.

2.2 Suites asymptotiquement orthonormales dans K_Θ

Dans [T1], je me suis intéressé, en collaboration avec I. Chalendar et D. Timotin, à la propriété de suites (ou bases) asymptotiquement orthonormales formées de noyaux reproduisants de K_Θ, avec Θ une fonction intérieure (scalaire). Rappelons que si $(x_n)_{n\geq 1}$ est une suite d'un espace de Hilbert H, on dit que $(x_n)_{n\geq 1}$ est une :

(a) *suite asymptotiquement orthonormale* dans H (abrégée par SAO) s'il existe un entier N_0 tel que, pour tout $N \geq N_0$, il existe des constantes $c_N, C_N > 0$ vérifiant pour toute suite finie $(a_n)_{n\geq 1}$ de nombres complexes :

$$c_N \sum_{n\geq N}^{\infty} |a_n|^2 \leq \left\| \sum_{n\geq N}^{\infty} a_n x_n \right\|^2 \leq C \sum_{n\geq N}^{\infty} |a_n|^2,$$

et $\lim_{N\to+\infty} c_N = \lim_{N\to+\infty} C_N = 1$;

(b) *suite basique asymptotiquement orthonormale* dans H (abrégée par SBAO) si $(x_n)_{n\geq 1}$ est une SAO avec $N_0 = 1$;

(c) *base asymptotiquement orthonormale* de H (abrégé par BAO) si $(x_n)_{n\geq 1}$ est une SBAO complète dans H.

Autrement dit, une BAO est très proche d'être une base orthonormale et en particulier, une BAO est une base de Riesz.

Dans [228], Volberg a montré que si $(\lambda_n)_{n\geq 1}$ est une suite de Blaschke de $\mathbb{D}$ et si B désigne le produit de Blaschke associé, alors la suite de noyaux reproduisants normalisée $(h_{\lambda_n})_{n\geq 1}$ est une BAO de K_B si et seulement si

$$\lim_{n\to+\infty} |B_n(\lambda_n)| = 1. \tag{2.5}$$

Les suites de Blaschke vérifiant (2.5) sont également apparues dans d'autres contextes dans la littérature (voir par exemple [218, 162, 110, 99]). En particulier, dans [218], C. Sundberg et T. Wolff montrent que parmi les suites de Carleson, les suites vérifiant (2.5) sont caractérisées par la propriété d'être des suites d'interpolation libre pour $H^\infty \cap VMO$. En utilisant la matrice de Gram, on peut donner une caractérisation différente. Notons Γ la matrice de Gram associée à la suite $(h_{\lambda_n})_{n\geq 1}$ et $(e_n)_{n\geq 1}$ la base orthonormale canonique de ℓ^2. Si $(h_{\lambda_n})_{n\geq 1}$ est une BAO, alors un fait général (voir [89, Lemma 3.7]) implique que Γ défini un opérateur borné et inversible sur ℓ^2 de la forme $\Gamma = Id + K$, avec K un opérateur compact. Comme $Ke_n \to 0$, $n \to +\infty$, on obtient que $(\Gamma - Id)e_n \to 0$, $n \to +\infty$. On montre dans [T1] que la réciproque est aussi vraie.

Proposition 2.2.1 *Soient $(\lambda_n)_{n\geq 1}$ une suite de Blaschke de points distincts de $\mathbb{D}$ et B le produit de Blaschke associé. Les assertions suivantes sont équivalentes :*

(i) *$(h_{\lambda_n})_{n\geq 1}$ est une BAO de K_B ;*

(ii) *$(\Gamma - Id)e_n \to 0$, $n \to +\infty$.*

Il est bien connu (voir par exemple [153]) que les suites de Carleson $\Lambda = (\lambda_n)_{n\geq 1}$ peuvent être caractérisées par le fait qu'elles sont séparées (par rapport à la métrique pseudo-hyperbolique) et que la mesure $\sum_{n\geq 1}(1-|\lambda_n|^2)\delta_{\lambda_n}$ est une mesure de Carleson. Nous avons montré qu'une caractérisation similaire peut être donnée pour les suites vérifiant (2.5). Pour énoncer le résultat, nous avons besoin d'introduire quelques notations. Pour tout $z \in \mathbb{D}$, I_z est l'arc contenu dans $\mathbb{T}$ de centre $z/|z|$ et de longueur $1-|z|$. Pour $I \subset \mathbb{T}$,

$$S_I := \{z \in \mathbb{D} : \frac{z}{|z|} \in I, |z| \geq 1 - |I|\};$$

pour $c > 0$, cI est l'intervalle de même centre et de longueur $c|I|$.

Proposition 2.2.2 *Soit $(\lambda_n)_{n\geq 1}$ une suite de Blaschke de points distincts de $\mathbb{D}$. Les assertions suivantes sont équivalentes :*

(i) *$(\lambda_n)_{n\geq 1}$ vérifie la condition (2.5) ;*

(ii) *pour tout $A \geq 1$, on a*

$$\lim_{n\to+\infty} \frac{1}{|I_{\lambda_n}|} \sum_{\substack{p\neq n \\ \lambda_p \in S_{AI_{\lambda_n}}}} (1-|\lambda_p|) = 0.$$

Comme conséquence de cette proposition, nous donnons deux résultats qui clarifient un peu la géométrie des suites satisfaisant (2.5).

Proposition 2.2.3 (a) *Soit $(\lambda_n)_{n\geq 1}$ une suite de $\mathbb{D}$ telle que la suite $(|\lambda_n|)_{n\geq 1}$ est une suite croissante vers 1. Si*

$$\gamma := \lim_{k\to+\infty} \frac{1-|\lambda_{k+1}|}{1-|\lambda_k|} = 0,$$

alors $(\lambda_n)_{n\geq 1}$ vérifie la condition (2.5). Si, en plus $(\lambda_n)_{n\geq 1} \subset [0,1[$, alors la réciproque est vraie.

(b) *Supposons que $(r_n)_{n\geq 1}$ soit une suite d'entiers positifs distincts, $0 < r_n < 1$, telle que $\sum_{n\geq 1}(1-r_n) < +\infty$. Alors il existe $t_n \geq 0$ tel que la suite $(r_n e^{it_n})_{n\geq 1}$ satisfait la condition (2.5).*

Ces deux résultats sont à comparer avec des résultats analogues pour les suites de Carleson (voir [153, Chap. VII]).

On se donne maintenant une fonction intérieure Θ et $\Lambda = (\lambda_n)_{n\geq 1}$ une suite de Blaschke de points distincts de $\mathbb{D}$. On s'intéresse à la propriété de BAO pour les suites de noyaux reproduisants normalisés $(h^\Theta_{\lambda_n})_{n\geq 1}$. Le résultat suivant donne une condition nécessaire.

Proposition 2.2.4 *Si $(h^\Theta_{\lambda_n})_{n\geq 1}$ est une SAO, alors la suite $(\lambda_n)_{n\geq 1}$ vérifie la condition (2.5).*

La preuve est basée sur la proposition 2.2.1 et l'estimation suivante $|\Gamma^\Theta_{n,p}| \geq |\Gamma_{n,p}|$, où $\Gamma = (\Gamma_{n,p})_{n,p\geq 1}$ est la matrice de Gram associée à $(h_{\lambda_n})_{n\geq 1}$ et $\Gamma^\Theta = (\Gamma^\Theta_{n,p})_{n,p\geq 1}$ celle associée à $(h^\Theta_{\lambda_n})_{n\geq 1}$.

En général, il n'est pas possible d'espérer obtenir une réciproque à la proposition 2.2.4 sans hypothèse supplémentaire. En effet, supposons que la suite $(\lambda_n)_{n\geq 1}$ vérifie la condition (2.5) et converge non-tangentiellement vers un point $\zeta \in \mathbb{T}$ à travers lequel Θ se prolonge analytiquement. Alors, on peut voir facilement que $(k^\Theta_{\lambda_n})_{n\geq 1}$ converge en norme vers $k^\Theta_\zeta \neq 0$, où

$$k^\Theta_\zeta(z) = \frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\overline{\zeta}z}.$$

Donc en particulier, $(h^\Theta_{\lambda_n})_{n\geq 1}$ ne peut pas être une suite de Riesz (car sinon $(h^\Theta_{\lambda_n})_{n\geq 0}$ doit converger faiblement vers 0). Cependant, sous la condition (2.4), nous avons donné une réciproque à la proposition 2.2.4.

Théorème 2.2.5 *Supposons que $(\lambda_n)_{n\geq 1}$ vérifie les conditions* (2.4) *et (2.5). Alors soit*

(i) *$(h^\Theta_{\lambda_n})_{n\geq 1}$ est une SBAO,*

soit

(ii) *il existe $p \geq 2$ tel que $(h^\Theta_{\lambda_n})_{n\geq p}$ est une BAO de K_Θ.*

Le théorème 2.2.5 améliore en fait un résultat de [89, Lemma 3.9].

Le cas (ii) du théorème 2.2.5 correspond aux suites $(h^\Theta_{\lambda_n})_{n\geq 1}$ qui ne sont pas minimales ; un exemple peut être construit en considérant pour Θ un facteur intérieur non trivial de B. D'un autre coté, les suites minimales de noyaux reproduisants de K_Θ ont été étudiées par I. Boricheva dans [42]. En combinant [42, théorème 4.7] et le théorème 2.2.5, on obtient alors la caractérisation suivante.

Corollaire 2.2.6 *Sous l'hypothèse* (2.4)*, les assertions suivantes sont équivalentes :*

(i) *la suite* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *est une SBAO dans* K_Θ *;*

(ii) $(\lambda_n)_{n\geq 1}$ *vérifie la condition (2.5) et la suite* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *est minimale ;*

(iii) $(\lambda_n)_{n\geq 1}$ *vérifie la condition (2.5) et il existe* $f \in H^\infty$*,* $f \neq 0$*, tel que* $\|\Theta + Bf\|_\infty \leq 1$.

Il est intéressant de comparer ce corollaire avec le théorème 2.1.1 qui montre que $(h^\Theta_{\lambda_n})_{n\geq 1}$ est une suite de Riesz si et seulement si $(\lambda_n)_{n\geq 1}$ vérifie la condition de Carleson et dist $(\Theta\overline{B}, H^\infty) < 1$. Cette dernière condition est bien sûr plus forte que la dernière condition du corollaire 2.2.6 ; d'un autre coté, la condition (2.5) est plus forte que la condition de Carleson.

Dans le cas où Θ n'est pas un produit de Blaschke, on peut dire un peu plus.

Proposition 2.2.7 *Soit* Θ *une fonction intérieure, avec une partie singulière non triviale, et supposons* (2.4)*. Les assertions suivantes sont équivalentes :*

(i) $(\lambda_n)_{n\geq 1}$ *vérifie la condition (2.5) ;*

(ii) $(h^\Theta_{\lambda_n})_{n\geq 1}$ *est une SBAO.*

De plus, dans ce cas, $\dim(K_\Theta \ominus \mathrm{Span}(h^\Theta_{\lambda_n} : n \geq 1)) = +\infty$.

La preuve de cette proposition adapte en fait un argument de [116, Théorème 3.2]. Nous avons également donné plusieurs résultats de stabilité qu'on peut comparer avec des résultats de stabilité pour les bases de Riesz obtenus dans [89].

Théorème 2.2.8 *Supposons que* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *est une SBAO et que* (2.4) *est satisfaite. Si* $\Lambda' = (\lambda'_n)_{n\geq 1}$ *est une suite de points distincts de* $\mathbb{D}$ *qui satisfait*

$$\limsup_{n\to\infty} |b_{\lambda_n}(\lambda'_n)| < \frac{1 - dist\ (\Theta\bar{B}, H^\infty)}{1 + dist\ (\Theta\bar{B}, H^\infty)}, \tag{2.6}$$

alors $(h^\Theta_{\lambda'_n})_{n\geq 1}$ *est une SBAO.*

2.3 Suites surcomplètes dans K^p_Θ

Dans [T2], avec I. Chalendar et J. Partington, nous nous sommes intéressés au problème suivant : *étant donné* Θ *une fonction intérieure,* $1 < p < +\infty$ *et* $(\lambda_n)_{n\geq 1}$

une suite de points distincts de $\mathbb{D}$, peut-on trouver une condition nécessaire et suffisante pour que la suite $(k^\Theta_{\lambda_n})_{n\geq 1}$ soit surcomplète dans K^p_Θ ?

Ici K^p_Θ désigne le sous-espace de H^p défini par

$$K^p_\Theta = H^p \cap \Theta\overline{H^p_0} = \{f \in H^p : \langle f, \Theta g\rangle = 0, \forall g \in H^q\},$$

où H^p est l'espace de Hardy du disque unité avec la norme $\|\cdot\|_p$, p et q sont des exposants conjugués et $H^p_0 = \{f \in H^p : f(0) = 0\}$. D'autre part, rappelons qu'une suite $(x_n)_{n\geq 1}$ d'un espace de Banach X est dite *surcomplète* si toute sous-suite infinie de $(x_n)_{n\geq 1}$ est complète dans X.

Comme on peut le voir facilement avec le théorème de Hahn-Banach, la surcomplétude de $(k^\Theta_{\lambda_n})_{n\geq 1}$ dans K^p_Θ est équivalente à l'assertion suivante : si $f \in K^q_\Theta$, $f(\lambda_{n_p}) = 0$ pour une sous-suite infinie $(\lambda_{n_p})_{p\geq 1}$ de $(\lambda_n)_{n\geq 1}$, alors $f \equiv 0$. Ainsi le problème de surcomplétude peut se lire comme un problème sur les ensembles de zéros des fonctions de K^q_Θ. D'autre part, ce problème est bien sûr relié au problème de complétude dont nous avons déjà parlé dans l'introduction de ce chapitre. Trouver un critère géométrique explicite pour la complétude des noyaux reproduisants dans K^p_Θ semble pour le moment hors d'atteinte et il nous a donc semblé naturel de nous intéresser à cette propriété de surcomplétude pour laquelle il devrait être plus facile d'obtenir des résultats. Citons cependant une avancée récente obtenue par Makarov–Poltoratski [145] qui généralise un résultat profond de Beurling–Malliavin sur la complétude des systèmes d'exponentielles.

Afin de formuler nos résultats, nous devons introduire deux ensembles intimement liés à la théorie des fonctions intérieures. Tout d'abord, si Θ est intérieure, on note par $\sigma(\Theta)$ *le spectre* de Θ, c'est à dire le complémentaire dans $\overline{\mathbb{D}}$ de l'ensemble des points $\zeta \in \overline{\mathbb{D}}$ tels que Θ^{-1} se prolonge analytiquement à travers un voisinage de ζ. En particulier, on a

$$\sigma(\Theta) \cap \mathbb{T} = \{\zeta \in \mathbb{T} : \liminf_{\substack{z\to\zeta \\ z\in\mathbb{D}}} |\Theta(z)| = 0\}. \tag{2.7}$$

De plus, si

$$\Theta(z) = \prod_{n\geq 1} \frac{|a_n|}{a_n}\frac{a_n - z}{1-\overline{a_n}z} \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu_S(\zeta)\right), \tag{2.8}$$

est la factorisation canonique de la fonction intérieure Θ, alors on a

$$\sigma(\Theta) = \mathrm{clos}(a_n) \cup \mathrm{supp}\mu_S. \tag{2.9}$$

D'autre part, on note par E_Θ l'ensemble défini par

$$E_\Theta := \left\{\zeta_0 \in \mathbb{T} : \sum_{n\geq 1} \frac{1-|a_n|^2}{|\zeta_0 - a_n|^2} + \int_{\mathbb{T}} \frac{d\mu_S(\zeta)}{|\zeta - \zeta_0|^2} < +\infty\right\}. \tag{2.10}$$

Cet ensemble correspond aux points $\zeta_0 \in \mathbb{T}$ où Θ admet une dérivée angulaire au sens de Carathéodory, c'est-à-dire Θ et Θ' admettent des limites radiales (finies) en ζ_0 et $|\Theta(\zeta_0)| = 1$. Nous reviendrons au chapitre 3 sur cet ensemble.

Remarquons que si Θ est un produit de Blaschke fini, alors K^p_Θ est de dimension finie et donc si $(\lambda_n)_{n\geq 1}$ est une suite de points distincts, $(k^\Theta_{\lambda_n})_{n\geq 1}$ est toujours surcomplète dans K^p_Θ. Par conséquent, dans la suite de la section 2.3, nous supposerons toujours que $(\lambda_n)_{n\geq 1}$ est une suite de points distincts de $\mathbb{D}$ et Θ n'est pas un produit de Blaschke fini. En particulier, $\sigma(\Theta) \cap \mathbb{T} \neq \emptyset$.

On obtient alors le résultat suivant qui donne des conditions nécessaires et une condition suffisante en terme des ensembles E_Θ et $\sigma(\Theta)$:

Théorème 2.3.1 *Soient $p \in [2, \infty)$, $(\lambda_n)_{n\geq 1}$ une suite infinie de points distincts de $\mathbb{D}$. Alors nous avons les implications suivantes :*

(SC) $$\inf_{n\geq 1} \textit{dist}\ (\lambda_n, \sigma(\Theta) \cap \mathbb{T}) > 0$$
$$\Downarrow$$
(OVC) $$(k_\Theta(., \lambda_n))_{n\geq 1} \text{ est surcomplète dans } K^p_\Theta$$
$$\Downarrow$$
(NC_1) $$(k_\Theta(., \lambda_n))_{n\geq 1} \text{ est relativement compact (pour la norme) dans } K^p_\Theta$$
$$\Downarrow$$
(NC_2) $$\sup_{n\geq 1} \frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2} < \infty$$
$$\Downarrow$$
(NC_3) $$\inf_{n\geq 1} \textit{dist}\ (\lambda_n, \mathbb{T} \setminus E_\Theta) > 0$$

De plus, pour $p \in (1, 2)$, $(SC) \Rightarrow (OVC) \Rightarrow (NC_1)$ restent vraies.

Donnons une idée de la preuve de ce résultat. Tout d'abord, l'implication $(SC) \Longrightarrow (OVC)$ est une conséquence immédiate du principe des zéros isolés pour les fonctions analytiques et de la propriété suivante du spectre d'une fonction intérieure : si Θ est une fonction intérieure, alors toute fonction $f \in K^q_\Theta$ se prolonge analytiquement à travers $\mathbb{T} \setminus \sigma(\Theta)$.
L'implication (essentielle) $(OVC) \Longrightarrow (NC1)$ découle du résultat général suivant.

Théorème 2.3.2 *Soient X un espace de Banach réflexif et $(x_n)_{n\geq 1}$ une suite bornée de vecteurs de X, deux à deux distincts. Si $(x_n)_{n\geq 1}$ ne contient aucune sous-suite uniformément minimale (donc en particulier si $(x_n)_{n\geq 1}$ est surcomplète dans X), alors $(x_n)_{n\geq 1}$ est relativement compact (pour la norme) dans X.*

Ce point clé dans la preuve du théorème 2.3.1 est lui basé sur un lemme de C. Bessaga et A. Pelczynski, qui affirme que si $(y_n)_{n\geq 1}$ est une suite de X qui tend faiblement vers 0 et qui satisfait $\inf_{n\geq 1} \|y_n\| > 0$, alors $(y_n)_{n\geq 1}$ contient une sous-suite qui est une base de Schauder de son enveloppe linéaire fermée.

Finalement, l'implication $(NC2) \Longrightarrow (NC3)$ du théorème 2.3.1 provient du théorème de Carathéodory qui implique que $\zeta_0 \in E_\Theta$ si et seulement si

$$\liminf_{\substack{z \in \mathbb{D} \\ z \to \zeta_0}} \frac{1-|\Theta(z)|^2}{1-|z|^2} < +\infty.$$

□

Dans le cas où $E_\Theta = \mathbb{T} \backslash \sigma(\Theta)$, le théorème 2.3.1 donne donc une caractérisation des suites surcomplètes de K^p_Θ, $p \geq 2$. Nous nous sommes alors intéressés à la classe des fonctions Θ intérieures telles que $E_\Theta = \mathbb{T} \setminus \sigma(\Theta)$. Rappelons d'abord qu'une suite $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ est une *suite de Stolz* s'il existe un sous-ensemble fini e de $\mathbb{T}$ et une constante positive c telle que, pour tout $n \geq 1$, on a

$$\text{dist}\,(\lambda_n, e) \leq c\,\text{dist}\,(\lambda_n, \mathbb{T}).$$

Si $(\lambda_n)_{n\geq 1}$ est une suite de Stolz et ζ est un point d'accumulation de la suite $(\lambda_n)_{n\geq 1}$, alors il existe une sous-suite $(\lambda_{n_p})_{p\geq 1}$ qui converge non-tangentiellement vers ζ.

Corollaire 2.3.3 *Soient $p \in [2, \infty)$ et $(\lambda_n)_{n\geq 1}$ une suite infinie de points distincts de $\mathbb{D}$. Soit $\Theta = BS$ une fonction intérieure, avec B un produit de Blaschke associé à une suite $(a_n)_{n\geq 1}$ et S une fonction intérieure singulière associée à une mesure μ. Supposons que $(a_n)_{n\geq 1}$ est une suite de Stolz et que μ est à support fini. Alors*

$$(k_\Theta(\cdot, \lambda_n))_{n\geq 1} \text{ est surcomplète dans } K^p_\Theta \Leftrightarrow (SC) \Leftrightarrow (NC_1) \Leftrightarrow (NC_2) \Leftrightarrow (NC_3).$$

La preuve de ce corollaire consiste à montrer que les hypothèses sur Θ impliquent que $E_\Theta = \mathbb{T} \setminus \sigma(\Theta)$ et le résultat suit alors immédiatement du théorème 2.3.1.

2.4 Bases orthogonales et bases de Riesz dans $\mathcal{H}(b)$

Dans [T3] et [T4], je me suis intéressé au problème des bases de noyaux reproduisants dans l'espace de de Branges–Rovnyak $\mathcal{H}(b)$. L'article [T3] concerne le cas scalaire et l'article [T4] (avec N. Chevrot et D. Timotin) traite du cas vectoriel.

Le premier problème auquel je me suis intéressé dans [T3] est celui de caractériser les suites de noyaux reproduisants $(k^b_{\lambda_n})_{n\geq 1}$ qui forment une base orthogonale de $\mathcal{H}(b)$. Rappelons que dans [51], D. Clark a obtenu un critère dans le cas où b est intérieure, ce qui correspond au cas des espaces modèles K_b. Comme dans le cas intérieur, il est facile de voir qu'il ne peut pas exister de bases orthogonales $(k^b_{\lambda_n})_{n\geq 1}$ avec des pôles λ_n contenus dans le disque unité $\mathbb{D}$. Dans certains cas, il est cependant possible de considérer des noyaux reproduisants avec des pôles sur le cercle unité. Pour une fonction b dans la boule unité de H^∞, on peut aussi définir un ensemble E_b analogue à (2.10) (voir (3.6) pour la définition précise) et qui correspond aux points de $\mathbb{T}$ où b a une dérivée angulaire au sens de Carathéodory. Dans ce cas, si $\zeta_0 \in E_b$, alors la fonction

$$k^b_{\zeta_0}(z) := \frac{1 - \overline{b(\zeta_0)}b(z)}{1 - \bar{\zeta}_0 z}, \qquad (z \in \mathbb{D}),$$

est dans $\mathcal{H}(b)$; toute fonction f dans $\mathcal{H}(b)$ a une limite radiale en ζ_0 et on a

$$f(\zeta_0) = \langle f, k^b_{\zeta_0}\rangle_b . \tag{2.11}$$

Si on veut une base orthogonale de noyaux reproduisants, il faut donc choisir la suite des pôles $(\lambda_n)_{n\geq 1}$ telle que $\lambda_n \in E_b$ et $b(\lambda_n) = \lambda \in \mathbb{T}$, $n \geq 1$.

On montre alors dans [T3] le résultat suivant.

Théorème 2.4.1 *Soient $b \in H^\infty$, $\|b\|_\infty \leq 1$ et $\lambda \in \mathbb{T}$. Supposons que la famille $\{k^b_\zeta : \zeta \in E_b, b(\zeta) = \lambda\}$ forme une base orthogonale de $\mathcal{H}(b)$. Alors b est intérieure.*

La preuve de ce résultat suit la méthode utilisée par Clark pour obtenir son critère dans le cas intérieur. L'idée est que si $\zeta \in E_b$ et $b(\zeta) = \lambda$, alors k^b_ζ représente un vecteur propre pour l'opérateur

$$U_\lambda := X^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k^b_0 \otimes S^* b,$$

où $X := S^*|\mathcal{H}(b)$. On utilise alors le fait que U_λ est unitairement équivalent à Z_{μ_λ}, l'opérateur de multiplication par la variable indépendante dans $L^2(\mu_\lambda)$, où μ_λ est la mesure sur $\mathbb{T}$ dont l'intégrale de Poisson est la partie réelle de $\dfrac{1+\bar{\lambda}b}{1-\bar{\lambda}b}$. On obtient alors que si la famille $\{k^b_\zeta : \zeta \in E_b, b(\zeta) = \lambda\}$ forme une base orthogonale de $\mathcal{H}(b)$, alors l'opérateur Z_{μ_λ} est diagonalisable, ce qui implique que μ_λ est une mesure purement atomique. Il est alors facile de voir que ceci n'arrive que si b est intérieure. □

Dans [T3], j'ai aussi considéré le problème des suites de noyaux reproduisants normalisées $(h^b_{\lambda_n})_{n\geq 1}$ qui forment une suite (base) de Riesz de $\mathcal{H}(b)$. J'ai notamment obtenu deux critères que j'ai par la suite généralisés au cas vectoriel dans [T4]. Je vais donc directement présenter les résultats principaux dans le cas vectoriel. Mentionnons néanmoins que les techniques utilisées dans les deux articles sont différentes comme nous y reviendrons plus tard.

On se donne donc une fonction $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$, $(\lambda_n)_{n\geq 1}$ une suite de Blaschke de $\mathbb{D}$ et $(e_n)_{n\geq 1} \subset E_*$, $\|e_n\| = 1$. On va supposer en plus que $\dim E_* < +\infty$. En effet, si $\dim E_* = +\infty$ et $(e_n)_{n\geq 1}$ est une base orthonormale de E_*, alors la famille de noyaux reproduisants normalisée $(h_{\lambda_n,e_n})_{n\geq 1}$ est une suite orthonormale dans $H^2(E_*)$ quelque soit la suite $(\lambda_n)_{n\geq 1}$. Dans un certain sens, si E_* est un espace de Hilbert de dimension infinie, il existe trop de liberté sur les vecteurs $(e_n)_{n\geq 1}$ pour espérer obtenir un critère satisfaisant pour les suites (ou bases) de Riesz. C'est la raison pour laquelle, les résultats obtenus jusqu'à présent dans le cas vectoriel supposent que E_* est un espace de Hilbert de dimension finie (voir [224, 23, 89]) et nous travaillerons donc nous aussi sous cette hypothèse. Dans ce cas, on a

$$\text{Span}(k_{\lambda_n,e_n} : n \geq 1) = H^2(E_*) \ominus BH^2(E_*) = K_B,$$

où la fonction intérieure $B \in H^\infty(E_* \to E_*)$ est un produit de Blaschke-Potapov (voir [158] pour la définition).

Le point de départ est toujours le même (voir la sous-section 2.1.3) et est basé sur l'idée de regarder la famille de noyaux reproduisants $(k^b_{\lambda_n,e_n})_{n\geq 1}$ de $\mathcal{H}(b)$ comme une distorsion de la famille de noyaux reproduisants $(k_{\lambda_n,e_n})_{n\geq 1}$ de $H^2(E_*)$. On va donc aussi supposer que la norme de ces deux familles de noyaux est comparable, ce qui revient à supposer la condition suivante

$$\sup_{n\geq 1} \|b(\lambda_n)^* e_n\| < 1. \tag{2.12}$$

Alors comme dans le cas intérieur scalaire, on montre dans [T3] (cas scalaire) et [T4] (cas vectoriel) qu'on peut ramener l'étude des suites (ou bases) de Riesz de noyaux reproduisants à une étude de l'inversibilité d'un certain opérateur.

Théorème 2.4.2 *Soient $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$, $(\lambda_n)_{n\geq 1}$ une suite de Blaschke dans $\mathbb{D}$ et $(e_n)_{n\geq 1} \subset E_*$, $\|e_n\| = 1$. Supposons que $\dim E_* < +\infty$ et que la condition (2.12) est satisfaite. Alors les assertions suivantes sont équivalentes :*

a) *la suite $(h^b_{\lambda_n,e_n})_{n\geq 1}$ est une suite (respectivement base) de Riesz de $\mathcal{H}(b)$;*

b) *la suite $(h_{\lambda_n,e_n})_{n\geq 1}$ est une base de Riesz de K_B et l'opérateur*

$$(Id - T_b T_b^*)|K_B : K_B \longrightarrow \mathcal{H}(b)$$

est un isomorphisme sur son image (respectivement sur $\mathcal{H}(b)$).

Mentionnons qu'Ivanov [23, Theorem II.2.12] donne un critère géométrique pour que la suite $(h_{\lambda_n,e_n})_{n\geq 1}$ soit une base de Riesz de K_B. Par conséquent, il reste à étudier l'inversibilité de l'opérateur $(Id - T_b T_b^*)|K_B : K_B \longrightarrow \mathcal{H}(b)$, que nous appelons dans la suite l'*opérateur de distorsion*. Rappelons que dans le cas où b est intérieure, on peut ramener l'inversibilité de cet opérateur à l'inversibilité du Toeplitz T_{b^*B}. En utilisant le théorème de Devinatz–Widon (ou le théorème de Nehari), ceci donne alors le théorème 2.1.1 et le théorème obtenu dans [89]. Dans le cas d'une fonction b scalaire (non nécessairement intérieure), le point clé de la méthode utilisée dans [T3] pour étudier l'inversibilité de l'opérateur de distorsion est le lemme suivant.

Lemme 2.4.3 *Soit b un point extrême de la boule unité de H^∞. Alors*

$$\mathit{Span}\left(\frac{b - b(\lambda)}{z - \lambda} : \lambda \in \mathbb{D}\right) = \mathcal{H}(b).$$

Nous reviendrons plus tard sur ce lemme (que nous avons aussi généralisé dans [T4]) mais il n'est plus vrai en général dans le cas vectoriel. Ainsi les deux méthodes utilisées jusqu'à maintenant dans la théorie ne s'appliquent pas au cas vectoriel. Nous avons donc employé une autre méthode, dans un certain sens plus géométrique, basée sur le modèle fonctionnel abstrait.

Dans [T4], nous avons en fait discuté du problème général de l'inversibilité de l'opérateur de distorsion

$$(Id - T_bT_b^*)|K_\Theta : K_\Theta \longrightarrow \mathcal{H}(b),$$

où Θ est une fonction intérieure quelconque dans $H^\infty(F \to E_*)$ et b est une fonction analytique contractive dans $H^\infty(E \to E_*)$. Autrement dit, dans les résultats qui suivent, Θ n'est plus nécessairement un produit de Blaschke-Potapov. On obtient alors le critère suivant.

Théorème 2.4.4 *L'opérateur de distorsion $(Id - T_bT_b^*)|K_\Theta : K_\Theta \longrightarrow \mathcal{H}(b)$ est*

(a) *un isomorphisme sur son image si et seulement si*

$$\textit{dist}\,(\Theta^* b, H^\infty(E \to F)) < 1;$$

(b) *un isomorphisme sur $\mathcal{H}(b)$ si et seulement si dist $(\Theta^* b, H^\infty(E \to F)) < 1$ et l'opérateur*

$$\Gamma_b := (P_+ b^*\Theta \quad P_+\Delta) : \begin{array}{c} H^2(F) \\ \oplus \\ \textit{clos}(\Delta H^2(E)) \end{array} \longrightarrow H^2(E).$$

est borné inférieurement.

La preuve de ce résultat utilise fortement le plongement fonctionnel abstrait introduit par Nikolski–Vasyunin [152] (voir la sous-section 1.2.2 pour les définitions et les notations). Si

$$\Pi = (\pi_*, \pi) : \begin{pmatrix} L^2(E_*) \\ L^2(E) \end{pmatrix} \longrightarrow \mathcal{H}$$

est un plongement fonctionnel abstrait tel que $\pi_*^*\pi = b$, alors le point clé de la preuve du théorème 2.4.4 consiste à remarquer que

$$Id - T_bT_b^* = \pi_*^* P_{\mathbb{H}'} \pi_* | H^2(E_*).$$

Comme π_*^* est une isométrie partielle de K_b sur $\mathcal{H}(b)$ et que $\ker \pi_*^* | K_b = \mathbb{H}'' = K_b \ominus \mathbb{H}'$, on voit que l'inversibilité de l'opérateur de distorsion peut se ramener à l'inversibilité de l'opérateur $P_{\mathbb{H}'}|\pi_* K_\Theta$. En utilisant des arguments géométriques standards sur les projections orthogonales dans un espace de Hilbert, on obtient alors le théorème 2.4.4. □

Remarquons que la condition pour l'inversibilité à gauche du théorème 2.4.4 généralise celle du théorème 2.1.1 ; la condition pour l'inversibilité elle ne peut pas être réduite à une condition fonctionnelle simple. Néanmoins, elle permet d'obtenir plusieurs corollaires intéressants qui montrent que dans plusieurs cas, la fonction b a une forme spéciale.

Corollaire 2.4.5 *Si clos$(\Delta H^2(E)) =$ clos$(\Delta L^2(E))$ et l'opérateur de distorsion est inversible, alors b est intérieure.*

Corollaire 2.4.6 *Supposons que $\dim F = \dim E_*$. Si l'opérateur de distorsion est inversible, alors b est $*$-intérieure. Si de plus, $\dim E = \dim E_*$, alors b est intérieure.*

En particulier le corollaire 2.4.6 peut être appliqué à notre situation des bases de noyaux reproduisants puisque la fonction intérieure Θ est un produit de Blaschke-Potapov B et donc on a $\dim F = \dim E_*$.

Corollaire 2.4.7 *Supposons que $\dim E = \dim E_* = 1$. L'opérateur de distorsion est inversible si seulement si on est dans une des deux situations suivantes :*
(i) b est intérieure, dist $(\bar{\Theta}b, H^\infty) < 1$ et dist $(\bar{b}\Theta, H^\infty) < 1$.
(ii) $F = \{0\}$ et $\|b\|_\infty < 1$.

En particulier, on obtient que si b est une fonction scalaire de la boule unité de H^∞ et si $\mathcal{H}(b)$ possède une base de Riesz formée de noyaux reproduisants qui satisfait la condition (2.12), alors nécessairement b est intérieure. Ce résultat doit être comparé avec le théorème 2.4.1.

Dans le cas non extrême, nous avions montré dans [T3] que la situation est drastiquement différente, puisqu'on ne peut jamais avoir de base de Riesz (complète) formée de noyaux reproduisants.

Théorème 2.4.8 *Soient $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ et $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. Supposons que b n'est pas un point extrême de la boule unité de H^∞. Les assertions suivantes sont équivalentes :*

(i) *la suite $(k^b_{\lambda_n})_{n\geqslant 1}$ est minimale ;*

(ii) *la suite $(k^b_{\lambda_n})_{n\geqslant 1}$ n'est pas complète dans $\mathcal{H}(b)$.*

De plus, dans ce cas, nous avons :

$$\dim(\mathcal{H}(b) \ominus \mathrm{Span}(k^b_{\lambda_n} : n\geqslant 1)) = +\infty.$$

Le point clé dans la preuve de ce résultat est le fait que, dans le cas où b est un point non extrême de la boule unité de H^∞, alors $\mathcal{H}(b)$ est invariant par S.

Dans [T4], nous discutons aussi de la complétude dans $\mathcal{H}(b)$ de la famille $\hat{k}^b_{\lambda,e}$ définie par

$$\hat{k}^b_{\lambda,e}(z) = \frac{b(z) - b(\lambda)}{z - \lambda} e, \qquad (\lambda \in \mathbb{D}, e \in E).$$

Avec les noyaux reproduisants, cette famille est le seul exemple d'éléments de $\mathcal{H}(b)$ avec une formule explicite. De plus, ils apparaissent naturellement dans le modèle fonctionnel et dans d'autres questions liées (voir [28] et [93]). La question à laquelle nous nous sommes intéressés est donc la suivante : a-t-on

$$\mathrm{Span}(\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E) = \mathcal{H}(b)\,?$$

Rappelons que dans [T3] (voir lemme 2.4.3), nous avons montré que si b est un point extrême de la boule unité de H^∞, alors la famille $\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D},\, e \in E\}$ est complète dans $\mathcal{H}(b)$. Dans [T4], nous avons donné le critère suivant pour le cas scalaire non extrême.

Théorème 2.4.9 *Supposons que b n'est pas un point extrême de la boule unité de H^∞. Alors la famille $\{\hat{k}^b_\lambda : \lambda \in \mathbb{D}\}$ est complète dans $\mathcal{H}(b)$ si et seulement si b n'admet pas de pseudo-prolongement analytique à travers $\mathbb{T}$.*

Rappelons qu'on dit qu'une fonction $\varphi \in H^2$ admet un *pseudo-prolongement analytique* à travers $\mathbb{T}$ s'il existe $f, g \in \cup_{p>0} H^p$ tels que $\varphi = \bar{f}/\bar{g}$ p.p. sur $\mathbb{T}$. R. Douglas, H. Shapiro et A. Shields [67] ont montré que φ admet un pseudo-prolongement analytique à travers $\mathbb{T}$ si et seulement si φ n'est pas S^*-cyclique, c'est-à-dire

$$\mathrm{Span}(S^{*n}\varphi : n \geq 0) \neq H^2.$$

Pour dépasser le cas scalaire, nous devons utiliser le langage du modèle fonctionnel. On obtient alors le résultat général suivant.

Théorème 2.4.10 *Soit $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$. Les assertions suivantes sont équivalentes :*

1. *$\mathrm{Span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b)$.*
2. *$\mathbb{H}' \cap \mathbb{H}''_* = \{0\}$.*
3. *$\mathbb{H}'' \vee \mathbb{H}'_* = K_b$.*

Dans l'énoncé du théorème 2.4.10, les espaces $\mathbb{H}'$, $\mathbb{H}''$, $\mathbb{H}'_*$ et $\mathbb{H}''_*$ qui interviennent sont les transcriptions de Sz.-Nagy–Foias des espaces définis dans (1.7) et (1.8).

Mentionnons pour finir deux corollaires intéressants.

Corollaire 2.4.11 *Soit $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$. Si $clos(\Delta_* H^2_-(E_*)) = clos(\Delta_* L^2(E_*))$, alors*

$$Span\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b).$$

En particulier, si b^* est intérieure alors les conoyaux sont complets dans $\mathcal{H}(b)$.

Corollaire 2.4.12 *Soit b un point extrême de la boule unité de $H^\infty(E \to E_*)$. Les assertions suivantes sont équivalentes :*

(i) *$Span\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b)$.*

(ii) *$clos(\Delta_* H^2_-(E_*)) = clos(\Delta_* L^2(E_*))$.*

Chapitre 3

Propriétés fonctionnelles des espaces de de Branges-Rovnyak

Dans ce chapitre, nous allons présenter les travaux [T5–T8] qui portent sur le comportement au bord des fonctions des espaces de de Branges-Rovnyak.

3.1 Introduction

Rappelons que si $\mathbb{D}$ est le disque unité ouvert du plan complexe, alors $H^2 = H^2(\mathbb{D})$ désigne l'espace de Hardy usuel, formé des fonctions f analytiques sur $\mathbb{D}$ et qui satisfont $\|f\|_2 < +\infty$, où

$$\|f\|_2^2 := \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, \frac{d\theta}{2\pi}.$$

Comme les fonctions de H^2 sont analytiques à l'intérieur du disque, il est naturel de se demander ce qui se passe à la frontière $\partial\mathbb{D} = \mathbb{T}$. Au début du $XX^{\text{ième}}$ siècle, P. Fatou [86] a probablement été le premier à s'intéresser au comportement au bord de ces fonctions. En particulier, il a montré que toute fonction de H^2 possède des limites non-tangentielles (finies) en presque tout point de la frontière. En général, on ne peut pas dire beaucoup plus concernant le comportement au bord de ces fonctions. En particulier, si $\zeta_0 \in \mathbb{T}$, il est facile de construire une fonction f appartenant à H^2 et telle que f n'a pas de limite radiale (finie) en ζ_0. Si on restreint la classe de fonctions, on peut naturellement espérer obtenir plus d'informations. M. Livshitz [139] et Moeller [148] ont effectué le premier pas dans cette direction en considérant le cas des espaces modèles K_Θ. Rappelons que si Θ est intérieure, alors K_Θ désigne le sous-espace de l'espace de Hardy H^2 défini par

$$K_\Theta = H^2 \ominus \Theta H^2 = H^2 \cap \Theta \overline{H_0^2},$$

où $H_0^2 := \{f \in H^2 : f(0) = 0\} = zH^2$. On note également M_Θ l'opérateur modèle de K_Θ dans lui-même, défini comme la compression du shift sur K_Θ. Autrement dit,

$$M_\Theta(f) := P_\Theta(zf), \qquad f \in K_\Theta,$$

où P_Θ est la projection orthogonale de H^2 sur K_Θ. Dans le résultat suivant, on voit que le comportement au bord des fonctions de K_Θ est étroitement lié au comportement au bord de la fonction Θ elle-même.

Théorème 3.1.1 (Livshitz–Moeller) *Soit $\zeta_0 \in \mathbb{T}$. Les assertions suivantes sont équivalentes :*

(i) *toute fonction f de K_Θ a un prolongement analytique à travers un voisinage de ζ_0 ;*

(ii) *l'opérateur $Id - \overline{\zeta_0} M_\Theta$ est inversible ;*

(iii) *Θ a un prolongement analytique à travers un voisinage de ζ_0.*

L'équivalence entre (ii) et (iii) peut être formulée par l'égalité suivante des spectres :

$$\sigma(\Theta) \cap \mathbb{T} = \sigma(M_\Theta) \cap \mathbb{T},$$

(voir (2.7) pour la définition du spectre d'une fonction intérieure). En fait, on a $\sigma(\Theta) = \sigma(M_\Theta)$ et ce résultat a ensuite été étendu par Sz.-Nagy–Foias et Helson qui ont montré que les spectres d'une contraction et de sa fonction caractéristique coincident (voir par exemple [153]).

En utilisant (2.9), on peut reformuler l'assertion (iii). Plus précisément, si la factorisation canonique de Θ est donnée par (2.8) et si on note par $\mu_\Theta := \mu_B + \mu_S$, où

$$\mu_B := \sum_{n=1}^{+\infty} (1 - |a_n|^2)\delta_{\{a_n\}},$$

alors l'assertion (iii) est équivalente à l'existence d'un voisinage $\mathcal{V}_{\zeta_0}$ de ζ_0 tel que $\mu_\Theta(\mathcal{V}_{\zeta_0}) = 0$. Autrement dit, toutes les fonctions de K_Θ peuvent se prolonger analytiquement à travers un point ζ_0 de $\mathbb{T}$ si et seulement si la mesure μ_Θ d'un voisinage de ζ_0 est nulle. Cette formulation rend plausible le principe général suivant : *si la mesure μ_Θ est suffisamment petite près d'un point ζ_0, alors les fonctions de K_Θ doivent être lisses près de ce point.*

Ceci est confirmé par le résultat suivant d'Ahern–Clark [3] sur l'existence des valeurs au bord pour les dérivées des fonctions de K_Θ.

Théorème 3.1.2 (Ahern-Clark) *Soit Θ une fonction intérieure dont la factorisation canonique est donnée par* (2.8). *Soient N un entier positif et $\zeta_0 \in \mathbb{T}$. Les assertions suivantes sont équivalentes :*

(i) *pour toute fonction f de K_Θ, $f(z), f'(z), \ldots, f^{(N)}(z)$ ont une limite (finie) non-tangentielle en ζ_0 ;*

(ii) *$M_\Theta^N(k_0^\Theta)$ appartient à l'image de l'opérateur $(Id - \overline{\zeta_0} M_\Theta)^{N+1}$;*

(iii) *on a*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{\zeta_0}a_n|^{2N+2}} + \int_0^{2\pi} \frac{d\mu_S(e^{i\vartheta})}{|1-\overline{\zeta_0}e^{i\vartheta}|^{2N+2}} < \infty.$$

De plus, sous la condition (iii), *la fonction* $k^{\Theta}_{\zeta_0,N}$*, définie par*

$$k^{\Theta}_{\zeta_0,N}(z) := \frac{N!z^N - \Theta(z)\sum_{j=0}^{N}\binom{N}{j}\overline{\Theta^{(j)}(\zeta_0)}(N-j)!z^{N-j}(1-\overline{\zeta_0}z)^j}{(1-\overline{\zeta_0}z)^{N+1}} \qquad (z \in \mathbb{D}),$$

appartient à K_Θ *et on a*

$$f^{(N)}(\zeta_0) = \int_{\mathbb{T}} f(\zeta)\overline{k^{\Theta}_{\zeta_0,N}(\zeta)}\, dm(\zeta), \tag{3.1}$$

pour toute fonction f *de* K_Θ.

Dans [57], W. Cohn a donné un analogue du résultat précédent pour les fonctions de $K^p_\Theta = H^p \cap \Theta\overline{zH^p}$.

Remarquons bien sûr que les théorèmes 3.1.1 et 3.1.2 admettent des analogues dans le demi-plan supérieur $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im m(z) > 0\}$. Récemment, K. Dyakonov [78] et A. Baranov [30] ont utilisé la formule intégrale (3.1) d'Ahern–Clark–Cohn pour obtenir des inégalités de type Bernstein dans les espaces K^p_Θ (dans le cadre du demi-plan supérieur).

Rappelons que, dans le cas où Θ est une fonction intérieure dans $\mathbb{C}_+$, alors $K^p_\Theta = H^p(\mathbb{C}_+) \cap \Theta\overline{H^p(\mathbb{C}_+)}$, où $H^p(\mathbb{C}_+)$ est l'espace de Hardy du demi-plan supérieur. Si $\Theta_a(z) = e^{iaz}$, $a > 0$, alors

$$K^p_{\Theta_a} = H^p(\mathbb{C}_+) \cap PW^p_a,$$

où PW^p_a désigne l'espace de Paley–Wiener des fonctions entières de type exponentiel au plus a, et dont la restriction à l'axe réel appartient à $L^p(\mathbb{R})$. L'inégalité de Bernstein classique affirme que, pour $1 \le p \le +\infty$, on

$$\|f'\|_p \le a\|f\|_p, \qquad f \in PW^p_a.$$

Cette inégalité a été généralisée par de nombreux auteurs dans des directions différentes. Il est impossible de donner une liste exhaustive de références mais je voudrais citer [40, 98, 167, 178, 177, 221] et [131, Lecture 28]. L'une des directions a été d'étendre l'inégalité de Bernstein aux espaces modèles. Dans [132, 133], M.B. Levin a montré que si Θ est une fonction intérieure et si $\Theta'(x)$ existe et est fini (dans un sens non-tangentiel), alors pour chaque fonction f de K^∞_Θ, la dérivée $f'(x)$ existe aussi dans un sens non-tangentiel et on a

$$\left|\frac{f'(x)}{\Theta'(x)}\right| \le \|f\|_\infty.$$

La dérivation dans les espaces modèles K^p_Θ a aussi été étudié par K. Dyakonov [79, 78]. En particulier, il a montré l'équivalence suivante :

(i) il existe une constante C telle que

$$\|f'\|_p \leq C\|f\|_p, \qquad f \in K^p_\Theta;$$

(ii) $\Theta' \in L^\infty(\mathbb{R})$.

Enfin, A. Baranov [30] a obtenu des inégalités de type Bernstein à poids pour les espaces modèles K^p_Θ qui généralisent les résultats précédents de Levin et Dyakonov. Plus précisément, pour une fonction intérieure générale Θ, il a prouvé des estimations de la forme

$$\|f^{(n)}\omega_{p,n}\|_{L^p(\mu)} \leq C\|f\|_p, \qquad (f \in K^p_\Theta),$$

où $n \geq 1$, μ est une mesure de Carleson sur le demi-plan supérieur fermé $\overline{\mathbb{C}_+}$ et $\omega_{p,n}$ est un poids relié à la norme des noyaux reproduisants dans l'espace modèle K^2_Θ. Ce poids compense en fait la possible croissance de la dérivée près du bord. Dans [T5]–[T7], nous avons discuté des analogues des résultats de Livshitz–Moeller, Ahern–Clark, Dyakonov et Baranov, dans le cadre des espaces de de Branges–Rovnyak $\mathcal{H}(b)$.

3.2 Prolongement analytique et continu pour les fonctions de $\mathcal{H}(b)$

Rappelons que si b désigne une fonction de la boule unité de H^∞ du disque unité, alors on peut factoriser b (à une constante unimodulaire près) sous la forme suivante

$$b(z) = \prod_{n=1}^{+\infty} \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \right) \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \right), \qquad (z \in \mathbb{D}), \tag{3.2}$$

où $(a_n)_{n\geq 1}$ est une suite de Blaschke correspondant aux zéros de la fonction b et σ est une mesure positive de la forme $d\sigma = d\mu_S + |\log|b||\, dm$, avec μ_S mesure positive singulière par rapport à la mesure de Lebesgue dm. On voit que dans le cas où b est intérieure, on retrouve la factorisation (2.8). Comme dans le cas intérieur, on peut définir le *spectre* de b, noté $\sigma(b)$, comme le complémentaire dans $\overline{\mathbb{D}}$ des points $\lambda \in \overline{\mathbb{D}}$ tels que

- si $\lambda \in \mathbb{D}$, alors $b(\lambda) \neq 0$;
- si $\lambda \in \mathbb{T}$, alors b se prolonge analytiquement à travers un voisinage $\mathcal{O}_\lambda$ de λ et $|b| = 1$ sur $\mathcal{O}_\lambda$.

Rappelons aussi que $\mathcal{H}(b)$ est un sous-espace S^*-invariant et S^* agit comme une contraction sur $\mathcal{H}(b)$ (voir [203, Chapitre 2]). Si on note $X := S^*|\mathcal{H}(b)$, l'opérateur X^* va alors être l'analogue de l'opérateur modèle M_Θ qui appara^dans les théorèmes 3.1.1 et 3.1.2.

Dans [T5], en collaboration avec J. Mashreghi, j'ai obtenu l'analogue suivant du résultat de Livshitz–Moeller.

Théorème 3.2.1 *Soit b une fonction dans la boule unité de H^∞ et soit I un arc ouvert de $\mathbb{T}$. Les assertions suivantes sont équivalentes :*

(i) *toute fonction de $\mathcal{H}(b)$ se prolonge analytiquement à travers I ;*

(ii) $I \subset {}^c\sigma(X^*)$ *;*

(iii) *b se prolonge analytiquement à travers I et $|b| = 1$ sur I ;*

(iv) *toute fonction de $\mathcal{H}(b)$ se prolonge continûment à travers I ;*

(v) *b se prolonge continûment à travers I et $|b| = 1$ sur I.*

En fait, l'équivalence entre les trois premières assertions a été obtenue par Sarason [203, Chapitre V] dans le cas où b est un point extrême de la boule unité de H^∞ et notre contribution principale concerne plutôt les points (iv) et (v). Même si ce théorème est valide sans hypothèse d'extrémalité, on voit en fait avec (iii) que si l'une de ces assertions est vérifiée, alors

$$\int_{\mathbb{T}} \log(1 - |b|^2)\, dm = -\infty,$$

et donc b est un point extrême. Autrement dit, si b n'est pas un point extrême et si I est un arc ouvert quelconque de $\mathbb{T}$, alors nécessairement il existe une fonction $f \in \mathcal{H}(b)$ telle que f ne se prolonge pas continûment à travers I.

3.3 Dérivées au bord radiales pour les fonctions de $\mathcal{H}(b)$

Dans [T5], nous avons également obtenu un critère pour que les fonctions de $\mathcal{H}(b)$, ainsi que leurs dérivées jusqu'à un certain ordre, admettent des limites radiales (finies) en un point ζ_0 de $\mathbb{T}$.

Théorème 3.3.1 *Soit b une fonction dans la boule unité de H^∞ admettant la factorisation canonique* (3.2)*. Soient $\zeta_0 \in \mathbb{T}$ et $n \in \mathbb{N}$. Les assertions suivantes sont équivalentes :*

(i) *pour toute fonction $f \in \mathcal{H}(b)$, $f(z), f'(z), \ldots, f^{(N)}(z)$ ont des limites finies lorsque z tend radialement vers ζ_0 ;*

(ii) $\left\|\partial^N k_z^b / \partial \overline{z}^N\right\|_b$ *est borné lorsque z tend radialement vers ζ_0 ;*

(iii) $X^{*N}k_0^b$ *appartient à l'image de l'opérateur* $(Id-\overline{\zeta_0}X^*)^{N+1}$;

(iv) *on a*

$$\sum_n \frac{1-|a_n|^2}{|\zeta_0-a_n|^{2N+2}}+\int_0^{2\pi}\frac{d\mu_S(e^{it})}{|\zeta_0-e^{it}|^{2N+2}}+\int_0^{2\pi}\frac{\left|\log|b(e^{it})|\right|}{|\zeta_0-e^{it}|^{2N+2}}\,dm(e^{it})<+\infty.$$

Signalons que ce résultat généralise le théorème 3.1.2. Mentionnons également que Sarason [203, page 58] a obtenu un autre critère en terme d'une certaine mesure de Poisson. Plus précisément, étant donné $\lambda\in\mathbb{T}$, la fonction $\Re\mathrm{e}\left(\frac{\lambda+\mathrm{b}}{\lambda-\mathrm{b}}\right)$ est harmonique et positive et donc elle est l'intégrale de Poisson d'une certaine mesure μ_λ positive sur $\mathbb{T}$. Par conséquent, on a

$$\frac{\lambda+b(z)}{\lambda-b(z)}=\int_{\mathbb{T}}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu_\lambda(e^{i\theta})\;+i\Im\mathrm{m}\frac{\lambda+b(0)}{\lambda-b(0)}.$$

Alors Sarason a montré que pour toute fonction $f\in\mathcal{H}(b)$, $f(z),f'(z),\dots,f^{(N)}(z)$ ont des limites finies lorsque z tend non tangentiellement vers ζ_0 si et seulement s'il existe $\lambda\in\mathbb{T}$ tel que

$$\int_{\mathbb{T}}\left|e^{i\theta}-\zeta_0\right|^{-2m-2}\,d\mu_\lambda(e^{i\theta})<+\infty. \tag{3.3}$$

Récemment, V. Bolotnikov et A. Kheifets ont également donné un autre résultat dans cette direction. Sous la condition

$$\liminf_{z\to\zeta_0}\frac{\partial^{2n}}{\partial z^n\partial\bar{z}^n}\left(\frac{1-|b(z)|^2}{1-|z|^2}\right)<+\infty, \tag{3.4}$$

ils montrent dans [41] que pour toute fonction $f\in\mathcal{H}(b)$, $f(z),f'(z),\dots,f^{(N)}(z)$ ont des limites finies lorsque z tend non tangentiellement vers ζ_0.

Les méthodes employées dans [203, 41] sont assez différentes des nôtres et nous donnons maintenant quelques éléments de la preuve du théorème 3.3.1.

L'implication (i) $\Longrightarrow$ (ii) est basée sur la formule suivante

$$f^{(N)}(z)=\langle f,\frac{\partial^N k_z^b}{\partial\bar{z}^N}\rangle_b$$

et le théorème de Banach-Steinhaus. Pour (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i), on utilise un lemme adapté d'un résultat d'Ahern-Clark.

Lemme 3.3.2 *Soient* $S_1,\dots,S_p$ *des contractions sur un espace de Hilbert* X *qui commutent. Soit* $(\lambda_1,\dots,\lambda_p)\in\mathbb{T}^p$ *tel que* $Id-\lambda_jS_j$ *est injectif et soit* $(\lambda_1^{(n)},\dots,\lambda_p^{(n)})\in\mathbb{D}^p$ *qui tend non-tangentiellement vers* $(\lambda_1,\dots,\lambda_p)$, *lorsque* $n\to+\infty$. *Alors, pour tout* $y\in X$, *la suite*

$$w_n:=(Id-\lambda_1^{(n)}S_1)^{-1}\dots(Id-\lambda_p^{(n)}S_p)^{-1}y$$

est uniformément borné si et seulement si y appartient à l'image de l'opérateur $(Id-\lambda_1 S_1)\dots(Id-\lambda_p S_p)$; dans ce cas, w_n tend faiblement vers

$$w_0 := (Id-\lambda_1 S_1)^{-1}\dots(Id-\lambda_p S_p)^{-1}y.$$

En utilisant la formule $k_\lambda^b = (Id-\bar{\lambda}X^*)^{-1}k_0^b$, on obtient que

$$\frac{\partial^N k_z^b}{\partial \bar{z}^N} = N!(Id-\bar{z}X^*)^{-(N+1)}X^{*N}k_0^b.$$

Comme $\sigma_p(X^*)\subset\mathbb{D}$ (voir [T3,Lemma 2.2]), on sait que $Id-\bar{\zeta}_0X^*$ est injectif et on peut appliquer le lemme 3.3.2 pour obtenir que (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Il reste l'équivalence avec (iv) qui est la partie difficile du théorème. Dans le cas où $b=\Theta$ est une fonction intérieure, Ahern–Clark ont remarqué que la condition (iv) est équivalente au problème d'interpolation suivant : il existe $k,g\in H^2$ tel que

$$(1-\bar{\zeta}_0 z)^{N+1}k(z) - N!z^N = \Theta(z)g(z).$$

Cette reformulation, basée sur la décomposition orthogonale $H^2 = K_\Theta\oplus\Theta H^2$, est cruciale dans la preuve d'Ahern–Clark. Evidemment cela n'est plus vraie dans le cas où K_Θ est remplacé par $\mathcal{H}(b)$ car alors cet espace n'est plus fermé pour la norme usuelle. Ceci induit une vraie difficulté que nous avons réussi à contourner malgré tout en utilisant et en adaptant d'autres résultats d'Ahern-Clark [4]. En particulier, nous avons utilisé le fait suivant : sous l'hypothèse (iv), alors, pour tout $0\le j\le 2N+1$, les deux limites

$$\lim_{r\to 1^-} b^{(j)}(r\zeta_0) \quad \text{et} \quad \lim_{R\to 1^+} b^{(j)}(R\zeta_0)$$

existent et sont égales. Ici on définit b en dehors du disque unité par

$$b(z) := \frac{1}{\overline{b(\frac{1}{\bar{z}})}}, \qquad |z|>1, z\neq\frac{1}{\bar{a}_n}.$$

□

Les théorèmes 3.2.1 et 3.3.1 admettent bien sûr un analogue dans le demi-plan supérieur $\mathbb{C}_+$. Dans la suite de ce chapitre (sauf mention express), nous allons supposer que b est une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$ et $\mathcal{H}(b)$ désigne l'espace de de Branges–Rovnyak correspondant (les définitions sont les mêmes que pour le disque unité).

Dans [T6], je me suis intéressé avec J. Mashreghi à l'obtention d'une formule intégrale pour les dérivées au bord des fonctions de $\mathcal{H}(b)$ qui généralise la formule (3.1). La difficulté essentielle ici est que contrairement au cas intérieur, le produit scalaire dans $\mathcal{H}(b)$ n'est pas directement donné par une intégrale. Avant tout, je dois rappeler quelques faits bien connus sur les espaces de de Branges–Rovnayk. En fait, dans [203], ces résultats sont formulés dans le cas du disque unité ; cependant les mêmes résultats avec des preuves similaires sont vrais dans le cas du demi-plan supérieur. Le premier rappel concerne le lien entre les espaces $\mathcal{H}(b)$ et $\mathcal{H}(\bar{b})$. Pour $f \in H^2(\mathbb{C}_+)$, on a [203, page 10]

$$f \in \mathcal{H}(b) \Longleftrightarrow T_{\bar{b}} f \in \mathcal{H}(\bar{b}).$$

De plus, si $f_1, f_2 \in \mathcal{H}(b)$, alors

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle T_{\bar{b}} f_1, T_{\bar{b}} f_2 \rangle_{\bar{b}}. \tag{3.5}$$

Mentionnons également une propriété essentielle de l'espace $\mathcal{H}(\bar{b})$ [203, page 16]. Soit $\rho(t) := 1 - |b(t)|^2$, $t \in \mathbb{R}$, et soit $L^2(\rho)$ l'espace de Hilbert formé des fonctions mesurables $f : \mathbb{R} \longrightarrow \mathbb{C}$ telles que $\|f\|_\rho < \infty$, où

$$\|f\|_\rho^2 := \int_{\mathbb{R}} |f(t)|^2 \rho(t)\, dt.$$

L'opérateur $\widetilde{T}_\rho : L^2(\rho) \longrightarrow H^2(\mathbb{C}_+)$, défini par

$$\widetilde{T}_\rho(q) := P_+(q\rho),$$

est une isométrie partielle de $L^2(\rho)$ sur $\mathcal{H}(\bar{b})$, dont le noyau est $L^2(\rho) \ominus H^2(\rho)$, où $H^2(\rho)$ désigne l'enveloppe linéaire fermée dans $L^2(\rho)$ des noyaux de Cauchy k_w, $w \in \mathbb{C}_+$.

Etant donné b une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$, on considère l'ensemble

$$E_n(b) := \{x \in \mathbb{R} : S_n(x) < +\infty\}, \tag{3.6}$$

où

$$S_n(x) := \sum_{r=1}^{+\infty} \frac{\Im\mathrm{m}\, z_r}{|x - z_r|^n} + \int_{\mathbb{R}} \frac{d\mu_S(t)}{|x - t|^n} + \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x - t|^n}\, dt.$$

Ici $(z_r)_{r \geq 1}$ est la suite des zéros de b et μ_S est la mesure singulière positive associée à b dans sa décomposition canonique en facteur intérieur-extérieur.

D'après le théorème 3.3.1, on sait que si $x_0 \in E_{2n+2}(b)$, alors pour toute fonction $f \in \mathcal{H}(b)$ et tout entier $0 \leq j \leq n$, la limite radiale

$$f^{(j)}(x_0) := \lim_{t \to 0^+} f^{(j)}(x_0 + it)$$

existe (et est finie). De plus, si $x_0 \in E_n(b)$, alors b et toutes ses dérivées jusqu'à l'ordre $n - 1$ ont des limites radiales en x_0 (voir [4]). On notera alors pour $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$

$$k_{z_0,n}^b(z) := \frac{1 - b(z) \displaystyle\sum_{j=0}^{n} \frac{\overline{b^{(j)}(z_0)}}{j!} (z - \overline{z_0})^j}{(z - \overline{z_0})^{n+1}}, \qquad z \in \mathbb{C}_+, \tag{3.7}$$

et

$$k^{\rho}_{z_0,n}(t) := \frac{\displaystyle\sum_{j=0}^{n} \frac{\overline{b^{(j)}(z_0)}}{j!}(t-\overline{z_0})^j}{(t-\overline{z_0})^{n+1}}, \qquad t \in \mathbb{R}. \tag{3.8}$$

On obtient alors le résultat suivant

Théorème 3.3.3 *Soient b une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$, n un entier positif et $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$. Alors $k^b_{z_0,n} \in \mathcal{H}(b)$, $k^\rho_{z_0,n} \in L^2(\rho)$. De plus, si $f \in \mathcal{H}(b)$ et $g \in H^2(\rho)$ tel que $T_{\bar{b}} f = \widetilde{T}_\rho(g)$, alors on a*

$$f^{(n)}(z_0) = \frac{n!}{2i\pi}\left(\int_{\mathbb{R}} f(t)\overline{k^b_{z_0,n}(t)}\,dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{z_0,n}(t)}\,dt\right). \tag{3.9}$$

Si b est intérieure, on voit que le deuxième terme dans la formule (3.9) disparait et on retombe sur la formule d'Ahern-Clark (3.1).

Dans le cas où $z_0 \in \mathbb{C}_+$, la formule (3.9) est une conséquence simple de la formule (3.5). Pour $z_0 \in E_{2n+2}(b)$, le résultat est plus délicat et le point clé de la preuve est de démontrer que

$$f^{(n)}(z_0) = \langle f, k^b_{z_0,n}\rangle_b, \tag{3.10}$$

puis

$$T_{\bar{b}} k^b_{z_0,n} = \widetilde{T}_\rho k^\rho_{z_0,n}. \tag{3.11}$$

Une conséquence de (3.10) et du théorème 3.3.3 est que si $x_0 \in E_{2n+2}(b)$, alors $k^b_{w,n}$ tend faiblement vers $k^b_{x_0,n}$ dans $\mathcal{H}(b)$, lorsque w tend radialement vers x_0. Il est naturel de se demander si cette convergence faible peut-être remplacée par une convergence forte.

En fait, cette question est apparue dans le cas intérieur dans [3, Remark VII]. De plus, dans [203], Sarason montre que k^b_w tend fortement dans $\mathcal{H}(b)$ vers $k^b_{z_0}$, lorsque w tend radialement vers un point $z_0 \in E_2(b)$. Autrement dit la réponse à notre question est positive pour le cas $n=0$. Dans [T6], nous avons montré le résultat suivant.

Théorème 3.3.4 *Soient b une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$, $n \in \mathbb{N}$ et $x_0 \in E_{2n+2}(b)$. Alors*

$$\left\|k^b_{w,n} - k^b_{x_0,n}\right\|_b \longrightarrow 0, \quad \text{lorsque } w \text{ tend radialement vers } x_0.$$

La preuve est assez technique. L'un des points remarquables est une formule de combinatoire amusante et qui nous semble intéressante par ailleurs.

Proposition 3.3.5 *Soient $n, r \in \mathbb{N}$, $0 \le r \le 2n+1$ et définissons*

$$A_{n,r} := (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p-\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \binom{n-p+\ell}{\ell}.$$

Alors

$$A_{n,r} = \begin{cases} -2^n, & 0 \le r \le n \\ 2^n, & n+1 \le r \le 2n+1. \end{cases} \tag{3.12}$$

La preuve de cette proposition se fait en utilisant la théorie des séries hypergéométriques.

Pour démontrer le théorème 3.3.4, on remarque tout d'abord que puisqu'on a déjà la convergence faible, pour obtenir la convergence en norme, il suffit de démontrer que

$$\|k_{w,n}^b\|_b \to \|k_{x_0,n}^b\|_b \quad \text{lorsque } w \text{ tend radialement vers } x_0. \tag{3.13}$$

On utilise alors des formules explicites pour calculer les normes $\|k_{w,n}^b\|_b$ et $\|k_{x_0,n}^b\|_b$ et des calculs simples mais “fastidieux” montrent alors que (3.13) se ramène à la formule (3.12). □

3.4 Inégalités de Bernstein à poids dans les espaces $\mathcal{H}(b)$

Comme nous l'avons remarqué dans [T5, Theorem 4.1], si on a une inégalité du type

$$\|f'\|_2 \le C\|f\|_b, \qquad (f \in \mathcal{H}(b)),$$

alors nécessairement la fonction b est intérieure. Ceci est une conséquence du théorème 3.2.1. Ainsi le résultat de K. Dyakonov [78] concernant l'inégalité de Bernstein dans les espaces K_Θ^p ne peut pas être étendu aux espaces $\mathcal{H}(b)$ généraux.

Dans [T7], je me suis alors intéressé en collaboration avec A. Baranov et J. Mashreghi à des inégalités de type Bernstein à poids. Pour obtenir ces inégalités, nous avons tout d'abord modifié quelque peu la formule intégrale (3.9).

Proposition 3.4.1 *Soit b une fonction de la boule unité de $H^\infty(\mathbb{C}_+)$. Soient $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, $n \in \mathbb{N}$ et*

$$\mathfrak{K}_{z_0,n}^{\rho}(t) := \overline{b(z_0)}\, \frac{\sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j\, \overline{b^j(z_0)}\, b^j(t)}{(t-\overline{z_0})^{n+1}}, \qquad t \in \mathbb{R}. \tag{3.14}$$

Alors $(k_{z_0}^b)^{n+1} \in H^2(\mathbb{C}_+)$ *et* $\mathfrak{K}_{z_0,n}^\rho \in L^2(\rho)$. *De plus, pour toute fonction* $f \in \mathcal{H}(b)$, *on a*

$$(3.15)\quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\left(\int_{\mathbb{R}} f(t)\overline{(k_{z_0}^b)^{n+1}(t)}\,dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{\mathfrak{K}_{z_0,n}^\rho(t)}\,dt\right),$$

où $g \in H^2(\rho)$ *est telle que* $T_{\overline{b}}f = \widetilde{T}_\rho g$.

On peut alors introduire les poids qui vont intervenir dans nos inégalités. Soient $1 < p \leq 2$, q son exposant conjugué et $n \in \mathbb{N}$. Alors pour $z \in \overline{\mathbb{C}_+}$, on définit

$$w_{p,n}(z) := \min\left\{ \|(k_z^b)^{n+1}\|_q^{-pn/(pn+1)},\ \|\rho^{1/q}\mathfrak{K}_{z,n}^\rho\|_q^{-pn/(pn+1)} \right\};$$

on suppose que $w_{p,n}(x) = 0$ dès que $x \in \mathbb{R}$ et au moins une des fonctions $(k_x^b)^{n+1}$ ou $\rho^{1/q}\mathfrak{K}_{x,n}^\rho$ n'est pas $L^q(\mathbb{R})$.

En utilisant des estimations simples, on montre que si $x \in E_{2n+2}(b)$, alors $\rho^{1/q}\mathfrak{K}_{x,n}^\rho$ est dans $L^q(\mathbb{R})$. Il est naturel d'espérer obtenir aussi que si $x \in E_{2n+2}(b)$, alors $(k_x^b)^{n+1}$ est dans $L^q(\mathbb{R})$. C'est vraie si b est intérieure par un résultat de Cohn [56] et pour une fonction générale b et $q = 2$ par [T6, Lemma 3.2]. Cependant, nous ne savons pas si ce résultat reste vrai en général.

Remarquons cependant que si $f \in \mathcal{H}(b)$ et $1 < p \leq 2$, alors $(f^{(n)}w_{p,n})(x)$ est bien définie sur $\mathbb{R}$. En effet, il suit du théorème 3.3.1 que $f^{(n)}(x)$ et $w_{p,n}(x)$ sont finis si $S_{2n+2}(x) < +\infty$. Si $S_{2n+2}(x) = +\infty$, alors $\|(k_x^b)^{n+1}\|_2 = +\infty$. D'où $\|(k_x^b)^{n+1}\|_q = +\infty$ ce qui, par définition, implique que $w_{p,n}(x) = 0$, et on peut donc supposer que $(f^{(n)}w_{p,n})(x) = 0$.

Dans le cas où b est intérieure, on a $\rho(t) = 0$ ($t \in \mathbb{R}$) et le second terme dans la définition de $\omega_{p,n}$ disparait ; on retombe alors sur les poids considérés par A. Baranov dans [30].

Pour formuler notre résultat principal, rappelons que si μ est une mesure Borélienne positive sur le demi-plan supérieur fermé $\overline{\mathbb{C}}_+$, alors μ est dite *mesure de Carleson* s'il existe une constante $C_\mu > 0$ telle que

$$(3.16)\qquad \mu(\,S(x,h)\,) \leqslant C_\mu\, h,$$

pour tout carré $S(x,h) = [x, x+h] \times [0,h]$, $x \in \mathbb{R}$, $h > 0$. On notera $\mathcal{C}$ l'ensemble des mesures de Carleson. Rappelons que le célèbre théorème de Carleson affirme que $\mu \in \mathcal{C}$ si et seulement si $H^p(\mathbb{C}_+) \subset L^p(\mu)$ pour un (tout) $p > 0$.

Théorème 3.4.2 *Soient* $\mu \in \mathcal{C}$, $n \in \mathbb{N}$, $1 < p \leq 2$, *et*

$$(T_{p,n}f)(z) = f^{(n)}(z)w_{p,n}(z), \qquad f \in \mathcal{H}(b).$$

Si $1 < p < 2$, *alors* $T_{p,n}$ *est un opérateur borné de* $\mathcal{H}(b)$ *dans* $L^2(\mu)$, *c'est-à-dire qu'il existe une constante* $C = C(\mu, p, n) > 0$ *telle que*

$$\|f^{(n)}w_{p,n}\|_{L^2(\mu)} \leqslant C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

Si $p=2$, alors $T_{2,n}$ est un opérateur de type faible $(2,2)$ comme opérateur de $\mathcal{H}(b)$ dans $L^2(\mu)$, c'est-à-dire qu'il existe une constante $C=C(\mu,n)>0$ telle que

$$\mu\left\{z\in\overline{\mathbb{C}}_+ : |T_{2,n}f(z)|>a\right\}\leq \frac{C}{a^2}\|f\|_b^2,\qquad f\in\mathcal{H}(b).$$

La preuve de ce théorème repose sur la représentation (3.15) et suit essentiellement la preuve du cas intérieur. Elle utilise notamment l'estimation suivante pour le poids : pour $1<p\leq 2$ et $n\in\mathbb{N}$, il existe une constante $A=A(p,n)>0$ telle que

$$w_{p,n}(z)\geq A\,\frac{(\Im\mathrm{m}\, z)^n}{(1-|b(z)|)^{\frac{pn}{q(pn+1)}}},\qquad z\in\mathbb{C}_+. \tag{3.17}$$

On peut également obtenir des estimations sur notre poids $w_{p,n}$ qui font intervenir les ensembles de niveaux de la fonction b. Pour $\varepsilon\in(0,1)$, on pose

$$\Omega(b,\varepsilon):=\{z\in\mathbb{C}_+ : |b(z)|<\varepsilon\},$$

$$\widetilde{\Omega}(b,\varepsilon):=\sigma_+(b)\cup\Omega(b,\varepsilon),$$

et

$$\tilde{d}_\varepsilon(x):=\mathrm{dist}\,(x,\widetilde{\Omega}(b,\varepsilon)),.$$

avec

$$\sigma_+(b):=\left\{x\in\mathbb{R} : \liminf_{\substack{z\longrightarrow x\\ z\in\mathbb{C}_+}}|b(z)|<1\right\}.$$

En particulier, $\mathrm{clos}\sigma_+(b)=\sigma(b)\cap\mathbb{R}$, où $\sigma(b)$ est le spectre de b défini à la section 3.2.

En utilisant des estimations fines sur $|b'(x)|$, on montre alors le résultat suivant :

Lemme 3.4.3 *Pour chaque $p>1$, $n\geq 1$ et $\varepsilon\in(0,1)$, il existe $C=C(\varepsilon,p,n)>0$ telle que*

$$\big(\tilde{d}_\varepsilon(x)\big)^n\leqslant C\,w_{p,n}(x+iy), \tag{3.18}$$

pour tout $x\in\mathbb{R}$ et $y\geq 0$.

Ce lemme combiné avec le théorème 3.4.2 donne alors immédiatement l'inégalité de type Bernstein suivante.

Corollaire 3.4.4 *Pour chaque $\varepsilon\in(0,1)$ et $n\in\mathbb{N}$, il existe une constante $C=C(\varepsilon,n)$ telle que*

$$\|f^{(n)}\tilde{d}_\varepsilon^n\|_2\leqslant C\|f\|_b,\qquad f\in\mathcal{H}(b).$$

3.5 Applications des inégalités de type Bernstein

Dans [T7], nous avons également donné quelques applications des inégalités de Bernstein. La première application est liée au problème de plongement de type Carleson pour les espaces modèles K_Θ^p. Plus précisément, étant donné une fonction intérieure Θ dans le demi-plan supérieur $\mathbb{C}_+$, on veut décrire la classe de mesures de Borel μ dans $\overline{\mathbb{C}}_+$ telles que $K_\Theta^p \subset L^p(\mu)$. Le théorème du graphe fermé implique évidemment que ceci est équivalent à l'existence d'une constante $C > 0$ telle que

$$\|f\|_{L^p(\mu)} \leq C\|f\|_p, \qquad (f \in K_\Theta^p).$$

Ce problème a été posé par Cohn dans [53]. Dans le cas limite où $\Theta \equiv 0$, on retrouve les mesures de Carleson. En dépit du nombre important de résultats profonds (voir par exemple [53, 56, 150, 230]), cette question est toujours ouverte dans le cas général. L'étude de la compacité du plongement est également intéressante et a été considéré dans [49, 55, 228]. Une méthode basée sur des inégalités de Bernstein à poids a été introduite dans [30]. Non seulement cette méthode a permis de donner des preuves unifiées de tous les résultats connus jusqu'à présent, mais elle a également conduit à des généralisations de ces résultats.

Dans [T7], nous donnons des résultats de plongement pour les espaces de de Branges–Rovnyak. Notamment nous obtenons la généralisation suivante d'un résultat de Treil–Volberg [230].

Théorème 3.5.1 *Soient μ une mesure de Borel dans $\overline{\mathbb{C}_+}$ et $\varepsilon \in (0,1)$.*

(a) *Supposons que $\mu(S(x,h)) \leqslant Kh$ pour tous les carrés de Carleson $S(x,h)$ qui satisfont*

$$S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset.$$

Alors $\mathcal{H}(b) \subset L^2(\mu)$, c'est-à-dire qu'il existe une constante $C > 0$ telle que

$$\|f\|_{L^2(\mu)} \leqslant C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

(b) *Supposons que $\mu(S(x,h))/h \to 0$ dès que $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$ et $h \to 0$ ou $\mathrm{dist}(S(x,h),0) \to +\infty$. Alors le plongement $\mathcal{H}(b) \subset L^2(\mu)$ est compact.*

Dans le théorème 3.5.1, on doit donc vérifier la condition de Carleson uniquement sur une sous-classe de carrés. Géométriquement, cela signifie que si on est loin du spectre de b, alors la mesure μ dans le théorème 3.5.1 peut être beaucoup plus grande qu'une mesure de Carleson standard. Cela s'explique par le fait que les fonctions de $\mathcal{H}(b)$ ont beaucoup de régularité en dehors des points du spectre. D'un autre coté, si $|b(x)| \leq \delta < 1$ presque partout sur un intervalle $I \subset \mathbb{R}$, alors les fonctions de $\mathcal{H}(b)$ se comportent sur I essentiellement comme n'importe quelle fonction de $H^2(\mathbb{C}_+)$.

La preuve du théorème 3.5.1 est assez technique et utilise l'inégalité de Bernstein obtenue dans le théorème 3.4.2. Le théorème 3.5.1 est une généralisation d'un résultat de Baranov [30] pour les espaces modèles K^p_Θ. La preuve dans notre cas général diffère quelque peu du cas intérieur et un certain nombres de difficultées supplémentaires sont apparues. Notamment, dans le cas intérieur, Baranov utilise un résultat d'Alexandrov qui affirme que l'ensemble des fonctions de K^2_Θ continues dans $\overline{\mathbb{D}}$ est dense dans K^2_Θ. On ne sait pas si ce résultat reste vrai dans le cas des espaces $\mathcal{H}(b)$.

Pour une classe de fonctions b qui généralise une classe de fonctions introduite par Cohn dans [53], la réciproque du (a) dans le théorème 3.5.1 est aussi vraie.

Théorème 3.5.2 *Soit b une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$ et supposons qu'il existe $\varepsilon \in (0,1)$ tel que $\Omega(b,\varepsilon)$ est connexe, non borné et tel que $\sigma_+(b) \subset \operatorname{clos}\Omega(b,\varepsilon)$. Soit μ une mesure de Borel sur $\overline{\mathbb{C}_+}$. Les assertions suivantes sont équivalentes :*

(a) $\mathcal{H}(b) \subset L^2(\mu)$.

(b) *Il existe $C > 0$ tel que $\mu(S(x,h)) \leqslant Ch$, pour tout carré $S(x,h)$ tel que $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$.*

(c) *Il existe $C > 0$ tel que*

$$\int_{\overline{\mathbb{C}}_+} \frac{\Im\mathrm{m}\, z}{|\zeta - \overline{z}|^2} d\mu(\zeta) \leqslant \frac{C}{1-|b(z)|}, \qquad z \in \mathbb{C}_+. \tag{3.19}$$

Ce résultat généralise certains résultats de [53, 230]. Remarquons que si b vérifie les conditions du théorème 3.5.2, alors il suffit de vérifier l'inégalité

$$\|f\|_{L^2(\mu)} \leq C\|f\|_b$$

pour les noyaux reproduisants de l'espace $\mathcal{H}(b)$ pour l'obtenir pour toutes les fonctions de $\mathcal{H}(b)$. Récemment, Nazarov et Volberg [150] ont montré que cela n'est pas vrai en général.

La deuxième application de nos inégalités de Bernstein concerne le problème de la stabilité des bases de Riesz de noyaux reproduisants. Comme mentionné dans l'introduction du chapitre 2, le problème de la stabilité des bases d'exponentielles et de noyaux reproduisants dans K^2_Θ est un problème ancien et il existe une grande littérature sur ce sujet [164, 23, 70, 117, 120, 234, 89]. Récemment, A. Baranov [31] a proposé une méthode pour étudier cette stabilité des bases de noyaux reproduisants dans K^2_Θ basée sur les inégalités de Bernstein. Dans [T7], nous avons repris et adapté sa méthode pour obtenir des résultats de stabilité pour les noyaux reproduisants dans $\mathcal{H}(b)$. Avant d'énoncer les résultats, nous devons préciser quelques notations.

Pour $\lambda \in \mathbb{C}_+ \cup E_2(b)$, on note h^b_λ le noyau reproduisant normalisé au point λ, c'est à dire, $h^b_\lambda = k^b_\lambda / \|k^b_\lambda\|_b$. On se donne alors une suite $(h^b_{\lambda_n})_{n\geq 1}$ qui forme une base (ou suite) de Riesz de $\mathcal{H}(b)$ et on considère des ensembles $G_n \subset \overline{\mathbb{C}}_+$ tels que $\lambda_n \in G_n$ et si $G := \bigcup_n G_n$, alors G satisfait les deux propriétés suivantes :

(i) il existe deux constantes positives c et C telles que

$$c \leq \frac{\|k^b_{z_n}\|_b}{\|k^b_{\lambda_n}\|_b} \leq C, \qquad z_n \in G_n.$$

(ii) pour chaque $z_n \in G_n$, la mesure $\nu = \sum_n \delta_{[\lambda_n, z_n]}$ est une mesure de Carleson et de plus, les constantes de Carleson C_ν de ces mesures ν sont uniformément bornées par rapport à z_n. Ici $[\lambda_n, z_n]$ désigne le segment dont les extrémités sont λ_n et z_n, et $\delta_{[\lambda_n, z_n]}$ est la mesure de Lebesgue de cet intervalle.

On peut montrer qu'il existe toujours des ensembles G_n satisfaisant ces propriétés. Par exemple, on peut considérer

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < r \Im\mathrm{m}\, \lambda_n\},$$

pour $r > 0$ suffisamment petit.

Enfin, on notera $\omega_p(z) := \omega_{p,1}(z) = \min(\|(k^b_z)^2\|_q^{-p/(p+1)}, \|\rho^{1/q} \mathfrak{K}^\rho_{z,1}\|_q^{-p/(p+1)})$ (voir section 3.4).

Théorème 3.5.3 *Soit $(\lambda_n)_{n\geq 1} \subset \mathbb{C}_+ \cup E_2(b)$ telle que $(h^b_{\lambda_n})_{n\geq 1}$ forme une base de Riesz de $\mathcal{H}(b)$ et soit $p \in [1, 2)$. Alors pour tout ensemble $G = \bigcup_n G_n$ satisfaisant* (i) *et* (ii)*, il existe $\varepsilon > 0$ tel que le système de noyaux reproduisants $(h^b_{\mu_n})_{n\geq 1}$ est une base de Riesz de $\mathcal{H}(b)$ dès que $\mu_n \in G_n$ et*

$$\sup_{n\geq 1} \frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n, \mu_n]} w_p(z)^{-2} |dz| < \varepsilon. \tag{3.20}$$

Donnons quelques éléments de la preuve. Tout d'abord, un raisonnement classique dans la théorie des bases de Riesz permet de voir qu'il suffit de montrer l'estimation suivante

$$\sum_{n=1}^{\infty} |\langle f, h^b_{\lambda_n} - \widetilde{h}^b_{\mu_n} \rangle_b|^2 \leq \varepsilon \|f\|_b^2, \qquad f \in \mathcal{H}(b), \tag{3.21}$$

pour $\varepsilon > 0$ suffisamment petit, où $\widetilde{h}^b_{\mu_n}$ est défini par

$$\widetilde{h}^b_{\mu_n} = \frac{k^b_{\mu_n}}{\|k^b_{\lambda_n}\|_b}.$$

D'autre part, avec l'hypothèse (3.20), on montre qu'on peut supposer que f est continue sur l'intervalle fermé $[\lambda_n, \mu_n]$ et différentiable sur l'intervalle ouvert $]\lambda_n, \mu_n[$. Ceci permet alors d'écrire que

$$|\langle f, h^b_{\lambda_n} - \widetilde{h}^b_{\mu_n} \rangle_b|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k^b_{\lambda_n}\|_b^2} = \frac{1}{\|k^b_{\lambda_n}\|_b^2} \left| \int_{[\lambda_n, \mu_n]} f'(z)\, dz \right|^2.$$

Il suffit alors d'appliquer le théorème 3.4.2 avec la mesure $\nu := \sum_n \delta_{[\lambda_n, \mu_n]}$. □

En combinant le théorème 3.5.3 et l'estimation (3.17), on obtient le résultat suivant qui généralise [89, Theorem 3.3] et un résultat de [31].

Corollaire 3.5.4 *Soient* $(\lambda_n) \subset \mathbb{C}_+$, $(h^b_{\lambda_n})_{n\geq 1}$ *une base de Riesz dans* $\mathcal{H}(b)$, *et soit* $\gamma > 1/3$. *Alors il existe* $\varepsilon > 0$ *tel que le système* $(h^b_{\mu_n})_{n\geq 1}$ *est une base de Riesz dès que*

$$\left|\frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}}\right| \leq \varepsilon(1 - |b(\lambda_n)|)^{\gamma}. \tag{3.22}$$

Chapitre 4

Produits de Blaschke

Dans ce chapitre, nous allons présenter les travaux [T8,T9] qui portent sur le comportement au bord des dérivées des produits de Blaschke.

4.1 Introduction

Rappelons que $\mathbb{D}$ désigne le disque unité ouvert du plan complexe, $\mathbb{T}$ le cercle unité et pour $0 < p \leq +\infty$, H^p est l'espace de Hardy du disque unité. Etant donnée une suite $(z_n)_{n\geq 1} \subset \mathbb{D}$ satisfaisant la condition de Blaschke $\sum_n (1-|z_n|) < +\infty$, on considère

$$B(z) = \prod_{n\geq 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}$$

le produit de Blaschke associé. Alors il est bien connu que B définit une fonction holomorphe et bornée dans $\mathbb{D}$, qui possède des limites radiales de module 1 presque partout sur $\mathbb{T}$. De plus, B se prolonge analytiquement sur $\mathbb{T} \setminus \text{clos}(z_n : n \geq 1)$. La question centrale dans [T8] est la suivante :

a-t-on $B' \in H^p$, pour un certain $p > 0$?

Cette question trouve son origine dans les travaux de Protas [173] et Ahern–Clark [5] et une question analogue pour les fonctions intérieures singulières a également été étudiée par J. Caughran–A Shields [46], M. Cullen [60] et H. Allen–C. Belna [16].

Bien sûr, si B est un produit de Blaschke fini (c'est-à-dire si $(z_n)_{n\geq 1}$ est une suite finie), alors le problème se trivialise car dans ce cas, B se prolonge analytiquement à travers $\mathbb{T}$ sur un disque qui contient le disque unité fermé. Ainsi on obtient immédiatement que $B' \in H^\infty$. Par conséquent, dans toute la suite, on suppose que la suite $(z_n)_{n\geq 1}$ est une suite infinie. Dans ce cas, un célèbre théorème de Privalov (voir [72, page 42]) implique que $B' \notin H^1$. Même si B' n'appartient pas à H^1, peut-on donner l'asymptotique de

$$\int_0^{2\pi} |B'(re^{i\theta})|\, d\theta \tag{4.1}$$

lorsque $r \to 1^-$? Plus généralement, si $p \in]0,1]$, que peut-on dire de

$$\int_0^{2\pi} |B'(re^{i\theta})|^p\, d\theta \tag{4.2}$$

lorsque $r \to 1^-$? Si on calcule la dérivée logarithmique de B un calcul élémentaire montre que

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\overline{z_n}\, z)(z-z_n)}, \qquad (z \in \mathbb{D}).$$

D'où

$$|B'(re^{i\theta})| \leq \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|\,1-\overline{z_n}\, re^{i\theta}\,|^2}, \qquad (re^{i\theta} \in \mathbb{D}). \tag{4.3}$$

Des estimations classiques montrent alors que

$$\int_0^{2\pi} |B'(re^{i\theta})|\, d\theta = \frac{o(1)}{1-r}, \qquad (r \to 1). \tag{4.4}$$

En général, si la condition de Blaschke est la seule restriction sur les zéros de B, on ne peut pas dire plus. Cependant, si on impose des restrictions supplémentaires sur la convergence des zéros du produit de Blaschke, on peut espérer obtenir des estimations plus précises sur le comportement asymptotique de l'intégrale (4.1). La restriction la plus commune est

$$\sum_{n=1}^{\infty} (1-|z_n|)^{\alpha} < \infty,$$

pour $\alpha \in (0,1)$. Nous noterons $C(\alpha)$ cette condition.

Le premier résultat dans cette direction a été obtenu par Protas [173] qui a démontré le théorème suivant.

Théorème 4.1.1 (Protas, 1973) *Si $0 < \alpha < \frac{1}{2}$ et si la suite de zéros $(z_n)_{n\geq 1}$ satisfait la condition $C(\alpha)$, alors $B' \in H^{1-\alpha}$.*

Protas a également donné une condition suffisante pour que la dérivée du produit de Blaschke appartienne à un espace de Bergman à poids. Rappelons que pour $0 < p < \infty$ et $\gamma > -1$, A^p_γ désigne l'espace de Bergman à poids, défini comme l'ensemble des fonctions $f : \mathbb{D} \to \mathbb{C}$ analytiques et telles que $\|f\|_{p,\gamma} < \infty$, où

$$\|f\|_{p,\gamma} = \left(\int_{\mathbb{D}} (1-|z|^2)^{\gamma} |f(z)|^p\, dm_2(z) \right)^{\frac{1}{p}},$$

avec dm_2 la mesure de Lebesgue planaire.

Théorème 4.1.2 (Protas, 1973) *Si $0 < \alpha < 1$ et si la suite de zéros $(z_n)_{n\geq 1}$ satisfait la condition $C(\alpha)$, alors $B' \in A^1_{\alpha-1}$.*

Notons que si $\alpha > 1$, alors $\Theta' \in A^1_{\alpha-1}$ pour toute fonction intérieure Θ quelconque [71, Theorem 5].

P. Ahern et D. Clark [5, 6] ont montré que les résultats de D. Protas sont optimaux dans le sens suivant :

Théorème 4.1.3 (Ahern–Clark, 1974 & 1976) (a) *il existe un produit de Blaschke B dont les zéros $(z_n)_{n\geq 1}$ satisfont la condition $C(\alpha)$ mais B' n'appartient à aucun espace H^p avec $p > 1-\alpha$;*

(b) *il existe un produit de Blaschke B dont les zéros $(z_n)_{n\geq 1}$ satisfont la condition $C(1/2)$ mais B' n'appartient pas à $H^{1/2}$;*

(c) *il existe un produit de Blaschke B dont les zéros $(z_n)_{n\geq 1}$ satisfont la condition $C(\alpha)$ mais B' n'appartient à aucun espace A^1_γ avec $\gamma < 1-\alpha$.*

P. Ahern et D. Clark ont également étudié la réciproque des résultats de Protas. Autrement dit, ils ont donné des conditions nécessaires pour que la dérivée d'un produit de Blaschke soit dans H^p ou dans A^1_γ. Plus précisément, on a le résultat suivant :

Théorème 4.1.4 (Ahern–Clark, 1974 & 1976) *Soient $0 < \alpha < \frac{1}{2}$ et B un produit de Blaschke.*

(a) *Si $B' \in H^{1-\alpha}$, alors la suite $(z_n)_{n\geq 1}$ des zéros de B satisfait la condition $C(\frac{\alpha}{1-\alpha})$.*

(b) *Si $B' \in A^1_{\alpha-1}$ alors la suite $(z_n)_{n\geq 1}$ des zéros de B satisfait la condition $C(\nu)$, pour tout $\nu > \frac{\alpha}{1-\alpha}$.*

(c) *Il existe un produit de Blaschke B satisfaisant $B' \in H^{1-\alpha}$ et dont la suite de zéros $(z_n)_{n\geq 1}$ ne vérifie la condition $C(\beta)$ pour aucun β, $0 < \beta < \frac{\alpha}{1-\alpha}$.*

Dans le même temps, C. Linden [136] a obtenu une généralisation du théorème 4.1.1 pour les dérivées d'ordre supérieur de B.

Théorème 4.1.5 (Linden, 1976) *Soient $\ell \in \mathbb{N}^*$, et $0 < \alpha < \frac{1}{\ell+1}$. Supposons que la suite des zéros $(z_n)_{n\geq 1}$ satisfait la condition $C(\alpha)$, alors $B^{(\ell)} \in H^{\frac{1-\alpha}{\ell}}$.*

Comme le montre le (c) du théorème 4.1.4, la réciproque du théorème 4.1.1 est fausse en général. Cependant, W. Cohn [54] a montré que cette réciproque est vraie pour une classe importante de produits de Blaschke, à savoir les produits de Blaschke d'interpolation. Rappelons que si $(z_n)_{n\geq 1}$ est une suite de Blaschke de $\mathbb{D}$, on dit que $(z_n)_{n\geq 1}$ est une suite de *Carleson* si

$$\inf_{n\geq 1} \prod_{j\neq n} \left| \frac{z_n - z_j}{1-\overline{z_n} z_j} \right| > 0. \tag{4.5}$$

On dit que B est un produit de *Blaschke d'interpolation* si la suite des zéros de B est une suite de Carleson.

Comme on l'a vu avec le théorème 4.1.3, sous la condition $C(\alpha)$, on n'a pas nécessairement $B' \in H^p$, pour $p > 1-\alpha$. Autrement dit, on sait que l'intégrale dans (4.2) va tendre vers $+\infty$ mais on peut essayer de donner une asymptotique. Récemment plusieurs auteurs [126, 127, 174, 100] se sont attachés à cette question. Ainsi, Y. Gotoh [100] a obtenu le résultat suivant.

Théorème 4.1.6 (Gotoh, 2007) *Soient B un produit de Blaschke associé à une suite de zéros $(z_n)_{n\geq 1}$, $\ell \in \mathbb{N}^*$, $0 < \alpha < 1$. Supposons que $(z_n)_{n\geq 1}$ vérifie la condition $C(\alpha)$. Alors, pour tout $p \geq \alpha$, $p > \frac{1-\alpha}{\ell}$, on a*

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^p \, d\theta = \frac{o(1)}{(1-r)^{\ell p+\alpha-1}}, \qquad (r \to 1). \tag{4.6}$$

En 2002, M. Kutbi [127] a obtenu ce résultat pour $0 < \alpha < 1/(l+1)$, puis, en 2004, D. Protas [174] a obtenu l'asymptotique (4.6) pour $\ell = 1$ et $\frac{1}{2} \leq \alpha \leq 1$.

Dans [T8], nous avons étudié des estimations asymptotiques de

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^p \, d\theta,$$

lorsque $r \to 1^-$, en remplaçant la condition $C(\alpha)$ par une condition plus générale du type

$$\sum_{n\geq 1}^{\infty} h(1-|z_n|) < +\infty, \tag{4.7}$$

où h est une fonction positive satisfaisant certaines conditions de "régularité" à préciser. Bien sûr, si nous voulons obtenir des résultats satisfaisant, nous devons au moins supposer que la condition (4.7) est plus forte que la condition de Blaschke, ce qui revient à supposer que $h(t) \geq t$, $t \to 0$.

Une autre direction de recherche concerne des estimations sur la dérivée logarithmique des produits de Blaschke. Rappelons que si f est une fonction méromorphe dans le disque unité $\mathbb{D}$, alors l'ordre de f est défini par

$$\sigma = \limsup_{r\to 1^-} \frac{\log^+ T(r)}{\log 1/(1-r)},$$

où

$$T(r) = \frac{1}{\pi} \int_{\{|z|<r\}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \, \log\left(\frac{r}{|z|}\right) \, dx\, dy$$

est la fonction caractéristique de Nevanlinna de f [151]. Les fonctions méromorphes d'ordre fini ont été étudiées de façon intensive et ont trouvé de nombreuses applications par exemple dans les équations différentielles linéaires. Bien souvent, un rôle important est joué par la dérivée logarithmique de ces fonctions méromorphes et on a besoin de connaître des estimations précises sur cette dérivée lorsqu'on s'approche du bord de $\mathbb{D}$. Plus précisément, considérons l'équation différentielle suivante

$$g^{(n)} + A_{n-1}g^{(n-1)} + \cdots + A_0 g = 0, \tag{4.8}$$

où les coefficients A_j sont des fonctions analytiques dans $\mathbb{D}$. On s'intéresse à la croissance des solutions de (4.8) en fonction de la croissance des coefficients A_j. Il apparait alors naturellement très utile pour ce problème de connaitre des estimations précises sur le comportement de $A_m^{(k)}/A_m^{(j)}$, $k > j > 0$, lorsqu'on s'approche du bord $\mathbb{T}$ (voir [101, 102, 111]). En particulier, J. Heittokangas obtient dans [111] le résultat suivant et donne ensuite une application de ces estimations au problème de l'étude de l'équation (4.8).

Théorème 4.1.7 (Heittokangas, 2000) *Soient f une fonction méromorphe dans le disque unité $\mathbb{D}$ d'ordre fini σ, $\varepsilon > 0$ et $k, j \in \mathbb{N}$ tel que $k > j \geq 0$. Supposons que $f^{(j)} \not\equiv 0$. Alors*

(a) *il existe un sous-ensemble $E_1 \subset (0, 1)$ satisfaisant*

$$\int_{E_1} \frac{dr}{1-r} < \infty,$$

tel que, pour tout $z \in \mathbb{D}$, $|z| \notin E_1$, on a

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| = \frac{O(1)}{(1-|z|)^{(3\sigma+4+\varepsilon)(k-j)}}, \qquad (|z| \to 1^-); \tag{4.9}$$

(b) *il existe un sous-ensemble $E_2 \subset [0, 2\pi)$, de mesure de Lebesgue nulle, tel que si $\theta \in [0, 2\pi) \setminus E_2$, alors pour tout $z \in \mathbb{D}$ satisfaisant $\arg z = \theta$, on a l'estimation* (4.9).

Puis dans [48, Corollary 3.2], I. Chyzhykov, G. Gundersen, et J. Heittokangas montrent que l'exposant $(3\sigma + 4 + \varepsilon)(k - j)$ dans (4.9) peut être remplacé par $(\sigma + 2 + \varepsilon)(k - j)$ et ils donnent un exemple pour montrer que cette estimation est la meilleure possible. En particulier, si B est un produit de Blaschke dans $\mathbb{D}$, alors c'est une fonction holomorphe dans $\mathbb{D}$, d'ordre fini $\sigma = 0$ et on obtient comme conséquence de [48, Corollary 3.2], le résultat suivant :

Corollaire 4.1.8 (Chyzhykov-Gundersen-Heittokangas, 2003) *Soit $\varepsilon > 0$ et B un produit de Blaschke dans $\mathbb{D}$. Alors*

(a) *il existe un sous-ensemble $E_1 \subset (0, 1)$ satisfaisant*

$$\int_{E_1} \frac{dr}{1-r} < \infty,$$

tel que, pour tout $z \in \mathbb{D}$, $|z| \notin E_1$, *on a*

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{O(1)}{(1-|z|)^{2+\varepsilon}}, \qquad (|z| \to 1^-); \tag{4.10}$$

(b) *il existe un sous-ensemble* $E_2 \subset [0, 2\pi)$, *de mesure de Lebesgue nulle, tel que si* $\theta \in [0, 2\pi) \setminus E_2$, *alors pour tout* $z \in \mathbb{D}$ *satisfaisant* $\arg z = \theta$, *on a l'estimation* (4.10).

Dans [T9], nous avons étudié une généralisation du corollaire 4.1.8 lorsqu'on remplace la condition de Blaschke par la condition (4.7).

4.2 Comportement des intégrales moyennes des dérivées des produits de Blaschke

Le résultat principal de [T8] est le suivant.

Théorème 4.2.1 *Soit* h *une fonction continue et positive sur* $(0, 1)$ *et supposons qu'il existe* $q \in (1/2, 1]$ *tel que* $h(t)/t^q$ *est décroissante et* $h(t)/t^{1-q}$ *est croissante sur* $(0, 1)$. *Soit* B *un produit de Blaschke associé à une suite de zéros* $z_n = r_n e^{i\theta_n}$, $n \geq 1$, *satisfaisant*

$$\sum_{n=1}^{\infty} h(1 - r_n) < \infty.$$

Alors pour tout $p \geq q$,

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{O(1)}{(1-r)^{p-1}\, h(1-r)}, \qquad (r \to 1). \tag{4.11}$$

De plus, si

$$\lim_{t \to 0} h(t)/t^{1-q} = 0,$$

alors

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{o(1)}{(1-r)^{p-1}\, h(1-r)}, \qquad (r \to 1). \tag{4.12}$$

Le point clé de la preuve du théorème 4.2.1 est le lemme élémentaire suivant.

Lemme 4.2.2 *Soient* h *une fonction continue et positive sur* $(0, 1)$ *et supposons qu'il existe* $p, q > 0$ *tels que* $h(t)/t^p$ *est décroissante et* $h(t)/t^{p-q}$ *est croissante sur* $(0, 1)$. *Alors si* $(r_n)_{n \geq 1}$ *est une suite de nombres réels dans* $(0, 1)$ *telle que*

$$\sum_{n=1}^{\infty} h(1 - r_n) < \infty,$$

on a

$$(4.13) \qquad \sum_{n=1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^q} = \frac{O(1)}{(1-r)^{q-p}\, h(1-r)}, \qquad (r \to 1).$$

De plus, si $\lim_{t\to 0^+} \frac{h(t)}{t^{p-q}} = 0$*, alors on peut remplacer dans l'estimation* (4.13) *le* $O(1)$ *par un* $o(1)$.

Donnons maintenant la preuve du théorème 4.2.1. Comme $q \leq 1$, on obtient avec (4.3)

$$|B'(re^{i\theta})|^q \leq \sum_{n=1}^{\infty} \frac{(1-r_n^2)^q}{|\, 1 - rr_n e^{i(\theta-\theta_n)}\,|^{2q}}.$$

En utilisant l'estimation classique suivante

$$\int_0^{2\pi} \frac{d\theta}{|1-re^{i\theta}|^{\nu}} \asymp \frac{1}{(1-r)^{\nu-1}}, \qquad (\nu > 1),$$

on en déduit que

$$(4.14) \qquad \int_0^{2\pi} |B'(re^{i\theta})|^q \, d\theta \leq C \sum_{n=1}^{\infty} \frac{(1-r_n)^q}{(1-rr_n)^{2q-1}}.$$

Il suffit alors d'appliquer le lemme 4.2.2 pour obtenir les estimations (4.11) et (4.12) pour $p = q$. Pour $p > q$, on utilise la propriété classique suivante

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < +\infty\,,$$

valable pour toute fonction $f \in H^\infty$. □

Comme on le voit, le théorème 4.2.1 repose donc principalement sur le lemme 4.2.2 dont la preuve est élémentaire. L'intérêt de notre travail est que non seulement ceci va nous permettre de retrouver les résultats de Protas, Linden, Kutbi, Gotoh et Cohn mais cela va aussi nous donner des généralisations de tous ces résultats et ceci de façon élémentaire.

Corollaire 4.2.3 *Soient* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \ldots \alpha_n \in \mathbb{R}$ *et*

$$h(t) := t^{\alpha} \left(\log \frac{1}{t}\right)^{\alpha_1} \cdots \left(\log_n \frac{1}{t}\right)^{\alpha_n},$$

où $\log_n = \log\log\ldots\log$*,* n *fois. Soit* B *un produit de Blaschke associé à une suite de zéros* $(z_n)_{n\geq 1}$ *satisfaisant*

$$\sum_{n=1}^{\infty} h(1-|z_n|) < +\infty.$$

Alors l'estimation suivante

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{O(1)}{(1-r)^{\alpha+p-1} \left(\log \frac{1}{1-r}\right)^{\alpha_1} \dots \left(\log_n \frac{1}{1-r}\right)^{\alpha_n}}, \qquad (r \to 1^-)$$

a lieu dans les cas suivants :

<u>cas 1</u> : $\alpha \in (0,1)$, $p > \max(\alpha, 1-\alpha)$, $\alpha_i \in \mathbb{R}$;

<u>cas 2</u> : $\alpha \in (0, \frac{1}{2})$, $p = 1-\alpha$, $\alpha_1 < 0$, $\alpha_i \in \mathbb{R}$;

<u>cas 3</u> : $\alpha \in (0, \frac{1}{2})$, $p = 1-\alpha$, $\alpha_i = 0$, $i \geq 1$;

<u>cas 4</u> : $\alpha \in (\frac{1}{2}, 1)$, $p = \alpha$, $\alpha_1 > 0$, $\alpha_i \in \mathbb{R}$;

<u>cas 5</u> : $\alpha \in (\frac{1}{2}, 1)$, $p = \alpha$, $\alpha_1 = 0$, $i \geq 1$.

Ce corollaire découle immédiatement du théorème 4.2.1 et il redonne également des résultats connus. Ainsi le cas 3 correspond au théorème 4.1.1 ; le cas 1 généralise le théorème de Kutbi [127] ; les cas 1 et 5 redonnent le théorème de Protas [174].

Avec une méthode similaire, on donne également des estimations pour les dérivées supérieures [T8, Theorem 4.1]. Ceci permet ainsi de retrouver les théorèmes 4.1.5 et 4.1.6. La seule différence est qu'au départ, à la place de (4.14), on utilise l'estimation suivante

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^p \, d\theta \leq C(p,\ell) \sum_{n=1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^{(\ell+1)p-1}}, \qquad (\frac{1}{\ell+1} < p \leq \frac{1}{\ell}),$$

De même, toujours avec la même méthode, on peut donner des estimations pour la dérivée dans l'espace de Bergman à poids [T8, Theorem 5.1]. On obtient alors une généralisation du théorème 4.1.2.

Dans [54], Cohn montre que si $(z_n)_{n\geq 1}$ est une suite de Carleson satisfaisant

$$\sum_{n=1}^{\infty} (1-|z_n|)^{1-p} < \infty,$$

pour un certain $p \in (2/3, 1)$, alors $f' \in H^{2p/(p+2)}$ pour toute $f \in K_B$. Toujours basé en utilisant le lemme 4.2.2, on démontre le théorème suivant qui généralise ce résultat.

Théorème 4.2.4 *Soient h une fonction continue et positive sur $(0,1)$ et B un produit de Blaschke associé à une suite de Carleson $(z_n)_{n\geq 1}$ satisfaisant*

$$\sum_{n=1}^{\infty} h(1-|z_n|) < \infty.$$

Supposons qu'il existe $p \in (2/3, 1)$ *tel que* $h(t)/t^{p/2}$ *est décroissante et* $h(t)/t^{1-p}$ *est croissante sur* $(0, 1)$*. Alors, pour toute fonction* $f \in K_B$*, on a*

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{(\,(1-r)^{p-1}h(1-r)\,)^{1/p}}, \qquad (r \to 1),$$

avec $\sigma = 2p/(p+2)$ *et* C *une constante absolue.*

La preuve est basée sur le fait que si $(z_n)_{n\geq 1}$ est une suite de Carleson, alors la suite des noyaux reproduisants normalisés associée $(h_{z_n})_{n\geq 1}$ forme une base de Riesz de K_B (voir chapitre 2).

4.3 Comportement de la dérivée logarithmique des produits de Blaschke

Dans la suite de ce chapitre, on va supposer que h vérifie les conditions suivantes :

a) h est continue, positive et croissante sur $(0, 1)$, avec $h(0^+) = 0$;

b) $h(t)/t$ est décroissante.

Sous l'hypothèse que B est un produit de Blaschke dont la suite des zéros $(z_n)_{n\geq 1}$ vérifie

$$\sum_{n=1}^{\infty} h(1 - |z_n|) < +\infty,$$

nous avons donné dans [T9],des estimations sur la croissance de la dérivée logarithmique $B'(z)/B(z)$, lorsque z tend vers le cercle unité $\mathbb{T}$, en évitant un ensemble exceptionnel. Encore une fois, le point clé est le lemme 4.2.2.

Le premier résultat concerne des ensembles exceptionnels circulaires.

Théorème 4.3.1 *Soit* B *un produit de Blaschke dont la suite des zéros* $(z_n)_{n\geq 1}$ *vérifie*

$$\sum_{n=1}^{\infty} h(1 - |z_n|) < +\infty.$$

Pour tout $\beta \geq 1$*, il existe un ensemble exceptionnel* $E_1 \subset (0, 1)$ *tel que*

$$\int_{E_1} \frac{dt}{(1-t)^\beta} < \infty$$

et

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^\beta \; h^2(1-|z|)}$$

lorsque $|z| \to 1^-$ *et* $|z| \notin E_1$*.*

En particulier, si

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} < \infty, \tag{4.15}$$

alors, pour tout $\beta \geq 1$, il existe un ensemble exceptionnel $E_1 \subset (0,1)$ tel que

$$\int_{E_1} \frac{dt}{(1-t)^{\beta}} < \infty \tag{4.16}$$

et

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+2\alpha}} \tag{4.17}$$

lorsque $|z| \to 1^-$ et $|z| \notin E_1$.

Si $(|z_n|)_{n\geq 1}$ est une suite de Carleson, alors

$$1-|z_{n+1}| \leq c\,(1-|z_n|)$$

pour une constante $c < 1$ [54, Theorem 9.2]. D'où (4.15) est satisfaite pour tout $\alpha > 0$. Par conséquent, pour tout $\beta \geq 1$ et tout $\varepsilon > 0$, il existe un ensemble exceptionnel E_1 satisfaisant (4.16) et tel que

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+\varepsilon}} \tag{4.18}$$

lorsque $|z| \to 1^-$ et $|z| \notin E_1$. Il serait intéressant de savoir si dans l'estimation (4.18), on peut remplacer ε par zéro.

Le deuxième résultat principal concerne des ensembles exceptionnels de type "radial".

Théorème 4.3.2 *Soit B un produit de Blaschke dont la suite des zéros $(z_n)_{n\geq 1}$ vérifie*

$$\sum_{n=1}^{\infty} h(1-|z_n|) < +\infty.$$

Alors il existe un sous-ensemble $E_2 \subset [0,2\pi)$, de mesure de Lebesgue nulle, tel que si $\theta \in [0,2\pi) \setminus E_2$, alors pour tout $z \in \mathbb{D}$ satisfaisant $\arg z = \theta$ on a

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)\;h(1-|z|)}, \qquad (|z| \to 1^-).$$

En particulier, si

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} < \infty,$$

alors il existe un sous-ensemble $E_2 \subset [0, 2\pi)$, de mesure de Lebesgue nulle, tel que si $\theta \in [0, 2\pi) \setminus E_2$, alors pour tout $z \in \mathbb{D}$, satisfaisant $\arg z = \theta$, on a

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^{1+\alpha}}, \qquad (|z| \to 1^-). \tag{4.19}$$

Si on compare nos estimations (4.17) et (4.19) avec l'estimation de Chyzhykov–Gundersen–Heittokangas (voir corollaire 4.1.8), on voit que notre estimation est meilleure si α est petit ($0 \leq \alpha \leq \frac{1}{2}$) et moins bonne sinon. D'autre part, mentionnons que J. Heittokangas [112, Theorem 1.2] a obtenu de façon indépendante l'estimation (4.19). Il donne également une amélioration de l'estimation (4.17). Plus précisément, sous l'hypothèse que B est un produit de Blaschke dont la suite de zéros $(z_n)_{n\geq 1}$ satisfait la condition (4.15), alors, pour tout $\varepsilon > 0$, il existe un sous-ensemble $E_1 \subset [0, 1)$ tel que

$$\int_{E_1} \frac{dr}{1-r} < \infty$$

et

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{O(1)}{(1-|z|)^{1+\alpha} \left(\log \frac{1}{1-|z|} \right)^{2+\varepsilon}}, \tag{4.20}$$

lorsque $|z| \to 1$, $|z| \notin E_1$. La preuve de ce résultat de Heittokangas est basée sur un lemme de Cartan. Nous espérons qu'en combinant notre méthode et le lemme de Cartan, nous pourrons étendre l'estimation (4.20) à notre cadre général (c'est-à-dire en remplaçant la condition (4.15) par la condition plus générale (4.7)).

Chapitre 5

Autour de certaines classes d'opérateurs

Ce chapitre regroupe les travaux [T10–T12] autour de trois questions en théorie des opérateurs. Même si la publication [T10] traite d'une question sans rapport avec les chapitres précédents, elle a néanmoins pour dénominateur commun avec le chapitre 2 (et particulièrement la prépublication [T4]) le modèle fonctionnel de Sz.-Nagy–Foias. En revanche, les travaux [T11] et [T12] sont dans l'esprit et les techniques assez différents de ce qui précède.

5.1 Les opérateurs complexes symétriques

5.1.1 Introduction

Rappelons que *les opérateurs complexes symétriques* sur un espace de Hilbert complexe, séparable sont caractérisés par l'existence d'une base orthonormale par rapport à laquelle leur matrice est symétrique. Leur théorie est donc directement reliée à la théorie des matrices symétriques, qui est un sujet classique en algèbre linéaire. Ces matrices ou opérateurs apparaissent naturellement dans des branches variées des mathématiques ou de la physique (on pourra consulter [95] pour un survey sur l'histoire du sujet et ses connexions avec les autres domaines). L'intérêt pour les opérateurs complexes symétriques s'est récemment ravivé avec le travail de Garcia–Putinar [93, 94, 95].

D'un point de vue "théorie des opérateurs", il est préférable d'avoir une définition plus intrinsèque de ces opérateurs qui ne dépend pas d'une base. Ceci amène donc à introduire la notion de conjugaison : soit H un espace de Hilbert complexe séparable ; une *conjugaison* C sur H est une application antilinéaire, isométrique et involutive. Autrement dit, C vérifie les relations suivantes :

$$C^2 = Id, \quad \langle Cf, Cg\rangle = \langle g, f\rangle,\ \ f, g \in \mathcal{H}.$$

L'exemple le plus simple de conjugaison est bien sûr la conjugaison complexe sur l'espace de Hilbert $\mathbb{C}$, $z \longmapsto \overline{z}$. C'est en fait l'exemple canonique comme le montre le fait général suivant [94] : si C est une conjugaison sur H, alors il existe une base orthonormale $(e_n)_{n\geq 1}$ de H telle que $Ce_n = e_n$, $n \geq 1$.

Etant donnée C une conjugaison sur H, un opérateur $T \in \mathcal{L}(H)$ est dit *C-symétrique* si

$$T = CT^*C,$$

où T^* désigne l'adjoint de T. On dit que T est *complexe symétrique* s'il existe une conjugaison C sur H telle que T est C-symétrique.

Remarquons que cette définition coïncide avec celle donnée précédemment. En effet, si C est une conjugaison sur H et si $(e_n)_{n\geq 1}$ est la base orthonormale de H telle que $Ce_n = e_n$, alors la matrice de CT^*C relativement à cette base est donnée par

$$[CT^*C]_{jk} = \langle CT^*Ce_j, e_k\rangle = \langle CT^*e_j, Ce_k\rangle = \langle e_k, T^*e_j\rangle = \langle Te_k, e_j\rangle = [T]_{kj},$$

ce qui montre que la matrice de CT^*C relativement à la base $(e_n)_{n\geq 1}$ est simplement la transposée de la matrice de T relativement à cette même base. On obtient donc que T est complexe symétrique si et seulement s'il existe une base orthonormale $(e_n)_{n\geq 1}$ de H par rapport à laquelle la matrice de T est symétrique.

Comme il a été remarqué dans [94], les exemples d'opérateurs complexes symétriques sont très variés : matrices symétriques et en particulier matrices de Hankel, matrices de Toeplitz, opérateurs normaux, certains types d'opérateurs de Volterra. Un exemple étudié en détail par Garcia–Putinar est constitué par l'opérateur modèle de Sz.-Nagy–Foias M_φ, associé à une fonction intérieure φ non constante. Ce dernier exemple a notamment été utilisé par Garcia–Putinar pour traiter d'autres questions comme l'étude des perturbations de rang 1 de M_φ, la paramétrisation explicite des vecteurs non-cycliques de S^* ou encore la résolution du problème de synthèse de Darlington.

Dans [T10], en collaboration avec N. Chevrot et D. Timotin, j'ai exploré en profondeur des généralisations de ce dernier exemple. Le contexte naturel est le modèle fonctionnel des contractions complètement non-unitaires développé par Sz. Nagy et Foias [220].

5.1.2 Un critère pour les contractions complexes symétriques

Le résultat principal de [T10] est un critère qui caractérise les contractions complexes symétriques. Pour cette caractérisation, nous utilisons le langage du modèle fonctionnel de Sz.-Nagy–Foias et plus précisément la notion de fonction caractéristique. Nous renvoyons le lecteur aux sections 1.2 et 1.3 pour toutes les définitions et la construction du modèle fonctionnel.

Théorème 5.1.1 *Soit T une contraction sur un espace de Hilbert H. Les assertions suivantes sont équivalentes :*

(i) *T est complexe symétrique ;*

(ii) *il existe une application antilinéaire $J : \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$, isométrique, surjective et telle que*

$$\Theta_T(z) = J\Theta_T(z)^* J, \qquad \forall z \in \mathbb{D};$$

(iii) *il existe un espace de Hilbert $\mathcal{E}$, une conjugaison J' sur $\mathcal{E}$ et une fonction analytique contractive pure $\Theta : \mathbb{D} \longrightarrow \mathcal{L}(\mathcal{E})$ tels que Θ coïncide avec Θ_T et $\Theta(z)$ est J'-symétrique, pour tout $z \in \mathbb{D}$.*

Donnons une idée de la preuve de ce résultat. L'équivalence entre (ii) et (iii) découle facilement des définitions.

La preuve de l'implication (i) $\Longrightarrow$ (ii) n'est pas difficile : si T est complexe symétrique, alors il existe une conjugaison C sur H telle que $T = CT^*C$. En utilisant les propriétés de C, on montre alors que $C\mathcal{D}_T = \mathcal{D}_{T^*}$. Si on pose $J := C|\mathcal{D}_T$, alors en utilisant la définition de la fonction caractéristique (voir (1.2)), on vérifie que la propriété (ii) est satisfaite.

L'implication (ii) $\Longrightarrow$ (i) est la partie intéressante et non triviale de la preuve. Tout d'abord, en utilisant la décomposition de Sz.-Nagy–Foias–Langer sur les contractions (voir la sous-section 1.2.1), on montre qu'on peut supposer que T est complétement non unitaire. Dans ce cas, on sait (voir la sous-section 1.3.1) que T est unitairement équivalent à l'opérateur modèle $M_{\Theta_T} : K_{\Theta_T} \longrightarrow K_{\Theta_T}$ défini par

$$M_{\Theta_T}(f \oplus g) := P_{\Theta_T}(zf \oplus zg),$$

Il s'agit donc de montrer que M_{Θ_T} est complexe symétrique. On considère alors $\tilde{J} : L^2(\mathcal{D}_T) \longrightarrow L^2(\mathcal{D}_{T^*})$ défini par

$$(\tilde{J}f)(z) := \bar{z}J(f(\bar{z})),$$

et on vérifie que $\tilde{J}$ est antilinéaire, isométrique et surjective. On pose ensuite

$$C(\pi f + \pi_* g) = \pi_*(\tilde{J}f) + \pi(\tilde{J}^{-1}g), \qquad f \in L^2(\mathcal{D}_T),\ g \in L^2(\mathcal{D}_{T^*}),$$

où $\pi : L^2(\mathcal{D}_T) \longrightarrow \mathcal{H}$ et $\pi_* : L^2(\mathcal{D}_{T^*}) \longrightarrow \mathcal{H}$ sont les deux isométries définies par (1.10). On montre que C est une conjugaison sur $\mathcal{H}$ qui laisse invariant K_{Θ_T} et son orthogonal. Alors si on pose $C' := C|K_{\Theta_T}$, on obtient que C' est une conjugaison sur K_{Θ_T} et M_{Θ_T} est C'-symétrique. $\square$

On en déduit alors facilement les deux corollaires.

Corollaire 5.1.2 *Soit T une contraction telle que $\partial_T = \partial_{T^*} = 1$. Alors T est complexe symétrique.*

Corollaire 5.1.3 *Tout opérateur agissant sur un espace de dimension 2 est complexe symétrique.*

Signalons que le corollaire 5.1.2 apparaît déjà dans [93, 95] mais avec l'hypothèse supplémentaire que $T \in C_{00}$ (ce qui correspond au cas où la fonction caractéristique de T est intérieure). Le corollaire 5.1.3 se trouve aussi dans [95] avec une preuve différente. Enfin remarquons que d'après le théorème 5.1.1, si une contraction T est complexe symétrique, alors nécessairement on a $\partial_T = \partial_{T^*}$. En fait, c'est aussi une conséquence d'un résultat plus général (voir [95]) qui dit que si un opérateur T (non nécessairement contractif) est complexe symétrique, alors $\dim\ker(T) = \dim\ker(T^*)$.

Dans la suite de [T10], nous avons discuté en profondeur du cas des contractions $T \in C_{00}$ telles que $\partial_T = \partial_{T^*} = 2$.

Donnons tout d'abord une définition : étant donnée $\Theta : \mathbb{D} \longrightarrow \mathcal{L}(H, H_*)$ une fonction analytique contractive, on dit que Θ est *symétrisable* si la matrice de $\Theta(z)$ par rapport à des bases orthonormales fixées de H et de H_* est symétrique pour tout $z \in \mathbb{D}$. En utilisant le théorème 5.1.1, il n'est pas difficile de montrer qu'une contraction T est complexe symétrique si et seulement si sa fonction caractéristique Θ_T est symétrisable. Notons aussi que d'après le corollaire 5.1.3, si $\partial_T = \partial_{T^*} = 2$, alors pour tout $z \in \mathbb{D}$, on peut trouver des bases orthonormales de $\mathcal{D}_T$ et $\mathcal{D}_{T^*}$ par rapport auxquelles la matrice de $\Theta_T(z)$ est symétrique, mais la difficulté ici pour obtenir des fonctions caractéristiques symétrisables est que le choix des bases ne doit pas dépendre de z.

Rappelons d'abord un résultat de [94] qui donne une paramétrisation des fonctions intérieures à valeurs matricielles 2×2.

Proposition 5.1.4 (Garcia, 2005) *Soit φ une fonction intérieure non constante, $a, b, c, d \in H^2$ et*

$$\Theta(z) = \begin{pmatrix} a(z) & -b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Alors Θ est une fonction intérieure dont le déterminant vaut φ si et seulement si

(i) $a, b, c, d \in K_{z\varphi} (= H^2 \ominus z\varphi H^2)$;

(ii) $d = C(a)$ *et* $c = C(b)$;

(iii) $|a|^2 + |b|^2 = 1$ *p.p. sur* $\mathbb{T}$.

Ici C désigne la conjugaison sur $K_{z\varphi}$ définie par

$$C(f) = \overline{f}\varphi, \qquad (f \in K_{z\varphi}). \tag{5.1}$$

On a alors obtenu la caractérisation suivante des fonctions intérieures à valeurs matricielles 2×2 et qui sont symétrisables.

Théorème 5.1.5 *Une fonction intérieure à valeurs matricielles 2×2*

$$\Theta(z) := \begin{pmatrix} a(z) & -b(z) \\ C(b)(z) & C(a)(z) \end{pmatrix}$$

est symétrisable si et seulement s'il existe $(\gamma, \theta) \neq (0,0)$ *tels que*

$$C(\gamma a + \theta b) = \gamma a + \theta b,$$

où C *est défini par* (5.1) *et* φ *est le déterminant de* Θ.

Comme exemple d'application, nous considérons alors les contractions $T \in \mathcal{L}(H)$ du type

$$T = \begin{pmatrix} M_u & X \\ 0 & M_v \end{pmatrix}$$

où u et v sont deux fonctions intérieures non constantes, M_u (resp. M_v) représente l'opérateur modèle associé à u (resp. à v) et $H = K_u \oplus K_v$. On montre que si $\|T\| \leq 1$ alors

(i) $X = D_{M_u^*} Y D_{M_v}$, $Y : \mathcal{D}_{M_v} \longmapsto \mathcal{D}_{M_u^*}$ est une contraction.

(ii) Si $\|Y\| = 1$, on a $\partial_T = \partial_{T^*} = 1$ et sinon $\partial_T = \partial_{T^*} = 2$.

(iii) $T \in C_{00}$.

Le résultat suivant donne alors un critère pour que T soit complexe symétrique.

Théorème 5.1.6 *La contraction* T *est complexe symétrique si et seulement si l'une des trois conditions suivantes est réalisée :*

(a) $Y = 0$.

(b) $\|Y\| = 1$.

(c) $0 < \|Y\| < 1$ *et il existe* $\lambda \in \mathbb{D}$ *et* $\mu \in \mathbb{T}$ *tels que* $v = \mu \dfrac{\lambda - u}{1 - \overline{\lambda} u}$.

La preuve est une application des résultats précédents :
• si $Y = 0$, alors $T = M_u \oplus M_v$ est complexe symétrique comme somme directe d'opérateurs complexes symétriques.
• si $\|Y\| = 1$, alors $\partial_T = \partial_{T^*} = 1$ et le corollaire 5.1.2 implique que T est complexe symétrique.
• si $0 < \|Y\| < 1$, alors $\partial_T = \partial_{T^*} = 2$. En utilisant la théorie de Sz.-Nagy–Foias, on montre alors que la fonction caractéristique de T coïncide avec la fonction intérieure

$$\Theta(z) = \begin{pmatrix} \alpha & -\beta u(z) \\ \overline{\beta} v(z) & \overline{\alpha} u(z) v(z) \end{pmatrix},$$

où $|\alpha|^2 + |\beta|^2 = 1$. Il y a alors encore un peu de travail mais le résultat découle des théorèmes 5.1.5 et 5.1.1. □

5.2 Combinaisons linéaires d'opérateurs algébriques

5.2.1 Introduction

La deuxième question présentée dans ce chapitre concerne les propriétés spectrales des combinaisons linéaires d'idempotents. Dans [124], J. Koliha, V. Rakočević et I. Straškraba se sont intéressés à l'inversibilité des différences et sommes de deux matrices idempotentes. J. Baksalary et O. Baksalary [24] ont alors prouvé que, si P_1, P_2 sont deux matrices idempotentes, alors la non inversibilité de P_1+P_2 est équivalente à la non inversibilité de n'importe quelle combinaison linéaire $c_1P_1 + c_2P_2$, où $c_1c_2 \neq 0$, $c_1 + c_2 \neq 0$. Récemment, H. Du, X. Yao et C. Deng [69] ont donné (avec une preuve très compliquée) une généralisation de ce résultat pour des opérateurs idempotents sur un espace de Hilbert. Dans [122], Koliha et Rakočević étendent le résultat de Baksalary–Baksalary en prouvant que la dimension du noyau (respectivement du rang) de $c_1P_1 + c_2P_2$ ne dépend pas du choix des nombres complexes $c_1, c_2 \neq 0$, $c_1 + c_2 \neq 0$. Dans [123], Koliha et Rakočević prouvent que la propriété de Fredholm ou de semi-Fredholm pour une combinaison linéaires d'idempotents sur un espace de Banach ne dépend pas non plus du choix des coefficients.

Rappelons que si X est un espace de Banach complexe et si $T \in \mathcal{L}(X)$, on note $\ker T$ son noyau, $\mathrm{Im}\ T$ son image, $\alpha(T) = \dim(\ker T)$ et $\beta(T) = \dim(X/\mathrm{Im}\ T)$. Un opérateur $T \in \mathcal{L}(X)$ est dit *semi-Fredholm* si $\mathrm{Im}\ T$ est fermé et au moins un des deux nombres $\alpha(T)$ ou $\beta(T)$ est fini. Pour un tel opérateur, on définit l'*indice* de T comme $i(T) = \alpha(T) - \beta(T)$. Notons par $\Phi_+(X)$ (resp. $\Phi_-(X)$) l'ensemble des *opérateurs semi-Fredholm supérieurement* (resp. *opérateurs semi-Fredholm inférieurement*), c'est à dire l'ensemble de tous les opérateurs semi-Fredholm pour lesquels $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$). Un opérateur $T \in \mathcal{L}(X)$ est dit *Fredholm* si $T \in \Phi(X) := \Phi_+(X) \cap \Phi_-(X)$.

On définit Γ le sous-ensemble de $\mathbb{C}^2$ par

$$\Gamma = \{(c_1, c_2) \in \mathbb{C}^2 : c_1 \neq 0, c_2 \neq 0, c_1 + c_2 \neq 0\}.$$

Alors on peut formuler précisément le résultat principal de [123].

Théorème 5.2.1 (Koliha-Rakočević, 2007) *Soient X un espace de Banach et $P_1, P_2 \in \mathcal{L}(X)$ deux opérateurs idempotents.*

(i) *Supposons qu'il existe $(c_1, c_2) \in \Gamma$ tel que $c_1P_1 + c_2P_2$ est semi-Fredholm supérieurement. Alors $z_1P_1 + z_2P_2$ est semi-Fredholm supérieurement pour tout $(z_1, z_2) \in \Gamma$. De plus, $\alpha(z_1P_1 + z_2P_2)$ est constant sur Γ.*

(ii) *Supposons qu'il existe $(c_1, c_2) \in \Gamma$ tel que $c_1P_1 + c_2P_2$ est semi-Fredholm inférieurement. Alors $z_1P_1 + z_2P_2$ est semi-Fredholm inférieurement pour tout $(z_1, z_2) \in \Gamma$. De plus, $\beta(z_1P_1 + z_2P_2)$ est constant sur Γ.*

La preuve de ce théorème est simple et basée sur les deux observations suivantes : si P_1, P_2 sont deux idempotents et si $(c_1, c_2) \in \Gamma$, alors

- $Id - P_1$ envoie injectivement $\ker(c_1P_1 + c_2P_2)$ dans $\ker((Id - P_1)P_2) \cap \ker(P_1)$.
- $(1 + c_1c_2^{-1})Id - P_2$ envoie injectivement $\ker((Id - P_1)P_2) \cap \ker(P_1)$ dans $\ker(c_1P_1 + c_2P_2)$.

De ces deux observations, on en déduit donc que

$$\dim(\ker(c_1P_1 + c_2P_2)) = \dim(\ker((Id - P_1)P_2) \cap \ker(P_1)), \tag{5.2}$$

ce qui prouve que $\alpha(c_1P_1 + c_2P_2)$ est constant sur Γ. Pour en déduire le théorème 5.2.1, Koliha–Rakočević utilisent alors un procédé introduit par B. Sadovskii [193] et plus tard (indépendamment) par J. Buoni, R. Harte et T. Wickstead [43]. Plus précisément, si X est un espace de Banach, alors

$$\ell^\infty(X) := \{\mathbf{x} = (x_n)_{n\in\mathbb{N}} : x_n \in X,\ \|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\| < \infty\},$$

et $m(X)$ est défini comme l'espace des suites $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ de X telle que tout sous-suite $(x_{n_k})_{k\in\mathbb{N}}$ a une sous-suite convergente. De façon équivalente, on a $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in m(X)$ si et seulement si $\{x_n : n \in \mathbb{N}\}$ est une partie totalement bornée de X (pour la définition, voir [192, page 72]). Il est facile de voir que $\ell^\infty(X)$ est un espace de Banach et $m(X)$ est un sous-espace fermé de $\ell^\infty(X)$. Par conséquent, l'espace quotient

$$\widetilde{X} := \ell^\infty(X)/m(X)$$

est un espace de Banach. Tout opérateur $T \in \mathcal{L}(X)$ détermine un opérateur $\widetilde{T} \in \mathcal{L}(\widetilde{X})$ défini par

$$\widetilde{T}((x_n)_{n\in\mathbb{N}} + m(X)) := (Tx_n)_{n\in\mathbb{N}} + m(X), \qquad (x_n)_{n\in\mathbb{N}} \in \ell^\infty(X).$$

L'application $T \longmapsto \widetilde{T}$ est un homomorphisme continu d'algèbre de $\mathcal{L}(X)$ dans $\mathcal{L}(\widetilde{X})$. On a alors le résultat essentiel suivant

Théorème 5.2.2 (Sadovskii, 1972 & Buoni–Harte–Wickstead, 1977) *Soit $T \in \mathcal{L}(X)$. Les assertions suivantes sont équivalentes :*

(i) *T est semi-Fredholm supérieurement ;*

(ii) *$\widetilde{T}$ est injectif ;*

(iii) *$\widetilde{T}$ est borné inférieurement.*

En utilisant le théorème 5.2.2 et (5.2), il est alors facile d'en déduire le théorème de Koliha–Rakočević.

Indépendamment, dans [68], H. Du, C. Deng, M. Mbekhta et V. Müller ont obtenu le résultat suivant.

Théorème 5.2.3 (Du–Deng–Mbekhta–Müller, 2007) *Soient X un espace de Banach et $P, Q \in \mathcal{L}(X)$ deux idempotents. Soient $(c_1, c_2) \in \Gamma$. Si $c_1P + c_2Q$ est inversible (respectivement inversible à gauche, inversible à droite, injectif, borné inférieurement, surjectif, Fredholm, semi-Fredholm supérieurement, semi-Fredholm inférieurement, inversible essentiellement à gauche, inversible essentiellement à droite, a un inverse généralisé), alors $z_1P + z_2Q$ a la même propriété pour tout $(z_1, z_2) \in \Gamma$.*

La preuve est basée sur l'utilisation de la topologie du "gap" et la "co-norme".

Motivé par tous ces résultats, en collaboration avec I. Chalendar et D. Timotin, nous avons cherché à voir dans [T11] si on pouvait étendre ces résultats de stabilité à des opérateurs algébriques possédant le même polynôme minimal.

5.2.2 Stabilité des propriétés spectrales

Dans [T11], nous nous sommes intéressés à la question suivante : *soient X un espace de Banach complexe et $T, S \in \mathcal{L}(X)$ deux opérateurs algébriques avec le même polynôme minimal p ; est-ce que les propriétés spectrales de $T - zS$ (injectivité, surjectivité, dimension du noyau et de l'image, image fermée,...) restent les mêmes en dehors d'un ensemble fini de valeurs pour le paramètre complexe z ?*

D'après les théorèmes 5.2.1 et 5.2.3, on sait que c'est vrai si $p(z) = z^2 - 1$. C'est aussi trivialement vrai si $p(z) = z - a$, $a \in \mathbb{C}$. Nous avons montré dans [T11] que ceux sont essentiellement les deux seuls cas où on peut obtenir des résultats de stabilité de ce type.

Théorème 5.2.4 *Soit p un polynôme unitaire de degré $d \geq 1$. Les assertions suivantes sont équivalentes :*

i) *$p(z) = z - a$ ou $p(z) = z^2 - bz$ avec $b \neq 0$;*
ii) *il existe un ensemble fini F tel que pour toute matrice S, T dont le polynôme minimal est p, alors $z \mapsto \dim \ker(T - zS)$ est constant sur $\mathbb{C} \setminus F$.*

La preuve est très simple : l'implication i) $\Longrightarrow$ (ii) découle facilement du théorème 5.2.1 ou 5.2.3. La preuve de ii) $\Longrightarrow$ (i) consiste à construire différents contre exemples dans les cas où

- $p(z) = (z - a)(z - b)$, $a, b \in \mathbb{C} \setminus \{0\}$, $a \neq b$;
- $p(z) = (z - a)^2$, $a \neq 0$;
- $p(z) = z^2$.

Pour les deux premiers cas, des matrices 2×2 suffisent, pour le troisième cas, il faut prendre des matrices 4×4. Le cas général s'en déduit alors facilement avec des sommes directes d'opérateurs. □

Dans les cas où les opérateurs $T, S \in \mathcal{L}(X)$ commutent, il est plus facile d'obtenir des résultats de stabilité.

Théorème 5.2.5 *Soient $T, S \in \mathcal{L}(X)$ deux opérateurs algébriques qui commutent, avec comme polynôme minimal correspondant p, q. Supposons que les racines de p sont $\lambda_i,\ i = 1, \ldots, m$ et celles de q sont $\mu_j,\ j = 1, \ldots, n$. On définit l'ensemble*

$$F = \left\{ \frac{\lambda_i}{\mu_j} : 1 \leq i \leq m, 1 \leq j \leq n, \mu_j \neq 0 \right\}.$$

Alors, pour tout $z \notin F$, $T - zS$ est simultanément inversible à gauche ou non.

La preuve est basée sur la notion de spectre de Harte de T et S et sur un théorème spectral [105].

Remarque 5.2.6 Notons que si T, S sont des opérateurs algébriques qui commutent, alors $T - zS$ est aussi algébrique, puisque les algèbres engendrées par T et par S sont de dimension finie, tandis que l'algèbre engendrée par $T - zS$ est contenu dans le produit de ces deux algèbres. Comme le spectre d'un opérateur algébrique coïncide avec son spectre ponctuel, l'injectivité est équivalente à l'inversibilité à droite, à l'inversibilité à gauche, à l'inversibilité ou encore à la propriété d'être borné inférieurement (toutes ces propriétés sont équivalentes au fait que $0 \notin \sigma(T)$). On peut donc reformuler le théorème 5.2.5 en remplaçant l'inversibilité à gauche par une de ces propriétés.

En utilisant alors le théorème 5.2.2, on obtient comme corollaire :

Corollaire 5.2.7 *Avec les notations ci-dessus, pour tout $z \notin F$, l'opérateur $T - zS$ est simultanément semi-Fredholm inférieurement, semi-Fredholm supérieurement, Fredholm, essentiellement inversible à droite, essentiellement inversible à gauche, essentiellement inversible.*

5.3 Opérateurs singuliers et sous-espaces invariants

5.3.1 Introduction

La troisième question présentée dans ce chapitre concerne le problème du sous-espace invariant. Rappelons que si X est un espace de Banach, E un sous-espace fermé de X et T un opérateur borné sur X, alors on dit que E est un *sous-espace invariant non trivial* pour T si $T(E) \subseteq E$ et $\{0\} \neq E \neq X$. Le problème du sous-espace invariant est le suivant : étant donné X un espace de Banach complexe, séparable et de dimension infinie, tout opérateur $T \in \mathcal{L}(X)$ possède-t-il un sous-espace invariant non trivial ?

On sait depuis le contre exemple d'Enflo en 1976 que la réponse à la question est négative. Depuis, de nombreux autres contre-exemples ont été donnés notamment par Beauzamy et Read. D'un autre coté, une branche importante des mathématiques s'est développée pour trouver des classes d'opérateurs qui possèdent un sous-espace invariant non trivial. Dans cette direction, l'un des résultats les plus célèbres est sans doute dû à Lomonosov en 1973 : il prouve [141] que si $T \in \mathcal{L}(X)$ est compact, $T \neq 0$, alors il existe un sous-espace fermé M de X, $\{0\} \neq M \neq X$, tel que $S(M) \subseteq M$, pour tout opérateur $S \in \mathcal{L}(X)$ qui commute avec T. En particulier, on obtient que tout opérateur qui commute avec un opérateur compact non nul possède un sous-espace invariant non trivial. Pełczyński a alors demandé quelle est la situation pour les opérateurs strictement singuliers, qui sont dans une certaine mesure assez proche des opérateurs compacts.

Rappelons que $T : X \longrightarrow Y$ est dit *strictement singulier* si pour tout $\varepsilon > 0$ et tout sous-espace de dimension infinie $E \subseteq X$, il existe un vecteur x dans la sphère unité de E tel que $\|Tx\| < \varepsilon$. De plus, on dira que T est *finiment strictement singulier* si pour tout $\varepsilon > 0$, il existe un entier $n \in \mathbb{N}^*$ tel que pour tout sous-espace $E \subseteq X$ avec $\dim(E) \geq n$, il existe un vecteur x dans la sphère unité de E tel que $\|Tx\| < \varepsilon$. Les opérateurs finiment strictement singuliers sont aussi appelés dans la littérature *super-strictement singuliers*. Notons que

$$\text{compact} \quad \Rightarrow \quad \text{finiment strictement singulier} \quad \Rightarrow \quad \text{strictement singulier},$$

et chacune de ces propriétés définit un sous-espace fermé (et même un idéal d'opérateurs) de $\mathcal{L}(X, Y)$. Pour plus d'informations sur les opérateurs strictement et finiment strictement singuliers, on pourra consulter [137, 147, 204, 21, 171, 168].

Dans [182], Read a répondu à la question de Pełczyński, en construisant un opérateur strictement singulier sans sous-espace invariant non trivial. Dans [T12], nous nous sommes intéressés à un analogue de la question de Pełczyński pour les opérateurs finiment strictement singuliers. Autrement dit,

est-ce que tout opérateur finiment strictement singulier a un sous-espace invariant non trivial ?

Des résultats partiels dans cette direction ont été obtenus dans [171, 21]. Dans [T12], nous répondons à cette question par la négative, en montrant que l'opérateur considéré par Read dans [182] est en fait finiment strictement singulier.

5.3.2 Opérateurs finiment strictement singuliers entre espaces de James

Il est assez délicat de décrire exactement la construction de l'opérateur considéré par Read dans [182] ; nous allons donc dans ce qui suit, uniquement rappeler ce dont nous avons besoin pour prouver que cet opérateur est finiment strictement singulier.

Tout d'abord, cet opérateur agit sur une somme directe (hilbertienne) infinie qui fait intervenir les espaces de James. Rappelons que l'espace de James J_p, $1 \leq p < \infty$, est l'espace des suites réelles $x = (x_n)_{n=1}^{\infty}$ dans c_0 satisfaisant

$\|x\|_{J_p} < \infty$ où

$$\|x\|_{J_p} := \Big(\sup\Big\{\sum_{i=1}^{n-1}|x_{k_{i+1}} - x_{k_i}|^p \ : \ 1 \leqslant k_1 < \cdots < k_n,\ n \in \mathbb{N}\Big\}\Big)^{\frac{1}{p}}$$

est la norme dans J_p (ici c_0 est l'espace des suites convergentes vers 0). Pour plus d'informations sur les espaces de James, le lecteur pourra consulter [118, 214, 137, 140, 169] ; signalons seulement que l'espace J_2 a été introduit comme premier exemple d'espace de Banach non réflexif et qui est isomorphe isométriquement à son bidual. L'opérateur de Read T est alors défini sur l'espace $X := (\ell_2 \oplus Y)_{\ell_2}$, où Y lui-même est défini comme une somme hilbertienne (infinie) d'espaces de James

$$Y := \left(\bigoplus_{i=1}^{+\infty} J_{p_i}\right)_{\ell_2},$$

avec (p_i) une suite strictement croissante de nombres réels dans $(2, +\infty)$, vérifiant certaines propriétés de croissance. Read montre que T est une perturbation compacte de $0 \oplus W_1$, où $W_1 : Y \longrightarrow Y$ agit comme un shift à poids, c'est-à-dire

$$W_1(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \beta_2 x_2, \dots), \qquad (x_i \in J_{p_i}),$$

avec $\beta_i \to 0$, $i \to +\infty$. Notons qu'on devrait plutôt écrire $\beta_i i_{p_i,p_{i+1}} x_i$ à la place de $\beta_i x_i$, avec $i_{p,q}$ qui désigne l'injection canonique de J_p dans J_q, $1 \le p < q < +\infty$. On obtient alors dans [T12] le résultat suivant.

Théorème 5.3.1 *L'opérateur de Read T est finiment strictement singulier et sans sous-espace invariant non trivial.*

Donnons une idée de la preuve de ce théorème. Comme T est une perturbation compacte de $0 \oplus W_1$, il suffit de montrer que W_1 est finiment strictement singulier. Si on considère $V_n : Y \longrightarrow Y$, défini par

$$V_n(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \beta_2 x_2, \dots, \beta_n x_n, 0, \dots), \qquad (x_i \in J_{p_i}),$$

alors comme $\beta_i \to 0$, on a $\|V_n - W_1\| \to 0$ et donc tout revient à prouver que V_n est finiment strictement singulier. Finalement, le le théorème 5.3.1 découle du résultat suivant :

Théorème 5.3.2 *Si $1 \le p < q < +\infty$, l'inclusion canonique $i_{p,q} : J_p \longrightarrow J_q$ est finiment strictement singulière.*

Notons qu'il existe un résultat analogue pour l'inclusion canonique de ℓ^p dans ℓ^q (voir [204, Proposition 3.3]), qui est basé sur le lemme suivant dû à Milman [147].

Lemme 5.3.3 (Milman, 1970) *Soient $k \in \mathbb{N}^*$ et E un sous-espace de dimension k dans c_0. Alors il existe un vecteur $x = (x_i)_{i \geq 1} \in E$, de norme 1 et tel qu'il existe $i_1 < i_2 < \cdots < i_k$ avec*

$$|x_{i_s}| = 1, \qquad (1 \leq s \leq k).$$

La preuve du lemme 5.3.3 est très simple ; elle se fait par récurrence et utilise le théorème des valeurs intermédiaires.

Pour prouver le théorème 5.3.2, nous utilisons un analogue du lemme de Milman. Pour formuler précisément notre résultat, nous introduisons une définition. Une suite finie ou infinie de nombres réels dans $[-1, 1]$ sera appelé un *zigzag* d'ordre k si elle possède une sous-suite de la forme $(-1, 1, -1, 1, -1, \dots)$ de longueur k. On a alors :

Théorème 5.3.4 *Soient $k \in \mathbb{N}^*$ et E un sous-espace de dimension k dans c_0. Alors E possède un zigzag d'ordre k.*

Ce résultat est la clé des théorèmes 5.3.1 et 5.3.4 mais la preuve est beaucoup plus compliquée que la preuve du lemme de Milman. La première étape consiste à montrer qu'il suffit de prouver le résultat pour les sous-espaces de $\mathbb{R}^n$. Puis nous donnons alors deux preuves différentes. La première est basée sur les propriétés combinatoires des polytopes. La seconde est basée sur la géométrie algébrique. Donnons une idée de la deuxième méthode. Tout d'abord, notons pour $k \geq 1$,

$$\Gamma_k = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1 \text{ et } x \text{ a au moins } k \text{ coordonnées alternées } \pm 1\}$$

La clé de la preuve consiste à construire par récurrence une projection orthogonale P_{k-1} sur un sous-espace π_{k-1} de dimension $n-k+1$ telle que P_{k-1} envoie homéomorphiquement Γ_k sur $\partial\Delta_{k-1}$, où Δ_{k-1} est un ensemble convexe, équilibré et qui contient 0 dans son intérieur. Si on compose alors P_{k-1} par l'application $x \longmapsto \frac{x}{\|x\|}$, on obtient alors une application $\phi : \Gamma_k \longrightarrow S_\infty^{n-k}$, qui est un homéomorphisme et qui satisfait la relation $\phi(-x) = -\phi(x)$ (ici S_∞^m désigne la sphère unité de $\mathbb{R}^{m+1}$ pour la norme infinie). Maintenant pour prouver le théorème 5.3.4, supposons par l'absurde que E est un sous-espace de $\mathbb{R}^n$ de dimension k et qui ne contient pas de zigzag d'ordre k. Alors $E \cap \Gamma_k = \emptyset$ et donc la projection de Γ_k sur $E^\perp$ ne contient pas 0. En composant cette projection par l'application $x \longmapsto \frac{x}{\|x\|}$, on obtient alors une application continue $\psi : \Gamma_k \longrightarrow S_\infty^{n-k-1}$ qui satisfait $\psi(-x) = -\psi(x)$. Par conséquent, l'application $\Phi := \psi \circ \phi^{-1} : S_\infty^{n-k} \longrightarrow S_\infty^{n-k-1}$ est continue et satisfait $\Phi(-x) = -\Phi(x)$. Un résultat classique de topologie algébrique nous dit alors qu'une telle application n'existe pas et on obtient une contradiction. □

Outre l'application à la question de l'existence de sous-espaces invariants non triviaux pour les opérateurs finiment strictement singuliers, nous pensons que le théorème 5.3.4 présente un intérêt propre et devrait trouver dans le futur d'autres applications.

Bibliographie

[1] Ahern, P. R., and Clark, D. N. On functions orthogonal to invariant subspaces. *Acta Math. 124* (1970), 191–204.

[2] Ahern, P. R., and Clark, D. N. On star-invariant subspaces. *Bull. Amer. Math. Soc. 76* (1970), 629–632.

[3] Ahern, P. R., and Clark, D. N. Radial limits and invariant subspaces. *Amer. J. Math. 92* (1970), 332–342.

[4] Ahern, P. R., and Clark, D. N. Radial nth derivatives of Blaschke products. *Math. Scand. 28* (1971), 189–201.

[5] Ahern, P. R., and Clark, D. N. On inner functions with H^p-derivative. *Michigan Math. J. 21* (1974), 115–127.

[6] Ahern, P. R., and Clark, D. N. On inner functions with B^p derivative. *Michigan Math. J. 23*, 2 (1976), 107–118.

[7] Aleksandrov, A. B. Invariant subspaces of shift operators. An axiomatic approach. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 113* (1981), 7–26, 264. Investigations on linear operators and the theory of functions, XI.

[8] Aleksandrov, A. B. Inner functions and related spaces of pseudocontinuable functions. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 170*, Issled. Linein. Oper. Teorii Funktsii. 17 (1989), 7–33, 321.

[9] Aleksandrov, A. B. On the maximum principle for pseudocontinuable functions. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 217*, Issled. po Linein. Oper. i Teor. Funktsii. 22 (1994), 16–25, 218.

[10] Aleksandrov, A. B. A simple proof of the Volberg-Treil theorem on the embedding of covariant subspaces of the shift operator. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 217*, Issled. po Linein. Oper. i Teor. Funktsii. 22 (1994), 26–35, 218.

[11] Aleksandrov, A. B. On the existence of angular boundary values of pseudocontinuable functions. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222*, Issled. po Linein. Oper. i Teor. Funktsii. 23 (1995), 5–17, 307.

[12] Aleksandrov, A. B. Isometric embeddings of co-invariant subspaces of the shift operator. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 232*, Issled. po Linein. Oper. i Teor. Funktsii. 24 (1996), 5–15, 213.

[13] Aleksandrov, A. B. Embedding theorems for coinvariant subspaces of the shift operator. II. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 262*, Issled. po Linein. Oper. i Teor. Funkts. 27 (1999), 5–48, 231.

[14] Aleksandrov, A. B. On embedding theorems for coinvariant subspaces of the shift operator. I. In *Complex analysis, operators, and related topics*, vol. 113 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 2000, pp. 45–64.

[15] Alexander, J. W., and Redheffer, R. M. The excess of sets of complex exponentials. *Duke Math. J. 34* (1967), 59–72.

[16] Allen, H. A., and Belna, C. L. Singular inner functions with derivative in B^p. *Michigan Math. J. 19* (1972), 185–188.

[17] Alpay, D. *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, vol. 6 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1998.

[18] Alpay, D., and Dym, H. Hilbert spaces of analytic functions, inverse scattering and operator models. I. *Integral Equations Operator Theory 7*, 5 (1984), 589–641.

[19] Alpay, D., and Dym, H. Hilbert spaces of analytic functions, inverse scattering and operator models. II. *Integral Equations Operator Theory 8*, 2 (1985), 145–180.

[20] Anderson, J. M., and Rovnyak, J. On generalized Schwarz-Pick estimates. *Mathematika 53*, 1 (2006), 161–168 (2007).

[21] Androulakis, G., Dodos, P., Sirotkin, G., and Troitsky, V. Classes of strictly singular operators and theirs products. Israel J. Math., to appear.

[22] Aronszajn, N., and Smith, K. T. Invariant subspaces of completely continuous operators. *Ann. of Math. (2) 60* (1954), 345–350.

[23] Avdonin, S. A., and Ivanov, S. A. *Families of exponentials.* Cambridge University Press, Cambridge, 1995. The method of moments in controllability problems for distributed parameter systems, Translated from the Russian and revised by the authors.

[24] Baksalary, J. K., and Baksalary, O. M. Nonsingularity of linear combinations of idempotent matrices. *Linear Algebra Appl. 388* (2004), 25–29.

[25] Ball, J. A. *Unitary perturbations of contractions.* PhD thesis, Univ. of Virginia, 1973.

[26] Ball, J. A. Models for noncontractions. *J. Math. Anal. Appl. 52*, 2 (1975), 235–254.

[27] Ball, J. A., and Cohen, N. de Branges-Rovnyak operator models and systems theory : a survey. In *Topics in matrix and operator theory (Rotterdam, 1989)*, vol. 50 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1991, pp. 93–136.

[28] Ball, J. A., and Kriete, III, T. L. Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators. *Integral Equations Operator Theory 10*, 1 (1987), 17–61.

[29] Baranov, A. D. Weighted Bernstein inequalities and embedding theorems for model subspaces. *Algebra i Analiz 15*, 5 (2003), 138–168.

[30] Baranov, A. D. Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings. *J. Funct. Anal. 223*, 1 (2005), 116–146.

[31] Baranov, A. D. Stability of bases and frames of reproducing kernels in model spaces. *Ann. Inst. Fourier (Grenoble) 55*, 7 (2005), 2399–2422.

[32] Baranov, A. D. Completeness and Riesz bases of reproducing kernels in model subspaces. *Int. Math. Res. Not.* (2006), Art. ID 81530, 34.

[33] Baranov, A. D., and Khavin, V. P. Admissible majorants for model subspaces and arguments of inner functions. *Funktsional. Anal. i Prilozhen. 40*, 4 (2006), 3–21, 111.

[34] Beauzamy, B. Un opérateur sans sous-espace invariant : simplification de l'exemple de P. Enflo. *Integral Equations Operator Theory 8*, 3 (1985), 314–384.

[35] Bernstein, A. R., and Robinson, A. Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos. *Pacific J. Math. 16* (1966), 421–431.

[36] Bessaga, C., and Pelczyński, A. On bases and unconditional convergence of series in Banach space. *Studia Math. 17* (1958), 151–164.

[37] Beurling, A. On two problems concerning linear transformations in Hilbert space. *Acta Math. 81* (1949), 239–255.

[38] Beurling, A., and Malliavin, P. On Fourier transforms of measures with compact support. *Acta Math. 107* (1962), 291–309.

[39] Beurling, A., and Malliavin, P. On the closure of characters and the zeros of entire functions. *Acta Math. 118* (1967), 79–93.

[40] Boas, Jr., R. P., and Schaeffer, A. C. Variational methods in entire functions. *Amer. J. Math. 79* (1957), 857–884.

[41] Bolotnikov, V., and Kheifets, A. A higher order analogue of the Carathéodory-Julia theorem. *J. Funct. Anal. 237*, 1 (2006), 350–371.

[42] Boricheva, I. Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2. *J. Funct. Anal. 161*, 2 (1999), 397–417.

[43] Buoni, J. J., Harte, R., and Wickstead, T. Upper and lower Fredholm spectra. I. *Proc. Amer. Math. Soc. 66*, 2 (1977), 309–314.

[44] Cargo, G. T. Angular and tangential limits of Blaschke products and their successive derivatives. *Canad. J. Math. 14* (1962), 334–348.

[45] Carleson, L. An interpolation problem for bounded analytic functions. *Amer. J. Math. 80* (1958), 921–930.

[46] Caughran, J. G., and Shields, A. L. Singular inner factors of analytic functions. *Michigan Math. J. 16* (1969), 409–410.

[47] Chan, K. C., and Seubert, S. M. Reducing subspaces of compressed analytic Toeplitz operators on the Hardy space. *Integral Equations Operator Theory 28*, 2 (1997), 147–157.

[48] Chyzhykov, I., Gundersen, G. G., and Heittokangas, J. Linear differential equations and logarithmic derivative estimates. *Proc. London Math. Soc. (3) 86*, 3 (2003), 735–754.

[49] Cima, J. A., and Matheson, A. L. On Carleson embeddings of star-invariant supspaces. *Quaest. Math. 26*, 3 (2003), 279–288.

[50] Cima, J. A., and Ross, W. T. *The backward shift on the Hardy space*, vol. 79 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[51] Clark, D. N. One dimensional perturbations of restricted shifts. *J. Analyse Math. 25* (1972), 169–191.

[52] Clark, D. N. Concrete model theory for a class of operators. *J. Functional Analysis 14* (1973), 269–280.

[53] Cohn, B. Carleson measures for functions orthogonal to invariant subspaces. *Pacific J. Math. 103*, 2 (1982), 347–364.

[54] Cohn, W. S. On the H^p classes of derivatives of functions orthogonal to invariant subspaces. *Michigan Math. J. 30*, 2 (1983), 221–229.

[55] Cohn, W. S. Carleson measures and operators on star-invariant subspaces. *J. Operator Theory 15*, 1 (1986), 181–202.

[56] Cohn, W. S. Radial limits and star invariant subspaces of bounded mean oscillation. *Amer. J. Math. 108*, 3 (1986), 719–749.

[57] Cohn, W. S. On fractional derivatives and star invariant subspaces. *Michigan Math. J. 34*, 3 (1987), 391–406.

[58] Cohn, W. S. A maximum principle for star invariant subspaces. *Houston J. Math. 14*, 1 (1988), 23–37.

[59] Crăsnaru, O. Hilbert spaces inside H^2_d. *An. Univ. Craiova Ser. Mat. Inform. 28* (2001), 127–133.

[60] Cullen, M. R. Derivatives of singular inner functions. *Michigan Math. J. 18* (1971), 283–287.

[61] de Branges, L. A proof of the Bieberbach conjecture. *Acta Math. 154*, 1-2 (1985), 137–152.

[62] de Branges, L., and Rovnyak, J. The existence of invariant subspaces. *Bull. Amer. Math. Soc. 70* (1964), 718–721.

[63] de Branges, L., and Rovnyak, J. Correction to "The existence of invariant subspaces". *Bull. Amer. Math. Soc. 71* (1965), 396.

[64] de Branges, L., and Rovnyak, J. Canonical models in quantum scattering theory. In *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965).* Wiley, New York, 1966, pp. 295–392.

[65] de Branges, L., and Rovnyak, J. *Square summable power series.* Holt, Rinehart and Winston, New York, 1966.

[66] de Leeuw, K., and Rudin, W. Extreme points and extremum problems in H_1. *Pacific J. Math. 8* (1958), 467–485.

[67] Douglas, R. G., Shapiro, H. S., and Shields, A. L. Cyclic vectors and invariant subspaces for the backward Shift Operator. *Ann. Inst, Fourier (Grenoble) 20* (1970), 37–76.

[68] Du, H., Deng, C., Mbekhta, M., and Müller, V. On spectral properties of linear combinations of idempotents. *Studia Math. 180*, 3 (2007), 211–217.

[69] Du, H., Yao, X., and Deng, C. Invertibility of linear combinations of two idempotents. *Proc. Amer. Math. Soc. 134*, 5 (2006), 1451–1457 (electronic).

[70] Duffin, R. J., and Eachus, J. J. Some notes on an expansion theorem of Paley and Wiener. *Bull. Amer. Math. Soc. 48* (1942), 850–855.

[71] Duren, P., Romberg, B., and Shields, A. Linear functionals on H^p spaces with $0 < p < 1$. *J. Reine Angew. Math. 238* (1969), 32–60.

[72] Duren, P. L. *Theory of H^p spaces.* Academic Press, 1970.

[73] Dyakonov, K., and Khavinson, D. Smooth functions in star-invariant subspaces. In *Recent advances in operator-related function theory*, vol. 393 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2006, pp. 59–66.

[74] Dyakonov, K. M. Interpolating functions of minimal norm, star-invariant subspaces and kernels of Toeplitz operators. *Proc. Amer. Math. Soc. 116*, 4 (1992), 1007–1013.

[75] Dyakonov, K. M. Division and multiplication by inner functions and embedding theorems for star-invariant subspaces. *Amer. J. Math. 115*, 4 (1993), 881–902.

[76] Dyakonov, K. M. Embedding theorems for star-invariant subspaces generated by smooth inner functions. *J. Funct. Anal. 157*, 2 (1998), 588–598.

[77] Dyakonov, K. M. Continuous and compact embeddings between star-invariant subspaces. In *Complex analysis, operators, and related topics*, vol. 113 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 2000, pp. 65–76.

[78] Dyakonov, K. M. Differentiation in star-invariant subspaces. I. Boundedness and compactness. *J. Funct. Anal. 192*, 2 (2002), 364–386.

[79] Dyakonov, K. M. Differentiation in star-invariant subspaces. II. Schatten class criteria. *J. Funct. Anal. 192*, 2 (2002), 387–409.

[80] Dyakonov, K. M. Zero sets and multiplier theorems for star-invariant subspaces. *J. Anal. Math. 86* (2002), 247–269.

[81] Dyakonov, K. M. Two theorems on star-invariant subspaces of BMOA. *Indiana Univ. Math. J. 56*, 2 (2007), 643–658.

[82] Dym, H. *J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, vol. 71 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.

[83] Elsner, J. Zulässige Abänderungen von Exponentialsystemen im $L^p(-A,A)$. *Math. Z. 120* (1971), 211–220.

[84] Enflo, P. On the invariant subspace problem in Banach spaces. In *Séminaire Maurey–Schwartz (1975–1976) Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. 14-15.* Centre Math., École Polytech., Palaiseau, 1976, p. 7.

[85] Enflo, P. On the invariant subspace problem for Banach spaces. *Acta Math. 158*, 3-4 (1987), 213–313.

[86] Fatou, P. Séries trigonométriques et séries de Taylor. *Acta Math. 30*, 1 (1906), 335–400.

[87] Fillmore, P. A. The shift operator. *Amer. Math. Monthly 81* (1974), 717–723.

[88] Foias, C. A remark on the universal model for contractions of G. C. Rota. *Com. Acad. R. P. Romine 13* (1963), 349–352.

[89] Fricain, E. Bases of reproducing kernels in model spaces. *J. Operator Theory 46*, 3, suppl. (2001), 517–543.

[90] Fricain, E. Complétude des noyaux reproduisants dans les espaces modèles. *Ann. Inst. Fourier (Grenoble) 52*, 2 (2002), 661–686.

[91] Frostman, O. Sur les produits de Blaschke. *Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 12*, 15 (1942), 169–182.

[92] Fujii, N., Nakamura, A., and Redheffer, R. M. On the excess of sets of complex exponentials. *Proc. Amer. Math. Soc. 127*, 6 (1999), 1815–1818.

[93] Garcia, S. Conjugation and Clark operators. *Contemporary Mathematics 393* (2006), 67–111.

[94] Garcia, S. R. Conjugation, the backward shift, and Toeplitz kernels. *J. Operator Theory 54*, 2 (2005), 239–250.

[95] Garcia, S. R., and Putinar, M. Complex symmetric operators and applications. *Trans. Amer. Math. Soc. 358*, 3 (2006), 1285–1315 (electronic).

[96] Garnett, J. B. *Bounded analytic functions*, first ed., vol. 236 of *Graduate Texts in Mathematics*. Springer, New York, 2007.

[97] Golovin, V. D. Biorthogonal expansions in linear combinations of exponential functions in L^2. *Zap. Meh.-Mat. Fak. Har'kov. Gos. Univ. i Har'kov. Mat. Obšč. (4) 30* (1964), 18–29.

[98] Gorin, E. A. Bernstein inequalities from the perspective of operator theory. *Vestnik Kharkov. Gos. Univ.*, 205 (1980), 77–105, 140.

[99] Gorkin, P., Lingenberg, H.-M., and Mortini, R. Homeomorphic disks in the spectrum of H^∞. *Indiana Univ. Math. J. 39*, 4 (1990), 961–983.

[100] Gotoh, Y. On integral means of the derivatives of Blaschke products. *Kodai Math. J. 30*, 1 (2007), 147–155.

[101] Gundersen, G. G. Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates. *J. London Math. Soc. (2) 37*, 1 (1988), 88–104.

[102] Gundersen, G. G., Steinbart, E. M., and Wang, S. The possible orders of solutions of linear differential equations with polynomial coefficients. *Trans. Amer. Math. Soc. 350*, 3 (1998), 1225–1247.

[103] Guyker, J. The de Branges-Rovnyak model with finite-dimensional coefficients. *Trans. Amer. Math. Soc. 347*, 4 (1995), 1383–1389.

[104] Halmos, P. R. Shifts on Hilbert spaces. *J. Reine Angew. Math. 208* (1961), 102–112.

[105] Harte, R. A spectral mapping theorem for holomorphic functions. *Math. Z. 154*, 1 (1977), 67–69.

[106] Hartmann, A., Sarason, D., and Seip, K. Surjective Toeplitz operators. *Acta Sci. Math. (Szeged) 70*, 3-4 (2004), 609–621.

[107] Havin, V., and Mashreghi, J. Admissible majorants for model subspaces of H^2. I. Slow winding of the generating inner function. *Canad. J. Math. 55*, 6 (2003), 1231–1263.

[108] Havin, V., and Mashreghi, J. Admissible majorants for model subspaces of H^2. II. Fast winding of the generating inner function. *Canad. J. Math. 55*, 6 (2003), 1264–1301.

[109] Havin, V., Mashregi, J., and Nazarov, F. L. The Beurling-Malliavin multiplier theorem : the seventh proof. *Algebra i Analiz 17*, 5 (2005), 3–68.

[110] Hedenmalm, H. Thin interpolating sequences and three algebras of bounded functions. *Proc. Amer. Math. Soc. 99*, 3 (1987), 489–495.

[111] Heittokangas, J. On complex differential equations in the unit disc. *Ann. Acad. Sci. Fenn. Math. Diss.*, 122 (2000), 54. Dissertation, University of Joensuu, Joensuu, 2000.

[112] Heittokangas, J. Growth estimates for logarithmic derivatives of Blaschke products and of functions in the Nevanlinna class. *Kodai Math. J. 30*, 2 (2007), 263–279.

[113] Helson, H. *Lectures on invariant subspaces.* Academic Press, New York, 1964.

[114] Helson, H., and Lowdenslager, D. Invariant subspaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960).* Jerusalem Academic Press, Jerusalem, 1961, pp. 251–262.

[115] Hruščëv, S. V. Perturbation theorems for bases consisting of exponentials and the Muckenhoupt condition. *Dokl. Akad. Nauk SSSR 247*, 1 (1979), 44–48.

[116] Hruščëv, S. V., Nikolskiĭ, N. K., and Pavlov, B. S. Unconditional bases of exponentials and of reproducing kernels. In *Complex analysis and spectral theory (Leningrad, 1979/1980)*, vol. 864 of *Lecture Notes in Math.* Springer, Berlin, 1981, pp. 214–335.

[117] Ingham, A. E. Some trigonometrical inequalities with applications to the theory of series. *Math. Z. 41*, 1 (1936), 367–379.

[118] James, R. C. A non-reflexive Banach space isometric with its second conjugate space. *Proc. Nat. Acad. Sci. U. S. A. 37* (1951), 174–177.

[119] Jury, M. T. Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators. *Proc. Amer. Math. Soc. 135*, 11 (2007), 3669–3675 (electronic).

[120] Kadeč, M. Ĭ. The exact value of the Paley-Wiener constant. *Dokl. Akad. Nauk SSSR 155* (1964), 1253–1254.

[121] Kato, T. *Perturbation theory for linear operators*, second ed. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[122] Koliha, J. J., and Rakočević, V. The nullity and rank of linear combinations of idempotent matrices. *Linear Algebra Appl. 418*, 1 (2006), 11–14.

[123] Koliha, J. J., and Rakočević, V. Stability theorems for linear combinations of idempotents. *Integral Equations Operator Theory 58*, 4 (2007), 597–601.

[124] Koliha, J. J., Rakočević, V., and Straškraba, I. The difference and sum of projectors. *Linear Algebra Appl. 388* (2004), 279–288.

[125] Kriete, T. L. Fourier transforms and chains of inner functions. *Duke Math. J. 40* (1973), 131–143.

[126] Kutbi, M. A. Integral means for the first derivative of Blaschke products. *Kodai Math. J. 24*, 1 (2001), 86–97.

[127] Kutbi, M. A. Integral means for the n'th derivative of Blaschke products. *Kodai Math. J. 25*, 3 (2002), 191–208.

[128] Lax, P. D. Translation invariant spaces. *Acta Math. 101* (1959), 163–178.

[129] Levin, B. J. On bases of exponential functions in L^2. *Zap. Har'kov. Gos. Univ. i Har'kov. Mat. Obšč. 27* (1961), 39–48.

[130] Levin, B. J. *Distribution of zeros of entire functions.* American Mathematical Society, Providence, R.I., 1964.

[131] Levin, B. Y. *Lectures on entire functions*, vol. 150 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko.

[132] Levin, M. B. An estimate of the derivative of a meromorphic function on the boundary of the domain. *Dokl. Akad. Nauk SSSR 216* (1974), 495–497.

[133] Levin, M. B. Estimation of the derivative of a meromorphic function on the boundary of the domain. *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, Vyp. 24 (1975), 68–85, ii.

[134] Levinson, N. On the closure of $\{e^{i\lambda_n x}\}$. *Duke Math. J. 2*, 3 (1936), 511–516.

[135] Levinson, N. *Gap and Density Theorems.* American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940.

[136] Linden, C. N. H^p-derivatives of Blaschke products. *Michigan Math. J. 23*, 1 (1976), 43–51.

[137] Lindenstrauss, J., and Tzafriri, L. *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[138] Livshic, M. S. On a class of linear operators on hilbert space. *Matem. Sbornik 19* (1946), 239–260. (Russian).

[139] Livshitz, M. S. On a certain class of linear operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S. 19(61)* (1946), 239–262.

[140] Lohman, R. H., and Casazza, P. G. A general construction of spaces of the type of R. C. James. *Canad. J. Math. 27*, 6 (1975), 1263–1270.

[141] Lomonosov, V. I. Invariant subspaces of the family of operators that commute with a completely continuous operator. *Funkcional. Anal. i Priložen. 7*, 3 (1973), 55–56.

[142] Lotto, B. A., and Sarason, D. Multiplicative structure of de Branges's spaces. *Rev. Mat. Iberoamericana 7*, 2 (1991), 183–220.

[143] Lotto, B. A., and Sarason, D. Multipliers of de Branges-Rovnyak spaces. *Indiana Univ. Math. J. 42*, 3 (1993), 907–920.

[144] Lotto, B. A., and Sarason, D. Multipliers of de Branges-Rovnyak spaces. II. In *Harmonic analysis and hypergroups (Delhi, 1995)*, Trends Math. Birkhäuser Boston, Boston, MA, 1998, pp. 51–58.

[145] Makarov, N., and Poltoratski, A. Meromorphic inner functions, Toeplitz kernels and the uncertainty principle. In *Perspectives in analysis*, vol. 27 of *Math. Phys. Stud.* Springer, Berlin, 2005, pp. 185–252.

[146] Makarov, N., and Vasyunin, V. A model for noncontractions and stability of the continuous spectrum. In *Complex analysis and spectral theory (Leningrad, 1979/1980)*, vol. 864 of *Lecture Notes in Math.* Springer, Berlin, 1981, pp. 365–412.

[147] Milman, V. D. Operators of class C_0 and C_0^*. *Teor. Funkciĭ Funkcional. Anal. i Priložen.*, 10 (1970), 15–26.

[148] Moeller, J. W. On the spectra of some translation invariant spaces. *J. Math. Anal. Appl. 4* (1962), 276–296.

[149] Müller, V. *Spectral theory of linear operators and spectral systems in Banach algebras*, vol. 139 of *Operator Theory : Advances and Applications.* Birkhäuser Verlag, Basel, 2003.

[150] Nazarov, F., and Volberg, A. The Bellman function, the two-weight Hilbert transform, and embeddings of the model spaces K_θ. *J. Anal. Math. 87* (2002), 385–414. Dedicated to the memory of Thomas H. Wolff.

[151] Nevanlinna, R. *Analytic functions.* Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162. Springer-Verlag, New York, 1970.

[152] Nikolski, N., and Vasyunin, V. Elements of spectral theory in terms of the free function model. I. Basic constructions. In *Holomorphic spaces (Berkeley, CA, 1995)*, vol. 33 of *Math. Sci. Res. Inst. Publ.* Cambridge Univ. Press, Cambridge, 1998, pp. 211–302.

[153] Nikolski, N. K. *Treatise on the shift operator*, vol. 273 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986. Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[154] Nikolski, N. K. *Operators, functions, and systems : an easy reading. Vol. 1*, vol. 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.

[155] Nikolski, N. K. *Operators, functions, and systems : an easy reading. Vol. 2*, vol. 93 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Model operators and systems, Translated from the French by Andreas Hartmann and revised by the author.

[156] Nikolski, N. K., and Vasyunin, V. I. Notes on two function models. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, vol. 21 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 1986, pp. 113–141.

[157] Nikolski, N. K., and Vasyunin, V. I. A unified approach to function models, and the transcription problem. In *The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988)*, vol. 41 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1989, pp. 405–434.

[158] Nikolskiĭ, N. K. Invariant subspaces in operator theory and function theory. In *Mathematical analysis, Vol. 12 (Russian)*. Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1974, pp. 199–412, 468. (loose errata).

[159] Nikolskiĭ, N. K. Bases of exponentials and values of reproducing kernels. *Dokl. Akad. Nauk SSSR 252*, 6 (1980), 1316–1320.

[160] Nikolskiĭ, N. K., and Khrushchëv, S. V. A functional model and some problems of the spectral theory of functions. *Trudy Mat. Inst. Steklov. 176* (1987), 97–210, 327. Translated in Proc. Steklov Inst. Math. **1988**, no. 3, 101–214, Mathematical physics and complex analysis (Russian).

[161] Nikolskiĭ, N. K., and Pavlov, B. S. Bases of eigenvectors of completely nonunitary contractions, and the characteristic function. *Izv. Akad. Nauk SSSR Ser. Mat. 34* (1970), 90–133.

[162] Nikolskiĭ, N. K., and Volberg, A. L. Tangential and approximate free interpolation. In *Analysis and partial differential equations*, vol. 122 of *Lecture Notes in Pure and Appl. Math.* Dekker, New York, 1990, pp. 277–299.

[163] Ortega-Cerdà, J., and Seip, K. Fourier frames. *Ann. of Math. (2) 155*, 3 (2002), 789–806.

[164] Paley, R. E. A. C., and Wiener, N. *Fourier transforms in the complex domain*, vol. 19 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1987. Reprint of the 1934 original.

[165] Pavlov, B. S. The basis property of a system of exponentials and the condition of Muckenhoupt. *Dokl. Akad. Nauk SSSR 247*, 1 (1979), 37–40.

[166] Peterson, D. R. The excess of sets of complex exponentials. *Proc. Amer. Math. Soc. 44* (1974), 321–325.

[167] Petrosyan, A. Some extremal problems for analytic functions. *Complex Variables Theory Appl. 39*, 2 (1999), 137–159.

[168] Plichko, A. Superstrictly singular and superstrictly cosingular operators. In *Functional analysis and its applications*, vol. 197 of *North-Holland Math. Stud.* Elsevier, Amsterdam, 2004, pp. 239–255.

[169] Plichko, A. N., and Maslyuchenko, V. K. Quasireflexive locally convex spaces without Banach subspaces. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, 44 (1985), 78–84.

[170] Poltoratskiĭ, A. G. Boundary behavior of pseudocontinuable functions. *Algebra i Analiz 5*, 2 (1993), 189–210.

[171] Popov, A. Schreier singular operators. Houston J. Math., to appear.

[172] Potapov, V. P. The multiplicative structure of J-contractive matrix functions. *Amer. Math. Soc. Transl. (2) 15* (1960), 131–243.

[173] Protas, D. Blaschke products with derivative in H^p and B^p. *Michigan Math. J. 20* (1973), 393–396.

[174] Protas, D. Mean growth of the derivative of a Blaschke product. *Kodai Math. J. 27*, 3 (2004), 354–359.

[175] Pták, V., and Vrbová, P. An abstract analogon of the de Branges-Rovnyak functional model. *Integral Equations Operator Theory 16*, 4 (1993), 565–599.

[176] Radjavi, H., and Rosenthal, P. *Invariant subspaces*, second ed. Dover Publications Inc., Mineola, NY, 2003.

[177] Rahman, Q. I., and Schmeisser, G. L^p inequalities for entire functions of exponential type. *Trans. Amer. Math. Soc. 320*, 1 (1990), 91–103.

[178] Rahman, Q. I., and Tariq, Q. M. On Bernstein's inequality for entire functions of exponential type. *Comput. Methods Funct. Theory 7*, 1 (2007), 167–184.

[179] Read, C. J. A solution to the invariant subspace problem. *Bull. London Math. Soc. 16*, 4 (1984), 337–401.

[180] Read, C. J. A solution to the invariant subspace problem on the space l_1. *Bull. London Math. Soc. 17*, 4 (1985), 305–317.

[181] Read, C. J. A short proof concerning the invariant subspace problem. *J. London Math. Soc. (2) 34*, 2 (1986), 335–348.

[182] Read, C. J. Strictly singular operators and the invariant subspace problem. *Studia Math. 132*, 3 (1999), 203–226.

[183] Redheffer, R. M. Elementary remarks on completeness. *Duke Math. J. 35* (1968), 103–116.

[184] Redheffer, R. M. Completeness of sets of complex exponentials. *Advances in Math. 24*, 1 (1977), 1–62.

[185] Redheffer, R. M., and Young, R. M. Completeness and basis properties of complex exponentials. *Trans. Amer. Math. Soc. 277*, 1 (1983), 93–111.

[186] Rosenblum, M., and Rovnyak, J. *Hardy classes and operator theory.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1985. Oxford Science Publications.

[187] Rota, G. C. Note on the invariant subspaces of linear operators. *Rend. Circ. Mat. Palermo (2) 8* (1959), 182–184.

[188] Rota, G.-C. On models for linear operators. *Comm. Pure Appl. Math. 13* (1960), 469–472.

[189] Rovnyak, J. Ideals of square summable power series. *Proc. Amer. Math. Soc. 13* (1962), 360–365.

[190] Rovnyak, J. *Some Hilbert Spaces of Analytic Functions.* PhD thesis, Yale University, 1963.

[191] Rudin, W. *Real and complex analysis*, third ed. McGraw-Hill Book Co., New York, 1987.

[192] Rudin, W. *Functional analysis*, second ed. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, 1991.

[193] Sadovskiĭ, B. N. Limit-compact and condensing operators. *Uspehi Mat. Nauk 27*, 1(163) (1972), 81–146.

[194] Sand, M. Operator ranges and non-cyclic vectors for the backward shift. *Integral Equations Operator Theory 22*, 2 (1995), 212–231.

[195] Sarason, D. A remark on the Volterra operator. *J. Math. Anal. Appl. 12* (1965), 244–246.

[196] Sarason, D. Generalized interpolation in H^∞. *Trans. Amer. Math. Soc. 127* (1967), 179–203.

[197] Sarason, D. Doubly shift-invariant spaces in H^2. *J. Operator Theory 16*, 1 (1986), 75–97.

[198] Sarason, D. Shift-invariant spaces from the Brangesian point of view. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, vol. 21 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 1986, pp. 153–166.

[199] Sarason, D. Angular derivatives via Hilbert space. *Complex Variables Theory Appl. 10*, 1 (1988), 1–10.

[200] Sarason, D. Nearly invariant subspaces of the backward shift. In *Contributions to operator theory and its applications (Mesa, AZ, 1987)*, vol. 35 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1988, pp. 481–493.

[201] Sarason, D. Exposed points in H^1. I. In *The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988)*, vol. 41 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1989, pp. 485–496.

[202] Sarason, D. Exposed points in H^1. II. In *Topics in operator theory : Ernst D. Hellinger memorial volume*, vol. 48 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1990, pp. 333–347.

[203] Sarason, D. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

[204] Sari, B., Schlumprecht, T., Tomczak-Jaegermann, N., and Troitsky, V. G. On norm closed ideals in $L(l_p, l_q)$. *Studia Math. 179*, 3 (2007), 239–262.

[205] Schreiber, M. Unitary dilations of operators. *Duke Math. J. 23* (1956), 579–594.

[206] Sedleckiĭ, A. M. The stability of the completeness and of the minimality in L^2 of a system of exponential functions. *Mat. Zametki 15* (1974), 213–219.

[207] Sedleckiĭ, A. M. Complete and minimal systems of exponentials in the spaces $L^p(-\pi, \pi)$, $1 \leq p \leq 2$. *Trudy Moskov. Orden. Lenin. Ènerget. Inst.*, Vyp. 290 (1976), 63–69.

[208] Sedleckiĭ, A. M. Excesses of systems of exponential functions. *Izv. Akad. Nauk SSSR Ser. Mat. 44*, 1 (1980), 203–218, 240.

[209] Shapiro, H. S. Reproducing kernels and Beurling's theorem. *Trans. Amer. Math. Soc. 110* (1964), 448–458.

[210] Shapiro, H. S., and Shields, A. L. On some interpolation problems for analytic functions. *Amer. J. Math. 83* (1961), 513–532.

[211] Shapiro, H. S., and Shields, A. L. Interpolation in Hilbert spaces of analytic functions. *Studia Math. (Ser. specjalna) Zeszyt 1* (1963), 109–110.

[212] Shapiro, J. E. Relative angular derivatives. *J. Operator Theory 46*, 2 (2001), 265–280.

[213] Shapiro, J. E. More relative angular derivatives. *J. Operator Theory 49*, 1 (2003), 85–97.

[214] Singer, I. *Bases in Banach spaces. I.* Springer-Verlag, New York, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154.

[215] Singer, I. *Bases in Banach spaces. II.* Editura Academiei Republicii Socialiste România, Bucharest, 1981.

[216] Štraus, A. V. Characteristic functions of linear operators. *Izv. Akad. Nauk SSSR Ser. Mat. 24* (1960), 43–74.

[217] SUÁREZ, F. D. Multipliers of de Branges-Rovnyak spaces in H^2. *Rev. Mat. Iberoamericana 11*, 2 (1995), 375–415.

[218] SUNDBERG, C., AND WOLFF, T. H. Interpolating sequences for QA_B. *Trans. Amer. Math. Soc. 276*, 2 (1983), 551–581.

[219] SZ.-NAGY, B. Sur les contractions de l'espace de hilbert. *Acta Sci. Math. (Szeged) 15* (1953), 87–92.

[220] SZ.-NAGY, B., AND FOIAS, C. *Analyse harmonique des opérateurs de l'espace de Hilbert.* Masson et Cie, Paris, 1967.

[221] TOTIK, V. Derivatives of entire functions of higher order. *J. Approx. Theory 64*, 2 (1991), 209–213.

[222] TREIL, S. Extreme points of the unit ball of the operator Hardy space $H^\infty(E \to E)$. *Zapiski Nauchn. Semin. LOMI 149* (1986), 160–164. (Russian) English transl : J. Soviet Math. 42 (1988), no. 2, 1653-1656.

[223] TREIL, S. R. Angles between co-invariant subspaces, and the operator corona problem. The Szőkefalvi-Nagy problem. *Dokl. Akad. Nauk SSSR 302*, 5 (1988), 1063–1068.

[224] TREIL, S. R. Geometric methods in spectral theory of vector-valued functions : some recent results. In *Toeplitz operators and spectral function theory*, vol. 42 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 1989, pp. 209–280.

[225] TREIL, S. R. Hankel operators, embedding theorems and bases of co-invariant subspaces of the multiple shift operator. *Algebra i Analiz 1*, 6 (1989), 200–234.

[226] VASYUNIN, V. I. The number of Carleson series. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 65* (1976), 178–182, 208. Investigations on linear operators and the theory of functions, VII.

[227] VASYUNIN, V. I. The construction of the B. Szőkefalvi-Nagy and C. Foiaş functional model. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 73* (1977), 16–23, 229 (1978). Investigations on linear operators and the theory of functions, VIII.

[228] VOLBERG, A. L. Thin and thick families of rational fractions. In *Complex analysis and spectral theory (Leningrad, 1979/1980)*, vol. 864 of *Lecture Notes in Math.* Springer, Berlin, 1981, pp. 440–480.

[229] VOLBERG, A. L. Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason. *J. Operator Theory 7*, 2 (1982), 209–218.

[230] VOLBERG, A. L., AND TREIL, S. R. Embedding theorems for invariant subspaces of the inverse shift operator. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149*, Issled. Linein. Teor. Funktsii. XV (1986), 38–51, 186–187.

[231] von Neumann, J. Allgemeine eigenwerttheorie hermitescher funktionaloperatoren. *Math. Ann. 102* (1929), 49–131.

[232] Wold, H. *A study in the analysis of stationary time series*. Almqvist and Wiksell, Stockholm, 1954. 2d ed, With an appendix by Peter Whittle.

[233] Young, R. M. A perturbation theorem for complete sets of complex exponentials. *Proc. Amer. Math. Soc. 55*, 2 (1976), 318–320.

[234] Young, R. M. *An introduction to nonharmonic Fourier series*, vol. 93 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.

[235] Young, R. M. On complete biorthogonal systems. *Proc. Amer. Math. Soc. 83*, 3 (1981), 537–540.

Chapitre 6

Annexes

6.1 Annexes sur le chapitre “Géométrie des espaces de de Branges-Rovnyak”

6.1.1 Référence [T1]

Auteurs
I. Chalendar, E. Fricain et D. Timotin

Titre
Functional models and asymptotically orthonormal sequences.

Paru dans
Ann. Inst. Fourier (Grenoble), 53 (2003), no. 5, 1527–1549.

Functional models and asymptotically orthonormal sequences

Isabelle Chalendar[*], Emmanuel Fricain[†] and Dan Timotin[‡]

1 Introduction

A canonical orthonormal basis in the Hilbert space $L^2(-\pi, \pi)$ is formed by the exponentials $\exp int$, $n \in \mathbb{Z}$. Starting with the works of Paley-Wiener ([12]) and Levinson ([8]), a whole direction of research has investigated other families of exponentials, looking for properties as completeness, minimality, or being an unconditional basis. In this context, functional models have been used in [7], together with some other tools from operator theory on a Hilbert space. The model spaces are subspaces of the Hardy space H^2, invariant under the adjoints of multiplications; their theory is connected to dilation theory for contractions on Hilbert spaces (see [14, 9]). The approach has been proved fruitful; it has allowed the recapture of all the classical results and has lead to many generalizations.

We are interested in investigating, along the line of research from [7], the case when the basis is asymptotically close to an orthogonal one (see definition below). This is a particular case of unconditional basis, where more rigidity is required, but the conclusions obtained are usually more precise. A basic result appears in Volberg's paper [15], where it is shown that the usual Carleson condition for an interpolation set can be adapted to obtain a characterization of asymptotically orthonormal sequences of reproducing kernels;

[*]Institut Girard Desargues, UFR de Mathématiques, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cedex, France. E-mail: chalenda@igd.univ-lyon1.fr.

[†]Institut Girard Desargues, UFR de Mathématiques, Université Claude Bernard Lyon 1, 69622 Villeurbanne Cedex, France. E-mail: fricain@igd.univ-lyon1.fr.

[‡]Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 70700, Romania. E-mail: dtimotin@imar.ro.

further developments can be found in [3]. We intend to provide a comprehensive treatment of this subject, emphasizing the parallel with unconditional bases.

The plan of the paper is the following. The next two sections contain preliminary material. The case of reproducing kernels for the whole Hardy space is treated in Section 4; we give an equivalent form of Volberg's condition and prove some related results. Section 5 investigates the relevance of Volberg's condition for model spaces; the main results are Theorem 5.2 and Corollary 5.6, which allow the characterization of asymptotically orthonormal sequences of reproducing kernels. Perturbation results are obtained in Section 6. In the last two sections we discuss the important case of exponentials, as well as some other examples.

2 Preliminaries

For most of the definitions and facts below, one can use [9] as a main reference.

Let $\mathcal{H}$ be a complex Hilbert space. A sequence $(x_n)_{n\geq 1} \subset \mathcal{H}$ is called:

- *complete* if $\operatorname{Span}\{x_n : n \geq 1\} = \mathcal{H}$;
- *minimal* if for all $n \geq 1$, $x_n \notin \operatorname{Span}\{x_m : m \neq n\}$;
- *Riesz* if there are positive constants c, C verifying, for all finite complex sequences $(a_n)_{n\geq 1}$,

$$c \sum_{n\geq 1} |a_n|^2 \leq \Big\| \sum_{n\geq 1} a_n x_n \Big\|^2 \leq C \sum_{n\geq 1} |a_n|^2. \tag{2.1}$$

A Riesz sequence is minimal, but the converse is in general not true.

The *Gram matrix* of the sequence $(x_n)_{n\geq 1}$ is $\Gamma = (\langle x_n, x_m \rangle)_{n,m\geq 1}$. Riesz sequences are characterized by the fact that Γ defines an invertible operator on ℓ^2.

The basic Hilbert space in which our objects live is the Hardy space H^2 of the open unit disc $\mathbb{D}$; this is the Hilbert space of analytic functions $f(z) = \sum_{n\geq 0} a_n z^n$ defined in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, such that $\sum_{n\geq 0} |a_n|^2 < \infty$. Alternately, it can be identified with a closed subspace of the Lebesgue space $L^2(\mathbb{T})$ on the unit circle, by associating to each analytic function its radial limit. The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞. Any $\phi \in H^\infty$ acts as a multiplication operator on H^2, that we will denote by T_ϕ.

Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is $k_\lambda(z) = \frac{1}{1-\bar\lambda z}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$. If $\phi \in H^\infty$, then k_λ is an eigenvector for T_ϕ^*, and $T_\phi^* k_\lambda = \overline{\phi(\lambda)} k_\lambda$. By normalizing k_λ we obtain $h_\lambda = \frac{k_\lambda}{\|k_\lambda\|} = \sqrt{1-|\lambda|^2} k_\lambda$.

Suppose now Θ is an inner function. We define the corresponding *model space* by the formula $K_\Theta = H^2 \ominus \Theta H^2$; the orthogonal projection onto K_Θ is denoted by P_Θ. In K_Θ the reproducing kernel for a point $\lambda \in \mathbb{D}$ is the function

$$k_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar\lambda z} \tag{2.2}$$

and the normalized reproducing kernel

$$h_\lambda^\Theta(z) = \sqrt{\frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2}} k_\lambda^\Theta(z). \tag{2.3}$$

Note that, according to (2.2), we have the orthogonal decomposition

$$k_\lambda = k_\lambda^\Theta + \Theta\overline{\Theta(\lambda)} k_\lambda. \tag{2.4}$$

Suppose $\Lambda = (\lambda_n)_{n\geq 1}$ is a *Blaschke sequence* of distinct points in $\mathbb{D}$ (which means that $\sum_{n\geq 1} 1 - |\lambda_n| < \infty$). As usual, we denote by $B = B_\Lambda = \prod_{n\geq 1} b_{\lambda_n}$ the associated Blaschke product, and $B_n = B/b_{\lambda_n}$; here $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1-\overline{\lambda_n} z}$. As B is an inner function, we may consider the model space K_B; it is well known that $(h_{\lambda_n})_{n\geq 1}$ is a complete minimal system in K_B. It is a Riesz basis if and only if it satisfies the Carleson condition

$$\delta(\Lambda) = \inf_{n\geq 1} |B_n(\lambda_n)| > 0;$$

we will write in this case $\Lambda \in (C)$ and say that Λ is *a Carleson sequence*. Also, the sequence Λ is called *separated* if $\inf_{n\neq m} |b_{\lambda_n}(\lambda_m)| > 0$.

In connection with Blaschke products, we will have the opportunity to use the following two formulas. If $\lambda, \mu \in \mathbb{D}$, then

$$\left|\frac{\lambda - \mu}{1 - \bar\lambda\mu}\right|^2 = 1 - \frac{(1-|\lambda|^2)(1-|\mu|^2)}{|1-\bar\lambda\mu|^2}; \tag{2.5}$$

the denominator in the right hand side is given by

$$|1-\bar\lambda\mu|^2 = (1-|\lambda\mu|)^2 + 4|\lambda\mu|\sin^2\frac{\theta}{2}, \tag{2.6}$$

where $\theta \in (-\pi, \pi]$ is the argument of $\bar\lambda\mu$.

The following two Lemmas are proved in [7], II.

Lemma 2.1. *If* $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ *is a minimal but not complete sequence in* K_Θ*, then, for all* $\mu \in \mathbb{D}\setminus\Lambda$, $\{h^{\Theta}_{\mu}, h^{\Theta}_{\lambda_n} : n \geq 1\}$ *is still a minimal sequence.*

Lemma 2.2. *If* Θ_1, Θ_2 *are two inner functions, then* $\operatorname{dist}(\bar{\Theta}_1\Theta_2, H^\infty) = \|P_{\Theta_1}T_{\Theta_2}|K_{\Theta_1}\|$*, and this quantity is strictly smaller than 1 if and only if* $P_{\Theta_2}|K_{\Theta_1}$ *is an isomorphism onto its image.*

We end the preliminaries with a lemma pertaining to Riesz sequences of normalized reproducing kernels.

Lemma 2.3. *Let* $(\lambda_n)_{n\geq 1}$ *be a Blaschke sequence of distinct points in* $\mathbb{D}$*,* B *the corresponding Blaschke product, and* Θ *an inner function. Suppose that* $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ *is a Riesz sequence in* K_Θ*, and denote by* c, C *the corresponding constants appearing in (2.1). Then*

$$\|P_B T_\Theta | K_B\| \leq \sqrt{\frac{C}{c}} \sup_{n\geq 1} |\Theta(\lambda_n)|$$

Proof. The subspace K_B is spanned by the eigenvectors h_{λ_n}, $n \geq 1$ of T^*_Θ, and $T^*_\Theta h_{\lambda_n} = \overline{\Theta(\lambda_n)} h_{\lambda_n}$. Take a sum (with a finite number of nonzero terms) $\sum_{n\geq 1} a_n h_{\lambda_n}$; we have

$$\Big\| \sum_{n\geq 1} a_n h_{\lambda_n} \Big\|^2 \geq c \sum_{n\geq 1} |a_n|^2$$

and

$$\begin{aligned} \Big\| T^*_\Theta \Big(\sum_{n\geq 1} a_n h_{\lambda_n} \Big) \Big\|^2 &= \Big\| \sum_{n\geq 1} \overline{\Theta(\lambda_n)} a_n h_{\lambda_n} \Big\|^2 \leq C \sum_{n\geq 1} |\Theta(\lambda_n)|^2 |a_n|^2 \\ &\leq C (\sup_{n\geq 1} |\Theta(\lambda_n)|)^2 \sum_{n\geq 1} |a_n|^2, \end{aligned}$$

whence

$$\Big\| T^*_\Theta \Big(\sum_{n\geq 1} a_n h_{\lambda_n} \Big) \Big\|^2 \leq \frac{(\sup_{n\geq 1} |\Theta(\lambda_n)|)^2 C}{c} \Big\| \sum_{n\geq 1} a_n h_{\lambda_n} \Big\|^2.$$

Since $(P_B T_\Theta | K_B)^* = T^*_\Theta | K_B$, the lemma is proved. □

3 Asymptotically orthonormal bases

We will say that $(x_n)_{n\geq 1}$ is an *asymptotically orthonormal sequence* in $\mathcal{H}$ (abbreviated AOS) if there exists $N_0 \in \mathbb{N}$, such that for all $N \geq N_0$, there are constants $c_N, C_N > 0$ verifying

$$c_N \sum_{n\geq N} |a_n|^2 \leq \Big\| \sum_{n\geq N} a_n x_n \Big\|^2 \leq C_N \sum_{n\geq N} |a_n|^2, \tag{3.1}$$

and $\lim_{N\to\infty} c_N = 1 = \lim_{N\to\infty} C_N$.

If $N_0 = 1$, then one says that $(x_n)_{n\geq 1}$ is an *asymptotically orthonormal basic sequence* (abbreviated AOB). Obviously this is equivalent to $(x_n)_{n\geq 1}$ being an AOS as well as a Riesz sequence.

The following simple lemma is a basic tool.

Lemma 3.1. *If $(x_n)_{n\geq 1} \subset \mathcal{H}$, then $(x_n)_{n\geq 1}$ is an AOB if and only if it is minimal and an AOS.*

Proof. If $(x_n)_{n\geq 1}$ is an AOB, then it is a Riesz sequence, and therefore minimal. Conversely, if $(x_n)_{n\geq 1}$ is an AOS, then $(x_n)_{n\geq N_0}$ is a Riesz sequence for some N_0. Now minimality ensures that we can add the first finite number of vectors and still preserve this property. □

As in the case of Riesz sequences, several equivalent characterizations are available for AOB's, as shown in the next proposition ([3], Section 3).

Proposition 3.2. *Let $(x_n)_{n\geq 1}$ be a sequence in $\mathcal{H}$. The following are equivalent:*

(i) *$(x_n)_{n\geq 1}$ is an AOB;*

(ii) *there exist a separable Hilbert space $\mathcal{K}$, an orthonormal basis $(e_n)_{n\geq 1} \subset \mathcal{K}$ and $U, K : \mathcal{K} \to \mathcal{H}$, U unitary, K compact, $U + K$ left invertible, such that $(U + K)(e_n) = x_n$;*

(iii) *the Gram matrix Γ associated to $(x_n)_{n\geq 1}$ defines a bounded invertible operator of the form $I + K$, with K compact.*

One can obtain complete AOB's by slightly perturbing orthonormal bases; this fact is made precise in the following lemma.

Lemma 3.3. *Let $\mathcal{H}$ be a Hilbert space, $(x_n)_{n\geq 1}$ an orthonormal basis in $\mathcal{H}$, and $(x'_n)_{n\geq 1}$ a sequence in $\mathcal{H}$, such that $\sum_{n\geq 1} \|x_n - x'_n\|^2 < 1$. Then $(x'_n)_{n\geq 1}$ is a complete AOB in $\mathcal{H}$.*

Proof. Consider the operator $\Phi : \mathcal{H} \to \mathcal{H}$, defined by $\Phi(x_n) = x'_n$. The condition in the statement implies that $I_{\mathcal{H}} - \Phi$ is Hilbert-Schmidt, of norm strictly smaller than 1. Thus Φ is of the form unitary plus compact and invertible; from Proposition 3.2 it follows that $(x'_n)_{n\geq 1}$ is an AOB. On the other hand, since Φ is invertible, $(x_n)_{n\geq 1}$ complete implies $(x'_n)_{n\geq 1}$ complete. □

We should note the following way to obtain AOB's.

Lemma 3.4. *Let $(x_n)_{n\geq 1}$ be a normalized sequence in $\mathcal{H}$, tending weakly to 0. There exists a subsequence $(x_{k_n})_{n\geq 1}$ which is an AOB.*

Proof. Choose recursively the sequence (x_{k_n}) by requiring that $\sum_{m<n} |\langle x_{k_n}, x_{k_m}\rangle|^2 \leq \frac{1}{2^{n+2}}$. Then $\sum_{m\neq n} |\langle x_{k_n}, x_{k_m}\rangle|^2 \leq 1/2$, whence, if Γ' is the Gram matrix associated to (x_{k_n}), then $\Gamma' - I$ has Hilbert-Schmidt norm smaller than $1/2$. Applying Proposition 3.2 to Γ' implies that $(x_{k_n})_{n\geq 1}$ is an AOB. $\square$

In particular, a Riesz sequence tends weakly to 0, and thus it contains AOB's as subsequences.

4 Reproducing kernels and AOB's

Suppose $\Lambda = (\lambda_n)_{n\geq 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$. Since the reproducing kernels $(k_{\lambda_n})_{n\geq 1}$ are complete and minimal in K_B, if $(h_{\lambda_n})_{n\geq 1}$ is an AOS, then it is also a complete AOB in K_B. Such sequences are characterized by the following theorem of Volberg ([15]).

Theorem A. *The sequence $(h_{\lambda_n})_{n\geq 1}$ is a complete AOB in K_B if and only if*

$$\lim_{n\to\infty} |B_n(\lambda_n)| = 1. \tag{4.1}$$

Blaschke sequences that satisfy (4.1) have already appeared in literature (see, for instance, [5, 4, 11, 13]). In particular, it follows from results in [13] that, among Carleson sequences, they are characterized by the possibility of free interpolation with functions in $H^\infty \cap VMO$. We will adopt the terminology in [13] and call a sequence $(\lambda_n)_{n\geq 1}$ that satisfies (4.1) a *thin interpolating sequence* (or just *thin sequence*); we will write $(\lambda_n) \in (\vartheta)$. Thus thin interpolating sequences correspond to AOS of normalized reproducing kernels.

A different characterization can be stated by using the Gram matrix.

Proposition 4.1. *If $\Lambda = (\lambda_n)_{n\geq 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$, then the following are equivalent:*

(i) *$(h_{\lambda_n})_{n\geq 1}$ is a complete AOB in K_B;*

(ii) $(\Gamma - I)e_n \to 0$.

Proof. (i)⇒(ii). By Proposition 3.2, (iii), it follows that $\Gamma = I + K$ with K compact; since $Ke_n \to 0$, $(\Gamma - I)e_n \to 0$.

(ii)⇒(i). By hypothesis $(\Gamma - I)e_n \to 0$. But

$$\|(\Gamma - I)e_n\|^2 = \sum_{p\neq n, p\geq 1} |\Gamma_{n,p}|^2 = \sum_{p\neq n, p\geq 1} \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n\lambda_p|^2} = \sum_{p\neq n, p\geq 1} (1-|b_{\lambda_p}(\lambda_n)|^2) \quad (4.2)$$

(we have used (2.5)).

In particular, there is some N such that for $n \geq N$ we have $\|(\Gamma - I)e_n\|^2 < 1/2$, and therefore $|b_{\lambda_p}(\lambda_n)|^2 > 1/2$ if p or n are larger than N. Since the points λ_n are distinct, the whole sequence Λ is separated, and there exists $\varepsilon > 0$, such that $|b_{\lambda_p}(\lambda_n)| \geq \varepsilon$ for all $n \neq p$. Therefore $1 - |b_{\lambda_p}(\lambda_n)|^2 \geq -c \log |b_{\lambda_p}(\lambda_n)| \geq 0$ for some $c > 0$. It follows that

$$\|(\Gamma - I)e_n\|^2 \geq -c \log |B_n(\lambda_n)|$$

whence $|B_n(\lambda_n)| \to 1$; by Theorem A it follows that $(h_{\lambda_n})_{n\geq 1}$ is a complete AOB in K_B. □

It is well known that Carleson sequences $\Lambda = (\lambda_n)_{n\geq 1}$ can also be characterized by the fact that they are separated and the measure $\sum (1 - |\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure. A similar characterization can be obtained for thin interpolating sequences, as suggested by Lemma 7.1 in [13]. We need some notations: for any $z \in \mathbb{D}$, I_z will be the interval $I_z \subset \mathbb{T}$ of center $\frac{z}{|z|}$ and length $1 - |z|$. For $I \subset \mathbb{T}$, $S_I = \{z \in \mathbb{D} : z/|z| \in I \text{ and } |z| \geq 1 - |I|\}$; while, for $C > 0$, CI is the interval with the same center and length $C|I|$.

Proposition 4.2. *Suppose $\Lambda = (\lambda_n)_{n\geq 1}$ is a Blaschke sequence. The following are equivalent:*

(i) Λ is a thin interpolating sequence;

(ii) for any $A \geq 1$,

$$\lim_{n\to\infty} \frac{1}{|I_{\lambda_n}|} \sum_{\substack{p\neq n \\ \lambda_p \in S_{AI_{\lambda_n}}}} (1 - |\lambda_p|) = 0. \quad (4.3)$$

Proof. (i)⇒(ii). Fix $A \geq 1$; from (2.6) it follows easily that there is some constant $a > 0$ such that, if $z \in S_{AI_{\lambda_n}}$, then $|1 - \bar{\lambda}_n z|^2 \leq a(1 - |\lambda_n|)^2$. Therefore, if $\lambda_p \in S_{AI_{\lambda_n}}$, then $\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n\lambda_p|^2} \geq \frac{1-|\lambda_p|}{a(1-|\lambda_n|)}$. Consequently, (4.3) follows from Proposition 4.1, (ii).

(ii)$\Rightarrow$(i). We show first that $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure. From (4.3) it follows that we may suppose (by deleting a finite number of terms, if necessary) that for all λ_n we have

$$\sum_{\substack{p\neq n\\ \lambda_p\in S_{5I_{\lambda_n}}}}(1-|\lambda_p|)\leq |I_{\lambda_n}|.$$

and therefore

$$\sum_{\lambda_p\in S_{5I_{\lambda_n}}}(1-|\lambda_p|)\leq 2|I_{\lambda_n}|. \tag{4.4}$$

Fix the interval $I\subset\mathbb{T}$, and define $\sigma=\{n\in\mathbb{N}:\lambda_n\in S_I\}$. If $n\in\sigma$, then $I_{\lambda_n}\subset 2I$. By Vitali's covering lemma (see for instance [6], V.17), there is $\sigma'\subset\sigma$, such that the intervals I_{λ_n} are disjoint for $n\in\sigma'$, while $\bigcup_{n\in\sigma}I_{\lambda_n}\subset\bigcup_{n\in\sigma'}5I_{\lambda_n}$; the last inclusion implies that $\{\lambda_n\in S_I\}\subset\bigcup_{n\in\sigma'}S_{5I_{\lambda_n}}$. Then, using (4.4), it follows that

$$\sum_{\lambda_n\in S_I}(1-|\lambda_n|)\leq\sum_{n\in\sigma'}\sum_{\lambda_p\in S_{5I_{\lambda_n}}}(1-|\lambda_p|)\leq 2\sum_{n\in\sigma'}|I_{\lambda_n}|\leq 4m(I);$$

thus $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is indeed a Carleson measure.

Fix $0<\varepsilon<1$, $A\geq 1$, and choose $n\in\mathbb{N}$, such that

$$\sum_{\substack{p\neq n\\ \lambda_p\in S_{AI_{\lambda_n}}}}(1-|\lambda_p|)\leq\varepsilon(1-|\lambda_n|) \tag{4.5}$$

We write

$$\sum_{p\neq n}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}=\sum_{\substack{p\neq n\\ \lambda_p\in S_{AI_{\lambda_n}}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}+\sum_{\lambda_p\notin S_{AI_{\lambda_n}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}$$

Since $|1-\bar\lambda_n\lambda_p|\geq 1-|\lambda_n|$, it follows from (4.5) that the first sum is bounded by 4ε.

The second sum can be written as

$$\sum_{k=0}^{\infty}\sum_{\lambda_p\in S_{2^{k+1}AI_{\lambda_n}}\setminus S_{2^kAI_{\lambda_n}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}. \tag{4.6}$$

By (2.6), there is some constant $C>0$ such that, for $z\notin S_{2^kAI_{\lambda_n}}$ one has $\frac{(1-|\lambda_n|^2)}{|1-\bar\lambda_n z|^2}\leq\frac{C}{2^{2k}A^2(1-|\lambda_n|)}$. On the other hand, since $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure, there exists

$C' > 0$, such that $\sum_{\lambda_p \in S_{2^{k+1}AI_{\lambda_n}}} (1-|\lambda_p|^2) \le C' 2^{k+1} A(1-|\lambda_n|)$. It follows then that (4.6) is bounded by $4CC'/A$. On the whole, we obtain

$$\sum_{p \neq n, p \geq 1} \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n \lambda_p|^2} \le 4(\varepsilon + CC'/A).$$

This can be made as small as possible by choosing ε and A; by Proposition 4.1 it follows that Λ is a thin interpolating sequence. □

As a consequence, we mention the following two results that help to clarify the geometry of thin sequences; they are suggested by corresponding results related to Carleson sequences (see [9], VII.3).

Proposition 4.3. (i) *Suppose* $\Lambda = (\lambda_n)_{n\geq 1}$ *is an increasing sequence in* $\mathbb{D}$ *such that* $\lim_{n\to\infty} |\lambda_n| = 1$. *If*

$$\gamma := \lim_{k\to\infty} \frac{1-|\lambda_{k+1}|}{1-|\lambda_k|} = 0,$$

then Λ *is a thin interpolating sequence. If, moreover,* $\Lambda \subset [0,1)$, *then* Λ *is a thin sequence if and only if* $\gamma = 0$.

(ii) *Suppose* $(r_n)_{n\geq 1}$ *is a sequence of distinct positive numbers,* $0 < r_n < 1$, *such that* $\sum_{n\geq 1}(1-r_n) < \infty$. *Then there exist* $t_n \geq 0$ *such that* $(r_n e^{it_n})_{n\geq 1}$ *is a thin interpolating sequence.*

Proof. (i) Fix $0 < \varepsilon < 1, A \geq 1$, and choose N such that for all $n \geq N$ we have

$$\frac{1-|\lambda_n|}{1-|\lambda_{n-1}|} < \varepsilon/A.$$

It follows that, if $n \geq N$ and $k < n$, then $\lambda_k \notin S_{AI_n}$.

On the other hand, if $k > n$, $1-|\lambda_k| \le (\varepsilon/A)^{k-n}(1-|\lambda_n|)$, and $\sum_{k>n}(1-|\lambda_k|) \le \frac{\varepsilon/A}{1-\varepsilon/A}(1-|\lambda_n|)$. Therefore

$$\sum_{\substack{k\neq n \\ \lambda_k \in S_{AI_{\lambda_n}}}} (1-|\lambda_k|) \le \frac{\varepsilon/A}{1-\varepsilon/A}(1-|\lambda_n|);$$

it follows by Proposition 4.2 that Λ is a thin sequence.

If $\Lambda \in (0,1)$, then

$$1 - \frac{1-\lambda_{k+1}}{1-\lambda_k} = \frac{\lambda_{k+1}-\lambda_k}{1-\lambda_k} \geq \frac{\lambda_{k+1}-\lambda_k}{1-\lambda_k\lambda_{k+1}} = b_{\lambda_k}(\lambda_{k+1}) \geq |B_k(\lambda_k)|.$$

Therefore, if $\lim_{k\to\infty} |B_k(\lambda_k)| = 1$, then $\lim_{k\to\infty} \frac{1-\lambda_{k+1}}{1-\lambda_k} = 0$.

(ii) We may suppose r_n is increasing. Choose numbers $b_n > 0$, such that $\sum_{n\geq 1} b_n < \infty$, and $(1-r_n)/b_{n+1} \to 0$. Since the thin interpolating property is not changed by adding a finite number of distinct points, we may suppose that $\sum_{n\geq 1} b_n < \pi/2$. We will then define $t_n = \sum_{k=1}^n b_k$, and $\lambda_n = r_n e^{it_n}$.

If $A > 0$ is given, then the condition $(1-r_n)/b_{n+1} \to 0$ implies that, for n sufficiently large, $S_{AI_{\lambda_n}} \cap \Lambda = \{\lambda_n\}$. Then (4.3) is trivially verified, whence Λ is a thin sequence. □

It should be mentioned that (ii) in the above proposition has already been noticed in [11].

We end this section by quoting a stability result from [3], Section 3, where it has been proved that thin sequences are stable with respect to "small" perturbations.

Theorem B. *Let $\Lambda = (\lambda_n)_{n\geq 1}$, $\Lambda' = (\lambda'_n)_{n\geq 1}$ be two sequences in $\mathbb{D}$. If $\sup_{n\geq 1} |b_{\lambda_n}(\lambda'_n)| < 1$, then $\Lambda \in (\vartheta)$ if and only if $\Lambda' \in (\vartheta)$.*

5 Projection onto a model space

Suppose now that Θ is an inner function, while Λ is a Blaschke sequence of distinct points in $\mathbb{D}$. We are interested in the AOB property for the corresponding sequences of normalized reproducing kernels $(h^{\Theta}_{\lambda_n})_{n\geq 1}$, as defined by (2.3). It turns out that Volberg's condition (4.1) is necessary also in this context, as is shown by the next result ([3], Section 3). Below we will give a simpler proof.

Proposition 5.1. *If $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ is an AOS, then $(\lambda_n)_{n\geq 1}$ is a thin interpolating sequence.*

Proof. By applying formula (2.5), we have

$$\begin{aligned} |\Gamma^{\Theta}_{n,p}|^2 &= |\Gamma_{n,p}|^2 \frac{|1-\overline{\Theta(\lambda_n)}\Theta(\lambda_p)|^2}{(1-|\Theta(\lambda_n)|^2)(1-|\Theta(\lambda_p)|^2)} \\ &= \frac{|\Gamma_{n,p}|^2}{1-|b_{\Theta(\lambda_n)}(\Theta(\lambda_p))|^2} \geq |\Gamma_{n,p}|^2. \end{aligned}$$

Since Proposition 3.2, (iii), implies $\|(\Gamma^{\Theta} - I)e_n\|^2 = \sum_{p\neq n} |\Gamma^{\Theta}_{n,p}|^2 \to 0$, it follows that $\|(\Gamma - I)e_n\|^2 = \sum_{p\neq n} |\Gamma_{n,p}|^2 \to 0$. Proposition 4.1 implies then that $(h_{\lambda_n})_{n\geq 1}$ is an AOB in K_B. □

There is no hope to obtain, without supplementary conditions, a converse to Proposition 5.1. Indeed, suppose (λ_n) is a thin sequence, converging nontangentially to a point $\zeta \in \mathbb{T}$, while Θ can be analytically extended on an neighborhood of ζ. It follows from Theorem C in Section 8 below that $(h^\Theta_{\lambda_n})$ is in this case norm convergent, and thus cannot be even a Riesz sequence.

We will therefore try to obtain partial converses to Proposition 5.1. It is then natural, in view of the theory of Riesz bases developed in [7], to work under the supplementary condition $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$. (Note that in the previous example we have $|\Theta(\lambda_n)| \to |\Theta(\zeta)| = 1$.)

Theorem 5.2. *Suppose* $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$. *If* $(\lambda_n)_{n\geq 1}$ *is a thin interpolating sequence, then either*

(i) $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB,*

or

(ii) *there exists* $p \geq 2$ *such that* $(h^\Theta_{\lambda_n})_{n\geq p}$ *is a complete AOB in* K_Θ*.*

Proof. The condition on $(h_{\lambda_n})_{n\geq 1}$ implies the existence of positive constants $(c_N)_{N\geq N_0}$, $(C_N)_{N\geq N_0}$, tending to 1, such that

$$c_N \sum_{n\geq N} |a_n|^2 \leq \Big\| \sum_{n\geq N} a_n h_{\lambda_n} \Big\|^2 \leq C_N \sum_{n\geq N} |a_n|^2. \tag{5.1}$$

According to (2.4), we have, applying (5.1),

$$\begin{aligned} \Big\| \sum_{n\geq N} a_n h^\Theta_{\lambda_n} \Big\|^2 &= \Big\| \sum_{n\geq N} \frac{a_n}{\sqrt{1-|\Theta(\lambda_n)|^2}} h_{\lambda_n} \Big\|^2 - \Big\| \sum_{n\geq N} \frac{a_n \overline{\Theta(\lambda_n)}}{\sqrt{1-|\Theta(\lambda_n)|^2}} h_{\lambda_n} \Big\|^2 \\ &\leq C_N \sum_{n\geq N} \frac{|a_n|^2}{1-|\Theta(\lambda_n)|^2} - c_N \sum_{n\geq N} \frac{|a_n|^2 |\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \\ &= C_N \sum_{n\geq N} |a_n|^2 + (C_N - c_N) \sum_{n\geq N} \frac{|a_n|^2 |\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \\ &\leq C_N \sum_{n\geq N} |a_n|^2 + (C_N - c_N) \sup_n \frac{|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \sum_{n\geq N} |a_n|^2. \end{aligned}$$

Since $C_N \to 1$, $C_N - c_n \to 0$, while $\sup_n \frac{|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} < \infty$, we can find constants $C'_N \to 1$, such that

$$\Big\| \sum_{n\geq N} a_n h^\Theta_{\lambda_n} \Big\|^2 \leq C'_N \sum_{n\geq N} |a_n|^2.$$

A similar argument shows the existence of $c'_N \to 1$, such that

$$\Big\| \sum_{n\geq N} a_n h^\Theta_{\lambda_n} \Big\|^2 \geq c'_N \sum_{n\geq N} |a_n|^2.$$

It follows that $(h^\Theta_{\lambda_n})_{n\geq 1}$ is an AOS; hence there exists $m \geq 1$ such that $(h^\Theta_{\lambda_n})_{n\geq m}$ is an AOB.

Let p be the smallest positive integer with the property that $(h^\Theta_{\lambda_n})_{n\geq p}$ is an AOB. If $p = 1$ we are in case (i) of the statement. Otherwise, Lemmas 3.1 and 2.1 imply that $(h^\Theta_{\lambda_n})_{n\geq p}$ is complete in K_Θ. The theorem is thus proved. □

It should be mentioned that a weaker version of Theorem 5.2 appears in Lemma 3.9 in [3].

Corollary 5.3. *Suppose that* $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$ *and* Λ *is a thin interpolating sequence. Then* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB if and only if it is minimal.*

Case (ii) in Theorem 5.2 corresponds to $(h^\Theta_{\lambda_n})_{n\geq 1}$ not minimal; an example can be obtained by taking Θ to be a proper inner divisor of B. Minimality of sequences of reproducing kernels has been investigated in [1]; using Theorem 4.7 therein, we obtain the following characterization.

Corollary 5.4. *Suppose that* $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$. *Then* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB if and only if* $(\lambda_n)_{n\geq 1}$ *is a thin interpolating sequence and there exists* $f \in H^\infty$, $f \neq 0$, *such that* $\|\Theta + Bf\|_\infty \leq 1$.

It is instructive to compare Corollary 5.4 with a result in [9], VIII.6, where it is proved that under the hypothesis $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$, $(h^\Theta_{\lambda_n})_{n\geq 1}$ is a Riesz sequence if and only if $\Lambda \in (C)$ and $\operatorname{dist}(\Theta\bar{B}, H^\infty) < 1$. This last condition is obviously stronger than the last requirement of Corollary 5.4. On the other hand, the thin interpolating condition is much more restrictive than Carleson's.

We can say more in case Θ is not a Blaschke product. The next result adapts an argument in [7], Theorem 3.2.

Proposition 5.5. *Let* Θ *be an inner function with a nontrivial singular part, and suppose* $\sup_{n\geq 1} |\Theta(\lambda_n)| < 1$. *If the sequence* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB in* K_Θ, *then its span has infinite codimension.*

Proof. Suppose that $\sup_{n\geq 1}|\Theta(\lambda_n)| = \eta < 1$. We shall write $\Theta = \beta S$, with β a Blaschke product and S singular, nonconstant. Let us also denote $B^{(N)} = \prod_{n\geq N} b_{\lambda_n}$.

By Proposition 5.1, $(h_{\lambda_n})_{n\geq 1}$ is an AOB (and in particular a Riesz sequence). If c_N, C_N are the constants in (3.1), then applying Lemma 2.3 to Θ and $B^{(N)}$ it follows that $\|P_{B^{(N)}}T_\Theta|K_{B^{(N)}}\| \leq (C_N/c_N)^{1/2}\eta$. Since $C_N/c_N \to 1$, we may find $N \in \mathbb{N}$, such that $\|P_{B^{(N)}}T_\Theta|K_{B^{(N)}}\| < 1$, which, according to Lemma 2.2, implies that $P_\Theta|K_{B^{(N)}}$ is an isomorphism on its image.

Now, if we define $\Theta' = \beta S^{1/2}$, Θ' is also an inner function, and

$$|\Theta'(\lambda_n)| \leq |\beta(\lambda_n)|^{1/2}|S(\lambda_n)|^{1/2} = |\Theta(\lambda_n)|^{1/2} \leq \eta^{1/2}.$$

If we apply the same argument to Θ', it follows that we can find $N \in \mathbb{N}$, such that both $P_\Theta|K_{B^{(N)}}$ and $P_{\Theta'}|K_{B^{(N)}}$ are isomorphisms on their images.

But we have

$$P_{\Theta'}|K_{B^{(N)}} = (P_{\Theta'}|K_\Theta)(P_\Theta|K_{B^{(N)}}).$$

The operator on the left is one-to-one, while the image of $(P_\Theta|K_{B^{(N)}})$ is closed. Therefore this image cannot intersect $\mathrm{Ker}(P_{\Theta'}|K_\Theta)$, which is infinite dimensional. But the image of $(P_\Theta|K_{B^{(N)}})$ is the space spanned by $h^\Theta_{\lambda_n}$ for $n \geq N$; it follows that the space spanned by all the $h^\Theta_{\lambda_n}$ $(n \geq 1)$ also has infinite codimension. □

In this case one can improve Corollary 5.4.

Corollary 5.6. *Suppose that Θ has a nontrivial singular part and* $\sup_{n\geq 1}|\Theta(\lambda_n)| < 1$. *The following assertions are equivalent:*

(i) Λ *is a thin interpolating sequence;*

(ii) $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB.*

Moreover, in this case, $\mathrm{Span}\{h^\Theta_{\lambda_n} : n \geq 1\}$ *has infinite codimension in K_Θ.*

Proof. If Λ is a thin sequence, Proposition 5.5 shows that we are in Case (i) of Theorem 5.2; consequently $(h^\Theta_{\lambda_n})_{n\geq 1}$ is an AOB. The converse is contained in Proposition 5.1. □

6 Stability of AOB's

We will next study the stability of AOB's with respect to small perturbations.

Theorem 6.1. *Suppose that* $\sup_{n\geq 1}|\Theta(\lambda_n)|<1$ *and* $(h^\Theta_{\lambda_n})_{n\geq 1}$ *is an AOB. If* $\Lambda'=(\lambda'_n)_{n\geq 1}$ *is a sequence of distinct points in* $\mathbb{D}$ *that satisfies*

$$\limsup_{n\to\infty}|b_{\lambda_n}(\lambda'_n)|<\frac{1-\operatorname{dist}(\Theta\bar{B},H^\infty)}{1+\operatorname{dist}(\Theta\bar{B},H^\infty)},\tag{6.1}$$

then $(h^\Theta_{\lambda'_n})_{n\geq 1}$ *is an AOB.*

Proof. Fix $N\geq 1$, and define

$$\gamma_n=\begin{cases}\lambda_n \text{ if } n<N,\\ \lambda'_n \text{ if } n\geq N;\end{cases}$$

and Φ the Blaschke product associated to $(\gamma_n)_{n\geq 1}$. Proposition 5.1 implies that Λ is a thin sequence, whence, by Theorem B, Λ' and $(\gamma_n)_{n\geq 1}$ are both thin sequences. If $g,h\in H^\infty$, then the equality $\Theta\bar{\Phi}-gh=\Theta\bar{B}(B\bar{\Phi}-g)+(\Theta\bar{B}-h)g$ implies

$$\|\Theta\bar{\Phi}-gh\|_\infty\leq\|B\bar{\Phi}-g\|_\infty+\|(\Theta\bar{B}-h)g\|_\infty,$$

which shows that

$$\operatorname{dist}(\Theta\bar{\Phi},H^\infty)\leq\operatorname{dist}(\Theta\bar{B},H^\infty)+(1+\operatorname{dist}(\Theta\bar{B},H^\infty))\operatorname{dist}(B\bar{\Phi},H^\infty).$$

Now, if $B^{(N)}=\prod_{n\geq N}b_{\lambda_n}$, $\Phi^{(N)}=\prod_{n\geq N}b_{\lambda'_n}$, then $B\bar{\Phi}=B^{(N)}\overline{\Phi^{(N)}}$. Suppose C_N and c_N are the constants associated to Λ' as in 3.1, while $\varepsilon_N=\sup_{n>N}|b_{\lambda_n}(\lambda'_n)|$; one has then obviously $\sup_{n\geq N}|B^{(N)}(\lambda'_n)|\leq\varepsilon_N$. Applying Lemmas 2.2 and 2.3, it follows that

$$\operatorname{dist}(B\bar{\Phi},H^\infty)=\operatorname{dist}(B^{(N)}\overline{\Phi^{(N)}},H^\infty)=\|P_{\Phi^{(N)}}T_{B^{(N)}}|K_{\Phi^{(N)}}\|\leq\varepsilon_N(C_N/c_N)^{1/2}.$$

Consequently,

$$\operatorname{dist}(\Theta\bar{\Phi},H^\infty)\leq\operatorname{dist}(\Theta\bar{B},H^\infty)+(1+\operatorname{dist}(\Theta\bar{B},H^\infty))\varepsilon_N(C_N/c_N)^{1/2}.$$

The hypothesis implies that, for N sufficiently large, $\varepsilon_N(C_N/c_N)^{1/2}<\frac{1-\operatorname{dist}(\Theta\bar{B},H^\infty)}{1+\operatorname{dist}(\Theta\bar{B},H^\infty)}$ and therefore $\operatorname{dist}(\Theta\bar{\Phi},H^\infty)<1$. There exists thus $f\in H^\infty$, $f\neq 0$, such that $\|\Theta-\Phi f\|_\infty<1$, and therefore $\sup_{n\geq 1}|\Theta(\gamma_n)|\leq\operatorname{dist}(\Theta\bar{\Phi},H^\infty)<1$. It follows by Corollary 5.4 that $(h^\Theta_{\gamma_n})_{n\geq 1}$ is an AOB. Applying repeatedly Lemma 2.1, we obtain that $(h^\Theta_{\lambda'_n})_{n\geq 1}$ is an AOB. ◻

In the particular case where Θ has a nontrivial singular part, we can improve the stability constant in Theorem 6.1.

Proposition 6.2. *Suppose that Θ has a nontrivial singular part, $\sup_{n\geq 1}|\Theta(\lambda_n)| < 1$ and $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ is an AOB. If $\Lambda' = (\lambda'_n)_{n\geq 1}$ is a sequence of distinct points in $\mathbb{D}$ that satisfies*

$$\limsup_{n\to\infty} |b_{\lambda_n}(\lambda'_n)| < \frac{1 - \limsup_{n\to\infty}|\Theta(\lambda_n)|}{1 + \limsup_{n\to\infty}|\Theta(\lambda_n)|},$$

then $(h^{\Theta}_{\lambda'_n})_{n\geq 1}$ is an AOB.

Proof. By Theorem B, $\Lambda' \in (\vartheta)$. On the other hand,

$$\left|\frac{\Theta(\lambda'_n) - \Theta(\lambda_n)}{b_{\lambda_n}(\lambda'_n)}\right| \leq \left\|\frac{\Theta - \Theta(\lambda_n)}{b_{\lambda_n}}\right\|_\infty \leq 1 + |\Theta(\lambda_n)|,$$

whence

$$|\Theta(\lambda'_n)| \leq |\Theta(\lambda_n)| + (1 + |\Theta(\lambda_n)|)|b_{\lambda_n}(\lambda'_n)|.$$

Therefore $\sup_n |\Theta(\lambda'_n)| < 1$; Corollary 5.6 implies that $(h^{\Theta}_{\lambda'_n})_{n\geq 1}$ is an AOB. ☐

It is also possible to complement these results by studying the completeness of the perturbed sequence. As concerns the effect of small perturbations on Riesz basis, the following theorem was proved in [3].

Theorem ([3], 3.1). *Suppose that $\sup_{n\geq 1}|\Theta(\lambda_n)| < 1$. If $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ is a Riesz basis in K_Θ, then there exists $\varepsilon = \varepsilon(\Theta, \Lambda) < 1$ such that for all sequences $\Lambda' = (\lambda'_n)_{n\geq 1}$ in $\mathbb{D}$ satisfying $|b_{\lambda_n}(\lambda'_n)| \leq \varepsilon$, we have $(h^{\Theta}_{\lambda'_n})_{n\geq 1}$ is a Riesz basis in K_Θ.*

Combining this result with theorem 6.1, we obtain the following consequence for complete AOB's.

Corollary 6.3. *Suppose that $\sup_{n\geq 1}|\Theta(\lambda_n)| < 1$. If $(h^{\Theta}_{\lambda_n})_{n\geq 1}$ is a complete AOB in K_Θ, then there exists $\varepsilon = \varepsilon(\Theta, \Lambda) < 1$ such that for all sequences $\Lambda' = (\lambda'_n)_{n\geq 1}$ in $\mathbb{D}$ satisfying*

$$|b_{\lambda_n}(\lambda'_n)| \leq \varepsilon,$$

we have $(h^{\Theta}_{\lambda'_n})_{n\geq 1}$ is a complete AOB in K_Θ.

A few words are in order concerning the different stability constants appearing in this section. The analogue for Riesz sequences of Theorem 6.1 appears in [3], Theorem 3.3. The right hand side of (6.1) is replaced therein by

$$\frac{\delta(\Lambda)^6}{8}\frac{1-\operatorname{dist}(\Theta\bar{B},H^\infty)}{1+\operatorname{dist}(\Theta\bar{B},H^\infty)}.$$

For AOB's, one should have expected a similar result, with $\delta(\Lambda)$ replaced by 1; Theorem 6.1 is therefore a sensible improvement.

As concerns completeness, there exists also an explicit upper bound for the constant $\varepsilon(\Theta,\Lambda)$ which appears in Theorem 3.1 of [3]; namely, we must have:

$$\varepsilon<\min\{\frac{\delta}{2},\frac{1-\sup_{n\geq1}|\Theta(\lambda_n)|}{2}\}$$

as well as

$$\frac{2\varepsilon}{\delta/2-\varepsilon}\|\Gamma\|^{1/2}\left(128\frac{1+\varepsilon}{1-\varepsilon}(1-6\log\delta)\frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)^{1/2}<1,$$

where $\delta=\inf_{n\geq1}|B_n(\lambda_n)|$ and Γ is the Gram matrix associated to $(h^\Theta_{\lambda_n})_{n\geq1}$. One can see that this is much more complicated than the bound given by formula (6.1).

7 Bases of exponentials

The study of bases of exponentials in $L^2(0,a)$ has provided the original motivation for the development of the functional model approach in [7]. It is therefore natural to discuss in more detail AOB's of exponentials. Some preliminaries are needed to translate the problem into the language of model spaces. Note also that, as is customary, the index set will now be $\mathbb{Z}$ rather than $\mathbb{N}^*$.

If $\mathbb{C}_+=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$, then we define $\phi:\mathbb{C}_+\to\mathbb{D}$ by $\phi(z)=\frac{z-i}{z+i}$ (ϕ is a conformal map from $\mathbb{C}_+$ to $\mathbb{D}$). The operator

$$(Uf)(z)=\frac{1}{\pi(z+i)}f(\phi(z)) \tag{7.1}$$

maps H^2 unitarily onto $H^2(\mathbb{C}_+)$, the Hardy space of the upper half-plane. The corresponding transformation for functions in H^∞ is

$$f\mapsto f\circ\phi; \tag{7.2}$$

it maps inner functions in $\mathbb{D}$ into inner functions in $\mathbb{C}_+$. We have then $UK_\Theta = H^2(\mathbb{C}_+) \ominus (\Theta \circ \phi)H^2(\mathbb{C}_+)$, and $U(k_\lambda^\Theta)$ is the reproducing kernel for the point $\phi(\lambda)$.

The Blaschke factor corresponding to $\mu \in \mathbb{C}_+$ is

$$b_\mu^+(z) = \frac{z-\mu}{z-\bar{\mu}}$$

and the Blaschke product with zeros $(\mu_n)_{n\in\mathbb{Z}}$ is

$$B^+(z) = \prod_{n\in\mathbb{Z}} c_{\mu_n} b_{\mu_n}^+(z),$$

the coefficients c_{μ_n} being chosen as to make all terms positive in i. The thin interpolating condition, which we will denote by (ϑ_+), becomes

$$\lim_{|n|\to\infty} \prod_{m\neq n} |b_{\mu_m}^+(\mu_n)| = 1.$$

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform. Then $\mathcal{F}U$ maps H^2 unitarily onto $L^2(0,\infty)$. If $\Theta_a(z) = e^{a\frac{z+1}{z-1}}$, then $\mathcal{F}U$ maps K_{Θ_a} unitarily onto $L^2(0,a)$; the normalized reproducing kernel $h_\lambda^{\Theta_a}$ ($\lambda \in \mathbb{D}$) is mapped into $\chi_\mu^a(t) = \kappa_a(\mu)e^{i\mu t}$, where $\mu = -\overline{\phi^{-1}(\lambda)}$, and $\kappa_a(\mu) = \left(\frac{2\operatorname{Im}\mu}{1-e^{-2a\operatorname{Im}\mu}}\right)^{1/2}$. We also have $\Theta_a(\lambda) = e^{ia\bar{\mu}}$.

The results from the previous sections concerning reproducing kernels can then be adapted to the case of exponentials $e^{i\mu t}$, with $\mu \in \mathbb{C}_+$; note that the relevant inner function Θ_a is singular. The next theorem deals, however, with a more general class of exponentials.

Theorem 7.1. *Let $(\mu_n)_{n\in\mathbb{Z}}, (\mu'_n)_{n\in\mathbb{Z}}$ be two sequences of distinct complex numbers. If $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$, and $\lim_{|n|\to\infty} |\mu_n - \mu'_n| = 0$, then $(e^{i\mu'_n t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$.*

Proof. Fix $N \geq 1$, and define

$$\gamma_n = \begin{cases} \mu_n \text{ if } |n| \leq N, \\ \mu'_n \text{ if } |n| > N; \end{cases}$$

and V by $V(e^{i\mu_n t}) = e^{i\gamma_n t}$ for $n \in \mathbb{Z}$. For $(a_n)_{n\in\mathbb{Z}} \in \ell^2$, we have:

$$\begin{aligned}
\big\|(V-I)(\sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t})\big\| &= \big\|(V-I)(\sum_{|n|>N} a_n e^{i\mu_n t})\big\| = \big\|\sum_{|n|>N} a_n(e^{i\mu'_n t} - e^{i\mu_n t})\big\| \\
&= \big\|\sum_{|n|>N} a_n e^{i\mu_n t} \sum_{k\geq 1} \frac{(i(\mu'_n-\mu_n))^k t^k}{k!}\big\| \\
&\leq \sum_{k\geq 1} \frac{1}{k!}\big\|t^k \sum_{|n|>N} (i(\mu'_n-\mu_n))^k a_n e^{i\mu_n t}\big\| \\
&\leq C_N \sum_{k\geq 1} \frac{a^k}{k!} \left(\sum_{|n|>N} |\mu'_n-\mu_n|^{2k}|a_n|^2\right)^{1/2} \\
&\leq C_N(e^{a\sup_{|n|>N}|\mu'_n-\mu_n|}-1)(\sum_{|n|>N}|a_n|^2)^{1/2},
\end{aligned}$$

C_N being the constant in (3.1) corresponding to the AOB $(e^{i\mu_n t})_{n\in\mathbb{Z}}$. Since $\sup_{|n|>N}|\mu'_n - \mu_n| \to 0$ and $C_N \to 1$ for $N \to \infty$, it follows that if N is large enough, then $\|V-I\| < 1$ and thus V is invertible. If P_m is the orthogonal projection onto $\operatorname{Span}\{e^{i\mu_n t} : |n| \geq m\}$, similar computations for $m \geq N$ show then that $\|(V-I)P_m\| \to 0$, and therefore $V-I$ is compact. Proposition 3.2 shows that $(e^{i\gamma_n t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$.

Now the two sequences of complex numbers $(\mu'_n)_{n\in\mathbb{Z}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ differ by a finite number of terms, and therefore $(e^{i\mu'_n t})_{n\in\mathbb{Z}}$ is an AOS. On the other hand, $\|e^{i\mu_n t}\| \to 1$ implies $\operatorname{Im}\mu_n \to 0$; thus $(\mu'_n)_{n\in\mathbb{Z}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ are both contained in a strip, say $|\operatorname{Im} z| < A$. Multiplication by e^{-At} is an invertible operator on $L^2(0,1)$; thus $(e^{i(\gamma_n+iA)t})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,1)$. An application of Lemma 2.1 implies that $(e^{i(\mu'_n+iA)t})_{n\in\mathbb{Z}}$ is also a Riesz basis, and therefore the same is true about $(e^{i\mu'_n t})_{n\in\mathbb{Z}}$; the proof is complete. □

In the case $\mu_n = 2\pi n$, one can compare Theorem 7.1 to Kadec's Theorem (see [7], I.5), which states, for real sequences (μ'_n), that Riesz bases are preserved under the requirement $|\mu'_n - 2\pi n| < 1/4$. Such a uniform bound is not adequate for AOB's; indeed, since in $L^2(0,1)$

$$\langle e^{2i\pi\mu_n t}, e^{2i\pi\mu_{n+1}t}\rangle = \frac{e^{2i\pi(\mu_n-\mu_{n+1})}-1}{2i\pi(\mu_n-\mu_{n+1})},$$

it follows that $\langle e^{2i\pi\mu_n t}, e^{2i\pi\mu_{n+1}t}\rangle \to 0$ implies $\mu_n - \mu_{n+1} - [\mu_n - \mu_{n+1}] \to 0$.

Suppose now that there is $\eta > 0$ such that $\operatorname{Im}\mu_n > \eta$ for all $n \in \mathbb{Z}$. In this case AOB of (normalized) exponentials in $L^2(0,a)$ $(a > 0)$ are exactly characterized by the corresponding condition (ϑ_+).

Proposition 7.2. *If* $\operatorname{Im}\mu_n > \eta > 0$ *for all* $n \in \mathbb{Z}$*, then the following are equivalent:*

(i) $(\mu_n)_{n\in\mathbb{Z}}$ *is thin interpolating;*

(ii) $(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ *is an AOB in* $L^2(0,a)$ *for all* $a > 0$*;*

(iii) $(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ *is an AOB in* $L^2(0,a)$ *for some* $a > 0$*.*

Proof. If we translate the problem in the disc, then the inner function Θ_a is singular, and if $\lambda_n = \phi(-\bar{\mu}_n)$, then $|\Theta_a(\lambda_n)| = e^{-a\operatorname{Im}\mu_n}$. Therefore $\sup_{n\in\mathbb{Z}} |\Theta_a(\lambda_n)| < 1$, and the results in the statement are a consequence of Corollary 5.6. □

One should remark that in this case the Volberg condition is independent of $a > 0$. This should be compared with the situation for Riesz sequences of exponentials (see, for instance, [10], D.5): in case $\inf_{n\in\mathbb{Z}} \operatorname{Im}\mu_n > -\infty$, if $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,a)$, then $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,a')$ for all $a' \geq a$, but usually not for $a' < a$.

Finally, a stability result can be obtained by translating Proposition 6.2.

Corollary 7.3. *Suppose* $\operatorname{Im}\mu_n > \eta > 0$ *for all* $n \in \mathbb{Z}$*, and* $(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ *is an AOB in* $L^2(0,a)$ *for some* $a > 0$*. If* $(\mu'_n)_{n\in\mathbb{Z}}$ *is a sequence of distinct points in* $\mathbb{C}_+$ *that satisfies*

$$\limsup_{|n|\to\infty} \left| \frac{\mu_n - \mu'_n}{\bar{\mu}_n - \mu'_n} \right| < \frac{1 - \limsup_{|n|\to\infty} e^{-a\operatorname{Im}\mu_n}}{1 + \limsup_{|n|\to\infty} e^{-a\operatorname{Im}\mu_n}},$$

then $(\chi^a_{\mu'_n})_{n\in\mathbb{Z}}$ *is an AOB in* $L^2(0,a)$*.*

8 Examples

As noticed in the previous section, bases of exponentials are related to a singular inner function Θ, with corresponding one-point supported measure. In this section we will give some examples related to other inner functions.

Since complete AOB's are asymptotically close to orthonormal bases, it is natural to try to obtain examples by perturbing orthonormal bases. If we take $\lambda, \lambda' \in \mathbb{D}$, then $\langle k^\Theta_\lambda, k^\Theta_{\lambda'}\rangle = \frac{1-\overline{\Theta(\lambda)}\Theta(\lambda')}{1-\bar{\lambda}\lambda'} \neq 0$, and thus the reproducing kernels themselves cannot be orthogonal. However, we may obtain orthogonal bases of reproducing kernels in case the evaluations on the boundary $\mathbb{T}$ of $\mathbb{D}$ are continuous; the precise statement appears in Theorem C below. Suppose $a_n \in \mathbb{D}$ are the zeros of the Blaschke factor of Θ, while σ is the positive singular measure on $\mathbb{T}$ corresponding to the singular factor of Θ. We define $E_\Theta \subset \mathbb{T}$ by the formula

$$E_\Theta = \Big\{\zeta \in \mathbb{T} \mid \sum_{k=1}^{\infty} \frac{1-|a_k|}{|\zeta - a_k|^2} + \int_0^{2\pi} \frac{d\sigma(t)}{|\zeta - e^{it}|^2} < \infty \Big\}. \tag{8.1}$$

The following theorem appears in [2].

Theorem C. (i) *If $\zeta \in E_\Theta$, then Θ has a nontangential limit $\Theta(\zeta)$ in ζ, of modulus* 1. *The function $k_\zeta^\Theta(z) := \frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar\zeta z}$ belongs to K_Θ, and $k_\lambda^\Theta \to k_\zeta^\Theta$ if $\lambda \to \zeta$ nontangentially. Moreover, any function $f \in K_\Theta$ has a nontangential limit $f(\zeta)$ in ζ, and $f(\zeta) = \langle f, k_\zeta^\Theta \rangle$.*

(ii) *If $\mathbb{T} \setminus E_\Theta$ is at most countable, then there exists a sequence $\zeta_n \in E_\Theta$ such that $\left(\frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|}\right)$ is an orthonormal basis of K_Θ.*

We may therefore obtain a large class of examples of complete AOB's in K_Θ formed by reproducing kernels.

Corollary 8.1. *If $\mathbb{T} \setminus E_\Theta$ is at most countable, then there exist sequences $(\lambda_n)_{n\ge1}$ in $\mathbb{D}$ such that $(h_{\lambda_n}^\Theta)_{n\ge1}$ is a complete AOB in K_Θ.*

Proof. By Theorem C, (ii), take a sequence $\zeta_n \in E_\Theta$ such that $\left(\frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|}\right)$ is an orthonormal basis of K_Θ. By (i) of the same proposition, it obviously follows that if $\zeta \in E_\Theta$, and $\lambda \to \zeta$ nontangentially, then $h_{\lambda_n}^\Theta \to \frac{k_\zeta^\Theta}{\|k_\zeta^\Theta\|}$. Choose then $\lambda_n \in \mathbb{D}$, $\lambda_n/\|\lambda_n\| = \zeta_n$, such that $\sum_{n\ge1} \left\|h_{\lambda_n}^\Theta - \frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|}\right\|^2 < 1$. The required conclusion follows then by applying Lemma 3.3. □

One should note that the choice of λ_n can obviously be made such that $|\Theta(\lambda_n) - \Theta(\zeta_n)| \to 0$; it follows then that $|\Theta(\lambda_n)| \to 1$, and therefore we are not in the context of the results in Section 5.

In case $\Theta = \Theta_a$, $E_{\Theta_a} = \mathbb{T}\setminus\{1\}$, and thus obviously satisfies the hypotheses of Theorem C and Corollary 8.1. Actually, Clark's paper [2] indeed has the bases of exponentials as a starting point. A different type of example, adapted from [7], shows that complete AOB's can appear in a case when $E_\Theta = \emptyset$.

Take first a sequence of positive integers q_n, $n \ge 1$, such that $q_{n+1} - q_n \to \infty$. Choose then another sequence of positive integers p_n, $n \ge 1$, subject to the conditions

$$\sum_{n\ge1} \frac{p_n}{2^{q_n}} < \infty \tag{8.2}$$

$$\sum_{n\ge1} \frac{p_n \log p_n}{2^{q_n}} = \infty \tag{8.3}$$

Choose p_n equidistant points on the circle centered in the origin and having radius $1-\frac{1}{2^{q_n}}$; the union of all these points (for $n \geq 1$) will be denoted by Λ. We will also denote $r_n = 1-\frac{1}{2^{q_n}}$

We have $\sum_{\lambda\in\Lambda}(1-|\lambda|) = \sum_{n\geq 1} p_n \frac{1}{2^{q_n}} < \infty$; thus Λ satisfies the Blaschke condition and we may form the corresponding product B. Take $A > 0$; for sufficiently large n, if $\lambda \in \Lambda$ has absolute value $1-\frac{1}{2^{q_n}}$, then $(S_{AI_\lambda} \cap \Lambda) \setminus \{\lambda\}$ contains only points on the circles of radii strictly larger than $|\lambda|$. On each of these circles, the number of these points is of order $p_k \times A|I_\lambda| = p_k A(1-|\lambda|)$. Therefore

$$\frac{1}{1-|\lambda|} \sum_{\substack{\mu\neq\lambda \\ \mu\in S_{AI_\lambda}}} (1-|\mu|)$$

can be estimated by $A\sum_{k\geq n+1} \frac{p_k}{2^{q_k}}$, and thus tends to 0 by (8.2). Therefore Λ is a thin sequence by Proposition 4.2.

On the other hand, $E_B = \emptyset$. Actually, as in [7], more can be proved, namely that $\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} = \infty$ for all $\zeta \in \mathbb{T}$. Indeed, for $\zeta \in \mathbb{T}$, we have

$$\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} = \sum_{n\geq 1} \frac{1}{2^{q_n}} \sum_{|\lambda|=r_n} \frac{1}{|\zeta-\lambda|}.$$

For each fixed n, if $|\lambda| = r_n$, then, with the possible exception of two points, $|\zeta-\lambda|$ is comparable to $|r_n\zeta - \lambda|$. The other points λ on this circle are at distances to ζ comparable to $j \cdot \frac{2\pi}{p_n}$, with $j = 1, 2, \ldots, p_n - 2$. Therefore

$$\sum_{|\lambda|=r_n} \frac{1}{|\zeta-\lambda|} \geq C \sum_{1}^{p_n-2} \frac{1}{\frac{j}{p_n}} \leq C p_n \log p_n.$$

Then

$$\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} \geq C \sum_{n\geq 1} \frac{1}{2^{q_n}} p_n \log p_n = \infty$$

as required.

Acknowledgements. The authors thank the referee for its useful remarks and comments.

References

[1] Boricheva, I. Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2. *Journal of Functional Analysis 161* (1999), 397–417.

[2] Clark, D. One dimensional perturbations of restricted shifts. *J. Analyse Math. 25* (1972), 169–191.

[3] Fricain, E. Bases of reproducing kernels in model spaces. *J. Operator Theory 46* (2001), 517–543.

[4] Gorkin, P., Lingenberg, H.-M., and Mortini, R. Homeomorphic disks in the spectrum of H^∞. *Indiana Univ. Math. J. 39* (1990), 961–983.

[5] Hedenmalm, H. Thin interpolating sequences and three algebras of bounded functions. *Proc. Amer. Math. Soc. 99* (1987), 489–495.

[6] Hewitt, E., and Stromberg, K. *Real and Abstract Analysis*. Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[7] Hruščev, S. V., Nikolski, N. K., and Pavlov, B. S. Unconditional bases of exponentials and of reproducing kernels. In *Complex Analysis and Spectral Theory*, V. P. Havin and N. K. Nikolski, Eds., Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg New-York, 1981, pp. 214–335.

[8] Levinson, N. *Gap and Density Theorems*, vol. 26. Amer. Math. Soc. Colloquium Publ., New-York, 1940.

[9] Nikolski, N. K. *Treatise on the Shift Operator*. Springer-Verlag, Berlin, 1986. Grundlehren der mathematischen Wissenschafte vol. 273.

[10] Nikolski, N. K. *Operators, Functions, and Systems: An Easy Reading: Volume 2: Model Operators and Systems*. American Mathematical Society, 2002. Mathematical Surveys and Monographs, 93.

[11] Nikolski, N. K., and Volberg, A. L. Tangential and approximate free interpolation. In *Analysis and partial differential equations*, C. Sadosky, Ed., Lecture Notes in Pure and Applied. Math. Dekker, New-York, 1990, pp. 277–299.

[12] Paley, R. E., and Wiener, N. *Fourier Transforms in the Complex Domain*, vol. 19. Amer. Math. Soc. Colloquium Publ., Providence, 1934.

[13] Sundberg, C., and Wolff, T. H. Interpolating sequences for QA_B. *Trans. Amer. Math. Soc. 276* (1983), 551–581.

[14] Sz-Nagy, B., and Foias, C. *Harmonic Analysis of Operators on Hilbert Spaces.* North-Holland Publishing Co., Amsterdam-London, 1970.

[15] Volberg, A. L. Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason. *J. Operator Theory 7* (1982), 209–218.

6.1.2 Référence [T2]

Auteurs
I. Chalendar, E. Fricain et J. Partington

Titre
Overcompleteness of sequences of reproducing kernels in model spaces.

Paru dans
Integral Equations Operator Theory, 56 (2006), no. 1, 45–56.

Integr. equ. oper. theory 56 (2006), 45–56

0378-620X/010045-12, *published online* December 20, 2005
DOI 10.1007/s00020-005-1413-1

Integral Equations and Operator Theory

Overcompleteness of Sequences of Reproducing Kernels in Model Spaces

I. Chalendar, E. Fricain and J. R. Partington

Abstract. We give necessary conditions and sufficient conditions for sequences of reproducing kernels $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ to be overcomplete in a given model space K^p_Θ where Θ is an inner function in H^∞, $p \in (1,\infty)$, and where $(\lambda_n)_{n\geq 1}$ is an infinite sequence of pairwise distinct points of $\mathbb{D}$. Under certain conditions on Θ we obtain an exact characterization of overcompleteness. As a consequence we are able to describe the overcomplete exponential systems in $L^2(0,a)$.

Mathematics Subject Classification (2000). Primary 46E22, 46B15.

Keywords. Overcompleteness, hypercompleteness, reproducing kernel, model space.

1. Introduction

Given a Banach space X and a sequence $(x_n)_{n\geq 1} \subset X$, the question of completeness of sequences $(x_n)_{n\geq 1}$ in X is classical and appears in many problems. In this paper, we deal with a stronger property than completeness.

Definition 1.1. Let X be a Banach space. An infinite sequence $(x_n)_{n\geq 1}$ whose terms are pairwise distinct is *overcomplete* in X if every infinite subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ is complete in X, i.e. $\operatorname{span}\{x_{n_k} : k \geq 1\} = X$, where span denotes the closed linear hull.

One might expect that overcomplete sequences were rare, but in fact V. Klee [13] proved that every separable Banach space contains an overcomplete sequence. Such sequences (also known as hypercomplete or densely-closed sequences) have been much studied in the theory of the geometry of Banach spaces, originally because of their links with the existence of bases. See the book of Singer [17] for further details.

In this paper, we study the following problem due to N. Nikolski and considered previously in [9].

Problem 1. Find necessary and sufficient conditions concerning the inner function Θ and the sequence $(\lambda_n)_{n\geq 1}$ of $\mathbb{D}$ in order to obtain overcompleteness of $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ in the model space K^p_Θ.

In fact overcompleteness of $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ in K^p_Θ is equivalent to the following assertion: if $f\in K^q_\Theta$ satisfies $f(\lambda_{n_p})=0$ for $(\lambda_{n_p})_{p\geq 1}$ an infinite subsequence of $(\lambda_n)_{n\geq 1}$, then $f=0$.

The characterization of overcompleteness is linked to the same problem for completeness, which is rather difficult, even in the special case of sequences of exponential type (see [3, 14] for partial results in this direction).

The plan of the paper is the following. The next section contains preliminary material on Hardy spaces and inner functions. In Section 3, we study reflexive Banach spaces X of holomorphic functions on a domain Ω admitting evaluations E_λ at points $\lambda\in\Omega$. We give necessary conditions and sufficient conditions for the overcompleteness of $(E_{\lambda_n})_{n\geq 1}$ in X. The main result of this section is the following:

if $X\cap H^\infty(\Omega)$ is dense in X, then the overcompleteness of $(E_{\lambda_n})_{n\geq 1}$ implies the strong relative compactness of $(E_{\lambda_n})_{n\geq 1}$.

In Section 4, we provide a characterization of the overcomplete sequences of exponentials, i.e.

$$(e^{i\mu_n t})_{n\geq 1} \text{ is overcomplete in } L^2(0,a) \Longleftrightarrow \sup_{n\geq 1}|\mu_n|<\infty.$$

The main result of Section 5 is a geometric necessary and sufficient condition for the overcompleteness of $k_\Theta(\cdot,\lambda_n)_{n\geq 1}$ in reflexive spaces K^p_Θ, holding for a wide class of inner functions Θ. We also study the links between overcompleteness of sequences of reproducing kernels and properties of minimality or uniform minimality of all their infinite subsequences. We conclude with some illustrative examples analysed using the theory of Toeplitz operators.

2. Preliminaries

For $1\leq p\leq +\infty$, H^p will denote the standard Hardy space of the open unit disk $\mathbb{D}$ in $\mathbb{C}$, which we identify with the subspace of functions $f\in L^p(\mathbb{T})$ for which $\hat{f}(n)=0$ for all $n<0$ [5, 10]. Here $\mathbb{T}$ denotes the unit circle with normalized Lebesgue measure. Recall that a function $\Theta\in H^\infty$ is called *inner* if $|\Theta(\zeta)|=1$ for almost $\zeta\in\mathbb{T}$. We associate with each inner function Θ the model space K^p_Θ defined by

$$K^p_\Theta := H^p\cap\Theta\overline{H^p_0}=\{f\in H^p : \langle f,\Theta g\rangle = 0,\ g\in H^q\},$$

where $\overline{H^p_0}=\{\bar{f} : f\in H^p : f(0)=0\}$ and where p and q are conjugate exponents.

For $p\in(1,\infty)$, Beurling's theorem ([10], Chap. II) states that every nontrivial closed invariant subspace of H^p for $S^* : f\longmapsto \dfrac{f-f(0)}{z}$ is of the form K^p_Θ. The study of the subspaces K^p_Θ is relevant in various subjects such as rational approximation [8, 11, 16], Toeplitz operators [4, 6] and spectral theory for general linear operators [15]. The reproducing kernels in the subspaces K^q_Θ are the functions $k_\Theta(.,\lambda) \in K^p_\Theta$ such that $f(\lambda) = \langle f, k_\Theta(.,\lambda)\rangle$ for $\lambda \in \mathbb{D}$ and $f \in K^q_\Theta$. By [12] they are given by

$$k_\Theta(z,\lambda) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}.$$

Recall that if Θ is an inner function in H^∞, then Θ has a canonical decomposition of the form

$$\Theta(z) = e^{i\alpha} z^N \prod_{n\geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z} \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right) \tag{2.1}$$

where $\alpha \in \mathbb{R}$, $a_n \neq 0$, $\sum_{n\geq 1}(1 - |a_n|) < \infty$ and where μ is a non-negative singular measure.

Definition 2.1. Let Θ be an inner function in H^∞. The spectrum of Θ is denoted by $\sigma(\Theta)$ and is defined to be the complement in $\overline{\mathbb{D}}$ of the set $\{\xi \in \overline{\mathbb{D}} : \frac{1}{\Theta}$ can be analytically continued in a (full) neighbourhood of $\xi\}$.

It follows from [15], p. 63, that $\sigma(\Theta) \cap \mathbb{T} = \{\xi \in \mathbb{T} : \liminf_{z\to\xi} |\Theta(z)| = 0\}$ and if Θ has the canonical decomposition (2.1), then $\sigma(\Theta) = \text{clos}\{a_n : n \geq 1\} \cup \text{supp}\,\mu$, where $\text{supp}(\mu)$ denotes the support of μ and clos denotes the closure.

A useful fact concerning the spectrum of an inner function is contained in the following proposition.

Proposition 2.2 ([15]**, p. 65).** *Let Θ be an inner function and $p \in (1,\infty)$. The set $\mathbb{T} \setminus \sigma(\Theta)$ coincides with the set of points ξ such that every function in the model space K^p_Θ admits an analytic continuation across ξ.*

We shall also require another set associated with Θ, defined as follows.

Definition 2.3. Let Θ be an inner function with the canonical decomposition (2.1). Then define the *Ahern–Clark set* E_Θ [1] by:

$$E_\Theta := \left\{ \zeta \in \mathbb{T} : \sum_{n\geq 1} \frac{1 - |a_n|^2}{|\zeta - a_n|^2} + 2\int_{\mathbb{T}} \frac{d\mu(t)}{|t - \zeta|^2} < +\infty \right\}.$$

Note that $\mathbb{T} \setminus \sigma(\Theta) \subset E_\Theta$, but as we shall see later these sets can be distinct. Also recall that the set E_Θ is an open set relative to $\mathbb{T}$. When Θ is an inner function on $\mathbb{D}$ and ζ_0 is a point in $\mathbb{T}$, one says that Θ has an *angular derivative in the sense of Carathéodory* at ζ_0 if Θ has a non-tangential limit at ζ_0 of modulus 1 and in addition the derivative Θ' of Θ has a non-tangential limit at ζ_0. We have the following characterization of such points:

Proposition 2.4. *Let Θ be an inner function and $\zeta_0 \in \mathbb{T}$. Then the following assertions are equivalent:*

(i) *Θ has an angular derivative in the sense of Carathéodory at ζ_0.*

(ii) $\liminf_{\substack{z\in\mathbb{D}\\ z\to\zeta_0}} \frac{1-|\Theta(z)|^2}{1-|z|^2} < \infty$.

(iii) $\zeta_0 \in E_\Theta$.

The equivalence between (*i*) and (*ii*) follows from Carathéodory's Theorem [16] and for the equivalence between (*ii*) and (*iii*) see [7].

Finally, we need the notion of minimal sequences.

Definition 2.5. Let $(x_n)_{n\geq 1}$ be a sequence of a Banach space X. Then $(x_n)_{n\geq 1}$ is called *minimal* if for every $n \geq 1$, we have $x_n \notin \operatorname{span}\{x_k : k \neq n\}$. Moreover, $(x_n)_{n\geq 1}$ is called *uniformly minimal* if $\inf_{n\geq 1} \operatorname{dist}(x_n/\|x_n\|, \operatorname{span}\{x_k : k \neq n\}) > 0$.

A standard application of the Hahn–Banach theorem gives the following characterization of minimality and uniform minimality ([15], p. 131).

Proposition 2.6. *Let $(x_n)_{n\geq 1}$ be a sequence of a Banach space X.*

1. *$(x_n)_{n\geq 1}$ is minimal if and only if there exists a sequence $(x_n^*)_{n\geq 1}$ in X^* satisfying $< x_n, x_k^* >= \delta_{n,k}$ where $\delta_{n,k}$ is the Kronecker symbol. Such a sequence is called a* biorthogonal sequence *of $(x_n)_{n\geq 1}$.*
2. *$(x_n)_{n\geq 1}$ is uniformly minimal if and only if there exists a biorthogonal sequence $(x_n^*)_{n\geq 1}$ of $(x_n)_{n\geq 1}$ such that* $\sup_{n\geq 1} \|x_n\|\|x_n^*\| < \infty$.

3. Overcomplete sequences in reflexive Banach spaces

First of all, we recall a useful lemma.

Lemma 3.1 ([2])**.** *Let $(y_n)_{n\geq 1}$ be a sequence in a Banach space X satisfying the condition $\inf_{n\geq 1} \|y_n\| > 0$ and such that $(y_n)_{n\geq 1}$ tends weakly to 0. Then $(y_n)_{n\geq 1}$ has a subsequence $(y_{n_p})_{p\geq 1}$ which is a basic sequence, i.e., a Schauder basis in its span.*

Now, we can give a general necessary condition for overcompleteness.

Theorem 3.2. *Let X be a reflexive Banach space and $(x_n)_{n\geq 1} \subset X$ a bounded infinite sequence of pairwise distinct vectors. If $(x_n)_{n\geq 1}$ does not contain a uniformly minimal subsequence (so, in particular if $(x_n)_{n\geq 1}$ is overcomplete in X), then $(x_n)_{n\geq 1}$ is strongly relatively compact.*

Proof. Suppose that $(x_n)_{n\geq 1}$ is not strongly relatively compact. As $(x_n)_{n\geq 1}$ is bounded, we can find $y \in X$ and a subsequence $(x_{n_k})_{k\geq 1}$ tending weakly to y such that $\inf_{k\geq 1} \|x_{n_k} - y\| > 0$.

First case: $y = 0$. Using Lemma 3.1, we obtain a subsequence of $(x_{n_k})_{k\geq 1}$ which forms a basis in its span. In particular this subsequence is uniformly minimal, which proves that $(x_n)_{n\geq 1}$ is not overcomplete in X.

Second case: $y \neq 0$. Using once more Lemma 3.1, we can find a subsequence $(x_{n_{k_p}} - y)_{p\geq 1}$ which is a basic sequence. It follows that $\bigcap_{i\geq 1}$ span $\{x_{n_{k_p}} - y : p \geq i\} = \{0\}$. Indeed, since $(x_{n_{k_p}} - y)_{p\geq 1}$ is a basic sequence, for any $z \in \text{span}\{x_{n_{k_p}} - y : p \geq 1\}$, there exists a unique scalar sequence $(a_{n_p})_{p\geq 1}$ such that $z = \sum_{p\geq 1} a_{n_p}(x_{n_{k_p}} - y)$. The minimality of $(x_{n_{k_p}} - y)_{p\geq 1}$ implies that $a_{n_p} = 0$ for $p \geq 1$ if, in addition, $z \in \bigcap_{i\geq 1}$ span $\{x_{n_{k_p}} - y : p \geq i\} = \{0\}$.

Since $y \neq 0$, there exists $i_0 \in \mathbb{N}$ such that $y \notin$ span $\{x_{n_{k_p}} - y : p \geq i_0\}$. Hence we get that $\mathfrak{X} = (y, x_{n_{k_p}} - y)_{p\geq i_0}$ is a basic sequence, and thus a uniformly minimal sequence. Let $(y^*, (x_{n_{k_p}} - y)^*)_{p\geq i_0}$ be the biorthogonal sequence of $\mathfrak{X}$ such that $\sup_{p\geq i_0} \|x_{n_{k_p}} - y\| \|(x_{n_{k_p}} - y)^*\| < \infty$. One can check that $((x_{n_{k_p}} - y)^*)_{p\geq i_0}$ is also a biorthogonal sequence for $(x_{n_{k_p}})_{p\geq i_0}$. Since $(x_{n_{k_p}})_{p\geq i_0}$ is bounded and $\inf_{p\geq i_0} \|x_{n_{k_p}} - y\| > 0$, it follows that $\sup_{p\geq i_0} \|x_{n_{k_p}}\| \|(x_{n_{k_p}} - y)^*\| < \infty$. Therefore, $(x_{n_{k_p}})_{p\geq i_0}$ is uniformly minimal. In particular, $(x_n)_{n\geq 1}$ is not overcomplete, which ends the proof. ☐

In the rest of the section, we consider a reflexive complex Banach space X and Ω a domain in $\mathbb{C}$. Moreover suppose that the mapping $f \longmapsto f$ is well-defined and continuous from X into $\text{Hol}(\Omega)$ (the space of holomorphic function on Ω equipped with the topology of the uniform convergence on compact subsets). It is a well-known fact that the evaluations $E_\lambda : f \mapsto f(\lambda)$ for $\lambda \in \Omega$, are continuous. In this context, we can relax the hypothesis under which we can give a necessary condition for overcompleteness.

Theorem 3.3. *Suppose that $X \cap H^\infty(\Omega)$ is dense in X and let $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in Ω. If $(E_{\lambda_n})_{n\geq 1}$ does not contain a uniformly minimal subsequence (so, in particular if $(E_{\lambda_n})_{n\geq 1}$ is overcomplete in X^*), then $(E_{\lambda_n})_{n\geq 1}$ is strongly relatively compact.*

Proof. By Theorem 3.2, it suffices to show that $\sup_{n\geq 1} \|E_{\lambda_n}\| < +\infty$. Assume that $\sup_{n\geq 1} \|E_{\lambda_n}\| = +\infty$ and let $(y_n)_{n\geq 1}$ be defined by $y_n = E_{\lambda_n}/\|E_{\lambda_n}\|$. For all $f \in H^\infty(\Omega) \cap X$, we have $|\langle f, y_n\rangle| = |f(\lambda_n)|/\|E_{\lambda_n}\| \leq \|f\|_\infty/\|E_{\lambda_n}\| \to 0$ as $n \to \infty$. Since $H^\infty(\Omega) \cap X$ is dense in X, we get that $(y_n)_{n\geq 1}$ tends weakly to 0 and using Lemma 3.1, we find a subsequence $(y_{n_p})_{p\geq 1}$ which is a basic sequence and in particular is uniformly minimal. Hence $(E_{\lambda_{n_p}})_{p\geq 1}$ cannot be overcomplete in X^*. ☐

An obvious sufficient condition for overcompleteness is given by the following proposition, which follows immediately from the principle of isolated zeros.

Proposition 3.4. *Let $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in Ω. If the closure of $(\lambda_n)_{n\geq 1}$ is a subset of Ω, then $(E_{\lambda_n})_{n\geq 1}$ is overcomplete in X^*.*

4. Overcomplete sequences in K^p_Θ, $1 < p < \infty$

Before investigating overcompleteness in the reflexive model spaces K^p_Θ, it is natural to consider the problem in H^p where the reproducing kernels are $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, for $\lambda \in \mathbb{D}$.

Theorem 4.1. *Let $p \in (1,\infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$. The sequence $(k_{\lambda_n})_{n\geq 1}$ is overcomplete in H^p if and only if $\sup_{n\geq 1}|\lambda_n| < 1$.*

Proof. In order to apply the results of Section 3, set $\Omega = \mathbb{D}$, $X = H^q$ where p and q are conjugate. In this context, for $\lambda \in \mathbb{D}$, E_λ can be identified with k_λ. By Proposition 3.4, the condition $\sup_{n\geq 1}|\lambda_n| < 1$ implies that $(k_{\lambda_n})_{n\geq 1}$ is overcomplete in H^p. Conversely, by Theorem 3.3 the overcompleteness of $(k_{\lambda_n})_{n\geq 1}$ implies in particular that $\sup_{n\geq 1}\|k_{\lambda_n}\|_p < \infty$. Now, it is known ([15], p. 188) that $\|k_{\lambda_n}\|_p \asymp \frac{1}{(1-|\lambda_n|^2)^{1/q}}$. Therefore, $\sup_{n\geq 1}\|k_{\lambda_n}\|_p < \infty$ if and only if $\sup_{n\geq 1}|\lambda_n| < 1$. □

The study of sequences of reproducing kernels in the model spaces K^p_Θ is often considered under the geometrical condition $\sup|\Theta(\lambda_n)| < 1$ [12]. In this case we have the following result.

Theorem 4.2. *Let $p \in (1,\infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$. Suppose $\sup\limits_{n\geq 1}|\Theta(\lambda_n)| < 1$; then $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if $\sup\limits_{n\geq 1}|\lambda_n| < 1$.*

Proof. Set $\Omega = \mathbb{D}$, $X = K^q_\Theta$ where p and q are conjugate. For $\lambda \in \mathbb{D}$, the evaluation E_λ on X can be identified with $k_\Theta(\cdot,\lambda)$. By Proposition 3.4, the second condition is sufficient for the overcompleteness. By Theorem 3.3, overcompleteness implies in particular that $\sup_{n\geq 1}\|k_\Theta(\cdot,\lambda_n)\|_p < \infty$. But we have

$$\|k_\Theta(\cdot,\lambda_n)\|^p_p \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-\overline{\Theta(\lambda_n)}\Theta(e^{it})}{1-\overline{\lambda_n}e^{it}}\right|^p dt \geq (1-|\Theta(\lambda_n)|)^p \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\overline{\lambda_n}e^{it}|^p}dt.$$

Since $\sup_{n\geq 1}|\Theta(\lambda_n)| < 1$, there is a positive constant c such that $\|k_\Theta(\cdot,\lambda_n)\|^p_p \geq c\|k_{\lambda_n}\|^p_p$. It follows that $\sup_{n\geq 1}\|k_{\lambda_n}\|_p < \infty$, and hence $\sup_{n\geq 1}|\lambda_n| < 1$, as shown in the proof of Theorem 4.1. □

The study of bases of exponentials in $L^2(0,a)$ provided the original motivation for the development of the functional model approach in [12]. In the remainder of this section we discuss in more detail overcompleteness of exponentials. Some preliminaries are needed to translate the problem into the language of model spaces.

If $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, then we define the conformal mapping $\phi : \mathbb{C}_+ \to \mathbb{D}$ by $\phi(z) = \frac{z-i}{z+i}$. The operator $(Uf)(z) = \frac{1}{\pi(z+i)}f(\phi(z))$ maps H^2 unitarily onto the Hardy space $H^2(\mathbb{C}_+)$. The corresponding transformation for functions in H^∞ is $f \mapsto f\circ\phi$; it maps inner functions in $\mathbb{D}$ into inner functions in $\mathbb{C}_+$. We have then $UK_\Theta = H^2(\mathbb{C}_+) \ominus (\Theta \circ \phi) H^2(\mathbb{C}_+)$, and $U(k_\lambda^\Theta)$ is the reproducing kernel for the point $\phi(\lambda)$.

The Blaschke factor corresponding to $\mu \in \mathbb{C}_+$ is $b_\mu^+(z) = \frac{z-\mu}{z-\bar{\mu}}$ and the Blaschke product with zeros $(\mu_n)_{n\geq 1}$ is $B^+(z) = \prod_{n\geq 1} c_{\mu_n} b_{\mu_n}^+(z)$, the coefficients c_{μ_n} being chosen as to make all terms positive at $z = i$.

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform. Then $\mathcal{F}U$ maps H^2 unitarily onto $L^2(0, \infty)$. If $\Theta_a(z) = e^{a\frac{z+1}{z-1}}$, then $\mathcal{F}U$ maps K_{Θ_a} unitarily onto $L^2(0, a)$; the reproducing kernel $k_{\Theta_a}(\cdot, \lambda)$ ($\lambda \in \mathbb{D}$) is mapped (up to a nonzero constant) into $e^{i\mu t}$, where $\mu = -\overline{\phi^{-1}(\lambda)}$. Note that $|\Theta_a(\lambda_n)| = e^{-a \operatorname{Im} \mu_n}$ and thus $\sup_{n\geq 1} |\Theta_a(\lambda_n)| < 1$ if and only if $\inf_{n\geq 1} \operatorname{Im} \mu_n > 0$.

Therefore, the previous results can then be adapted to the case of exponentials $e^{i\mu_n t}$, with $\inf_{n\geq 1} \operatorname{Im} \mu_n > 0$. Nevertheless we will see that the hypothesis $\inf_{n\geq 1} \operatorname{Im} \mu_n > 0$ can be removed.

Theorem 4.3. *Let $a > 0$ and $(\mu_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{C}$. Then $(e^{i\mu_n t})_{n\geq 1}$ is overcomplete in $L^2(0, a)$ if and only if $\sup_{n\geq 1} |\mu_n| < \infty$.*

Proof. Consider the sequence $(\mu_n^*)_{n\geq 1}$ defined as follows:

$$\mu_n^* = \begin{cases} \mu_n & \text{if } \operatorname{Im} \mu_n \geq 0, \\ \overline{\mu_n} & \text{if } \operatorname{Im} \mu_n < 0. \end{cases}$$

We will prove that

$$(e^{i\mu_n t})_{n\geq 1} \text{ is overcomplete in } L^2(0, a) \iff (e^{i\mu_n^* t})_{n\geq 1} \text{ is overcomplete in } L^2(0, a). \tag{4.1}$$

First we remark that for every infinite subset Λ of $\mathbb{N}^*$, considering the anti-linear bijection T defined by $Tf(t) = \overline{f(-t + a)}$ on $L^2(0, a)$, we have:

$$(e^{i\mu_n t})_{n\in\Lambda} \text{ overcomplete in } L^2(0, a) \iff (e^{i\overline{\mu_n} t})_{n\in\Lambda} \text{ overcomplete in } L^2(0, a). \tag{4.2}$$

If $\{n \geq 1 : \operatorname{Im} \mu_n < 0\}$ is finite or $\{n \geq 1 : \operatorname{Im} \mu_n \geq 0\}$ is finite, (4.1) follows from (4.2) and that fact that adding or deleting a finite set does not change the overcompleteness property. Otherwise, (4.1) follows from (4.2) and the fact that the union of two overcomplete sequences is overcomplete.

Let $\delta > 0$. Now, considering the unitary operator U on $L^2(0, a)$ defined by $Uf(t) = e^{i\delta t} f(t)$, we have:

$$(e^{i\mu_n^* t})_{n\in\Lambda} \text{ overcomplete in } L^2(0, a) \iff (e^{i(\mu_n^* + \delta)t})_{n\in\Lambda} \text{ overcomplete in } L^2(0, a). \tag{4.3}$$

Since $\inf_{n\geq 1} \operatorname{Im}(\mu_n^* + \delta) > 0$, by Theorem 4.2 and the translation of our problem into the language of model spaces, we get:

$$(e^{i(\mu_n^* + \delta)t})_{n\in\Lambda} \text{ overcomplete in } L^2(0, a) \iff \sup_{n\geq 1} |\mu_n^* + \delta| < \infty \iff \sup_{n\geq 1} |\mu_n| < \infty.$$

Using (4.1) and (4.3), the proof of the theorem follows. □

5. Overcompleteness in K^p_Θ in terms of $\sigma(\Theta)$ and E_Θ

The following result shows that we may assume, in the sequel, that Θ is an inner function which is not a finite Blaschke product and thus $\sigma(\Theta)\cap\mathbb{T}\neq\emptyset$.

Proposition 5.1. *Let $p\in(1,\infty)$, $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{D}$ and let Θ be a finite Blaschke product. Then $(k_\Theta(.,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ.*

Proof. Set $\Omega=\{z\in\mathbb{C}:|z|<R\}$ where $\frac{1}{R}=\max\{z\in\mathbb{D}:\Theta(z)=0\}<1$ and $X=K^q_\Theta$ where p and q are conjugate. For $\lambda\in\mathbb{D}$, the evaluation E_λ on X can be identified with $k_\Theta(\cdot,\lambda)$. Since $\mathrm{clos}(\{\lambda_n:n\geq 1\})\subset\{z\in\mathbb{C}:|z|\leq 1\}\subset\Omega$, by Proposition 3.4, $(k_\Theta(.,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ. $\square$

Proposition 5.2. *Let $p\in[2,\infty)$, $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points in $\mathbb{D}$. We have the following sequence of implications:*

(SC) $\quad \inf_{n\geq 1}\mathrm{dist}(\lambda_n,\sigma(\Theta)\cap\mathbb{T})>0$

$\Downarrow$

(OVC) $\quad (k_\Theta(.,\lambda_n))_{n\geq 1}$ *is overcomplete in* K^p_Θ

$\Downarrow$

(NC_1) $\quad (k_\Theta(.,\lambda_n))_{n\geq 1}$ *is strongly relatively compact in* K^p_Θ

$\Downarrow$

(NC_2) $$\sup_{n\geq 1}\frac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2}<\infty$$

$\Downarrow$

(NC_3) $\quad \inf_{n\geq 1}\mathrm{dist}(\lambda_n,\mathbb{T}\setminus E_\Theta)>0$

Moreover, for $p\in(1,2)$, $(SC)\Rightarrow(OVC)\Rightarrow(NC_1)$ remains true.

Proof. Let $p\in(1,\infty)$. Set $\Omega=\mathbb{C}\setminus(\sigma(\Theta)\cup\{\frac{1}{\bar{z}}:\Theta(z)=0\})$ and $X=K^q_\Theta$ where p and q are conjugate. Using Proposition 2.2, X embeds continuously into $\mathrm{Hol}(\Omega)$. Then $(SC)\implies(OVC)$ and $(OVC)\implies(NC_1)$ applying respectively Proposition 3.4 and Theorem 3.3.

Now take $p\in[2,\infty)$. If (NC_1) is satisfied, then $\sup_{n\geq 1}\|k_\Theta(\cdot,\lambda_n)\|_p<\infty$, since relatively compact sets are bounded. Since $p\geq 2$ we have:

$$\sup_{n\geq 1}\frac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2}=\sup_{n\geq 1}\|k_\Theta(.,\lambda_n)\|_2^2\leq\sup_{n\geq 1}\|k_\Theta(.,\lambda_n)\|_p^2<\infty,$$

which implies that (NC_2) is satisfied. To prove that $(NC_2)\implies(NC_3)$, take ζ_0 be a limit point of $(\lambda_n)_{n\geq 1}$ in $\mathbb{T}$. Then since $\liminf_{\substack{z\in\mathbb{D}\\ z\to\zeta_0}}\frac{1-|\Theta(z)|^2}{1-|z|^2}\leq\sup_{n\geq 1}\frac{1-|\Theta(\lambda_n)|^2}{1-|\lambda_n|^2}<\infty$, it follows from Proposition 2.4 that $\zeta_0\in E_\Theta$. Since $\mathbb{T}\setminus E_\Theta$ is closed, there exists $\delta>0$ such that for every n, $\mathrm{dist}(\lambda_n,\mathbb{T}\setminus E_\Theta)\geq\delta$. $\square$

In the case where $E_\Theta=\mathbb{T}\setminus\sigma(\Theta)$, Proposition 5.2 provides a characterization of overcomplete sequence of reproducing kernels in K^p_Θ for $p\geq 2$. The next theorem provides an explicit class of inner functions Θ for which $E_\Theta = \mathbb{T}\backslash\sigma(\Theta)$. First, recall that a sequence $(\alpha_n)_{n\geq 1} \subset \mathbb{D}$ is a *Stolz sequence* if there exists a finite subset e of $\mathbb{T}$ and a positive constant $c >$ such that for all $n \geq 1$, $\operatorname{dist}(\alpha_n, e) \leq c \operatorname{dist}(\alpha_n, \mathbb{T})$. If $(\alpha_n)_{n\geq 1}$ is a Stolz sequence and ζ is a limit point of $(\alpha_n)_{n\geq 1}$ then there exists a subsequence $(\alpha_{n_p})_{p\geq 1}$ and a Stolz angle

$$\Delta_\zeta := \{z \in \mathbb{D} : |\arg(1-\overline{\zeta}z)| < \alpha, \, |z-\zeta| < \rho\} \qquad \left(0 < \alpha < \frac{\pi}{2}, \, \rho < 2\cos\alpha\right),$$

such that $(\alpha_{n_p})_{p\geq 1} \subset \Delta_\zeta$ and $\lim_{p\to+\infty} \alpha_{n_p} = \zeta$. In other words, this means that $(\alpha_{n_p})_{p\geq 1}$ converges nontangentially to ζ.

Theorem 5.3. *Let $p \in [2, \infty)$ and $(\lambda_n)_{n\geq 1}$ be an infinite sequence of pairwise distinct points of $\mathbb{D}$. Let Θ be an inner function with the canonical decomposition* (2.1). If $(a_n)_{n\geq 1}$ is a Stolz sequence and if μ has a finite support, then

$$(k_\Theta(\cdot, \lambda_n))_{n\geq 1} \text{ is overcomplete in } K^p_\Theta \Leftrightarrow (SC) \Leftrightarrow (NC_1) \Leftrightarrow (NC_2) \Leftrightarrow (NC_3).$$

Proof. By Proposition 5.2, it is sufficient to prove that $\mathbb{T} \setminus E_\Theta = \mathbb{T} \cap \sigma(\Theta)$, or, equivalently, that $\mathbb{T} \setminus \sigma(\Theta) = E_\Theta$. The inclusion $\mathbb{T} \setminus \sigma(\Theta) \subset E_\Theta$ is true for any inner function Θ and follows from the definitions of $\sigma(\Theta)$ and E_Θ. Note also that $E_\Theta = E_B \cap E_{S_\mu}$ and $\sigma(\Theta) = \sigma(B) \cup \sigma(S_\mu)$. Therefore it suffices to prove that $E_B \subset \mathbb{T} \setminus \sigma(B)$ and $E_{S_\mu} \subset \mathbb{T} \setminus \sigma(S_\mu)$. Write $\mu = \sum_{\lambda\in\operatorname{supp}(\mu)} c_\lambda \delta_\lambda$ where $\operatorname{supp}(\mu)$ is the support of μ, $c_\lambda > 0$ and δ_λ is the Dirac measure at λ. If $\zeta_0 \in E_{S_\mu}$, then $\int_{\mathbb{T}} \frac{d\mu(t)}{|t-\zeta_0|^2} < \infty$, that is, $\sum_{\lambda\in\operatorname{supp}(\mu)} \frac{c_\lambda}{|\lambda-\zeta_0|^2} < \infty$. Since the support of μ is finite, we conclude that $\inf_{\lambda\in\operatorname{supp}(\mu)} |\lambda-\zeta_0| \inf_{\lambda\in\sigma(S_\mu)} |\lambda-\zeta_0| > 0$, and thus $\zeta_0 \in \mathbb{T}\backslash\sigma(S_\mu)$.

It remains to check that $E_B \subset \mathbb{T} \setminus \sigma(B)$. Take $\zeta_0 \in E_B \cap \sigma(B)$. Since, $\zeta_0 \in E_B$, using Proposition 2.4, we know that B has a nontangential limit at ζ_0 with $|B(\zeta_0)| = 1$. Moreover, since $\zeta_0 \in \sigma(B) \cap \mathbb{T}$, there exists a sequence $(\alpha_n)_{n\geq 1}$ which tends to ζ_0 and satisfying $B(\alpha_n) = 0$ for $n \geq 1$. Since $(\alpha_n)_{n\geq 1}$ is a Stolz sequence, it follows that $B(\zeta_0) = 0$, which is absurd. □

Note that $k_\Theta(\cdot, \lambda_n)$ strongly converges in K^2_Θ if $\lambda_n \to \zeta \in E_\Theta$ nontangentially [1, 16]. Now, assuming that the sequence $(\lambda_n)_{n\geq 1}$ is a Stolz sequence, the conditions (NC_1), (NC_2) and (NC_3) are obviously equivalent with $p = 2$.

We now give a characterization of overcomplete sequences of reproducing kernels $(k_\Theta(\cdot, \lambda_n))_{n\geq 1}$ for some particular Blaschke products Θ whose sets of zeros are not necessarily Stolz sequences. If Θ is inner and $\alpha \in \mathbb{D}$, then we define $\Theta_\alpha = \frac{\Theta-\alpha}{1-\overline{\alpha}\Theta}$. Then Θ_α is also an inner function and according to theorem of Frostman, for almost all $\alpha \in \mathbb{D}$, it is actually a Blaschke product.

Proposition 5.4. *Let $p \in [2, \infty)$ and $(\lambda_n)_{n\geq 1}$ is an infinite sequence of pairwise distinct points of $\mathbb{D}$. Let Θ be a Blaschke product and suppose that there exists $\alpha \in \mathbb{D}$ and a singular inner function S with finite support such that $\Theta = S_\alpha$. Then*

$$(k_\Theta(\cdot, \lambda_n))_{n\geq 1} \textit{ is overcomplete in } K^p_\Theta \Leftrightarrow (SC) \Leftrightarrow (NC_1) \Leftrightarrow (NC_2) \Leftrightarrow (NC_3).$$

Proof. It is not difficult to check that the formula $U(f) = \sqrt{1-|\alpha|^2}\frac{f}{1-\overline{\alpha}\Theta}$ defines a unitary operator $U : K^p_S \to K^p_\Theta$ which maps (up to a nonzero constant) $k_S(\cdot,\lambda_n)$ into $k_\Theta(\cdot,\lambda_n)$. Therefore $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if $(k_S(\cdot,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_S. Moreover it follows from the very definition of the spectrum and Proposition 2.4 that $E_S = E_\Theta$ and $\sigma(S)\cap\mathbb{T} = \sigma(\Theta)\cap\mathbb{T}$. Applying Theorem 5.3, we conclude the proof of the proposition. □

Let $S(z) = e^{\frac{z-1}{z+1}}$, a singular inner function whose support is $\{-1\}$. For almost every $\alpha\in\mathbb{D}$, S_α is a Blaschke product. An easy calculation shows that the set of zeros of S_α, say $(a_n)_{n\geq 1}$, satisfies the equation

$$\left|a_n - \frac{\ln|\alpha|}{1-\ln|\alpha|}\right| = \frac{1}{1-\ln|\alpha|},$$

which means that the sequence $(a_n)_{n\geq 1}$ is on a circle tangent to $\mathbb{T}$ and thus $(a_n)_{n\geq 1}$ is not a Stolz sequence. Theorem 5.3 does not apply; however, Proposition 5.4 gives a criterion for overcompleteness in K_{S_α}.

In the introduction we have already mentioned the links between overcompleteness and minimality and uniform minimality. The next theorem gives the precise statements.

Theorem 5.5. *Let $p\in(1,\infty)$ and $(\lambda_n)_{n\geq 1}$ an infinite sequence of pairwise distinct points in $\mathbb{D}$.*

1. *The sequence $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is overcomplete in K^p_Θ if and only if it has no infinite subsequence which is minimal.*
2. *The sequence $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is strongly relatively compact in K^p_Θ if and only if it is bounded and has no infinite subsequence which is uniformly minimal.*

Proof. 1. By definition, an overcomplete sequence in a Banach space does not contain any infinite minimal subsequence. Conversely, if $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is not overcomplete, there exists an infinite subsequence $(k_\Theta(\cdot,\lambda_{n_p}))_{p\geq 1}$ which is not complete in K^p_Θ. By the Hahn–Banach theorem, there exists $g\in K^q_\Theta\setminus\{0\}$ such that $g(\lambda_{n_p}) = 0, p\geq 1$. Now, if m_p is the multiplicity of the zero at λ_{n_p} of g, the function Ψ_{n_p} defined by $\Psi_{n_p} = \frac{g}{(b_{\lambda_{n_p}})^{m_p}}$, with $b_{\lambda_{n_p}}(z) = \frac{z-\lambda_{n_p}}{1-\overline{\lambda_{n_p}}z}$, belongs to K^q_Θ ([15], p. 211). By construction $(\frac{\Psi_{n_p}}{\Psi_{n_p}(\lambda_{n_p})})_{p\geq 1}$ is a biorthogonal sequence of $(k_\Theta(\cdot,\lambda_{n_p}))_{p\geq 1}$. Therefore, the infinite subsequence $(k_\Theta(\cdot,\lambda_{n_p}))_{p\geq 1}$ is minimal.
2. By Theorem 3.2, if $(x_n)_{n\geq 1}$ is a bounded sequence in a reflexive Banach space which does not contain any uniformly minimal sequence is necessarily strongly relatively compact. Conversely, first note that

$$\|k_\Theta(\cdot,\lambda_n)\| \geq \left|\left\langle \frac{P_\Theta 1}{\|P_\Theta 1\|_q}, k_\Theta(\cdot,\lambda_n)\right\rangle\right| \frac{|1-\Theta(0)\overline{\Theta(\lambda_n)}|}{\|P_\Theta 1\|_q} \geq \frac{1-|\Theta(0)|}{\|P_\Theta 1\|_q}.$$

Therefore, there exists $c>0$ such that $\inf_{n\geq 1}\|k_\Theta(\cdot,\lambda_n)\|\geq c$. It follows that

$$\operatorname{dist}\left(\frac{k_\Theta(\cdot,\lambda_n)}{\|k_\Theta(\cdot,\lambda_n)\|}, \operatorname{span}\{k_\Theta(\cdot,\lambda_k) : k\neq n\}\right) \leq \inf_{k\neq n}\frac{\|k_\Theta(\cdot,\lambda_n)-k_\Theta(\cdot,\lambda_k)\|}{c}.$$

Thus, if $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is strongly relatively compact, it is clear that $(k_\Theta(\cdot,\lambda_n))_{n\geq 1}$ is bounded and cannot have a uniformly minimal infinite subsequence. □

By means of examples we obtain further information on the links between some of the conditions considered.

Proposition 5.6. *The condition* (NC_3) *is strictly weaker than* (NC_2)*; furthermore, the condition* (NC_1) *is strictly weaker than* (SC)*.*

Proof. We first construct an example where (SC) is not valid but (NC_1) is satisfied. Let

$$a_n = \frac{\frac{1}{n} + i(\frac{1}{2^n} - 1)}{\frac{1}{n} + i(\frac{1}{2^n} + 1)}$$

for $n \geq 1$. Since $1 - |a_n|^2 \asymp \frac{1}{2^n}$, $(a_n)_{n\geq 1}$ is a Blaschke sequence. Let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence which converges to -1 and which satisfies the Stolz condition. Denote by B the Blaschke product associated with $(\lambda_n)_{n\geq 1}$. Since $\sigma(B)\cap\mathbb{T} = \{-1\}$ and $\lim_{n\to\infty} a_n = -1$, applying Theorem 5.3, it follows that $(k_B(\cdot,a_n))_{n\geq 1}$ is not overcomplete in K_B^2. Therefore there exists a subsequence $(a_{n_p})_{p\geq 1}$ of $(a_n)_{n\geq 1}$ such that $(k_B(\cdot,a_{n_p}))_{p\geq 1}$ is not complete in K_B^2. By Lemma 97 of [15], this is equivalent to the condition that $\ker T_{\overline{B}\Theta_1} \neq \{0\}$ where Θ_1 is the Blaschke product associated with $(a_{n_p})_{p\geq 1}$. By Coburn's lemma [15, Lemma 43, p. 318], it follows that $\{0\} = \ker T^*_{\overline{B}\Theta_1} = \ker T_{\overline{\Theta_1}B}$. Applying once more Lemma 97 of [15], we deduce that the sequence $(k_{\Theta_1}(\cdot,\lambda_n))_{n\geq 1}$ is complete in $K^2_{\Theta_1}$. Obviously, we have $\sigma(\Theta_1) = \{-1\}$. Nevertheless we have $E_{\Theta_1} = \mathbb{T}$. Indeed, since $\mathbb{T}\setminus\sigma(\Theta_1) \subset E_{\Theta_1}$, we get $\mathbb{T}\setminus\{-1\} \subset E_{\Theta_1}$. By Definition 2.3, $-1 \in E_{\Theta_1}$ if and only if

$$\sum_{p\geq 1} \frac{1 - |a_{n_p}|^2}{|1 + a_{n_p}|^2} < \infty\,.$$

But this convergence follows from the estimate $1 - |a_{n_p}|^2 \asymp \frac{1}{2^{n_p}}$ and the existence of a constant $c > 0$ such that $|1 + a_{n_p}|^2 \geq \frac{c}{n_p^2}$. Therefore, we get $E_{\Theta_1} = \mathbb{T}$. Now, since $(\lambda_n)_{n\geq 1}$ is a Stolz sequence, $(k_{\Theta_1}(\cdot,\lambda_n))_{n\geq 1}$ converges in norm in $K^2_{\Theta_1}$, and then satisfies the condition (NC_1) but (SC) is not valid.

Moreover, if one takes Θ_1 defined as previously and $\lambda_n = a_n$, then $\Theta_1(\lambda_n) = 0$, which implies that (NC_2) does not hold, whereas (NC_3) is valid since $E_{\Theta_1} = \mathbb{T}$. □

Acknowledgments. The authors are grateful to the European Research Training Network in Analysis and Operators, Contract No. HPRN-CT-2000-00116, for financial support. They also wish to thank the referee for some valuable comments.

References

[1] P. Ahern and D. Clark, *Radial limits and invariant subspaces.* Amer. J. Math. **2** (1970), 332–342.

[2] C. Bessaga and A. Pelczyński, *On bases and unconditional convergence of series in Banach space*. Studia Math. **17** (1958), 151–164.

[3] A. Beurling and P. Malliavin, *On the closure of characters and the zeros of entire functions*. Acta Math. **118** (1967), 79–93.

[4] R.G. Douglas, H.S. Shapiro, and A.L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*. Ann. Inst. Fourier (Grenoble) **20** (1970), 37–76.

[5] P. L. Duren, *Theory of H^p spaces*. Academic Press, New York, 1970.

[6] K. M. Dyakonov, *Smooth functions in the range of a Hankel operator*. Indiana Univ. Math. J. **43** (1994), 805–838.

[7] ———, *Factorization of smooth analytic functions via Hilbert–Schmidt operators*. St. Petersburg Math. J. **8** (1997), no. 4, 543–569.

[8] ———, *Kernels of Toeplitz operators via Bourgain's factorization theorem*. J. Funct. Anal. **170** (2000), no. 1, 93–106.

[9] E. Fricain, *Propriétés géométriques des suites de noyaux reproduisants dans les espaces modèles*. Thesis, University of Bordeaux I, 1999.

[10] J. B. Garnett, *Bounded analytic functions*. Academic Press, New-York, 1981.

[11] E. Hayashi, *Classification of nearly invariant subspaces of the backward shift*. Proc. Amer. Math. Soc. **110** (1990), 441–448.

[12] S. V. Hruščev, N. K. Nikolski, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*. Complex Analysis and Spectral Theory (V. P. Havin and N. K. Nikolski, eds.), Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg New-York, 1981, pp. 214–335.

[13] V. Klee, *On the borelian and projective types of linear subspaces*. Math. Scand. **6** (1958), 189–199.

[14] P. Koosis, *Leçons sur le théorème de Beurling et Malliavin*. Les Publications CRM, Montréal, 1996.

[15] N. K. Nikolski, *Treatise on the shift operator*. Springer-Verlag, Berlin, 1986.

[16] D. E. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*. Lecture Notes in the Mathematical Sciences, vol. 10, J. Wiley and Sons, Inc., New York, 1994.

[17] I. Singer, *Bases in Banach spaces II*. Springer-Verlag, Berlin, 1981.

I. Chalendar and E. Fricain
Institut Girard Desargues
UFR de Mathématiques, Université Claude Bernard Lyon 1
F-69622 Villeurbanne Cedex, France
e-mail: `chalenda@igd.univ-lyon1.fr`
`fricain@igd.univ-lyon1.fr`

J. R. Partington
School of Mathematics, University of Leeds
Leeds LS2 9JT, United Kingdom
e-mail: `J.R.Partington@leeds.ac.uk`

Submitted: March 10, 2005

6.1.3 Référence [T3]

Auteurs
E. Fricain

Titre
Bases of reproducing kernels in de Branges spaces.

Paru dans
J. Funct. Anal., 226 (2005), no. 2, 373–405.

Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Journal of Functional Analysis 226 (2005) 373–405

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Bases of reproducing kernels in de Branges spaces

E. Fricain*

Institut Girard Desargues, UFR de Mathématiques, Université Claude Bernard Lyon 1, 21, avenue Claude Bernard, FR-69622 Villeurbanne Cedex, France

Received 23 August 2004; accepted 15 November 2004
Communicated by G. Pisier
Available online 5 March 2005

Abstract

This paper deals with geometric properties of sequences of reproducing kernels related to de-Branges spaces. If b is a nonconstant function in the unit ball of H^∞, and T_b is the Toeplitz operator, with symbol b, then the de-Branges space, $\mathscr{H}(b)$, associated to b, is defined by $\mathscr{H}(b) = (Id - T_b T_{\overline{b}})^{1/2} H^2$, where H^2 is the Hardy space of the unit disk. It is equipped with the inner product such that $(Id - T_b T_{\overline{b}})^{1/2}$ is a partial isometry from H^2 onto $\mathscr{H}(b)$. First, following a work of Ahern–Clark, we study the problem of orthogonal basis of reproducing kernels in $\mathscr{H}(b)$. Then we give a criterion for sequences of reproducing kernels which form an unconditional basis in their closed linear span. As far as concerns the problem of complete unconditional basis in $\mathscr{H}(b)$, we show that there is a dichotomy between the case where b is an extreme point of the unit ball of H^∞ and the opposite case.

MSC: 46E22; 47B32; 47B38; 30H05; 46J15; 46B15

Keywords: de-Branges spaces; Riesz bases; Reproducing kernels

1. Introduction

This paper is devoted to geometric properties of sequences of reproducing kernels in de-Branges spaces. These spaces, first studied by de Branges and Rovnyak [6], are

* Fax: +33 4 72 43 16 87.
E-mail address: fricain@igd.univ-lyon1.fr.

0022-1236/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2004.11.014

(not necessarily closed) subspaces of the Hardy space H^2 of the unit disk, $\mathbb{D}$. Recall first that

$$H^2 := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ analytic} : \sup_{0 \leqslant r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) < \infty \right\},$$

where $\mathbb{T}$ is the unit circle and *dm* is the normalized Lebesgue measure on $\mathbb{T}$. As usual, H^2 will be identified (via radial limits) with the space of $L^2 = L^2(\mathbb{T})$ functions whose negatively indexed Fourier coefficients vanish. Norm and inner product in L^2 or H^2 will be denoted by $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle_2$, respectively.

Let P_+ denote the orthogonal projection of L^2 onto H^2. For $\varphi \in L^\infty$, let T_φ denote the Toeplitz operator with symbol φ defined on H^2 by $T_\varphi f = P_+(\varphi f)$. The de-Branges space, $\mathcal{H}(\varphi)$, associated to φ consists of those H^2 functions which belong to the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_\varphi := \langle P_{Ker(Id - T_\varphi T_{\overline{\varphi}})^\perp} f_1, P_{Ker(Id - T_\varphi T_{\overline{\varphi}})^\perp} g_1 \rangle_2,$$

where $f = (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f_1$, $g = (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g_1$ and $P_{Ker(Id - T_\varphi T_{\overline{\varphi}})^\perp}$ denotes the orthogonal projection of H^2 onto $Ker(Id - T_\varphi T_{\overline{\varphi}})^\perp$. Note that $\mathcal{H}(\varphi)$ is contained contractively in H^2 and the inner product is defined in order to make $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$ a partial isometry of H^2 onto $\mathcal{H}(\varphi)$. The norm of $\mathcal{H}(\varphi)$ will be denoted by $\|\cdot\|_\varphi$.

For $\lambda \in \mathbb{D}$, we let k_λ denote the kernel function for the functional on H^2 of evaluation at λ; it is given by $k_\lambda(z) = (1 - \overline{\lambda} z)^{-1}$ ($z \in \mathbb{D}$) and satisfies $f(\lambda) = \langle f, k_\lambda \rangle_2$ ($f \in H^2$). Since $\mathcal{H}(\varphi)$ is contained contractively in H^2, the restriction to $\mathcal{H}(\varphi)$ of evaluation at λ is a bounded linear functional on $\mathcal{H}(\varphi)$. It is thus induced, relative to the inner product in $\mathcal{H}(\varphi)$, by a vector k_λ^φ in $\mathcal{H}(\varphi)$. It is easy to see ([19, (II-3)]) that $k_\lambda^\varphi = (Id - T_\varphi T_{\overline{\varphi}}) k_\lambda$ and

$$f(\lambda) = \langle f, k_\lambda^\varphi \rangle_\varphi,$$

for all $f \in \mathcal{H}(\varphi)$. From now on, b will be a nonconstant function in the unit ball of H^∞, that is an holomorphic and bounded function in $\mathbb{D}$, with $\|b\|_\infty \leqslant 1$. Then since $T_{\overline{b}} k_\lambda = \overline{b(\lambda)} k_\lambda$, we have

$$k_\lambda^b = (Id - T_b T_{\overline{b}}) k_\lambda = \frac{1 - \overline{b(\lambda)} b}{1 - \overline{\lambda} z}.$$

It is easy to see that $\mathcal{H}(b)$ is a closed subspace of H^2 if and only if T_b is a partial isometry. That happens if and only if b is an inner function, that is a function in H^∞ whose radial limits are of modulus one almost everywhere. Then $\mathcal{H}(b)$ is the orthogonal complement of the Beurling invariant subspace bH^2, the typical nontrivial invariant subspace of the shift operator S. Hence, the space $\mathcal{H}(b)$, with b inner, are the nontrivial invariant subspaces of the backward shift S^*. In this case, starting with the work of Hruscev, Nikolski and Pavlov, a whole direction of research has investigated geometric properties of reproducing kernels in $\mathcal{H}(b)$ (see [4,9–11]). One of the motivation to study geometric properties of reproducing kernels in $\mathcal{H}(b)$ is the link being with nontrigonometric exponentials systems. Recall that in the special case where $b(z) = \exp(a\frac{z+1}{z-1})$, $a > 0$, the reproducing kernels k_λ^b, with $\lambda \in \mathbb{D}$, arise as the range of the exponential functions $\exp(-i\overline{\mu}w)\chi_{(0,a)}$, with $\mu = i\frac{1+\lambda}{1-\lambda}$, under a natural unitary map going from $L^2(0,a)$ to $\mathcal{H}(b)$. Geometric properties of family of exponentials arise in many problems such as scattering theory, controllability and analysis of convolution equations (see [3,11] for details). We intend to provide a comprehensive treatment of geometric properties of reproducing kernels of $\mathcal{H}(b)$, emphasizing the parallel with the particular case where b is an inner function.

We now recall some basic definitions concerning geometric properties of sequences in an Hilbert space. For most of the definitions and facts below, one can use [14] as a main reference.

Let $\mathcal{H}$ be a complex Hilbert space. If $(x_n)_{n\geqslant 1} \subset \mathcal{H}$, we denote by $Span(x_n : n\geqslant 1)$ the closure of the linear hull generated by $(x_n)_{n\geqslant 1}$. The sequence $(x_n)_{n\geqslant 1}$ is called:

(Co) *complete* if $Span(x_n : n\geqslant 1) = \mathcal{H}$;

(M) *minimal* if for all $n\geqslant 1$, $x_n \notin Span(x_m : m \neq n)$;

(UM) *uniformly minimal* if $\inf_{n\geqslant 1} dist\left(\frac{x_n}{\|x_n\|}, Span(x_m : m \neq n)\right) > 0$;

(UBS) *an unconditional basis in its closed linear span* if every element $x \in Span(x_n : n\geqslant 1)$ can be uniquely decomposed in an unconditional convergent series $x = \sum_{n\geqslant 1} a_n x_n$;

(RS) *a Riesz basis in its closed linear span* if there are positive constants c, C such that

$$c\sum_{n\geqslant 1}|a_n|^2 \leqslant \left\|\sum_{n\geqslant 1} a_n x_n\right\|^2 \leqslant C\sum_{n\geqslant 1}|a_n|^2, \tag{1}$$

finite complex sequences $(a_n)_{n\geqslant 1}$;

(UB) *an unconditional basis of* $\mathcal{H}$ if it is complete and an unconditional basis in its closed linear span.

Obviously we have

$$(\text{UB}) \Longrightarrow (\text{RS}) \Longrightarrow (\text{USB}) \Longrightarrow (\text{UM}) \Longrightarrow (M).$$

In general, all the converse implications are false but Köthe–Topelitz theorem asserts that if $\|x_n\| \asymp 1$, then (USB) $\Longleftrightarrow$ (RS).

The *Gram matrix* of the sequence $(x_n)_{n\geqslant 1}$ is $\Gamma = (\langle x_n, x_m\rangle)_{n,m\geqslant 1}$. Unconditional basis are characterized by the fact that Γ defines an invertible operator on ℓ^2.

We recall some well-known facts concerning reproducing kernels in H^2. Let $\Lambda = (\lambda_n)_{n\geqslant 1}$ be a sequence of distinct points in $\mathbb{D}$ and denote by $x_n = \dfrac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$ the normalized reproducing kernel. Then we have

- $(k_{\lambda_n})_{n\geqslant 1}$ is minimal if and only if $(\lambda_n)_{n\geqslant 1}$ is Blaschke sequence (which means that $\sum_{n\geqslant 1}(1-|\lambda_n|) < \infty$). As usual, we denote by $B = B_\Lambda = \prod_{n\geqslant 1} b_{\lambda_n}$, where $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}\frac{\lambda_n - z}{1-\overline{\lambda}_n z}$.
- If $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence, then $(k_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(B)$.
- $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$ if and only if it is uniformly minimal which is equivalent to $(\lambda_n)_{n\geqslant 1}$ satisfies the Carleson condition

$$\inf_{n\geqslant 1} |B_n(\lambda_n)| > 0,$$

where $B_n = B/b_{\lambda_n}$; we will write in this case $(\lambda_n)_{n\geqslant 1} \in (C)$.

In this paper, we intend to study the property of unconditional basis for sequences of reproducing kernels in $\mathcal{H}(b)$. The study of the spaces $\mathcal{H}(b)$ frequently bifurcates into two cases depending b is an extreme point of the unit ball of H^∞ or not. We will show that for the property of unconditional basis in $\mathcal{H}(b)$, there exists a dichotomy between the two cases. Recall that de Leeuw and Rudin [7] proved that b is an extreme point of the unit ball of H^∞ (abbreviated by b is extreme) if and only if

$$\int_{\mathbb{T}} \log(1-|b|^2)\, dm = -\infty.$$

We now precise some notations that will be used in this paper. For a positive finite Borel measure ν on $\mathbb{T}$ and q a function in $L^2(\nu)$, we let

$$(K_\nu q)(z) := \int_{\mathbb{T}} \frac{q(e^{i\theta})}{1-e^{-i\theta}z}\, d\nu(e^{i\theta}), \qquad z \in \mathbb{C}\setminus\mathbb{T},$$

and we think of K_ν as a linear transformation of $L^2(\nu)$ into the space of holomorphic functions in $\mathbb{C}\setminus\mathbb{T}$. Moreover, we let $H^2(\nu)$ be the closed linear span of z^n, $n\geqslant 0$, (for the norm of $L^2(\nu)$) and we denote by Z_ν the operator of multiplication by the independant variable on $H^2(\nu)$. If ν is absolutely continuous and ρ is its Radon–Nikodym derivative with respect to normalized Lebesgue measure, we write K_ρ in place of K_ν, $H^2(\rho)$ in place of $H^2(\nu)$ and Z_ρ in place of Z_ν. Notice that if $q \in L^2(\rho)$ then $q\rho \in L^2$ and

$$K_\rho q = P_+(q\rho).$$

The plan of the paper is the following: the next section deals with the problem of orthogonal basis of reproducing kernels in $\mathcal{H}(b)$. As for the classical case where b is inner, this problem depends on the spectral study of a rank one perturbation of X^*, where $X = S^*{}_{|\mathcal{H}(b)}$. In particular, we prove (Corollary 2.2) that if b is not an inner function, then $\mathcal{H}(b)$ does not possess orthogonal basis of reproducing kernels. In Section 3, we give a criterion for the property of unconditional basis in its closed linear span (Theorems 3.1 and 3.2). Then we give some applications of this criterion, which are generalizations of results concerning the classical case. In Section 4, we study the case where b is extreme and prove that $Id - T_b T_{\overline{b}}$ is an invertible operator from $\mathcal{H}(u)$ onto $\mathcal{H}(b)$, with u an inner function, if and only if $dist(\overline{u}b, H^\infty) < 1$ and $dist(\overline{zu}b, H^\infty) = 1$ (Theorem 4.1). Then we use this result to characterize sequences $(k^b_{\lambda_n})_{n\geqslant 1}$ which form an unconditional basis of $\mathcal{H}(b)$ (Theorem 4.2). In Section 5, we study the case where b is not an extreme point. Contrary to the extreme case, we show that $\mathcal{H}(b)$ cannot possess unconditional basis of reproducing kernels (Corollary 5.1). Then, we get a characterization of completeness (Proposition 5.2) and finally making further assumption on the multiplier of $\mathcal{H}(b)$, we give a result concerning summation basis of reproducing kernels (Theorem 5.1).

2. Orthogonal bases of reproducing kernel

It is clear that if $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$, then the family $(k^b_{\lambda_n})_{n\geqslant 1}$ cannot be orthogonal. In some cases, it is possible, however, to consider reproducing kernels with poles on the unit circle. Let

$$b(z) = z^N \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right)$$

be the canonical factorization of b, where $\sum_n (1-|a_n|) < \infty$ and where μ is a positive Borel measure on $\mathbb{T}$ and set

$$E_b := \left\{\zeta \in \mathbb{T} : \sum_n \frac{1-|a_n|^2}{|\zeta - a_n|^2} + \int_{\mathbb{T}} \frac{d\mu(t)}{|t-\zeta|^2} < +\infty\right\}.$$

Recall that we say that b has an angular derivative in the sense of Carathéodory at the point λ of $\mathbb{T}$ if b and b' have a nontangential limit at λ and $|b(\lambda)| = 1$. Then we have the following criterion for the inclusion $k^b_\lambda := \dfrac{1 - \overline{b(\lambda)} b}{1 - \overline{\lambda} z} \in \mathcal{H}(b)$, $\lambda \in \mathbb{T}$.

Theorem A (*Ahern and Clark [2] and Sarason [19]*). *Let* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ *and* $\lambda \in \mathbb{T}$. *Then the following assertions are equivalent*:

(i) *there is a complex number* c *of unit modulus such that the function* $\dfrac{1 - \overline{c} b(z)}{1 - \overline{\lambda} z}$ *is in* $\mathcal{H}(b)$;

(ii) $\lambda \in E_b$;
(iii) $\liminf_{z\to\lambda} \dfrac{1-|b(z)|}{1-|z|} < +\infty$;
(iv) *b has an angular derivative in the sense of Carathéodory at* λ;
(v) *every function in* $\mathcal{H}(b)$ *has a nontangential limit at the point* λ.

Moreover, in this case, the number c is unique and is given by $c = b(\lambda) := \lim_{r\to 1} b(r\lambda)$. *If* $k_\lambda^b := \dfrac{1-\overline{b(\lambda)}b}{1-\bar{\lambda}z}$, *then for all* $f \in \mathcal{H}(b)$, *we have*

$$f(\lambda) = \langle f, k_\lambda^b\rangle_b.$$

Let now $\lambda, \lambda' \in E_b$, $\lambda \neq \lambda'$ and assume that $b(\lambda) = b(\lambda') = \alpha$, $|\alpha| = 1$. Then

$$\langle k_\lambda^b, k_{\lambda'}^b\rangle_b = k_\lambda^b(\lambda') = \frac{1-\overline{b(\lambda)}b(\lambda')}{1-\bar{\lambda}\lambda'} = 0.$$

So if we want to get an orthogonal sequence of reproducing kernel $(k_{\lambda_n}^b)_{n\geqslant 1}$, we have to choose sequence $(\lambda_n)_{n\geqslant 1}$ such that $(\lambda_n)_{n\geqslant 1} \subset E_b$ and $b(\lambda_n) = \alpha$, $n\geqslant 1$, $|\alpha| = 1$. Following the work of Ahern–Clark [1] concerning the classical case where b is an inner function, we proceed first to a study of rank one perturbations of X^* which are isometry, where $X = S^*_{|\mathcal{H}(b)}$. Recall that if $\varphi \in H^\infty$, then $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$ and the norm of $T_{\overline{\varphi}}$ as an operator in $\mathcal{H}(b)$ does not exceed $\|\varphi\|_\infty$. Hence $S^* = T_{\bar{z}}$ acts as a contraction in $\mathcal{H}(b)$ (see [19, (II-7)]). Recall also that we have (see [19, (II-9)])

$$X^*h = Sh - \langle h, S^*b\rangle_b b \qquad (h \in \mathcal{H}(b)). \tag{2}$$

2.1. Spectral properties of rank one perturbation of X^*

In this subsection, we proceed to an investigation of spectral properties of rank one perturbations of X^* which are isometry. Actually, our study goes beyond what is necessary for our treatment of the existence of orthogonal basis. First we give results concerning spectral properties for X^*. We will see that these properties depend whether b is an extreme point or not (for the analogue results for X, see [19, (IV-5), (V-7) and (V-8)]).

Lemma 2.1. *Let* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ *and* $h \in \mathcal{H}(b)$. *Then*

$$\|X^*h\|_b = \|h\|_b \iff \langle h, S^*b\rangle_b = 0.$$

Proof. Using relation (2), we get

$$XX^*h = S^*(Sh - \langle h, S^*b\rangle_b b) = h - \langle h, S^*b\rangle_b S^*b.$$

Hence

$$\|X^*h\|_b^2 = \langle XX^*h, h\rangle_b = \|h\|_b^2 - |\langle h, S^*b\rangle_b|^2, \tag{3}$$

which gives the lemma. □

Lemma 2.2. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Then $\sigma_p(X^*) \subset \mathbb{D}$.*

Proof. The inclusion $\sigma_p(X^*) \subset \overline{\mathbb{D}}$ follows from the fact that X^* is a contraction. Assume that there exist $\lambda \in \mathbb{T} \cap \sigma_p(X^*)$ and let $h \in \mathcal{H}(b)$, $h \not\equiv 0$ such that $X^*h = \lambda h$. Then $\|X^*h\|_b = \|h\|_b$ and Lemma 2.1 implies that $\langle h, S^*b\rangle_b = 0$. Hence $X^*h = Sh = \lambda h$, which gives that $\lambda \in \sigma_p(S)$, which is absurd and proves the lemma. □

Proposition 2.1. (a) *If b is extreme then*

$$\sigma_p(X^*) = \{\lambda \in \mathbb{D} : b(\lambda) = 0\} \quad \textit{and} \quad \sigma(X^*) = \sigma_p(X^*) \cup \sigma(b),$$

where $\sigma(b) := \mathbb{T} \setminus \rho(b)$ and $\rho(b)$ denotes the set of points $\zeta \in \mathbb{T}$ such that there exist an open arc I, $\zeta \in I$ and b can be continued analytically across I with $|b| = 1$ on I. Moreover if $b(\lambda) = 0$, then

$$Ker(X^* - \lambda Id) = \mathbb{C}\left(\frac{b}{z-\lambda}\right). \tag{4}$$

(b) *If b is nonextreme then $\sigma(X^*) = \overline{\mathbb{D}}$.*

Proof. Recall that X^* is completely nonunitary and if Θ_{X^*} denotes the characteristic operator function of X^*, in the theory of Sz-Nagy and Foias, we have (see [17]) $\Theta_{X^*} = b$ (in the extreme case) and $\Theta_{X^*} = \begin{pmatrix} b \\ a \end{pmatrix}$ (in the nonextreme case). Spectral properties of X^* follow now from a theorem of Sz-Nagy and Foias (see [20, Theorem 4.1. p. 247]). It just remains to check equality (4). Let $\lambda \in \mathbb{D}$, $b(\lambda) = 0$ and $f \in Ker(X^* - \lambda Id)$, $f \not\equiv 0$. Then using (2), we have

$$(z - \lambda)f = \langle f, S^*b\rangle_b b,$$

which implies that $f \in \mathbb{C}(b/(z-\lambda))$. Thus $Ker(X^* - \lambda Id) \subset \mathbb{C}\left(\frac{b}{z-\lambda}\right)$ and an argument of dimension shows that there is equality. □

Rank one perturbations of X^* that we will interest in are defined as follows.

Definition 2.1. If λ is a complex number of modulus 1, define the operator U_λ of $\mathcal{H}(b)$ by

$$U_\lambda := X^* + \lambda(1 - \lambda\overline{b(0)})^{-1} k_0^b \otimes S^* b.$$

Proposition 2.2. *The operator U_λ is an isometry of $\mathcal{H}(b)$. Moreover, it is a unitary operator of $\mathcal{H}(b)$ if and only if b is extreme and in this case, the U_λ are the only one-dimensional perturbations of X^* which are unitary.*

Proof. Denote by μ_λ the measure on $\mathbb{T}$ whose Poisson integral is the real part of $\dfrac{1+\bar{\lambda}b}{1-\bar{\lambda}b}$, denote by $V_{\bar{\lambda}b}$ the transformation defined on $L^2(\mu_\lambda)$ by $V_{\bar{\lambda}b}q(z) = (1 - \bar{\lambda}b(z))K_{\mu_\lambda}q(z)$, and finally denote by Z_{μ_λ} the operator of multiplication by the independant variable on $H^2(\mu_\lambda)$. We know (see [19, (III-8)]) that we have

$$U_\lambda = V_{\bar{\lambda}b} Z_{\mu_\lambda} V_{\bar{\lambda}b}^{-1} \tag{5}$$

and moreover $V_{\bar{\lambda}b}$ is an isometry of $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$. Hence U_λ is clearly an isometry of $\mathcal{H}(b)$. We see also that this isometry is onto if and only if Z_{μ_λ} is onto, which is equivalent to $H^2(\mu_\lambda) = L^2(\mu_\lambda)$. But a theorem of Szegö says that $H^2(\mu_\lambda) = L^2(\mu_\lambda)$ if and only if the Radon–Nikodym derivative of the absolutely continuous part of μ_λ with respect to normalized Lebesgue measure is not log-integrable. Now a theorem of Fatou shows that this Radon–Nikodym derivative equals to $\dfrac{1-|b|^2}{|1-\bar{\lambda}b|^2}$. Since $\log|1-\bar{\lambda}b|^2$ is always integrable (being the logarithm of the modulus of the H^∞ function $1-\bar{\lambda}b$), we see that $H^2(\mu_\lambda) = L^2(\mu_\lambda)$ if and only if $\log(1-|b|^2)$ is not integrable, which is exactly the condition that b is extreme.

Now, assume that b is extreme and that $U := X^* + h \otimes k$, $h, k \in \mathcal{H}(b)$, is a unitary operator. If $f \perp k$, then we have $Uf = X^* f$, which gives $\|X^* f\|_b = \|f\|_b$. Lemma 2.1 implies that $f \perp S^* b$. It follows that there exist $c \in \mathbb{C}$ such that $k = cS^* b$, which gives $U = X^* + h_1 \otimes S^* b$, with $h_1 = \bar{c}h$. Taking the adjoint of this relation, we see that if $f \perp h_1$, then $\|Xf\|_b = \|f\|_b$. Now recall (see [19, (VIII-4)]) that $\|Xf\|_b^2 = \|f\|_b^2 - |f(0)|^2$, which gives $f(0) = 0$, that is $f \perp k_0^b$. It follows that there exist $c_1 \in \mathbb{C}$ such that $h_1 = c_1 k_0^b$ and thus $U = X^* + c_1 k_0^b \otimes S^* b$. It remains to show that there exist $\lambda \in \mathbb{T}$ such that $c_1 = \lambda(1 - \lambda\overline{b(0)})^{-1}$. Notice that for all $f \in \mathcal{H}(b)$, we have

$$\|f\|_b^2 = \|Uf\|_b^2 = \|X^* f\|_b^2 + |c_1|^2 |\langle f, S^* b\rangle_b|^2 \|k_0^b\|_b^2 + 2\,\mathrm{Re}\left(c_1 \langle f, S^* b\rangle_b \langle k_0^b, X^* f\rangle_b\right).$$

In particular for $f = S^* b$, using relation (3), we get

$$0 = -\|S^* b\|_b^2 + |c_1|^2 \|S^* b\|_b^2 (1 - |b(0)|^2) + 2\,\mathrm{Re}(c_1 \overline{(X^* S^* b)(0)}).$$

Since $X^*S^*b = SS^*b - \|S^*b\|_b^2 b$, it follows that $(X^*S^*b)(0) = -\|S^*b\|_b^2 b(0)$, which implies that

$$0 = -1 + |c_1|^2(1 - |b(0)|^2) - 2\,\mathrm{Re}(c_1\overline{b(0)}).$$

Now define $\lambda := \overline{c_1}^{-1} + b(0)$. Using the previous equality, easy computations show that $\lambda \in \mathbb{T}$ and $c_1 = \lambda(1 - \lambda\overline{b(0)})^{-1}$, which ends the proof of the proposition. □

The following lemma is a generalization of a result of Ahern–Clark [1] for the case where b is an inner function.

Lemma 2.3. *Let $\zeta \in \mathbb{T}$. The following assertions are equivalent*:

(i) *b has an angular derivative in the sense of Carathéodory at ζ*;
(ii) *$k_0^b \in Im(Id - \overline{\zeta}X^*)$.*

Moreover, in that case, we have $(Id - \overline{\zeta}X^*)k_\zeta^b = k_0^b$.

Proof. (i) $\Longrightarrow$ (ii): Since b has angular derivative in the sense of Carathéodory at ζ, we know (see [19, (VI-4)]) that k_z^b tends to k_ζ^b in norm as z tends nontangentially to ζ. Notice we have

$$\begin{aligned}\|(Id - \overline{z}X^*)k_z^b - (Id - \overline{\zeta}X^*)k_\zeta^b\| &= \|(Id - \overline{z}X^*)(k_z^b - k_\zeta^b) \\ &\quad + ((Id - \overline{z}X^*) - (Id - \overline{\zeta}X^*))k_\zeta^b\| \\ &\leqslant 2\|k_z^b - k_\zeta^b\| + |z - \zeta|\|X^*\|\|k_\zeta^b\|.\end{aligned}$$

Hence $(Id - \overline{z}X^*)k_z^b$ tends to $(Id - \overline{\zeta}X^*)k_\zeta^b$ as z tends nontangentially to ζ. Moreover we have

$$(Id - \overline{z}X^*)k_z^b = k_0^b,$$

(see [19, (V-8)]) which implies that $k_0^b = (Id - \overline{\zeta}X^*)k_\zeta^b$.

(ii) $\Longrightarrow$ (i): Assume that there exists $g \in \mathcal{H}(b)$ such that $k_0^b = (Id - \overline{\zeta}X^*)g$. We have

$$\begin{aligned}k_z^b &= (Id - \overline{z}X^*)^{-1}k_0^b \\ &= (Id - \overline{z}X^*)^{-1}(Id - \overline{\zeta}X^*)g \\ &= g + (\overline{z} - \overline{\zeta})(Id - \overline{z}X^*)X^*g,\end{aligned}$$

which gives that

$$\|k_z^b\| \leqslant \|g\|\left(1 + |z - \zeta|\|(Id - \overline{z}X^*)^{-1}\|\|X^*\|\right).$$

Using the fact that $\|(Id-\overline{z}X^*)^{-1}\|\leqslant(1-|z|)^{-1}$, we deduce that

$$\|k_z^b\|\leqslant\|g\|\left(1+\frac{|z-\zeta|}{1-|z|}\|X^*\|\right).$$

As $|z-\zeta|/(1-|z|)$ stays bounded as z tends nontangentially to ζ, we get that $\|k_z^b\|$ stays bounded as z tends nontangentially to ζ, which by Theorem A, implies that b has an angular derivative in the sense of Carathéodory at ζ. $\square$

Since U_λ is an isometry, its point spectrum is located on the unit circle. The notion of angular derivative will lead us to characterize it. This result was obtained by Ahern–Clark [1] for the case where b is inner.

Theorem 2.1. *Let $\lambda\in\mathbb{T}$. Then a complex number ζ is an eigenvalue of U_λ if and only if b has an angular derivative in the sense of Caratheodory at ζ and $b(\zeta)=\lambda$. Moreover we have $Ker(U_\lambda-\zeta Id)=\mathbb{C}k_\zeta^b$.*

Proof. Assume that b has an angular derivative in the sense of Caratheodory at ζ and $b(\zeta)=\lambda$. Using Lemma 2.3, we have $k_0^b=(Id-\overline{\zeta}X^*)k_\zeta^b$. Hence

$$(U_\lambda-\zeta Id)k_\zeta^b=(X^*-\zeta Id)k_\zeta^b+\lambda(1-\lambda\overline{b(0)})^{-1}\langle k_\zeta^b,S^*b\rangle_b k_0^b$$

$$=-\zeta k_0^b+\lambda(1-\lambda\overline{b(0)})^{-1}\langle k_\zeta^b,S^*b\rangle_b k_0^b.$$

Take now a sequence $(z_n)_n$ which tends nontangentially to ζ and notice that

$$\langle S^*b,k_\zeta^b\rangle_b=\lim_{n\to+\infty}\langle S^*b,k_{z_n}^b\rangle_b=\lim_{n\to+\infty}\frac{b(z_n)-b(0)}{z_n}=\frac{\lambda-b(0)}{\zeta}.$$

That implies

$$(U_\lambda-\zeta Id)k_\zeta^b=-\zeta k_0^b+\lambda(1-\lambda\overline{b(0)})^{-1}\zeta(\overline{\lambda}-\overline{b(0)})k_0^b=0,$$

which proves that $\zeta\in\sigma_p(U_\lambda)$ and $\mathbb{C}k_\zeta^b\subset Ker(U_\lambda-\zeta Id)$.

Reciprocally, let $\zeta\in\sigma_p(U_\lambda)$ and $f\in\mathcal{H}(b)$, $f\not\equiv 0$ such that $(U_\lambda-\zeta Id)f=0$. Then, we have $(X^*-\zeta Id)f=-\lambda(1-\lambda\overline{b(0)})^{-1}\langle f,S^*b\rangle_b k_0^b$. Notice that if $\langle f,S^*b\rangle_b=0$, then $\zeta\in\sigma_p(X^*)$, which is absurd thanks to Lemma 2.2. Hence $\langle f,S^*b\rangle_b\neq 0$, and there exists c$\in\mathbb{C}$, $c\neq 0$, such that $k_0^b=(Id-\overline{\zeta}X^*)(cf)$. Lemma 2.3 implies that b has an angular derivative in the sense of Carathédory at ζ and $k_0^b=(Id-\overline{\zeta}X^*)k_\zeta^b$. We deduce that $k_\zeta^b-cf\in Ker(X^*-\overline{\zeta}Id)$ and Lemma 2.2 implies that $k_\zeta^b=cf$.

Hence $k_\zeta^b \in Ker(U_\lambda - \zeta Id)$. But previous computations show that

$$(U_\lambda - \zeta Id)k_\zeta^b = \left(-\zeta + \lambda(1-\lambda\overline{b(0)})^{-1}\frac{\overline{b(\zeta)}-\overline{b(0)}}{\overline{\zeta}}\right)k_0^b,$$

which implies that $\lambda(\overline{b(\zeta)} - \overline{b(0)})(1-\lambda\overline{b(0)})^{-1} = 1$, hence $b(\zeta) = \lambda$. Moreover as $k_\zeta^b = cf$, we have that $Ker(U_\lambda - \zeta Id) \subset \mathbb{C}k_\zeta^b$. □

As in the classical case where b is inner, we can deduce from this result the description of the spectrum of U_λ.

Corollary 2.1. *Let* $\lambda \in \mathbb{T}$.

(a) *If b is extreme, then* $\sigma(U_\lambda) \subset \mathbb{T}$ *and*

$$\zeta \in \sigma(U_\lambda) \Longleftrightarrow \begin{array}{ll} \text{(i)} & \zeta \in \sigma(b) \\ \textit{or} & \\ \text{(ii)} & \zeta \in {}^c\sigma(b) \textit{ and } b(\zeta) = \lambda. \end{array}$$

(b) *If b is nonextreme, then* $\sigma(U_\lambda) = \overline{\mathbb{D}}$.

Proof. (a) Assume that b is extreme. Proposition 2.2 shows that U_λ is unitary, so $\sigma(U_\lambda) \subset \mathbb{T}$. Let $\zeta \in \sigma(U_\lambda)$, $\zeta \in {}^c\sigma(b)$. Using the fact that $\sigma(b) = \sigma(X^*) \cap \mathbb{T}$ (see Proposition 2.1) and the fact that U_λ is a rank-one perturbation of X^*, we deduce that $U_\lambda - \zeta Id$ is a Fredholm operator of index 0. As $\zeta \in \sigma(U_\lambda)$, we get that $\zeta \in \sigma_p(U_\lambda)$ and Theorem 2.1 implies that $b(\zeta) = \lambda$.

Reciprocally let $\zeta \in \sigma(b)$ and assume that $\zeta \in {}^c\sigma(U_\lambda)$. Using once more the fact that U_λ is a rank-one perturbation of X^*, we get that $X^* - \zeta Id$ is a Fredholm operator of index 0. Thanks to Lemma 2.2, we have that $Ker(X^* - \zeta Id) = \{0\}$. Hence $X^* - \zeta Id$ is invertible, which gives $\zeta \in {}^c\sigma(X^*) = {}^c\sigma(b)$, which is absurd.

On the other hand, let $\zeta \in {}^c\sigma(b)$ and $b(\zeta) = \lambda$. By definition, there exist an open arc I, $\zeta \in I$ such that b can be continued analytically across I and $|b| = 1$ on I. In particular, b has an angular derivative in the sense of Caratheodory at ζ and since $b(\zeta) = \lambda$, thanks to Theorem 2.1, we get that $\zeta \in \sigma_p(U_\lambda) \subset \sigma(U_\lambda)$.

(b) Assume that b is nonextreme. Since U_λ is an isometry, we clearly have $\sigma(U_\lambda) \subset \overline{\mathbb{D}}$. Now let $\zeta \in \mathbb{D}$ and assume that $\zeta \in {}^c\sigma(U_\lambda)$. Recall that when b is nonextreme, then $b \in \mathcal{H}(b)$ and the space $\mathcal{H}(b)$ is invariant under the unilateral shift S (see [19, (IV-5)]). Hence if we denote by $Y := S_{|\mathcal{H}(b)}$, we have, using formula (2),

$$U_\lambda = X^* + \lambda(1-\lambda\overline{b(0)})^{-1}k_0^b \otimes S^*b = Y - b \otimes S^*b + \lambda(1-\lambda\overline{b(0)})^{-1}k_0^b \otimes S^*b.$$

Thus we get that $Y - \zeta Id$ is a Fredholm operator of index 0. Since $Ker(Y - \zeta Id) = \{0\}$, the following lemma gives a contradiction; hence $\mathbb{D} \subset \sigma(U_\lambda)$, which ends the proof of the corollary. □

Lemma 2.4. *Assume that b is nonextreme and let $Y = S_{|\mathcal{H}(b)}$. Then for $\mu \in \mathbb{D}$, we have $Ker(Y^* - \overline{\mu}Id) = \mathbb{C}k_\mu^b$.*

Proof. For all $f \in \mathcal{H}(b)$, we have

$$\langle Y^* k_\mu^b, f\rangle_b = \langle k_\mu^b, Yf\rangle_b = \langle k_\mu^b, zf\rangle_b = \overline{\mu \overline{f(\mu)}} = \langle \overline{\mu} k_\mu^b, f\rangle_b,$$

which proves that $Y^* k_\mu^b = \overline{\mu} k_\mu^b$. Hence $k_\mu^b \in Ker(Y^* - \overline{\mu}Id)$.

Let now $f \in Ker(Y^* - \overline{\mu}Id)$ and $g \in (\mathbb{C}k_\mu^b)^\perp$. Define $h := \frac{g}{z-\mu}$. Since $g(\mu) = 0$, we get that $h \in \mathcal{H}(b)$ (see [19, (II-8)]). Hence

$$g = (z-\mu)h = (Y - \mu Id)h \in (Y - \mu Id)\mathcal{H}(b) \subset (Ker(Y^* - \overline{\mu}Id))^\perp.$$

That implies $\langle f, g\rangle_b = 0$, and thus $f \in ((\mathbb{C}k_\mu^b)^\perp)^\perp = \mathbb{C}k_\mu^b$. □

2.2. Orthogonal bases of reproducing kernels in $\mathcal{H}(b)$

Let $\lambda \in \mathbb{T}$. The function $\dfrac{1 - |b(z)|^2}{|\lambda - b(z)|^2}$ is a nonnegative harmonic function in $\mathbb{D}$, so it can be represented as a Poisson integral

$$\frac{1 - |b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{\zeta} z|^2}\, d\mu_\lambda(\zeta); \quad z \in \mathbb{D},$$

where μ_λ is a nonnegative Borel measure in $\mathbb{T}$.

The following result gives a criterion in terms of the measure μ_λ for the existence of an orthogonal basis of reproducing kernels in $\mathcal{H}(b)$. In the particular case where b is inner, this result was obtained by Ahern–Clark [1].

Theorem 2.2. *Let $\lambda \in \mathbb{T}$. The following assertions are equivalent*:

(i) *the family $\{k_\zeta^b : \zeta \in E_b, b(\zeta) = \lambda\}$ forms an orthogonal basis of $\mathcal{H}(b)$*;

(ii) *the measure μ_λ is purely atomic*.

Proof. If $\zeta \in E_b$, $b(\zeta) = \lambda$, then Theorem 2.1 implies that $U_\lambda k_\zeta^b = \zeta k_\zeta^b$. Hence the family $\{k_\zeta^b : \zeta \in E_b, b(\zeta) = \lambda\}$ forms an orthogonal system of eigenvectors of U_λ in $\mathcal{H}(b)$.

(i) $\Longrightarrow$ (ii): Since $\mathcal{H}(b)$ is separable, $\{\zeta \in E_b : b(\zeta) = \lambda\}$ is countable. Denote by $(\zeta_n)_{n \geqslant 1} := \{\zeta \in E_b : b(\zeta) = \lambda\}$. Since $V_{\overline{\lambda}b}$ is an isometry from $H^2(\mu_\lambda)$ onto $\mathcal{H}(b)$, the family $(V_{\overline{\lambda}b}^{-1} k_{\zeta_n}^b)_{n \geqslant 1}$ is an orthogonal basis of $H^2(\mu_\lambda)$. Moreover, using (5), we have

$$Z_{\mu_\lambda} V_{\overline{\lambda}b}^{-1} k_{\zeta_n}^b = V_{\overline{\lambda}b}^{-1} U_\lambda k_{\zeta_n} = \zeta_n V_{\overline{\lambda}b}^{-1} k_{\zeta_n}^b.$$

That means that $H^2(\mu_\lambda)$ has an orthogonal basis of eigenvectors of Z_{μ_λ}, the operator of multiplication by the independant variable on $L^2(\mu_\lambda)$. It is now a well-known fact that implies that $\mu_\lambda = \sum_{n\geqslant 1} a_n \delta_{\{\zeta_n\}}$, $a_n := \mu_\lambda(\zeta_n)$.

(ii) $\Longrightarrow$ (i): Assume that μ_λ is purely atomic, that is $\mu_\lambda = \sum_{n\geqslant 1} a_n \delta_{\{\zeta_n\}}$, with $a_n = \mu_\lambda(\{\zeta_n\}) > 0$. In particular, for all f in $H^2(\mu_\lambda) = L^2(\mu_\lambda)$, we have

$$\|f\|^2 = \sum_{n\geqslant 1} a_n |f(\zeta_n)|^2.$$

Using this equality, it is easy to see that $(\chi_{\{\zeta_n\}})_{n\geqslant 1}$ is an orthogonal basis of $L^2(\mu_\lambda)$ and we get that $(V_{\bar\lambda b}\chi_{\{\zeta_n\}})_{n\geqslant 1}$ is an orthogonal basis of $\mathcal{H}(b)$. Using once more (5), we have $U_\lambda(V_{\bar\lambda b}\chi_{\{\zeta_n\}}) = V_{\bar\lambda b} Z_{\mu_\lambda}\chi_{\{\zeta_n\}} = \zeta_n V_{\bar\lambda b}\chi_{\{\zeta_n\}}$. Theorem 2.1 implies that $\zeta_n \in E_b$, $b(\zeta_n) = \lambda$ and there exits $c_n \in \mathbb{C}^*$ such that $V_{\bar\lambda b}\chi_{\{\zeta_n\}} = c_n k^b_{\zeta_n}$. Hence $(k^b_{\zeta_n})_{n\geqslant 1}$ is an orthogonal basis of $\mathcal{H}(b)$. It remains to notice that $\{\zeta \in E_b : b(\zeta) = \lambda\} = (\zeta_n)_{n\geqslant 1}$. The inclusion $(\zeta_n)_{n\geqslant 1} \subset \{\zeta \in E_b : b(\zeta) = \lambda\}$ has already been proved. Assume that there exists $\zeta \in E_b$, $b(\zeta) = \lambda$, $\zeta \neq \zeta_n$, $n \geqslant 1$. Theorem A implies that $k^b_\zeta \in \mathcal{H}(b)$ and

$$\langle k^b_{\zeta_n}, k^b_\zeta \rangle = \frac{1 - \overline{b(\zeta_n)}b(\zeta)}{1 - \overline{\zeta_n}\zeta} = 0.$$

Hence $k^b_\zeta \in \mathcal{H}(b) \ominus Span(k^b_{\zeta_n} : n \geqslant 1)$, which is absurd. $\square$

Corollary 2.2. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that b is not an inner function. Then $\mathcal{H}(b)$ does not have an orthogonal basis of reproducing kernels.*

Proof. Let $\lambda \in \mathbb{T}$. Since b is not an inner function, there exists $A \in Bor(\mathbb{T})$, $m(A) > 0$ such that for all $\zeta \in A$, $|b(\zeta)| \neq 1$. Now if

$$\frac{1 - |b(z)|^2}{|\lambda - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar\zeta z|^2}\, d\mu_\lambda(\zeta); \qquad z \in \mathbb{D},$$

and if $\mu_\lambda^{(a)}$ denotes the absolutely component part of the measure μ_λ, we know that $\dfrac{1 - |b(\zeta)|^2}{|\lambda - b(\zeta)|^2} = \dfrac{d\mu_\lambda^{(a)}}{dm}(\zeta)$, for almost $\zeta \in A$ with respect to the Lebesgue measure. Hence $\mu_\lambda^{(a)} \neq 0$ and the measure μ_λ cannot be purely atomic. Theorem 2.2 implies that $\mathcal{H}(b)$ does not have an orthogonal basis of reproducing kernels. $\square$

3. Unconditional bases of reproducing kernels in $\mathcal{H}(b)$

Let us first remark that if $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $(k^b_{\lambda_n})_{n\geqslant 1}$ is minimal, then $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distinct points. Thus from now on, we assume that $(\lambda_n)_{n\geqslant 1}$ is a

Blaschke sequence of distinct points of the unit disk and we denote by B the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$.

The problem of unconditional basis of reproducing kernels of $\mathcal{H}(b)$, in the case where b is inner, was solved by Hruscev et al. [11]. They prove that if b is inner and $\sup_{n\geqslant 1}|b(\lambda_n)| < 1$, then $(k^b_{\lambda_n})_{n\geqslant 1}$ is an unconditional basis in its closed linear span (resp. of $\mathcal{H}(b)$) if and only if $(\lambda_n)_{n\geqslant 1} \in (C)$, $dist(\overline{B}b, H^\infty) < 1$ (resp. plus $dist(\overline{B}b, H^\infty) < 1$). The key point to get this criterion is the following formulae:

$$bJT_{b\overline{B}}J\overline{B} = Id_{H^2_-} \oplus P_{b|\mathcal{H}(B)}, \quad \text{in the space } BH^2_- = H^2_- \oplus \mathcal{H}(B), \tag{6}$$

where $P_b = (Id - T_bT_{\overline{b}})^{1/2}$ is the orthogonal projection of H^2 onto $\mathcal{H}(b) = H^2 \ominus bH^2$, $Jg = \overline{zg}$, $g \in L^2(\mathbb{T})$ (see [16, Lemma 4.4.4. p. 309]).

In the general case, formula (6) is no longer true. However, we will see that it can be possible to get some similar results for unconditional basis of reproducing kernels in their closed linear span. For complete unconditional basis, as we will see in Sections 4 and 5, the solution breaks down into two cases depending whether b is extreme or not.

From now on, we denote by $x_n := \dfrac{k_{\lambda_n}}{\|k_{\lambda_n}\|_2}$ (resp. by $x^b_n := \dfrac{k^b_{\lambda_n}}{\|k^b_{\lambda_n}\|_b}$) the normalized reproducing kernels of H^2 (resp. of $\mathcal{H}(b)$) associated to a sequence $(\lambda_n)_{n\geqslant 1}$.

3.1. A criterion for unconditional basis in its closed linear span

The next result shows that Carleson's condition is necessary for a sequence of reproducing kernels of $\mathcal{H}(b)$ to be an unconditional basis in its closed linear span. The proof is similar to the classical case where b is inner (see [14, Lecture VIII, p. 200]) and is left to the reader.

Proposition 3.1. *Assume that $(k^b_{\lambda_n})_{n\geqslant 1}$ forms an unconditional basis in its closed linear span. Then $(\lambda_n)_{n\geqslant 1} \in$ (C).*

The next result is the first step in our study of unconditional basis property.

Theorem 3.1. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\sup_{n\geqslant 1}|b(\lambda_n)| < 1. \tag{7}$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditional basis of* $\mathcal{H}(b)$ (*resp. in its closed linear span*);

(ii) (a) $(\lambda_n)_{n\geqslant 1} \in (C)$, (b) *the operator* $Id - T_bT_{\overline{b}}$ *is an isomorphism of* $\mathcal{H}(B)$ *onto* $\mathcal{H}(b)$ (*resp. onto its range*).

Proof. (i) $\Longrightarrow$ (ii): Proposition 3.1 implies that $(\lambda_n)_{n\geqslant 1} \in (C)$ and thus $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$. Moreover, condition (7) shows that (i) is equivalent to the fact that $((1-|b(\lambda_n)|^2)^{1/2}x_n^b)_{n\geqslant 1}$ forms a Riesz basis. But

$$(Id - T_bT_{\overline{b}})x_n = (1-|\lambda_n|^2)^{1/2}k_{\lambda_n}^b = (1-|b(\lambda_n)|^2)^{1/2}x_n^b.$$

Hence the operator $Id - T_bT_{\overline{b}}$ transforms a Riesz basis of $\mathcal{H}(B)$ onto a Riesz basis of $\mathcal{H}(b)$ (resp. of its closed linear span), so it is an isomorphism of $\mathcal{H}(B)$ onto $\mathcal{H}(b)$ (resp. onto its range).

(ii) $\Longrightarrow$ (i): From (a), we get that $(x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(B)$ and using (b), we have that $((Id - T_bT_{\overline{b}})x_n)_{n\geqslant 1}$ is a Riesz basis of $\mathcal{H}(b)$ (resp. of its closed linear span). Hence $(k_{\lambda_n}^b)_{n\geqslant 1}$ forms an unconditional basis of $\mathcal{H}(b)$ (resp. in its closed linear span). $\square$

We will now give a criterion for the left invertibily of $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(B)}$.

Lemma 3.1. *Let u be an inner function and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Then the following statements are equivalent*:

(i) *The operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range*;
(ii) *$dist(\overline{u}b, H^\infty) < 1$*;
(iii) $\|P_u b_{|\mathcal{H}(u)}\| < 1$.

Proof. The operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range if and only if there exists $c > 0$ such that

$$c\|f\|_2 \leqslant \|(Id - T_bT_{\overline{b}})f\|_b, \quad (f \in \mathcal{H}(u)).$$

Notice that

$$\|(Id - T_bT_{\overline{b}})f\|_b^2 = \|(Id - T_bT_{\overline{b}})^{1/2}f\|_2^2 = \langle (Id - T_bT_{\overline{b}})f, f\rangle_2 = \|f\|_2^2 - \|T_{\overline{b}}f\|_2^2.$$

Hence the operator $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto its range if and only if there exists $c > 0$ such that, for all f in $\mathcal{H}(u)$, we have

$$\|T_{\overline{b}}f\|_2^2 \leqslant (1-c^2)\|f\|_2^2,$$

which is equivalent to $\|T_{\overline{b}|\mathcal{H}(u)}\| < 1$. But $T_{\overline{b}|\mathcal{H}(u)} = P_+\overline{b}_{|\mathcal{H}(u)}$ and it is easy to see that $(T_{\overline{b}|\mathcal{H}(u)})^* = P_u b = uP_-\overline{u}b$. It follows that

$$\|T_{\overline{b}|\mathcal{H}(u)}\| = \|uP_-\overline{u}b\| = \|P_-\overline{u}b\| = \|H_{\overline{u}b}\| = dist(\overline{u}b, H^\infty),$$

which gives the equivalence of the first two statements. Now notice that $P_u b_{|uH^2} = 0$ and so $\|P_u b\| = \|P_u b_{|\mathcal{H}(u)}\|$, which gives the equivalence with the third assertion. $\square$

Using Theorem 3.1 and Lemma 3.1, we get the following criterion which generalizes the classical one (see [11, Theorems 2 and 3 bis, pp. 230–232]).

Theorem 3.2. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditional basis in its closed linear span*;

(ii) (a) $(\lambda_n)_{n\geqslant 1} \in (C)$, (b) $dist(\overline{B}b, H^\infty) < 1$.

3.2. Applications of Theorem 3.1

We now give some applications of our criterion. The proof of the following facts are similar to the case where b is inner (see [11, Theorems 3.2, 3.5]) and are left to the reader.

Corollary 3.1. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) *there exits $p \in \mathbb{N}$ sufficiently large such that $(k^{b^p}_{\lambda_n})_{n\geqslant 1}$ forms an unconditional basis in its closed linear span*;

(ii) $(\lambda_n)_{n\geqslant 1} \in (C)$.

Corollary 3.2. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ such that*

$$\lim_{n\to 0} |b(\lambda_n)| = 0.$$

Assume that $(\lambda_n)_{n\geqslant 1} \in (C)$. Then there exists $N \in \mathbb{N}$ sufficiently large such that $(k^b_{\lambda_n})_{n\geqslant N}$ forms an unconditional basis in its closed linear span.

Corollary 3.3. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that*

$$\lim_{n\to +\infty} |b(\lambda_n)| = 0.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditional basis in its closed linear span*;
(ii) $(k^b_{\lambda_n})_{n\geqslant 1}$ *is uniformly minimal*.

In the case where b is inner, Hruscev et al. [11, Theorem 3.2] show that if $b = BS$ is the canonical factorization of b, where B is a Blaschke product and S is a singular inner function, and if $S \neq const$ and $\lim_{n\to+\infty} |b(\lambda_n)| = 0$, then the Carleson's condition is sufficient for the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ to be an unconditional basis of its closed linear span. Moreover, we have $dim(\mathcal{H}(b)\ominus Span(k^b_{\lambda_n} : n\geqslant 1)) = +\infty$. Using Theorem 3.1, we can give an analogue of this result. But before, we will need two lemmas. The first one is an easy generalization of a result for the classical case (see [16, p. 313]) and the proof is left to the reader. The second one is also a generalization of a result for the classical case but is more complicated.

Lemma 3.2. *Let* $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ *and* $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. *Assume that*

$$Span(k^b_{\lambda_n} : n\geqslant 1) \not\subseteq \mathcal{H}(b).$$

Then for all $\mu \neq \lambda_n$, $n\geqslant 1$, *we have*

(a) $k^b_\mu \notin Span(k^b_{\lambda_n} : n\geqslant 1)$; *in particular* $(k^b_{\lambda_n})_{n\geqslant 1}$ *is minimal*.
(b) $Span(k^b_{\lambda_n}, k^b_\mu : n \neq p) \not\subseteq \mathcal{H}(b)$, $\forall p\geqslant 1$.
(c) $\{k^b_{\lambda_n}, k^b_\mu : n\geqslant 1\}$ *is minimal*.

Lemma 3.3. *Let* φ_1 *be an inner function*, $\varphi_2 \in H^\infty$, $\|\varphi_2\|_\infty \leqslant 1$, *and* $\varphi = \varphi_1\varphi_2$. *Then we have*

$$Ker(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(\varphi)} = \mathcal{H}(\varphi) \cap \varphi_1 H^2 = \varphi_1\mathcal{H}(\varphi_2).$$

Proof. Notice that $Id - T_\varphi T_{\overline{\varphi}} \geqslant Id - T_{\varphi_1}T_{\overline{\varphi}_1}$. Indeed, for all $f \in H^2$, we have

$$\begin{aligned}\langle (Id - T_\varphi T_{\overline{\varphi}})f, f\rangle_2 &= \|f\|_2^2 - \|P_+\overline{\varphi}f\|_2^2\\ &= \|f\|_2^2 - \|P_+\overline{\varphi_2}P_+\overline{\varphi_1}f\|_2^2\\ &\geqslant \|f\|_2^2 - \|P_+\overline{\varphi_1}f\|_2^2\\ &= \langle (Id - T_{\varphi_1}T_{\overline{\varphi}_1})f, f\rangle_2.\end{aligned}$$

Using a result of Douglas (see [19, (I-5)]), it follows that $\mathcal{H}(\varphi_1) \subset \mathcal{H}(\varphi)$. Hence, we have $\mathcal{H}(\varphi) \in Lat(Id - T_{\varphi_1}T_{\overline{\varphi}_1})$.

Since φ_1 is inner, we have $Ker(Id - T_{\varphi_1}T_{\overline{\varphi}_1}) = \varphi_1 H^2$ and it follows that

$$Ker(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(\varphi)} = \mathcal{H}(\varphi) \cap \varphi_1 H^2.$$

Let us show now that $H(\varphi) \cap \varphi_1 H^2 = \varphi_1 \mathcal{H}(\varphi_2)$. First let $f \in \mathcal{H}(\varphi) \cap \varphi_1 H^2$. Then there exists $g \in H^2$ such that $f = \varphi_1 g$ and we have

$$T_{\overline{\varphi}_2} g = P_+ \overline{\varphi}_2 g = P_+ \overline{\varphi} \varphi_1 g = T_{\overline{\varphi}} f.$$

Since $f \in \mathcal{H}(\varphi)$, $T_{\overline{\varphi}} f \in \mathcal{H}(\overline{\varphi})$ (see [19, (II-4)]). But $\mathcal{H}(\overline{\varphi}) = \mathcal{H}(\overline{\varphi}_2)$, and so $T_{\overline{\varphi}_2} g \in \mathcal{H}(\overline{\varphi}_2)$. Using once more [19, (II-4)], we get that $g \in \mathcal{H}(\varphi_2)$. Reciprocally, let $g \in \mathcal{H}(\varphi_2)$, and $f = \varphi_1 g$. Of course, $f \in \varphi_1 H^2$. On the other hand, we have

$$T_{\overline{\varphi}} f = P_+ \overline{\varphi} \varphi_1 g = P_+ \overline{\varphi}_2 g = T_{\overline{\varphi}_2} g,$$

and an other application of [19, (II-4)] show that $f \in \mathcal{H}(\varphi)$. $\square$

As it was mentioned, the next result generalizes Theorem 3.2 in [11].

Theorem 3.3. *Let $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ and let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$. Let u_0 be the inner factor and b_0 the outer factor of b. Assume that u_0 is non constant, that b is not a Blaschke product and furthermore that*

$$\lim_{n\to+\infty} |u_0(\lambda_n)| = 0.$$

Then the following statements are equivalent:

(i) $(k_{\lambda_n}^b)_{n\geqslant 1}$ *forms an unconditional basis in its closed linear span*;
(ii) $(\lambda_n)_{n\geqslant 1} \in (C)$.

Moreover in this case, we have $\dim(\mathcal{H}(b) \ominus \mathrm{Span}(k_{\lambda_n}^b : n \geqslant 1)) = +\infty$.

Proof. (i) $\Longrightarrow$ (ii): Is always true and follows from Proposition 3.1.

(ii) $\Longrightarrow$ (i): Write $u_0 = B_0 S_0$, where B_0 is a Blaschke product and S_0 is a singular inner function. Define

$$\varphi_1 = \begin{cases} S_0^{1/2} B_0 & \text{if } b_0 \equiv const, \\ u_0 & \text{if } b_0 \not\equiv const \end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} b_0 S_0^{1/2} & \text{if } b_0 \equiv const, \\ b_0 & \text{if } b_0 \not\equiv const. \end{cases}$$

In the two cases, we have $b = \varphi_1 \varphi_2$, φ_1 is an inner function and $\varphi_2 \in H^\infty$, $\|\varphi_2\|_\infty \leqslant 1$. Moreover, the assumptions on b imply that $\varphi_2 \not\equiv const$ and $\lim_{n\to+\infty} |\varphi_1(\lambda_n)| = \lim_{n\to+\infty} |b(\lambda_n)| = 0$. Consequently, it follows from Corollary 3.2 that there exists $N \in \mathbb{N}$ sufficiently large such that both family $(k_{\lambda_n}^{\varphi_1})_{n\geqslant N}$ and $(k_{\lambda_n}^b)_{n\geqslant N}$ form an unconditional bases in their closed linear span. Moreover, we see that the norms $\|k_{\lambda_n}^b\|_b$ and $\|k_{\lambda_n}^{\varphi_1}\|_{\varphi_1}$ are comparable with each other. Now notice that $Id - T_{\varphi_1} T_{\overline{\varphi}_1} =$

$(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}(Id - T_bT_{\overline{b}})$. Indeed, we have, for all $f \in H^2$

$$(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}(Id - T_bT_{\overline{b}})f = (Id - T_{\varphi_1}T_{\overline{\varphi}_1})f - T_bT_{\overline{b}}f + T_{\varphi_1}T_{\overline{\varphi}_1}T_bT_{\overline{b}}f$$

and

$$\begin{aligned} T_{\varphi_1}T_{\overline{\varphi}_1}T_bT_{\overline{b}}f &= \varphi_1 P_+\overline{\varphi}_1 P_+ bP_+\overline{b}f \\ &= \varphi_1 P_+\overline{\varphi}_1 bP_+\overline{b}f \\ &= \begin{cases} \varphi_1 P_+\overline{S_0^{1/2}B_0}S_0B_0P_+\overline{b}f & \text{if } b_0 \equiv cte, \\ \varphi_1 P_+\overline{u_0}b_0u_0P_+\overline{b}f & \text{if } b_0 \not\equiv cte, \end{cases} \\ &= \begin{cases} \varphi_1 S_0^{1/2}P_+\overline{b}f & \text{if } b_0 \equiv cte, \\ \varphi_1 b_0P_+\overline{b}f & \text{if } b_0 \not\equiv cte, \end{cases} \\ &= T_bT_{\overline{b}}f. \end{aligned}$$

Hence $Id - T_{\varphi_1}T_{\overline{\varphi}_1} = (Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}(Id - T_bT_{\overline{b}})$. This implies that

$$k_{\lambda_n}^{\varphi_1} = (Id - T_{\varphi_1}T_{\overline{\varphi}_1})k_{\lambda_n} = (Id - T_{\varphi_1}T_{\overline{\varphi}_1})(Id - T_bT_{\overline{b}})k_{\lambda_n} = (Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}k_{\lambda_n}^b.$$

Therefore, $(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)}$ is an isomorphism from $Span(k_{\lambda_n}^b : n \geqslant 1)$ onto $Span(k_{\lambda_n}^{\varphi_1} : n \geqslant 1) \subset \mathcal{H}(\varphi_1)$. Using Lemma 3.3, we get

$$dim(\mathcal{H}(b) \ominus Span(k_{\lambda_n}^b : n \geqslant N)) \geqslant dim\, Ker(Id - T_{\varphi_1}T_{\overline{\varphi}_1})_{|\mathcal{H}(b)} = dim(\varphi_1\mathcal{H}(\varphi_2)).$$

But $\varphi_2 \not\equiv const$ and thus

$$dim(\mathcal{H}(b) \ominus Span(k_{\lambda_n}^b : n \geqslant N)) = +\infty.$$

Applying repeatedly Lemma 3.2, it follows that $(k_{\lambda_n}^b)_{n \geqslant 1}$ is an unconditional basis in its closed linear span, which has infinite codimension. □

Theorem 3.2 gives also a criterion for a sequence $(k_{\lambda_n})_{n \geqslant 1}$ to be an unconditional basis in the closed subspace of $H^2(\mu)$ it generates.

Theorem 3.4. *Let $(\lambda_n)_{n \geqslant 1} \subset \mathbb{D}$ and let μ be a positive Borel measure on $\mathbb{T}$. Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ such that*

$$\frac{1 - |b(z)|^2}{|1 - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2}\, d\mu(e^{i\theta}).$$

Assume that

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

The following statements are equivalent:

(i) $(k_{\lambda_n})_{n\geqslant 1}$ *is an unconditional basis in the closed subspace of* $H^2(\mu)$ *it generates*;
(ii) $(k^b_{\lambda_n})_{n\geqslant 1}$ *is an unconditional basis in its closed linear span*;
(iii) (a) $(\lambda_n)_{n\geqslant 1} \in (C)$, (b) $dist(\overline{B}b, H^\infty) < 1$.

Proof. The equivalence of (ii) and (iii) follows from Theorem 3.2. To show that (i) $\Longleftrightarrow$ (ii), consider the linear map $V_b : L^2(\mu) \to Hol(\mathbb{D})$ defined by $V_b q(z) = (1-b(z))K_\mu q(z)$, $q \in L^2(\mu)$, $z \in \mathbb{D}$. We know that V_b is an isometry from $H^2(\mu)$ onto $\mathcal{H}(b)$ and $V_b k_{\lambda_n} = (1-\overline{b(\lambda_n)})^{-1} k^b_{\lambda_n}$ (see [19, (III-7)]). Hence

$$V_b\left(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_{L^2(\mu)}}\right) = (1-\overline{b(\lambda_n)})^{-1}\frac{\|k^b_{\lambda_n}\|_b}{\|k_{\lambda_n}\|_{L^2(\mu)}} x^b_n = \alpha_n x^b_n,$$

with $\alpha_n = (1-\overline{b(\lambda_n)})^{-1}\dfrac{\|k^b_{\lambda_n}\|_b}{\|k_{\lambda_n}\|_{L^2(\mu)}}$. Notice that $|\alpha_n| = 1$, $n\geqslant 1$ and it follows that $(k_{\lambda_n})_{n\geqslant 1}$ is an unconditional basis in the closed subspace of $H^2(\mu)$ it generates if and only if $(\alpha_n x^b_n)_{n\geqslant 1}$ is a Riesz basis in its closed linear span, which is equivalent to $(k^b_{\lambda_n})_{n\geqslant 1}$ is an unconditional basis in its closed linear span. □

4. The extreme case

In this section, we want to characterize sequences $(k^b_{\lambda_n})_{n\geqslant 1}$ which form an unconditional basis of $\mathcal{H}(b)$. So thanks to Theorem 3.1, this problem can be reduced to the fact that $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(B)$ onto $\mathcal{H}(b)$. Recall that in the classical case where b is inner, thanks to formula (6), we can reformulate this property in terms of the invertibility of $T_{\overline{B}b}$ and then get a criterion in terms of $dist(\overline{B}b, H^\infty)$ and $dist(\overline{b}B, H^\infty)$. In the general case, formula (6) is no longer true but nevertheless we can obtain a similar criterion. First, we will give two lemmas.

Lemma 4.1. *Let* $b \in H^\infty$ *and* $\lambda \in \mathbb{D}$. *Then we have*

$$Id - T_{\overline{z}b}T_{z\overline{b}} = Id - T_bT_{\overline{b}} - S^*b \otimes S^*b \tag{8}$$

and

$$(Id - T_bT_{\overline{b}})(Id - \lambda S^*) = (Id - \lambda S^*)(Id - T_bT_{\overline{b}}) - \lambda S^*b \otimes b. \tag{9}$$

Proof. Notice that $T_{\bar{z}b} = S^*T_b$; hence we have $Id - T_{\bar{z}b}T_{z\bar{b}} = Id - S^*T_bT_{\bar{b}}S$. But $SS^* = Id - \mathbb{1}\otimes\mathbb{1}$, which implies that

$$\begin{aligned} Id - T_{\bar{z}b}T_{z\bar{b}} &= Id - S^*T_b(SS^* + \mathbb{1}\otimes\mathbb{1})T_{\bar{b}}S \\ &= Id - T_bT_{\bar{b}} - S^*b\otimes S^*b. \end{aligned}$$

For formula (9), write

$$\begin{aligned} (Id - T_bT_{\bar{b}})(Id-\lambda S^*) - (Id-\lambda S^*)(Id - T_bT_{\bar{b}}) &= \lambda(S^*T_bST_{\bar{b}}S^* - S^*T_bT_{\bar{b}}) \\ &= \lambda S^*T_b(ST_{\bar{b}}S^* - T_{\bar{b}}) \\ &= \lambda S^*T_b(SS^* - Id)T_{\bar{b}} \\ &= -\lambda S^*b\otimes b. \qquad \square \end{aligned}$$

Lemma 4.2. *Let b be an extreme point of the unit ball of* H^∞. *Then*

$$Span\left(\frac{b-b(\lambda)}{z-\lambda} : \lambda\in\mathbb{D}\right) = \mathcal{H}(b).$$

Proof. Let $f\in\mathcal{H}(b)\ominus Span\left(\frac{b-b(\lambda)}{z-\lambda} : \lambda\in\mathbb{D}\right)$. Using the equality

$$\frac{b-b(\lambda)}{z-\lambda} = (1-\lambda S^*)^{-1}S^*b = \sum_{n\geqslant 0}\lambda^n S^{*n+1}b,$$

we get $\langle f, S^{*n+1}b\rangle_b = 0$, $\forall n\geqslant 0$. It follows from the relation between $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ (see [19, (II-4)]), that

$$\langle f, S^{*n+1}b\rangle_b = \langle f, S^{*n+1}\rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}S^{*n+1}b\rangle_{\bar{b}} = \langle f\bar{b}, \bar{z}^{n+1}\rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}S^{*n+1}b\rangle_{\bar{b}}.$$

Now recall that if ρ is the function $1-|b|^2$ on $\mathbb{T}$, then the operator $K_\rho : H^2(\rho)\to\mathcal{H}(\bar{b})$ defined by $K_\rho g := P_+(g\rho)$ is an isometry of $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$. Moreover we have $K_\rho J_\rho = T_\rho$, where J_ρ is the canonical injection from H^2 into $L^2(\rho)$ and

$$K_\rho Z_\rho^* = S^*K_\rho$$

(see [19, (III-2) and (III-3)]). Since $f\in\mathcal{H}(b)$, $T_{\bar{b}}f\in\mathcal{H}(\bar{b})$ and there exists $g\in H^2(\rho)$ such that $T_{\bar{b}}f = K_\rho g = P_+(g\rho)$. Moreover, notice that $T_{\bar{b}}S^*b = S^*T_{\bar{b}}b =$

$S^*(\mathbb{1} - (Id - T_{\overline{b}}T_b)\mathbb{1}) = -S^*T_\rho \mathbb{1} = -K_\rho Z_\rho^* \mathbb{1}$ and by induction

$$T_{\overline{b}} S^{*n+1} = -K_\rho Z_\rho^{*n+1} \mathbb{1}.$$

It follows that

$$\begin{aligned}\langle T_{\overline{b}} f, T_{\overline{b}} S^{*n+1} b\rangle_{\overline{b}} &= -\langle K_\rho g, K_\rho Z_\rho^{*n+1}\rangle_{\overline{b}} \\ &= -\langle g, Z_\rho^{*n+1}\mathbb{1}\rangle_\rho \\ &= -\langle Z_\rho^{n+1} g, \mathbb{1}\rangle_\rho \\ &= -\langle \rho g, \overline{z}^{n+1}\rangle_2.\end{aligned}$$

Finally, we get

$$\langle f, S^{*n+1} b\rangle_b = \langle f\overline{b}, \overline{z}^{n+1}\rangle_2 - \langle \rho g, \overline{z}^{n+1}\rangle_2,$$

which implies that $\langle f\overline{b} - \rho g, \overline{z}^{n+1}\rangle_2 = 0$, $\forall n \geqslant 0$. That means that $f\overline{b} - \rho g \in H^2$. But since $T_{\overline{b}} f = P_+(\rho g)$, we also have $f\overline{b} - \rho g \in H^2_-$, and thus $f\overline{b} = \rho g$. Notice now that $|\rho g|$ is not log-integrable. Indeed, we have

$$\log|\rho g| \leqslant \log^+ |\rho g|^{1/2} + \tfrac{1}{2}\log\rho,$$

and the first term on the right side is integrable, whereas the second term has integral $-\infty$ because b is extreme. That implies that $\log|fb| = \log|\rho g| \notin L^1$. But $fb \in H^2$, thus $fb \equiv 0$, that is $f \equiv 0$, which ends the proof. □

Theorem 4.1. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$, and let u be an inner function. Assume that b is an extreme point of the unit ball of H^∞ and that $(Id - T_b T_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible. Then the following statements are equivalent*:

(i) *$Id - T_b T_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto $\mathcal{H}(b)$*;
(ii) *$Ker(Id - T_b T_{\overline{b}})_{|\mathcal{H}(zu)} \neq \{0\}$*;
(iii) *$(Id - T_b T_{\overline{b}})_{|\mathcal{H}(zu)}$ is not left invertible*;
(iv) *$S^* b \in (Id - T_b T_{\overline{b}})\mathcal{H}(u)$*;
(v) *$dist(\overline{zu} b, H^\infty) = 1$*;

Proof. Notice that (ii) $\Longrightarrow$ (iii) is trivial and (iii) $\Longleftrightarrow$ (v) follows from Lemma 3.1.

(i) $\Longrightarrow$ (ii): There exists $f \in \mathcal{H}(u)$ such that $(Id - T_b T_{\overline{b}})u = (Id - T_b T_{\overline{b}})f$. Define $g := f - u$. It is easy to see that $g \in \mathcal{H}(zu) = H^2 \ominus zuH^2$ and thus $g \in Ker((Id - T_b T_{\overline{b}})_{|\mathcal{H}(zu)})$. Moreover, $g \not\equiv 0$ (because otherwise $u = f \in \mathcal{H}(u)$), which proves that $(Id - T_b T_{\overline{b}})_{|\mathcal{H}(zu)}$ is not injective.

(iii) $\Longrightarrow$ (iv): Using the fact that $S^*{}_{|\mathcal{H}(b)}$ is a contraction, we get

$$\inf_{\substack{f\in\mathcal{H}(zu)\\ \|f\|_2=1}} \|S^*(Id - T_bT_{\overline{b}})f\|_b = 0.$$

Writing now $f = SS^*f + f(0)$, we have

$$S^*(Id - T_bT_{\overline{b}})f = S^*(Id - T_bT_{\overline{b}})SS^*f + f(0)S^*k_0^b.$$

But $S^*(Id - T_bT_{\overline{b}})S = Id - T_{\overline{z}b}T_{z\overline{b}}$ and $S^*k_0^b = S^*(1 - \overline{b(0)}b) = -\overline{b(0)}S^*b$, which gives

$$S^*(Id - T_bT_{\overline{b}})f = (Id - T_{\overline{z}b}T_{z\overline{b}})S^*f - f(0)\overline{b(0)}S^*b.$$

Now it follows from (8) that

$$S^*(Id - T_bT_{\overline{b}})f = (Id - T_bT_{\overline{b}})S^*f - \left(\langle S^*f, S^*b\rangle_2 + f(0)\overline{b(0)}\right) S^*b,$$

which implies

$$\inf_{\substack{f\in\mathcal{H}(zu)\\ \|f\|_2=1}} \left\|(Id - T_bT_{\overline{b}})S^*f - \left(\langle S^*f, S^*b\rangle_2 + f(0)\overline{b(0)}\right) S^*b\right\|_b = 0.$$

Thus there exists a sequence $(f_n)_{n\geqslant 1} \subset \mathcal{H}(zu)$, $\|f_n\|_2 = 1$, such that

$$\lim_{n\to+\infty}((Id - T_bT_{\overline{b}})S^*f_n - \left(\langle S^*f_n, S^*b\rangle_2 + f_n(0)\overline{b(0)}\right) S^*b) = 0.$$

Notice that the sequence of complex numbers $a_n := \langle S^*f_n, S^*b\rangle_2 + f_n(0)\overline{b(0)}$ is bounded. Hence we can find a subsequence $(a_{n_p})_{p\geqslant 1}$ which converges, say to c. So we have $\lim_{p\to+\infty}(Id - T_bT_{\overline{b}})S^*f_{n_p} = cS^*b$. Since $f_{n_p} \in \mathcal{H}(zu)$, we have $S^*f_{n_p} \in \mathcal{H}(u)$ and thus $cS^*b \in \overline{(Id - T_bT_{\overline{b}})\mathcal{H}(u)}$. Using the fact that $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible, we get that $cS^*b \in (Id - T_bT_{\overline{b}})\mathcal{H}(u)$. Moreover, we have

$$\|S^*f_{n_p}\|_2^2 = \|f_{n_p}\|_2^2 - |f_{n_p}(0)|^2 = 1 - |f_{n_p}(0)|^2. \tag{10}$$

First case: $\delta := \sup_{p\geqslant 1} |f_{n_p}(0)| < 1$. Using the left invertibility of $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(u)}$, there exists $k > 0$ such that

$$\|(Id - T_bT_{\overline{b}})f\|_b \geqslant k\|f\|_2 \qquad \forall f \in \mathcal{H}(u).$$

It now follows, using (10) that

$$|c|^2\|S^*b\|_b^2 = \lim_{p\to+\infty}\|(Id - T_bT_{\overline{b}})S^*f_{n_p}\|_b^2 \geqslant k^2 \limsup_{p\to+\infty}\|S^*f_{n_p}\|_2^2 \geqslant k^2(1-\delta^2) > 0,$$

which implies that $c \neq 0$ and thus $S^*b \in (Id - T_bT_{\overline{b}})\mathcal{H}(u)$.

Second case: $\sup_{p\geqslant 1}|f_{n_p}(0)| = 1$ and $b(0) \neq 0$. We can assume that the sequence $(f_{n_p}(0))_{p\geqslant 1}$ is convergent, say to λ. Since $|\lambda| = 1$, we have, using (10) $\lim_{p\to+\infty}\|S^*f_{n_p}\|_2 = 0$, which implies, in particular that

$$\lim_{p\to+\infty}\langle S^*f_{n_p}, S^*b\rangle_2 = 0.$$

It now follows that $\lim_{p\to+\infty} a_{n_p} = \lambda\overline{b(0)}$. Thus $c = \lambda\overline{b(0)} \neq 0$ and $S^*b \in (Id - T_bT_{\overline{b}})\mathcal{H}(u)$.

Third case: $b(0) = 0$. Then $b_1 := \overline{z}b \in H^\infty$ and applying Lemma 3.1, we get that $(Id - T_{b_1}T_{\overline{b_1}})_{|\mathcal{H}(u)}$ is not left invertible. Hence

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \|(Id - T_{b_1}T_{\overline{b_1}})f\|_{b_1} = 0.$$

But $\mathcal{H}(b_1) \subset \mathcal{H}(b)$, and closed graph theorem gives

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \|(Id - T_{b_1}T_{\overline{b_1}})f\|_{b} = 0.$$

Using now (8), we have

$$\inf_{\substack{f\in\mathcal{H}(u)\\ \|f\|_2=1}} \left\|(Id - T_bT_{\overline{b}})f - \langle f, S^*b\rangle_2 S^*b\right\|_b = 0.$$

Since $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(u)}$ is left invertible, we get as above that $S^*b \in (Id - T_bT_{\overline{b}})\mathcal{H}(u)$.

(iv) $\Longrightarrow$ (i): Let $\lambda \in \mathbb{D}$ and $f \in \mathcal{H}(u)$ such that $S^*b = (Id - T_bT_{\overline{b}})f$. Then we have

$$\frac{b - b(\lambda)}{z - \lambda} = (Id - \lambda S^*)^{-1}S^*b = (Id - \lambda S^*)^{-1}(Id - T_bT_{\overline{b}})f.$$

But thanks to (9), we have

$$(Id - \lambda S^*)^{-1}(Id - T_bT_{\overline{b}}) = (Id - T_bT_{\overline{b}})(Id - \lambda S^*)^{-1} - \lambda(Id - \lambda S^*)^{-1}S^*b \otimes (Id - \overline{\lambda}S)^{-1}b,$$

which gives

$$\frac{b-b(\lambda)}{z-\lambda} = (Id - T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f - \lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 (Id-\lambda S^*)^{-1}S^*b$$
$$= (Id - T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f - \lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 \frac{b-b(\lambda)}{z-\lambda}.$$

Thus

$$\left(1+\lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2\right)\frac{b-b(\lambda)}{z-\lambda} = (Id - T_bT_{\overline{b}})(Id-\lambda S^*)^{-1}f.$$

Notice that $(Id-\lambda S^*)^{-1}f \in \mathcal{H}(u)$. Moreover if $c := 1+\lambda\langle f, (Id-\overline{\lambda}S)^{-1}b\rangle_2 = 0$, then $(Id-\lambda S^*)^{-1}f \in \mathcal{H}(u)\cap Ker(Id - T_bT_{\overline{b}})$, which implies by left invertibility of $(Id - T_bT_{\overline{b}})_{|\mathcal{H}(u)}$ that $f=0$, which is absurd. Thus $c\neq 0$ and

$$\frac{b-b(\lambda)}{z-\lambda} \in (Id - T_bT_{\overline{b}})\mathcal{H}(u).$$

Using Lemma 4.2, we get that $\mathcal{H}(b) = (Id - T_bT_{\overline{b}})\mathcal{H}(u)$, which proves that $Id - T_bT_{\overline{b}}$ is an isomorphism of $\mathcal{H}(u)$ onto $\mathcal{H}(b)$. □

We can now give our criterion for unconditional basis in $\mathcal{H}(b)$.

Theorem 4.2. *Let $(\lambda_n)_{n\geqslant 1}\subset \mathbb{D}$ and let $b\in H^\infty$, $\|b\|_\infty \leqslant 1$. Assume that b is an extreme point of the unit ball of H^∞ and*

$$\sup_{n\geqslant 1}|b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) $(k_{\lambda_n})_{n\geqslant 1}$ *is an unconditional basis of* $\mathcal{H}(b)$;
(ii) (a) $(\lambda_n)_{n\geqslant 1}\in (C)$. (b) $dist(\overline{B}b, H^\infty) < 1$. (c) $dist(\overline{zB}b, H^\infty) = 1$.

Proof. It suffices to combine Theorem 3.1, Lemma 3.1 and Theorem 4.1. □

To finish this section, we would like to give a generalization of Theorem 9 in [11], which underlines the link between spectral properties of the model operator and geometric properties of reproducing kernels.

First of all, recall that when b is extreme then

$$\sigma_p(X) = \{\overline{\lambda}\in\mathbb{D} : b(\lambda)=0\},\quad Ker(X-\overline{\lambda}) = \mathbb{C}k_\lambda$$

and

$$\sigma_p(X^*) = \{\lambda \in \mathbb{D} : b(\lambda) = 0\}, \quad Ker(X^* - \lambda) = \mathbb{C}\frac{b}{z - \lambda},$$

(see [19] for the result for X and Proposition 2.1 for X^*).

Assume that b has an infinite sequence $(\lambda_n)_{n\geqslant 1}$ of zeros and let B be the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$ and let $b_1 = \overline{B}b$. Then the following result gives a criterion for the sequence of eigenvectors of X and X^* forms an unconditional basis of $\mathcal{H}(b)$.

Theorem 4.3. *With the previous notations, the following statements are equivalent*:

(i) $(k_{\lambda_n})_{n\geqslant 1} \cup \left(\dfrac{b}{z - \lambda_n}\right)_{n\geqslant 1}$ *forms an unconditional basis of* $\mathcal{H}(b)$;

(ii) $\sup_{n\geqslant 1} |b_1(\lambda_n)| < 1$ *and* $(k^{b_1}_{\lambda_n})_{n\geqslant 1}$ *forms an unconditional basis of* $\mathcal{H}(b_1)$;

(iii) $(\lambda_n)_{n\geqslant 1} \in (C)$, $dist(\overline{B}b_1, H^\infty) < 1$, $dist(\overline{zB}b_1, H^\infty) = 1$.

Proof. (ii) $\Longleftrightarrow$ (iii): Notice that if b is extreme then b_1 is also extreme. Moreover if $dist(\overline{B}b_1, H^\infty) < 1$, then there exists $h \in H^\infty$ such that $\|b_1 - Bh\|_\infty < 1$, and we have

$$\sup_{n\geqslant 1} |b_1(\lambda_n)| = \sup_{n\geqslant 1} |(b_1 - Bh)(\lambda_n)| \leqslant \|b_1 - Bh\|_\infty < 1.$$

Now it suffices to apply Theorem 4.2.

For (i) $\Longleftrightarrow$ (ii), we will need the following lemmas.

Lemma 4.3. *With the previous notations, we have*

(a) $\mathcal{H}(b) = \mathcal{H}(B) \oplus^\perp B\mathcal{H}(b_1)$.
(b) $\mathcal{H}(b) = \mathcal{H}(b_1) \oplus^\perp b_1\mathcal{H}(B)$.

Moreover, T_B (resp. T_{b_1}) acts as an isometry of $\mathcal{H}(b_1)$ (resp. of $\mathcal{H}(B)$) into $\mathcal{H}(b)$.

Lemma 4.4. *With the previous notations, the sequence $(k^{b_1}_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b_1)$ if and only if*

$$\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}^{\mathcal{H}(b)}.$$

Lemma 4.5. *Let H be an Hilbert space and X, Y be two closed subspaces of H. Assume that $(x_n)_{n\geqslant 1}$ (resp. $(y_n)_{n\geqslant 1}$) is an unconditional basis of X (resp. Y). Then $(x_n)_{n\geqslant 1} \cup (y_n)_{n\geqslant 1}$ is an unconditional basis of H if and only if*

$$\overline{X + Y} = H \quad and \quad \langle X, Y\rangle > 0.$$

(i) $\Longrightarrow$ (ii): Recall (see [14, Lecture IV]) that

$$Span(k_{\lambda_n} : n \geqslant 1) = \mathcal{H}(B) = Span\left(\frac{B}{z-\lambda_n} : n \geqslant 1\right).$$

Then it follows from Lemma 4.3 that

$$Span\left(\frac{b}{z-\lambda_n} : n \geqslant 1\right) = b_1\mathcal{H}(B).$$

Now Lemma 4.5 implies that $\langle \mathcal{H}(B), b_1\mathcal{H}(B)\rangle > 0$ and $\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}$. Thus, using Lemma 4.4, we get that $(k_{\lambda_n}^{b_1})_{n\geqslant 1}$ is complete in $\mathcal{H}(b_1)$. Since $(k_{\lambda_n})_{n\geqslant 1}$ is an unconditional basis of $\mathcal{H}(B)$, it remains, thanks to Theorem 3.1, to show that $(Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$ is an isomorphism onto its range. But it follows from Lemma 4.3 that $\mathcal{H}(b_1)$ is a closed subspace of $\mathcal{H}(b)$ and then we can consider $P_{\mathcal{H}(b_1)}$ the orthogonal projection of $\mathcal{H}(b)$ onto $\mathcal{H}(b_1)$. Now notice that

$$k_{\lambda_n} = k_{\lambda_n}^b = \frac{1-\overline{b_1(\lambda_n)}b_1}{1-\overline{\lambda_n}z} + \overline{b_1(\lambda_n)}\frac{b_1}{1-\overline{\lambda_n}z} = (Id - T_{b_1}T_{\overline{b}_1})k_{\lambda_n} + \overline{b_1(\lambda_n)}b_1k_{\lambda_n},$$

which implies, using the fact that $b_1\mathcal{H}(B) = (\mathcal{H}(b_1))^\perp$,

$$P_{\mathcal{H}(b_1)}k_{\lambda_n} = (Id - T_{b_1}T_{\overline{b}_1})k_{\lambda_n}.$$

Consequently we have $P_{\mathcal{H}(b_1)|\mathcal{H}(B)} = (Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$. Since $\langle \mathcal{H}(B), (\mathcal{H}(b_1))^\perp\rangle > 0$, it follows, from [14, Lemma on Close Subspaces, Lecture VIII, p. 201], that $P_{\mathcal{H}(b_1)}$, and thus $Id - T_{b_1}T_{\overline{b}_1}$, is an isomorphism of $\mathcal{H}(B)$ onto its range.

(ii) $\Longrightarrow$ (i): Using Proposition 3.1, we have $(\lambda_n)_{n\geqslant 1} \in (C)$. It follows that $(k_{\lambda_n})_{n\geqslant 1}$ and $\left(\frac{B}{z-\lambda_n}\right)_{n\geqslant 1}$ form an unconditional basis of $\mathcal{H}(B)$ (see [14, Lecture VI]). Thanks to Lemma 4.3, we get that $\left(\frac{b}{z-\lambda_n}\right)_{n\geqslant 1}$ forms an unconditional basis of $b_1\mathcal{H}(B)$. Using Lemma 4.5, it remains to show that

$$\langle \mathcal{H}(B), b_1\mathcal{H}(B)\rangle > 0 \quad \text{and} \quad \mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}.$$

But we know that $(Id - T_{b_1}T_{\overline{b}_1})_{|\mathcal{H}(B)}$ is an isomorphism onto its range, which implies that $P_{\mathcal{H}(b_1)|\mathcal{H}(B)}$ is also an isomorphism onto its range. Now using once more lemma on close subspaces from [14], we get that

$$\langle \mathcal{H}(B), \mathcal{H}(b_1)^\perp\rangle > 0 \quad \text{and} \quad \mathcal{H}(b) = \overline{\mathcal{H}(B) + \mathcal{H}(b_1)^\perp},$$

which ends the proof because $\mathcal{H}(b_1)^\perp = b_1\mathcal{H}(B)$. $\square$

Proof of Lemma 4.3. (a) Follows from [19, (II-6)].

(b) Let $A := T_b$, $A_1 := T_{b_1}$ and $A_2 := T_B$. Using [19, (I-10)], we have

$$\mathcal{H}(b) = \mathcal{H}(A) = \mathcal{H}(A_1) + A_2\mathcal{H}(A_1) = \mathcal{H}(b_1) + b\mathcal{H}(B).$$

Moreover, we have

$$\mathcal{H}(b) = \mathcal{H}(b_1) \oplus^{\perp} b_1\mathcal{H}(B) \Longleftrightarrow \mathcal{H}(b_1) \cap b_1\mathcal{H}(B) = \{0\}.$$

But $\mathcal{H}(b_1) \cap b_1\mathcal{H}(B) \subset \mathcal{H}(b_1) \cap b_1 H^2 = \mathcal{H}(b_1) \cap \mathcal{M}(b_1) = T_{b_1}\mathcal{H}(\overline{b}_1)$ (see [19, (II-5)]). Now let $f \in \mathcal{H}(b_1) \cap b_1\mathcal{H}(B)$. Then there exists $h \in \mathcal{H}(B)$ and $g \in \mathcal{H}(\overline{b}_1)$ such that $f = b_1 g = b_1 h$. Thus $g = h$. Since $h \in \mathcal{H}(B)$, h is not a cyclic vector of S^* (see [8]). It is known that when b is extreme, the nonzero functions in $\mathcal{H}(\overline{b})$ are cyclic vectors of S^* (see [19, (V-2)]). Thus $g \equiv 0$ and $f \equiv 0$. The fact that T_{b_1} acts as an isometry of $\mathcal{H}(B)$ into $\mathcal{H}(b)$ follows from [19, (I-11)]. □

Proof of Lemma 4.4. Recall that if M and N are two closed subspaces of an Hilbert space H, then $H = \overline{M + N^{\perp}}$ if and only if $M^{\perp} \cap N = \{0\}$ (see [14, Lemma on Close Subspaces, Lecture VIII, p. 201]). Moreover, thanks to Lemma 4.3, if $M = b_1\mathcal{H}(B)$ and $N = B\mathcal{H}(b_1)$, we have $M^{\perp} = \mathcal{H}(b_1)$ and $N^{\perp} = \mathcal{H}(B)$. Thus

$$\mathcal{H}(b) = \overline{\mathcal{H}(B) + b_1\mathcal{H}(B)}^{\mathcal{H}(b)} \Longleftrightarrow \mathcal{H}(b_1) \cap B\mathcal{H}(b_1) = \{0\}.$$

On the other hand, if $f \in \mathcal{H}(b_1)$ and $f(\lambda_n) = 0$, $n \geqslant 1$, then $\frac{f}{B} \in \mathcal{H}(b_1)$. Thus

$$\mathcal{H}(b_1) \ominus Span(k_{\lambda_n}^{b_1} : n \geqslant 1) = \mathcal{H}(b_1) \cap B\mathcal{H}(b_1),$$

which gives the result. □

Proof of Lemma 4.5. It suffices to use the link between the angle and the skew projections (see [15] or [14]). □

5. The nonextreme case

In this section, we discuss the nonextreme case. As we shall see, contrary to the extreme case, there cannot exist basis of reproducing kernels in $\mathcal{H}(b)$.

First, recall that if $(k_{\lambda_n}^b)_{n \geqslant 1}$ is not complete in $\mathcal{H}(b)$ then it is minimal (see Lemma 3.2). The following result shows that the converse is also true in the nonextreme case. The key point is the fact that $\mathcal{H}(b)$ is invariant under the shift.

Proposition 5.1. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. Assume that b is nonextreme. The following statements are equivalent*:

(i) $(k_{\lambda_n})_{n\geqslant 1}$ *is minimal*;
(ii) $(k_{\lambda_n})_{n\geqslant 1}$ *is not complete in* $\mathcal{H}(b)$.

Moreover, in this case, we have

$$dim(\mathcal{H}(b) \ominus Span(k^b_{\lambda_n} : n \geqslant 1)) = +\infty.$$

Proof. Thanks to Lemma 3.2, it suffices to prove that if $(k_{\lambda_n})_{n\geqslant 1}$ is minimal, then $dim(\mathcal{H}(b) \ominus Span(k^b_{\lambda_n} : n \geqslant 1)) = +\infty$. Suppose that $dim(\mathcal{H}(b) \ominus Span(k^b_{\lambda_n} : n \geqslant 1)) = N < +\infty$. Then it implies the existence of a sequence of reproducing kernels which is minimal and complete in $\mathcal{H}(b)$. Indeed, we can assume that $N \geqslant 1$. Applying repeatedly Lemma 3.2, we get that if $(\mu_i)_{1\leqslant i\leqslant N} \subset \mathbb{D}$, with $\mu_i \neq \mu_j$, $i \neq j$ and $\mu_i \neq \lambda_n$, then $(k^b_{\lambda_n}, k^b_{\mu_i})_{\substack{n\geqslant 1\\ N\geqslant i\geqslant 1}}$ is minimal and complete in $\mathcal{H}(b)$. In particular, it implies the existence of a function $h \in \mathcal{H}(b)$ such that $h(\lambda_1) = 0$ and $h(\lambda_n) = h(\mu_i) = 0$, $n \geqslant 2$, $1 \leqslant i \leqslant N$. Now consider $f := (z - \lambda_1)h$. Since $S\mathcal{H}(b) \subset \mathcal{H}(b)$ in the nonextreme case (see [19, (IV-5)]), we have

$$f \in \mathcal{H}(b) \ominus Span(k^b_{\lambda_n}, k^b_{\mu_i} : n \geqslant 1, N \geqslant i \geqslant 1).$$

Since $h \not\equiv 0$, we have $f \not\equiv 0$, which contradicts the completeness of $(k^b_{\lambda_n}, k^b_{\mu_i})_{\substack{n\geqslant 1\\ N\geqslant i\geqslant 1}}$. □

Corollary 5.1. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $(\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$. Assume that b is nonextreme and that*

$$\sup_{n\geqslant 1} |b(\lambda_n)| < 1.$$

Then the following statements are equivalent:

(i) $(k^b_{\lambda_n})_{n\geqslant 1}$ *forms an unconditional basis in its closed linear span*;
(ii) (a) $(\lambda_n)_{n\geqslant 1} \in (C)$, (b) $dist(\overline{B}b, H^\infty) < 1$.

Moreover in this case, we have

$$dim(\mathcal{H}(b) \ominus Span(k^b_{\lambda_n} : n \geqslant 1)) = +\infty.$$

Proof. It suffices to combine Proposition 5.1 and Theorem 3.2. □

We can precise a little more Proposition 5.1 and get a characterization of completeness (and thus of minimality).

Proposition 5.2. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $(\lambda_n)_{n\geqslant 1} \in \mathbb{D}$. Assume that b is nonextreme.*

(a) *If b is pseudocontinuable, then the following statements are equivalent*:
  (i) *the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$;*
  (ii) $\sum_{n\geqslant 1}(1-|\lambda_n|) = +\infty$.

(b) *If b is not pseudocontinuable, then the following statements are equivalent*:
  (i) *the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$;*
  (ii) $S^*b \in Span(k^b_{\lambda_n} : n \geqslant 1)$.

Recall that a function f in H^2 is pseudocontinuable (across $\mathbb{T}$) if there exist functions $f_1, f_2 \in H^\infty$ such that $f = \overline{f_1}/\overline{f_2}$ a.e. on $\mathbb{T}$. Douglas, Shapiro and Shields show that a function $f \in H^2$ is pseudocontinuable if and only if f is not S^*-cyclic (see [8]).

Proof. (a) Assume that b is nonextreme and pseudocontinuable.

(ii) $\Longrightarrow$ (i): Follows from the fact that $\mathcal{H}(b) \subset H^2$.

(i) $\Longrightarrow$ (ii): Assume that $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$ and that $(\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence. Denote by B the Blaschke product associated to $(\lambda_n)_{n\geqslant 1}$. Since b is pseudocontinuable, there exists a nonconstant inner function u such that $b \in \mathcal{H}(u)$. Then it follows that $b = \bar{z}\bar{h}u$, where $h \in H^2$. We will show that $k^b_{\lambda_n} \in \mathcal{H}(uB)$, $n \geqslant 1$. For all polymial p, we have

$$\begin{aligned}\langle k^b_{\lambda_n}, Bup\rangle_2 &= \langle k_{\lambda_n}, Bup\rangle_2 - \overline{b(\lambda_n)}\langle bk_{\lambda_n}, Bup\rangle_2\\ &= -\overline{b(\lambda_n)}\langle \bar{z}\bar{h}uk_{\lambda_n}, Bup\rangle_2\\ &= -\overline{b(\lambda_n)}\langle k_{\lambda_n}, zhBp\rangle_2 = 0.\end{aligned}$$

Hence, using the density of polynomials in H^2, we get that $k^b_{\lambda_n} \in \mathcal{H}(uB)$, $n \geqslant 1$. Thus, we have

$$Span_{\mathcal{H}(b)}(k^b_{\lambda_n} : n \geqslant 1) \subset \overline{\mathcal{H}(uB)}^{\mathcal{H}(b)} \subset \mathcal{H}(uB),$$

because $\mathcal{H}(b)$ is contained contractively in H^2. Since the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is complete in $\mathcal{H}(b)$, we get that $\mathcal{H}(b) \subset \mathcal{H}(uB)$. But since b is nonextreme, the polynomials belong to $\mathcal{H}(b)$ (see [19, (IV-2)]) and thus to $\mathcal{H}(uB)$. It follows that $H^2 \subset \mathcal{H}(uB)$, which is absurd.

(b) Assume that b is nonextreme and not pseudocontinuable.

(i) $\Longrightarrow$ (ii): Is trivial.

(ii) $\Longrightarrow$ (i): Using the equality $Xk^b_{\lambda_n} = \overline{\lambda_n}k^b_{\lambda_n} - \overline{b(\lambda_n)}S^*b$ (see [19, (II-9)]), we get that $Span(k^b_{\lambda_n} : n \geqslant 1)$ is invariant under X. But we know that invariant subspaces of the operator X, when b is nonextreme, are just the intersections of $\mathcal{H}(b)$ with the invariant subspaces of S^* (see [18]). Hence there is an inner function u such that

$$Span(k^b_{\lambda_n} : n \geqslant 1) = \mathcal{H}(b) \cap \mathcal{H}(u).$$

But then the fact that $S^*b \in \mathcal{H}(u)$ implies that $b \in \mathcal{H}(uz)$, which is absurd unless $u \equiv 0$ (because b is not pseudocontinuable). Hence $Span(k^b_{\lambda_n} : n \geqslant 1) = \mathcal{H}(b)$. $\square$

Remark 5.1. For the extreme case, an analogue of this result is far from being known, even in the particular case where $b(z) = \exp(-a\frac{1+z}{1-z})$, $a > 0$.

If $(k_{\lambda_n})_{n\geqslant 1}$ is a minimal sequence, then it is well-known that there exists a summable method V such that $(k_{\lambda_n})_{n\geqslant 1}$ is a V-basis of $\mathcal{H}(B)$ (see [14, Lecture VIII, p. 194]). If we make assumption on multipliers of $\mathcal{H}(b)$, we can give an analogue of this result.

First of all, recall that we say that a function $\varphi \in H^\infty$ is a multiplier of $\mathcal{H}(b)$ if $\mathcal{H}(b)$ is invariant under T_φ. From the closed graph Theorem, it follows that T_φ is a bounded operator of $\mathcal{H}(b)$. We denote in this case, $M_\varphi := T_{\varphi|\mathcal{H}(b)}$.

Many authors study multipliers of $\mathcal{H}(b)$ (see for instance [12,13] or [5]). In particular, it is proved in [12] that if b is extreme, then $\mathcal{H}(b)$ does not have inner multipliers.

Theorem 5.1. *Let $b \in H^\infty$, $\|b\|_\infty \leqslant 1$ and $(\lambda_n)_{n\geqslant 1} \in \mathbb{D}$. Assume that b is nonextreme and that B is a multiplier of $\mathcal{H}(b)$. Then the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is minimal. Moreover, if $(\varphi_n)_{n\geqslant 1}$ is the unique biorthogonal of $(k^b_{\lambda_n})_{n\geqslant 1}$, with $\varphi_n \in Span(k^b_{\lambda_n} : n \geqslant 1)$, then for all function $f \in Span(k^b_{\lambda_n} : n \geqslant 1)$, we have*

$$f = \sum_{n\geqslant 1} \overline{B^p(\lambda_n)} \langle f, \varphi_n \rangle_b k^b_{\lambda_n},$$

where $B^{(p)} = \prod_{n\geqslant p} b_{\lambda_n}$. In particular, we have

$$Span(\varphi_n : n \geqslant 1) = Span(k^b_{\lambda_n} : n \geqslant 1).$$

Proof. Recall that when b is nonextreme, the polynomials belong to $\mathcal{H}(b)$ (see [19, (IV-2)]), thus $\mathbb{1} \in \mathcal{H}(b)$. Since B is a multiplier of $\mathcal{H}(b)$, we get that $B \in \mathcal{H}(b)$. It follows that

$$B_n = \frac{B}{b_{\lambda_n}} = P_+(\overline{b_{\lambda_n}} B) = T_{\overline{b_{\lambda_n}}} B \in \mathcal{H}(b),$$

because $\mathcal{H}(b)$ is invariant under $T_{\overline{\varphi}}$, for all $\varphi \in H^\infty$. Moreover, we have

$$\left\langle \frac{B_n}{B_n(\lambda_n)}, k^b_{\lambda_p} \right\rangle_b = \frac{B_n(\lambda_p)}{B_n(\lambda_n)} = \delta_{n,p},$$

which implies that $\left(\frac{B_n}{B_n(\lambda_n)}\right)_{n\geqslant 1}$ is a biorthogonal of $(k^b_{\lambda_n})_{n\geqslant 1}$. Thus the sequence $(k^b_{\lambda_n})_{n\geqslant 1}$ is minimal.

Now we will show that $B^{(p)}$ is multiplier of $\mathcal{H}(b)$ and that $\|M_{B^{(p)}}\|\leqslant\|M_B\|$, for all $p\geqslant 1$.

First notice that if $\widetilde{B}^{(p)} := \prod_{n<p} b_{\lambda_n}$, then for all $f\in\mathcal{H}(b)$, we have

$$B^{(p)}f = P_+(\overline{\widetilde{B}^{(p)}}Bf) = T_{\overline{\widetilde{B}^{(p)}}}(Bf).$$

Since B is a multiplier, we have $Bf\in\mathcal{H}(b)$ and thus $B^{(p)}f\in\mathcal{H}(b)$. Moreover, we have

$$\|B^{(p)}f\|_b\leqslant\|\widetilde{B}^{(p)}\|_\infty\|M_B\|\|f\|_b = \|M_B\|\|f\|_b,$$

because the norm of $T_{\overline{\widetilde{B}^{(p)}}}$ as an operator of $\mathcal{H}(b)$ does not exceed $\|\widetilde{B}^{(p)}\|_\infty$ (see [19, (II-7)]). That proves that $B^{(p)}$ is multiplier of $\mathcal{H}(b)$ and $\|M_{B^{(p)}}\|\leqslant\|M_B\|$, for all $p\geqslant 1$. On the other hand, since $B^{(p)}$ is multiplier of $\mathcal{H}(b)$, we have

$$M^*_{B^{(p)}}(k^b_{\lambda_n}) = \overline{B^{(p)}(\lambda_n)}k^b_{\lambda_n},$$

(see [19, (II-10)]) and thus $\lim_{p\to+\infty} M^*_{B^{(p)}}(k^b_{\lambda_n}) = k^b_{\lambda_n}$, $n\geqslant 1$. Since $\|M^*_{B^{(p)}}\| = \|M_{B^{(p)}}\|\leqslant\|M_B\|$, Banach–Steinhauss Theorem implies that, for all $f\in \mathit{Span}(k^b_{\lambda_n} : n\geqslant 1)$, we have $\lim_{p\to+\infty} M^*_{B(p)}f = f$. But now it is easy to see that, for all $f\in \mathit{Lin}(k^b_{\lambda_n} : n\geqslant 1)$, we have

$$M^*_{B^{(p)}}f = \sum_{n=1}^{p-1}\overline{B^{(p)}(\lambda_n)}\langle f,\varphi_n\rangle_b k^b_{\lambda_n}.$$

By density, we get this equality for all $f\in \mathit{Span}(k^b_{\lambda_n} : n\geqslant 1)$, which implies, letting p tends to ∞ that

$$f = \sum_{n\geqslant 1}\overline{B^p(\lambda_n)}\langle f,\varphi_n\rangle_b k^b_{\lambda_n}. \qquad \Box$$

Acknowledgments

This work was done while the author was visiting the Department of Mathematics and Statistics of the University of Laval (Quebec). He is grateful to the Department members, specially to T. Ransford, for their warm hospitality.

References

[1] P.R. Ahern, D.N. Clark, Radial limits and invariant subspaces, Amer. J. Math. 92 (2) (1970) 332–342.
[2] P.R. Ahern, D.N. Clark, On inner functions with H^p-derivative, Michigan Math. J. 21 (1974) 115–127.
[3] S.A. Avdonin, S.A. Ivanov, Families of Exponentials, Cambridge University Press, Cambridge, 1995.
[4] I. Chalendar, E. Fricain, D. Timotin, Functional models and asymptotically orthonormal sequences, Ann. Inst. Fourier 53 (5) (2003) 1527–1549.
[5] B.M. Davis, J.E. McCarthy, Multipliers of de Branges' spaces, Michigan Math. J. 38 (1991) 225–240.
[6] L. de Branges, J. Rovnyak, Square Summable Power Series, Holt, Rinehart and Winston, New York, 1966.
[7] K. de Leeuw, W. Rudin, Extreme points and extreme problems in H^1, Pacific J. Math. 8 (1958) 467–485.
[8] R.G. Douglas, H.S. Shapiro, A.L. Shields, Cyclic vectors and invariant subspaces for the backward Shift Operator, Ann. Inst. Fourier 20 (1970) 37–76.
[9] E. Fricain, Bases of reproducing kernels in model spaces, J. Operator Theory 46 (2001) 517–543.
[10] E. Fricain, Complétude des noyaux reproduisants dans les espaces modèles, Ann. Inst. Fourier 52 (2) (2002) 661–686.
[11] S.V. Hruscev, N.K. Nikolski, B.S. Pavlov, Unconditional bases of exponentials and reproducing kernels, Lect. Notes Math. 864 (1981) 214–335.
[12] B.A. Lotto, Inner multipliers of de Branges spaces, Integral Equations Operator Theory 13 (1990) 216–230.
[13] B.A. Lotto, D. Sarason, Multipliers of de Branges–Rovnyak spaces, Indiana Univ. Math. J. 42 (3) (1993) 907–920.
[14] N.K. Nikolski, Treatise on the Shift Operator, vol. 273, Springer, Berlin, 1986.
[15] N.K. Nikolski, Operators, Functions, and Systems: An Easy Reading, Volume 1: Hardy, Hankel, and Toeplitz, vol. 92, Mathematical Surveys and Monograph, American Mathematical Society, Providence, RI, 2002.
[16] N.K. Nikolski, Operators, Functions, and Systems: An Easy Reading, Volume 2: Model Operators and Systems, vol. 93, Mathematical Surveys and Monograph, American Mathematical Society, Providence, RI, 2002.
[17] D. Sarason, Shift-invariant spaces from the Brangesian point of view, in: The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof, vol. 21, Mathematical Surveys and Monographs, 1985, pp. 153–166.
[18] D. Sarason, Doubly shift-invariant spaces in H^2, J. Operator Theory 16 (1986) 75–97.
[19] D. Sarason, Sub-Hardy Hilbert Spaces in the Unit Disk, vol. 10, University of Arkansas Lecture Notes in the Mathematical Sciences, Wiley-Interscience Publication, New York, 1995.
[20] B. Szökefalvi-Nagy, C. Foias, Analyse harmonique des opérateurs de l'espace de Hilbert, Akadémiai Kiado, Budapest, 1967.

Further reading

[21] I. Boricheva, Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2, J. Funct. Anal. 161 (1999) 397–417.
[22] N. Nikolski, V. Vasyunin, Notes on two function models, in: The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof, vol. 21, Mathematical Surveys and Monographs, 1985, pp. 113–141.

6.1.4 Référence [T4]

Auteurs
N. Chevrot, E. Fricain et D. Timotin

Titre
On certain Riesz families in vector-valued de Branges-Rovnyak spaces.

Soumis

ON CERTAIN RIESZ FAMILIES IN VECTOR-VALUED DE BRANGES–ROVNYAK SPACES

NICOLAS CHEVROT, EMMANUEL FRICAIN, AND DAN TIMOTIN

Abstract. We obtain criteria for the Riesz basis property for families of reproducing kernels in vector-valued de Branges–Rovnyak spaces $\mathcal{H}(b)$. It is shown that in several situations the property implies a special form for the function b. We also study the completeness of a related family.

1. Introduction

Starting with the works of Paley–Wiener ([PW34]), a whole direction of research has investigated families of exponentials in $L^2(\mathbb{R})$, looking for properties as completeness, minimality, or being an unconditional basis. A classical result is here Ingham's Theorem [Ing34], which says roughly that a small perturbation of the standard exponentials e^{int} remains a basis in $L^2(-\pi,\pi)$. This line of approach, the consideration of a given family of exponentials as a small perturbation of one that is known to be complete or a Riesz basis has subsequently yielded many stability results. One should also mention that the theory of geometric properties of scalar or vector valued exponential families has found applications in various areas such as convolution equations, string scattering theory or controllability of dynamical systems (see [HNP81, AI95, Nik02b] for a survey on exponentials systems and their applications).

In this context, functional models have been used in [HNP81], allowing the use of tools from operator theory on a Hilbert space. The model spaces are subspaces of the Hardy space H^2, invariant under the adjoints of multiplications; their theory is

2000 *Mathematics Subject Classification.* Primary: 46E15, 46E22 Secondary: 30D55, 47A15.

Key words and phrases. de Branges–Rovnyak spaces, bases of reproducing kernels, completeness.

connected to dilation theory for contractions on Hilbert spaces (see [SNF67]). The approach has proved fruitful, leading to the recapture of all the classical results as well as to several generalizations.

This investigation has been pursued with respect to families of reproducing kernels in vector-valued and scalar model spaces in [Fri01], [Fri02] and [CFT03], and in scalar de Branges–Rovnyak spaces in [Fri05]. Again the main goal is to obtain criteria for a family of reproducing kernels to be complete, minimal or Riesz basis. We also mention an interesting paper of A. Baranov [Bar06] whose criteria is based on recent work of Ortega-Cerdà and Seip [OCS02].

In this paper we investigate similar problems in the context of vector-valued de Branges–Rovnyak spaces. It appears that the functional methods used in [Fri01, Fri05] are no more appropriate in this situation, and we have to find a new approach. This is essentially done by using in more detail the structure of the model theory of contractions [SNF67], and especially its relation to vector valued de Branges–Rovnyak spaces as emphasized in [NV85]. This new approach throws some light also on the scalar-valued case; we show, for instance, that if the scalar-valued de Branges–Rovnyak space $\mathcal{H}(b)$ admits a Riesz basis of reproducing kernels, then necessarily b is inner. Similar methods are used to investigate properties of a different family of functions, called difference quotients; in particular, in the scalar case, we show that the difference quotients are complete in $\mathcal{H}(b)$ if and only if either b is an extreme point of the unit ball of H^∞ or b is not pseudocontinuable.

The plan of the paper is the following. The next section contains some preliminary material. The connection of the de Branges–Rovnyak spaces to the functional model for contractions is described in detail in Section 3, following [NV85, NV98], where a main notion is that of *abstract functional embedding*, introduced in [Vas77]. Section 4 shows how the problems concerning bases of reproducing kernels can be reduced, under suitable hypotheses, to the study of the invertibility of a certain operator (the *distortion* operator). Criteria for this invertibility are given in Section 5, which contains the main results of the paper. A different type of criterium appears in

Section 6, while Section 7 contains some interesting examples. Finally, Section 8 studies completeness properties of the difference quotients.

2. Preliminaries

2.1. **Hardy spaces and de Branges–Rovnyak spaces.** If E is a separable complex Hilbert space, $L^2(E)$ is the usual L^2-space of E-valued functions f on the unit circle $\mathbb{T}$ with respect to the normalized measure m endowed with the norm

$$\|f\|_2^2 = \int_{\mathbb{T}} \|f(z)\|_E^2 \, dm(z).$$

The corresponding Hardy space $H^2(E)$ is defined as E-valued analytic functions on $\mathbb{D}$, $f(z) = \sum_{n\geq 0} a_n z^n$, $a_n \in E$, with $\|f\|_2 < +\infty$, where

$$\|f\|_2^2 = \sum_{n\geq 0} \|a_n\|_E^2.$$

Alternately, it is well-known that $H^2(E)$ can be regarded as the closed subspace of $L^2(E)$ consisting of functions whose negative Fourier coefficients vanish. The symbol P_+ (respectively P_-) stands for the Riesz orthogonal projection from $L^2(E)$ onto $H^2(E)$ (respectively onto $H^2_-(E) := L^2(E) \ominus H^2(E)$).

If E, E_* are two separable Hilbert spaces, we denote by $\mathcal{L}(E, E_*)$ the space of all bounded linear operators from E to E_*. Then $L^\infty(E \to E_*)$ is the Banach space of weakly measurable essentially bounded functions defined on $\mathbb{T}$ with values in $\mathcal{L}(E, E_*)$, endowed with the essential norm. The Banach space $H^\infty(E \to E_*)$ is formed by bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(E, E_*)$; taking (strong) radial limits identifies $H^\infty(E \to E_*)$ with a subspace of $L^\infty(E \to E_*)$.

If $\varphi \in L^\infty(E \to E_*)$, we will make a standard abuse of notation and denote by the same symbol φ the multiplication operator

$$\begin{array}{rccc} \varphi: & L^2(E) & \longrightarrow & L^2(E_*) \\ & f & \longmapsto & \varphi f \end{array}$$

defined by $(\varphi f)(\zeta) := \varphi(\zeta) f(\zeta)$, $\zeta \in \mathbb{T}$. The inclusion $\varphi H^2(E) \subset H^2(E_*)$ is equivalent to $\varphi \in H^\infty(E \to E_*)$, while $\|\varphi\| \leq 1$ (or $\|\varphi | H^2(E)\| \leq 1$) is equivalent to

$\|\varphi(\zeta)\| \le 1$ a.e. on $\mathbb{T}$. The symbol T_φ denotes the *Toeplitz* operator from $H^2(E)$ to $H^2(E_*)$ defined by

$$T_\varphi f := P_+(\varphi f).$$

Then $T_\varphi \in \mathcal{L}(H^2(E), H^2(E_*))$, $\|T_\varphi\| = \|\varphi\|_\infty$, and $T_\varphi^* = T_{\varphi^*}$, where $\varphi^* \in L^\infty(E_* \to E)$ is defined by $\varphi^*(\zeta) := (\varphi(\zeta))^*$, $\zeta \in \mathbb{T}$.

We will also use occasionally the *Hankel* operator $\mathrm{H}_\varphi : H^2(E) \to H^2_-(E_*)$ defined by

$$\mathrm{H}_\varphi(f) = P_-(\varphi f).$$

We have then

$$\varphi f = T_\varphi f + \mathrm{H}_\varphi f, \qquad \|\varphi f\|^2 = \|T_\varphi f\|^2 + \|\mathrm{H}_\varphi f\|^2. \tag{2.1}$$

The vector valued Nehari Theorem says that $\|\mathrm{H}_\varphi\| = \mathrm{dist}(\varphi, H^\infty(E \to E_*))$.

Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \le 1$. The *de Branges–Rovnyak space* $\mathcal{H}(b)$, associated to b, is the vector space of those $H^2(E_*)$ functions which are in the range of the operator $(Id - T_b T_b^*)^{1/2}$; it becomes a Hilbert space when equipped with the inner product

$$\langle (Id - T_b T_b^*)^{1/2} f, (Id - T_b T_b^*)^{1/2} g\rangle_b := \langle f, g\rangle_2,$$

where $f, g \in H^2(E_*) \ominus \ker(Id - T_b T_b^*)^{1/2}$ (see [dBR66, BK87]; [Sar95] contains an extensive presentation of the scalar case). Note that $\mathcal{H}(b)$ is contained contractively in $H^2(E_*)$ and the inner product is defined in order to make $(Id - T_b T_b^*)^{1/2}$ a coisometry from $H^2(E_*)$ to $\mathcal{H}(b)$. The norm of $\mathcal{H}(b)$ will be denoted by $\|\cdot\|_b$.

An important particular case is obtained for b an inner function, that is, a function in $H^\infty(E \to E_*)$ such that $b(\zeta)$ is an isometry for almost all $\zeta \in \mathbb{T}$. Then $\mathcal{H}(b)$ is a *closed* subspace of $H^2(E_*)$, and $\|\cdot\|_b$ coincides with the induced norm; more precisely, we have $\mathcal{H}(b) = H^2(E_*) \ominus bH^2(E)$. By the Lax–Halmos Theorem, these are the nontrivial subspaces of $H^2(E_*)$ which are invariant for the backward shift $S^*|H^2(E_*)$. They are traditionally denoted by K_b; thus, in this case, we have $\mathcal{H}(b) = K_b$.

For further use, remember that b is called $*$-*inner* if $b(\zeta)$ is a coisometry for almost all $\zeta \in \mathbb{T}$. This is equivalent to $\tilde{b}(\zeta) := b(\bar{\zeta})^*$ being inner.

2.2. **Reproducing kernels.** If $\lambda \in \mathbb{D}$ and $e \in E_*$, the function $k_{\lambda,e}(z) = \frac{1}{1-\overline{\lambda}z}e$ belongs to $H^2(E_*)$ and is a reproducing kernel for this space; that is, for any $f \in H^2(E_*)$ we have $\langle f(\lambda), e\rangle_{E_*} = \langle f, k_{\lambda,e}\rangle_{H^2(E_*)}$. Since $\mathcal{H}(b)$ is contained contractively in $H^2(E_*)$, this formula defines also a bounded linear functional on $\mathcal{H}(b)$, which, according to Riesz's theorem, is given by the inner product in $\mathcal{H}(b)$ with a vector $k^b_{\lambda,e} \in \mathcal{H}(b)$; thus, for all $f \in \mathcal{H}(b)$, $\langle f, k^b_{\lambda,e}\rangle_b = \langle f(\lambda), e\rangle_{E_*}$. A computation similar to the case of scalar de Branges–Rovnyak spaces (see [Sar95, Ch.2]) yields the formula

$$k^b_{\lambda,e}(z) = (Id - T_bT_b^*)k_{\lambda,e} = \frac{1}{1-\overline{\lambda}z}(Id - b(z)b(\lambda)^*)e \tag{2.2}$$

for the reproducing kernels in $\mathcal{H}(b)$. Also, it follows easily that

$$\|k_{\lambda,e}\|_2^2 = \frac{\|e\|^2}{1-|\lambda|^2}, \qquad \|k^b_{\lambda,e}\|_b^2 = \frac{\|e\|^2 - \|b(\lambda)^*e\|^2}{1-|\lambda|^2}. \tag{2.3}$$

We denote by $\kappa_{\lambda,e}$ and $\kappa^b_{\lambda,e}$ the normalized reproducing kernels of $H^2(E_*)$ and $\mathcal{H}(b)$ respectively; that is

$$\kappa_{\lambda,e}(z) = \frac{\sqrt{1-|\lambda|^2}}{(1-\overline{\lambda}z)\|e\|}e$$

and

$$\kappa^b_{\lambda,e}(z) = \frac{\sqrt{1-|\lambda|^2}}{(1-\overline{\lambda}z)\sqrt{\|e\|^2 - \|b(\lambda)^*e\|^2}}(Id - b(z)b(\lambda)^*)e.$$

We will also discuss properties of another interesting family of elements of the de Branges–Rovnyak space $\mathcal{H}(b)$: the so-called *difference quotients*, defined by

$$\hat{k}^b_{\lambda,e} = \frac{1}{z-\lambda}(b(z)-b(\lambda))e, \quad \lambda \in \mathbb{D}, \quad e \in E. \tag{2.4}$$

3. A geometric approach to the de Branges–Rovnyak space

The function-theoretical approach, as developed for the scalar case in [Sar95], is no more adequate when dealing with vector-valued de Branges–Rovnyak spaces. We will use a more geometric description, connected to the model theory for contractions. The main source for this point of view is [NV98] (see also [NV85], as well as the exposition of [Nik02b]).

We start with an *abstract functional embedding (AFE)*. This is a linear mapping

$$\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K,$$

satisfying the following properties:

(1) the restrictions π and π_* are isometries;
(2) $\pi H^2(E) \perp \pi_* H^2_-(E_*)$;
(3) the range of Π is dense in K;
(4) $\pi_*^*\pi$ commutes with the shift operator and maps $H^2(E)$ into $H^2(E_*)$; hence we know (see [Nik02b, Lemma 1.2.3]) that $\pi_*^*\pi = b$, with b being a contractive $H^\infty(E \to E_*)$ function.

Set $\Delta = (Id - b^*b)^{1/2}$ and $\Delta_* = (Id - bb^*)^{1/2}$. Since

$$\|(\pi - \pi_* b)f\|_K = \|\Delta f\|_2, \qquad \|(\pi_* - \pi b^*)g\|_K = \|\Delta_* g\|_2,$$

for every $f \in L^2(E)$ and $g \in L^2(E_*)$, the equalities

$$\tau\Delta = \pi - \pi_* b, \qquad \tau_*\Delta_* = \pi_* - \pi b^*$$

determine the partial isometries

$$\tau : L^2(E) \longrightarrow K, \qquad \tau_* : L^2(E_*) \longrightarrow K$$

with initial spaces $\operatorname{clos}(\Delta L^2(E))$ and $\operatorname{clos}(\Delta_* L^2(E_*))$ respectively. It is easy to see that

$$\tau^*\pi = \Delta, \quad \tau^*\pi_* = 0, \quad \tau_*^*\pi = 0, \quad \tau_*^*\pi_* = \Delta_* \tag{3.1}$$

and

$$Id = \pi\pi^* + \tau_*\tau_*^* = \pi_*\pi_*^* + \tau\tau^*. \tag{3.2}$$

In particular, we get from (3.1) and (3.2) the following decompositions:

$$K = \pi(L^2(E)) \oplus \tau_*(L^2(E_*)) = \pi_*(L^2(E_*)) \oplus \tau(L^2(E)). \tag{3.3}$$

Note that, in contrast to [NV98], we do not include the purity of $\pi_*^*\pi$ in the definition (since we are not interested in the correspondence with the model contraction).

It is shown that the operator U_Π defined by the relation $U_\Pi \Pi = \Pi z$ is a unitary on K.

Now for a given AFE Π, we define $\mathbb{H} = K \ominus (\pi(H^2(E)) \oplus \pi_*(H^2_-(E_*))$; thus

$$K = \mathbb{H} \oplus \pi(H^2(E)) \oplus \pi_*(H^2_-(E_*)), \tag{3.4}$$

and

$$P_{\mathbb{H}} = Id - \pi P_+ \pi^* - \pi_* P_- \pi_*^*. \tag{3.5}$$

The space $\mathbb{H}$ is further decomposed as

$$\mathbb{H} = \mathbb{H}' \oplus \mathbb{H}'' = \mathbb{H}'_* \oplus \mathbb{H}''_*,$$

where $\mathbb{H}'' = \mathbb{H} \cap \tau(L^2(E)) = \mathbb{H} \cap \tau(\operatorname{clos}(\Delta L^2(E)))$, $\mathbb{H}' = \mathbb{H} \ominus \mathbb{H}''$, and $\mathbb{H}''_* = \mathbb{H} \cap \tau_*(L^2(E_*)) = \mathbb{H} \cap \tau_*(\operatorname{clos}(\Delta L^2(E_*)))$, $\mathbb{H}'_* = \mathbb{H} \ominus \mathbb{H}''_*$. Note also that (3.3) and (3.4) imply that actually $\mathbb{H}'' = \mathbb{H} \cap \pi_*(H^2(E_*))^\perp$ and $\mathbb{H}''_* = \mathbb{H} \cap \pi(H^2_-(E))^\perp$.

The following simple lemma will be used several times in the sequel.

Lemma 3.1. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^* \pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. Then, we have*

$$\mathbb{H}'' = \tau\left(\operatorname{clos}(\Delta L^2(E)) \ominus \operatorname{clos}(\Delta H^2(E))\right)$$

and

$$\mathbb{H}''_* = \tau_*\left(\operatorname{clos}(\Delta_* L^2(E_*)) \ominus \operatorname{clos}(\Delta_* H^2_-(E_*))\right).$$

Consequently, $\mathbb{H} = \mathbb{H}'$ if and only if $\operatorname{clos}(\Delta H^2(E)) = \operatorname{clos}(\Delta L^2(E))$, and $\mathbb{H} = \mathbb{H}'_*$ if and only if $\operatorname{clos}(\Delta_* H^2_-(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$.

Proof. Suppose $f = \tau g$ with $g \in \operatorname{clos}(\Delta L^2(E))$. By (3.1), it follows that $f \perp \pi_* L^2(E_*)$; in particular, $f \perp \pi_* H^2_-(E_*)$. Then

$$f \in \mathbb{H}'' \Leftrightarrow f \in \mathbb{H} \Leftrightarrow f \perp \pi(H^2(E))$$

Using again (3.1), one obtains that $f \perp \pi(H^2(E))$ is equivalent to

$$0 = \langle \tau g, \pi h \rangle = \langle g, \tau^* \pi h \rangle = \langle g, \Delta h \rangle$$

for all $h \in H^2(E)$, which proves the first assertion of the Lemma. The second assertion follows from similar arguments, while for the last part we have only to remember that τ and τ_* are isometries on $\operatorname{clos}(\Delta L^2(E))$ and $\operatorname{clos}(\Delta_* L^2(E_*))$ respectively. $\square$

In the end of this section we connect the abstract functional embeddings with the de Branges–Rovnyak spaces.

Lemma 3.2. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^* \pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. Then*

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}} \pi_* | H^2(E_*) = \pi_*^* P_{\mathbb{H}'} \pi_* | H^2(E_*).$$

Proof. Using (3.5) as well as the relations $\pi_*^* \pi_* = Id$ and $\pi_*^* \pi = b$, we obtain

$$\pi_*^* P_{\mathbb{H}} \pi_* = \pi_*^* \pi_* - \pi_*^* \pi P_+ \pi^* \pi_* - \pi_*^* \pi_* P_- \pi_*^* \pi_* = P_+ - b P_+ b^*.$$

whence

$$\pi_*^* P_{\mathbb{H}} \pi_* | H^2(E_*) = (P_+ - b P_+ b^*) | H^2(E_*) = Id - T_b T_b^*.$$

The second equality follows since $\mathbb{H} \ominus \mathbb{H}'$ is contained in the kernel of π_*^* according to (3.1). $\square$

Proposition 3.3. *Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE and let $b = \pi_*^* \pi$ be the contractive $H^\infty(E \to E_*)$ function associated to Π. The operator π_*^* is a coisometry from $\mathbb{H}$ onto $\mathcal{H}(b)$, with $\ker \pi_*^* | \mathbb{H} = \mathbb{H}''$. In particular, if $\operatorname{clos}(\Delta L^2(E)) = \operatorname{clos}(\Delta H^2(E))$, then $\pi_*^* : \mathbb{H} \to \mathcal{H}(b)$ is unitary.*

Proof. First, using (3.4), we see that $\pi_*^* \mathbb{H} \subset H^2(E_*)$. Then, if we put $A = \pi_*^* | \mathbb{H} : \mathbb{H} \to H^2(E_*)$ and $B = (Id - T_b T_b^*)^{1/2}$, Lemma 3.2 shows that $AA^* = B^2$. There exists therefore a partial isometry $U : \mathbb{H} \longrightarrow H^2(E_*)$, with initial space $(\ker A)^\perp$ and final space the closure of the range of B, such that $A = BU$. The definition of the norm of $\mathcal{H}(b)$ implies then that $A = \pi_*^* | \mathbb{H}$ is a partial isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$. Its kernel is

$$\ker \pi_*^* | \mathbb{H} = \mathbb{H} \cap \ker \pi_*^* = \mathbb{H} \cap (\operatorname{Im}\ \pi_*)^\perp = \mathbb{H}'',$$

whence the proof is ended using Lemma 3.1. ☐

Remark 3.4. In [NV85] (see also [Nik02a, pp. 84-86]), some conditions equivalent to $\mathrm{clos}(\Delta H^2(E)) = \mathrm{clos}(\Delta L^2(E))$ are given; in particular, one of them is the density of the polynomials in $L^2(E, \Delta)$. In [Tre86], S. Treil shows that b is an extreme point in the unit ball of $H^\infty(E \to E_*)$ if and only if either $\mathrm{clos}(\Delta H^2(E)) = \mathrm{clos}(\Delta L^2(E))$ or $\mathrm{clos}(\Delta_* H^2_-(E_*)) = \mathrm{clos}(\Delta_* L^2(E_*))$, which is equivalent (according to Lemma 3.1) to $\mathbb{H} = \mathbb{H}'$ or $\mathbb{H} = \mathbb{H}'_*$. In the scalar case $\dim E = \dim E_* = 1$, b is extreme if and only if $\log(1 - |b|)$ is not integrable on $\mathbb{T}$ (see [dLR58]).

In Section 8 we will use the Sz.-Nagy–Foiaş transcription of the model. More precisely, starting from any contractive function $b \in H^\infty(E \to E_*)$, we can construct an abstract functional embedding Π_b by defining $K = L^2(E_*) \oplus \mathrm{clos}(\Delta L^2(E))$, where $\Delta = (Id - b^* b)^{1/2}$, and

$$\Pi_b(f \oplus g) = (g + bf) \oplus \Delta f, \quad f \in L^2(E), g \in L^2(E_*). \tag{3.6}$$

We have then

$$\mathbb{H} = \big(H^2(E_*) \oplus \mathrm{clos}(\Delta L^2(E))\big) \ominus \{bf \oplus \Delta f : f \in H^2(E)\}. \tag{3.7}$$

This explicit transcription is related to the construction of the Sz-Nagy–Foiaş model for contractions on Hilbert spaces [SNF67].

4. Riesz bases of reproducing kernels

The main problems that we intend to study are the following: given $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$, given a sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ and a sequence $(e_n)_{n\geq 1} \subset E_*$, $\|e_n\| = 1$, $n \geq 1$, find criteria for the sequence $(\kappa^b_{\lambda_n, e_n})_{n\geq 1}$ to form

(P1): a Riesz basis of its closed linear hull;

(P2): a Riesz basis of $\mathcal{H}(b)$.

We will not, however, study these problems in the most general form. First, note that if $\dim E_* = +\infty$ and $(e_n)_{n\geq 1}$ is an orthonormal sequence in E_*, then $(\kappa_{\lambda_n, e_n})_{n\geq 1}$ is an orthonormal sequence in $H^2(E_*)$, for any choice of sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{D}$. In some sense, if E_* is an infinite dimensional Hilbert space, there is too much freedom for the vectors e_n to hope to get a satisfactory criterion for Riesz basis. That is why interesting results have usually been obtained under the condition dim $E_* < +\infty$ (see [Tre89, AI95]). This condition will be assumed in the next two sections.

Now, it is easy to see that if $(\kappa^b_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis, then $(\kappa_{\lambda_n,e_n})_{n\geq 1}$ is minimal, which implies that $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence [AI95, page 65-67]. Therefore we will also suppose, in the sequel, that the sequence $(\lambda_n)_{n\geq 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$. We have then $\mathrm{span}(\kappa_{\lambda_n,e_n} : n \geq 1) = H^2(E_*) \ominus BH^2(E_*) = K_B$, where the inner function $B \in H^\infty(E_* \to E_*)$ is a Blaschke–Potapov product (see, for instance, [Nik74]).

At this point, the technique originating in [Nik80] (and which is used in [Fri01, Fri05]) regards the family $(\kappa^b_{\lambda_n,e_n})_{n\geq 1}$ as a "distortion" of $(\kappa_{\lambda_n,e_n})_{n\geq 1}$. It is then assumed that $Id - T_bT_b^*$ does not distort very much the norms of the reproducing kernels, in the sense that

$$\sup_{n\geq 1} \frac{\|k_{\lambda_n,e_n}\|_2}{\|(Id - T_bT_b^*)k_{\lambda_n,e_n}\|_b} < +\infty.$$

Using (2.3), we see that this condition is equivalent to

$$\sup_{n\geq 1} \|b(\lambda_n)^* e_n\| < 1. \tag{4.1}$$

Under this further condition, we can state the following result.

Theorem 4.1. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$, let $(\lambda_n)_{n\geq 1}$ be a Blaschke sequence in $\mathbb{D}$ and let $(e_n)_{n\geq 1} \subset E_*$, $\|e_n\| = 1$. Assume that dim $E_* < +\infty$ and that condition (4.1) is satisfied. Then the following are equivalent:*

a) *the sequence $(\kappa^b_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$);*

b) *the sequence $(\kappa_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of K_B and the operator*

$$(Id - T_bT_b^*)|K_B : K_B \longrightarrow \mathcal{H}(b)$$

is an isomorphism onto its range (resp. onto $\mathcal{H}(b)$).

Proof. a) $\Rightarrow$ b) By formula (2.2) we have $(Id - T_bT_b^*)(k_{\lambda_n,e_n}) = k^b_{\lambda_n,e_n}$ and condition (4.1) implies that $\|k^b_{\lambda_n,e_n}\|_b \asymp \|k_{\lambda_n,e_n}\|_2$. It follows then (see, for instance, [HNP81, page 228]) that the uniform minimality of $(k^b_{\lambda_n,e_n})_{n\geq 1}$ implies the uniform minimality of $(k_{\lambda_n,e_n})_{n\geq 1}$. But, according to a result of S. Treil [Tre89], since $\dim E_* < \infty$, the latter is equivalent to the fact that the sequence $(\kappa_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of K_B. Since the operator $(Id - T_bT_b^*)|K_B$ maps one Riesz basis onto another, it is an isomorphism of K_B onto $\operatorname{span}(\kappa^b_{\lambda_n,e_n} : n \geq 1)$.

b) $\Rightarrow$ a) Conversely, if $(Id - T_bT_b^*)|K_B$ is an isomorphism onto its range and the sequence $(\kappa_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of K_B, then $((Id - T_bT_b^*)\kappa_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of its closed linear hull. But

$$(Id - T_bT_b^*)\kappa_{\lambda_n,e_n} = \frac{\|k^b_{\lambda_n,e_n}\|_b}{\|k_{\lambda_n,e_n}\|_2}\kappa^b_{\lambda_n,e_n},$$

and since $\frac{\|k^b_{\lambda_n,e_n}\|_b}{\|k_{\lambda_n,e_n}\|_2}$ is bounded from below and above, we obtain that the sequence $(\kappa^b_{\lambda_n,e_n})_{n\geq 1}$ is a Riesz basis of its closed linear hull. Moreover, if $(Id - T_bT_b^*)|K_B$ is an isomorphism onto $\mathcal{H}(b)$, we have

$$\operatorname{span}(\kappa^b_{\lambda_n,e_n} : n \geq 1) = \operatorname{span}((Id - T_bT_b^*)k_{\lambda_n,e_n} : n \geq 1) = \mathcal{H}(b). \qquad \square$$

Remark 4.2. Till now the elaborated theory [Nik80, Fri01, Fri05] works under condition (4.1) only. This is not surprising in view of the method used, which is based on projecting a basis from K_B; therefore the first thing to require is that the size of the individual elements of the base should not be changed too drastically. In the particular case of exponential families, this condition means that the imaginary parts of the frequencies of the exponentials are bounded below, which is the case for families arising from control theory. It should be mentioned however that in [Bar06] Baranov gives certain criteria for a family of reproducing kernels to be a Riesz basis in a model subspace associated to a meromorphic inner function, without using the assumption (4.1).

Remark 4.3. Theorem 4.1 reduces the problem of finding Riesz bases in $\mathcal{H}(b)$ to the case of K_B. To apply it, we should be able first to decide when a reproducing sequence of kernels in $H^2(E_*)$ forms a Riesz sequence. Such a criterion has been given by S. Ivanov (see [AI95, page 73]); we need some further notations in order to state it.

We define, for $\lambda \in \mathbb{D}$ and $r > 0$, the pseudo-hyperbolic disc

$$\omega(\lambda, r) := \{z \in \mathbb{D} : |b_\lambda(z)| < r\}, \quad \text{where } b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Then, for a sequence $\Lambda = (\lambda_n)_{n\geq 1}$ in $\mathbb{D}$, we set

$$G(\Lambda, r) = \bigcup_{n\geq 1} \omega(\lambda_n, r).$$

For $m \geq 1$, we denote by $G_m(\Lambda, r)$ the connected components of the set $G(\Lambda, r)$ and we write

$$E_m(r) := \{n \geq 1 : \lambda_n \in G_m(\Lambda, r)\}.$$

Finally, we put $N := \dim E_*$. Then the sequence $(\kappa_{\lambda_n, e_n})_{n\geq 1}$ is a Riesz basis of its closed linear hull if and only if the two following conditions are satisfied:

a) the sequence $(\lambda_n)_{n\geq 1}$ is the union of at most N Carleson sets;
b) there exists $r > 0$ such that

$$\inf_{m\geq 1} \min_{n\in E_m(r)} \alpha(e_n, \operatorname{span}(e_p : p \in E_m(r),\, p \neq n)) > 0,$$

where $\alpha(e_n, Y)$ denotes the angle between the vector e_n and the subspace Y.

We will call $(Id - T_b T_b^*)|K_B : K_B \to \mathcal{H}(b)$ the *distortion operator*. By Theorem 4.1 and Remark 4.3, problems P1 and P2 are reduced, in case condition (4.1) is satisfied, to the following: find criteria for the distortion operator to be

(P1′): an isomorphism onto its range;

(P2′): an isomorphism onto $\mathcal{H}(b)$.

These problems will be addressed in the next section.

We end this section by stating a stability result. The proof is similar to the analogous result for model subspaces (i.e. the inner case) obtained in [Fri05, Theorem 3.4], and will therefore be omitted.

Theorem 4.4. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \le 1$, $\Lambda = (\lambda_n)_{n\ge1}$ be a Blaschke sequence in $\mathbb{D}$ and let $(e_n)_{n\ge1} \subset E_*$, $\|e_n\| = 1$ such that the sequence $(\kappa^b_{\lambda_n,e_n})_{n\ge1}$ is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$). Assume that dim $E_* < +\infty$ and that condition (4.1) is satisfied. Then there exists $\varepsilon > 0$ such that any sequence $(\kappa^b_{\mu_n,a_n})_{n\ge1}$ satisfying*

$$|b_{\lambda_n}(\mu_n)| \le \varepsilon \quad \text{and} \quad \|a_n - e_n\| \le \varepsilon, \quad n \ge 1,$$

is a Riesz basis of its closed linear hull (resp. of $\mathcal{H}(b)$).

Let us also mention that in the scalar de Branges–Rovnyak spaces, using a different approach based on Bernstein type inequalities, a stability result was found in [BFM] without the assumption (4.1). However, the techniques used therein do not seem adaptable to the vector case.

5. The distortion operator

We will discuss in this section the invertibility of the distortion operator $(Id - T_bT_b^*)|K_\Theta : K_\Theta \to \mathcal{H}(b)$, for a general inner function $\Theta \in H^\infty(F \to E_*)$ and $b \in H^\infty(E \to E_*)$ contractive. As noted above, the methods used in the scalar case in [HNP81, Fri05] and in the inner vector case in [Fri01], are no more appropriate, and we have to use a different approach, based on the AFE introduced in Section 3.

We start by reminding a simple lemma, whose proof we omit.

Lemma 5.1. *Suppose we have two orthogonal decompositions of a Hilbert space $\mathfrak{H}$:*

$$\mathfrak{H} = \mathfrak{X}_1 \oplus \mathfrak{X}_2 = \mathfrak{Y}_1 \oplus \mathfrak{Y}_2.$$

Then the following statements are all equivalent:

(1) $P_{\mathfrak{Y}_1}|\mathfrak{X}_1$ *is surjective.*
(2) $P_{\mathfrak{X}_1}|\mathfrak{Y}_1$ *is bounded below.*
(3) $\|P_{\mathfrak{X}_2}|\mathfrak{Y}_1\| < 1$.
(4) $\|P_{\mathfrak{Y}_1}|\mathfrak{X}_2\| < 1$.
(5) $P_{\mathfrak{Y}_2}|\mathfrak{X}_2$ *is bounded below.*

(6) $P_{\mathfrak{X}_2}|\mathfrak{Y}_2$ *is surjective.*

Here for a (closed) subspace E of $\mathfrak{H}$, the notation P_E denotes the orthogonal projection of $\mathfrak{H}$ onto E.

The first result gives the answer to Problem (P1′).

Theorem 5.2. *Let* $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$ *and let* $\Theta \in H^\infty(F \to E_*)$ *be an inner function. The distortion operator is an isomorphism onto its range if and only if* $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$.

Proof. Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE such that $\pi_*^*\pi = b$. Recall that according to Lemma 3.2, we have

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}} \pi_* | H^2(E_*).$$

Moreover by Proposition 3.3, π_*^* is a partial isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$ with kernel equals to $\mathbb{H}''$. Since $P_{\mathbb{H}}\pi_* L^2(E_*) \subset (\ker \pi_*^* | \mathbb{H})^\perp$, we have that $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is an isomorphism onto its range if and only if $P_{\mathbb{H}}|\pi_* K_\Theta$ is bounded below. Applying Lemma 5.1, this last assertion is equivalent to

$$\|P_{K \ominus \mathbb{H}} | \pi_* K_\Theta\| < 1.$$

Now, $K \ominus \mathbb{H} = \pi(H^2(E)) \oplus \pi_*(H^2_-(E_*))$, and the second term in the orthogonal sum is orthogonal to $\pi_* K_\Theta$. Thus the condition is equivalent to $\|P_{\pi(H^2(E))}|\pi_* K_\Theta\| < 1$, or, passing to the adjoint, $\|P_{\pi_* K_\Theta}|\pi(H^2(E))\| < 1$.

But we obviously have

$$\|P_{\pi_* K_\Theta}|\pi(H^2(E))\| = \|\pi_* P_{K_\Theta} \pi_*^* \pi | H^2(E)\| = \|P_{K_\Theta} b | H^2(E)\|$$

while, using the vector valued Nehari Theorem,

$$\|P_{K_\Theta} b | H^2(E)\| = \|\Theta P_- \Theta^* b | H^2(E)\| = \|\mathrm{H}_{\Theta^* b}\| = \operatorname{dist}(\Theta^* b, H^\infty(E \to F)).$$

This string of equalities proves the theorem. □

The next theorem is an answer to Problem (P2′); it is not, however, as explicit as the answer to Problem (P1′).

Theorem 5.3. *The distortion operator is an isomorphism onto* $\mathcal{H}(b)$ *if and only if* $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$ *and the operator*

$$\Gamma_b := (P_+ b^*\Theta \quad P_+\Delta) : \begin{matrix} H^2(F) \\ \oplus \\ \operatorname{clos}(\Delta H^2(E)) \end{matrix} \longrightarrow H^2(E).$$

is bounded below.

Proof. Let $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ be an AFE such that $\pi_*^*\pi = b$. It follows from Proposition 3.3 that the operator π_*^* is an isometry from $\mathbb{H}'$ onto $\mathcal{H}(b)$; therefore, using Lemma 3.2, we get that

$$Id - T_b T_b^* = \pi_*^* P_{\mathbb{H}'} \pi_* | H^2(E_*),$$

and $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is an isomorphism onto $\mathcal{H}(b)$ if and only if $P_{\mathbb{H}'}|\pi_* K_\Theta$ is bounded below and surjective. According to Theorem 5.2, $P_{\mathbb{H}'}|\pi_* K_\Theta$ is bounded below if and only if $\operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1$; thus it remains to show that $P_{\mathbb{H}'}|\pi_* K_\Theta$ is surjective if and only if Γ_b is bounded below.

Now, since

$$\mathbb{H}' \oplus \pi_*(H^2_-(E_*)) = K \ominus \left[\pi(H^2(E)) \oplus \mathbb{H}''\right],$$

$$\pi_*(K_\Theta) \oplus \pi_*(H^2_-(E_*)) = \pi_*(L^2(E_*)) \ominus \pi_*(\Theta H^2(F)),$$

it follows that $P_{\mathbb{H}'}|\pi_* K_\Theta$ is surjective if and only if $P_{K \ominus [\pi(H^2(E)) \oplus \mathbb{H}'']}|\pi_*(L^2(E_*)) \ominus \pi_*(\Theta H^2(F))$ is surjective. Apply then Lemma 5.1 to the case $\mathfrak{X}_1 = \pi_*(L^2(E_*)) \ominus \pi_*(\Theta H^2(F))$, $\mathfrak{X}_2 = \pi_*(\Theta H^2(F)) \oplus \pi_*(L^2(E_*))^\perp$, $\mathfrak{Y}_1 = K \ominus \left[\pi(H^2(E)) \oplus \mathbb{H}''\right]$, $\mathfrak{Y}_2 = \left[\pi(H^2(E)) \oplus \mathbb{H}''\right]$, the surjectivity of $P_{\mathfrak{Y}_1}|\mathfrak{X}_1$ is equivalent to $P_{\mathfrak{Y}_2}|\mathfrak{X}_2$ bounded below.

Since $\mathbb{H}'' \subset \pi_*(L^2(E_*))^\perp$, this last condition is equivalent to

$$P_{\pi(H^2(E))}|\pi_*(\Theta H^2(F)) \oplus \left[\pi_*(L^2(E_*))^\perp \ominus \mathbb{H}''\right]$$

bounded below. Now we note that $P_{\pi H^2(E)} = \pi P_+ \pi^*$ and according to (3.3) and Lemma 3.1, we have

$$\pi_*(L^2(E_*))^\perp \ominus \mathbb{H}'' = \tau(L^2(E)) \ominus \mathbb{H}'' = \tau(\operatorname{clos}(\Delta L^2(E))) \ominus \mathbb{H}'' = \tau(\operatorname{clos}(\Delta H^2(E))).$$

Therefore, $P_{\mathbb{H}'}|\pi_* K_\Theta$ is surjective if and only if

$$\pi P_+ \pi^* \left(\pi_* \Theta \quad \tau\right) : H^2(F) \oplus \operatorname{clos}(\Delta H^2(E)) \longrightarrow K$$

is bounded below. But it follows from (3.1) that

$$\pi P_+ \pi^* \left(\pi_* \Theta \quad \tau\right) = \pi \left(P_+ \pi^* \pi_* \Theta \quad P_+ \pi^* \tau\right) = \pi \left(P_+ b^* \Theta \quad P_+ \Delta\right) = \pi \Gamma_b.$$

Since π is an isometry, we obtain the desired conclusion. □

Although the necessary and sufficient condition given by Theorem 5.3 cannot be reduced to a simple functional condition, we can obtain some useful corollaries.

Corollary 5.4. *If* $\operatorname{clos}(\Delta H^2(E)) = \operatorname{clos}(\Delta L^2(E))$ *and the distortion operator is invertible, then b is inner.*

Proof. If Γ_b is bounded below, then for any $f \in \operatorname{clos}(\Delta L^2(E)) = \operatorname{clos}(\Delta H^2(E))$, we have

$$c\|f\|_2 = c\|\overline{z}^n f\|_2 \leq \|\Gamma_b(0 \oplus \overline{z}^n f)\|_2 = \|P_+ \Delta \overline{z}^n f\|_2 = \|P_+ \overline{z}^n \Delta f\|_2.$$

Since the right side of the last inequality tends to 0 as $n \to +\infty$, we obtain $\operatorname{clos}(\Delta L^2(E)) = \{0\}$, which is equivalent to b inner. □

An interesting result can be obtained in the case the inner function Θ has full range.

Corollary 5.5. *Suppose* $\dim F = \dim E_*$. *If the distortion operator is invertible, then b is $*$-inner.*

If also $\dim E = \dim E_*$, *then b is inner.*

Proof. If Γ_b is bounded below, then $(b^* \Theta \quad \Delta)$ is bounded below as an operator from $H^2(F) \oplus \operatorname{clos}(\Delta L^2(E))$ to $L^2(E)$. Since for any $f \in L^2$ and $\epsilon > 0$ one can find $g \in H^2$ and $N \in \mathbb{N}$ such that $\|z^N f - g\|_2 < \epsilon$, a standard argument shows that $(b^* \Theta \quad \Delta)$ is bounded below from $L^2(F) \oplus \operatorname{clos}(\Delta L^2(E))$ to $L^2(E)$. If $\dim F = \dim E_*$, then Θ

inner implies that $\Theta L^2(F) = L^2(E_*)$, and thus

$$(b^* \quad \Delta) : \begin{matrix} L^2(E_*) \\ \oplus \\ \operatorname{clos}(\Delta L^2(E)) \end{matrix} \longrightarrow L^2(E)$$

is bounded below. But the adjoint of this last operator is an isometry. Since a coisometry that is bounded below is necessarily unitary, it is easily seen that multiplication with b must be a coisometry from $L^2(E)$ to $L^2(E_*)$, whence b is $*$-inner. The last assertion is then obvious. □

In particular, Corollary 5.5 can be applied to our original problem, namely the Riesz property of a family of reproducing kernels in $\mathcal{H}(b)$. Indeed, in that case the inner function Θ is actually a Blaschke–Potapov product corresponding to a Blaschke sequence (λ_n), which verifies the condition $\dim F = \dim E_*$.

Corollary 5.6. *Suppose* $\dim E = \dim E_* = 1$. *Then the distortion operator is invertible exactly in the two following cases:*

(i) b *is inner,* $\operatorname{dist}(\bar\Theta b, H^\infty) < 1$ *and* $\operatorname{dist}(\bar b \Theta, H^\infty) < 1$.

(ii) $F = \{0\}$ *and* $\|b\|_\infty < 1$.

Proof. From Corollary 5.5 it follows that the invertibility of the distortion operator implies either b inner or $F = \{0\}$. If b is inner, then the conditions in (i) are known to be equivalent to the invertibility of the distortion operator (see, for instance, [Nik86]). If $\{F\} = \{0\}$, then $K_\Theta = H^2$, and the invertibility of the distortion operator is equivalent to $\|b\|_\infty < 1$. □

Remark 5.7. Provided condition (4.1) is satisfied, Corollary 5.6 generalizes Proposition 5.1 in [Fri05], where it is shown that in the scalar nonextreme case we cannot have bases of reproducing kernels for $\mathcal{H}(b)$.

The next corollary discusses the connection between different conditions that are related to our original problem.

Corollary 5.8. *Consider the assertions*

(i) *the operator* $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ *is invertible;*

(ii) $\operatorname{dist}(\Theta^* b, H^\infty(E,F)) < 1$ *and* $T_{b^*\Theta}$ *is left invertible;*

(iii) $\operatorname{dist}(\Theta^* b, H^\infty(E,F)) < 1$ *and* $\operatorname{dist}(b^*\Theta, H^\infty(E,F)) < 1$.

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

If b *is inner (*$b^*b = Id$*), then* (ii) $\Rightarrow$ (i)*, while, if* b *is* $*$*-inner, then* (iii) $\Rightarrow$ (ii).

Proof. For $f \in H^2(F)$ we have

$$\|\Gamma_b(f \oplus 0)\|_2 = \|T_{b^*\Theta} f\|_2 \tag{5.1}$$

and (i) $\Rightarrow$ (ii) follows immediately from Theorem 5.3. If b is inner, then $\Gamma_b = T_{b^*\Theta}$ and we use again Theorem 5.3 to conclude that (ii) $\Rightarrow$ (i).

Since we have, by (2.1),

$$\|f\|_2 \geq \|b^*\Theta f\|_2^2 = \|T_{b^*\Theta} f\|_2^2 + \|\mathrm{H}_{b^*\Theta} f\|_2^2,$$

the vector valued Nehari Theorem yields (ii) $\Rightarrow$ (iii). If b is $*$-inner, the first inequality becomes an equality, giving the converse. □

In general, none of the implications in Corollary 5.8 can be reversed; examples will be given in the Section 7.

Theorem 5.3 may be compared to a basic result in the scalar case, namely the *Theorem on Close Subspaces* in [Nik86, page 201] (and its complement on page 204), where conditions are given for the projection from K_θ to $K_{\theta'}$ to be an isomorphism (θ, θ' scalar inner functions). For instance, an equivalent condition therein is the invertibility of the scalar Toeplitz operator $T_{\theta\bar{\theta}'}$.

As noted above, the theorem is not very convenient to apply directly, and our main use of it is to obtain necessary conditions on b for the invertibility of the distortion operator, as in Corollaries 5.4, 5.5, or 5.6. In the general case, it does not seem however that the invertibility of the distortion operator can be reduced to the cases when b is inner or those that can be trivially obtained from it; see Examples 7.3 and 7.4 below.

6. A different characterization

In the scalar case, another equivalent condition for the projection from K_θ to $K_{\theta'}$ to be an isomorphism is given by the two relations $\operatorname{dist}(\theta\bar{\theta}', H^\infty) < 1$, $\operatorname{dist}(\bar{\theta}\bar{\theta}', H^\infty) = 1$. A similar condition for scalar de Branges spaces appears in [Fri05, Theorem 4.1]. We will obtain below an alternate answer along that line to Problem (P2′); however, the formulation in the case of vector-valued de Branges–Rovnyak spaces is less elegant.

Some notations are needed: for any $x \in F$, let P_x be the orthogonal projection onto the subspace generated by x and define $\theta_x \in H^\infty(F \to F)$ by $\theta_x(z) := zP_x + (Id - P_x)$. It is immediate that θ_x is an inner function in $H^\infty(F \to F)$ and we have $K_{\theta_x} = \mathbb{C}x$ (the subspace of constant functions equal to a multiple of x).

Proposition 6.1. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \le 1$ and let $\Theta \in H^\infty(F \to E_*)$ be an inner function. Assume that the operator $Id - T_bT_b^* : K_\Theta \to \mathcal{H}(b)$ is left invertible. Then the following assertions are equivalent:*

(i) *$Id - T_bT_b^* : K_\Theta \to \mathcal{H}(b)$ is an isomorphism;*

(ii) *for all $x \in F$, we have $\operatorname{dist}(\theta_x^*\Theta^* b, H^\infty(E \to F)) = 1$.*

Proof. Once more, we will use an AFE $\Pi = (\pi, \pi_*) : L^2(E) \oplus L^2(E_*) \to K$ such that $\pi_*^*\pi = b$. By Lemma 3.2 and Proposition 3.3, the assertion (i) is equivalent to the invertibility of $P_{\mathbb{H}'}\pi_*|K_\Theta : K_\Theta \to \mathbb{H}'$, while from Theorem 5.2 it follows that

$$\text{(ii)} \iff \text{for all } x \in F,\ \ P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x} : K_{\Theta\theta_x} \to \mathbb{H}' \text{ is not left invertible.}$$

We will also use repeatedly the equality

$$K_{\Theta\theta_x} = K_\Theta \oplus \Theta K_{\theta_x} = K_\Theta \oplus \mathbb{C}\Theta x. \tag{6.1}$$

(i) $\Longrightarrow$ (ii) Since (6.1) implies $K_\Theta \subsetneq K_{\Theta\theta_x}$, it follows that if $P_{\mathbb{H}'}\pi_*|K_\Theta$ is invertible, then $P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x}$ is not one-to-one. Therefore it cannot be left invertible.

(ii) $\Rightarrow$ (i) We argue by contradiction, assuming that $P_{\mathbb{H}'}\pi_*|K_\Theta$ is not invertible. Since this operator is left invertible, that means that $P_{\mathbb{H}'}\pi_*K_\Theta$ is not dense in $\mathbb{H}'$; there exists thus $\chi \in \mathbb{H}'$, $\chi \ne 0$ such that $\chi \perp \pi_*K_\Theta$.

Since $P_{\mathbb{H}'}\pi_*|K_\Theta$ is left invertible and $P_{\mathbb{H}'}\pi_*|K_{\Theta\theta_x}$ is not, this last operator is not one-to-one, and we may choose $g_x \in K_{\Theta\theta_x} \setminus K_\Theta$ such that $P_{\mathbb{H}'}\pi_* g = 0$. Since $g_x \in K_{\Theta\theta_x} \subset H^2(E_*)$, it follows that $\pi_* g_x \in \pi_* H^2(E_*) \subset (\pi_* H^2_-(E_*))^\perp = \mathbb{H} \oplus \pi H^2(E)$. However, by definition we have $\pi_* g_x \in \mathbb{H}'^\perp$, while, using (3.3) and the definition of $\mathbb{H}''$, we also have $\pi_* g_x \in \mathbb{H}''^\perp$. Therefore $\pi_* g_x \in \mathbb{H}^\perp$, whence $\pi_* g_x \in \pi(H^2(E))$. But the space $\pi H^2(E)$ is U_Π-invariant, which implies that $U_\Pi^k \pi_* g_x \in \pi H^2(E)$. In particular, we have $\pi_*(z^k g_x) = U_\Pi^k(\pi_* g_x) \perp \chi$ (since $\pi(H^2(E)) \perp \mathbb{H}'$).

We claim now that

$$\operatorname{span}\big(K_\Theta, z^k g_x : k \geq 0,\, x \in F\big) = H^2(E_*). \tag{6.2}$$

To prove it, let $f \in H^2(E_*)$ and assume that $f \perp \operatorname{span}\big(K_\Theta, z^k g_x : k \geq 0,\, x \in F\big)$. Since $f \perp K_\Theta$, there exists $f_1 \in H^2(F)$ such that $f = \Theta f_1$. We will show by induction that for all $k \geq 0$, $f_1^{(k)}(0) = 0$, which of course will imply that $f_1 \equiv 0$, and thus the truth of (6.2).

First, by (6.1) there exists $g_x^\Theta \in K_\Theta$ and $\lambda_x \in \mathbb{C}^*$ such that

$$g_x = g_x^\Theta + \lambda_x \Theta x.$$

We have then

$$0 = \langle f, g_x\rangle = \langle \Theta f_1, g_x^\Theta + \lambda_x \Theta x\rangle_2 = \overline{\lambda}_x \langle f_1, x\rangle_2 = \overline{\lambda_x}\langle f_1(0), x\rangle_F.$$

Since $\lambda_x \neq 0$, that implies that $\langle f_1(0), x\rangle_F = 0$ for all $x \in F$, whence $f_1(0) = 0$.

Assume now that $f_1^{(k)}(0) = 0$. That means that there exists $f_{k+2} \in H^2(F)$ such that $f_1 = z^{k+1} f_{k+2}$. Therefore

$$0 = \langle f, z^{k+1} g_x\rangle = \langle \Theta z^{k+1} f_2, z^{k+1} g_x\rangle_2 = \langle \Theta f_{k+2}, g_x\rangle_2.$$

As before we deduce that $f_{k+2}(0) = 0$, which implies that $f_1^{(k+1)}(0) = 0$. The property for f_1 follows now by induction, concluding the proof of (6.2).

Since π_* is an isometry, (6.2) implies that

$$\operatorname{span}\big(\pi_* K_\Theta, \pi_*(z^k g_x) : k \geq 0,\, x \in F\big) = \pi_* H^2(E_*).$$

Recall that by construction $\chi \perp \pi_* K_\Theta$, while we have shown that $\chi \perp \pi_*(z^k g_x)$, for all $k \geq 0$ and for all $x \in F$. Consequently $\chi \perp \pi_* H^2(E_*)$. On the other hand, since $\chi \in \mathbb{H}'$, we also have $\chi \perp \pi_* H^2_-(E_*)$, whence $\chi \perp \pi_* L^2(E_*)$. Finally, we obtain that $\chi \in \mathbb{H} \cap (\pi_* L^2(E_*))^\perp = \mathbb{H}''$. Therefore $\chi \in \mathbb{H}' \cap \mathbb{H}'' = \{0\}$ which is absurd and ends the proof of the Proposition.

□

The next result is then a consequence of Theorem 5.2 and Proposition 6.1.

Theorem 6.2. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$ and let $\Theta \in H^\infty(F \to E_*)$ be an inner function. Then the operator $(Id - T_b T_b^*)|K_\Theta$ is an isomorphism from K_Θ onto $\mathcal{H}(b)$ if and only if*

$$\begin{cases} \operatorname{dist}(\Theta^* b, H^\infty(E \to F)) < 1, & \\ \operatorname{dist}(\theta_x^* \Theta^* b, H^\infty(E \to F)) = 1, & \forall x \in F. \end{cases}$$

In the scalar case $\dim E = \dim E_* = \dim F = 1$, we have $\theta_x(z) = z$; thus Theorem 6.2 generalizes part of [Fri05, Theorem 4.1] and of the Theorem on Close Subspaces in [Nik86].

7. Some examples and remarks

The first two examples show that the two implications in Corollary 5.8 cannot be reversed even in the scalar case $\dim E = \dim E_* = \dim F = 1$.

Example 7.1. Define

$$f(e^{i\vartheta}) := \begin{cases} 1 & \text{if } \vartheta \in [0, \pi] \\ 1/2 & \text{if } \vartheta \in]\pi, 2\pi[, \end{cases}$$

and consider the outer function g, positive at the origin and with modulus equals to $|f|$ a.e. on $\mathbb{T}$. Set $b = \Theta g$, where Θ is any inner function. Since $\log(1 - |b|)$ is not integrable, b is an extreme point of the unit ball of H^∞. Since $f, f^{-1} \in L^\infty$, it is immediate that g is invertible in H^∞, whence $T_{\bar{b}\Theta} = T_{\bar{g}} = T_g^*$ is invertible. Also, $\bar{\Theta} b = g \in H^\infty$, so $\operatorname{dist}(\bar{\Theta} b, H^\infty) = 0 < 1$.

On the other hand, Corollary 5.6 shows that the distortion operator $Id - T_b T_b^* : K_\Theta \longrightarrow \mathcal{H}(b)$ cannot be invertible. Consequently, the implication (i) $\Rightarrow$ (ii) in Corollary 5.8 cannot be reversed.

Example 7.2. Let $h : \mathbb{D} \to \mathbb{D}$ be the conformal transform of the disk $\mathbb{D}$ onto the simply connected domain

$$\Omega = \Big\{ z \in \mathbb{C} : |z| < 1,\ -\frac{1}{4} < \Re\mathrm{e} z < 0 \Big\}.$$

If we regard h as an element of H^∞, then $|h(e^{it})| = 1$ on an arc of positive measure, while 0 is in the essential range of h; also, $|\Re\mathrm{e} h| \le \frac{1}{4}$ everywhere.

Let Θ be an arbitrary inner function, and define $b = \Theta h$; b is then an extreme function. Since $\bar\Theta b = h \in H^\infty$, $\mathrm{dist}(\bar\Theta b, H^\infty) = 0 < 1$. Also, $\bar b \Theta = \bar h$; since $|\bar h + h| = 2|\Re\mathrm{e} h| \le 1/2$ everywhere, it follows that $\mathrm{dist}(\bar b \Theta, H^\infty) \le 1/2 < 1$. Thus condition (iii) of Corollary 5.8 is satisfied.

On the other hand, 0 is in the essential range of $\bar b \Theta = h$; it is then known (see, for instance, [Nik02a, B.4.2]) that $T_{\bar b \Theta}$ cannot be left invertible. Thus the implication (ii) $\Rightarrow$ (iii) in Corollary 5.8 cannot be reversed.

The next example is an application of Theorem 5.3.

Example 7.3. Suppose $b \in H^\infty(\mathbb{C}^2 \to \mathbb{C}^2)$, $b(z) = \begin{pmatrix} \alpha(z) & \gamma(z) \\ \beta(z) & \delta(z) \end{pmatrix}$, with $|\alpha|^2 + |\beta|^2 \equiv 1$; $\Theta \in H^\infty(\mathbb{C}, \mathbb{C}^2)$, $\Theta(z) = \begin{pmatrix} \theta_1(z) \\ \theta_2(z) \end{pmatrix}$. Then $\Delta = \begin{pmatrix} 0 & 0 \\ 0 & \Delta' \end{pmatrix}$, with $\Delta' = (1 - |\gamma|^2 - |\delta|^2)^{1/2}$, and $b^*\Theta = \begin{pmatrix} \bar\alpha\theta_1 + \bar\beta\theta_2 \\ \bar\gamma\theta_1 + \bar\delta\theta_2 \end{pmatrix}$. Obviously Γ_b bounded below implies Δ' invertible; we may then identify $\mathrm{clos}(\Delta H^2(\mathbb{C}^2))$ with $\Delta' H^2$. With this last identification, we obtain

$$\Gamma_b = \begin{pmatrix} T_{\bar\alpha\theta_1 + \bar\beta\theta_2} & 0 \\ T_{\bar\gamma\theta_1 + \bar\delta\theta_2} & P_+\Delta' \end{pmatrix} = \begin{pmatrix} T_{\bar\alpha\theta_1 + \bar\beta\theta_2} & 0 \\ T_{\bar\gamma\theta_1 + \bar\delta\theta_2} & T_{\Delta'^2} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \Delta'^{-1} \end{pmatrix} : \begin{matrix} H^2 \\ \oplus \\ \Delta' H^2 \end{matrix} \to H^2(\mathbb{C}^2)$$

The second operator in the product above is invertible, so the boundedness below condition is transferred to the first operator, which acts on $H^2(\mathbb{C}^2)$. Moreover, the Hartmann–Wintner Theorem implies that $T_{\Delta'^2}$ is invertible. Thus Γ_b is bounded below if and only if $\begin{pmatrix} T_{\bar\alpha\theta_1+\bar\beta\theta_2} & 0 \\ T_{\bar\gamma\theta_1+\bar\delta\theta_2} & I \end{pmatrix}$ is bounded below. This last condition is easily seen to be equivalent to $T_{\bar\alpha\theta_1+\bar\beta\theta_2}$ bounded below.

Summing up, the distortion operator is invertible if and only if the following three conditions are satisfied:

(1) $\operatorname{dist}((\bar\theta_1\alpha+\bar\theta_2\beta \quad \bar\theta_1\gamma+\bar\theta_2\delta), H^\infty(\mathbb{C}^2\to\mathbb{C}))<1$;
(2) the function $(1-|\gamma|^2-|\delta|^2)^{1/2}$ is bounded below;
(3) the Toeplitz operator $T_{\bar\alpha\theta_1+\bar\beta\theta_2}$ is bounded below.

One can easily find concrete examples that satisfy conditions (1)–(3).

The following example presents some other interesting cases of de Branges spaces for which one can obtain the invertibility of the distortion operator.

Example 7.4. Consider a $*$-inner function $b=(\alpha \quad \beta)\in H^\infty(\mathbb{C}^2\to\mathbb{C})$; that is, $|\alpha|^2+|\beta|^2=1$. Then

$$(I-T_bT_b^*)f=\alpha P_-(\bar\alpha f)+\beta P_-(\bar\beta f).$$

Therefore the image of $(I-T_bT_b^*)$ as well as the image of $(I-T_bT_b^*)^{1/2}$ are contained in the invariant subspace for T_z^* generated by $T_z^*\alpha$ and $T_z^*\beta$. In particular, if α and β are rational, then $\mathcal{H}(b)$ is finite dimensional, and thus equal as a set to K_Θ for some Blaschke product Θ. But in general the norm on $\mathcal{H}(b)$ is different from the usual H^2 norm on K_Θ, and the distortion operator corresponding to Θ is invertible, but not equal to the identity.

On the other hand, if we take $b=(1/\sqrt{2} \quad 1/\sqrt{2}B)\in H^\infty(\mathbb{C}^2\to\mathbb{C})$ with B an infinite Blaschke product, then $(I-T_bT_b^*)^{1/2}=1/\sqrt{2}P_{K_B}$. Thus $\mathcal{H}(b)$ is just K_B with the norm divided by $\sqrt{2}$, and the corresponding distortion operator is again invertible.

One can combine the two previous cases. Consider α,β,Θ,B as above, and take $b=(\alpha B \quad \beta)$. Applying [Sar95, I-10], it follows that, as a linear space, $\mathcal{H}(b)=K_\Theta+\alpha K_B$. We leave to the reader the task of showing that one can choose α and B

such that the reproducing kernels in $\mathcal{H}(b)$ corresponding to zeros of B and Θ form a Riesz basis in $\mathcal{H}(b)$.

One might expect that if b and Θ are sufficiently close, in the sense that

$$\|b - \Theta\|_\infty < 1, \tag{7.1}$$

then the distortion operator should be invertible. In the scalar case, if b is inner, then it was pointed in [Nik86, p. 202] that this condition is indeed sufficient. If b is a vector-valued inner function, then condition (7.1) remains sufficient to ensure the invertibility of the distortion operator. Indeed, it follows from (7.1) that $\|1 - \Theta^* b\|_\infty < 1$, whence $T_{\Theta^* b} = Id + T_{1-\Theta^* b}$ is invertible. In particular, $T_{b^*\Theta} = (T_{\Theta^* b})^*$ is left invertible; therefore, condition (ii) of Corollary 5.8 is satisfied, which implies that the distortion operator is invertible.

However, even in the general case of an arbitrary extreme point b in the unit ball of H^∞, condition (7.1) is no longer sufficient to ensure the invertibility of the distortion operator. Actually, it seems improbable that a condition expressed only in terms of functions might be found. The next example shows that indeed no condition similar to (7.1) is sufficient.

Example 7.5. For $\varepsilon > 0$, let h be the conformal transform of the disk $\mathbb{D}$ onto the simply connected domain

$$\Omega = \left\{ z \in \mathbb{C} : |z| < 1,\, \Re e z > 1 - \frac{\varepsilon}{2} \right\}.$$

If we regard h as an element of H^∞, then $|h| = 1$ on an arc of positive measure but h is not inner. Moreover, on $\mathbb{T}$, we have

$$|1 - h|^2 = 1 + |h|^2 - 2\Re e h \le 2(1 - \Re e h) < \varepsilon.$$

Now take $b = \Theta h$ with Θ is an arbitrary inner function. We have $\|\Theta - b\|_\infty = \|1 - h\|_\infty \le \varepsilon$, while Corollary 5.6 implies that the distortion operator $Id - T_b T_b^* : K_\Theta \to \mathcal{H}(b)$ is not invertible. Consequently, no condition of closeness in the H^∞ norm can ensure the invertibility of the distortion operator.

8. Completeness of the difference quotients

Recall that the difference quotients are the elements $\hat{k}^b_{\lambda,e}$ of $\mathcal{H}(b)$ defined by (2.4). For $\dim E = \dim E_* = 1$ and b an extreme point in the unit ball of H^∞, the set $\{\hat{k}^b_\lambda : \lambda \in \mathbb{D}\}$ (which does not depend on e in this case) has been shown to be complete in $\mathcal{H}(b)$ in [Fri05]; this completeness is further used therein to obtain results about properties of reproducing kernels. If, moreover, b is inner, then the completeness of the difference quotients can easily be obtained by noting that the mapping $f \mapsto \bar{z}b\bar{f}$ is an antilinear surjective isometry which maps k^b_λ onto $\hat{k}^b_\lambda$.

The study of the completeness of the family of the difference quotients in the general case may present independent interest. Together with the kernels, the difference quotients represent the main examples of "concrete" elements of the de Branges space $\mathcal{H}(b)$; they also appear in the study of model spaces and related questions (see, for instance, [BK87, Gar06]). We devote this section to the investigation of their completeness.

We start with an equivalent condition.

Lemma 8.1. *Let $b \in H^\infty(E \to E_*)$. Then the following two conditions are equivalent:*

(1) $\operatorname{span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b)$.
(2) $\operatorname{span}\{S^{*n+1}be : n \geq 0, e \in E\} = \mathcal{H}(b)$.

Proof. As in the scalar case [Sar95, II-8] it is easily seen that, for $\lambda \in \mathbb{D}$ and $f \in H^2(E)$, we have

$$\frac{f(z) - f(\lambda)}{z - \lambda} = (Id - \lambda S^*)^{-1} S^* f.$$

In particular, applying this formula to $f(z) := b(z)e$, we obtain

$$\frac{b(z) - b(\lambda)}{z - \lambda} e = (Id - \lambda S^*)^{-1} S^* be = \sum_{n=0}^{\infty} \lambda^n S^{*n+1} be. \tag{8.1}$$

Now according to (8.1), we have $f \in \mathcal{H}(b) \ominus \operatorname{span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\}$ if and only

$$\sum_{n=0}^{\infty} \lambda^n \langle S^{*n+1} be, f\rangle_b = 0, \qquad (\lambda \in \mathbb{D}, e \in E),$$

and, since the function $\lambda \longmapsto \sum_{n=0}^{\infty} \lambda^n \langle S^{*n+1} be, f\rangle_b$ is analytic in a neighbourhood of 0, this is equivalent to

$$\langle S^{*n+1} be, f\rangle_b = 0, \qquad (n \geq 0, e \in E),$$

which gives the result. □

The scalar case has been discussed in [Fri05]; we give a different proof of Lemma 4.2 therein, which seems to us of independent interest. Note that in Corollary 8.6 below we will obtain a more general result.

Theorem 8.2 (Lemma 4.2, [Fri05])**.** *Let b be an extreme point of the unit ball of H^∞. Then*

$$\operatorname{span}\{\hat{k}^b_\lambda : \lambda \in \mathbb{D}\} = \mathcal{H}(b).$$

Proof. As in the case of b inner, we will construct an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$ which maps k^b_λ onto $\hat{k}^b_\lambda$. Consider the transcription of Sz.-Nagy–Foiaş of the AFE associated to b (see Section 3), and let $W : K \longrightarrow K$ be the operator defined (on a dense set) by

$$W(\pi f + \pi_* g) = \pi J g + \pi_* J f,$$

with $J : L^2 \longrightarrow L^2$ the antilinear map defined by $Jf = \bar{z}\bar{f}$. Then standard arguments show that W is an antilinear surjective isometry and we have $W\mathbb{H} = \mathbb{H}$. Now, for $f \in \mathbb{H}$, set

$$\Omega(\pi_*^* f) = \pi_*^*(Wf).$$

Since b is an extreme point of the unit ball of H^∞, it follows from Proposition 3.3 that π_*^* is an isometry from $\mathbb{H}$ onto $\mathcal{H}(b)$, and then we can easily verify that Ω is an antilinear surjective isometry from $\mathcal{H}(b)$ onto $\mathcal{H}(b)$. For $\lambda \in \mathbb{D}$, recall that k_λ denotes the reproducing kernel of H^2 and consider the function $g_\lambda \in K$ defined by

$$g_\lambda = k_\lambda^b \oplus \left(-\overline{b(\lambda)}\Delta k_\lambda\right).$$

An easy computation using (3.7) shows that $g_\lambda \in \mathbb{H}$ and we obviously have $\pi_*^*(g_\lambda) = k_\lambda^b$. Therefore we obtain that

$$\Omega(k_\lambda^b) = \Omega(\pi_*^* g_\lambda) = \pi_*^* W g_\lambda.$$

Note that $g_\lambda = \pi_*(k_\lambda) + \pi(-\overline{b(\lambda)}k_\lambda)$, whence

$$\begin{aligned} Wg_\lambda =& W(\pi_*(k_\lambda) + \pi(-\overline{b(\lambda)}k_\lambda)) \\ =& \pi J k_\lambda + \pi_* J(-\overline{b(\lambda)}k_\lambda) \\ =& \pi\left(\frac{1}{z-\lambda}\right) - \pi_*\left(\frac{b(\lambda)}{z-\lambda}\right) \\ =& \hat{k}_\lambda^b \oplus \frac{\Delta}{z-\lambda}. \end{aligned}$$

Finally we obtain $\Omega(k_\lambda^b) = \hat{k}_\lambda^b$ and the proof is complete. □

For the nonextreme scalar case, we have to recall that a function f in the Nevanlinna class of the unit disc $\mathbb{D}$ is said to be *pseudocontinuable* (across $\mathbb{T}$) if there exist $g, h \in \bigcup_{p>0} H^p$ such that

$$f = \overline{h}/\overline{g}$$

a.e. on $\mathbb{T}$. The function $\widetilde{f} := \overline{h}/\overline{g}$ is the (nontangential) boundary function of the meromorphic function $\widetilde{f}(z) := \overline{h}(\frac{1}{\overline{z}})/\overline{g}(\frac{1}{\overline{z}})$ defined for $|z| > 1$, which is called a pseudocontinuation of f. R. Douglas, H. Shapiro and A. Shields have obtained [DSS70] the following characterization: a function $f \in H^2$ is pseudocontinuable if and only if it is not S^*-cyclic, that is $\mathrm{span}(S^{*n}f : n \geq 0) \neq H^2$.

Theorem 8.3. *Suppose b is not an extreme point in the unit ball of H^∞. Then*

$$\mathrm{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b) \iff b \textit{ is not pseudocontinuable.}$$

Proof. Assume that $\operatorname{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b)$ but b is pseudocontinuable. Then there exists a nonconstant inner function u such that $b \in \mathcal{H}(u)$. Since $\mathcal{H}(u)$ is S^*-invariant, $S^{*n+1}b \in \mathcal{H}(u)$ for all $n \geq 0$. As $\mathcal{H}(b)$ is contained continuously in H^2, we deduce that

$$\operatorname{span}_{\mathcal{H}(b)}\left(S^{*n+1}b : n \geq 0\right) \subset \operatorname{clos}_{\mathcal{H}(b)}\mathcal{H}(u) \subset \mathcal{H}(u),$$

and it follows from Lemma 8.1 that $\mathcal{H}(b) \subset \mathcal{H}(u)$. Now since b is not an extreme point in the unit ball of H^∞, we know that the polynomials belong to $\mathcal{H}(b)$ [Sar95, Chap. IV] and consequently to $\mathcal{H}(u)$. Hence $H^2 \subset \mathcal{H}(u)$, which is absurd. Thus, if the difference quotients are complete in $\mathcal{H}(b)$, then b is not pseudocontinuable.

Conversely, assume b is not pseudocontinuable. Note that $\operatorname{span}_{\mathcal{H}(b)}\left(S^{*n+1}b : n \geq 0\right)$ is a closed S^*-invariant subspace of $\mathcal{H}(b)$. But we know from [Sar86] the description of these subspaces when b is not an extreme point: they are just the intersection of $\mathcal{H}(b)$ with the invariant subspaces of S^*. Hence there is an inner function u such that

$$\operatorname{span}_{\mathcal{H}(b)}\left(S^{*n+1}b : n \geq 0\right) = \mathcal{H}(b) \cap \mathcal{H}(u).$$

But $S^*b \in \mathcal{H}(u)$ implies that $b \in \mathcal{H}(uz)$, which is impossible unless $u \equiv 0$ (because b is not pseudocontinuable). Hence

$$\operatorname{span}_{\mathcal{H}(b)}\left(S^{*n+1}b : n \geq 0\right) = \mathcal{H}(b),$$

and applying once more Lemma 8.1, we obtain that the difference quotients are complete in $\mathcal{H}(b)$. □

Example 8.4. As a consequence of Theorem 8.3, it is simple to give two examples of de Branges–Rovnyak spaces (both corresponding to nonextreme functions b), with the completeness of the difference quotients false for the first and true for the second. Note first that, if $\sup_{z\in\mathbb{T}} |b(z)| < 1$, then $\log(1 - |b|)$ is integrable, and thus b is not extreme. This condition is satisfied by both functions $b_1(z) := 1/(z-3)$ and $b_2(z) := \exp((z-2)^{-1})$. The first is pseudocontinuable, and thus the difference quotients are not complete in $\mathcal{H}(b_1)$, while the second is not, whence the difference quotients are complete in $\mathcal{H}(b_2)$.

To go beyond the scalar case, we will use the transcription of the Sz.-Nagy–Foiaş of the (abstract) model introduced in Subsection 3. We have then the following general result.

Theorem 8.5. *Suppose $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$. The following assertions are equivalent:*

(1) $\operatorname{span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b)$.
(2) $\mathbb{H}' \cap \mathbb{H}''_* = \{0\}$.
(3) $\mathbb{H}'' \vee \mathbb{H}'_* = \mathbb{H}$.

Proof. (1) $\Leftrightarrow$ (2) Denote $\xi_n = S^{*n+1}be \oplus \Delta \bar{z}^{n+1}e \in K$; obviously $\pi_*^*\xi_n = S^{*n+1}be$. Moreover, using (3.5), one can easily check that $P_{\mathbb{H}}\pi(\bar{z}^{n+1}e) = \xi_n$, whence $\xi_n \in \mathbb{H}$ and

$$\operatorname{span}(\xi_n) = \operatorname{clos}(P_{\mathbb{H}}\pi H^2_-(E)). \tag{8.2}$$

We know from Proposition 3.3 that π_*^* is a coisometry from $\mathbb{H}$ onto $\mathcal{H}(b)$ with kernel $\mathbb{H}'' = \mathbb{H} \ominus \mathbb{H}'$. Denoting $\eta_n = P_{\mathbb{H}'}\xi_n$, we have

$$\pi_*^*\eta_n = \pi_*^*(P_{\mathbb{H}'} + P_{\mathbb{H}\ominus\mathbb{H}'})\xi_n = \pi_*^*P_{\mathbb{H}}\xi_n = \pi_*^*\xi_n = S^{*n+1}be.$$

It follows that π_*^* is a unitary from $\operatorname{span}(\eta_n)$ onto $\operatorname{span}(S^{*n+1}be : n \geq 0, e \in E)$. Then, according to Lemma 8.1, the difference quotients are not complete iff there exists a non-null vector χ in $\mathbb{H}'$ that is orthogonal to all η_n; or equivalently, that is orthogonal to all ξ_n. By (8.2), this is equivalent to being orthogonal to $\pi(H^2_-(E))$, which is the same as saying that $\chi \in \mathbb{H}''_*$. The equivalence is thus proved.

(2) $\Leftrightarrow$ (3) follows easily from the definition. □

Corollary 8.6. *Let $b \in H^\infty(E \to E_*)$, $\|b\|_\infty \leq 1$. If $\operatorname{clos}(\Delta_* H^2_-(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$, then*

$$\operatorname{span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b).$$

In particular, if b is $*$-inner, then the difference quotients are complete.

Proof. According to Lemma 3.1, the hypothesis implies that $\mathbb{H}''_* = \{0\}$ and the conclusion follows from Theorem 8.5. □

Corollary 8.7. *Let b be an extreme point of the unit ball of $H^\infty(E \to E_*)$. Then the two following conditions are equivalent:*

(i) $\operatorname{span}\{\hat{k}^b_{\lambda,e} : \lambda \in \mathbb{D}, e \in E\} = \mathcal{H}(b)$.

(ii) $\operatorname{clos}(\Delta_* H^2_-(E_*)) = \operatorname{clos}(\Delta_* L^2(E_*))$.

Proof. (ii) $\Rightarrow$ (i) follows from Corollary 8.6. As for (i) $\Rightarrow$ (ii), we know from [Tre86] (see Remark 3.4) that b is an extreme point of the unit ball of $H^\infty(E \to E)$ if and only if $\mathbb{H} = \mathbb{H}'$ or $\mathbb{H} = \mathbb{H}'_*$. Assume that (ii) is not satisfied, which means by Lemma 3.1 that $\mathbb{H} \neq \mathbb{H}'_*$. Then we necessarily have $\mathbb{H} = \mathbb{H}'$, whence $\mathbb{H}'' = \{0\}$. But if the difference quotients are complete, then by Theorem 8.5, we obtain $\mathbb{H} = \mathbb{H}'_*$—a contradiction. □

We see from Example 8.4 that extremality is not a necessary condition for the completeness of the difference quotients.

Acknowledgements

The authors thank the anonymous referee for extremely useful comments and suggestions, which have lead to a significantly improved version of the paper.

References

[AI95] S.A. Avdonin and S.A. Ivanov. *Families of Exponentials*. Cambridge Univ. Press, Cambridge, 1995.

[Bar06] A. Baranov. Completeness and Riesz basis of reproducing kernels in model subspaces. *Int. Math. Res. Not.*, Art. ID 81530:34 pp., 2006.

[BFM] A. Baranov, E. Fricain, and J. Mashreghi. Weighted norm inequalities for de Branges-Rovnyak spaces and their applications. *preprint*.

[BK87] J.A. Ball and T.L. Kriete. Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators. *Integral Equations Operator Theory*, 10:17–61, 1987.

[CFT03] I. Chalendar, E. Fricain, and D. Timotin. Functional models and asymptotically orthonormal sequences. *Ann. Inst. Fourier (Grenoble)*, 53(5):1527–1549, 2003.

[dBR66] L. de Branges and J. Rovnyak. Canonical models in quantum scattering theory. In *Perturbation theory and its applications in quantum mecanics*, pages 295–392. C. H. Wilcox, Wiley, New York, 1966.

[dLR58] K. de Leueuw and W. Rudin. Extreme points and extremum problems in H_1. *Pacific Journal of Mathematics*, 8(3):467–485, 1958.

[DSS70] R.G. Douglas, H.S. Shapiro, and A. Shields. Cyclic vectors and invariant subspaces for the backward shift operator. *Ann. Inst. Fourier (Grenoble)*, 20(1):37–76, 1970.

[Fri01] E. Fricain. Bases of reproducing kernels in model spaces. *J. Operator Theory*, 46:517–543, 2001.

[Fri02] E. Fricain. Complétude des noyaux reproduisants dans les espaces modèles. *Ann. Inst. Fourier (Grenoble)*, 52(2):661–686, 2002.

[Fri05] E. Fricain. Bases of reproducing kernels in the de Branges spaces. *Journal of Functional Analysis*, 226(2):373–405, 2005.

[Gar06] S.R. Garcia. Conjugation and Clark operators. *Contemporary Mathematics*, 393:67–111, 2006.

[HNP81] S.V. Hruschev, N.K. Nikolski, and B.S. Pavlov. Unconditional bases of exponentials and reproducing kernels. *Lect. Notes Math.*, 864:214–335, 1981.

[Ing34] A.E. Ingham. A note on Fourier transforms. *J. London Math. Soc.*, 9:29–32, 1934.

[Nik74] N. K. Nikolskiĭ. Invariant subspaces in operator theory and function theory. In *Mathematical analysis, Vol. 12 (Russian)*, pages 199–412, 468. (loose errata). Akad. Nauk SSSR Vsesojuz. Inst. Naučn. i Tehn. Informacii, Moscow, 1974.

[Nik80] N.K. Nikolski. Bases of exponentials and values of reproducing kernels. *Doklady Akad. Nauk SSSR*, 252(6):1316–1320, 1980 (Russian) English Transl: Soviet Math. Dokl., 21(3): 937-941, 1980.

[Nik86] N.K. Nikolski. *Treatise on the Shift operator*, volume 273. Springer Verlag, Berlin-Heidelberg, 1986.

[Nik02a] N.K. Nikolski. *Operators, Functions, and Systems: An Easy Reading, Volume 1: Hardy, Hankel and Toeplitz*, volume 92. Mathematical Surveys and Monograph, American Mathematical Society, 2002.

[Nik02b] N.K. Nikolski. *Operators, Functions, and Systems: An Easy Reading, Volume 2: Model Operators and Systems*, volume 93. Mathematical Surveys and Monograph, American Mathematical Society, 2002.

[NV85] N. Nikolski and V. Vasyunin. Notes on Two Function Models. In *The Bieberbach Conjecture, Proceedings of the Symposium on the Occasion of the Proof*, volume 21, pages 113–141. Mathematical Surveys and Monographs, 1985.

[NV98] N. K. Nikolski and V. Vasyunin. Elements of spectral theory in terms of the free function model. I. Basic constructions. In *Holomorphic spaces (Berkeley, CA, 1995)*, volume 33 of *Math. Sci. Res. Inst. Publ.*, pages 211–302. Cambridge Univ. Press, Cambridge, 1998.

[OCS02] J. Ortega-Cerdà and K. Seip. Fourier frames. *Annals of Mathematics. Second Series*, 155(3):789–806, 2002.

[PW34] R.E. Paley and N. Wiener. *Fourier Transforms in the Complex Domain.* Amer. Math. Soc. Collquium Publ., Providence, 1934.

[Sar86] D. Sarason. Doubly shift-invariant spaces in H^2. *J. Operator Theory*, 16:75–97, 1986.

[Sar95] D. Sarason. *Sub-Hardy Hilbert Spaces in the Unit Disk*, volume 10. University of Arkansas Lecture Notes in the Mathematical Sciences, Willey-Interscience Publication, 1995.

[SNF67] B. Szökefalvi-Nagy and C. Foias. *Analyse harmonique des opérateurs de l'espace de Hilbert.* Akadémiai Kiado, Budapest, 1967. English Transl.: Harmonic analysis of operators on Hilbert space, North Holland, New-York, 1970.

[Tre86] S. Treil. Extreme points of the unit ball of the operator Hardy space $H^\infty(E \to E)$. *Zapiski Nauchn. Semin. LOMI*, 149:160–164, 1986. (Russian) English transl: J. Soviet Math. 42 (1988), no. 2, 1653-1656.

[Tre89] S.R. Treil. Geometric methods in spectral theory of vector-valued functions: some recent results. In *Oper. Theory Adv. Appl.*, volume 42, pages 209–280. Birkhauser, Basel, 1989.

[Vas77] V. I. Vasjunin. The construction of the B. Szőkefalvi-Nagy and C. Foiaş functional model. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 73:16–23, 229 (1978), 1977. Investigations on linear operators and the theory of functions, VIII.

Nicolas Chevrot, Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne

E-mail address: `chevrot@math.univ-lyon1.fr`

Emmanuel Fricain, Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne

E-mail address: `fricain@math.univ-lyon1.fr`

Dan Timotin, Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania.

E-mail address: Dan.Timotin@imar.ro

6.2 Annexes sur le chapitre "Propriétés fonctionnelles des espaces de de Branges-Rovnyak"

6.2.1 Référence [T5]

Auteurs
E. Fricain et J. Mashreghi

Titre
Boundary behavior of functions in the de Branges-Rovnyak spaces.

Paru dans
Compl. anal. oper. theory, 2 (2008), 87–97.

Compl. anal. oper. theory 2 (2008), 87–97

1661-8254/010087-11, *published online* September 21, 2007
DOI 10.1007/s11785-007-0028-8

Complex Analysis and Operator Theory

Boundary Behavior of Functions in the de Branges–Rovnyak Spaces

Emmanuel Fricain and Javad Mashreghi

Abstract. This paper deals with the boundary behavior of functions in the de Branges–Rovnyak spaces. First, we give a criterion for the existence of radial limits for the derivatives of functions in the de Branges–Rovnyak spaces. This criterion generalizes a result of Ahern–Clark. Then we prove that the continuity of all functions in a de Branges–Rovnyak space on an open arc I of the boundary is enough to ensure the analyticity of these functions on I. We use this property in a question related to Bernstein's inequality.

Mathematics Subject Classification (2000). Primary: 46E15, 46E22, Secondary: 30D55, 47A15, 30B40.

Keywords. de Branges–Rovnyak spaces, model subspaces of H^2, boundary behavior, shift operator.

1. Introduction

For $0 < p \leq \infty$, let $H^p(\mathbb{D})$ denote the classical Hardy space of analytic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. As usual, we also treat $H^p(\mathbb{D})$ as a closed subspace of $L^p(\mathbb{T}, m)$, where $\mathbb{T} := \partial\mathbb{D}$ and m is the normalized arc length measure on $\mathbb{T}$. Let b be in the unit ball of $H^\infty(\mathbb{D})$. Then the canonical factorization of b is $b = BF$, where

$$B(z) = \gamma \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}\,, \qquad (z \in \mathbb{D})\,,$$

is the Blaschke product with zeros a_n in the unit disc $\mathbb{D}$ satisfying the Blaschke condition $\sum_n (1-|a_n|) < +\infty$, γ is a constant of modulus one, and F is of the form

$$F(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) \right), \qquad (z \in \mathbb{D})\,,$$

This work was supported by NSERC (Canada) and FQRNT (Québec). A part of this work was done while the first author was visiting McGill University. He would like to thank this institution for its warm hospitality.

where $d\sigma = -\log|b|\,dm + d\mu$ and $d\mu$ is a positive singular measure on $\mathbb{T}$. In the definition of B, we assume that $|a_n|/a_n = 1$ whenever $a_n = 0$. In this paper, we study some aspects of the de Branges–Rovnyak spaces

$$\mathcal{H}(b) := (Id - T_b T_{\overline{b}})^{1/2} H^2 \,.$$

Here T_φ denotes the Toeplitz operator defined on H^2 by $T_\varphi(f) = P_+(\varphi f)$, where P_+ is the (Riesz) orthogonal projection of $L^2(\mathbb{T})$ onto H^2. In general, $\mathcal{H}(b)$ is not closed with respect to the norm of $H^2(\mathbb{D})$. However, it is a Hilbert space when equipped with the inner product

$$\left\langle\, (Id - T_b T_{\overline{b}})^{1/2} f,\, (Id - T_b T_{\overline{b}})^{1/2} g \,\right\rangle_b = \langle f, g\rangle_2 \,,$$

where f and g are chosen so that

$$f, g \perp \ker\, (Id - T_b T_{\overline{b}})^{1/2} \,.$$

As a very special case, if $|b| = 1$ a.e. on $\mathbb{T}$, or equivalently when b is an inner function for the unit disc, then $Id - T_b T_{\overline{b}}$ is an orthogonal projection and the $\mathcal{H}(b)$ norm coincides with the H^2 norm. In this case, $\mathcal{H}(b)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{D})$, which coincides with the shift-coinvariant subspace $K_b := H^2 \ominus bH^2$.

This paper treats two questions related to the boundary behavior of functions in $\mathcal{H}(b)$. The first of these concerns the existence of radial limits for the derivatives of functions in the de Branges–Rovnyak spaces. More precisely, given a non-negative integer N, we are interested in finding a characterization of points $\zeta_0 \in \mathbb{T}$ such that every function f in $\mathcal{H}(b)$ and its derivatives up to order N have radial limits at ζ_0. Ahern and Clark [1] studied this question when b is an inner function and they got a characterization in terms of the zeros sets (a_n) and the measure μ. In Section 3, we show that their methods in [1, 2] can be extended in order to obtain similar results for the general de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an arbitrary element of the unit ball of H^∞. Let us also mention that Sarason [11, page 58] has obtained another criterion in terms of the measure whose Poisson integral is the real part of $\frac{\lambda+b}{\lambda-b}$, with $\lambda \in \mathbb{T}$. Recently, Bolotnikov and Kheifets [3] gave a result, in some sense more algebraic, in terms of the Schwarz-Pick matrix.

Our second theme is related to the analytic continuation of functions in $\mathcal{H}(b)$ through a given open arc of $\mathbb{T}$. In [7], in the case where b is an inner function, Helson proved that every function in K_b has an analytic continuation through an open arc I of $\mathbb{T}$ if and only if b has an analytic continuation through I. Then, in [11, page 42], Sarason extended this result to the de Branges–Rovnyak spaces $\mathcal{H}(b)$, when b is an extreme point of the unit ball of H^∞. In the last section, we study the question of continuity on the open arc I for functions in $\mathcal{H}(b)$. In particular, we show that the continuity on some open arc of the boundary of all functions in $\mathcal{H}(b)$ implies the analyticity on this arc. We apply this remarkable property to discuss a possible generalization of the Bernstein's inequality obtained by Dyakonov [5] in the model space K_b.

2. Preliminaries

We first recall some basic well-known facts concerning the reproducing kernel in $\mathcal{H}(b)$. For any $\lambda \in \mathbb{D}$, the linear functional $f \longmapsto f(\lambda)$ is bounded on $H^2(\mathbb{D})$ and thus, by Riesz' theorem, it is induced by a unique element k_λ of $H^2(\mathbb{D})$. On the other hand, by Cauchy's formula, we have

$$f(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\vartheta})}{1-\lambda e^{-i\vartheta}}\, d\vartheta\,, \qquad (f \in H^2(\mathbb{D})\,,\, \lambda \in \mathbb{D})\,,$$

and thus

$$k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}\,, \qquad (z \in \mathbb{D})\,.$$

Now, since $\mathcal{H}(b)$ is contained contractively in $H^2(\mathbb{D})$, the restriction to $\mathcal{H}(\mathbb{D})$ of the evaluation functional at $\lambda \in \mathbb{D}$ is a bounded linear functional on $\mathcal{H}(\mathbb{D})$. Hence, relative to the inner product in $\mathcal{H}(b)$, it is induced by a vector k_λ^b in $\mathcal{H}(b)$. In other words, for all $f \in \mathcal{H}(b)$, we have

$$f(\lambda) = \langle f, k_\lambda^b\rangle_b\,.$$

But if $f = (Id - T_bT_{\overline{b}})^{1/2} f_1 \in \mathcal{H}(b)$, we have

$$\left\langle f, (Id - T_bT_{\overline{b}})k_\lambda\right\rangle_b = \left\langle f_1, (Id - T_bT_{\overline{b}})^{1/2}k_\lambda\right\rangle_2 = \langle f, k_\lambda\rangle_2 = f(\lambda)\,,$$

which implies that

$$k_\lambda^b = (Id - T_bT_{\overline{b}})k_\lambda\,.$$

Finally, using the well known result $T_{\overline{b}}k_w = \overline{b(w)}k_w$, we obtain

$$k_\lambda^b(z) = \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}\,, \qquad (z \in \mathbb{D})\,.$$

We know (see [11, page 11]) that $\mathcal{H}(b)$ is invariant under the backward shift operator S^* and, in the following, we use extensively the contraction $X := S^*|\mathcal{H}(b)$. Its adjoint satisfies the important formula

$$X^*h = Sh - \langle h, S^*b\rangle_b\, b\,, \tag{2.1}$$

for all $h \in \mathcal{H}(b)$ (see [11, page 12]).

We end this section by recalling the definition of the spectrum of a function b in the unit ball of $H^\infty(\mathbb{D})$ (see [9, page 103]). A point $\lambda \in \overline{\mathbb{D}}$ is said to be regular (for b) if either $\lambda \in \mathbb{D}$ and $b(\lambda) \neq 0$, or $\lambda \in \mathbb{T}$ and b admits an analytic continuation across a neighbourhood $V_\lambda = \{z : |z-\lambda| < \varepsilon\}$ of λ with $|b| = 1$ on $V_\lambda \cap \mathbb{T}$. The spectrum of b, denoted by $\sigma(b)$, is then defined as the complement in $\overline{\mathbb{D}}$ of all regular points of b.

3. Existence of derivatives for functions of de Branges–Rovnyak spaces

We first begin with a lemma which is essentially due to Ahern–Clark [1, Lemma 2.1].

Lemma 3.1. *Let $S_1, \dots, S_p$ be bounded commuting operators of norm less or equal to 1 on a Hilbert space X. Let $(\lambda_1, \dots, \lambda_p) \in \mathbb{T}^p$ such that $Id - \lambda_j S_j$ is one to one. Furthermore, let $(\lambda_1^{(n)}, \dots, \lambda_p^{(n)}) \in \mathbb{D}^p$ tend nontangentially to $(\lambda_1, \dots, \lambda_p)$ as $n \to +\infty$. Then, for any $y \in X$, the sequence $w_n := (Id - \lambda_1^{(n)} S_1)^{-1} \dots (Id - \lambda_p^{(n)} S_p)^{-1} y$ is uniformly bounded if and only if y belongs to the range of the operator $(Id - \lambda_1 S_1) \dots (Id - \lambda_p S_p)$, in which case, w_n tends weakly to $w_0 := (Id - \lambda_1 S_1)^{-1} \dots (Id - \lambda_p S_p)^{-1} y$.*

Proof. If $\|S_j\| < 1$, then the operator $Id - \lambda_j S_j$ is invertible and $(Id - \lambda_j^{(n)} S_j)^{-1}$ tends to $(Id - \lambda_j S_j)^{-1}$ in operator norm, as $n \to +\infty$. Therefore, we see that we can assume that all operators S_j are of norm equal to 1. This case is precisely the result of Ahern–Clark. □

The following result gives several criterions for the existence of derivatives of functions in $\mathcal{H}(b)$ and it generalizes the Ahern–Clark result.

Theorem 3.2. *Let b be a point in the unit ball of $H^\infty(\mathbb{D})$ and let*

$$b(z) = \gamma \prod_n \left(\frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z} \right) \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right) \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|b(\zeta)| \, dm(\zeta) \right) \tag{3.1}$$

be its canonical factorization. Let $\zeta_0 \in \mathbb{T}$ and let N be a non-negative integer. Then the following are equivalent.

(i) *for every function $f \in \mathcal{H}(b)$, $f(z), f'(z), \dots, f^{(N)}(z)$ have finite limits as z tends radially to ζ_0;*

(ii) *for every function $f \in \mathcal{H}(b)$, $|f^{(N)}(z)|$ remains bounded as z tends radially to ζ_0;*

(iii) *$\left\| \partial^N k_z^b / \partial \overline{z}^N \right\|_b$ is bounded as z tends radially to ζ_0;*

(iv) *$X^{*N} k_0^b$ belongs to the range of $(Id - \overline{\zeta_0} X^*)^{N+1}$;*

(v) *we have*

$$\sum_n \frac{1 - |a_n|^2}{|\zeta_0 - a_n|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^{2N+2}} + \int_0^{2\pi} \frac{\left| \log|b(e^{it})| \right|}{|\zeta_0 - e^{it}|^{2N+2}} \, dm(e^{it}) < +\infty \, .$$

Proof. (i) $\Longrightarrow$ (ii): it is obvious.

(ii) $\Longrightarrow$ (iii): for a point z in $\mathbb{D}$, the function $\frac{\partial^N k_z^b}{\partial \overline{z}^N}$ is easily seen to be the kernel function in $\mathcal{H}(b)$ for the functional of evaluation of the Nth derivative at z:

$$f^{(N)}(z) = \left\langle f, \frac{\partial^N k_z^b}{\partial \overline{z}^N} \right\rangle_b , \qquad \forall f \in \mathcal{H}(b) \, . \tag{3.2}$$

Therefore, the implication (ii) $\Longrightarrow$ (iii) follows from the principle of uniform boundedness.

The equivalence of (i) and (iii) is not new and can be found in [11, page 58].

(iii) $\Longrightarrow$ (iv): using the fact that $k_z^b = (Id - \overline{z}X^*)^{-1}k_0^b$ (see [11, page 42]), we easily get

$$\frac{\partial^N k_z^b}{\partial \overline{z}^N} = N!(Id - \overline{z}X^*)^{-(N+1)}X^{*N}k_0^b\,. \tag{3.3}$$

We know from [6, Lemma 2.2] that $\sigma_p(X^*) \subset \mathbb{D}$ and thus the operator $Id - \overline{\zeta_0}X^*$ is one-to-one. By assumption, $(Id - \overline{z_n}X^*)^{-(N+1)}X^{*N}k_0^b$ is uniformly bounded for any sequence $z_n \in \mathbb{D}$ tending radially to ζ_0. Hence, by Lemma 3.1, $X^{*N}k_0^b$ belongs to the range of $(Id - \overline{\zeta_0}X^*)^{N+1}$.

(iv) $\Longrightarrow$ (i): using once more Lemma 3.1 with $p = N+1$, $S_1 = \cdots = S_p = X^*$, $\lambda_1 = \cdots = \lambda_p = \overline{\zeta}_0$ and $y = X^{*N}k_0^b$, we see that (iv) implies that $(Id - \overline{z_n}X^*)^{-(N+1)}X^{*N}k_0^b$ tends weakly to $(Id - \overline{\zeta_0}X^*)^{-(N+1)}X^{*N}k_0^b$, for any sequence $z_n \in \mathbb{D}$ tending radially to ζ. Hence (3.2) and (3.3) imply that, for every function f in $\mathcal{H}(b)$, $f^{(N)}(z)$ has a finite limit as z tends radially to ζ_0. Now of course, for every $0 \le j \le N$, (iv) ensures that $X^{*j}k_0^b$ belongs to the range of $(Id - \overline{\zeta_0}X^*)^{j+1}$ and similar arguments show that, for every function f in $\mathcal{H}(b)$, $f^{(j)}(z)$ has a finite limit as z tends radially to ζ_0.

(v) $\Longrightarrow$ (iii): without loss of generality we assume that $\zeta_0 = 1$. Using Leibnitz' rule, by straightforward computations we obtain

$$k_{\omega,N}^b(z) := \frac{\partial^N k_\omega^b}{\partial \overline{\omega}^N}(z) = \frac{h_{\omega,N}^b(z)}{(1-\overline{\omega}z)^{N+1}}\,, \tag{3.4}$$

with

$$h_{\omega,N}^b(z) = N!z^N - b(z)\sum_{j=0}^N \binom{N}{j}\overline{b^{(j)}(\omega)}(N-j)!z^{N-j}(1-\overline{\omega}z)^j\,. \tag{3.5}$$

Hence, by (3.2), we have

$$\left\|\frac{\partial^N k_\omega^b}{\partial \overline{\omega}^N}\right\|_b^2 = (k_{\omega,N}^b)^{(N)}(\omega)\,,$$

and thus, we need to prove that $(k_{r,N}^b)^{(N)}(r)$ is bounded as $r \to 1^-$.

But the condition (v) clearly implies that

$$\sum_n \frac{1-|a_n|^2}{|\zeta_0 - a_n|^j} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|\zeta_0 - e^{it}|^j} + \int_0^{2\pi} \frac{\left|\log|b(e^{it})|\right|}{|\zeta_0 - e^{it}|^j}\, dm(e^{it}) < +\infty\,,$$

for $0 \le j \le 2N+2$ and then it follows from [2, Lemma 4] that

$$\lim_{r\to 1^-} b^{(j)}(r) \quad \text{and} \quad \lim_{R\to 1^+} b^{(j)}(R)$$

exist and are equal. Here we extend the function b outside the unit disk by the formula (3.1), which represents an analytic function for $|z| > 1$, $z \neq 1/\overline{a_n}$. We denote this function also by b and it is easily verified that it satisfies

$$b(z) = \frac{1}{\overline{b(1/\overline{z})}}\,, \qquad \forall z \in \mathbb{C}\,. \tag{3.6}$$

Therefore, there exists $R_0 > 1$ such that b has $2N+1$ continuous derivatives on $[0, R_0]$. Now take $R_0^{-1} < r < 1$. Noting that b can have only a finite number of real zeros, we can assume that the interval $(R_0^{-1}, 1)$ is free of zeros. Then straightforward computations using (3.5) and (3.6) show that $h_{r,N}^b$ and its first N derivatives must vanish at $z = 1/r$. Therefore we can write, for $s \in (0,1)$,

$$\begin{aligned} h_{r,N}^b(s) &= \int_0^1 \frac{d}{dt} h_{r,N}^b\left(\frac{1}{r} + t\left(s - \frac{1}{r}\right)\right) dt \\ &= \left(s - \frac{1}{r}\right) \int_0^1 (h_{r,N}^b)'\left(\frac{1}{r} + t\left(s - \frac{1}{r}\right)\right) dt \\ &= \left(s - \frac{1}{r}\right)^2 \int_0^1\int_0^1 (h_{r,N}^b)''\left(\frac{1}{r} + tu\left(s - \frac{1}{r}\right)\right) t\, du\, dt\,. \end{aligned}$$

Continuing this procedure, we get

$$h_{r,N}^b(s) = \left(s - \frac{1}{r}\right)^{N+1} \int_0^1\int_0^1 \dots \int_0^1 (h_{r,N}^b)^{(N+1)} \left(\frac{1}{r} + t_1 t_2 \dots t_{N+1}\left(s - \frac{1}{r}\right)\right) m(t)\, dt_1 \,\dots\, dt_{N+1}\,,$$

where $m(t)$ is a monomial in $t_1, \dots, t_{N+1}$. Hence, using (3.4), we obtain

$$k_{r,N}^b(s) = \frac{1}{r^{N+1}} \int_0^1\int_0^1 \dots \int_0^1 \left(h_{r,N}^b\right)^{(N+1)} \left(\frac{1}{r} + t_1 t_2 \dots t_{N+1}\left(s - \frac{1}{r}\right)\right) m(t)\, dt_1 \,\dots\, dt_{N+1}\,.$$

But, thanks to properties of b, we can differentiate under the integral sign to get

$$\left(k_{r,N}^b\right)^{(N)}(s) = \frac{1}{r^{N+1}} \int_0^1\int_0^1 \dots \int_0^1 \left(h_{r,N}^b\right)^{(2N+1)} \left(\frac{1}{r} + t_1 t_2 \dots t_{N+1}\left(s - \frac{1}{r}\right)\right) v(t)\, dt_1 \,\dots\, dt_{N+1}\,,$$

where $v(t)$ is a monomial in $t_1, \dots, t_{N+1}$. Since $(h_{r,N}^b)^{(2N+1)}$ is bounded on $(0, R_0)$, we deduce that $|(k_{r,N}^b)^{(N)}(r)| \leq \frac{1}{r^{N+1}} \|(h_{r,N}^b)^{(2N+1)}\|_\infty$, which is bounded as $r \to 1^-$.

(iii) $\Longrightarrow$ (v): here we also assume that $\zeta_0 = 1$. According to [1, Lemma 4.2] we can take a sequence $(B_j)_{j\geq 1}$ of Blaschke products converging uniformly to b

on compact subsets of $\mathbb{D}$ and such that

$$\sum_k \frac{1-|a_{j,k}|^2}{|1-ra_{j,k}|^{2N+2}} \underset{j\to+\infty}{\longrightarrow} \sum_k \frac{1-|a_k|^2}{|1-ra_k|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it}-r|^{2N+2}} + \int_0^{2\pi} \frac{|\log|b(e^{it})||}{|e^{it}-r|^{2N+2}}\,dm(e^{it}),$$

where $(a_{j,k})_{k\geq 1}$ is the sequence of zeros of B_j. As before, let $k^b_{\omega,N} := \partial^N k^b_\omega/\partial\overline{\omega}^N$ and let $k^{B_j}_{\omega,N} := \partial^N k^{B_j}_\omega/\partial\overline{\omega}^N$. Hence, we have

$$k^{B_j}_{\omega,N}(z) = \frac{N!z^N - B_j(z)\sum_{p=0}^{N}\binom{N}{p}\overline{B_j^{(p)}(\omega)}(N-p)!z^{N-p}(1-\overline{\omega}z)^p}{(1-\overline{\omega}z)^{N+1}} \tag{3.7}$$

and thus $k^{B_j}_{\omega,N}$ tends to $k^b_{\omega,N}$ uniformly on compact subsets of $\mathbb{D}$. Therefore,

$$\lim_{j\to+\infty}\left(k^{B_j}_{\omega,N}\right)^{(N)}(\omega) = \left(k^b_{\omega,N}\right)^{(N)}(\omega).$$

But,

$$\left\|\frac{\partial^N k^b_\omega}{\partial\overline{\omega}^N}\right\|_b^2 = \left(k^b_{\omega,N}\right)^{(N)}(\omega), \qquad \text{and} \qquad \left\|\frac{\partial^N k^{B_j}_\omega}{\partial\overline{\omega}^N}\right\|_2^2 = \left(k^{B_j}_{\omega,N}\right)^{(N)}(\omega),$$

and condition (iii) implies that there exists $C_1 > 0$ such that, for all $0 < r < 1$, we have $|(k^b_{r,N})^{(N)}(r)| \leq C_1$. Therefore, for all $0 < r < 1$, there exists $j_r \in \mathbb{N}$, such that for $j \geq j_r$, we have

$$\left\|\frac{\partial^N k^{B_j}_r}{\partial r^N}\right\|_2^2 = \left|\left(k^{B_j}_{r,N}\right)^{(N)}(r)\right| \leq C_1 + 1.$$

Moreover, using (3.7), we see that

$$(1-rz)^{N+1}\frac{\partial^N k^{B_j}_r}{\partial r^N}(z) = N!z^N - B_j(z)g_j(z),$$

where $g_j \in H^2$. Hence, it follows from [1, Theorem 3.1] that there is a constant K (independent of r) such that

$$\sum_k \frac{1-|a_{j,k}|^2}{|1-ra_{j,k}|^{2N+2}} \leq K, \qquad (j \geq j_r),$$

Letting $j \to +\infty$, we obtain

$$\sum_k \frac{1-|a_k|^2}{|1-ra_k|^{2N+2}} + \int_0^{2\pi} \frac{d\mu(e^{it})}{|e^{it}-r|^{2N+2}} + \int_0^{2\pi} \frac{\big|\log|b(e^{it})|\big|}{|e^{it}-r|^{2N+2}}\,dm(e^{it}) \leq K$$

for all $r \in (0,1)$. Now we let $r \to 1^-$, we get the desired condition (v). $\square$

4. Continuity and analytic continuation for functions of the de Branges–Rovnyak spaces

In this section, we study the continuity and analyticity of functions in the de Branges–Rovnyak spaces $\mathcal{H}(b)$ on an open arc of $\mathbb{T}$. As we will see the theory bifurcates into two opposite cases depending whether b is an extreme point of the unit ball of $H^\infty(\mathbb{D})$ or not. Let us recall that if X is a linear space and S is a convex subset of X, then an element $x \in S$ is called an extreme point of S if it is not a proper convex combination of any two distinct points in S. Then, it is well known (see [4, page 125]) that a function f is an extreme point of the unit ball of $H^\infty(\mathbb{D})$ if and only if

$$\int_{\mathbb{T}} \log\big(1 - |f(\zeta)|\big)\, d\zeta = -\infty\,.$$

The following result is a generalization of results of Helson [7] and Sarason [11]. The equivalence of (i), (ii) and (iii) were proved in [11, page 42] under the assumption that b is an extreme point. Our contribution is the last two parts. The mere assumption of continuity implies analyticity and this observation has interesting applications.

Theorem 4.1. *Let b be in the unit ball of $H^\infty(\mathbb{D})$ and let I be an open arc of $\mathbb{T}$. Then the following are equivalent:*

(i) *b has an analytic continuation across I and $|b| = 1$ on I;*
(ii) *I is contained in the resolvent set of X^*;*
(iii) *any function f in $\mathcal{H}(b)$ has an analytic continuation across I;*
(iv) *any function f in $\mathcal{H}(b)$ has a continuous extension to $\mathbb{D} \cup I$;*
(v) *b has a continuous extension to $\mathbb{D} \cup I$ and $|b| = 1$ on I.*

Proof. (i) $\Longrightarrow$ (ii): since $|b| = 1$ on an open interval, it is clear that b is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. In that case, we know that the characteristic function of the operator X^* (in the theory of Sz-Nagy and Foias) is b (see [10]). But then this theory tells us that $\sigma(X^*) = \sigma(b)$ (see [9, Theorem 2.3.4., page 102]). Therefore, if b has an analytic continuation across I and $|b| = 1$ on I, then I is contained in the complement of $\sigma(b)$ and thus I is contained in the resolvent set of X^*.

(ii) $\Longrightarrow$ (iii): for $f \in \mathcal{H}(b)$, we have

$$f(\omega) = \left\langle f, k_\omega^b \right\rangle_b = \left\langle f, (Id - \overline{\omega} X^*)^{-1} k_0^b \right\rangle_b .$$

Now if I is contained in the resolvent set of X^*, then the vector valued function $\omega \longmapsto (Id - \omega X^*)^{-1} k_0^b$, thought of as an $\mathcal{H}(b)$-valued function, can be continued analytically across I and thus the condition (iii) follows.

(iii) $\Longrightarrow$ (iv): is clear.

(iv) $\Longrightarrow$ (v): let $\omega_0 \in \mathbb{D}$ such that $b(\omega_0) \neq 0$. Since $\frac{1 - \overline{b(\omega_0)} b(z)}{1 - \overline{\omega_0} z}$ belongs to $\mathcal{H}(b)$, it has a continuous extension to $\mathbb{D} \cup I$. Therefore b also has a continuous extension to $\mathbb{D} \cup I$. Now let ζ_0 be a point of I. An application of the principle of uniform boundedness shows that the functional on $\mathcal{H}(b)$ of evaluation at ζ_0 is bounded. Let $k^b_{\zeta_0}$ denote the corresponding kernel function. The family k^b_ω tends weakly to $k^b_{\zeta_0}$ as ω tends to ζ_0 from $\mathbb{D}$. Thus, for any $z \in \mathbb{D}$, we also have

$$k^b_{\zeta_0}(z) = \left\langle k^b_{\zeta_0}, k^b_z \right\rangle_b = \lim_{\omega\to\zeta_0} \left\langle k^b_\omega, k^b_z \right\rangle_b = \lim_{\omega\to\zeta_0} \frac{1-\overline{b(\omega)}b(z)}{1-\overline{\omega}z} = \frac{1-\overline{b(\zeta_0)}b(z)}{1-\overline{\zeta_0}z}\,.$$

In particular, the function $\frac{1-\overline{b(\zeta_0)}b(z)}{z-\zeta_0}$ is in $H^2(\mathbb{C}_+)$, which is possible only if $|b(\zeta_0)| = 1$. Hence we get that $|b| = 1$ on I.

(v) $\Longrightarrow$ (i): follows from standard facts based on the Schwarz's reflection principle. □

As we have seen in the proof of Theorem 4.1, any of the conditions (i) – (v) implies that b is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. Thus, the continuity (or equivalently, the analytic continuation) of b or of the elements of $\mathcal{H}(b)$ on the boundary completely depend on b being an extreme point or not. If b is not an extreme point of the unit ball of $H^\infty(\mathbb{D})$ and if I is an open arc of $\mathbb{T}$, then there exists necessarily a function $f \in \mathcal{H}(b)$ such that f has not a continuous extension to $\mathbb{D}\cup I$. On the opposite case, if b is an extreme point such that b has continuous extension to $\mathbb{D}\cup I$ with $|b| = 1$ on I, then all the functions $f \in \mathcal{H}(b)$ are continuous on I (and even can be continued analytically across I).

Theorem 4.1 shows that the de Branges–Rovnyak spaces $\mathcal{H}(b)$ have a remarkable property, i.e., continuity on an open arc of $\mathbb{T}$ of all functions of $\mathcal{H}(b)$ is enough to imply the analyticity of these functions. This property enables us to show that the result of Dyakonov [5] concerning the Bernstein's inequality in the model spaces is sharp in the sense that we cannot extend it to all de Branges–Rovnyak spaces. The definition of de Branges–Rovnyak spaces of the upper half plane is similar to its counterpart for the unit disc. First, we make precise a little more the transfer of the unit disc to the upper half plane $\mathbb{C}_+$. We consider γ the conformal map from $\mathbb{C}_+$ onto $\mathbb{D}$ defined by

$$\gamma(z) = \frac{z-i}{z+i}\,, \qquad z \in \mathbb{C}_+\,,$$

and we denote by U the (unitary) map from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$ defined by

$$(Uf)(x) := \frac{1}{\sqrt{\pi}}\frac{1}{x+i}f\left(\frac{x-i}{x+i}\right), \qquad x \in \mathbb{R}, f \in L^2(\mathbb{T})\,. \tag{4.1}$$

Then it is well known (see [8, pages 247-248]) that U maps $H^2(\mathbb{D})$ onto $H^2(\mathbb{C}_+)$. Moreover, if $\varphi \in L^\infty(\mathbb{T})$, then

$$UT_\varphi = T_{\varphi\circ\gamma}U\,. \tag{4.2}$$

Now let b be in the unit ball of $H^\infty(\mathbb{D})$ and let $b_1 = b\circ\gamma$. Then, using (4.2), basic arguments show that U maps unitarily $\mathcal{H}(b)$ onto $\mathcal{H}(b_1)$. Using this unitary transform, we can obviously state the analogue of Theorem 3.2 and Theorem 4.1 in the upper half plane $\mathbb{C}_+$.

Corollary 4.2. *Let b_1 be a point of the unit ball of $H^\infty(\mathbb{C}_+)$. Then the following are equivalent:*

(i) *the operator $f \longrightarrow f'$ is a bounded operator from $\mathcal{H}(b_1)$ into $H^2(\mathbb{C}_+)$;*
(ii) *b_1 is an inner function and $b_1' \in H^\infty(\mathbb{C}_+)$.*

Proof. Using [5, Theorem 1], the only thing to prove is that if (i) holds, then b_1 is inner. But, if for any function f in $\mathcal{H}(b_1)$, we have $f' \in H^2(\mathbb{C}_+)$, then in particular, f has a continuous extension to $\mathbb{C}_+ \cup \mathbb{R}$. Thus, using the analogue of Theorem 4.1 in the upper half plane, we see that b_1 has a continuous extension to $\mathbb{C}_+ \cup \mathbb{R}$ and $|b_1| = 1$ on $\mathbb{R}$, which means b_1 is an inner function. □

Acknowledgements

The authors deeply thank the anonymous referee for his/her valuable remarks and suggestions which improved the quality of this paper. We also thank A. Baranov for several valuable discussions.

References

[1] P. R. Ahern and D. N. Clark, Radial limits and invariant subspaces. *Amer. J. Math.* 92 (1970), 332–342.

[2] P. R. Ahern and D. N. Clark, Radial nth derivatives of Blaschke products. *Math. Scand.* 28 (1971), 189–201.

[3] V. Bolotnikov and A. Kheifets, A higher order analogue of the Carathéodory-Julia theorem. *J. Funct. Anal.* 237 (2006), 350–371.

[4] P. L.Duren, *Theory of H^p spaces.* Academic Press, 1970.

[5] K. M. Dyakonov, Differentiation in star-invariant subspaces I: boudedness and compactness. *J. Funct. Anal.* 192 (2002), 364–386.

[6] E. Fricain, Bases of reproducing kernels in de Branges spaces. *J. Funct. Anal.* 226 : 2 (2005), 373–405.

[7] H. Helson, *Lectures on Invariant Subspaces.* Academic Press, 1964.

[8] S. V. Hruščev, N. K. Nikolski, and B. S. Pavlov, Unconditional bases of exponentials and of reproducing kernels. In *Complex Analysis and Spectral Theory*, V. P. Havin and N. K. Nikolski, Eds., Lectures Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New-York, 1981, pp. 214–335.

[9] N. K. Nikolski, *Operators, Functions, and Systems: An Easy Reading, Vol.* 2*: Model Operators and Systems.* Mathematical Surveys and Monographs, Vol. 93, A.M.S., 2002.

[10] D. Sarason, Shift-invariant spaces from the Brangesian point of view. In *The Bieberbach Conjecture-Proceedings of the Symposium on the Occasion of the Proof.* Amer. Math. Soc., Providence, 1986, pp. 153–166.

[11] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk.* University of Arkansas Lecture Notes in the Mathematical Sciences 10, John Wiley & Sons Inc., New York, 1994.

Emmanuel Fricain
Institut Camille Jordan
Université de Lyon
Université Lyon 1
CNRS UMR 5208
43, boulevard du 11 Novembre 1918
F-69622 Villeurbanne
France
e-mail: fricain@math.univ-lyon1.fr

Javad Mashreghi
Département de mathématiques et de statistique
Université Laval
Québec, QC, G1K 7P4
Canada
e-mail: Javad.Mashreghi@mat.ulaval.ca

Communicated by Mihai Putinar.

Received: May 10, 2007.

Revised: August 8, 2007.

Accepted: August 8, 2007.

6.2.2 Référence [T6]

Auteurs
E. Fricain et J. Mashreghi

Titre
Integral representations of the n-th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel.

A Paraître dans
Ann. Inst. Fourier (Grenoble).

INTEGRAL REPRESENTATION OF THE n-TH DERIVATIVE IN DE BRANGES-ROVNYAK SPACES AND THE NORM CONVERGENCE OF ITS REPRODUCING KERNEL

EMMANUEL FRICAIN, JAVAD MASHREGHI

Abstract. In this paper, we give an integral representation for the boundary values of derivatives of functions of the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is in the unit ball of $H^\infty(\mathbb{C}_+)$. In particular, we generalize a result of Ahern–Clark obtained for functions of the model spaces K_b, where b is an inner function. Using hypergeometric series, we obtain a nontrivial formula of combinatorics for sums of binomial coefficients. Then we apply this formula to show the norm convergence of reproducing kernel $k_{\omega,n}^b$ of the evaluation of n-th derivative of elements of $\mathcal{H}(b)$ at the point ω as it tends radially to a point of the real axis.

1. Introduction

Let $\mathbb{C}_+$ denote the upper half plane in the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space consisting of analytic functions f on $\mathbb{C}_+$ which satisfy

$$\|f\|_2 := \sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)|^2 \, dx \right)^{1/2} < +\infty.$$

P. Fatou [12] proved that, for any function f in $H^2(\mathbb{C}_+)$ and for almost all x_0 in $\mathbb{R}$,

$$f^*(x_0) := \lim_{t \to 0^+} f(x_0 + it)$$

exists. Moreover, we have $f^* \in L^2(\mathbb{R})$, $\mathcal{F}f^* = 0$ on $(-\infty, 0)$, where $\mathcal{F}$ is the Fourier–Plancherel transformation, and $\|f^*\|_2 = \|f\|_2$. Of course the boundary points where the radial limit exists depend on the function f. However we cannot say more about the

2000 *Mathematics Subject Classification.* Primary: 46E22, Secondary: 47A15, 33C05, 05A19.

Key words and phrases. de Branges-Rovnyak spaces, model subspaces of H^2, integral representation, hypergeometric functions.

This work was supported by funds from NSERC (Canada) and the Jacques Cartier Center (France).

boundary behavior of a typical element of $H^2(\mathbb{C}_+)$. Then many authors, e.g. [16, 1, 2, 14], have studied this question by restricting the class of functions. A particularly interesting class of subspaces of $H^2(\mathbb{C}_+)$ consists of de Branges–Rovnyak spaces.

For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi(f) := P_+(\varphi f), \qquad (f \in H^2(\mathbb{C}_+)),$$

where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $\varphi \in L^\infty(\mathbb{R})$, $\|\varphi\|_\infty \leq 1$, the de Branges–Rovnyak space $\mathcal{H}(\varphi)$, associated with φ, consists of those $H^2(\mathbb{C}_+)$ functions which are in the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space when equipped with the inner product

$$\langle (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f, (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g\rangle_\varphi = \langle f, g\rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker\, (Id - T_\varphi T_{\overline{\varphi}})^{1/2}$.

These spaces (and more precisely their general vector-valued version) appeared first in L. de Branges and J. Rovnyak [7, 8] as universal model spaces for Hilbert space contractions. As a special case, when b is an inner function (that is $|b| = 1$ a.e. on $\mathbb{R}$), the operator $(Id - T_b T_{\overline{b}})$ is an orthogonal projection and $\mathcal{H}(b)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{C}_+)$ which coincides with the so-called model spaces $K_b = H^2(\mathbb{C}_+) \ominus bH^2(\mathbb{C}_+)$. Thanks to the pioneer works of Sarason, e.g. [18], we know that de Branges-Rovnyak spaces have an important role to be played in numerous questions of complex analysis and operator theory. We mention a recent paper of A. Hartmann, D. Sarason and K. Seip [15] who give a nice characterization of surjectivity of Toeplitz operator and the proof involves the de Branges-Rovnyak spaces. We also refer to works of J. Shapiro [19, 20] concerning the notion of angular derivative for holomorphic self-maps of the unit disk. See also a paper of J. Anderson and J. Rovnyak [3], where generalized Schwarz-Pick estimates are given and a paper of M. Jury [17], where composition operators are studied by methods based on $\mathcal{H}(b)$ spaces.

In the case where b is an inner function, H. Helson [16] studied the problem of analytic continuation across the boundary for functions in K_b. Then, still when b is an inner function, P. Ahern and D. Clark [1] characterized those points x_0 of $\mathbb{R}$ where every function

f of K_b and all its derivatives up to order n have a radial limit. More precisely, if $b = BI_\mu$ is the canonical factorization of the inner function b into Blaschke product B associated with the sequence $(z_k)_k$ and singular inner part I_μ associated with the singular measure μ, then every function $f \in K_b$ and its derivatives up to order n have finite radial limits at x_0 if and only if

$$\sum_k \frac{\Im\mathrm{m}(z_k)}{|x_0 - z_k|^{2n+2}} + \int_{\mathbb{R}} \frac{d\mu(t)}{|t - x_0|^{2n+2}} < +\infty. \tag{1.1}$$

Recently, we [14] gave an extension of the preceding results of Helson and of Ahern–Clark. See also the paper of E. Fricain [13] where the orthogonal and Riesz basis of $\mathcal{H}(b)$ spaces, which consist of reproducing kernels, are studied.

Now, using Cauchy formula, it is easy to see that if b is inner, $\omega \in \mathbb{C}_+$, n is a non-negative integer and $f \in K_b$, then we have

$$f^{(n)}(\omega) = \int_{\mathbb{R}} f(t)\, \overline{k^b_{\omega,n}(t)}\, dt, \tag{1.2}$$

where

$$\frac{k^b_{\omega,n}(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z) \displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(\omega)}}{p!}(z - \overline{\omega})^p}{(z - \overline{\omega})^{n+1}}, \qquad (z \in \mathbb{C}_+). \tag{1.3}$$

A natural question is to ask if one can extend the formula (1.2) at boundary points x_0. If x_0 is a real point which does not belong to the boundary spectrum of b, then b and all functions of K_b are analytic through a neighborhood of x_0 and then it is obvious that the formula (1.2) is valid at the point x_0. On the other hand, if x_0 satisfies the condition (1.1), then Ahern–Clark [1] showed that the formula (1.2) is still valid at the point $x_0 \in \mathbb{R}$. Recently, K. Dyakonov [10, 11] and then A. Baranov [5] used this formula to get some Bernstein type inequalities in the model spaces K_b.

In this paper, our first goal is to obtain an analogue of formula (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an *arbitrary* function in the unit ball of $H^\infty(\mathbb{C}_+)$ (not necessarily inner). We will provide an integral representation for $f^{(n)}(\omega)$, $\omega \in \mathbb{C}_+$, and also show that under certain conditions the formula remains valid if $\omega = x_0 \in \mathbb{R}$. However, if one tries to generalize techniques used in the model spaces K_b in order to obtain such a representation for the derivative of functions in $\mathcal{H}(b)$, some difficulties appear mainly due to the fact that the evaluation functional in $\mathcal{H}(b)$ (contrary to the model spaces K_b) is not a usual integral operator. Nevertheless, we will overcome this difficulty and provide an integral formula similar to (1.2) for functions in $\mathcal{H}(b)$.

Our second goal is to prove the norm convergence of reproducing kernels of evaluation functional of the n-th derivative as we approach a boundary point. If $n = 0$, for de Branges–Rovnyak spaces of the unit disc, Sarason [18, page 48] showed that

$$\|k^b_{z_0}\|_b^2 = z_0\overline{b(z_0)}b'(z_0), \qquad (z_0 \in \mathbb{T}).$$

We first obtain

$$\|k^b_{x_0,n}\|_b^2 = \frac{n!^2}{2i\pi}\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!}\frac{b^{(2n+1-p)}(x_0)}{(2n+1-p)!}, \qquad (x_0 \in \mathbb{R}),$$

which is an analogue (and generalization) of Sarason's formula for the reproducing kernel of the n-th derivative for de Branges–Rovnyak spaces of the upper half plane. Then we apply this identity to show that $\|k^b_{\omega,n} - k^b_{x_0,n}\|_b \to 0$ as ω tends radially to x_0. Again if $n = 0$, this result is due to Sarason. In establishing the norm convergence we naturally face with the (nontrivial) finite sum

$$(-1)^{r+1}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-2)^{p-\ell}\binom{r}{n-\ell}\binom{2n+1-r}{p}\binom{n-p+\ell}{\ell}, \tag{1.4}$$

with $n, r \in \mathbb{N}$, $0 \le r \le 2n+1$. Using hypergeometric series we show that this sum is equal to $\pm 2^n$, where the choice of sign depends on r.

We mention a recent and very interesting work of V. Bolotnikov and A. Kheifets [6] who obtained an analogue of the classical Carathéodory–Julia theorem on boundary derivatives. Using different techniques, the authors also obtained a condition which guarantees that we can write an analogue of formula (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$. More precisely, this condition is

$$\liminf_{\omega \to x_0} \frac{\partial^{2n}}{\partial\omega^n\partial\overline{\omega}^n}\left(\frac{1-|b(\omega)|^2}{\Im\mathrm{m}\,\omega}\right) < +\infty \tag{1.5}$$

and it is stated that this is equivalent to the existence of the boundary Schwarz-Pick matrix at point x_0. They also got the norm convergence (under their condition). Comparing condition (1.5) with our condition (2.2) is under further investigation.

The plan of the paper is the following. In the next section, we give some preliminaries concerning the de Branges-Rovnyak spaces. In the third section, we establish some integral formulas for the n-th derivatives of functions in $\mathcal{H}(b)$. The fourth section contains the part of combinatorics of this paper. In particular, we show how we can compute the sum (1.4) and get an interesting and quite surprising formula. Finally, in the last section, we apply this formula of combinatorics to solve an important problem of norm convergence for the kernels $k^b_{\omega,n}$ corresponding to the n-th derivative at points ω for functions in $\mathcal{H}(b)$. More precisely, we prove that $k^b_{\omega,n}$ tends in norm to $k^b_{x_0,n}$ as ω tends radially to x_0. We also get some interesting relations between the derivatives of the function b at point x_0.

2. PRELIMINARIES

We first recall two general facts about the de Branges-Rovnyak spaces. As a matter of fact, in [18], these results are formulated for the unit disc. However, the same results with similar proofs also work for the upper half plane. The first one concerns the relation between $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$. For $f \in H^2(\mathbb{C}_+)$, we have [18, page 10]

$$f \in \mathcal{H}(b) \Longleftrightarrow T_{\overline{b}} f \in \mathcal{H}(\overline{b}).$$

Moreover, if $f_1, f_2 \in \mathcal{H}(b)$, then

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \langle T_{\overline{b}} f_1, T_{\overline{b}} f_2\rangle_{\overline{b}}. \tag{2.1}$$

We also mention an integral representation for functions in $\mathcal{H}(\overline{b})$ [18, page 16]. Let $\rho(t) := 1 - |b(t)|^2$, $t \in \mathbb{R}$, and let $L^2(\rho)$ stand for the usual Hilbert space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ with $\|f\|_\rho < \infty$, where

$$\|f\|^2_\rho := \int_{\mathbb{R}} |f(t)|^2 \rho(t)\, dt.$$

For each $w \in \mathbb{C}_+$, the Cauchy kernel k_w belongs to $L^2(\rho)$. Hence, we define $H^2(\rho)$ to be the span in $L^2(\rho)$ of the functions k_w ($w \in \mathbb{C}_+$). If q is a function in $L^2(\rho)$, then $q\rho$ is in $L^2(\mathbb{R})$, being the product of $q\rho^{1/2} \in L^2(\mathbb{R})$ and the bounded function $\rho^{1/2}$. Finally, we define the operator $C_\rho : L^2(\rho) \longrightarrow H^2(\mathbb{C}_+)$ by

$$C_\rho(q) := P_+(q\rho).$$

Then C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\overline{b})$ whose initial space equals to $H^2(\rho)$ and it is an isometry if and only if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$.

In [14], we have studied the boundary behavior of functions of the de Branges–Rovnyak spaces and we mention some parts of [14, Theorem 3.1] that we need here.

Theorem 2.1. *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$ and let*

$$b(z) = \prod_k e^{i\alpha_k} \frac{z - z_k}{z - \overline{z_k}} \ \exp\left(-\frac{1}{i\pi}\int_{\mathbb{R}} \frac{tz+1}{(t-z)(t^2+1)}\, d\mu(t)\right) \ \exp\left(\frac{1}{i\pi}\int_{\mathbb{R}} \frac{tz+1}{t-z}\frac{\log|b(t)|}{t^2+1}\, dt\right)$$

be its canonical factorization. Then, for $x_0 \in \mathbb{R}$ and for a non-negative integer n, the following are equivalent:

(i) *for every function $f \in \mathcal{H}(b)$, $f(x_0+it), f'(x_0+it), \dots f^{(n)}(x_0+it)$ have finite limits as $t \to 0^+$;*

(ii) *we have*

$$\sum_k \frac{\Im\mathrm{m}(z_k)}{|x_0 - z_k|^{2n+2}} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x_0 - t|^{2n+2}} + \int_{\mathbb{R}} \frac{|\log|b(t)||}{|x_0 - t|^{2n+2}}\, dt < +\infty. \tag{2.2}$$

For $f \in \mathcal{H}(b)$, $x_0 \in \mathbb{R}$ and for a non-negative integer n, if $f^{(n)}(x_0+it)$ has a finite limit as $t \to 0^+$, then we define

$$f^{(n)}(x_0) := \lim_{t\to 0^+} f^{(n)}(x_0+it).$$

Moreover, under the condition (2.2), we know that for $0 \le j \le 2n+1$,

$$\lim_{t\to 0^+} b^{(j)}(x_0+it) \tag{2.3}$$

exists (see [2, Lemma 4]) and we denote this limit by $b^{(j)}(x_0)$.

Remark 2.2. Let $x_0 \in \mathbb{R}$ and suppose that x_0 does not belong to the spectrum $\sigma(b)$ of b, which means (by definition) that, for some $\eta > 0$, b is analytic on $B(x_0,\eta) := \{z \in \mathbb{C} :$

$|z - x_0| < \eta\}$ and $|b(x)| = 1$ on $(x_0 - \eta, x_0 + \eta)$. Denote by $a_p := \frac{b^{(p)}(x_0)}{p!}$, $p \geq 0$. Since

$$b(x) = \sum_{p=0}^{\infty} a_p (x - x_0)^p, \qquad x \in (x_0 - \eta, x_0 + \eta),$$

we get

$$1 = |b(x)|^2 = b(x)\overline{b(x)} = \sum_{r=0}^{\infty} c_r (x - x_0)^r,$$

where $c_r = \sum_{p=0}^{r} a_p \, \overline{a_{r-p}}$. Hence

$$c_0 = |a_0|^2 = 1 \qquad \text{and} \qquad \sum_{p=0}^{r} a_p \, \overline{a_{r-p}} = 0, \quad (\forall r \geq 1).$$

As we will see in the proof of Theorem 3.3, the condition (2.2) implies that

$$|a_0|^2 = 1 \qquad \text{and} \qquad \sum_{p=0}^{r} a_p \, \overline{a_{r-p}} = 0, \quad (1 \leq r \leq n).$$

Therefore, the condition (2.2) is somehow a weaker version of the assumption $x_0 \notin \sigma(b)$.

The next result gives a (standard) Taylor formula at a point on the boundary.

Lemma 2.3. *Let h be a holomorphic function in the upper-half plane $\mathbb{C}_+$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$. Assume that $h^{(n)}$ has a radial limit at x_0. Then $h, h', \ldots, h^{(n-1)}$ have radial limits at x_0 and*

$$h(\omega) = \sum_{p=0}^{n} \frac{h^{(p)}(x_0)}{p!} (\omega - x_0)^p + (\omega - x_0)^n \varepsilon(\omega), \qquad (\omega \in \mathbb{C}_+),$$

with $\lim_{t \to 0^+} \varepsilon(x_0 + it) = 0$.

Proof: The case $n = 1$ is contained in [18, Chap. VI]. To establish the general case one assumes as the induction hypothesis that the property is true for $n - 1$. Applying the induction hypothesis to h', we see that $h', h^{(2)}, \ldots, h^{(n)}$ have a radial limit at x_0 and

$$h'(\omega) = \sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (\omega - x_0)^p + (\omega - x_0)^{n-1} \varepsilon_1(\omega),$$

with $\lim\limits_{t\to 0^+} \varepsilon_1(x_0+it)=0$. Since h' has a radial limit at x_0, by the case $n=1$, $h(x_0)=\lim\limits_{t\to 0} h(x_0+it)$ exists and an application of Cauchy's theorem shows that

$$h(\omega)=h(x_0)+\int_{[x_0,\omega]} h'(u)\,du,$$

for all $\omega=x_0+it$, $t>0$. Hence we have

$$\begin{aligned} h(\omega) &= h(x_0)+\int_{[x_0,\omega]}\left(\sum_{p=0}^{n-1}\frac{h^{(p+1)}(x_0)}{p!}(u-x_0)^p+(u-x_0)^{n-1}\varepsilon_1(u)\right)\,du \\ &= \sum_{p=0}^{n}\frac{h^{(p)}(x_0)}{p!}(\omega-x_0)^p+\int_{[x_0,\omega]}(u-x_0)^{n-1}\varepsilon_1(u)\,du. \end{aligned}$$

Finally, let

$$\varepsilon(\omega)=\frac{1}{(\omega-x_0)^n}\int_{[x_0,\omega]}(u-x_0)^{n-1}\varepsilon_1(u)\,du.$$

It is clear that $\lim\limits_{t\to 0^+}\varepsilon(x_0+it)=0$.

□

3. Integral representations

We first begin by proving an integral representation for the derivatives of elements of $\mathcal{H}(b)$ at points ω in the upper half plane. Since ω is away from the boundary, the representation is easy to establish. Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$. Recall that for $\omega\in\mathbb{C}_+$, the function

$$k_\omega^b(z)=\frac{i}{2\pi}\frac{1-\overline{b(\omega)}b(z)}{z-\overline{\omega}}, \qquad (z\in\mathbb{C}_+),$$

is the reproducing kernel of $\mathcal{H}(b)$, that is

$$f(\omega)=\langle f,k_\omega^b\rangle_b, \qquad (f\in\mathcal{H}(b)). \tag{3.1}$$

Now let $\omega \in \mathbb{C}_+$ and let n be a non-negative integer. In order to get an integral representation for the nth derivative of f at point ω for functions in the de-Branges-Rovnyak spaces, we need to introduce the following kernels

$$\frac{k_{\omega,n}^b(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z) \displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(\omega)}}{p!}(z-\overline{\omega})^p}{(z-\overline{\omega})^{n+1}}, \qquad (z \in \mathbb{C}_+), \tag{3.2}$$

and

$$\frac{k_{\omega,n}^\rho(t)}{n!} := \frac{i}{2\pi} \frac{\displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(\omega)}}{p!}(t-\overline{\omega})^p}{(t-\overline{\omega})^{n+1}}, \qquad (t \in \mathbb{R}). \tag{3.3}$$

For $n = 0$, we see that $k_{\omega,0}^b = k_\omega^b$ and $k_{\omega,0}^\rho = \overline{b(\omega)} k_\omega$. Moreover, we also see that the kernel $k_{\omega,n}^b$ coincides with those of the inner case defined by formula (1.3).

Proposition 3.1. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\overline{b}} f = C_\rho(g)$. Then, for all $\omega \in \mathbb{C}_+$ and for any non-negative integer n, we have $k_{\omega,n}^b \in \mathcal{H}(b)$ and $k_{\omega,n}^\rho \in H^2(\rho)$ and*

$$f^{(n)}(\omega) = \langle f, k_{\omega,n}^b \rangle_b = \int_{\mathbb{R}} f(t) \overline{k_{\omega,n}^b(t)}\, dt + \int_{\mathbb{R}} g(t)\rho(t) \overline{k_{\omega,n}^\rho(t)}\, dt. \tag{3.4}$$

Proof: According to (3.1) and (2.1), we have

$$f(\omega) = \langle f, k_\omega^b \rangle_b = \langle f, k_\omega^b \rangle_2 + \langle T_{\overline{b}} f, T_{\overline{b}} k_\omega^b \rangle_{\overline{b}}.$$

But using the fact that $k_\omega^b = k_\omega - \overline{b(\omega)} b k_\omega$ and that $T_{\overline{b}} k_\omega = \overline{b(\omega)} k_\omega$, we obtain

$$T_{\overline{b}} k_\omega^b = \overline{b(\omega)} \left(k_\omega - P_+(|b|^2 k_\omega) \right) = \overline{b(\omega)} P_+ \left((1-|b|^2) k_\omega \right) = \overline{b(\omega)} C_\rho(k_\omega),$$

which implies that

$$f(\omega) = \langle f, k_\omega^b \rangle_2 + b(\omega) \langle C_\rho(g), C_\rho(k_\omega) \rangle_{\overline{b}}.$$

Since C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\overline{b})$, with initial space equals to $H^2(\rho)$, we conclude that

$$f(\omega) = \langle f, k_\omega^b \rangle_2 + b(\omega) \langle g, k_\omega \rangle_\rho = \langle f, k_{\omega,0}^b \rangle_2 + \langle \rho g, k_{\omega,0}^\rho \rangle_2,$$

which gives the representation (3.4) for $n = 0$.

Now straightforward computations show that

$$\frac{\partial^n k_{\omega,0}^b}{\partial\overline{\omega}^n} = k_{\omega,n}^b \quad \text{and} \quad \frac{\partial^n k_{\omega,0}^\rho}{\partial\overline{\omega}^n} = k_{\omega,n}^\rho.$$

Since $k_{\omega,0}^b \in \mathcal{H}(b)$ and $k_{\omega,0}^\rho \in H^2(\rho)$, we thus have $k_{\omega,n}^b \in \mathcal{H}(b)$ and $k_{\omega,n}^\rho \in H^2(\rho)$, $n \geq 0$. The representation (3.4) follows now by induction and by differentiating under the integral sign, which is justified by the dominated convergence theorem. □

In the following, we show that (3.4) is still valid at the boundary points x_0 which satisfy (2.2). We will need the boundary analogues of the kernels (3.2) and (3.3), i.e.

$$\frac{k_{x_0,n}^b(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z) \displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!}(z-x_0)^p}{(z-x_0)^{n+1}}, \qquad (z \in \mathbb{C}_+), \tag{3.5}$$

and

$$\frac{k_{x_0,n}^\rho(t)}{n!} := \frac{i}{2\pi} \frac{\displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!}(t-x_0)^p}{(t-x_0)^{n+1}}, \qquad (t \in \mathbb{R} \setminus \{x_0\}). \tag{3.6}$$

The following result shows that, under condition (2.2), $k_{x_0,n}^b$ is the kernel function in $\mathcal{H}(b)$ for the functional of the n-th derivative at x_0.

Lemma 3.2. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$. Assume that x_0 satisfies the condition (2.2). Then $k_{x_0,n}^b \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, we have*

$$f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b. \tag{3.7}$$

Proof: According to Theorem 2.1, the condition (2.2) guarantees that, for every function $f \in \mathcal{H}(b)$, $f^{(n)}(\omega)$ tends to $f^{(n)}(x_0)$, as ω tends radially to x_0. Therefore, an application of the uniform boundedness principle shows that the functional $f \longmapsto f^{(n)}(x_0)$ is bounded on $\mathcal{H}(b)$. Hence, by Riesz' theorem, there exists $\varphi_{x_0,n} \in \mathcal{H}(b)$ such that

$$f^{(n)}(x_0) = \langle f, \varphi_{x_0,n} \rangle_b, \qquad (f \in \mathcal{H}(b)).$$

Since

$$f^{(n)}(\omega) = \langle f, \frac{\partial^n k^b_{\omega,0}}{\partial\overline{\omega}^n}\rangle_b = \langle f, k^b_{\omega,n}\rangle_b, \qquad (f \in \mathcal{H}(b)),$$

we see that $k^b_{\omega,n}$ tends weakly to $\varphi_{x_0,n}$, as ω tends radially to x_0. Thus, for $z \in \mathbb{C}_+$, we can write

$$\begin{aligned}\varphi_{x_0,n}(z) &= \langle \varphi_{x_0,n}, k^b_z\rangle_b = \lim_{t\to 0^+} \langle k^b_{x_0+it,n}, k^b_z\rangle_b = \lim_{t\to 0^+} k^b_{x_0+it,n}(z)\\ &= \lim_{t\to 0^+} n!\frac{i}{2\pi}\frac{1-b(z)\displaystyle\sum_{p=0}^{n}\frac{\overline{b^{(p)}(x_0+it)}}{p!}(z-x_0+it)^p}{(z-x_0+it)^{n+1}}\\ &= n!\frac{i}{2\pi}\frac{1-b(z)\displaystyle\sum_{p=0}^{n}\frac{\overline{b^{(p)}(x_0)}}{p!}(z-x_0)^p}{(z-x_0)^{n+1}},\end{aligned}$$

which implies that $\varphi_{x_0,n} = k^b_{x_0,n}$. Hence $k^b_{x_0,n} \in \mathcal{H}(b)$ and for every function $f \in \mathcal{H}(b)$ we have

$$f^{(n)}(x_0) = \langle f, k^b_{x_0,n}\rangle_b.$$

□

For $n = 0$, Lemma 3.2 appears in [18, Chap. V], in the context of the unit disc. The problem with the representation (3.7) is that the inner product in $\mathcal{H}(b)$ is not an explicit integral formula and thus it is not convenient to use it. That is why we prefer to have an integral formula of type (3.4).

If x_0 satisfies the condition (2.2) we also have $k^\rho_{x_0,n} \in L^2(\rho)$. Indeed, according to (3.6), it suffices to prove that $(t-x_0)^{-j} \in L^2(\rho)$, for $1 \le j \le n+1$. Since $\rho \le 1$, it is enough to verify this fact in a neighborhood of x_0, say $I_{x_0} = [x_0-1, x_0+1]$. But according to the condition (2.2), we have

$$\int_{I_{x_0}} \frac{1-|b(t)|^2}{|t-x_0|^{2j}}\,dt \le 2\int_{I_{x_0}} \frac{|\log|b(t)||}{|t-x_0|^{2j}}\,dt \le 2\int_{I_{x_0}} \frac{|\log|b(t)||}{|t-x_0|^{2(n+1)}}\,dt < +\infty.$$

Theorem 3.3. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\overline{b}}f = C_\rho(g)$. Then, for every point $x_0 \in \mathbb{R}$*

satisfying the condition (2.2), we have

$$f^{(n)}(x_0) = \int_{\mathbb{R}} f(t)\overline{k^b_{x_0,n}(t)}\,dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{x_0,n}(t)}\,dt. \tag{3.8}$$

Proof: Recall that according to (2.3), the condition (2.2) guarantees that $b^{(j)}(x_0)$ exists for $0 \leq j \leq 2n+1$. Moreover, Lemma 3.2 implies that $k^b_{x_0,p} \in \mathcal{H}(b)$, for $0 \leq p \leq n$. First of all, we prove that

$$h_{x_0,n}(z) := \frac{b(z) - \displaystyle\sum_{p=0}^{n} \frac{b^{(p)}(x_0)}{p!}(z-x_0)^p}{(z-x_0)^{n+1}}, \qquad (z \in \mathbb{C}_+),$$

satisfies

$$h_{x_0,n} = 2i\pi \sum_{p=0}^{n} \frac{b^{(n-p)}(x_0)}{(n-p)!p!} k^b_{x_0,p}. \tag{3.9}$$

To simplify a little bit the next computations, we put $a_p := \frac{b^{(p)}(x_0)}{p!}$, $0 \leq p \leq n$. According to (3.2), we have

$$\begin{aligned}
2i\pi \sum_{p=0}^{n} a_{n-p} \frac{k^b_{x_0,p}(z)}{p!} &= \sum_{p=0}^{n} a_{n-p} \left(\frac{\displaystyle\sum_{j=0}^{p} \overline{a_j}(z-x_0)^j b(z) - 1}{(z-x_0)^{p+1}} \right) \\
&= \frac{\displaystyle\sum_{p=0}^{n} a_{n-p}(z-x_0)^{n-p} \left(b(z) \sum_{j=0}^{p} \overline{a_j}(z-x_0)^j - 1 \right)}{(z-x_0)^{n+1}} \\
&= \frac{b(z) \left(\displaystyle\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} \right) - \displaystyle\sum_{k=0}^{n} a_k (z-x_0)^k}{(z-x_0)^{n+1}}.
\end{aligned}$$

Therefore, we see that (3.9) is equivalent to

$$\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} = 1. \tag{3.10}$$

But, putting $j = \ell - n + p$, we obtain

$$\sum_{p=0}^{n}\sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} = \sum_{\ell=0}^{n}\left(\sum_{p=n-\ell}^{n} a_{n-p}\overline{a_{\ell-n+p}}\right)(z-x_0)^{\ell}$$

$$= \sum_{\ell=0}^{n}\left(\sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q}\right)(z-x_0)^{\ell}.$$

Consequently, (3.10) is equivalent to

$$|b(x_0)|^2 = 1 \quad \text{and} \quad \sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q} = 0, \qquad (1 \le \ell \le n). \tag{3.11}$$

Now if we define

$$\varphi(z) := 1 - b(z)\sum_{p=0}^{n}\overline{a_p}(z-x_0)^p, \qquad (z \in \mathbb{C}_+),$$

then φ is holomorphic in $\mathbb{C}_+$ and according to (2.3), φ and its derivatives up to $2n+1$ have radial limits at x_0. An application of Lemma 2.3 shows that we can write

$$\varphi(z) = \sum_{p=0}^{n}\frac{\varphi^{(p)}(x_0)}{p!}(z-x_0)^p + o((z-x_0)^n),$$

as z tends radially to x_0. Assume that there exists $k \in \{0,\dots,n\}$ such that $\varphi^{(k)}(x_0) \ne 0$ and let

$$j_0 := \min\{0 \le p \le n : \varphi^{(p)}(x_0) \ne 0\}.$$

Hence, as $t \to 0^+$,

$$|k^b_{x_0,n}(x_0+it)| \sim \frac{1}{2\pi}\frac{|\varphi^{(j_0)}(x_0)|}{j_0!}t^{j_0-(n+1)},$$

which implies that $\lim\limits_{t\to 0^+}|k^b_{x_0,n}(x_0+it)| = +\infty$. This is a contradiction with the fact that $k^b_{x_0,n}$ belongs to $\mathcal{H}(b)$ and has a finite radial limit at x_0. Therefore we necessarily have $\varphi^{(\ell)}(x_0) = 0$, $0 \le \ell \le n$. But $\varphi(x_0) = 1 - b(x_0)\overline{b(x_0)} = 1 - |b(x_0)|^2$ and if we use the Leibniz' rule to compute the derivative of φ, for $1 \le \ell \le n$, we get

$$\varphi^{(\ell)}(x_0) = -\sum_{p=0}^{\ell}\overline{a_p}\binom{\ell}{p}p!\,b^{(\ell-p)}(x_0) = -\ell!\sum_{p=0}^{\ell}\overline{a_p}a_{\ell-p},$$

which gives (3.11). Hence (3.9) is proved. According to Lemma 3.2, (3.9) implies $h_{x_0,n} \in \mathcal{H}(b)$. Now for almost all $t \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{b(t)}\frac{k^b_{x_0,n}(t)}{n!} &= \frac{i}{2\pi}\frac{\overline{b(t)} - |b(t)|^2 \sum_{p=0}^{n} \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} \\
&= \frac{i}{2\pi}(1-|b(t)|^2)\frac{\sum_{p=0}^{n} \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} + \frac{i}{2\pi}\frac{\overline{b(t)} - \sum_{p=0}^{n} \overline{a_p}(t-x_0)^p}{(t-x_0)^{n+1}} \\
&= \rho(t)\frac{k^\rho_{x_0,n}}{n!} + \frac{i}{2\pi}\overline{h_{x_0,n}(t)}.
\end{aligned}$$

Since $h_{x_0,n} \in \mathcal{H}(b) \subset H^2(\mathbb{C}_+)$, we get that $P_+(\overline{b}k^b_{x_0,n}) = P_+(\rho k^\rho_{x_0,n})$, which can be written as $T_{\overline{b}}k^b_{x_0,n} = C_\rho k^\rho_{x_0,n}$. It follows from (2.1) and Lemma 3.2 that

$$\begin{aligned}
f^{(n)}(x_0) &= \langle f, k^b_{x_0,n}\rangle_b \\
&= \langle f, k^b_{x_0,n}\rangle_2 + \langle T_{\overline{b}}f, T_{\overline{b}}k^b_{x_0,n}\rangle_{\overline{b}} \\
&= \langle f, k^b_{x_0,n}\rangle_2 + \langle g, k^\rho_{x_0,n}\rangle_\rho \\
&= \int_{\mathbb{R}} f(t)\overline{k^b_{x_0,n}(t)}\,dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{x_0,n}(t)}\,dt,
\end{aligned}$$

which proves the relation (3.8).

□

If b is inner, then it is clear that the second integral in (3.8) is zero and we obtain the formula of Ahern–Clark (1.2).

4. A formula of combinatorics

We first recall some well-known facts concerning hypergeometric series (see [4, 21]).

The ${}_2F_1$ hypergeometric series is a power series in z defined by

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{p=0}^{+\infty} \frac{(a)_p(b)_p}{p!(c)_p} z^p, \tag{4.1}$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, and

$$(t)_p := \begin{cases} 1, & \text{if } p = 0, \\ t(t+1)\dots(t+p-1), & \text{if } p \geq 1. \end{cases}$$

We see that the hypergeometric series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0, 1, 2, \dots)$. It is clear that the radius of convergence of the ${}_2F_1$ series is equal to 1. One can show that when $\Re\mathrm{e}(\mathrm{c} - \mathrm{a} - \mathrm{b}) \leq -1$ this series is divergent on the entire unit circle, when $-1 < \Re\mathrm{e}(\mathrm{c} - \mathrm{a} - \mathrm{b}) \leq 0$ this series converges on the unit circle except for $z = 1$ and when $0 < \Re\mathrm{e}(\mathrm{c} - \mathrm{a} - \mathrm{b})$ this series is (absolutely) convergent on the entire unit circle (see [4, Theorem 2.1.2]).

We note that a power series $\sum_p \alpha_p z^p$ $(\alpha_0 = 1)$ can be written as a hypergeometric series ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right]$ if and only if

$$\frac{\alpha_{p+1}}{\alpha_p} = \frac{(p+a)(p+b)}{(p+1)(p+c)}. \tag{4.2}$$

Finally we recall two useful well-known formulas [4, page 68] for the hypergeometric series:

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right] \qquad \text{(Euler's formula)}, \tag{4.3}$$

and

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \tfrac{1}{2}\right] = 2^a {}_2F_1\left[\begin{matrix} a, c-b \\ c \end{matrix}; -1\right], \qquad \Re\mathrm{e}(b-a) > -1, \quad \text{(Pfaff's formula)}. \tag{4.4}$$

Now we state the result which we use in the last section.

Proposition 4.1. *Let $n, r \in \mathbb{N}$, $0 \leq r \leq 2n+1$ and define*

$$A_{n,r} := (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p-\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \binom{n-p+\ell}{\ell}. \tag{4.5}$$

Then

$$A_{n,r} = \begin{cases} -2^n, & 0 \leq r \leq n \\ 2^n, & n+1 \leq r \leq 2n+1. \end{cases}$$

For the proof of this result, we need the following lemma.

Lemma 4.2. *For $m \in \mathbb{N}$, we have*

$$\sum_{k=0}^{m} \binom{m}{k} (z-1)^{-k} {}_2F_1\left[\begin{matrix} a, b-k \\ c \end{matrix}; z\right] = \frac{(c-a)_m}{(c)_m} \left(\frac{z}{z-1}\right)^m {}_2F_1\left[\begin{matrix} a, b \\ c+m \end{matrix}; z\right]. \tag{4.6}$$

Proof: First note that (4.6) is equivalent to

$$\sum_{k=0}^{m} \binom{m}{k} (1-z)^{-k} (-1)^{m-k} {}_2F_1\left[\begin{matrix} a, b-k \\ c \end{matrix}; z\right] = \frac{(c-a)_m}{(c)_m} \left(\frac{z}{1-z}\right)^m {}_2F_1\left[\begin{matrix} a, b \\ c+m \end{matrix}; z\right], \tag{4.7}$$

and denote by LH the left hand side of the inequality (4.7). Applying transformation (4.3), we obtain

$$LH = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b+k \\ c \end{matrix}; z\right].$$

Now we introduce the operator of difference Δ defined by $\Delta f(x) = f(x+1) - f(x)$. Then it is well-known and easy to verify that

$$\Delta^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(x+k).$$

Using this formula, we see that $LH = (1-z)^{c-a-b} \Delta^m f(c-b)$, with

$$f(x) := {}_2F_1\left[\begin{matrix} c-a, x \\ c \end{matrix}; z\right].$$

But now we can compute $\Delta^m f(x)$. Indeed, we have

$$\begin{aligned} \Delta f(x) &= \sum_{k=0}^{+\infty} \frac{(c-a)_k}{(c)_k} ((x+1)_k - (x)_k) \frac{z^k}{k!} \\ &= \sum_{k=1}^{+\infty} \frac{(c-a)_k}{(c)_k} (x+1)_{k-1} \frac{z^k}{(k-1)!} \\ &= \frac{(c-a)}{c}\, z\, {}_2F_1\left[\begin{matrix} c-a+1, x+1 \\ c+1 \end{matrix}; z\right], \end{aligned}$$

and by induction, it follows that

$$\Delta^m f(x) = \frac{(c-a)_m}{(c)_m} z^m {}_2F_1\left[\begin{matrix} c-a+m, x+m \\ c+m \end{matrix}; z\right].$$

Therefore, we get

$$LH = (1-z)^{c-a-b}\frac{(c-a)_m}{(c)_m}z^m {}_2F_1\left[\begin{matrix} c-a+m, c-b+m \\ c+m \end{matrix}; z\right].$$

Applying once more Euler's formula, we obtain (4.7).

□

Proof of Proposition 4.1: Changing ℓ into $n-\ell$ in the second sum of (4.5), we see that

$$A_{n,r} = (-1)^{r+1}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-2)^{p+\ell-n}\binom{r}{\ell}\binom{2n+1-r}{p}\binom{2n-p-\ell}{n-\ell}. \tag{4.8}$$

Hence

$$\begin{aligned} A_{n,2n+1-r} &= (-1)^{2n+1-r+1}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-2)^{p+\ell-n}\binom{2n+1-r}{\ell}\binom{r}{p}\binom{2n-p-\ell}{n-\ell} \\ &= -(-1)^{r+1}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-2)^{p+\ell-n}\binom{2n+1-r}{\ell}\binom{r}{p}\binom{2n-p-\ell}{n-p} \\ &= -A_{n,r}. \end{aligned}$$

Therefore, it is sufficient to show $A_{n,r} = -2^n$ for $0 \le r \le n$ and then the result for $n+1 \le r \le 2n+1$ will follow immediately.

We will now assume that $0 \le r \le n$. Changing p to $n-p$ in the first sum of (4.8) and permuting the two sums, we get

$$A_{n,r} = (-1)^{r+1}\sum_{\ell=0}^{n}(-2)^{l}\binom{r}{\ell}\sum_{p=0}^{n}(-2)^{-p}\binom{2n+1-r}{n-p}\binom{n+p-\ell}{n-\ell}.$$

According to (4.1) and (4.2), we see that

$$\sum_{p=0}^{n}(-2)^{-p}\binom{2n+1-r}{n-p}\binom{n+p-\ell}{n-\ell} = \binom{2n+1-r}{n}{}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2}\right],$$

which implies

$$A_{n,r} = (-1)^{r+1}\binom{2n+1-r}{n}\sum_{\ell=0}^{n}(-2)^{\ell}\binom{r}{\ell}{}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2}\right].$$

Since $r \le n$ and since $\binom{r}{\ell} = 0$ if $r < \ell$, the sum (in the last equation) ends at $\ell = r$ and we can apply Lemma 4.2 with $m = r$, $a = -n$, $b = n+1$, $c = n+2-r$ and $z = \frac{1}{2}$. Therefore

$$\begin{aligned} A_{n,r} =& (-1)^{r+1}\binom{2n+1-r}{n}\frac{(2n+2-r)_r}{(n+2-r)_r}(-1)^r {}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \frac{1}{2}\right] \\ =& -\frac{(2n+1)!}{(n+1)!n!}\ {}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \frac{1}{2}\right]. \end{aligned}$$

We now use formula (4.4) which gives

$$\begin{aligned} {}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \frac{1}{2}\right] &= 2^{-n} {}_2F_1\left[\begin{matrix}-n, 1\\ n+2\end{matrix}; -1\right] \\ &= 2^{-n}\sum_{i=0}^{\infty}\frac{(-n)_i(1)_i}{(n+2)_i}\frac{(-1)^i}{i!} \\ &= 2^{-n}\sum_{i=0}^{n}\binom{n}{i}\frac{i!(n+1)!}{(n+i+1)!}, \end{aligned}$$

where we have used $(1)_i = i!$, $\dfrac{(-n)_i(-1)^i}{i!} = \dbinom{n}{i}$ and $(n+2)_i = \dfrac{(n+i+1)!}{(n+1)!}$. Hence

$${}_2F_1\left[\begin{matrix}-n, n+1\\ n+2\end{matrix}; \frac{1}{2}\right] = 2^{-n}(n+1)!n!\sum_{i=0}^{n}\frac{1}{(n+i+1)!(n-i)!}$$

which implies

$$A_{n,r} = -2^{-n}\sum_{i=0}^{n}\binom{2n+1}{i}.$$

But

$$\sum_{i=0}^{n}\binom{2n+1}{i} = \frac{1}{2}\sum_{i=0}^{2n+1}\binom{2n+1}{i} = 2^{2n}$$

which ends the proof.

□

5. Norm convergence for the reproducing kernels.

In Section 3, we saw that if $x_0 \in \mathbb{R}$ satisfies (2.2), then $k^b_{\omega,n}$ tends *weakly* to $k^b_{x_0,n}$ in $\mathcal{H}(b)$ as ω approaches radially to x_0. It is natural to ask if this weak convergence can be replaced by norm convergence. In other words, is it true that $\|k^b_{\omega,n} - k^b_{x_0,n}\|_b \to 0$ as ω tends radially to x_0?

In [1], Ahern and Clark said that they can prove this result for the case where b is inner and $n = 0$. For general functions b in the unit ball of H^∞, Sarason [18, Chap. V] got this norm convergence for the case $n = 0$. In this section, we prove the general case.

Since we already have weak convergence, to prove the norm convergence, it is sufficient to prove that $\|k^b_{\omega,n}\|_b \to \|k^b_{x_0,n}\|_b$ as ω tends radially to x_0. Therefore we need to compute $\|k^b_{x_0,n}\|_b$. For $n = 0$, in the context of the unit disc, Sarason [18, Chap. V] proved that $\|k^b_{z_0}\|_b^2 = z_0\overline{b(z_0)}b'(z_0)$, $z_0 \in \mathbb{T}$. We can give an analogue of this formula showing that the norm of $k^b_{x_0,n}$ can be expressed in terms of the derivatives of b at x_0.

Proposition 5.1. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$ satisfying the condition (2.2). Then*

$$\|k^b_{x_0,n}\|_b^2 = \frac{n!^2}{2i\pi}\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!}\frac{b^{(2n+1-p)}(x_0)}{(2n+1-p)!}.$$

Proof: Following the notations of Section 3, we define

$$\varphi(z) = 1 - b(z)\sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!}(z-x_0)^p.$$

Then, by (2.3) and Lemma 2.3, as z tends radially to x_0, we have

$$k^b_{x_0,n}(z) = \frac{in!}{2\pi}(z-x_0)^{-n-1}\left(\sum_{p=0}^{2n+1} \frac{\varphi^{(p)}(x_0)}{p!}(z-x_0)^p + o((z-x_0)^{2n+1})\right).$$

As we have shown in the proof of Theorem 3.3, $\varphi^{(k)}(x_0) = 0$ if $0 \le k \le n$. Hence

$$k^b_{x_0,n}(z) = \frac{in!}{2\pi}\sum_{p=0}^{n} \frac{\varphi^{(p+n+1)}(x_0)}{(p+n+1)!}(z-x_0)^p + o((z-x_0)^n).$$

Using once more Lemma 2.3, we can also write

$$k^b_{x_0,n}(z)=\sum_{p=0}^{n}\frac{(k^b_{x_0,n})^{(p)}(x_0)}{p!}(z-x_0)^p+\,o((z-x_0)^n),$$

which implies

$$(k^b_{x_0,n})^{(p)}(x_0)=\frac{in!}{2\pi}\frac{p!}{(p+1+n)!}\varphi^{(p+n+1)}(x_0).$$

But, according to Lemma 3.2, we have $\|k^b_{x_0,n}\|_b^2=(k^b_{x_0,n})^{(n)}(x_0)$ and we get

$$\|k^b_{x_0,n}\|_b^2=\frac{in!^2}{2\pi}\frac{\varphi^{(2n+1)}(x_0)}{(2n+1)!}.$$

Finally, the result follows by Leibniz' rule.

□

The next result provides an affirmative answer to the question of norm convergence.

Theorem 5.2. *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0\in\mathbb{R}$ satisfying the condition (2.2). Then*

$$\left\|k^b_{\omega,n}-k^b_{x_0,n}\right\|_b\longrightarrow 0,\quad \textit{as } \omega \textit{ tends radially to } x_0.$$

Proof: We denote by $a_p(\omega):=\frac{b^{(p)}(\omega)}{p!}$ and $a_p:=a_p(x_0)$. We recall that

$$k^b_{\omega,n}(z)=\frac{in!}{2\pi}\left(\frac{1}{(z-\overline{\omega})^{n+1}}-\sum_{p=0}^{n}\overline{a_p(\omega)}(z-\overline{\omega})^{p-n-1}b(z)\right).$$

We have

$$\frac{\partial^n}{\partial z^n}\left(\frac{1}{(z-\overline{\omega})^{n+1}}\right)=(-1)^n\frac{(2n)!}{n!}\frac{1}{(z-\overline{\omega})^{2n+1}},$$

and by Leibniz' rule

$$\frac{\partial^n}{\partial z^n}\left((z-\overline{\omega})^{p-n-1}b(z)\right)=\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell\frac{(n-p+\ell)!}{(n-p)!}(z-\overline{\omega})^{p-n-\ell-1}b^{(n-\ell)}(z).$$

According to Proposition 3.4, we have $\|k^b_{\omega,n}\|_b^2=(k^b_{\omega,n})^{(n)}(\omega)$, which implies

(5.1)

$$\|k^b_{\omega,n}\|_b^2=\frac{in!}{2\pi}\frac{(-1)^n\frac{(2n)!}{n!}-\sum_{p=0}^{n}\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^\ell\frac{(n-p-\ell)!}{(n-p)!}(\omega-\overline{\omega})^{n+p-\ell}\overline{a_p(\omega)}b^{(n-\ell)}(\omega)}{(\omega-\overline{\omega})^{2n+1}}.$$

For $0 \leq s \leq n$, the function $b^{(s)}$ is analytic in the upper-half plane and its derivative of order $2n+1-s$, which coincides with $b^{(2n+1)}$, has a radial limit at x_0. According to Lemma 2.3, as ω tends radially to x_0, we have

$$b^{(s)}(\omega) = \sum_{r=s}^{2n+1} a_r \frac{r!}{(r-s)!}(\omega - x_0)^{r-s} + o((\omega - x_0)^{2n+1-s}).$$

Hence if we put $\omega = x_0 + it$, we get

$$(\omega - \overline{\omega})^s b^{(s)}(\omega) = 2^s \sum_{r=s}^{2n+1} a_r \frac{r!}{(r-s)!} i^r t^r + o(t^{2n+1}),$$

and thus

$$(\omega - \overline{\omega})^{n+p-\ell}\overline{a_p(\omega)} b^{(n-\ell)}(\omega) = \frac{(-1)^p}{p!} 2^{n+p-\ell} \left(\sum_{r=n-\ell}^{2n+1} a_r \frac{r!}{(r-n+\ell)!} i^r t^r + o(t^{2n+1}) \right) \times \left(\sum_{j=p}^{2n+1} \overline{a_j} \frac{j!}{(j-p)!} (-i)^j t^j + o(t^{2n+1}) \right).$$

We deduce from (5.1) that

$$\|k^b_{\omega,n}\|_b^2 = \frac{(-1)^n n!}{2^{2n+2}\pi} t^{-2n-1} \Bigg[(-1)^n \frac{(2n)!}{n!} - n! \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \times \left(\sum_{r=n-\ell}^{2n+1} a_r \binom{r}{n-\ell} i^r t^r \right) \left(\sum_{j=p}^{2n+1} \overline{a_j} \binom{j}{p} (-i)^j t^j \right) + o(t^{2n+1}) \Bigg],$$

and denoting by $c_n = \dfrac{(-1)^n n!}{2^{2n+2}\pi}$, we can write

$$\|k^b_{\omega,n}\|_b^2 = c_n t^{-2n-1} \left[(-1)^n \frac{(2n)!}{n!} - n! \sum_{s=0}^{2n+1} \lambda_{s,n} t^s + o(t^{2n+1}) \right],$$

with

$$\lambda_{s,n} := i^s \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \sum_{r=0}^{s} \binom{r}{n-\ell} \binom{s-r}{p} (-1)^{s-r} a_r \overline{a_{s-r}},$$

where we assumed that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

Now we recall that $k^b_{\omega,n}$ is weakly convergent as ω tends radially to x_0 and thus $\|k^b_{\omega,n}\|_b$ remains bounded. Therefore we necessarily have

$$(5.2)\qquad (-1)^n\frac{(2n)!}{n!}-n!\lambda_{0,n}=0 \qquad \text{and} \qquad \lambda_{s,n}=0, \quad (1\le s\le 2n),$$

which implies that

$$(5.3)\qquad \|k^b_{\omega,n}\|_b^2=-n!c_n\lambda_{2n+1,n}+o(1),$$

as ω tends radially to x_0. But

$$\begin{aligned}\lambda_{2n+1,n}&=(-1)^n i\sum_{r=0}^{2n+1}(-1)^{r+1}a_r\overline{a_{2n+1-r}}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-1)^{p+\ell}2^{n+p-\ell}\binom{n-p+\ell}{\ell}\binom{r}{n-\ell}\binom{2n+1-r}{p}\\ &=(-1)^n i2^n\sum_{r=0}^{2n+1}A_{n,r}a_r\overline{a_{2n+1-r}}.\end{aligned}$$

According to Proposition 4.1, we have $A_{n,r}=-2^n$ if $0\le r\le n$ and $A_{n,r}=2^n$ if $n+1\le r\le 2n+1$. Then we obtain

$$\lambda_{2n+1,n}=(-1)^n i2^{2n}\left(\sum_{r=n+1}^{2n+1}a_r\overline{a_{2n+1-r}}-\sum_{r=0}^{n}a_r\overline{a_{2n+1-r}}\right).$$

Now note that

$$\sum_{r=n+1}^{2n+1}a_r\overline{a_{2n+1-r}}=\sum_{r=0}^{n}\overline{a_r}a_{2n+1-r},$$

which means

$$\lambda_{2n+1,n}=(-1)^{n+1}2^{2n+1}\Im\mathrm{m}\left(\sum_{r=0}^{n}\overline{a_r}a_{2n+1-r}\right).$$

But Proposition 5.1 implies that

$$\lambda_{2n+1,n}=(-1)^{n+1}2^{2n+2}\frac{\pi}{n!^2}\|k^b_{x_0,n}\|_b^2,$$

and finally using (5.3) and the definition of c_n, we obtain

$$\|k^b_{\omega,n}\|_b^2=\|k^b_{x_0,n}\|_b^2+o(1),$$

which proves that $\|k^b_{\omega,n}\|_b\longrightarrow\|k^b_{x_0,n}\|_b$ as ω tends radially to x_0. Since $k^b_{\omega,n}$ tends also weakly to $k^b_{x_0,n}$ in $\mathcal{H}(b)$ as ω tends radially to x_0, we get the desired conclusion.

□

Remark 5.3. We have already seen in the proof of Theorem 3.3 that if x_0 satisfies the condition (2.2), then $|a_0| = 1$ and

$$\sum_{p=0}^{k} a_p \overline{a_{k-p}} = 0, \qquad (1 \le k \le n), \tag{5.4}$$

where $a_p := \frac{b^{(p)}(x_0)}{p!}$. In fact, we can prove that the relation (5.4) is also valid for $n+1 \le k \le 2n$, k even. Indeed, according to (5.2), for $n+1 \le s \le 2n$, we have

$$\begin{aligned} 0 = \lambda_{s,n} :=& i^s \sum_{p=0}^{n}\sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \sum_{r=0}^{s} \binom{r}{n-\ell}\binom{s-r}{p} (-1)^{s-r} a_r \overline{a_{s-r}} \\ =& (-i)^s 2^n \sum_{r=0}^{s} (-1)^r a_r \overline{a_{s-r}} \sum_{p=0}^{n}\sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{p-\ell} \binom{n-p+\ell}{\ell}\binom{r}{n-\ell}\binom{s-r}{p} \\ =& (-i)^s 2^n \sum_{r=0}^{s} a_r \overline{a_{s-r}} A_{n,r,s}, \end{aligned} \tag{5.5}$$

with

$$A_{n,r,s} := (-1)^r \sum_{p=0}^{n}\sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{p-\ell} \binom{n-p+\ell}{\ell}\binom{r}{n-\ell}\binom{s-r}{p}. \tag{5.6}$$

Using similar arguments as in the proof of Proposition 4.1, we show that for every $0 \le r \le n < s$, we have

$$A_{n,r,s} = \binom{s}{n} \frac{\Gamma\left(\frac{s-n+1}{2}\right)\Gamma\left(\frac{s-n+2}{2}\right)}{\Gamma\left(\frac{s-2n+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right)}. \tag{5.7}$$

In the proof of this identity, we use Bayley's Theorem [4] which says that

$${}_2F_1\left[\begin{matrix} a, 1-a \\ b \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{1-a+b}{2}\right)}.$$

Now using (5.6) it is easy to see that $A_{n,s-r,s} = (-1)^s A_{n,r,s}$, and with (5.5) and (5.7), we obtain

$$\binom{s}{n} \frac{\Gamma\left(\frac{s-n+1}{2}\right)\Gamma\left(\frac{s-n+2}{2}\right)}{\Gamma\left(\frac{s-2n+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right)} \left(\sum_{r=0}^{n} a_r \overline{a_{s-r}} + (-1)^s \sum_{r=n+1}^{s} a_r \overline{a_{s-r}} \right) = 0.$$

Now recall that the Gamma function is a meromorphic function in the complex plane without zeros and with poles at zero and the negative integers . Therefore we see that if

s is even $(n+1 \le s \le 2n)$, then

$$\sum_{r=0}^{s} a_r \overline{a_{s-r}} = 0.$$

But if s is odd $(n+1 \le s \le 2n)$, then $\frac{s-2n+1}{2}$ is zero or a negative integer and, using this argument, we are not able to conclude that $\sum_{r=0}^{s} a_r \overline{a_{s-r}} = 0$. This still remains as an open question.

Acknowledgments: We would like to thank F. Jouhet, E. Mosaki and J. Zeng for helpful discussions concerning the proof of Proposition 4.1 and A. Baranov for some suggestions to improve the redaction of this text. A part of this work was done while the first author was visiting McGill University. He would like to thank this institution for its warm hospitality.

References

[1] Ahern, P. R., and Clark, D. N. Radial limits and invariant subspaces. *Amer. J. Math. 92* (1970), 332–342.

[2] Ahern, P. R., and Clark, D. N. Radial nth derivatives of Blaschke products. *Math. Scand. 28* (1971), 189–201.

[3] Anderson, J. M., and Rovnyak, J. On generalized Schwarz-Pick estimates. *Mathematika 53*, 1 (2006), 161–168 (2007).

[4] Andrews, G. E., Askey, R., and Roy, R. *Special functions*, vol. 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[5] Baranov, A. D. Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings. *J. Funct. Anal. 223*, 1 (2005), 116–146.

[6] Bolotnikov, V., and Kheifets, A. A higher order analogue of the Carathéodory-Julia theorem. *J. Funct. Anal. 237*, 1 (2006), 350–371.

[7] de Branges, L., and Rovnyak, J. Canonical models in quantum scattering theory. In *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965).* Wiley, New York, 1966, pp. 295–392.

[8] de Branges, L., and Rovnyak, J. *Square summable power series*. Holt, Rinehart and Winston, New York, 1966.

[9] Duren, P. L. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York, 1970.

[10] Dyakonov, K. M. Entire functions of exponential type and model subspaces in H^p. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 190*, Issled. po Linein. Oper. i Teor. Funktsii. 19 (1991), 81–100, 186.

[11] Dyakonov, K. M. Differentiation in star-invariant subspaces. I. Boundedness and compactness. *J. Funct. Anal. 192*, 2 (2002), 364–386.

[12] Fatou, P. Séries trigonométriques et séries de Taylor. *Acta Math. 30*, 1 (1906), 335–400.

[13] Fricain, E. Bases of reproducing kernels in de Branges spaces. *J. Funct. Anal. 226*, 2 (2005), 373–405.

[14] Fricain, E., and Mashreghi, J. Boundary behavior of functions of the de branges-rovnyak spaces. *Complex Analysis and Operator Theory* (to appear).

[15] Hartmann, A., Sarason, D., and Seip, K. Surjective Toeplitz operators. *Acta Sci. Math. (Szeged) 70*, 3-4 (2004), 609–621.

[16] Helson, H. *Lectures on invariant subspaces*. Academic Press, New York, 1964.

[17] Jury, M. T. Reproducing kernels, de Branges-Rovnyak spaces, and norms of weighted composition operators. *Proc. Amer. Math. Soc. 135*, 11 (2007), 3669–3675 (electronic).

[18] Sarason, D. *Sub-Hardy Hilbert spaces in the unit disk*. University of Arkansas Lecture Notes in the Mathematical Sciences, 10. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

[19] Shapiro, J. E. Relative angular derivatives. *J. Operator Theory 46*, 2 (2001), 265–280.

[20] Shapiro, J. E. More relative angular derivatives. *J. Operator Theory 49*, 1 (2003), 85–97.

[21] Slater, L. J. *Generalized hypergeometric functions*. Cambridge University Press, Cambridge, 1966.

Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne

E-mail address: `fricain@math.univ-lyon1.fr`

Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4.

E-mail address: `Javad.Mashreghi@mat.ulaval.ca`

6.2.3 Référence [T7]

Auteurs
A. Baranov, E. Fricain et J. Mashreghi

Titre
Weighted norm inequalities for de Branges-Rovnyak spaces and their applications.

Soumis

WEIGHTED NORM INEQUALITIES FOR DE BRANGES–ROVNYAK SPACES AND THEIR APPLICATIONS

ANTON BARANOV, EMMANUEL FRICAIN, JAVAD MASHREGHI

Abstract. Let $\mathcal{H}(b)$ denote the de Branges–Rovnyak space associated with a function b in the unit ball of $H^\infty(\mathbb{C}_+)$. We study the boundary behavior of the derivatives of functions in $\mathcal{H}(b)$ and obtain weighted norm estimates of the form $\|f^{(n)}\|_{L^2(\mu)} \le C\|f\|_{\mathcal{H}(b)}$, where $f \in \mathcal{H}(b)$ and μ is a Carleson-type measure on $\mathbb{C}_+ \cup \mathbb{R}$. We provide several applications of these inequalities. We apply them to obtain embedding theorems for $\mathcal{H}(b)$ spaces. These results extend Cohn and Volberg–Treil embedding theorems for the model (star-invariant) subspaces which are special classes of de Branges–Rovnyak spaces. We also exploit the inequalities for the derivatives to study stability of Riesz bases of reproducing kernels $\{k^b_{\lambda_n}\}$ in $\mathcal{H}(b)$ under small perturbations of the points λ_n.

1. Introduction

Let $\mathbb{C}_+$ denote the upper half-plane in the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space on $\mathbb{C}_+$. For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi f := P_+(\varphi f), \qquad f \in H^2(\mathbb{C}_+),$$

where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $\varphi \in L^\infty(\mathbb{R})$, $\|\varphi\|_\infty \le 1$, the de Branges–Rovnyak space $\mathcal{H}(\varphi)$ associated to φ consists of those functions in $H^2(\mathbb{C}_+)$ which are in the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space

2000 *Mathematics Subject Classification.* Primary: 46E15, 46E22, Secondary: 30D55, 47A15.

Key words and phrases. Bernstein's inequality, de Branges–Rovnyak space, model subspace, reproducing kernel, embedding theorem, Riesz basis.

This work was supported by funds from NSERC (Canada), Jacques Cartier Center (France) and RFBR (Russia).

when equipped with the inner product

$$\langle\, (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f,\, (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g\,\rangle_\varphi = \langle f, g\rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker\,(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. In what follows we always assume that $\varphi = b$ is an analytic function in the unit ball of $H^\infty(\mathbb{C}_+)$. In this case, if

$$k^b_\omega(z) := \frac{1 - \overline{b(\omega)}b(z)}{z - \overline{\omega}}, \qquad \omega \in \mathbb{C}_+, \tag{1.1}$$

then we have $\langle f, k^b_\omega\rangle_b = 2\pi i f(\omega)$ for all $f \in \mathcal{H}(b)$. In other words, $\mathcal{H}(b)$ is a reproducing kernel Hilbert space.

These spaces (and, more precisely, their general vector-valued version) were introduced by de Branges and Rovnyak [14, 15] as universal model spaces for Hilbert space contractions. Thanks to the pioneer works of Sarason, we know that de Branges–Rovnyak spaces play an important role in numerous questions of complex analysis and operator theory (e.g. see [4, 23, 37, 38, 39]). For the general theory of $\mathcal{H}(b)$ spaces we refer to [37].

In the special case where $b = \Theta$ is an inner function (that is, $|\Theta| = 1$ a.e. on $\mathbb{R}$), the operator $(Id - T_\Theta T_{\overline{\Theta}})^{1/2}$ is an orthogonal projection and $\mathcal{H}(\Theta)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{C}_+)$ which coincides with the so-called model subspace

$$K^2_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+) = H^2(\mathbb{C}_+) \cap \Theta\, \overline{H^2(\mathbb{C}_+)}$$

(for the model space theory see [29]). We mention one important particular class of model spaces. If $\Theta(z) = \exp(iaz)$, $a > 0$, then $\mathcal{H}(\Theta) = K^2_\Theta = H^2(\mathbb{C}_+) \cap PW^2_a$, where PW^2_a stands for the Paley–Wiener space of all entire functions of exponential type at most a, whose restrictions to $\mathbb{R}$ belong to $L^2(\mathbb{R})$. Then the famous Bernstein's inequality asserts that

$$\|f'\|_2 \le a\|f\|_2, \qquad f \in PW^2_a.$$

This classical and important inequality was extended by many authors in many different directions. It is impossible to give an exhaustive list of references, but we would like to mention [9, 22, 32, 34, 35, 40] and [26, Lecture 28].

Notably, one natural direction is to extend Bernstein's inequality to general model subspaces. In [27], Levin showed that if Θ is an inner function and $|\Theta'(x)| < \infty$, $x \in \mathbb{R}$, then for each function $f \in K_\Theta^\infty = H^\infty(\mathbb{C}_+) \cap \Theta\overline{H^\infty(\mathbb{C}_+)}$, the derivative $f'(x)$ exists in the sense of nontangential boundary values and

$$|f'(x)/\Theta'(x)| \le \|f\|_\infty.$$

Differentiation in the model spaces $K_\Theta^p := H^p(\mathbb{C}_+) \cap \Theta\overline{H^p(\mathbb{C}_+)}$, $1 < p < \infty$, was studied extensively by Dyakonov [16, 17], who showed that the Bernstein-type inequality $\|f'\|_p \le C\|f\|_p$, $f \in K_\Theta^p$, holds if and only if $\Theta' \in L^\infty(\mathbb{R})$. Recently, Baranov [5, 6, 8] has obtained weighted Bernstein-type inequalities for the model subspaces K_Θ^p, which generalized previous results of Levin and Dyakonov. More precisely, for a general inner function Θ, he proved estimates of the form

$$\|f^{(n)} w_{p,n}\|_{L^p(\mu)} \le C\,\|f\|_p, \qquad f \in K_\Theta^p, \tag{1.2}$$

where $n \ge 1$, μ is a Carleson measure in the closed upper half-plane and $w_{p,n}$ is some weight related to the norm of reproducing kernels of the space K_Θ^2 which compensates possible growth of the derivative near the boundary.

One of the main ingredients in the results of Dyakonov and Baranov was an integral formula for the derivatives of functions in K_Θ^p. Using Cauchy formula, it is easy to see that if Θ is inner, $\omega \in \mathbb{C}_+$, n is a non-negative integer and $f \in K_\Theta^p$, then we have

$$f^{(n)}(\omega) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(t)\,\overline{k_{\omega,n}^\Theta(t)}\,dt, \tag{1.3}$$

where

$$\frac{k_{\omega,n}^\Theta(z)}{n!} := \frac{1 - \Theta(z)\displaystyle\sum_{p=0}^{n} \frac{\overline{\Theta^{(p)}(\omega)}}{p!}(z-\overline{\omega})^p}{(z-\overline{\omega})^{n+1}}, \qquad z \in \mathbb{C}_+. \tag{1.4}$$

A natural question is whether one can extend the formula (1.3) to boundary points x_0. If $x_0 \in \mathbb{R}$ does not belong to the boundary spectrum $\sigma(\Theta)$ of Θ (see the definition in Section

5), then Θ and all functions of K^p_Θ are analytic through a neighborhood of x_0 and then it is obvious that (1.3) is valid for $z = x_0$. More generally, if x_0 satisfies

$$\sum_k \frac{\operatorname{Im} z_k}{|x_0 - z_k|^{(n+1)q}} + \int_{\mathbb{R}} \frac{d\mu(t)}{|t - x_0|^{(n+1)q}} < +\infty, \tag{1.5}$$

then, by the results of Ahern and Clark [1] (for $p = 2$) and Cohn [12] (for $p > 1$), the formula (1.3) is still valid at the point $x_0 \in \mathbb{R}$ for any $f \in K^p_\Theta$ (here $\{z_k\}$ is the sequence of zeros of Θ and μ is the singular measure associated to Θ). Recently Fricain and Mashreghi studied the boundary behavior of functions in de Branges–Rovnyak spaces $\mathcal{H}(b)$ and obtained a generalization of representation (1.3) [20, 21].

In the present paper de Branges–Rovnyak spaces are studied from the point of view of function theory. Namely, we are interested in boundary properties of the elements of $\mathcal{H}(b)$ and of their derivatives, and we establish a number of weighted Bernstein-type inequalities. Our first goal is to exploit the generalization of representation (1.3) and obtain an analogue of Bernstein-type inequality (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an *arbitrary function* in the unit ball of $H^\infty(\mathbb{C}_+)$ (not necessarily inner). It should be noted that the inner product in $\mathcal{H}(b)$ is not given by a usual integral formula. This fact causes certain difficulties. For example, we will see that one has to add one more term to formula (1.3) in the general case. In what follows we try to emphasize the points where there is a difference with the inner case, and suggest a few open questions.

Our second goal is to provide several applications of these Bernstein-type inequalities. The classical Carleson embedding theorem gives a simple geometrical condition on a measure μ in the closed upper half-plane such that the embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ holds. A similar question for model subspaces K^p_Θ was studied by Cohn [11] and then by Volberg and Treil [42]. An approach based on the (weighted norm) Bernstein inequalities for model subspaces K^p_Θ was suggested in [6]. Given b in the unit ball of $H^\infty(\mathbb{C}_+)$, we describe a class of Borel measures μ in $\mathbb{C}_+ \cup \mathbb{R}$ such that $\mathcal{H}(b) \subset L^2(\mu)$. We obtain a geometric condition on μ sufficient for such embedding. This result generalizes the previous results of Cohn and Volberg–Treil.

Another application concerns the problem of stability of Riesz bases consisting of reproducing kernels of $\mathcal{H}(b)$. This problem is connected with the famous problem of bases of exponentials in L^2 on an interval which goes back to Paley and Wiener [30]. Exponential bases were described by Pavlov [31] and by Hruschev, Nikolski and Pavlov in [24], where functional model methods have been used. This approach has been proved fruitful; it has allowed both to recapture all the classical results and to extend them to general model spaces (for a detailed presentation of the subject see [29]). Fricain has pursued this investigation with respect to bases of reproducing kernels in vector-valued model spaces [18] and in de Branges–Rovnyak spaces [19] where some criteria for a family of reproducing kernels to be a Riesz basis were obtained. However, the criteria mentioned above involve some properties of a given family of reproducing kernel that are rather difficult to verify. On the other hand, in many cases, the given family is a slight perturbation of another family of reproducing kernels that is known to be a basis. This gives rise to the following stability problem: *Given a Riesz basis of reproducing kernels* $(k^b_{\lambda_n})_{n\geq 1}$ *of* $\mathcal{H}(b)$*, characterize perturbations of frequencies* $(\lambda_n)_{n\geq 1}$ *which preserve the property to be a Riesz basis.*

This problem was also studied by many authors in the context of exponential bases (see e.g. [25, 36]) and of model subspaces K^2_Θ [7, 13, 18]. In the present paper, using the weighted norm inequalities (1.2) we extend the results about stability in pseudohyperbolic metrics from [7, 18] to de Branges-Rovnyak spaces.

The paper is organized as follows. Sections 2 and 3 contain some preliminaries concerning integral representations for the n-th derivative of functions in de Branges–Rovnyak spaces. In Section 4 we prove our first main result, a Bernstein-type inequality for $\mathcal{H}(b)$. Section 5 contains some estimates relating the weight $w_{p,n}$ involved in Bernstein inequalities to the distances to the level sets of $|b|$. Section 6 is devoted to embedding theorems.

Finally, in Section 7 we apply the Bernstein inequality to the problem of stability of Riesz basis of reproducing kernels in $\mathcal{H}(b)$.

In what follows, the letter C will denote a positive constant and we assume that its value may change. We write $f \asymp g$ if $C_1 g \le f \le C_2 g$ for some positive constants C_1, C_2. The set of integers $1, 2, \cdots$ will be denoted by $\mathbb{N}$.

2. Preliminaries

Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$ and let $b = BI_\mu O_b$ be its canonical factorization, where

$$B(z) = \prod_r e^{i\alpha_r} \frac{z - z_r}{z - \overline{z_r}}$$

is a Blaschke product, the singular inner function I_μ is given by

$$I_\mu(z) = \exp\left(iaz - \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) d\mu(t) \right)$$

with a positive singular measure μ and $a \ge 0$, and O_b is the outer function

$$O_b(z) = \exp\left(\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) \log|b(t)|\, dt \right).$$

Then the modulus of the angular derivative of b at a point $x \in \mathbb{R}$ is given by

$$|b'(x)| = a + \sum_r \frac{2\,\mathrm{Im}\, z_r}{|x - z_r|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(t)}{|x-t|^2} + \frac{1}{\pi} \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x-t|^2}\, dt. \tag{2.1}$$

Hence, we are motivated to define

$$S_n(x) := \sum_{r=1}^{+\infty} \frac{\mathrm{Im}\, z_r}{|x - z_r|^n} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x-t|^n} + \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|x-t|^n}\, dt, \tag{2.2}$$

and

$$E_n(b) := \{x \in \mathbb{R} : S_n(x) < +\infty\}.$$

The formula (2.1) explains why the quantity S_2 is of special interest.

We will need the following simple estimate.

Lemma 2.1. *For any $x \in \mathbb{R}$, $y > 0$, we have $|b'(x+iy)| \le |b'(x)|$.*

Proof. Let $z = x + iy$, $y > 0$, and assume that b is outer,

$$b(z) = \exp\left(\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)\log|b(t)|\,dt\right).$$

Then

$$b'(z) = -b(z)\frac{i}{\pi}\int_{\mathbb{R}}\frac{\log|b(t)|}{(t-z)^2}\,dt,$$

and clearly

$$|b'(z)| \leq \frac{1}{\pi}\int_{\mathbb{R}}\frac{|\log|b(t)||}{|t-z|^2}\,dt \leq \frac{1}{\pi}\int_{\mathbb{R}}\frac{|\log|b(t)||}{|t-x|^2}\,dt = |b'(x)|,$$

by (2.1). The estimates for inner factors are analogous and left to the reader (recall that $|b'(x)| = |O_b'(x)| + |I_\mu'(x)| + |B'(x)|$, $x \in \mathbb{R}$).

□

Ahern and Clark [2] showed that if $x_0 \in E_n(b)$, then b and all its derivatives up to order $n-1$ have (finite) nontangential limits at x_0. In [20], we showed that if $x_0 \in E_{2n+2}(b)$ where $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, then, for each $f \in \mathcal{H}(b)$ and for each $0 \leq j \leq n$, the nontangential limit

$$f^{(j)}(x_0) := \lim_{\substack{z \longrightarrow x_0 \\ \sphericalangle}} f^{(j)}(z)$$

exists. This is a generalization of the Ahern–Clark theorem [1] for the elements of model subspaces K^2_Θ, i.e. for the case when $b = \Theta$ is an inner function. Moreover, for every $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$ and for every function $f \in \mathcal{H}(b)$, we obtained in [21] the following integral representation for $f^{(n)}(z_0)$. Let $\rho(t) = 1 - |b(t)|^2$ and let $H^2(\rho)$ be the span of the Cauchy kernels k_z, $z \in \mathbb{C}_+$, in $L^2(\rho)$ (recall that $k_z(\omega) = (\omega - \overline{z})^{-1}$). Consider the operator

$$\begin{array}{rcl} \widetilde{T}_\rho : L^2(\rho) & \longrightarrow & H^2(\mathbb{C}_+) \\ q & \longmapsto & P_+(q\rho). \end{array}$$

We know from [37, II-3, III-2] that if $f \in \mathcal{H}(b)$ then there exists a (unique) function g in $H^2(\rho)$ such that $T_{\overline{b}}f = \widetilde{T}_\rho g$. It was shown in [21] that, for $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, $n \in \mathbb{Z}_+$,

we have

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\left(\int_{\mathbb{R}} f(t)\overline{k^b_{z_0,n}(t)}\,dt+\int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{z_0,n}(t)}\,dt\right), \tag{2.3}$$

where $k^b_{z_0,n}$ is the function in $\mathcal{H}(b)$ defined by

$$k^b_{z_0,n}(z):=\frac{1-b(z)\displaystyle\sum_{j=0}^n\frac{\overline{b^{(j)}(z_0)}}{j!}(z-\overline{z_0})^j}{(z-\overline{z_0})^{n+1}},\qquad z\in\mathbb{C}_+, \tag{2.4}$$

and $k^\rho_{z_0,n}$ is the function in $L^2(\rho)$ defined by

$$k^\rho_{z_0,n}(t):=\frac{\displaystyle\sum_{j=0}^n\frac{\overline{b^{(j)}(z_0)}}{j!}(t-\overline{z_0})^j}{(t-\overline{z_0})^{n+1}},\qquad t\in\mathbb{R}. \tag{2.5}$$

Note that if b is inner, then $\rho\equiv 0$ and thus (2.3) reduces to (1.3) which was the key representation formula used in [5, 6, 16, 17] to obtain Bernstein-type inequalities for model subspaces K^p_Θ. If $n=0$ then $k^b_{z_0,0}$ corresponds to the reproducing kernel of $\mathcal{H}(b)$ defined in (1.1).

3. A new representation formula for the derivatives

We start with a slight modification of the representation (2.3) for $n\in\mathbb{N}$.

Proposition 3.1. *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$. Let $z_0\in\mathbb{C}_+\cup E_{2n+2}(b)$, $n\in\mathbb{N}$, and let*

$$\mathfrak{K}^\rho_{z_0,n}(t):=\overline{b(z_0)}\,\frac{\sum_{j=0}^n\binom{n+1}{j+1}(-1)^j\,\overline{b^j(z_0)}\,b^j(t)}{(t-\overline{z_0})^{n+1}},\qquad t\in\mathbb{R}. \tag{3.1}$$

Then $\left(k^b_{z_0}\right)^{n+1}\in H^2(\mathbb{C}_+)$ and $\mathfrak{K}^\rho_{z_0,n}\in L^2(\rho)$. Moreover, for every function $f\in\mathcal{H}(b)$, we have

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\left(\int_{\mathbb{R}} f(t)\overline{(k^b_{z_0})^{n+1}(t)}\,dt+\int_{\mathbb{R}} g(t)\rho(t)\overline{\mathfrak{K}^\rho_{z_0,n}(t)}\,dt\right), \tag{3.2}$$

where $g\in H^2(\rho)$ is such that $T_{\overline{b}}f=\widetilde{T}_\rho g$.

Proof. Let $a_j = b^{(j)}(z_0)/j!$. Then

$$\begin{aligned} k^b_{z_0,\ell}(z) &= \frac{1-\overline{b(z_0)}b(z) - b(z)\displaystyle\sum_{j=1}^{\ell}\overline{a_j}(z-\overline{z_0})^j}{(z-\overline{z_0})^{\ell+1}} \\ &= \frac{1-\overline{b(z_0)}b(z)}{(z-\overline{z_0})^{\ell+1}} - b(z)\sum_{j=1}^{\ell}\frac{\overline{a_j}}{(z-\overline{z_0})^{\ell+1-j}}. \end{aligned}$$

Hence, multiplying by $(1-\overline{b(z_0)}b(z))^\ell$, we obtain

$$(k^b_{z_0})^{\ell+1}(z) = (1-\overline{b(z_0)}b(z))^\ell\, k^b_{z_0,\ell}(z) + b(z)\sum_{j=1}^{\ell}\overline{a_j}(1-\overline{b(z_0)}b(z))^{j-1}(k^b_{z_0})^{\ell+1-j}(z). \tag{3.3}$$

Since $z_0 \in \mathbb{C}_+ \cup E_{2n+2}(b)$, according to [21, Proposition 3.1 and Lemma 3.2], the functions $k^b_{z_0}$ and $k^b_{z_0,\ell}$ $(1 \le \ell \le n)$ belong to $\mathcal{H}(b)$. Hence, using the recurrence relation (3.3) and that $1-\overline{b(z_0)}b(z) \in H^\infty(\mathbb{C}_+)$, we see immediately by induction that $(k^b_{z_0})^{n+1} \in H^2(\mathbb{C}_+)$.

We prove now that $\mathfrak{K}^\rho_{z_0,n} \in L^2(\rho)$. Write $\mathfrak{K}^\rho_{z_0,n}(t) = (t-\overline{z_0})^{-(n+1)}\varphi(t)$, with

$$\varphi(t) = \overline{b(z_0)}\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\overline{b^j(z_0)}b^j(t).$$

Since $\varphi \in L^\infty(\mathbb{R})$, it is sufficient to prove that $(t-\overline{z_0})^{-(n+1)} \in L^2(\rho)$. If $z_0 \in \mathbb{C}_+$, this fact is trivial and if $z_0 \in E_{2n+2}(b)$, the inequality $1-x \leqslant |\log x|$, $x \in (0,1]$, implies

$$\int_{\mathbb{R}} \frac{1-|b(t)|^2}{|t-z_0|^{2n+2}}\,dt \leqslant 2\int_{\mathbb{R}}\frac{\big|\log|b(t)|\big|}{|t-z_0|^{2n+2}}\,dt < +\infty$$

which is the required result.

It remains to prove (3.2). Let ψ be any element of $H^2(\mathbb{C}_+)$. According to (2.3), we have

$$\begin{aligned} \frac{2\pi i}{n!}\,f^{(n)}(z_0) =& \langle f, k^b_{z_0,n}\rangle_2 + \langle \rho g, k^\rho_{z_0,n}\rangle_2 \\ =& \langle f, k^b_{z_0,n} - b\psi\rangle_2 + \langle \bar{b}f, \psi\rangle_2 + \langle \rho g, k^\rho_{z_0,n}\rangle_2. \end{aligned}$$

But we have $T_{\overline{b}}f = \widetilde{T}_\rho g$, which means that $\overline{b}f - \rho g \perp H^2(\mathbb{C}_+)$. Since $\psi \in H^2(\mathbb{C}_+)$, it follows that $\langle \overline{b}f, \psi\rangle_2 = \langle \rho g, \psi\rangle_2$. Hence the identity

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \langle f, k^b_{z_0,n} - b\psi\rangle_2 + \langle \rho g, k^\rho_{z_0,n} + \psi\rangle_2 \tag{3.4}$$

holds for each $\psi \in H^2(\mathbb{C}_+)$. A very specific ψ gives us the required representation. To find ψ note that, on one hand, we have

$$\begin{aligned} k^b_{z_0,n}(t) - (k^b_{z_0})^{n+1}(t) &= \frac{1 - b(t)\sum_{j=0}^n \overline{a_j}(t-\overline{z_0})^j - (1-\overline{b(z_0)}b(t))^{n+1}}{(t-\overline{z_0})^{n+1}} \\ &= \frac{1-(1-\overline{b(z_0)}b(t))^{n+1}}{(t-\overline{z_0})^{n+1}} - b(t)\frac{\sum_{j=0}^n \overline{a_j}(t-\overline{z_0})^j}{(t-\overline{z_0})^{n+1}} = b(t)\psi(t), \end{aligned}$$

where

$$\psi(t) = \frac{\sum_{j=1}^{n+1}(-1)^{j+1}\binom{n+1}{j}(\overline{b(z_0)})^j(b(t))^{j-1}}{(t-\overline{z_0})^{n+1}} - \frac{\sum_{j=0}^n \overline{a_j}(t-\overline{z_0})^j}{(t-\overline{z_0})^{n+1}}.$$

On the other hand, we easily see that

$$\begin{aligned} k^\rho_{z_0,n}(t) + \psi(t) &= \frac{\sum_{j=1}^{n+1}(-1)^{j+1}\binom{n+1}{j}(\overline{b(z_0)})^j(b(t))^{j-1}}{(t-\overline{z_0})^{n+1}} \\ &= \overline{b(z_0)}\frac{\sum_{j=0}^{n}(-1)^{j}\binom{n+1}{j+1}(\overline{b(z_0)})^j(b(t))^{j}}{(t-\overline{z_0})^{n+1}} = \mathfrak{K}^\rho_{z_0,n}(t). \end{aligned}$$

Therefore, (3.2) follows immediately from (3.4).

□

We now introduce the weight involved in our Bernstein-type inequalities. Let $1 < p \le 2$ and let q be its conjugate exponent. Let $n \in \mathbb{N}$. Then, for $z \in \overline{\mathbb{C}_+}$, we define

$$w_{p,n}(z) := \min\left\{ \|(k^b_z)^{n+1}\|_q^{-pn/(pn+1)},\ \|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^{-pn/(pn+1)} \right\};$$

we assume $w_{p,n}(x) = 0$, whenever $x \in \mathbb{R}$ and at least one of the functions $(k^b_x)^{n+1}$ or $\rho^{1/q}\mathfrak{K}^\rho_{x,n}$ is not in $L^q(\mathbb{R})$. In what follows we will write w_p for $w_{p,1}$.

The choice of the weight is motivated by representation (3.2) which shows that the quantity $\max\left\{\|(k^b_z)^{n+1}\|_2,\ \|\rho^{1/2}\mathfrak{K}^\rho_{z,n}\|_2\right\}$ is related to the norm of the functional $f \mapsto f'(z)$ on $\mathcal{H}(b)$. Moreover, we strongly believe that the norms of reproducing kernels are an important characteristic of the space $\mathcal{H}(b)$ which captures many geometric properties of b (see Section 5 for certain estimates confirming this point).

Using similar arguments as in the proof of Proposition 3.1, it is easy to see that $\rho^{1/q}\mathfrak{K}^{\rho}_{x,n} \in L^q(\mathbb{R})$ if $x \in E_{q(n+1)}(b)$. It is also natural to expect that $(k^b_x)^{n+1} \in L^q(\mathbb{R})$ for $x \in E_{q(n+1)}(b)$. This is true when b is an inner function, by a result of Cohn [12], and for a general function b with $q = 2$ by (3.3) and [20, Lemma 3.2]. However, it seems that the methods of [12] and [20] do not apply in the general case.

Question 3.2. Is it true that for $x \in \mathbb{R}$, $(k^b_x)^{n+1} \in L^q(\mathbb{R})$ if $x \in E_{q(n+1)}(b)$?

Remark 3.3. If $f \in \mathcal{H}(b)$ and $1 < p \le 2$, then $(f^{(n)}w_{p,n})(x)$ is well-defined on $\mathbb{R}$. It follows from the [20] that $f^{(n)}(x)$ and $w_{p,n}(x)$ are finite if $S_{2n+2}(x) < +\infty$. If $S_{2n+2}(x) = +\infty$, then $\|(k^b_x)^{n+1}\|_2 = +\infty$. Hence, $\|(k^b_x)^{n+1}\|_q = +\infty$ which, by definition, implies $w_{p,n}(x) = 0$, and thus we may assume $(f^{(n)}w_{p,n})(x) = 0$.

Remark 3.4. In the inner case, we have $\rho(t) \equiv 0$ and the second term in the definition of the weight $w_{p,n}$ disappears. It should be emphasized that in the general case both terms are essential: below we show (Example 4.2) that the norm $\|\rho^{1/q}\mathfrak{K}^{\rho}_{z,n}\|_q$ can not be majorized uniformly by the norm $\|(k^b_z)^{n+1}\|_q$.

Lemma 3.5. *For $1 < p \le 2$, $n \in \mathbb{N}$, there is a constant $A = A(p,n) > 0$ such that*

$$w_{p,n}(z) \ge A\,\frac{(\operatorname{Im} z)^n}{(1-|b(z)|)^{\frac{pn}{q(pn+1)}}}, \qquad z \in \mathbb{C}_+.$$

Proof. On one hand, note that

$$\begin{aligned}\|(k^b_z)^{n+1}\|_q^q &= \int_{\mathbb{R}} \left|\frac{1-\overline{b(z)}b(t)}{t-\overline{z}}\right|^{(n+1)q} dt \le \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \int_{\mathbb{R}} \left|\frac{1-\overline{b(z)}b(t)}{t-\overline{z}}\right|^2 dt \\ &= \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \|k^b_z\|_b^2 \le C\,\frac{1-|b(z)|}{(\operatorname{Im} z)^{(n+1)q-1}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^q &= \int_{\mathbb{R}} \left| \frac{b(z)\sum_{j=0}^n \binom{n+1}{j+1}(-1)^j \overline{b(z)}^j b^j(t)}{(t-\overline{z})^{n+1}} \right|^q (1-|b(t)|^2)\,dt \\ &\le \frac{C}{(\operatorname{Im} z)^{(n+1)q-2}} \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2}\,dt.\end{aligned}$$

If $|b(z)| < 1/2$, then we obviously have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2} dt \le C \frac{1-|b(z)|}{\operatorname{Im} z},$$

and if $|b(z)| \ge 1/2$, using $1-|b(t)| \le \big|\log|b(t)|\big|$, we get

$$\operatorname{Im} z \int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2} dt \le \operatorname{Im} z \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2} dt = \pi \log \frac{1}{|O_b(z)|} \asymp 1-|O_b(z)|,$$

since $|O_b(z)| \ge |b(z)| \ge 1/2$. We recall that O_b is the outer part of b. Therefore, in any case we have

$$\int_{\mathbb{R}} \frac{1-|b(t)|}{|t-z|^2} dt \le C \frac{1-|b(z)|}{\operatorname{Im} z},$$

and we get

$$\|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^q \le C\, \frac{1-|b(z)|}{(\operatorname{Im} z)^{(n+1)q-1}}.$$

To complete the proof, it suffices to note that $\frac{(n+1)q-1}{q} = n + \frac{1}{p} = \frac{np+1}{p}$.

□

Representation formulae discussed above reduce the study of differentiation in de Branges–Rovnyak spaces $\mathcal{H}(b)$ to the study of certain integral operators.

4. Bernstein-type inequalities

A Borel measure μ in the closed upper half-plane $\overline{\mathbb{C}_+}$ is said to be a Carleson measure if there is a constant $C_\mu > 0$ such that

$$\mu(\, S(x,h)\,) \le C_\mu\, h, \tag{4.1}$$

for all squares $S(x,h) = [x, x+h] \times [0,h]$, $x \in \mathbb{R}$, $h > 0$, with the lower side on the real axis. We denote the class of Carleson measures by $\mathcal{C}$. Recall that, according to a classical theorem of Carleson, $\mu \in \mathcal{C}$ if and only if $H^p(\mathbb{C}_+) \subset L^p(\mu)$ for some (all) $p > 0$.

One of our main results in this paper is the following Bernstein-type inequality.

Theorem 4.1. *Let $\mu \in \mathcal{C}$, let $n \in \mathbb{N}$, let $1 < p \le 2$, and let*

$$(T_{p,n} f)(z) = f^{(n)}(z) w_{p,n}(z), \qquad f \in \mathcal{H}(b).$$

If $1 < p < 2$, then $T_{p,n}$ is a bounded operator from $\mathcal{H}(b)$ to $L^2(\mu)$, that is, there is a constant $C = C(\mu, p, n) > 0$ such that

$$\|f^{(n)} w_{p,n}\|_{L^2(\mu)} \le C \|f\|_b, \qquad f \in \mathcal{H}(b). \tag{4.2}$$

If $p = 2$, then $T_{2,n}$ is of weak type $(2,2)$ as an operator from $\mathcal{H}(b)$ to $L^2(\mu)$.

Proof. According to Proposition 3.1, for all $z \in \overline{\mathbb{C}_+}$ and any function $f \in \mathcal{H}(b)$, we have

$$\frac{2\pi i}{n!} f^{(n)}(z) w_{p,n}(z) = w_{p,n}(z) \int_{\mathbb{R}} f(t) \overline{(k_{z_0}^b)^{n+1}(t)}\, dt + w_{p,n}(z) \int_{\mathbb{R}} g(t)\rho(t) \overline{\mathfrak{K}_{z,n}^{\rho}(t)}\, dt. \tag{4.3}$$

Let

$$w_{p,n}^{(1)}(z) := \|(k_z^b)^{n+1}\|_q^{-pn/(pn+1)}, \qquad w_{p,n}^{(2)}(z) := \|\rho^{1/q} \mathfrak{K}_{z,n}^{\rho}\|_q^{-pn/(pn+1)},$$

where we assume that $w_{p,n}^{(i)}(z) = 0$ if the corresponding integrand is not in $L^q(\mathbb{R})$, and put $h_i(z) = (w_{p,n}^{(i)}(z))^{1/n}$, $i = 1, 2$. We remind that

$$w_{p,n}(z) = \min\{ w_{p,n}^{(1)}(z),\, w_{p,n}^{(2)}(z)\}.$$

We split each of the two integrals in (4.3) into two parts, i.e.

$$\frac{2\pi i}{n!} f^{(n)}(z) w_{p,n}(z) = I_1 f(z) + I_2 f(z) + I_3 g(z) + I_4 g(z),$$

where

$$I_1 f(z) = w_{p,n}(z) \int_{|t-z| \ge h_1(z)} f(t) \overline{(k_z^b)^{n+1}(t)}\, dt,$$

$$I_2 f(z) = w_{p,n}(z) \int_{|t-z| < h_1(z)} f(t) \overline{(k_z^b)^{n+1}(t)}\, dt,$$

$$I_3g(z) = w_{p,n}(z)\int_{|t-z|\geq h_2(z)} g(t)\rho(t)\overline{\mathfrak{K}^\rho_{z,n}(t)}\,dt,$$

$$I_4g(z) = w_{p,n}(z)\int_{|t-z|< h_2(z)} g(t)\rho(t)\overline{\mathfrak{K}^\rho_{z,n}(t)}\,dt.$$

Note that by Lemma 3.5, $h_i(z)\geq A\,\mathrm{Im}\,z$, $z\in\mathbb{C}_+$, $i=1,2$. Hence,

$$\begin{aligned}|I_1f(z)| \leq& Ch_1^n(z)\int_{|t-z|\geq h_1(z)}\frac{|f(t)|}{|t-z|^{n+1}}\,dt\\ \leq& Ch_1(z)\int_{|t-z|\geq h_1(z)}\frac{|f(t)|}{|t-z|^2}\,dt,\end{aligned}$$

and

$$\begin{aligned}|I_3g(z)| \leq& Ch_2^n(z)\int_{|t-z|\geq h_2(z)}\frac{|g(t)|\rho^{1/2}(t)}{|t-z|^{n+1}}\,dt\\ \leq& Ch_2(z)\int_{|t-z|\geq h_2(z)}\frac{|g(t)|\rho^{1/2}(t)}{|t-z|^2}\,dt.\end{aligned}$$

Using [6, Theorem 3.1], we see that $I_1 : L^2(\mathbb{R}) \longrightarrow L^2(\mu)$ and $I_3 : L^2(\rho) \longrightarrow L^2(\mu)$ are bounded operators. To estimate the integral I_2f, put

$$K(z,t) := h_1^n(z)|(k_z^b)^{n+1}(t)|.$$

Then

$$\begin{aligned}\|K(z,\cdot)\|_q^{-p} =& (h_1(z))^{-pn}\|(k_z^b)^{n+1}\|_q^{-p}\\ =& (h_1(z))^{-pn}(w_{p,n}^{(1)}(z))^{(pn+1)/n} = h_1(z).\end{aligned}$$

Thus

$$|I_2f(z)| \leq h_1^n(z)\int_{|t-z|<h_1(z)}|f(t)||(k_z^b)^{n+1}(t)|\,dt = \int_{|t-z|<\|K(z,\cdot)\|_q^{-p}}|f(t)|K(z,t)\,dt.$$

Since $\|K(z,\cdot)\|_q^{-p} = h_1(z) \geq A\,\mathrm{Im}\,z$, we may apply [6, Theorem 3.2]. Therefore, the operator I_2 is of weak type $(2,2)$ as an operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $p=2$ and it is a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mu)$ if $1 < p < 2$. To estimate the integral $I_4 g$, we use the same technique and put

$$\kappa(z,t) := \frac{\rho^{1/q}(t)|\mathfrak{K}^\rho_{z,n}(t)|}{\|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^{pn/(pn+1)}}.$$

In other words, $\kappa(z,t) = w^{(2)}_{p,n}(z)\rho^{1/q}(t)|\mathfrak{K}^\rho_{z,n}(t)|$. Thus

$$\begin{aligned}|I_4 g(z)| &\le w^{(2)}_{p,n}(z)\int_{|t-z|<h_2(z)} |g(t)|\rho(t)|\mathfrak{K}^\rho_{z,n}(t)|\,dt\\ &= \int_{|t-z|<h_2(z)} |g(t)|\rho^{1/p}(t)\kappa(z,t)\,dt.\end{aligned}$$

But $\|\kappa(z,\cdot)\|_q^{-p} = (w^{(2)}_{p,n}(z))^{-p}\|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q^{-p} = h_2(z)$. Hence, we get

$$|I_4 g(z)| \le \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/p}(t)\kappa(z,t)\,dt.$$

Since $p \le 2$ and $\rho(t) \le 1$, we have

$$|I_4 g(z)| \le \int_{|t-z|<\|\kappa(z,\cdot)\|_q^{-p}} |g(t)|\rho^{1/2}(t)\kappa(z,t)\,dt,$$

and since $\|\kappa(z,\cdot)\|_q^{-p} = h_2(z) \ge A\,\operatorname{Im} z$, we may apply again [6, Theorem 3.2]. Therefore, the operator I_4 is of weak type $(2,2)$ as an operator from $L^2(\rho)$ to $L^2(\mu)$ if $p = 2$ and it is a bounded operator from $L^2(\rho)$ to $L^2(\mu)$ if $1 < p < 2$.

To conclude it remains to note that

$$\|f\|_b^2 = \|f\|_2^2 + \|g\|_\rho^2,$$

which implies that the operators $f \mapsto f$ from $\mathcal{H}(b)$ to $H^2(\mathbb{C}_+)$ and $f \mapsto g$ from $\mathcal{H}(b)$ to $L^2(\rho)$ are contractions.

□

Example 4.2. We show that for a general function b both terms in the definition of the weight $w_{p,n}$ are important. Obviously, for an inner b the norm $\|\rho^{1/q}\mathfrak{K}^\rho_{z,n}\|_q$ vanishes. However, for some outer functions b it may be essentially larger than $\|(k_z^b)^{n+1}\|_q$.

Let $\varepsilon \in (0,1)$ and let b be an outer function such that $|b(t)| = \varepsilon$ for $|t| < 1$ and $|b(t)| = 1$ for $|t| > 1$. Note that $b(z) = \exp\Big(-\frac{i}{\pi}\log\varepsilon\log\frac{z-1}{z+1}\Big)$, where log is the main branch of the logarithm in $\mathbb{C}\setminus(-\infty,0]$. We show that

$$\sup_{y>0}\frac{\|\rho^{1/q}\mathfrak{K}^{\rho}_{iy,1}\|_q}{\|(k^b_{iy})^2\|_q} \longrightarrow \infty \quad \text{as} \quad \varepsilon \longrightarrow 1-, \tag{4.4}$$

and so, the second term in the weight $w_{p,1}$ can be dominating. Note that $b(iy) \to \varepsilon$ and $b(t) \to \varepsilon$, as $y \to 0+$ and $|t| \le \sqrt{y}$. Hence, for a fixed ε and sufficiently small $y > 0$ we have

$$\int_{|t|\le\sqrt{y}} |k^b_{iy}(t)|^{2q}dt = \int_{|t|\le\sqrt{y}}\left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q}dt \le C(1-\varepsilon)^{2q}\int_{|t|\le\sqrt{y}}\frac{dt}{|t+iy|^{2q}}.$$

Thus

$$\int_{|t|\le\sqrt{y}}\left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q}dt \le C\frac{(1-\varepsilon)^{2q}}{y^{2q-1}}, \tag{4.5}$$

whereas

$$\int_{|t|>\sqrt{y}}\left|\frac{1-\overline{b(iy)}b(t)}{t+iy}\right|^{2q}dt \le Cy^{-q+1/2}. \tag{4.6}$$

On the other hand,

$$\mathfrak{K}^{\rho}_{iy,1}(t) = \overline{b(iy)}\frac{2-\overline{b(iy)}b(t)}{(t+iy)^2},$$

and so

$$\|\rho^{1/q}\mathfrak{K}^{\rho}_{iy,1}\|_q^q \asymp |b(iy)|^q\int_{\mathbb{R}}\frac{1-|b(t)|}{|t+iy|^{2q}} \asymp \frac{1-\varepsilon}{y^{2q-1}}.$$

Combining the last estimate with (4.5) and (4.6), we obtain (4.4).

Remark 4.3. It should be emphasized that the constants in the Bernstein-type inequalities corresponding to Theorem 4.1 depend only on p, n and the Carleson constant C_μ of the measure μ, but not on b (the properties of b are contained in the weight $w_{p,n}$ in the left-hand side of (4.2)).

Remark 4.4. All the results stated above have their natural analogues for the spaces $\mathcal{H}(b)$ in the unit disc. In particular, Theorem 4.1 remains true when we replace the kernels for the half-plane by the kernels for the disc. The case of inner functions in the disc is considered in detail in [8].

Remark 4.5. An important feature of the de Branges–Rovnyak spaces theory is the difference between the extreme (i.e. b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$) and the non-extreme cases. Our Bernstein inequality applies to both cases. However, in the extreme case one can expect more regularity near the boundary and this situation is more interesting for us.

5. Distances to the level sets

To apply Theorem 4.1, one should have effective estimates for the weight $w_{p,n}$, that is, for the norms of the reproducing kernels. In this section we relate the weight $w_{p,n}$ to the distances to the level sets of $|b|$. We start with some notations. Denote by $\sigma(b)$ the boundary spectrum of b, i.e.

$$\sigma(b) := \Big\{x \in \mathbb{R} : \liminf_{\substack{z \longrightarrow x \\ z \in \mathbb{C}_+}} |b(z)| < 1\Big\}.$$

Then, for $b = BI_\mu O_b$, $\operatorname{Clos}\sigma(b)$ is the smallest closed subset of $\mathbb{R}$ containing the limit points of the zeros of the Blaschke product B and the supports of the measures μ and $\log|b(t)|\,dt$. It is well known and easy to see that b and any element of $\mathcal{H}(b)$ has an analytic extension through any interval from the open set $\mathbb{R} \setminus \operatorname{Clos}\sigma(b)$.

For $\varepsilon \in (0,1)$, we put

$$\Omega(b,\varepsilon) := \{z \in \mathbb{C}_+ : |b(z)| < \varepsilon\},$$

and

$$\widetilde{\Omega}(b,\varepsilon) := \sigma(b) \cup \Omega(b,\varepsilon),$$

where $\sigma(b)$ is the boundary spectrum of b. Finally, for $x \in \mathbb{R}$, we introduce the following three distances

$$\begin{aligned} d_0(x) &:= \operatorname{dist}(x, \sigma(b)), \\ d_\varepsilon(x) &:= \operatorname{dist}(x, \Omega(b,\varepsilon)), \\ \tilde{d}_\varepsilon(x) &:= \operatorname{dist}(x, \widetilde{\Omega}(b,\varepsilon)). \end{aligned}$$

Note that whenever $b = \Theta$ is an inner function, for all $x \in \sigma(\Theta)$, we have

$$\liminf_{\substack{z \longrightarrow x \\ z \in \mathbb{C}_+}} |\Theta(z)| = 0,$$

and thus $d_\varepsilon(t) = \tilde{d}_\varepsilon(t)$, $t \in \mathbb{R}$. However, for an arbitrary function b in the unit ball of $H^\infty(\mathbb{C}_+)$, we have to distinguish between the distance functions d_ε and $\tilde{d}_\varepsilon$.

Lemma 5.1. *There exists a positive constant $C = C(\varepsilon)$ such that, for all $x \in \mathbb{R} \setminus \sigma(b)$,*

$$|b'(x)| \le C\big(\tilde{d}_\varepsilon(x)\big)^{-1}.$$

Proof. For the case of an inner function the inequality is proved in [6, Theorem 4.9]. For the general case, let $b = I_b O_b$ be the inner-outer factorization of b. Since $|b'(x)| = |I_b'(x)| + |O_b'(x)|$, $x \in \mathbb{R} \setminus \sigma(b)$, we may assume, without loss of generality, that b is outer. Recall that in this case

$$|b'(x)| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-x|^2}\, dt.$$

Fix $x \in \mathbb{R} \setminus \sigma(b)$ and suppose $0 < y < d_0(x)$. Let $z = x + iy$. Then

$$\log \frac{1}{|b(z)|} = \frac{y}{\pi} \int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\, dt = \frac{y}{\pi} \int_{|t-x| \ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\, dt.$$

Since $|t-z| \le |t-x| + y \le 2|t-x|$ whenever $|t-x| \ge d_0(x)$, we have

$$\log \frac{1}{|b(z)|} \ge \frac{y}{4\pi} \int_{|t-x| \ge d_0(x)} \frac{\big|\log|b(t)|\big|}{|t-x|^2}\, dt = \frac{y|b'(x)|}{4}.$$

Hence

$$|b(x+iy)| \le \exp\big(-y|b'(x)|/4\big), \tag{5.1}$$

provided that $0 < y < d_0(x)$.

Let $C = 4\log\varepsilon^{-1}$. If $|b'(x)| \le C/|d_0(x)|$, then the statement is valid since $\tilde{d}_\varepsilon(x) \le d_0(x)$. On the other hand, if $|b'(x)| > C/|d_0(x)|$, then we consider the point $z = x + iC/|b'(x)|$ for which $\operatorname{Im} z = C/|b'(x)| < d_0(x)$. Hence, by (5.1), we have $|b(z)| \le \varepsilon$ which immediately implies $\tilde{d}_\varepsilon(x) \le C/|b'(x)|$.

□

Lemma 5.2. *For each $p > 1$, $n \ge 1$ and $\varepsilon \in (0,1)$, there exists $C = C(\varepsilon, p, n) > 0$ such that*

$$\left(\tilde{d}_\varepsilon(x)\right)^n \le C\, w_{p,n}(x+iy), \tag{5.2}$$

for all $x \in \mathbb{R}$ and $y \ge 0$.

Proof. Let $z = x+iy$, $y \ge 0$. Assume that $x \in \mathbb{R} \setminus \sigma(b)$ (otherwise $\tilde{d}_\varepsilon(x) = 0$ and (5.2) is trivial). Since $-(n+1)q + 1 = -q\frac{np+1}{p}$, the estimate (5.2) is equivalent to

$$\int_{\mathbb{R}} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1}, \tag{5.3}$$

and

$$\int_{\mathbb{R}} \left| \frac{\overline{b(z)} \sum_{j=0}^{n} \binom{n+1}{j+1} (-1)^j \overline{b(z)}^j b^j(t)}{(t-\overline{z})^{n+1}} \right|^q \rho(t)\, dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1}. \tag{5.4}$$

Inequality (5.4) is obvious, since $\rho(t) = 0$ if $|t - x| < \tilde{d}_\varepsilon(x)$. To prove (5.3), we estimate separately the integrals over $\{t : |t-x| \le \tilde{d}_\varepsilon(x)/2\}$ and $\{t : |t-x| > \tilde{d}_\varepsilon(x)/2\}$. Obviously,

$$\int_{|t-x|>\tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \le C(\tilde{d}_\varepsilon(x))^{-(n+1)q+1}.$$

Since $|b(t)| = 1$ if $|t - x| \le \tilde{d}_\varepsilon(x)/2$, for the second integral we have

$$\begin{aligned}\int_{|t-x|\le\tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt &= \int_{|t-x|\le\tilde{d}_\varepsilon(x)/2} \left| \frac{b(t) - b(z)}{t - z} \right|^{(n+1)q} dt \\ &\le \tilde{d}_\varepsilon(x) \max |b'(u)|^{(n+1)q},\end{aligned}$$

where the maximum is taken over $u \in [t, z]$ with $|t - x| \le \tilde{d}_\varepsilon(x)/2$ (by $[t, z]$ we denote the straight line segment with the endpoints t and z). Note that for such u we have $|\mathrm{Re}\, u - x| \le \tilde{d}_\varepsilon(x)/2$. By Lemma 5.2, $|b'(u)| \le |b'(\mathrm{Re}\, u)|$, and hence,

$$\int_{|t-x|\le \tilde{d}_\varepsilon(x)/2} \left| \frac{1 - \overline{b(z)}b(t)}{t - \overline{z}} \right|^{(n+1)q} dt \le \tilde{d}_\varepsilon(x) \max_{|t-x|\le \tilde{d}_\varepsilon(x)/2} |b'(t)|^{(n+1)q}.$$

According to Lemma 5.1, $|b'(t)| \le C_1(\tilde{d}_\varepsilon(t))^{-1} \le C_2(\tilde{d}_\varepsilon(x))^{-1}$ whenever $|t - x| < \tilde{d}_\varepsilon(x)/2$ which leads to the required estimate.

□

Corollary 5.3. *For each $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, there exists $C = C(\varepsilon, n)$ such that*

$$\|f^{(n)}\tilde{d}_\varepsilon^n\|_2 \le C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

Proof. The statement follows immediately from Lemma 5.2 and Theorem 4.1.

□

We conclude this section with a a corollary of our Bernstein inequalities, concerning the regularity on the boundary for functions in $\mathcal{H}(b)$. This technical result will be used later.

Corollary 5.4. *Let $I = [x_0, x_0 + y_0]$ be a bounded interval on $\mathbb{R}$, $1 < p < 2$. Assume that*

$$\int_I w_p(x)^{-2} dx < +\infty. \tag{5.5}$$

Then we have

a) $]x_0, x_0 + y_0[\cap \sigma(b) = \emptyset$. *In particular, each function f in $\mathcal{H}(b)$ is differentiable on* $]x_0, x_0 + y_0[$.

b) *b is continuous on the Carleson square $S(I) = [x_0, x_0 + y_0] \times [0, y_0]$.*

Proof. a) According to Theorem 4.1, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}} |f'(x)w_p(x)|^2\, dx \le C\|f\|_b^2, \qquad f \in \mathcal{H}(b).$$

Then, using (5.5) and the Cauchy–Schwartz inequality, we get $f' \in L^1(I)$ for any $f \in \mathcal{H}(b)$. Now choose $z \in \mathbb{C}_+$ such that $b(z) \neq 0$ and take $f = k_z^b$. We have

$$f'(x) = -\overline{b(z)}\frac{b'(x)}{x-\overline{z}} - \frac{k_z^b(x)}{x-\overline{z}}$$

and, since $k_z^b \in L^1(I)$, we conclude that

$$\int_{x_0}^{x_0+y_0} |b'(x)|\,dx < +\infty. \tag{5.6}$$

Now it follows immediately from the formula (2.1) for $|b'(x)|$ that (5.6) implies $]x_0, x_0 + y_0[\, \cap\, \sigma(b) = \emptyset$. As a matter of fact, this is obvious for the outer and the singular inner factors since $\int_I (x-t)^{-2} dt = \infty$ for any $x \in I$; and if b is a Blaschke product with zeros z_r tending to $x \in]x_0, x_0 + y_0[$, then, for sufficiently large r,

$$\int_{x_0}^{x_0+y_0} \frac{2\,\mathrm{Im}\, z_r}{|x-z_r|^2} dx \geq \pi,$$

and so the integral in (5.6) diverges.

b) By statement a), b is continuous on $S(I)$ except possibly at the points x_0 and x_0+y_0. It remains to show that b is continuous at x_0 and x_0+y_0. Fix $x_1 \in]x_0, x_0+y_0[$ and define

$$b(x_0) := b(x_1) - \int_{x_0}^{x_1} b'(x)\,dx.$$

(Note that this definition of $b(x_0)$ does not seem to correspond to the classical one with non-tangential limits but, in fact, as we will see at the end, they coincide). Since b is differentiable on $]x_0, x_0 + y_0[$, this definition does not depend on the choice of x_1 and we see from (5.6) that $b(x)$ tends to $b(x_0)$ as $x \to x_0$ along I. Now let $z = x+iy \in S(I)$, with $x \in [x_0, x_0+y_0/2[$, $y \in]0, y_0/2[$. Write $b(z)-b(x_0) = b(x+iy)-b(x+y)+b(x+y)-b(x_0)$. Using the continuity of b at x_0 along I, we have $b(x+y) - b(x_0) \to 0$, as $x \to x_0$ and $y \to 0$. Moreover, since b is analytic on $\mathbb{C}_+ \cup\,]x_0, x_0 + y_0[$, we can write

$$b(x+y) - b(x+iy) = (1-i)y \int_0^1 b'(t(x+y) + (1-t)(x+iy))\,dt.$$

Applying Lemma 2.1, we get

$$|b(x+y) - b(x+iy)| \le \sqrt{2} \int_x^{x+y} |b'(u)|\, du.$$

According to (5.6), we deduce that $b(x+y) - b(x+iy) \to 0$, as $x \to x_0$ and $y \to 0$. Therefore, $b(z) \to b(x_0)$, as $z \to x_0$, $z \in S(I)$. □

6. Carleson-type embedding theorems

Weighted Bernstein-type inequalities of the form (1.2) turned out to be an efficient tool for the study of the so-called Carleson-type embedding theorems for the shift-coinvariant subspaces K^p_Θ. More precisely, given an inner function Θ, we want to describe the class of Borel measure μ in the closed upper half-plane $\overline{\mathbb{C}_+}$ such that the embedding $K^p_\Theta \subset L^p(\mu)$ takes place. In other words, we are interested in the class of Borel measure μ in $\overline{\mathbb{C}_+}$ such that there is a constant C satisfying

$$\|f\|_{L^p(\mu)} \le C\|f\|_p,$$

for all $f \in K^p_\Theta$. This problem was posed by Cohn in [11]. In spite of a number of beautiful results (see, e.g., [11, 12, 28, 42]), the question still remains open in the general case. Compactness of the embedding operator is also of interest and is considered in [10, 13, 41].

Methods based on the Bernstein-type inequalities allow to give unified proofs and essentially generalize almost all known results concerning these problems (see [6, 8]). Here we obtain an embedding theorem for de Branges–Rovnyak spaces. In the case of an inner function the first statement coincides with a well-known theorem due to Volberg and Treil [42].

A Carleson measure for the closed upper half-plane is called a *vanishing Carleson measure* if $\mu(S(x,h))/h \to 0$ whenever $h \to 0$ or $\operatorname{dist}(S(x,h), 0) \to \infty$. Vanishing Carleson measures in the closed unit disc are discussed, e.g., in [33]. An equivalent definition for a vanishing Carleson measure ν in the disc is that

$$\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}\zeta|^2} d\nu(\zeta) \longrightarrow 0, \qquad \text{as } |z| \to 1.$$

Changing the variables to the upper half-plane with $|w+i|^{-2} d\mu(w) = d\nu(\zeta)$, we obtain

$$\int_{\mathbb{C}_+} \frac{\operatorname{Im} z}{|w-\overline{z}|^2} d\mu(w) \longrightarrow 0,$$

whenever either $\operatorname{Im} z \to 0$ or $|z| \to +\infty$. It is easily seen that this condition is equivalent to the above definition of a vanishing Carleson measure. It is well known that an embedding $H^p(\mathbb{C}_+) \subset L^p(\mu)$ is compact if and only if μ is a vanishing Carleson measure.

Theorem 6.1. *Let μ be a Borel measure in $\overline{\mathbb{C}_+}$, and let $\varepsilon \in (0,1)$.*

(a) *Assume that $\mu(S(x,h)) \le Kh$ for all Carleson squares $S(x,h)$ satisfying*

$$S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset.$$

Then $\mathcal{H}(b) \subset L^2(\mu)$, that is, there is a constant $C > 0$ such that

$$\|f\|_{L^2(\mu)} \le C\|f\|_b, \qquad f \in \mathcal{H}(b).$$

(b) *Assume that μ is a vanishing Carleson measure for $\mathcal{H}(b)$, that is, $\mu(S(x,h))/h \to 0$ whenever $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$ and $h \to 0$ or $\operatorname{dist}(S(x,h),0) \to +\infty$. Then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

In Theorem 6.1 we need to verify the Carleson condition only on a *special* subclass of squares. Geometrically this means that when we are far from the spectrum $\sigma(b)$, the measure μ in Theorem 6.1 can be essentially larger than standard Carleson measures. The reason is that functions in $\mathcal{H}(b)$ have much more regularity at the points $x \in \mathbb{R} \setminus \operatorname{Clos} \sigma(b)$ where $|b(x)| = 1$. On the other hand, if $|b(x)| \le \delta < 1$, almost everywhere on some interval $I \subset \mathbb{R}$, then the functions in $\mathcal{H}(b)$ behave on I essentially the same as a general element of $H^2(\mathbb{C}_+)$ on that interval, and for any Carleson measure for $\mathcal{H}(b)$ its restriction to the square $S(I)$ is a standard Carleson measure.

We will see that, for a class of functions b, the sufficient condition of Theorem 6.1 is also necessary. However, it may be far from being necessary for certain functions b even in the model space setting.

By *a closed square* in $\overline{\mathbb{C}_+}$, we mean a set of the form

$$S(x_0, y_0, h) := \{x + iy : x_0 \le x \le x_0 + h,\ y_0 \le y \le y_0 + h\}, \tag{6.1}$$

where $x_0 \in \mathbb{R}$, $y_0 \ge 0$ and $h > 0$; by the *lower side* of the closed square $S(x_0, y_0, h)$ we mean the interval $\{x + iy_0 : x_0 \le x \le x_0 + h\}$.

We deduce Theorem 6.1 from the following more general result. Recall that

$$w_p(z) = w_{p,1}(z) = \min(\|(k_z^b)^2\|_q^{-p/(p+1)}, \|\rho^{1/q}\mathfrak{K}^\rho_{z,1}\|_q^{-p/(p+1)}).$$

Theorem 6.2. *Let $\{S_k\}_{k\ge1}$ be a sequence of closed squares in $\overline{\mathbb{C}_+}$, let I_k denote the lower side of the square S_k, and let δ_{I_k} be the Lebesgue measure on I_k. Assume that the squares S_k satisfy the following two conditions:*

$$\sum_k \delta_{I_k} \in \mathcal{C}, \tag{6.2}$$

and, for some p, $1 < p < 2$,

$$\sup_{k\ge1,\, y\ge0} |I_k| \int_{S_k\cap\{\operatorname{Im} z = y\}} w_p^{-2}(u)|du| < \infty. \tag{6.3}$$

Let μ be a Borel measure with $\operatorname{supp}\mu \subset \bigcup_k S_k$. Then

(a) *if $\mu(S_k) \le C|I_k|$, then $\mathcal{H}(b) \subset L^2(\mu)$.*
(b) *if, moreover, $I_k \cap \operatorname{Clos}\sigma(b) = \emptyset$, $k \ge 1$, and $\mu(S_k) = o(|I_k|)$, $k \to \infty$, then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

For the model subspaces a result, analogous to Theorem 6.2, was obtained in [6, Theorem 2.2]. For the sake of completeness, we include the proof.

Proof. (a) The idea of the proof is to replace the measure μ with some Carleson measure ν, and to estimate the difference between the norms $\|f\|_{L^2(\mu)}$ and $\|f\|_{L^2(\nu)}$ using the Bernstein-type inequality of Section 4.

It follows from Corollary 5.4 (b) that the set of functions $f \in \mathcal{H}(b)$ which are continuous on each of S_k is dense in $\mathcal{H}(b)$ (take the reproducing kernels k_z^b, $z \in \mathbb{C}^+$). Thus it is sufficient to prove the estimate $\|f\|_{L^2(\mu)} \le C\|f\|_b$ only for $f \in \mathcal{H}(b)$ continuous on $\bigcup_k S_k$. Now let $f \in \mathcal{H}(b)$ be continuous on each of S_k. Then there exist $w_k \in S_k$ such that

$$\|f\|^2_{L^2(\mu)} \le \sum_k |f(w_k)|^2 \mu(S_k) \le \sup_k \frac{\mu(S_k)}{|I_k|} \cdot \sum_k |f(w_k)|^2 |I_k|. \tag{6.4}$$

Statement (a) will be proved as soon as we show that

$$\sum_k |f(w_k)|^2 |I_k| \le C\|f\|_b^2 \tag{6.5}$$

where the constant C does not depend on f and on the choice of $w_k \in S_k$.

Consider the intervals $J_k = S_k \cap \{\operatorname{Im} z = \operatorname{Im} w_k\}$. Let $\nu = \sum_k \delta_{J_k}$. Then it follows from (6.2) that $\nu \in \mathcal{C}$ (and the Carleson constants C_ν of such measures ν are uniformly bounded). We have

$$\Big(\sum_k |f(w_k)|^2 |I_k|\Big)^{1/2} \le \|f\|_{L^2(\nu)} + \Big(\sum_k \int_{J_k} |f(z) - f(w_k)|^2 |dz|\Big)^{1/2}, \tag{6.6}$$

and $\|f\|_{L^2(\nu)} \le C_1 \|f\|_2 \le C_1 \|f\|_b$.

We estimate the last term in (6.6). For $z \in J_k$ denote by $[z, w_k]$ the straight line interval with the endpoints z and w_k. Then $f(z) - f(w_k) = \int_{[z,w_k]} f'(u)du$ (in the case $J_k \subset \mathbb{R}$ note that, by Corollary 5.4 (a), any $f \in \mathcal{H}(b)$ is differentiable on J_k except, may be, at the endpoints). So, by the Cauchy–Schwartz inequality,

$$\sum_k \int_{J_k} |f(z) - f(w_k)|^2 |dz| \le \sum_k \int_{J_k} \Big| \int_{J_k} |f'(u)||du| \Big|^2 |dz|$$

$$\le \sum_k |J_k| \Big(\int_{J_k} w_p^{-2}(u)|du| \Big) \Big(\int_{J_k} |f'(u)|^2 w_p^2(u)|du| \Big).$$

By (6.3), we obtain

$$\sum_k \int_{J_k} |f(z)-f(w_k)|^2|dz| \le C_2 \sum_k \int_{J_k} |f'(u)|^2 w_p^2(u)|du|$$

$$= C_2\|f'w_p\|^2_{L^2(\nu)} \le C_3\|f\|_b^2,$$

where the last inequality follows from Theorem 4.1.

(b) For a Borel set $E \subset \overline{\mathbb{C}_+}$ define the operator $\mathcal{I}_E : \mathcal{H}(b) \to L^2(\mu)$ by $\mathcal{I}_E f = \chi_E f$ where χ_E is the characteristic function of E. For $N \in \mathbb{N}$ put $F_N = \bigcup_{k=1}^N S_k$ and $\widehat{F}_N = \overline{\mathbb{C}_+} \setminus F_N$. As above we assume that $f \in \mathcal{H}(b)$ is continuous on $\bigcup_k S_k$. Then it follows from (6.4) and (6.5) that

$$\int_{\widehat{F}_N} |f|^2 d\mu \le C \sup_{k>N} \frac{\mu(S_k)}{|I_k|}\|f\|_b^2,$$

and so $\|\mathcal{I}_{\widehat{F}_N}\| \to 0$, $N \to \infty$. Statement (b) will be proved as soon as we show that $\mathcal{I}_{F_N}$ is a compact operator for any N (thus, our embedding operator $\mathcal{I}_{F_N} + \mathcal{I}_{\widehat{F}_N}$ may be approximated in the operator norm by compact operators $\mathcal{I}_{F_N}$). Clearly, it suffices to prove the compactness of $\mathcal{I}_{S_k}$ for each fixed k.

We approximate $\mathcal{I}_{S_k}$ by finite rank operators. For a given $\epsilon > 0$, partition the square S_k into finite union of squares $\{\tilde{S}_l\}_{l=1}^L$ with pairwise disjoint interiors so that

$$\left(\int_{[\zeta,z]} w_p^{-2}(u)|du| \right) < \epsilon \tag{6.7}$$

for any l, $1 \le l \le L$, and any $\zeta, z \in \tilde{S}_l$. Such a partition exists since $I_k \cap \operatorname{Clos}\sigma(b) = \emptyset$, $k \ge 1$. Indeed, b is analytic in a neighborhood of S_k, and the norms involved in the definition of $w_p(z)$ are continuous on S_k.

Now fix $\zeta_l \in \tilde{S}_l$ and consider the finite rank operator $T : \mathcal{H}(b) \to L^2(\mu)$, $(Tf)(z) = \sum_{l=1}^L f(\zeta_l)\chi_{\tilde{S}_l}(z)$. We show that $\|\mathcal{I}_{S_k} - T\|^2 \le C\epsilon$. As in the proof of (a), we have

$$\|(\mathcal{I}_{S_K} - T)f\|^2_{L^2(\mu)} = \sum_{l=1}^L \int_{\tilde{S}_l} |f(z) - f(\zeta_l)|^2 d\mu(z)$$

$$\leq \sum_{l=1}^{L} \int_{\tilde{S}_l} \left(\int_{[\zeta_l, z]} |f'(u)|^2 w_p^2(u)|du| \right) \cdot \left(\int_{[\zeta_l, z]} w_p^{-2}(u)|du| \right) d\mu(z).$$

By Theorem 4.1,

$$\int_{[\zeta_l, z]} |f'(u)|^2 w_p^2(u)|du| \leq C_1 \|f\|_b^2$$

where C_1 does not depend on $f \in \mathcal{H}(b)$, $1 \leq l \leq L$ and $z \in \tilde{S}_l$. Hence, by (6.7),

$$\|(\mathcal{I}_{S_K} - T)f\|_{L^2(\mu)}^2 \leq C_1 \epsilon \|f\|_b^2 \sum_{l=1}^{L} \mu(\tilde{S}_l) = C_1 \epsilon \mu(S_k) \|f\|_b^2.$$

We conclude that $\mathcal{I}_{S_K}$ may be approximated by finite rank operators and is, therefore, compact. □

We comment now on a couple of details of the proof where the situation differs from the inner case.

Remark 6.3. In the inner case $b = \Theta$ one can prove the estimate $\|f\|_{L^2(\mu)} \leq C\|f\|_2$ for functions f in K_Θ^2 which are continuous on the closed upper half-plane $\overline{\mathbb{C}_+}$ and then use a result of Aleksandrov [3] which says that such functions are dense in K_Θ^2. We do not know if this result is still valid in $\mathcal{H}(b)$. To avoid this difficulty, in the proof of Theorem 6.2, we used the density in $\mathcal{H}(b)$ of the functions continuous on all squares S_k.

Question 6.4. Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$. Is it true that the set of functions f in $\mathcal{H}(b)$, continuous on $\overline{\mathbb{C}_+}$, is dense in $\mathcal{H}(b)$?

Remark 6.5. In the inner case, in Theorem 6.2, the assumption (6.3) can be replaced by the weaker assumption (only for the lower side of the square)

$$\sup_{k \geq 1} |I_k| \int_{I_k} w_p^{-2}(u)|du| < \infty. \tag{6.8}$$

It was noticed in [6, Corollary 4.7] that in the inner case, for $q > 1$, there exists $C = C(q) > 0$ such that, for any $x \in \mathbb{R}$ and $0 \leq y_2 \leq y_1$, we have

$$\|k_{x+iy_1}^b\|_q \leq C(q) \|k_{x+iy_2}^b\|_q. \tag{6.9}$$

Thus, it follows from (6.9) that if the sequence $\{S_k\}$ satisfies (6.8), then it also satisfies (6.3).

Question 6.6. Does the monotonicity property (6.9) of the norms of the reproducing kernels along the rays parallel to imaginary axis remains true for a general b? (It is true for $q = 2$, but this is not the interesting case for us.)

Proof. of Theorem 6.1. (a) Consider the open set $E = \mathbb{R} \setminus \operatorname{Clos} \widetilde{\Omega}(b, \varepsilon)$. If $E = \emptyset$, then μ is a Carleson measure and $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu)$. So we may assume that $E \neq \emptyset$ and we can write it as a union of disjoint intervals Δ_l. Note that $\int_{\Delta_l} (\tilde{d}_\varepsilon(t))^{-1} dt = \infty$. Hence, partitioning the intervals Δ_l, we may represent E as a union of intervals I_k with mutually disjoint interiors such that

$$\int_{I_k} \left[\tilde{d}_\varepsilon(t)\right]^{-1} dt = \frac{1}{2}.$$

It follows that there exists $x_k \in I_k$ such that $\tilde{d}_\varepsilon(x_k) = 2|I_k|$. Hence, for any $x \in I_k$, $\tilde{d}_\varepsilon(x) \geq \tilde{d}_\varepsilon(x_k) - |I_k| = |I_k|$ and $\tilde{d}_\varepsilon(x) \leq 3|I_k|$. This implies

$$|I_k| \int_{I_k} \left[\tilde{d}_\varepsilon(t)\right]^{-2} dt \leq 1,$$

and using Lemma 5.2, we conclude that the intervals I_k satisfy (6.3). Condition (6.2) is obvious.

Let $S_k = S(I_k)$ be the Carleson square with the lower side I_k, let $F = \bigcup_k S_k$, and let $G = \overline{\mathbb{C}_+} \setminus F$. Put $\mu_1 = \mu|_F$ and $\mu_2 = \mu|_G$. We show that the measure μ_1 satisfies the conditions of Theorem 6.2 whereas μ_2 is a usual Carleson measure (and, thus, $\mathcal{H}(b) \subset H^2(\mathbb{C}_+) \subset L^2(\mu_2)$).

Let us show that $\mu_1(S_k) \leq C_2|I_k|$. Indeed, it follows from the estimate $|I_k| \leq \tilde{d}_\varepsilon(x) \leq 3|I_k|$, $x \in I_k$, that $S(6I_k) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$ (by $6I_k$ we denote the 6 times larger interval with the same center as I_k). By the hypothesis, $\mu_1(S_k) \leq \mu(S(6I_k)) \leq C|I_k|$. Hence, μ_1 satisfies the conditions of Theorem 6.2 (a), and so $\mathcal{H}(b) \subset L^2(\mu_1)$.

Now we show that $\mu_2 \in \mathcal{C}$. Assume that $S(I) \cap G \neq \emptyset$ for some interval $I \subset \mathbb{R}$, and let $z = x + iy \in S(I) \cap G$. If $x \in \operatorname{Clos} \widetilde{\Omega}(b, \varepsilon)$, then $S(2I) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$. Otherwise, if $x \in I_k$ for some k, then $\tilde{d}_\varepsilon(x) \leq 3|I_k| \leq 3|I|$ since $z \in S(I) \setminus S(I_k)$. Thus

$$S(6I) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset. \tag{6.10}$$

By the hypothesis, $\mu_2(S(I)) \leq \mu(S(6I)) \leq C|I|$, and so μ_2 is a Carleson measure.

(b) Let F, G, μ_1 and μ_2 be the same as above. We show that μ_1 satisfies the conditions of Theorem 6.2 (b), whereas μ_2 is a vanishing Carleson measure. Indeed, we can split the family $\{S_k\}$ into two families $\{S_k\}_{k \in K_1}$ and $\{S_k\}_{k \in K_2}$ such that $|I_k| \to 0$, $k \to \infty$, $k \in K_1$, whereas $\operatorname{dist}(I_k, 0) \to \infty$ when $k \to \infty$, $k \in K_2$. Since $S(6I_k) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$ we conclude that Theorem 6.2 (b) applies to μ_1 and the embedding $\mathcal{H}(b) \subset L^2(\mu_1)$ is compact. Finally, any Carleson square $S(I)$ with $S(I) \cap G \neq \emptyset$ satisfies (6.10), and so, by the assumptions of Theorem 6.1 (b), μ_2 is a vanishing Carleson measure.

□

We state an analogous result for the spaces in the unit disc (for the case of inner functions statement (b) is proved in [8]; it answers a question posed in [10]).

Theorem 6.7. *Let μ be a Borel measure in the closed unit disc $\overline{\mathbb{D}}$, and let $\varepsilon \in (0, 1)$.*

(a) *Assume that $\mu(S(x, h) \leq Ch$ for all Carleson squares $S(x, h)$ such that $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$. Then $\mathcal{H}(b) \subset L^2(\mu)$.*
(b) *If, moreover, $\mu(S(x, h))/h \to 0$ when $h \to 0$ and $S(x, h) \cap \widetilde{\Omega}(b, \varepsilon) \neq \emptyset$, then the embedding $\mathcal{H}(b) \subset L^2(\mu)$ is compact.*

For a class of functions b the converse to Theorem 6.1 is also true. We say that b satisfies the *connected level set condition* if the set $\Omega(b, \varepsilon)$ is connected for some $\varepsilon \in (0, 1)$. Our next result is analogous to certain results from [11] and to [42, Theorem 3].

Theorem 6.8. *Let b satisfy the connected level set condition for some $\varepsilon \in (0,1)$. Assume that $\Omega(b,\varepsilon)$ is unbounded and $\sigma(b) \subset \operatorname{Clos}\Omega(b,\varepsilon)$. Let μ be a Borel measure on $\overline{\mathbb{C}_+}$. Then the following statements are equivalent:*

(a) $\mathcal{H}(b) \subset L^2(\mu)$.

(b) *There exists $C > 0$ such that $\mu(S(x,h)) \le Ch$ for all Carleson squares $S(x,h)$ such that $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$.*

(c) *There exists $C > 0$ such that*

$$\int_{\overline{\mathbb{C}_+}} \frac{\operatorname{Im} z}{|\zeta - \overline{z}|^2} d\mu(\zeta) \le \frac{C}{1 - |b(z)|}, \qquad z \in \mathbb{C}_+. \tag{6.11}$$

Proof. The implication (b) $\Longrightarrow$ (a) holds for any b by Theorem 6.1, and the implication (a) $\Longrightarrow$ (c) is trivial (apply the inequality $\|f\|_{L^2(\mu)} \le C\|f\|_b$ to $f = k_z^b$). To prove that (c) $\Longrightarrow$ (b), we use an argument from [42]. Let $S(x,h)$ be a Carleson square such that $S(x,h) \cap \widetilde{\Omega}(b,\varepsilon) \neq \emptyset$. Since $\sigma(b) \subset \operatorname{Clos}\Omega(b,\varepsilon)$ it follows that $S(x,2h) \cap \Omega(b,\varepsilon) \neq \emptyset$. Choose $z_1 \in S(x,2h) \cap \mathbb{C}_+$ with $|b(z_1)| < \varepsilon$. Now consider $S(x,3h)$. Since $\Omega(b,\varepsilon)$ is connected and unbounded, there exists a point z_2 on the boundary of $S(x,3h)$ such that $|b(z_2)| < \varepsilon$. Hence, there exists a continuous curve γ connecting z_1 and z_2 and such that $|b| < \varepsilon$ on γ. Now let $z = x + ih$. Applying the theorem on two constants to the domain $\operatorname{Int} S(x,3h) \setminus \gamma$ we conclude that $|b(z)| \le \delta$ where $\delta \in (0,1)$ depends only on ε. Then inequality (6.11) implies

$$h \int_{S(x,h)} \frac{d\mu(\zeta)}{|\zeta - \overline{z}|^2} \le C(1-\delta)^{-1}.$$

It remains to note that $|\zeta - \overline{z}| \le C_1 h$, $\zeta \in S(x,h)$ to obtain $\mu(S(x,h)) \le C_2 h$.

□

Example 6.9. Examples are known of inner functions satisfying the connected level set condition. We would like to emphasize that there are also many outer functions satisfying the conditions of Theorem 6.8. For example, let $b(z) = \exp(\frac{i}{\pi} \log z)$, where $\log z$ is the main branch of the logarithm in $\mathbb{C} \setminus (-\infty, 0]$.

Remark 6.10. We see that if b satisfies the conditions of Theorem 6.8, then it suffices to verify the inequality $\|f\|_{L^2(\mu)} \le C\|f\|_b$ for the reproducing kernels of the space $\mathcal{H}(b)$ to get it for all functions f in $\mathcal{H}(b)$. Recently, Nazarov and Volberg [28] showed that it is no longer true in the general case.

7. Stability of bases of reproducing kernels

Another application of Bernstein inequalities for model subspaces K^p_Θ is considered in [7]; it is connected with stability of Riesz bases and frames of reproducing kernels $(k^\Theta_{\lambda_n})$ under small perturbations of the points λ_n. Riesz bases of reproducing kernels in de Branges–Rovnyak spaces $\mathcal{H}(b)$ were studied in [19]. Making use of Theorem 4.1 we extend the results of [7] to the spaces $\mathcal{H}(b)$.

For $\lambda \in \mathbb{C}_+ \cup E_2(b)$, we denote by κ^b_λ the normalized reproducing kernel at the point λ, that is, $\kappa^b_\lambda = k^b_\lambda/(2\pi i\,\|k^b_\lambda\|_b)$. Let $(\kappa^b_{\lambda_n})_{n\ge1}$ be a Riesz basis in $\mathcal{H}(b)$, let $\lambda_n \in G_n$ and let $G = \bigcup_n G_n \subset \overline{\mathbb{C}_+}$ satisfy the following properties.

(i) There exist positive constants c and C such that

$$c \le \frac{\|k^b_{z_n}\|_b}{\|k^b_{\lambda_n}\|_b} \le C, \qquad z_n \in G_n.$$

(ii) For any $z_n \in G_n$, the measure $\nu = \sum_n \delta_{[\lambda_n, z_n]}$ is a Carleson measure and, moreover, the Carleson constants C_ν of such measures (see (4.1)) are uniformly bounded with respect to z_n. Here $[\lambda_n, z_n]$ is the straight line interval with the endpoints λ_n and z_n, and $\delta_{[\lambda_n, z_n]}$ is the Lebesgue measure on the interval.

Remark 7.1. As in the inner case, it should be noted that for $\lambda_n \in \mathbb{C}_+$, there always exist non-trivial sets G_n satisfying (i) and (ii). More precisely, we can take

$$G_n := \{z \in \mathbb{C}_+ : |z - \lambda_n| < r \operatorname{Im} \lambda_n\},$$

for sufficiently small $r > 0$. Indeed, we know [19] that if $(\kappa^b_{\lambda_n})_{n\geq 1}$ is a Riesz basis in $\mathcal{H}(b)$, then $(\lambda_n)_{n\geqslant 1}$ is a Carleson sequence, that is,

$$\inf_{k\geqslant 1}\prod_{n\neq k}\left|\frac{\lambda_n-\lambda_k}{\lambda_n-\overline{\lambda}_k}\right|>0.$$

In particular, the measure $\nu := \sum_n \operatorname{Im}\lambda_n\,\delta_{\lambda_n}$ is a Carleson measure. Therefore, we see that G_n satisfy (ii). Moreover, using Lemma 7.3 below, we see that G_n satisfy also the condition (i).

Recall that $w_p(z) = \min(\|(k^b_z)^2\|_q^{-p/(p+1)}, \|\rho^{1/q}\mathfrak{K}^\rho_{z,1}\|_q^{-p/(p+1)})$.

Theorem 7.2. *Let $(\lambda_n)_{n\geq 1} \subset \mathbb{C}_+ \cup E_2(b)$ be such that $(\kappa^b_{\lambda_n})_{n\geq 1}$ is a Riesz basis in $\mathcal{H}(b)$ and let $p \in [1, 2)$. Then for any set $G = \bigcup_n G_n$ satisfying* (i) *and* (ii)*, there is $\varepsilon > 0$ such that the system of reproducing kernels $(\kappa^b_{\mu_n})_{n\geq 1}$ is a Riesz basis whenever $\mu_n \in G_n$ and*

$$\sup_{n\geq 1}\frac{1}{\|k^b_{\lambda_n}\|_b^2}\int_{[\lambda_n,\mu_n]} w_p(z)^{-2}|dz| < \varepsilon. \tag{7.1}$$

Proof. Since $\mu_n \in G_n$, the condition (i) implies that $\|k^b_{\mu_n}\|_b \asymp \|k^b_{\lambda_n}\|_b$ and thus $(\kappa^b_{\mu_n})_{n\geq 1}$ is a Riesz basis if and only if $(\widetilde{\kappa}^b_{\mu_n})_{n\geq 1}$ is a Riesz basis where

$$\widetilde{\kappa}^b_{\mu_n} = \frac{k^b_{\mu_n}}{2\pi i\,\|k^b_{\lambda_n}\|_b}.$$

In view of [7, Lemma 2.3], it suffices to check the estimate

$$\sum_{n=1}^{\infty}|\langle f, \kappa^b_{\lambda_n} - \widetilde{\kappa}^b_{\mu_n}\rangle_b|^2 \leq \varepsilon\|f\|_b^2, \qquad f \in \mathcal{H}(b), \tag{7.2}$$

for sufficiently small $\varepsilon > 0$. Now it follows from (7.1) and Corollary 5.4 (a) that any f in $\mathcal{H}(b)$ is differentiable in $]\lambda_n, \mu_n[$. Moreover, the set of functions in $\mathcal{H}(b)$ which are continuous on $[\lambda_n, \mu_n]$ is dense in $\mathcal{H}(b)$ (take the set of reproducing kernels). Therefore, we can prove (7.2) only for functions $f \in \mathcal{H}(b)$ continuous on $[\lambda_n, \mu_n]$. Then

$$|\langle f, \kappa^b_{\lambda_n} - \widetilde{\kappa}^b_{\mu_n}\rangle_b|^2 = \frac{|f(\lambda_n) - f(\mu_n)|^2}{\|k^b_{\lambda_n}\|_b^2} = \frac{1}{\|k^b_{\lambda_n}\|_b^2}\left|\int_{[\lambda_n,\mu_n]} f'(z)\,dz\right|^2.$$

By the Cauchy–Schwartz inequality and (7.1), we get

$$|\langle f, \kappa^b_{\lambda_n} - \widetilde{\kappa}^b_{\mu_n}\rangle_b|^2 \le \varepsilon \int_{[\lambda_n,\mu_n]} |f'(z)w_p(z)|^2|dz|.$$

It follows from assumption (ii) that $\nu := \sum_n \delta_{[\lambda_n,\mu_n]}$ is a Carleson measure with a constant C_ν which does not exceed some absolute constant depending only on G. Hence, according to Theorem 4.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty} |\langle f, \kappa^b_{\lambda_n} - \widetilde{\kappa}^b_{\mu_n}\rangle_b|^2 &\le \varepsilon \sum_{n=1}^{\infty} \int_{[\lambda_n,\mu_n]} |f'(z)w_p(z)|^2|dz| \\ &= \varepsilon \|f'w_p\|^2_{L^2(\nu)} \le C\,\varepsilon\,\|f\|^2_b,\end{aligned}$$

for a constant C which depends on G, (λ_n) and p. Then Lemma 2.3 of [7] implies that we can choose a sufficiently small $\varepsilon > 0$ such that $(\widetilde{\kappa}^b_{\mu_n})_{n\ge 1}$ is a Riesz basis in $\mathcal{H}(b)$.

□

Denote by $\rho(z,\omega)$ the pseudohyperbolic distance between z and ω,

$$\rho(z,\omega) := \left|\frac{z-\omega}{z-\overline{\omega}}\right|.$$

For the proof of the next corollary we need the following well-known property.

Lemma 7.3. *Let $b \in H^\infty(\mathbb{C}_+)$ with $\|b\|_\infty \le 1$ and $\varepsilon_0 \in (0,1)$. Then there exist constants $C_1, C_2 > 0$ (depending only on ε_0) such that for any $z,\omega \in \mathbb{C}_+$ satisfying $\rho(z,\omega) < \varepsilon_0$, we have*

$$C_1 \le \frac{1-|b(z)|}{1-|b(\omega)|} \le C_2. \tag{7.3}$$

Proof. For the case of an inner function, the proof can be found, e.g., in [7, Lemma 4.1]. Since for $0 \le t_1, t_2, s_1, s_2 < 1$, we have

$$\frac{1-t_1t_2}{1-s_1s_2} \le \frac{1-t_1}{1-s_1} + \frac{1-t_2}{1-s_2},$$

the inner and outer factors of b can be treated separately and we can assume that b is outer. It follows easily from $\rho(z,\omega) < \varepsilon_0$ that

$$|z-\omega| < \frac{2\varepsilon_0}{1-\varepsilon_0} \operatorname{Im}\omega \tag{7.4}$$

and

$$\frac{1-\varepsilon_0}{1+\varepsilon_0} < \frac{\operatorname{Im} z}{\operatorname{Im}\omega} < \frac{1+\varepsilon_0}{1-\varepsilon_0}.$$

Hence

$$\frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt \asymp \frac{\operatorname{Im}\omega}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-\omega|^2}\,dt.$$

Since b is outer, we have

$$\log|b(z)| = -\frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}} \frac{\big|\log|b(t)|\big|}{|t-z|^2}\,dt \asymp \log|b(\omega)|, \tag{7.5}$$

which implies $1-|b(z)| \asymp 1-|b(\omega)|$.

□

Corollary 7.4. *Let $(\lambda_n) \subset \mathbb{C}_+$, let $(\kappa^b_{\lambda_n})_{n\ge1}$ be a Riesz basis in $\mathcal{H}(b)$, and let $\gamma > 1/3$. Then there is $\varepsilon > 0$ such that the system $(\kappa^b_{\mu_n})_{n\ge1}$ is a Riesz basis whenever*

$$\left|\frac{\lambda_n-\mu_n}{\lambda_n-\overline{\mu_n}}\right| \le \varepsilon(1-|b(\lambda_n)|)^\gamma. \tag{7.6}$$

Proof. By Remark 7.1, for sufficiently small $r>0$, the sets $G_n = \{z : |z-\lambda_n| \le r \operatorname{Im}\lambda_n\}$ satisfy the conditions (i) and (ii). Let $(\mu_n)_{n\ge1}$ satisfy (7.6). Then, by (7.4), we have

$$|\lambda_n-\mu_n| \le \frac{2\varepsilon}{1-\varepsilon}(1-|b(\lambda_n)|)^\gamma \operatorname{Im}\lambda_n. \tag{7.7}$$

Therefore, if ε is sufficiently small, then $\mu_n \in G_n$. Without loss of generality, we can assume that $\gamma<1$ and since $\gamma>1/3$, there exists $1<p<2$ such that $2\frac{p-1}{p+1} = 1-\gamma$. Let q be the conjugate exponent of p and note that $\frac{2p}{q(p+1)} = 1-\gamma$.

Then it follows from Lemma 3.5 that there is a constant $C = C(p) > 0$ such that

$$w_p(z) \ge C\frac{\operatorname{Im} z}{(1-|b(z)|)^{\frac{p}{q(p+1)}}}, \qquad z \in \mathbb{C}_+.$$

Therefore, by Lemma 7.3, we have

$$w_p^{-2}(z) \le C_1 \frac{(1-|b(\lambda_n)|)^{1-\gamma}}{(\operatorname{Im}\lambda_n)^2}$$

for $z \in [\lambda_n, \mu_n]$. Hence,

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n,\mu_n]} w_p(z)^{-2}|dz| \le C_2 \frac{\operatorname{Im}\lambda_n}{1-|b(\lambda_n)|} |\lambda_n - \mu_n| \frac{(1-|b(\lambda_n)|)^{1-\gamma}}{(\operatorname{Im}\lambda_n)^2}$$

and using (7.7), we obtain

$$\frac{1}{\|k^b_{\lambda_n}\|_b^2} \int_{[\lambda_n,\mu_n]} w_p(z)^{-2}|dz| \le C_3\varepsilon.$$

To complete the proof, take a sufficiently small ε and apply Theorem 7.2.

□

Remark 7.5. It should be noted that all the statements remain valid if we are interested in the stability of Riesz sequences of reproducing kernels, that is, of systems of reproducing kernels which constitute Riesz bases in their closed linear spans.

Remark 7.6. In the case where

$$\sup_{n\ge 1} |b(\lambda_n)| < 1, \tag{7.8}$$

the stability condition (7.6) is equivalent to

$$\left| \frac{\lambda_n - \mu_n}{\lambda_n - \overline{\mu_n}} \right| \le \varepsilon,$$

and we essentially get the result of stability obtained in the inner case in [18]. Moreover, if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$ and if (7.8) is satisfied, then a criterion for $(\kappa^n_{\lambda_n})$ to be a Riesz basis of $\mathcal{H}(b)$ is given in [19]. On the other hand, in the non-extreme case, there are no Riesz bases of $\mathcal{H}(b)$ and the previous results (Theorem 7.2 and Corollary 7.4) apply only for Riesz sequences.

References

[1] P.R. Ahern, D.N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.

[2] P.R. Ahern, D.N. Clark, *Radial nth derivatives of Blaschke products*, Math. Scand. **28** (1971), 189–201.

[3] A.B. Aleksandrov, *Invariant subspaces of shift operators. An axiomatic approach*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 7-26; English transl.: J. Soviet Math **22** (1983), 115–129.

[4] J.M. Anderson, J. Rovnyak, *On generalized Schwarz–Pick estimates*, Mathematika **53** (2006), no. 1, 161–168.

[5] A.D. Baranov, *Weighted Bernstein-type inequalities and embedding theorems for shift-coinvariant subspaces*, Algebra i Analiz **15** (2003), no. 5, 138–168; English transl.: St. Petersburg Math. J. **15** (2004), no. 5, 733–752.

[6] A.D. Baranov, *Bernstein-type inequalities for the shift-coinvariant subspaces and their applications to Carleson embeddings*, J. Funct. Anal. **223** (2005), 116–146.

[7] A.D. Baranov, *Stability of bases and frames of reproducing kernels in model subspaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), 2399–2422.

[8] A.D. Baranov, *Compact and Schatten class embeddings of star-invariant subspaces in the disc*, arXiv:0712.0684v1 [math.CV].

[9] R.P. Boas, A.C. Schaeffer, *Variational methods in entire functions*, Amer. J. Math. **79** (1957), 857–884.

[10] J.A. Cima, A.L. Matheson, *On Carleson embeddings of star-invariant subspaces*, Quaest. Math. **26** (2003), 3, 279–288.

[11] W.S. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. **103** (1982), 347–364.

[12] W. S. Cohn, *Radial limits and star invariant subspaces of bounded mean oscillation*, Amer. J. Math. **108** (1986), 719–749.

[13] W.S. Cohn, *Carleson measures and operators on star-invariant subspaces*, J. Oper. Theory, **15** (1986), no. 1, 181–202.

[14] L. de Branges, J. Rovnyak, *Canonical models in quantum scattering theory*, pp. 295-392 in: Perturbation theory and its application in quantum mecanics, Madison, 1965, ed. C.H.Wilcox, Wiley, N.Y., 1966.

[15] L. de Branges, J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, N.Y., 1966.

[16] K.M. Dyakonov, *Entire functions of exponential type and model subspaces in H^p*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) **190** (1991), 81–100; Engl. transl.: J. Math. Sci. **71** (1994), 2222–2233.

[17] K.M. Dyakonov, *Differentiation in star-invariant subspaces I: boudedness and compactness*, J. Funct. Anal. **192** (2002), 364–386.

[18] E. Fricain, *Bases of reproducing kernels in model spaces*, J. Oper. Theory, **46** (2001), 3 (suppl.), 517–543.

[19] E. Fricain, *Bases of reproducing kernels in de Branges spaces*, J. Funct. Anal. **226** (2005), 2, 373–405.

[20] E. Fricain, J. Mashreghi, *Boundary behavior of functions of the de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory, to appear.

[21] E. Fricain, J. Mashreghi, *Integral representation of the n-th derivative in de Branges-Rovnyak spaces and the norm convergence of its reproducing kernel*, Annales de l'Institut Fourier, to appear.

[22] E.A. Gorin, *Bernstein inequalities from the operator theory point of view*, Vestni Kharkov. Univ. Prikl. Mat. Mekh. **45** (1980), 77–105; English transl.: Selecta Math. Soviet. **7** (1988).

[23] A. Hartmann, D. Sarason, K. Seip, *Surjective Toeplitz operators*, Acta Sci. Math. (Szeged) **70** (2004), no. 3-4, 609–621.

[24] S. Hruscev, N. Nikolski, B. Pavlov, *Unconditional Bases of Exponentials and Reproducing Kernels*, Vol. 864, Lecture Notes Math., Springer Berlin-Heidelberg, (1981), 214–335.

[25] M.I. Kadec, *The exact value of the Paley–Wiener constant*, Dokl. Akad. Nauk. SSSR, **155** (1964), 1253–1254; English transl.: Sov. Math. Dokl., **5** (1964), 559–561.

[26] B.Ya. Levin, *Lectures on entire functions*, Transl. Math. Monogr., Vol. 150, AMS, Providence, RI, 1996.

[27] M.B. Levin, *An estimate of the derivative of a meromorphic function on the boundary of domain*, Soviet Math. Dokl. **15** (1974), 831–834.

[28] F. Nazarov, A. Volberg, *The Bellman function, the two-weight Hilbert transform, and embeddings of the model space K_Θ*, J. Anal. Math. **87** (2002), 385-414.

[29] N.K. Nikolski, *Operators, Functions, and Systems: an Easy Reading*, Math. Surveys Monogr., Vol. 92-93, AMS, Providence, RI, 2002.

[30] R.E. Paley, N. Wiener, *Fourier Transforms in the Complex Domain*, Amer. Math. Soc. Colloq. Publ., Vol. 19, Providence, 1934.

[31] B.S. Pavlov, *Bases of exponentials and the Muchenhoupt condition*, Doklady Akad. Nauk SSSR **247** (1979), no. 1, 37–40; English transl.: Soviet Math. Dokl. **20** (1979), no. 4, 655–659.

[32] A. Petrosyan, *Some extremal problems for analytic functions*, Complex Variables Theory Appl. **39** (1999), no. 2, 137–159.

[33] S.C. Power, *Vanishing Carleson measures,* Bull. Lond. Math. Soc. **12** (1980), 207–210.

[34] Q.I. Rahman, G. Schmeisser, *L^p inequalities for entire functions of exponential type*, Trans. Amer. Math. Soc. **320** (1990), no. 1, 91–103.

[35] Q.I. Rahman, Q.M. Tariq, *On Bernstein's inequality for entire functions of exponential type*, Comput. Methods Funct. Theory **7** (2007), no. 1, 167–184.

[36] R.M. Redheffer, *Completeness of sets of complex exponentials*, Adv. Math. **24** (1977), no. 1, 1–62.

[37] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, University of Arkansas Lecture Notes in the Mathematical Sciences 10, John Wiley & Sons Inc., New York, 1994.

[38] J.E. Shapiro, *Relative angular derivatives*, J. Operator Theory **46** (2001), no. 2, 265–280.

[39] J.E. Shapiro, *More relative angular derivatives*, J. Operator Theory **49** (2003), no. 1, 85–97.

[40] V. Totik, *Derivatives of entire functions of higher order*, J. Approx. Theory **64** (1991), no. 2, 209–213.

[41] A.L. Volberg, *Thin and thick families of rational fractions*, Lect. Notes in Math. **864** (1981), 440-481.

[42] A.L. Volberg, S.R. Treil, *Embedding theorems for invariant subspaces of the inverse shift operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **149** (1986), 38–51; English transl.: J. Soviet Math. **42** (1988), 1562–1572.

Department of Mathematics and Mechanics, St. Petersburg State University, 28, Universitetskii pr., St. Petersburg, 198504, Russia

E-mail address: A.Baranov@ev13934.spb.edu

Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne

E-mail address: fricain@math.univ-lyon1.fr

Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4.

E-mail address: Javad.Mashreghi@mat.ulaval.ca

6.3 Annexes sur le chapitre "Produits de Blaschke"

6.3.1 Référence [T8]

Auteurs
E. Fricain et J. Mashreghi

Titre
Integral means of the derivatives of Blaschke products.

Paru dans
Glasgow Mathematical Journal, 50 (2008), no. 2, 233–249.

INTEGRAL MEANS OF THE DERIVATIVES OF BLASCHKE PRODUCTS

EMMANUEL FRICAIN, JAVAD MASHREGHI

Abstract. We study the rate of growth of some integral means of the derivatives of a Blaschke product and we generalize several classical results. Moreover, we obtain the rate of growth of integral means of the derivative of functions in the model subspace K_B generated by the Blaschke product B.

1. Introduction

Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc satisfying the Blaschke condition

$$\sum_{n=1}^{\infty}(1-|z_n|)<\infty. \tag{1.1}$$

Then, the product

$$B(z)=\prod_{n=1}^{\infty}\frac{|z_n|}{z_n}\,\frac{z_n-z}{1-\bar{z}_n\,z}$$

is a bounded analytic function on the unit disc $\mathbb{D}$ with zeros only at the points z_n, $n\geq 1$, [5, page 20]. Since the product converges uniformly on compact subsets of $\mathbb{D}$, the logarithmic derivative of B is given by

$$\frac{B'(z)}{B(z)}=\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{(1-\bar{z}_n\,z)(z-z_n)}, \qquad (z\in\mathbb{D}).$$

2000 *Mathematics Subject Classification.* Primary: 30D50, Secondary: 32A70.

Key words and phrases. Blaschke products, model space.

This work was supported by NSERC (Canada) and FQRNT (Québec). A part of this work was done while the first author was visiting McGill University. He would like to thank this institution for its warm hospitality.

Therefore,

$$|B'(re^{i\theta})| \leq \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|\,1-\bar{z}_n\, re^{i\theta}\,|^2}, \qquad (re^{i\theta} \in \mathbb{D}). \tag{1.2}$$

If (1.1) is the only restriction we put on the zeros of B, we can only say that

$$\begin{aligned} \int_0^{2\pi} |B'(re^{i\theta})|\, d\theta &\leq \sum_{n=1}^{\infty} (1-|z_n|^2) \int_0^{2\pi} \frac{d\theta}{|\,1-\bar{z}_n\, re^{i\theta}\,|^2} \\ &= \sum_{n=1}^{\infty} (1-|z_n|^2)\, \frac{2\pi}{(1-|z_n|^2 r^2)} \\ &\leq \frac{4\pi \sum_{n=1}^{\infty}(1-|z_n|)}{(1-r)}, \end{aligned}$$

which implies

$$\int_0^{2\pi} |B'(re^{i\theta})|\, d\theta = \frac{o(1)}{1-r}, \qquad (r \to 1). \tag{1.3}$$

However, assuming stronger restrictions on the rate of increase of the zeros of B give us more precise estimates about the rate of increase of integral means of B'_r as $r \to 1$. The most common restriction is

$$\sum_{n=1}^{\infty} (1-|z_n|)^{\alpha} < \infty \tag{1.4}$$

for some $\alpha \in (0,1)$. Protas [15] took the first step in this direction by proving the following results.

Let us mention that H^p, $0 < p < \infty$, stands for the classical Hardy space equipped with the norm

$$\|f\|_p = \lim_{r \to 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \ \frac{d\theta}{2\pi} \right)^{\frac{1}{p}},$$

and its cousin A^p_γ, $0 < p < \infty$ and $\gamma > -1$, stands for the (weighted) Bergman space equipped with the norm

$$\|f\|_{p,\gamma} = \left(\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^p \ \frac{r(1-r^2)^{\gamma} dr\ d\theta}{\pi/(1+\gamma)} \right)^{\frac{1}{p}}.$$

Theorem 1.1 (Protas)**.** *If $0 < \alpha < \frac{1}{2}$ and the Blaschke sequence $(z_n)_{n\geq 1}$ satisfies (1.4), then $B' \in H^{1-\alpha}$.*

Theorem 1.2 (Protas)**.** *If $0 < \alpha < 1$ and the Blaschke sequence $(z_n)_{n\geq 1}$ satisfies (1.4), then $B' \in A^1_{\alpha-1}$.*

Then, Ahern and Clark [1] showed that Theorem 1.1 is sharp in the sense that B' need not lie in any H^p with $p > 1-\alpha$. Later on, they also showed that the condition $\sum_{n=1}^{\infty}(1-|z_n|)^{1/2} < \infty$ is not enough to imply that $B' \in H^{1/2}$ [2]. At the same time, Linden [12] generalized Theorem 1.1 for higher derivatives of B. In the converse direction, Ahern and Clark [1] also obtained the following result.

Theorem 1.3 (Ahern–Clark)**.** *If $\frac{1}{2} < p < 1$, then there is a Blaschke product B with $B' \in H^p$, and such that its zeros satisfies*

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} = \infty$$

for all α with $0 < \alpha < (1-p)$.

However, Cohn [3] proved that for interpolating sequences the two conditions are equivalent.

Theorem 1.4 (Cohn)**.** *Let $0 < \alpha < \frac{1}{2}$, and let $(z_n)_{n\geq 1}$ be an interpolating Blaschke sequence. Then, $B' \in H^{1-\alpha}$ if and only if $(z_n)_{n\geq 1}$ satisfies (1.4).*

Recently, Kutbi [11] showed that under the hypothesis of Theorem 1.1,

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{o(1)}{(1-r)^{p+\alpha-1}}, \qquad (r \to 1), \tag{1.5}$$

for any $p > 1-\alpha$. In particular, for $p = 1$, we have

$$\int_0^{2\pi} |B'(re^{i\theta})| \, d\theta = \frac{o(1)}{(1-r)^{\alpha}}, \qquad (r \to 1),$$

which is a refinement of (1.3).

Then, Protas [16] proved that the estimate (1.5) is still valid if $1/2 < \alpha \leq 1$, $p \geq \alpha$ and the Blaschke sequence $(z_n)_{n\geq 1}$ satisfies (1.4). Finally, Gotoh [7] got an extension of Protas's results for higher derivatives of B.

A Blaschke sequence which satisfies the Carleson condition is called an interpolation, or Carleson, Blaschke sequence [10, page 200]. Let I be an inner function for the unit disc. In particular, I could be any Blaschke product. Then,

$$K_I := H^2 \ominus IH^2$$

is called the model subspace of H^2 generated by the inner function I [6, 8]. Cohn [3] obtained the following result about the derivative of functions in K_B.

Theorem 1.5 (Cohn)**.** *Let* $(z_n)_{n\geq 1}$ *be an interpolating Blaschke sequence, and let* $p \in (2/3, 1)$*. Then,* $B' \in H^p$ *if and only if* $f' \in H^{2p/(p+2)}$ *for all* $f \in K_B$*.*

In this paper, we replace the condition (1.4) by a more general assumption

$$\sum_{n=1}^{\infty} h(1 - |z_n|) < \infty, \tag{1.6}$$

where h is a positive continuous function satisfying certain smoothness conditions, and then we generalize all the preceding results. Since our sequence already satisfies the Blaschke condition, (1.6) will provide further information about the rate of increase of the zeros only if $h(t) \geq t$ as $t \to 0$.

In particular, we are interested in

$$h(t) = t^{\alpha} \ (\log 1/t)^{\alpha_1} \ (\log_2 1/t)^{\alpha_2} \ \cdots \ (\log_n 1/t)^{\alpha_n}, \tag{1.7}$$

where $\alpha \in (0, 1)$, $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$, and $\log_n = \log\log\cdots\log$ (n times) [13].

In the following, we will use the estimates

$$\int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\nu}} \asymp \frac{1}{(1-r)^{\nu-1}}, \qquad (\nu > 1),$$

$$\int_0^1 \int_0^{2\pi} \frac{(1-\rho^2)^{\gamma}}{|1 - r\rho e^{i\theta}|^{\nu}} \ \rho d\rho d\theta \asymp \frac{1}{(1-r)^{\nu-\gamma-2}}, \qquad (\nu - 2 > \gamma > -1),$$

as $r \to 1^-$. See [9, page 7]. Both relations can be proved using the fact that $|1 - re^{i\theta}| \asymp (1-r) + |\theta|$ as $r \to 1^-$.

2. An estimation lemma

In the following we assume that h is a continuous positive function defined on the interval $(0,1)$ with

$$\lim_{t\to 0^+} h(t) = 0.$$

Our prototype is the one given in (1.7). The following lemma has simple assumptions and also a very simple proof. However, it has many interesting applications in the rest of the paper.

Lemma 2.1. *Let $(r_n)_{n\ge 1}$ be a sequence in the interval $(0,1)$ such that*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty.$$

Let $p, q > 0$ be such that $h(t)/t^p$ is decreasing and $h(t)/t^{p-q}$ is increasing on $(0,1)$. Then,

$$\sum_{n=1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^q} = \frac{O(1)}{(1-r)^{q-p}\, h(1-r)}$$

as $r \to 1^-$. Moreover, if

$$\lim_{t\to 0^+} \frac{h(t)}{t^{p-q}} = 0,$$

then

$$\sum_{n=1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^q} = \frac{o(1)}{(1-r)^{q-p}\, h(1-r)}.$$

Proof. We have

$$\frac{(1-r_n)^p}{(1-rr_n)^q} = \left(\frac{(1-r_n)^p}{h(1-r_n)} \, \frac{h(1-rr_n)}{(1-rr_n)^p} \right) \left(\frac{h(1-r_n)}{(1-rr_n)^{q-p}\, h(1-rr_n)} \right).$$

By assumption

$$\frac{h(1-rr_n)}{(1-rr_n)^p} \le \frac{h(1-r_n)}{(1-r_n)^p},$$

and

$$(1-rr_n)^{q-p}\, h(1-rr_n) \ge (1-r)^{q-p}\, h(1-r).$$

Thus, for any $n \geq 1$,

$$\frac{(1-r_n)^p}{(1-rr_n)^q} \leq \frac{h(1-r_n)}{(1-r)^{q-p}\,h(1-r)}. \tag{2.1}$$

Given $\varepsilon > 0$, fix N such that

$$\sum_{n=N+1}^{\infty} h(1-r_n) < \varepsilon.$$

Hence, by (2.1),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^q} &= \sum_{n=1}^{N} \frac{(1-r_n)^p}{(1-rr_n)^q} + \sum_{n=N+1}^{\infty} \frac{(1-r_n)^p}{(1-rr_n)^q} \\ &\leq \sum_{n=1}^{N} (1-r_n)^{p-q} + \frac{\sum_{n=N+1}^{\infty} h(1-r_n)}{(1-r)^{q-p}\,h(1-r)} \\ &\leq C_N + \frac{\varepsilon}{(1-r)^{q-p}\,h(1-r)}, \end{aligned}$$

where C_N is independent of r. This inequality implies both assertions of the Lemma.

□

The Lemma is still valid if instead of “decreasing” and “increasing”, we assume that our functions are respectively “boundedly decreasing” and “boundedly increasing”. We say that φ is boundedly increasing if there is a constant $C > 0$ such that $\varphi(x) \leq C\varphi(y)$ whenever $x \leq y$. Similarly, φ is boundedly decreasing if there is a constant $C > 0$ such that $\varphi(x) \geq C\varphi(y)$ whenever $x \leq y$.

3. H^p MEANS OF THE FIRST DERIVATIVE

In this section we apply Lemma 2.1 to obtain a general estimate for the integral means of the first derivative of a Blaschke product. Special cases of the following theorem generalize Protas and Kutbi's results.

Theorem 3.1. *Let B be the Blaschke product formed with zeros $z_n = r_n e^{i\theta_n}$, $n \geq 1$, satisfying*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty$$

for a positive continuous function h. Suppose that there is $q \in (1/2, 1]$ such that $h(t)/t^q$ is decreasing and $h(t)/t^{1-q}$ is increasing on $(0, 1)$. Then, for any $p \geq q$,

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{O(1)}{(1-r)^{p-1}\, h(1-r)}, \qquad (r \to 1).$$

Moreover, if $\lim_{t\to 0} h(t)/t^{1-q} = 0$, then $O(1)$ can be replaced by $o(1)$.

Proof. Since $q \leq 1$, (1.2) implies

$$|B'(re^{i\theta})|^q \leq \sum_{n=1}^{\infty} \frac{(1-r_n^2)^q}{|\,1 - rr_n e^{i(\theta-\theta_n)}\,|^{2q}}.$$

Hence

$$\int_0^{2\pi} |B'(re^{i\theta})|^q \, d\theta \leq C \sum_{n=1}^{\infty} \frac{(1-r_n)^q}{(1-rr_n)^{2q-1}}. \tag{3.1}$$

(Here we used $2q > 1$.) Therefore, by Lemma 2.1,

$$\int_0^{2\pi} |B'(re^{i\theta})|^q \, d\theta \leq \frac{C}{(1-r)^{q-1}\, h(1-r)}.$$

Now recall that any function f in H^∞ is in the Bloch space $\mathcal{B}$ [4, page 44], that is

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < +\infty\,.$$

Hence, for any $p \geq q$,

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta \leq \frac{1}{(1-r)^{p-q}} \int_0^{2\pi} |B'(re^{i\theta})|^q \; d\theta \leq \frac{C}{(1-r)^{p-1}\, h(1-r)}.$$

Finally, as $r \to 1$, Lemma 2.1 also assures that C can be replaced by any small positive constant if $\lim_{t\to 0} h(t)/t^{1-q} = 0$. □

Now, we can apply Theorem 3.1 for the special function defined in (1.7).

Case I: If

$$\sum_{n=1}^{\infty} (1-r_n)^\alpha (\log \frac{1}{1-r_n})^{\alpha_1} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

then, for any

$$p > \max\{\alpha, 1-\alpha\}$$

we have

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{o(1)}{(1-r)^{\alpha+p-1}(\log \frac{1}{1-r})^{\alpha_1} \cdots (\log_m \frac{1}{1-r})^{\alpha_m}}, \qquad (r \to 1).$$

In particular, if

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha} < \infty,$$

with $\alpha \in (0, 1/2)$, then, for any $p > 1 - \alpha$,

$$\int_0^{2\pi} |B'(re^{i\theta})|^p \, d\theta = \frac{o(1)}{(1-r)^{p+\alpha-1}}, \qquad (r \to 1),$$

which is Kutbi's result. Moreover, if $\alpha \in [1/2, 1)$, the last estimate still holds for any $p > \alpha$, which is Protas's result [16].

Case II: If

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha} (\log_k \frac{1}{1-r_n})^{\alpha_k} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $\alpha \in (0, 1/2)$, $\alpha_k < 0$ and $\alpha_{k+1}, \cdots, \alpha_m \in \mathbb{R}$, then,

$$\int_0^{2\pi} |B'(re^{i\theta})|^{1-\alpha} \, d\theta = \frac{o(1)}{(\log_k \frac{1}{1-r})^{\alpha_k} \cdots (\log_m \frac{1}{1-r})^{\alpha_m}}, \qquad (r \to 1).$$

But, if

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha} < \infty,$$

with $\alpha \in (0, 1/2)$, then

$$\int_0^{2\pi} |B'(re^{i\theta})|^{1-\alpha} \, d\theta = O(1), \qquad (r \to 1),$$

i.e. $B' \in H^{1-\alpha}$, which is Protas' result [15].

Case III: If

$$\sum_{n=1}^{\infty} (1-r_n)^{\alpha} (\log_k \frac{1}{1-r_n})^{\alpha_k} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $\alpha \in (1/2, 1)$, $\alpha_k > 0$ and $\alpha_{k+1}, \cdots, \alpha_n \in \mathbb{R}$, then,

$$\int_0^{2\pi} |B'(re^{i\theta})|^{\alpha} \, d\theta = \frac{o(1)}{(1-r)^{2\alpha-1}(\log \frac{1}{1-r})^{\alpha_1} \cdots (\log_m \frac{1}{1-r})^{\alpha_m}}, \qquad (r \to 1).$$

However, if

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}<\infty,$$

with $\alpha\in(1/2,1)$, then we still have

$$\int_0^{2\pi}|B'(re^{i\theta})|^{\alpha}\,d\theta=\frac{o(1)}{(1-r)^{2\alpha-1}},\qquad (r\to 1).$$

4. H^p MEANS OF HIGHER DERIVATIVES

Straightforward calculation leads to

$$\int_0^{2\pi}|B^{(\ell)}(re^{i\theta})|^p\,d\theta\le C(p,\ell)\sum_{n=1}^{\infty}\frac{(1-r_n)^p}{(1-rr_n)^{(\ell+1)p-1}},\qquad (\frac{1}{\ell+1}<p\le\frac{1}{\ell}), \tag{4.1}$$

which is a generalization of (3.1). This observation along with Lemma 2.1 enable us to generalize the results of the preceding section for higher derivatives of a Blaschke product. The proof is similar to that of Theorem 3.1.

Theorem 4.1. *Let B be the Blaschke product formed with zeros $z_n=r_ne^{i\theta_n}$, $n\ge 1$, satisfying*

$$\sum_{n=1}^{\infty}h(1-r_n)<\infty$$

for a positive continuous function h. Suppose that there is $q\in(1/(\ell+1),1/\ell]$ such that $h(t)/t^q$ is decreasing and $h(t)/t^{1-\ell q}$ is increasing on $(0,1)$. Then, for any $p\ge q$,

$$\int_0^{2\pi}|B^{(\ell)}(re^{i\theta})|^p\,d\theta=\frac{O(1)}{(1-r)^{\ell p-1}\,h(1-r)},\qquad (r\to 1).$$

Moreover, if $\lim_{t\to 0}h(t)/t^{1-\ell q}=0$, then $O(1)$ can be replaced by $o(1)$.

Now, we can apply Theorem 4.1 for the special function defined in (1.7).

Case I: If

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}(\log\frac{1}{1-r_n})^{\alpha_1}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m}<\infty,$$

then, for any

$$p>\max\{\alpha,(1-\alpha)/\ell\}$$

we have

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^p\, d\theta = \frac{o(1)}{(1-r)^{\alpha+\ell p-1}(\log\frac{1}{1-r})^{\alpha_1}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}}, \qquad (r\to 1).$$

In particular, if

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}<\infty,$$

with $\alpha\in(0,1/(\ell+1))$, then, for any $p>(1-\alpha)/\ell$,

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^p\, d\theta = \frac{o(1)}{(1-r)^{\ell p+\alpha-1}}, \qquad (r\to 1),$$

which is Kutbi's result. Moreover, if $\alpha\in[1/(\ell+1),1)$, the last estimate still holds for any $p>\alpha$, which is Gotoh's result [7] .

Case II: If

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}(\log_k\frac{1}{1-r_n})^{\alpha_k}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m}<\infty,$$

with $\alpha\in(0,1/(1+\ell))$, $\alpha_k<0$ and $\alpha_{k+1},\cdots,\alpha_n\in\mathbb{R}$, then,

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^{(1-\alpha)/\ell}\, d\theta = \frac{o(1)}{(\log_k\frac{1}{1-r})^{\alpha_k}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}}, \qquad (r\to 1).$$

But, if

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}<+\infty,$$

with $\alpha\in(0,1/(1+\ell))$, then

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^{(1-\alpha)/\ell}\, d\theta = O(1), \qquad (r\to 1),$$

i.e. $B^{(\ell)}\in H^{(1-\alpha)/\ell}$ which is Linden's result [12].

Case III: If

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}(\log_k\frac{1}{1-r_n})^{\alpha_k}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m}<\infty,$$

with $\alpha\in(1/(1+\ell),1)$, $\alpha_k>0$ and $\alpha_{k+1},\cdots,\alpha_n\in\mathbb{R}$, then,

$$\int_0^{2\pi} |B^{(\ell)}(re^{i\theta})|^{\alpha}\, d\theta = \frac{o(1)}{(1-r)^{(\ell+1)\alpha-1}(\log_k\frac{1}{1-r})^{\alpha_k}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}}, \qquad (r\to 1).$$

However, if

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}<\infty,$$

with $\alpha \in (1/(\ell+1),1)$, then we still have

$$\int_0^{2\pi}|B^{(\ell)}(re^{i\theta})|^{\alpha}\,d\theta=\frac{o(1)}{(1-r)^{(\ell+1)\alpha-1}},\qquad (r\to 1).$$

5. A^p_γ MEANS OF THE FIRST DERIVATIVE

In this section we apply Lemma 2.1 to obtain a general estimate for the integral means of the first derivative of a Blaschke product. Special cases of the following theorem generalize Protas's results [15].

Theorem 5.1. *Let B be the Blaschke product formed with zeros $z_n = r_n e^{i\theta_n}$ satisfying*

$$\sum_{n=1}^{\infty}h(1-r_n)<\infty$$

for a positive continuous function h. Let $\gamma \in (-1,0)$. Suppose that there is $q \in (1+\gamma/2,1]$ such that $h(t)/t^q$ is decreasing and $h(t)/t^{2+\gamma-q}$ is increasing on $(0,1)$. Then, for any $p \geq q$,

$$\int_0^1\int_0^{2\pi}|B'(r\rho e^{i\theta})|^p\ \rho(1-\rho^2)^{\gamma}d\rho\ d\theta=\frac{O(1)}{(1-r)^{p-\gamma-2}\,h(1-r)},\qquad (r\to 1).$$

Moreover, if $\lim_{t\to 0}h(t)/t^{2+\gamma-q}=0$, then $O(1)$ can be replaced by $o(1)$.

Proof. We saw that

$$|B'(r\rho e^{i\theta})|^q\leq\sum_{n=1}^{\infty}\frac{(1-r_n^2)^q}{|\,1-rr_n\rho e^{i(\theta-\theta_n)}\,|^{2q}}.$$

Hence

$$\int_0^1\int_0^{2\pi}|B'(r\rho e^{i\theta})|^q\ \rho(1-\rho^2)^{\gamma}d\rho\ d\theta\leq C\sum_{n=1}^{\infty}\frac{(1-r_n)^q}{(1-rr_n)^{2q-\gamma-2}}. \tag{5.1}$$

(Here we used $2q-\gamma-2>0$.) Therefore, by Lemma 2.1,

$$\int_0^1\int_0^{2\pi}|B'(r\rho e^{i\theta})|^q\ \rho(1-\rho^2)^{\gamma}d\rho\ d\theta\leq\frac{C}{(1-r)^{q-\gamma-2}\,h(1-r)}.$$

Hence, for any $p \geq q$,

$$\begin{aligned}\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^p \ \rho(1-\rho^2)^\gamma d\rho \ d\theta \ &\leq \ \frac{1}{(1-r)^{p-q}} \int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^q \ \rho(1-\rho^2)^\gamma d\rho \ d\theta \\ &\leq \ \frac{C}{(1-r)^{p-\gamma-2}\, h(1-r)}.\end{aligned}$$

Finally, as $r \to 1$, Lemma 2.1 also assures that C can be replaced by any small positive constant if $\lim_{t\to 0} h(t)/t^{2+\gamma-q} = 0$. □

Now, we can apply Theorem 5.1 for the special function defined in (1.7).

Case I: If

$$\sum_{n=1}^{\infty}(1-r_n)^\alpha(\log\frac{1}{1-r_n})^{\alpha_1}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m} < \infty,$$

and if $\gamma \in (-1, \alpha-1)$, then, for any

$$p > \max\{\,\alpha,\, 2+\gamma-\alpha,\, 1+\gamma/2\,\},$$

we have

$$\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^p \ \rho(1-\rho^2)^\gamma d\rho \ d\theta = \frac{o(1)}{(1-r)^{\alpha+p-\gamma-2}\,(\log\frac{1}{1-r})^{\alpha_1}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}},$$

as $(r \to 1)$. In particular, if

$$\sum_{n=1}^{\infty}(1-r_n)^\alpha < \infty,$$

then

$$\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^p \ \rho(1-\rho^2)^\gamma d\rho \ d\theta = \frac{o(1)}{(1-r)^{p+\alpha-\gamma-2}}.$$

Case II: If

$$\sum_{n=1}^{\infty}(1-r_n)^\alpha(\log_k\frac{1}{1-r_n})^{\alpha_k}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $\alpha_k < 0$, then, for any $p \geq 1$,

$$\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^p \ \rho(1-\rho^2)^{\alpha-1} d\rho \ d\theta = \frac{o(1)}{(1-r)^{p-1}\,(\log_k\frac{1}{1-r})^{\alpha_k}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}}, \qquad (r\to 1).$$

Case III: If

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha} < \infty,$$

then, for any $p \geq 1$,

$$\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})|^p \;\; \rho(1-\rho^2)^{\alpha-1} d\rho \; d\theta = \frac{O(1)}{(1-r)^{p-1}}, \qquad (r \to 1).$$

In particular, for $p = 1$,

$$\int_0^1 \int_0^{2\pi} |B'(r\rho e^{i\theta})| \;\; \rho(1-\rho^2)^{\alpha-1} d\rho \; d\theta = O(1), \qquad (r \to 1).$$

which is the Protas' result [15].

Some other cases can also be considered here. But, since they are immediate consequence of Theorem 5.1, we do not proceed further. Moreover, using similar techniques, one can obtain estimates for the A^p_γ means of the higher derivatives for a Blaschke product satisfying the hypothesis of Theorem 5.1.

6. Interpolating Blaschke products

Cohn's theorems 1.4 and 1.5 imply that if $z_n = r_n e^{i\theta_n}$, $n \geq 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^{\infty}(1-r_n)^{1-p} < \infty$$

for some $p \in (2/3, 1)$, then $f' \in H^{2p/(p+2)}$ for all $f \in K_B$. The following result generalizes this fact.

Theorem 6.1. *Let $z_n = r_n e^{i\theta_n}$, $n \geq 1$, be a Carleson sequence satisfying*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty$$

for a positive continuous function h. Let B be the Blaschke product formed with zeros z_n, $n \geq 1$. Suppose that there is $p \in (2/3, 1)$ such that $h(t)/t^{p/2}$ is decreasing and $h(t)/t^{1-p}$ is increasing on $(0,1)$. Then, for all $f \in K_B$, we have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^{\sigma} \, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{(\,(1-r)^{p-1} h(1-r)\,)^{1/p}}, \qquad (r \to 1),$$

with $\sigma = 2p/(p+2)$ and C an absolute constant.

Proof. Since $(z_n)_{n\geq 1}$ is a Carleson sequence, we know that the functions

$$f_n(z) := \frac{(1-r_n)^{1/2}}{1-\overline{z}_n\, z}, \qquad (n \geq 1),$$

form a Riesz basis of K_B (see [14] for instance). Now, let $f = \sum_{n=1}^{N} \beta_n f_n$, $\beta_n \in \mathbb{C}$. Then

$$f'(z) = \sum_{n=1}^{N} \frac{\overline{z}_n\, \beta_n (1-r_n)^{1/2}}{(1-\overline{z}_n\, z)^2},$$

and thus we get

$$|f'(z)| \leq \sum_{n=1}^{N} \frac{|\beta_n|(1-r_n)^{1/2}}{|1-\overline{z}_n\, z|^2}.$$

Since $p \in (2/3, 1)$, we have $\sigma \in (1/2, 1)$ and we can write

$$|f'(z)|^\sigma \leq \sum_{n=1}^{N} \frac{|\beta_n|^\sigma (1-r_n)^{\sigma/2}}{|1-\overline{z}_n\, z|^{2\sigma}}.$$

Therefore,

$$\begin{aligned} \int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta \ &\leq\ \sum_{n=1}^{N} |\beta_n|^\sigma (1-r_n)^{\sigma/2} \int_0^{2\pi} \frac{d\theta}{|1-\overline{z}_n r e^{i\theta}|^{2\sigma}} \\ &\leq\ c \sum_{n=1}^{N} |\beta_n|^\sigma \frac{(1-r_n)^{\sigma/2}}{(1-rr_n)^{2\sigma-1}}. \end{aligned}$$

Let $p' = 2/\sigma$ and let q' be its conjugate exponent. Then Hölder's inequality implies that

$$\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta \leq c \left(\sum_{n=1}^{N} |\beta_n|^2 \right)^{1/p'} \left(\sum_{n=1}^{N} \frac{(1-r_n)^{\sigma q'/2}}{(1-rr_n)^{(2\sigma-1)q'}} \right)^{1/q'}.$$

But since $(f_n)_{n\geq 1}$ forms a Riesz basis of K_B, there exists a constant $c_1 > 0$ such that

$$\sum_{n=1}^{N} |\beta_n|^2 \leq c_1 \|f\|_2^2,$$

whence

$$\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta \le c_2\|f\|_2^\sigma \left(\sum_{n=1}^N \frac{(1-r_n)^{\sigma q'/2}}{(1-rr_n)^{(2\sigma-1)q'}}\right)^{1/q'}.$$

Now easy computations show that $q' = \frac{p+2}{2}$, $\sigma q' = p$, $(2\sigma-1)q' = 3p/2-1$ and therefore, by Lemma 2.1, we have

$$\sum_{n=1}^N \frac{(1-r_n)^{\sigma q'/2}}{(1-rr_n)^{(2\sigma-1)q'}} \le \frac{C}{(1-r)^{p-1}h(1-r)},$$

where C is a constant independent of N. We deduce that

$$\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta \le \frac{c_3\|f\|_2^\sigma}{((1-r)^{p-1}h(1-r))^{1/q'}}.$$

Since $1/\sigma q' = 1/p$, and using a density argument, we get that for all $f \in K_B$,

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \le \frac{c_3^{1/\sigma}\|f\|_2}{((1-r)^{p-1}h(1-r))^{1/p}}.$$

□

Now, we can apply Theorem 6.1 for the special function defined in (1.7).

Case I: If $z_n = r_n e^{i\theta_n}$, $n \ge 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^\infty (1-r_n)^\alpha (\log\frac{1}{1-r_n})^{\alpha_1} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $p \in (2/3, 1)$, $1-p < \alpha < p/2$, and $\alpha_1, \dots, \alpha_m \in \mathbb{R}$, then, for all $f \in K_B$, we have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \le \frac{C\,\|f\|_2}{\left((1-r)^{\alpha+p-1}(\log\frac{1}{1-r})^{\alpha_1}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}\right)^{1/p}},$$

with $\sigma = 2p/(p+2)$ and C an absolute constant.

Case II: If $z_n = r_n e^{i\theta_n}$, $n \ge 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^\infty (1-r_n)^{1-p} (\log_k\frac{1}{1-r_n})^{\alpha_k} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $p \in (2/3,1)$, $k \geq 1$, $\alpha_k, \alpha_{k+1}, \dots, \alpha_m \in \mathbb{R}$, and $\alpha_k < 0$, then, for all $f \in K_B$, we have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{\left((\log_k \frac{1}{1-r})^{\alpha_k} \cdots (\log_m \frac{1}{1-r})^{\alpha_m} \right)^{1/p}}, \qquad (r \to 1),$$

with $\sigma = 2p/(p+2)$ and C an absolute constant. However, if

$$\sum_{n=1}^{\infty} (1-r_n)^{1-p} < \infty,$$

then we still have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \leq C\,\|f\|_2, \qquad (r \to 1),$$

i.e. $f' \in H^{2p/(p+2)}$, for any $f \in K_B$, which is Cohn's result.

Case III: If $z_n = r_n e^{i\theta_n}$, $n \geq 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^{\infty} (1-r_n)^{p/2} (\log_k \frac{1}{1-r_n})^{\alpha_k} \cdots (\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $p \in (2/3,1)$, $k \geq 1$, $\alpha_k, \alpha_{k+1}, \dots, \alpha_m \in \mathbb{R}$, and $\alpha_k > 0$, then, for all $f \in K_B$, we have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{\left((1-r)^{3p/2-1} (\log_k \frac{1}{1-r})^{\alpha_k} \cdots (\log_m \frac{1}{1-r})^{\alpha_m} \right)^{1/p}},$$

with $\sigma = 2p/(p+2)$ and C an absolute constant. However, if

$$\sum_{n=1}^{\infty} (1-r_n)^{p/2} < \infty,$$

then we still have

$$\left(\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\sigma \, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{(1-r)^{(3p-2)/2p}}.$$

Using similar techniques we can obtain some results about the A^p_γ means of the derivatives of function in the model subspaces of H^2.

Theorem 6.2. *Let $z_n = r_n e^{i\theta_n}$, $n \ge 1$, be a Carleson sequence satisfying*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty$$

for a positive continuous function h, and let B be the Blaschke product formed with zeros z_n, $n \ge 1$. Let $p \in (2/3,1)$, $\sigma = 2p/(p+2)$ and $-1 < \gamma < 2\sigma - 2$ such that $h(t)/t^{p/2}$ is decreasing and $h(t)/t^{(1-p)+(1+\gamma)(1+p/2)}$ is increasing on $(0,1)$. Then, for all $f \in K_B$, we have

$$\left(\int_0^1\int_0^{2\pi} \left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\ \rho(1-\rho^2)^{\gamma}d\rho\,d\theta\right)^{1/\sigma} \le \frac{C\,\|f\|_2}{\big(\,(1-r)^{-(1-p)-(1+\gamma)(1+p/2)}\,h(1-r)\,\big)^{1/p}}$$

as $r \to 1^-$.

Proof. The beginning of the proof is as of Theorem 6.1 until

$$|f'(z)|^{\sigma} \le \sum_{n=1}^{N} \frac{|\beta_n|^{\sigma}(1-r_n)^{\sigma/2}}{|1-\overline{z}_n\,z|^{2\sigma}}.$$

Therefore, by Hölder's inequality (with $p' = 2/\sigma$ and q' its conjugate exponent) and by Lemma 2.1, we have

$$\begin{aligned}
\int_0^1\int_0^{2\pi} \left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\ \rho(1-\rho^2)^{\gamma}d\rho\,d\theta &\le \sum_{n=1}^{N}|\beta_n|^{\sigma}(1-r_n)^{\sigma/2}\int_0^1\int_0^{2\pi}\frac{\rho(1-\rho^2)^{\gamma}d\rho\,d\theta}{|1-\overline{z}_n r\rho e^{i\theta}|^{2\sigma}}\\
&\le c\sum_{n=1}^{N}|\beta_n|^{\sigma}\frac{(1-r_n)^{\sigma/2}}{(1-rr_n)^{2\sigma-\gamma-2}}\\
&\le c\left(\sum_{n=1}^{N}|\beta_n|^2\right)^{1/p'}\left(\sum_{n=1}^{N}\frac{(1-r_n)^{\sigma q'/2}}{(1-rr_n)^{(2\sigma-\gamma-2)q'}}\right)^{1/q'}\\
&\le c'\,\|f\|_2^{\sigma}\left(\sum_{n=1}^{N}\frac{(1-r_n)^{\sigma q'/2}}{(1-rr_n)^{(2\sigma-\gamma-2)q'}}\right)^{1/q'}\\
&\le \frac{c''\,\|f\|_2^{\sigma}}{\big((1-r)^{-(1-p)-(1+\gamma)(1+p/2)}\,h(1-r)\big)^{1/q'}}.
\end{aligned}$$

□

Now, we can apply Theorem 6.2 for the special function defined in (1.7).

Case I: If $z_n = r_n e^{i\theta_n}$, $n \geq 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}\,(\log\frac{1}{1-r_n})^{\alpha_1}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m}<\infty,$$

with $p \in (2/3, 1)$, $\sigma = 2p/(p+2)$, $-1 < \gamma < 2\sigma - 2$, and $(1-p)+(1+\gamma)(1+p/2) < \alpha < p/2$, then, for all $f \in K_B$, we have

$$\begin{aligned}&\left(\int_0^1\int_0^{2\pi}\left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\;\rho(1-\rho^2)^{\gamma}d\rho\,d\theta\right)^{1/\sigma}\\ \leq\;&\frac{C\,\|f\|_2}{\big(\,(1-r)^{\alpha-(1-p)-(1+\gamma)(1+p/2)}\;(\log\frac{1}{1-r})^{\alpha_1}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}\,\big)^{1/p}}\end{aligned}$$

as $r \to 1^-$.

Case II: If $z_n = r_n e^{i\theta_n}$, $n \geq 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^{\infty}(1-r_n)^{(1-p)+(1+\gamma)(1+p/2)}\,(\log_k\frac{1}{1-r_n})^{\alpha_k}\cdots(\log_m\frac{1}{1-r_n})^{\alpha_m}<\infty,$$

with $p \in (2/3, 1)$, $\sigma = 2p/(p+2)$, $-1 < \gamma < 2\sigma - 2$, $\alpha_k, \alpha_{k+1}, \ldots, \alpha_m \in \mathbb{R}$, and $\alpha_k < 0$, then, for all $f \in K_B$, we have

$$\left(\int_0^1\int_0^{2\pi}\left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\;\rho(1-\rho^2)^{\gamma}d\rho\,d\theta\right)^{1/\sigma}\leq\frac{C\,\|f\|_2}{\big(\,(\log_k\frac{1}{1-r})^{\alpha_k}\cdots(\log_m\frac{1}{1-r})^{\alpha_m}\,\big)^{1/p}}$$

as $r \to 1^-$. However, if

$$\sum_{n=1}^{\infty}(1-r_n)^{(1-p)+(1+\gamma)(1+p/2)}<\infty,$$

then, we still have

$$\left(\int_0^1\int_0^{2\pi}\left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\;\rho(1-\rho^2)^{\gamma}d\rho\,d\theta\right)^{1/\sigma}\leq C\,\|f\|_2,$$

which means that

$$f' \in A_{\gamma}^{2p/(p+2)},$$

and the differential operator

$$\begin{array}{rcl} K_B & \longrightarrow & A_\gamma^{2p/(p+2)}, \\ f & \longmapsto & f', \end{array}$$

is bounded.

Case III: If $z_n = r_n e^{i\theta_n}$, $n \geq 1$, is a Carleson sequence satisfying

$$\sum_{n=1}^{\infty}(1-r_n)^{p/2}\,(\log_k \frac{1}{1-r_n})^{\alpha_k}\cdots(\log_m \frac{1}{1-r_n})^{\alpha_m} < \infty,$$

with $p \in (2/3, 1)$, $\sigma = 2p/(p+2)$, $-1 < \gamma < 2\sigma - 2$, $\alpha_k, \alpha_{k+1}, \ldots, \alpha_m \in \mathbb{R}$ and $\alpha_k > 0$, then, for all $f \in K_B$, we have

$$\begin{aligned} &\left(\int_0^1 \int_0^{2\pi} \left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\ \rho(1-\rho^2)^{\gamma} d\rho\, d\theta\right)^{1/\sigma} \\ \leq\ & \frac{C\,\|f\|_2}{\Big((1-r)^{(3p/2-1)-(1+\gamma)(1+p/2)}\ (\log_k \frac{1}{1-r})^{\alpha_k}\cdots(\log_m \frac{1}{1-r})^{\alpha_m}\Big)^{1/p}} \end{aligned}$$

as $r \to 1^-$. However, if

$$\sum_{n=1}^{\infty}(1-r_n)^{p/2} < \infty,$$

then, we still have

$$\left(\int_0^1 \int_0^{2\pi} \left|f'(r\,\rho e^{i\theta})\right|^{\sigma}\ \rho(1-\rho^2)^{\gamma} d\rho\, d\theta\right)^{1/\sigma} \leq \frac{C\,\|f\|_2}{(1-r)^{\big((3p/2-1)-(1+\gamma)(1+p/2)\big)/p}}$$

as $r \to 1^-$.

References

[1] Ahern, P. R. and D. N. Clark, *On inner functions with H^p-derivative*, Michigan Math. J. 21 (1974), 115-127.

[2] Ahern, P. R. and D. N. Clark, *On inner functions with B^p-derivative*, Michigan Math. J. 23 (1976), 107-118.

[3] Cohn, W. S., *On the H^p classes of derivative of functions orthogonal to invariant subspaces*, Michigan Math. J. 30 (1983), 221-229.

[4] Duren, P. L. and A. Schuster, *Bergman Spaces*, Mathematical Surveys and Monographs, Volume 100, AMS, 2004.

[5] Duren, P. L., *Theory of H^p spaces*, Academic Press, 1970.

[6] Fricain, E., *Bases of reproducing kernels in model spaces*, J. Operator Theory 46 (2001), no. 3, 517–543.

[7] Gotoh, Y., *On integral means of the derivatives of Blaschke products*, Kodai Math. J., 30 (2007), 147-155.

[8] Havin, V, Mashreghi, J., *Admissible majorants for model subspaces of $H^2(\mathbb{R})$, Part I & II*, Canadian Journal of Mathematics, Vol. 55, No. 6 (2003), 1231-1263 and 1264-1301.

[9] Hedenmalm, H., Korenblum, B. and K. Zhu, *Theory of Bergman Spaces*, Graduate Text in Mathematics 199, Springer, 2000.

[10] Koosis, P., *Introduction to H^p Spaces*, Second Edition, Cambridge Tracts in Mathematics, 115, 1998.

[11] Kutbi, M. A., *Integral Means for the n'th Derivative of Blaschke Products*, Kodai Math. J., 25 (2002), 191-208.

[12] Linden, C. N., *H^p-derivatives of Blaschke products*, Michigan Math. J. 23 (1976), 43-51.

[13] Mashreghi, J., *Generalized Lipschitz functions*, Computational Methods and Function Theory, Vol. 5, No. 2 (2005), 431-444.

[14] Nikolski, N., *Treatise on the shift operator*, Springer-Verlag, Berlin etc., 1986.

[15] Protas, D., *Blaschke products with derivatives in H^p and B^p*, Michigan Math. J. 20 (1973), 393-396.

[16] Protas, D., *Mean growth of the derivative of a Blaschke product*, Kodai Math. J., 27 (2004), No. 3 354-359.

Université de Lyon; Université Lyon 1; Institut Camille Jordan CNRS UMR 5208; 43, boulevard du 11 Novembre 1918, F-69622 Villeurbanne.

E-mail address: fricain@math.univ-lyon1.fr

Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4.

E-mail address: Javad.Mashreghi@mat.ulaval.ca

6.3.2 Référence [T9]

Auteurs
E. Fricain et J. Mashreghi

Titre
Exceptional sets for the derivatives of Blaschke products

Paru dans
Proceedings of the St. Petersburg Mathematical Society, Amer. Math. Soc. Transl., 222 (2008), no. 2, 163-170.

EXCEPTIONAL SETS FOR THE DERIVATIVES OF BLASCHKE PRODUCTS

EMMANUEL FRICAIN, JAVAD MASHREGHI

ABSTRACT. We obtain growth estimates for the logarithmic derivative $B'(z)/B(z)$ of a Blaschke product as $|z| \to 1$ and z avoids some exceptional sets.

1. INTRODUCTION

Let f be a meromorphic function in the unit disc $\mathbb{D}$. Then its order is defined by

$$\sigma = \limsup_{r \to 1^-} \frac{\log^+ T(r)}{\log 1/(1-r)},$$

where

$$T(r) = \frac{1}{\pi} \int_{\{|z|<r\}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \log(\frac{r}{|z|}) \, dx \, dy$$

is the Nevanlinna characteristic of f [13]. Meromorphic functions of finite order have been extensively studied and they have numerous applications in pure and applied mathematics, e.g. in linear differential equations. In many applications a major role is played by the logarithmic derivative of meromorphic functions and we need to obtain sharp estimates for the logarithmic derivative as we approach to the boundary [7, 8]. In particular, the following result for the rate of growth of meromorphic functions of finite order in the unit disc has application in the study of linear differential equations [10, Theorem 5.1].

2000 *Mathematics Subject Classification.* Primary: 30D50, Secondary: 26A12.

Key words and phrases. Blaschke products, Nevanlinna class, logarithmic derivative.

This work was supported by NSERC (Canada) and FQRNT (Québec).

Theorem 1.1. *Let f be a meromorphic function in the unit disc $\mathbb{D}$ of finite order σ and let $\varepsilon > 0$. Then the following two statements hold.*

(a) There exists a set $E_1 \subset (0,1)$ which satisfies

$$\int_{E_1} \frac{dr}{1-r} < \infty,$$

such that, for all $z \in \mathbb{D}$ with $|z| \notin E_1$, we have

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{1}{(1-|z|)^{3\sigma+4+\varepsilon}}. \tag{1.1}$$

(b) There exists a set $E_2 \subset [0, 2\pi)$ whose Lebesgue measure is zero and a function $R(\theta) : [0,2\pi) \setminus E_2 \longrightarrow (0,1)$ such that for all $z = re^{i\theta}$ with $\theta \in [0,2\pi) \setminus E_2$ and $R(\theta) < r < 1$ the inequality (1.1) holds.

Clearly, the relation (1.1) can also be written as

$$\left| \frac{f'(z)}{f(z)} \right| = \frac{O(1)}{(1-|z|)^{3\sigma+4+\varepsilon}}$$

as $|z| \to 1$. But we should note that in case (b) it does not hold uniformly with respect to $|z|$.

Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc satisfying the Blaschke condition

$$\sum_{n=1}^{\infty} (1-|z_n|) < \infty. \tag{1.2}$$

Then the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \, \frac{z_n - z}{1 - \bar{z}_n \, z}$$

is an analytic function in the unit disc with order $\sigma = 0$ and

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n \, z)(z-z_n)}. \tag{1.3}$$

Thus Theorem 1.1 implies that, for any $\varepsilon > 0$,

$$\left| \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n \, z)(z-z_n)} \right| = \frac{O(1)}{(1-|z|)^{4+\varepsilon}}$$

as $|z| \to 1^-$ in any of the two manners explained above. In this paper, instead of (1.2), we pose more restrictive conditions on the rate of convergence of zeros z_n and instead we improve the exponent $4+\varepsilon$. The most common condition is

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} < \infty, \tag{1.4}$$

for some $\alpha \in (0,1]$. However, we consider a more general assumption

$$\sum_{n=1}^{\infty} h(1-|z_n|) < \infty, \tag{1.5}$$

where h is a positive continuous function satisfying certain smoothness conditions which will be described below. Our main prototype for h is

$$h(t) = t^{\alpha}\ (\log 1/t)^{\alpha_1}\ (\log_2 1/t)^{\alpha_2}\ \cdots\ (\log_n 1/t)^{\alpha_n}, \tag{1.6}$$

where $\log_n = \log\log\cdots\log$ (n times), $\alpha \in (0,1]$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero exponent among $\alpha_1, \alpha_2, \cdots, \alpha_n$ is positive [12].

The function h is usually defined in an open interval $(0,\epsilon)$. Of course, by extending its domain of definition, we may assume that h is defined on the interval $(0,1)$, or if required, on the entire positive real axis. Moreover, since a Blaschke sequence satisfies (1.2), the condition (1.5) will provide further information about the rate of increase of the zeros provided that $h(t) \ge C\,t$ as $t \to 0$.

The condition (1.4) has been extensively studied by many authors [1, 2, 3, 9, 11, 14] to obtain estimates for the integral means of the derivative of Blaschke products. We [6] have recently shown that many of these estimates can be generalized for Blaschke products satisfying (1.5).

2. Circular Exceptional Sets

The function h given in (1.6) satisfies the following conditions:

a) h is continuous, positive and increasing with $h(0+) = 0$;

b) $h(t)/t$ is decreasing;

In the following, we just need these conditions. Hence, we state our results for a general function h satisfying $a)$ and $b)$.

Theorem 2.1. *Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc satisfying*

$$\sum_{n=1}^{\infty} h(1-|z_n|) < \infty$$

and let B be the Blaschke product formed with zeros z_n, $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^\beta} < \infty$$

and that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^\beta \; h^2(1-|z|)}$$

as $|z| \to 1^-$ with $|z| \notin E$.

Proof. Without loss of generality, assume that $h(t) < 1$ for $t \in (0,1)$. Let

$$E = \bigcup_{n=1}^{\infty} \bigg(\, |z_n| - (1-|z_n|)^\beta h(1-|z_n|), \; |z_n| + (1-|z_n|)^\beta h(1-|z_n|) \bigg).$$

In the definition of E we implicitly assume that $|z_n| - (1-|z_n|)^\beta h(1-|z_n|) > 0$ in order to have $E \subset (0,1)$. Certainly this condition holds for large values of n. If it does not hold for some small values of n, we simply remove those intervals from the definition of E.

Let $z \in \mathbb{D}$ with $|z| \notin E$ and fix $0 < \delta \leq (1-|z|)/2$. By (1.3), we have

$$\frac{B'(z)}{B(z)} = \Bigg(\sum_{\big|\, |z|-|z_n| \,\big| \geq \delta} + \sum_{\big|\, |z|-|z_n| \,\big| < \delta} \Bigg) \frac{1-|z_n|^2}{(1-\bar{z}_n \, z)(z-z_n)}.$$

We use different techniques to estimate each sum. For the first sum we have

$$\sum_{\big|\, |z|-|z_n| \,\big| \geq \delta} \frac{1-|z_n|^2}{|1-\bar{z}_n \, z| \, |z-z_n|} \;\; \leq \;\; \frac{2}{\delta} \sum_{\big|\, |z|-|z_n| \,\big| \geq \delta} \frac{1-|z_n|}{1-|z_n| \, |z|}.$$

But

$$\frac{1-|z_n|}{1-|z|\,|z_n|} = \left(\frac{1-|z_n|}{h(1-|z_n|)}\,\frac{h(1-|z|\,|z_n|)}{1-|z|\,|z_n|} \right) \left(\frac{h(1-|z_n|)}{h(1-|z|\,|z_n|)} \right).$$

Since $h(t)$ is increasing and $h(t)/t$ is decreasing, we get

$$\frac{1-|z_n|}{1-|z|\,|z_n|} \le \frac{h(1-|z_n|)}{h(1-|z|)}$$

and thus

$$\sum_{\big|\,|z|-|z_n|\,\big|\ge\delta} \frac{1-|z_n|^2}{|1-\bar{z}_n\,z|\,|z-z_n|} \le \frac{2\sum_{\big|\,|z|-|z_n|\,\big|\ge\delta} h(1-|z_n|)}{\delta\; h(1-|z|)} \le \frac{C}{\delta\; h(1-|z|)}.$$

A generalized version of this estimation technique has been used in [6, Lemma 2.1]. To estimate the second sum, we see that

$$\begin{aligned} \left| \frac{1-|z_n|^2}{(1-\bar{z}_n\,z)(z-z_n)} \right| &\le \frac{2}{|z-z_n|} \le \frac{2}{(1-|z_n|)^\beta\, h(1-|z_n|)} \\ &\le \frac{C}{(1-|z|)^\beta\, h(1-|z|)}, \end{aligned}$$

and thus

$$\left| \sum_{\big|\,|z|-|z_n|\,\big|<\delta} \frac{1-|z_n|^2}{(1-\bar{z}_n\,z)(z-z_n)} \right| \le C\; \frac{n(|z|+\delta)-n(|z|-\delta)}{(1-|z|)^\beta\, h(1-|z|)},$$

where $n(t)$ is the number of points z_n lying in the disc $\{\, z \,:\, |z| \le t \,\}$. Therefore

$$\left| \frac{B'(z)}{B(z)} \right| \le \frac{C}{h(1-|z|)} \left(\frac{1}{\delta} + \frac{n(|z|+\delta)-n(|z|-\delta)}{(1-|z|)^\beta} \right) \tag{2.1}$$

provided that $z \in \mathbb{D}$ with $|z| \notin E$. The best choice of δ depends on the counting function $n(t)$. We make a choice for the most general case.

Assume that $\delta = (1-|z|)/2$. Our assumption (1.5) on the rate of increase of zeros z_n is equivalent to

$$\int_0^1 h(1-t)\; dn(t) < \infty,$$

and it is well known that this condition implies

$$n(t) = \frac{o(1)}{h(1-t)} \tag{2.2}$$

as $t \to 1^-$. Therefore,

$$n(|z|+\delta) - n(|z|-\delta) \leq \frac{o(1)}{h(1-|z|)}. \tag{2.3}$$

Hence, by (2.1) and (2.3), we get the promised growth for B'/B. To verify the size of E, note that

$$\begin{aligned}
\int_E \frac{dt}{(1-t)^\beta} &= \sum_{n=1}^{\infty} \int_{|z_n|-(1-|z_n|)^\beta h(1-|z_n|)}^{|z_n|+(1-|z_n|)^\beta h(1-|z_n|)} \frac{dt}{(1-t)^\beta} \\
&= \sum_{n=1}^{\infty} \int_{(1-|z_n|)-(1-|z_n|)^\beta h(1-|z_n|)}^{(1-|z_n|)+(1-|z_n|)^\beta h(1-|z_n|)} \frac{d\tau}{\tau^\beta} \\
&\leq \sum_{n=1}^{\infty} \frac{2(1-|z_n|)^\beta h(1-|z_n|)}{(\,(1-|z_n|)-(1-|z_n|)^\beta h(1-|z_n|)\,)^\beta} \\
&\leq C \sum_{n=1}^{\infty} h(1-|z_n|) < \infty.
\end{aligned}$$

□

Remark 1: As the counting function $n(t) = 1/(1-t)^\alpha$ suggests, the assumption

$$n(|z|+\delta) - n(|z|-\delta) \leq C\, \frac{\delta\, n(|z|)}{1-|z|} \tag{2.4}$$

is fulfilled by a wide class of distribution of zeros. If (2.4) holds, by (2.3) and (2.1) with

$$\delta = (1-|z|)^{\frac{1+\beta}{2}}\, h^{\frac{1}{2}}(1-|z|),$$

we obtain

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{O(1)}{(1-|z|)^{\frac{1+\beta}{2}}\, h^{\frac{3}{2}}(1-|z|)}$$

as $|z| \to 1^-$ with $|z| \notin E$.

Remark 2: Let us call φ almost increasing if $\varphi(x) \leq \text{Const}\,\varphi(y)$ provided that $x \leq y$. Almost decreasing functions are defined similarly. As it can be easily verified, Theorem 2.1 (and also Theorem 3.1) is still true if we assume that $h(t)$ is almost increasing and $h(t)/t$ is almost decreasing.

Corollary 2.2. *Let $\alpha \in (0,1]$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. Let $(z_n)_{n\geq 1}$ be a sequence in the unit disc with*

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha}\ (\log 1/(1-|z_n|))^{\alpha_1}\ \cdots\ (\log_n 1/(1-|z_n|))^{\alpha_n} < \infty$$

and let B be the Blaschke product formed with zeros z_n, $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^{\beta}} < \infty$$

and that

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+2\alpha}\,(\log 1/(1-|z|))^{2\alpha_1}\ \cdots\ (\log_n 1/(1-|z|))^{2\alpha_n}} \tag{2.5}$$

as $|z| \to 1^-$ with $|z| \notin E$.

In particular, if

$$\sum_{n=1}^{\infty}(1-|z_n|)^{\alpha} < \infty, \tag{2.6}$$

then, for any $\beta \geq 1$, there is an exceptional set $E \subset (0,1)$ such that

$$\int_E \frac{dt}{(1-t)^{\beta}} < \infty \tag{2.7}$$

and that

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{\beta+2\alpha}}$$

as $|z| \to 1^-$ with $|z| \notin E$. If $(|z_n|)_{n\geq 1}$ is an interpolating sequence then

$$1-|z_{n+1}| \leq c\,(1-|z_n|)$$

for a constant $c < 1$ [4, Theorem 9.2]. Hence, (2.6) is satisfied for any $\alpha > 0$ and thus, for any $\beta \geq 1$ and for any $\varepsilon > 0$, there is an exceptional set E satisfying (2.7) such that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^{\beta+\varepsilon}} \tag{2.8}$$

as $|z| \to 1^-$ with $|z| \notin E$. It is interesting to know if in (2.8) we are able to replace ε by zero.

3. Radial Exceptional Sets

Contrary to the preceding section, we now study the behavior of

$$\left| \frac{B'(re^{i\theta})}{B(re^{i\theta})} \right|$$

as $r \to 1$ for a *fixed* θ. We obtain an upper bound for the quotient B'/B as long as $e^{i\theta} \in \mathbb{T} \setminus E$ where E is an exceptional set of Lebesgue measure zero.

Theorem 3.1. *Let B be the Blaschke product formed with zeros $z_n = r_n e^{i\theta_n}$, $n \geq 1$, satisfying*

$$\sum_{n=1}^{\infty} h(1-r_n) < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)\ h(1-|z|)}$$

as $|z| \to 1^-$.

Proof. Let us consider the open set

$$U_n = \{\, z \in \mathbb{D} : (1-|z|) > C|z-z_n| \,\}$$

with $C > 1$, and we define

$$I_n = \{\, \zeta \in \mathbb{T} : \exists z \in U_n \ \&\ \zeta = z/|z| \,\}.$$

In other words, I_n is the radial projection of U_n on the unit circle $\mathbb{T}$. Then we know that

$$|I_n| \le C'(1 - r_n), \tag{3.1}$$

where C' is a constant just depending on C. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} I_k.$$

By (3.1), we see that $|E| = 0$.

Fix $z \in \mathbb{D}$ with $z/|z| \notin E$. Hence, there is N such that $z/|z| \notin I_k$ for all $k \ge N$. Let $R = (1 + |z|)/2$. Now, we write

$$\frac{B'(z)}{B(z)} = \left(\sum_{|z_n| \ge R} + \sum_{|z_n| < R,\, n \ge N} + \sum_{n=1}^{N-1} \right) \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)},$$

and as in the preceding case

$$\sum_{|z_n| \ge R} \frac{1 - |z_n|^2}{|1 - \bar{z}_n\, z|\, |z - z_n|} \le \frac{o(1)}{(1 - |z|)\ h(1 - |z|)}. \tag{3.2}$$

To estimate the second sum, we see that

$$\left| \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)} \right| \le \frac{2}{|z - z_n|} \le \frac{2C}{1 - |z|}, \qquad (|z| \notin E),$$

and thus, by (2.2),

$$\left| \sum_{|z_n| < R,\, n \ge N} \frac{1 - |z_n|^2}{(1 - \bar{z}_n\, z)(z - z_n)} \right| \le \frac{2C\, n(R)}{1 - |z|} \le \frac{o(1)}{(1 - |z|)\, h(1 - |z|)}. \tag{3.3}$$

Since the last sum is uniformly bounded (θ is fixed), (3.2) and (3.3) give the required result. □

Corollary 3.2. *Let $\alpha \in (0, 1]$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero number among $\alpha_1, \alpha_2, \cdots, \alpha_n$ is positive. Let B be the Blaschke product formed with* *zeros* $z_n = r_n e^{i\theta_n}$, $n \geq 1$, *satisfying*

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha}\ (\log 1/(1-r_n))^{\alpha_1}\ \cdots\ (\log_n 1/(1-r_n))^{\alpha_n} < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ *whose Lebesgue measure* $|E|$ *is zero such that for all* $z = re^{i\theta}$ *with* $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{1+\alpha}\,(\log 1/(1-|z|))^{\alpha_1}\ \cdots\ (\log_n 1/(1-|z|))^{\alpha_n}} \tag{3.4}$$

as $|z| \to 1^-$.

In particular, if

$$\sum_{n=1}^{\infty}(1-r_n)^{\alpha} < \infty,$$

then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left|\frac{B'(z)}{B(z)}\right| = \frac{o(1)}{(1-|z|)^{1+\alpha}} \tag{3.5}$$

as $|z| \to 1^-$.

Remark: Theorems 2.1 and 3.1 can be easily generalized to obtain estimates for

$$\frac{B^{(k)}(z)}{B^{(j)}(z)}$$

as $|z| \to 1^-$. This is a standard technique which can been find for example in [9, 11].

References

[1] Ahern, P. R. and D. N. Clark, *On inner functions with H^p-derivative*, Michigan Math. J. 21 (1974), 115-127.

[2] Ahern, P. R. and D. N. Clark, *On inner functions with B^p-derivative*, Michigan Math. J. 23 (1976), 107-118.

[3] Ahern, P. R., *On a theorem of Hayman concerning the derivative of function of bounded characteristic*, Pacific J. Math. 83 (1979), 297-301.

[4] Cohn, W. S., *On the H^p classes of derivative of functions orthogonal to invariant subspaces*, Michigan Math. J. 30 (1983), 221-229.

[5] Duren, P. L., *Theory of H^p spaces*, Academic Press, 1970.

[6] Fricain, E., Mashreghi, J., *Integral means of the derivatives of Blaschke products*, preprint.

[7] Gundersen, G., *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) 37 (1988) 88-104.

[8] Gundersen, G., Steinbart, E., Wang, S., *The possible orders of solutions of linear differential equations with polynomial coefficients*, Trans. Amer. Math. Soc., 350 (1998) 1225-1247.

[9] Kutbi, M. A., *Integral Means for the n'th Derivative of Blaschke Products*, Kodai Math. J., 25 (2002), 191-208.

[10] Heittokangas, J., *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. 122 (2000) 1-54.

[11] Linden, C. N., *H^p-derivatives of Blaschke products*, Michigan Math. J. 23 (1976), 43-51.

[12] Mashreghi, J., *Generalized Lipschitz functions*, Computational Methods and Function Theory, Vol. 5, No. 2 (2005), 431-444.

[13] Nevanlinna, R., *Analytic Functions*, Springer, Berlin, 1970.

[14] Protas, D., *Blaschke products with derivatives in H^p and B^p*, Michigan Math. J. 20 (1973), 393-396.

Université de Lyon, Lyon, F-69003, France; Université Lyon I, Institut Camille Jordan, Villeurbanne Cedex F-69622, France; CNRS, UMR5208, Villeurbanne, F-69622, France.

E-mail address: fricain@math.univ-lyon1.fr

Département de mathématiques et de statistique, Université Laval, Québec, QC, Canada G1K 7P4.

E-mail address: Javad.Mashreghi@mat.ulaval.ca

6.4 Annexes sur le chapitre “Autour de certaines classes d’opérateurs”

6.4.1 Référence [T10]

Auteurs
N. Chevrot, E. Fricain et D. Timotin

Titre
The characteristic function of a complex symmetric contraction.

Paru dans
Proc. Amer. Math. Soc., 135 (2007), no. 9, 2877–2886.

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 135, Number 9, September 2007, Pages 2877–2886
S 0002-9939(07)08803-X
Article electronically published on May 8, 2007

THE CHARACTERISTIC FUNCTION OF A COMPLEX SYMMETRIC CONTRACTION

NICOLAS CHEVROT, EMMANUEL FRICAIN, AND DAN TIMOTIN

(Communicated by Joseph A. Ball)

Abstract. It is shown that a contraction on a Hilbert space is complex symmetric if and only if the values of its characteristic function are all symmetric with respect to a fixed conjugation. Applications are given to the description of complex symmetric contractions with defect indices equal to 2.

1. Introduction

Complex symmetric operators on a complex Hilbert space are characterized by the existence of an orthonormal basis with respect to which their matrix is symmetric. Their theory is therefore connected with the theory of symmetric matrices, which is a classical topic in linear algebra. A more intrinsic definition implies the introduction of a conjugation in the Hilbert space, that is, a conjugate-linear, isometric and involutive map, with respect to which the symmetry is defined. Such operators or matrices appear naturally in many different areas of mathematics and physics; we refer to [5] for more about the history of the subject and its connections to other domains, as well as for an extended list of references.

The interest in complex symmetric operators has been recently revived by the work of Garcia and Putinar [3, 4, 5]. In their papers a general framework is established for such operators, and it is shown that large classes of operators on a Hilbert space can be studied in this framework. The examples are rather diverse: normal operators are complex symmetric, for instance, but also certain types of Volterra and Toeplitz operators, as well as the so-called compressed shift on the functional model spaces $H^2 \ominus \phi H^2$, where ϕ denotes a nonconstant inner function.

The purpose of this paper is to explore further the generalizations of this last example. The natural context is the model theory of completely nonunitary contractions developed by Sz. Nagy and Foias [6]. The main result is a criterion for a contraction to be complex symmetric in terms of its characteristic function. In the sequel some applications of this result are given.

The plan of the paper is the following. The next section presents preliminary material. Section 3 contains the announced criterion. In Section 4 one discusses 2×2 inner characteristic functions, and the results are applied in the last section in order to obtain a series of examples of complex symmetric contractions with defect indices 2.

Received by the editors April 8, 2006 and, in revised form, May 28, 2006.
2000 *Mathematics Subject Classification.* Primary 47A45, 47B15.
Key words and phrases. Complex symmetric operator, contraction, characteristic function.

2. Preliminaries

2.1. **Complex symmetric operators.** We first recall some basic facts from [3, 4, 5]. Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A *conjugation* C on $\mathcal{H}$ is a conjugate-linear, isometric and involutive map; thus $C^2 = I$, and $\langle Cf, Cg\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$.

For a fixed conjugation operator C on $\mathcal{H}$, we say that a linear operator T on $\mathcal{H}$ is *C-symmetric* if $T = CT^*C$. One sees immediately that if T is C-symmetric, then T^* is C-symmetric. Then $T \in \mathcal{L}(\mathcal{H})$ is called *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that T is C-symmetric. Among various examples of complex symmetric operators [5], we mention the class of normal operators; in particular, unitary operators are complex symmetric. Also, direct sums of complex symmetric operators are complex symmetric.

Complex symmetric operators can also be characterized in terms of certain matrix representations, as shown by the following result from [5].

Lemma 2.1. *Let C be a conjugation on $\mathcal{H}$. Then:*

(i) *There exists an orthonormal basis $(e_n)_{n=1}^{\dim \mathcal{H}}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n; such a basis is called a C-*real orthonormal basis *for $\mathcal{H}$.*

(ii) *$T \in \mathcal{L}(\mathcal{H})$ is C-symmetric if and only if there exists a C-real orthonormal basis $(e_n)_{n=1}^{\dim \mathcal{H}}$ for $\mathcal{H}$ such that*

$$\langle Te_n, e_m\rangle = \langle Te_m, e_n\rangle, \qquad \forall n, m \geq 1.$$

2.2. **Characteristic functions and model operators.** The characteristic function for a contraction and the construction of the basic functional model is developed by B. Sz.-Nagy and C. Foias [6], which is the main source for this subsection. In the sequel $\mathbb{D}$ and $\mathbb{T}$ denote the unit disc and the unit circle in the complex plane.

Suppose $T \in \mathcal{L}(\mathcal{H})$, $\|T\| \leq 1$. There is a unique decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_u$ such that $T\mathcal{H}_0 \subset \mathcal{H}_0$, $T\mathcal{H}_u \subset \mathcal{H}_u$ and $T_{|\mathcal{H}_u}$ is unitary, whereas $T_{|\mathcal{H}_0}$ is *completely nonunitary* (c.n.u.), that is, $T_{|\mathcal{H}_0}$ is not unitary on any of its invariant subspaces.

The operator $D_T = (I - T^*T)^{1/2}$ is called the *defect operator* of T. The *defect spaces* of T are $\mathcal{D}_T = \overline{D_T\mathcal{H}}$, $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$, and the *defect indices* $\partial_T = \dim \mathcal{D}_T$, $\partial_{T^*} = \dim \mathcal{D}_{T^*}$. Since $D_T = D_{T_0} \oplus 0$, $D_{T^*} = D_{T_0^*} \oplus 0$, we have $\mathcal{D}_T = \mathcal{D}_{T_0}$ and $\mathcal{D}_{T^*} = \mathcal{D}_{T_0^*}$.

We say that $T \in C_{0\cdot}$ if $T^n \to 0$ strongly, and $T \in C_{\cdot 0}$ if $T^* \in C_{0\cdot}$; also, $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

Suppose $\mathcal{E}, \mathcal{E}'$ are Hilbert spaces, and $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is a contraction-valued analytic function. One can decompose $\mathcal{E} = \mathcal{E}_p \oplus \mathcal{E}_u$, $\mathcal{E}' = \mathcal{E}'_p \oplus \mathcal{E}'_u$, such that:

— for all $z \in \mathbb{D}$, $\Theta(z)\mathcal{E}_p \subset \mathcal{E}'_p$, $\Theta(z)\mathcal{E}_u \subset \mathcal{E}'_u$;

— if $\Theta = \Theta_p \oplus \Theta_u$ is the corresponding decomposition of Θ, then Θ_p is *pure*, that is, $\|\Theta_p(0)h\| < \|h\|$ for all $h \in \mathcal{E}_p$, $h \neq 0$, while Θ_u is a unitary constant.

Θ_p is then called the pure part of Θ.

One says [6] that two contractive analytic functions $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, $\Theta' : \mathbb{D} \to \mathcal{L}(\mathcal{E}', \mathcal{E}'_*)$ *coincide* if there are unitaries $U : \mathcal{E} \to \mathcal{E}'$, $U_* : \mathcal{E}_* \to \mathcal{E}'_*$, such that $\Theta(z) = U_*^*\Theta'(z)U$ for all $z \in \mathbb{D}$.

The *characteristic function* of T is an operator valued function $\Theta_T(\lambda) : \mathcal{D}_T \to \mathcal{D}_{T^*}$ defined for $\lambda \in \mathbb{D}$ by

$$\Theta_T(\lambda) := -T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T | \mathcal{D}_T. \tag{2.1}$$

Θ_T is a pure contraction-valued analytic function on $\mathbb{D}$, and one easily sees that $\Theta_T = \Theta_{T_0}$.

For $\mathcal{E}$ a Hilbert space, we denote by $L^2(\mathcal{E})$ the Lebesgue space of measurable functions $f : \mathbb{T} \to \mathcal{E}$ of square integrable norm, and by $H^2(\mathcal{E}) \subset L^2(\mathcal{E})$ the Hardy space of functions whose negative Fourier coefficients vanish. P_+ is the orthogonal projection onto $H^2(\mathcal{E})$, and $P_- = I - P_+$.

If we are given an arbitrary contraction-valued analytic function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}')$ ($\mathcal{E}, \mathcal{E}_*$ Hilbert spaces), one defines the *model space* associated to Θ by

$$\mathfrak{K}_\Theta = \left(H^2(\mathcal{E}_*) \oplus \overline{(I - \Theta^*\Theta)^{1/2} L^2(\mathcal{E})}\right) \ominus \{\Theta_T f \oplus (I - \Theta^*\Theta)^{1/2} f : f \in H^2(\mathcal{E})\}, \tag{2.2}$$

and the *model operator* $\mathbf{T}_\Theta \in \mathcal{L}(\mathfrak{K}_\Theta)$ by

$$\mathbf{T}_\Theta(f \oplus g) = P_{\mathfrak{K}_\Theta}(zf \oplus zg) \tag{2.3}$$

($P_{\mathfrak{K}_\Theta}$ is the orthogonal projection onto $\mathfrak{K}_\Theta$). Then $\mathbf{T}_\Theta$ is a c.n.u. contraction, and its characteristic function coincides with the pure part of Θ.

If we start with a contraction T, and apply the previous constructions to Θ_T, the resulting operator $\mathbf{T}_{\Theta_T}$ is unitarily equivalent to T_0 (the completely non-unitary part of T).

A contractive analytic function Θ is called *inner* if its boundary values $\Theta(e^{it})$ are isometries a.e. on $\mathbb{T}$. If T is c.n.u., then $T \in C_{\cdot 0}$ if and only if Θ_T is inner.

3. The main theorem

Our main result gives a criterion for complex symmetric contractions.

Theorem 3.1. *Let T be a contraction on the Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(i) *T is complex symmetric.*

(ii) *There exists a conjugate-linear map $J : \mathcal{D}_T \to \mathcal{D}_{T^*}$ which is isometric, onto and satisfies*

$$\Theta_T(z) = J\Theta_T(z)^* J, \qquad \forall z \in \mathbb{D}. \tag{3.1}$$

(iii) *There exists a Hilbert space $\mathcal{E}$, a conjugation J' on $\mathcal{E}$, and a pure contractive analytic function $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E})$, whose values are J'-symmetric operators, such that Θ_T coincides with Θ.*

Proof. (i)⇒(ii) If T is complex symmetric, there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. Since C is involutive, we get $CT^* = TC$, $CT = T^*C$, and $C(I - T^*T) = (I - TT^*)C$. Thus $CD_T^2 = D_{T^*}^2 C$, and therefore $CD_T^{2n} = D_{T^*}^{2n} C$, $n \geq 0$. If $(p_n)_{n\geq 1}$ is a sequence of polynomials tending uniformly to $\sqrt{x}$ on $[0, 1]$, then $Cp_n(D_T^2) = p_n(D_{T^*}^2)C$, whence $CD_T = D_{T^*}C$. In particular, $C\mathcal{D}_T \subset \mathcal{D}_{T^*}$; since T^* is also C-symmetric, we actually have equality. Moreover, $CT^n = T^{*n}C$ for all $n \geq 1$ implies $C(I - \overline{z}T)^{-1} = (I - zT^*)^{-1}C$.

Now define $J := C|\mathcal{D}_T$. Then J is a conjugate-linear map from $\mathcal{D}_T$ onto $\mathcal{D}_{T^*}$ which is isometric, and the equalities above imply that $J\Theta_T(z)^*J = \Theta_T(z)$ for all $z \in \mathbb{D}$.

(ii)⇒(i) Assume first that T is completely nonunitary. We will prove that the model operator $\mathbf{T}_{\Theta_T} \in \mathcal{L}(\mathfrak{K}_{\Theta_T})$, as defined by (2.2) and (2.3), is complex symmetric. For simplicity, we will write in the sequel of the proof $\mathbf{T}$ and $\mathfrak{K}$ instead of $\mathbf{T}_{\Theta_T}$ and $\mathfrak{K}_{\Theta_T}$.

Let us introduce some supplementary notation. Define

$$\mathfrak{H} := L^2(\mathcal{D}_{T^*}) \oplus \overline{(I - \Theta_T^* \Theta_T)^{1/2} L^2(\mathcal{D}_T)}$$

and $\pi : L^2(\mathcal{D}_T) \to \mathfrak{H}$, $\pi_* : L^2(\mathcal{D}_{T^*}) \to \mathfrak{H}$ by

$$\pi(f) = \Theta_T f \oplus (I - \Theta_T^* \Theta_T)^{1/2} f, \qquad \pi_*(g) = g \oplus 0,$$

for $f \in L^2(\mathcal{D}_T)$ and $g \in L^2(\mathcal{D}_{T^*})$. Then π and π_* are isometries, $\mathfrak{H}$ is spanned by $\pi L^2(\mathcal{D}_T)$ and $\pi_* L^2(\mathcal{D}_{T^*})$, $\pi_*^* \pi = \Theta_T$, and $\mathfrak{K} = \mathfrak{H} \ominus (\pi H^2(\mathcal{D}_T) \oplus \pi_* H^2_-(\mathcal{D}_{T^*}))$. If P denotes the orthogonal projection (in $\mathfrak{H}$) onto $\mathfrak{K}$, then $P = I_{\mathfrak{H}} - \pi P_+ \pi^* - \pi_* P_- \pi_*^*$.

Let $Z \in \mathcal{L}(\mathfrak{H})$ be the unitary operator which acts as multiplication by z on both coordinates. Then $\pi(zf) = Z\pi f$, $\pi_*(zg) = Z\pi_* g$, and, according to (2.3), $\mathbf{T} = PZ|\mathfrak{K}$.

If $\widetilde{J} : L^2(\mathcal{D}_T) \to L^2(\mathcal{D}_{T^*})$ is defined by $(\widetilde{J}f)(z) = \overline{z} J(f(z))$, then $\widetilde{J}$ is conjugate-linear, isometric and onto; moreover

$$\widetilde{J} P_+ = P_- \widetilde{J}, \quad \widetilde{J} H^2(\mathcal{D}_T) = H^2_-(\mathcal{D}_{T^*}), \tag{3.2}$$

and $\widetilde{J}^{-1} g(z) = \bar{z} J^{-1} g(z)$ for $g \in L^2(\mathcal{D}_{T^*})$.

We define the conjugate-linear map $C : \mathfrak{H} \to \mathfrak{H}$ by the formula

$$C(\pi f + \pi_* g) := \pi_*(\widetilde{J} f) + \pi(\widetilde{J}^{-1} g), \qquad f \in L^2(\mathcal{D}_T), g \in L^2(\mathcal{D}_{T^*}).$$

We prove first that C is a conjugation on $\mathfrak{H}$ and that Z is C-symmetric. Since $\pi, \pi_*, \widetilde{J}, \widetilde{J}^{-1}$ are (linear or conjugate-linear) isometries, it follows that for all $f, h \in L^2(\mathcal{D}_T)$ and all $g, k \in L^2(\mathcal{D}_{T^*})$,

$$\begin{aligned} \langle C(\pi f + \pi_* g), C(\pi h + \pi_* k)\rangle &= \langle \pi(\widetilde{J}^{-1} g), \pi(\widetilde{J}^{-1} k)\rangle + \langle \pi_*(\widetilde{J} f), \pi_*(\widetilde{J} h)\rangle \\ &\quad + \langle \pi(\widetilde{J}^{-1} g), \pi_*(\widetilde{J} h)\rangle + \langle \pi_*(\widetilde{J} f), \pi(\widetilde{J}^{-1} k)\rangle \\ &= \langle k, g\rangle + \langle h, f\rangle + \langle \Theta_T \widetilde{J}^{-1} g, \widetilde{J} h\rangle + \langle \widetilde{J} f, \Theta_T \widetilde{J}^{-1} k\rangle. \end{aligned}$$

But $J\Theta_T(z)^* J = \Theta_T(z)$ implies $\Theta_T \widetilde{J}^{-1} = \widetilde{J} \Theta_T^*$, and therefore

$$\begin{aligned} \langle C(\pi f + \pi_* g), C(\pi h + \pi_* k)\rangle &= \langle k, g\rangle + \langle h, f\rangle + \langle \widetilde{J} \Theta_T^* g, \widetilde{J} h\rangle + \langle \widetilde{J} f, \widetilde{J} \Theta_T^* k\rangle \\ &= \langle k, g\rangle + \langle h, f\rangle + \langle h, \Theta_T^* g\rangle + \langle \Theta_T^* k, f\rangle \\ &= \langle k, g\rangle + \langle h, f\rangle + \langle h, \pi^* \pi_* g\rangle + \langle \pi^* \pi_* k, f\rangle \\ &= \langle \pi h + \pi_* k, \pi f + \pi_* g\rangle. \end{aligned}$$

Thus C is a well-defined isometric conjugate-linear map. It follows immediately from the definition that $C^2 = I_{\mathfrak{H}}$ and thus C is a conjugation on $\mathfrak{H}$.

If $f \in L^2(\mathcal{D}_T)$, then

$$\begin{aligned} CZC(\pi(f)) &= CZ\pi_*(\widetilde{J} f) = C\pi_*(z\widetilde{J} f) = C\pi_*(Jf) \\ &= \pi(\widetilde{J}^{-1} Jf) = \pi(\bar{z} J^{-1} Jf) = \pi(\bar{z} f) = Z^* \pi(f). \end{aligned}$$

Similarly one proves that $CZC(\pi_*(g)) = Z^* \pi_*(g)$ for $g \in L^2(\mathcal{D}_{T^*})$, and therefore $CZC = Z^*$; that is, Z is C-symmetric.

By (3.2), $C(\pi H^2(\mathcal{D}_T)) = \pi_* \widetilde{J} H^2(\mathcal{D}_T) = \pi_* H^2_-(\mathcal{D}_{T^*})$ and $C(\pi_* H^2_-(\mathcal{D}_{T^*})) = \pi H^2(\mathcal{D}_T)$. Since C is isometric, we have

$$C\mathfrak{K} = C\mathfrak{H} \ominus C\left(\pi H^2(\mathcal{D}_T) \oplus \pi_* H^2_-(\mathcal{D}_{T^*})\right) = \mathfrak{H} \ominus \left(\pi H^2(\mathcal{D}_T) \oplus \pi_* H^2_-(\mathcal{D}_{T^*})\right) = \mathfrak{K}.$$

Therefore the restriction C' of C to $\mathfrak{K}$ is a conjugation on $\mathfrak{K}$. Since C leaves $\mathfrak{K}$ and its orthogonal invariant, we have $C|\mathfrak{K} = PCP|\mathfrak{K}$ and $PC(I_{\mathfrak{H}} - P) = 0$. Therefore

$$\mathbf{T} = PZ|\mathfrak{K} = PCZ^*C|\mathfrak{K} = PCPZ^*PCP|\mathfrak{K} = C'\mathbf{T}^*C'.$$

Thus $\mathbf{T}$ is C'-symmetric. Since T is completely nonunitary, T is unitarily equivalent to $\mathbf{T}$ and is therefore also complex symmetric.

Now, let $T \in \mathcal{L}(\mathcal{H})$ be a general contraction satisfying condition (ii) in the statement of the theorem. If we decompose $T = T_0 \oplus T_u$, with T_0 c.n.u. and T_u unitary, then T_0 also satisfies (ii), and it is therefore complex symmetric by the above argument. Since T_u is unitary, it is complex symmetric. Therefore T, being the direct sum of two complex symmetric operators, is also complex symmetric.

(ii)⇒(iii) If (3.1) is satisfied, and C' is some conjugation on $\mathcal{D}_T$, then $U = JC' : \mathcal{D}_T \to \mathcal{D}_{T^*}$ is unitary and $C' = U^*J$. If $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{D}_T)$ is defined by $\Theta(z) = U^*\Theta_T(z)$, then

$$\Theta(z) = U^*J\Theta_T(z)^*J = U^*J(U^*\Theta_T(z))^*U^*J = C'\Theta(z)^*C'.$$

(iii)⇒(ii) If $U : \mathcal{E} \to \mathcal{D}_T$, $U_* : \mathcal{E} \to \mathcal{D}_{T^*}$ are unitary operators satisfying $\Theta_T(z) = U_*\Theta(z)U^*$ for all $z \in \mathbb{D}$, then $J = U_*J'U^*$ satisfies all requirements in (ii). □

Corollary 3.2. *A contraction T with $\partial_T = \partial_{T^*} = 1$ is complex symmetric.*

Proof. If $\partial_T = \partial_{T^*} = 1$, then Θ_T is scalar-valued, and we may identify $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ with $\mathbb{C}$. The natural conjugation J on $\mathbb{C}$ defined by $J(z) = \overline{z}$ then satisfies condition (iii) in Theorem 3.1, whence T is complex symmetric. □

For the case $T \in C_{00}$, Corollary 3.2 is proved in [5] and [4], where more of its consequences are developed. Also in [5] one can find the next result, for which we give a different proof.

Corollary 3.3. *Any operator on a 2-dimensional space is complex symmetric.*

Proof. Since the complex symmetry is preserved by multiplication with nonzero scalars, it is enough to assume $\|T\| = 1$. But then either T is unitary, or $\partial_T = \partial_{T^*} = 1$, in which case we may apply Corollary 3.2. □

It follows from Theorem 3.1 that if a contraction T is complex symmetric, then $\partial_T = \partial_{T^*}$. However, this is also a consequence of a more general result from [5], namely that if a (not necessarily contractive) operator T is complex symmetric, then $\dim\ker T = \dim\ker T^*$.

4. 2×2 INNER FUNCTIONS

As shown in Corollary 3.2, contractions with defect indices 1 are always complex symmetric. As an application of Theorem 3.1, we will discuss in this section the case $\partial_T = \partial_{T^*} = 2$. We assume moreover that the characteristic function Θ_T is inner, which is equivalent to $T \in C_{00}$.

Definition 4.1. Let $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ be a contractive analytic function. We say that Θ is *symmetrizable* if its matrix with respect to some fixed orthonormal bases (independent of z) in $\mathcal{E}$ and $\mathcal{E}_*$ is symmetric for all $z \in \mathbb{D}$.

According to Lemma 2.1 and Theorem 3.1, (iii), a contraction is complex symmetric if and only if its characteristic function is symmetrizable. We are interested in this section in 2×2 matrix-valued characteristic functions $\Theta(z)$. Note that Corollary 3.3 implies that, for all $z \in \mathbb{D}$, there exist $U_1(z), U_2(z)$ unitary such that $U_1(z)\Theta(z)U_2(z)$ is symmetric. But, in order to find symmetrizable analytic functions, the matrices U_1 and U_2 should not depend on z.

We recall the following result in [3] which gives a parametrization of 2×2 inner functions.

Proposition 4.2. *Suppose ϕ is a nonconstant inner function in H^∞, $a, b, c, d \in H^\infty$, and*

$$\Theta(z) = \begin{pmatrix} a(z) & -b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Then Θ is a 2×2 inner function and $\det \Theta = \phi$ if and only if

(i) *a, b, c, d belong to $\mathcal{H}(z\phi) = H^2 \ominus z\phi H^2$;*
(ii) *$d = C(a)$ and $c = C(b)$;*
(iii) *$|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$.*

Here C denotes the natural conjugation on $\mathcal{H}_{z\phi}$ defined by

$$C(f) = \overline{f}\phi \qquad (f \in \mathcal{H}_{z\phi}). \tag{4.1}$$

The following result characterizes the symmetrizable 2×2 matrix-valued inner functions.

Theorem 4.3. *A 2×2 inner function $\Theta(z) = \begin{pmatrix} a(z) & -b(z) \\ C(b)(z) & C(a)(z) \end{pmatrix}$ is symmetrizable if and only if there exists $(\gamma, \theta) \neq (0,0)$ such that $\gamma a + \theta b$ is a fixed point of C, where C is defined by (4.1), $\phi = \det \Theta$.*

Proof. Suppose there exists $(\gamma, \theta) \neq (0,0)$ such that $C(\gamma a + \theta b) = \gamma a + \theta b$; we may assume that $|\gamma|^2 + |\theta|^2 = 1$. Define the unitary matrix U by $U = \begin{pmatrix} \overline{\theta} & -\gamma \\ \overline{\gamma} & \theta \end{pmatrix}$. Then

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \Theta(z) U = \begin{pmatrix} -i\overline{\theta}a(z) + i\overline{\gamma}b(z) & i(\gamma a(z) + \theta b(z)) \\ i(\gamma a(z) + \theta b(z)) & -i\gamma C(b)(z) + i\theta C(a)(z) \end{pmatrix}$$

since $\overline{\theta}C(b)(z) + \overline{\gamma}C(a)(z) = C(\gamma a + \theta b)(z) = (\gamma a + \theta b)(z)$. Therefore Θ is symmetrizable.

Reciprocally, assume that Θ is symmetrizable. If a nontrivial linear combination of a, b is 0, then we are done, since of course 0 is a fixed point of C.

Suppose then that the system $\{a, b\}$ is linearly independent. By definition, there exist two unitary matrices U_1 and U_2 such that $U_1\Theta(z)U_2$ is symmetric for all $z \in \mathbb{D}$. Write

$$U_1 = \begin{pmatrix} \mu & -\overline{\lambda} \\ \lambda & \overline{\mu} \end{pmatrix}, \qquad U_2 = \begin{pmatrix} \theta & -\overline{\gamma} \\ \gamma & \overline{\theta} \end{pmatrix},$$

with $|\mu|^2 + |\lambda|^2 = 1$ and $|\theta|^2 + |\gamma|^2 = 1$. Straightforward computations show that

$$U_1\Theta(z)U_2 = \begin{pmatrix} * & X \\ Y & * \end{pmatrix},$$

with $X = -\mu(\overline{\gamma}a + \overline{\theta}b) - \overline{\lambda}C(-\gamma b + \theta a)$ and $Y = \lambda(\theta a - \gamma b) + \overline{\mu}C(\overline{\theta}b + \overline{\gamma}a)$. Then the symmetry of the matrix is equivalent to

$$-(\mu\overline{\gamma} + \lambda\theta)a - (\mu\overline{\theta} - \lambda\gamma)b = C\left((\mu\overline{\gamma} + \lambda\theta)a + (\mu\overline{\theta} - \lambda\gamma)b\right).$$

If we put $u := (\mu\overline{\gamma} + \lambda\theta)a + (\mu\overline{\theta} - \lambda\gamma)b$, then it follows that $C(u) = -u$, that is, $C(iu) = iu$, and iu is a fixed point of C. To conclude the proof, we need to show that $(\mu\overline{\gamma} + \lambda\theta, \mu\overline{\theta} - \lambda\gamma) \neq (0,0)$.

Suppose then that

$$\mu\overline{\gamma} + \lambda\theta = \mu\overline{\theta} - \lambda\gamma = 0. \tag{4.2}$$

If we multiply $\mu\overline{\gamma} = -\lambda\theta$ by $\overline{\theta}$ and $\mu\overline{\theta} = \lambda\gamma$ by $\overline{\gamma}$, and subtract, we obtain $\lambda(|\theta|^2 + |\gamma|^2) = 0$. But $|\theta|^2 + |\gamma|^2 = 1$, so $\lambda = 0$, whence $|\mu|^2 + |\lambda|^2 = 1$ yields $|\mu| = 1$. Then (4.2) implies $\gamma = \theta = 0$: a contradiction. □

Remark 4.4. Note that the fixed points of a conjugation C can easily be described by using Lemma 2.1 (i). They form the real vector space of all elements which have real Fourier coefficients with respect to a C-real orthonormal basis.

Remark 4.5. A closely related question would be to describe all symmetric 2×2-matrix valued analytic contractive inner functions $\Theta(z)$. This can be done along the lines of the solution of the Darlington sythesis problem in [3, Section 5], as follows. We first fix $\det \Theta$, which will be a nonconstant scalar inner function $\phi \in H^\infty$. Then we take a function $b \in H^2 \ominus z\phi H^2$, such that $Cb = b$ (C the conjugation $f \mapsto \phi\overline{f}$ on $H^2 \ominus z\phi H^2$). If b is inner, then $b^2 = \phi$, and

$$\Theta(z) = \begin{pmatrix} 0 & ib(z) \\ ib(z) & 0 \end{pmatrix}.$$

If b is not inner, then we take $a \in H^2 \ominus z\phi H^2$, such that $|a|^2 + |b|^2 = 1$ (such a's exist by [3, Proposition 5.2]). Then

$$\Theta(z) = \begin{pmatrix} a(z) & ib(z) \\ ib(z) & C(a)(z) \end{pmatrix}.$$

In [5, 8.2] one further discusses the parametrization of all rational solutions of a Darlington synthesis. Similarly, one could describe all rational symmetric 2×2-matrix valued analytic contractive inner functions $\Theta(z)$.

However, our interest is in rather complex symmetric contractions, and the characteristic function is only a method of studying them. If we want to parametrize, up to unitary equivalence, all complex symmetric contractions with defect indices 2, then we also need to determine when two symmetric characteristic functions coincide. This problem does not seem to have a neat solution.

5. An example

Consider two nonconstant scalar inner functions $u, v \in H^\infty$, and let $\mathbf{T}_u, \mathbf{T}_v$ be the corresponding model operators (the *compressed shifts* in the terminology of [3]). The corresponding model spaces are $\mathfrak{K}_u = H^2 \ominus uH^2$ and $\mathfrak{K}_v = H^2 \ominus vH^2$. As noted above, $\mathbf{T}_u$ and $\mathbf{T}_v$ are completely nonunitary contractions with characteristic functions u and v respectively. Their defect spaces are 1-dimensional, and it follows from Corollary 3.2 that they are both complex symmetric.

We will discuss the contractions of the form

$$T = \begin{pmatrix} \mathbf{T}_u & X \\ 0 & \mathbf{T}_v \end{pmatrix}; \tag{5.1}$$

thus $T \in \mathcal{L}(\mathcal{H})$, with $\mathcal{H} = \mathfrak{K}_u \oplus \mathfrak{K}_v$. The next lemma gathers some facts about this operator.

Lemma 5.1. *Suppose $T \in \mathcal{L}(\mathfrak{K}_u \oplus \mathfrak{K}_v)$ is a contraction. Then:*

(i) $X = D_{\mathbf{T}_u^*} Y D_{\mathbf{T}_v}$*, with* $Y : \mathcal{D}_{\mathbf{T}_v} \to \mathcal{D}_{\mathbf{T}_u^*}$ *a contraction.*
(ii) $\partial_T = \partial_{T^*} = 1$ *if* $\|Y\| = 1$*, and* $\partial_T = \partial_{T^*} = 2$ *otherwise.*
(iii) $T \in C_{00}$.

Note that, since $\dim \mathcal{D}_{\mathbf{T}_v} = \dim \mathcal{D}_{\mathbf{T}_u^*} = 1$, Y can actually be identified with a complex number of modulus not larger than 1.

Proof. The general form of the entries of a 2×2 contraction, as described, for instance, in [1, Theorem 1.3] or [2, IV.3], applied to the case when one of the entries is null, immediately yields (i), as well as an identification of $\mathcal{D}_T$ with $\mathcal{D}_{\mathbf{T}_v^*} \oplus \mathcal{D}_Y$, and of $\mathcal{D}_{T^*}$ with $\mathcal{D}_{\mathbf{T}_v} \oplus \mathcal{D}_{Y^*}$, whence (ii) follows.

Finally, (iii) is an instance of a more general fact: if $T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is a contraction, then $T_i \in C_{0\cdot}$ implies $T \in C_{0\cdot}$. Indeed, take $\epsilon > 0$, and a vector $x = x_1 \oplus x_2$. First choose k such that $\|T_2^k x_2\| < \epsilon$. If $T^k(0 \oplus x_2) = x_1' \oplus T_2^k x_2$, take k' such that $\|T_1^{k'}(x_1' \oplus T_1^k x_1)\| < \epsilon$. Then

$$\begin{aligned}\|T^{k+k'}x\| &= \|T^{k+k'}(x_1 \oplus 0) + T^{k+k'}(0 \oplus x_2)\| = \|(T_1^{k+k'}x_1 \oplus 0) + T^{k'}(x_1' \oplus T_2^k x_2)\| \\ &\le \|(T_1^{k'}(T_1^k x_1 + x_1') \oplus 0)\| + \|T^{k'}(0 \oplus T_2^k x_2)\| \le \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Since in our case $T_1 = \mathbf{T}_u$ and $T_2 = \mathbf{T}_v$ are both of class C_{00}, the result follows. ☐

The next theorem determines when is T complex symmetric.

Theorem 5.2. *T is complex symmetric precisely in the following cases:*

(i) $Y = 0$;
(ii) $\|Y\| = 1$;
(iii) $0 < \|Y\| < 1$ *and there exist* $\lambda \in \mathbb{D}$ *and* $\mu \in \mathbb{T}$ *such that* $v = \mu b_\lambda(u)$*, where* b_λ *denotes the elementary Blaschke factor defined by*

$$b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Proof. If $Y = 0$, then $T = \mathbf{T}_u \oplus \mathbf{T}_v$, and it is therefore complex symmetric as the direct sum of two complex symmetric operators. If $\|Y\| = 1$, then T has defect indices 1 by Lemma 5.1, and is therefore complex symmetric by Corollary 3.2. (One can then easily see, using (5.2) below, that Θ_T coincides with the scalar function uv.) We can thus suppose in the sequel that $0 < \|Y\| < 1$, and $\partial_T = \partial_{T^*} = 2$.

Since we intend to apply Theorem 3.1, we have to determine the characteristic function of T. This can be calculated directly, but in order to avoid some tedious computations, we prefer to use the theory of invariant subspaces of contractions and factorizations of the characteristic function, as developed in [6, Chapter VII].

First, note that $T \in C_{00}$ implies Θ_T inner. Since $\mathfrak{K}_u$ is an invariant subspace for T, it follows from Theorem VII.1.1 and Proposition VII.2.1 from [6] that one can factorize

$$\Theta_T(z) = \Theta_2(z)\Theta_1(z) \tag{5.2}$$

into two analytic inner functions, and that the characteristic functions of $\mathbf{T}_u$ and $\mathbf{T}_v$, that is, u and v, are equal to the pure parts of Θ_1 and Θ_2. Also, Θ_1 and Θ_2 both being inner, the dimensions of their range spaces are both equal to the dimension of the range of Θ_T.

It then follows that Θ_1 and Θ_2 must be 2×2 matrix valued inner functions, and their pure parts are u and v respectively. They coincide therefore with $\left(\begin{smallmatrix}1&0\\0&u\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\0&v\end{smallmatrix}\right)$ respectively. According to (5.2), we have 2×2 unitary matrices U_1, U_2, V_1, V_2 such that

$$\Theta_T = U_1 \begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix} U_2 V_1 \begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix} V_2.$$

If we write

$$U_2 V_1 = \begin{pmatrix}\alpha & -\beta\\ \bar\beta & \bar\alpha\end{pmatrix}$$

with α, β complex numbers satisfying $|\alpha|^2+|\beta|^2=1$, it follows that the characteristic function Θ_T coincides with the inner function

$$\Theta(z) = \begin{pmatrix}1 & 0\\ 0 & v\end{pmatrix}\begin{pmatrix}\alpha & -\beta\\ \bar\beta & \bar\alpha\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix} = \begin{pmatrix}\alpha & -\beta u(z)\\ \bar\beta v(z) & \bar\alpha u(z)v(z)\end{pmatrix}. \tag{5.3}$$

Note that condition $0<\|Y\|<1$ implies both α and β different from 0.

We now apply Theorem 4.3 in order to determine when Θ, as given by (5.3), is symmetrizable. Since $\det\Theta = uv$, this happens if and only if a linear combination of α and βu, not having both coefficients null, belongs to the fixed points of the conjugation C on $\mathfrak{K}_{zuv}$ given by $C(f)=uv\bar f$.

If this is the case, and we write the combination as $g=s+tu$, $s,t\in\mathbb{C}$ (and s,t are not both null, which implies also $g\neq 0$), then

$$C(g)=g \Leftrightarrow v(\bar s u+\bar t)=s+tu, \tag{5.4}$$

and thus

$$v=\frac{s+tu}{\bar s u+\bar t}.$$

We must have $t\neq 0$, since otherwise uv is constant, which is not possible. So we can write

$$v=\frac{t}{\bar t}\,\frac{\frac{s}{t}+u}{1+\frac{\bar s}{\bar t}u}.$$

But now if $|s|=|t|$, then $v=\frac{t}{\bar s}$ which is impossible. If $|s|>|t|$, then we see that v is at the same time analytic and coanalytic; whence v is constant — again a contradiction. So the only possibility is $|s|<|t|$. If we put $\lambda=-\frac{s}{t}$ and $\mu=-\frac{t}{\bar t}$ we get the desired conclusion that $v=\mu b_\lambda(u)$.

Conversely, suppose $v=\mu b_\lambda(u)$ with $|\lambda|<1$ and $|\mu|=1$. Write $\mu=-\frac{\zeta}{\bar\zeta}$, with $\zeta\neq 0$. Then

$$v=\frac{\zeta u-\lambda\zeta}{\bar\zeta-\overline{\zeta\lambda}u},$$

and if we define $s:=-\lambda\zeta$ and $t:=\zeta$, then

$$v(\bar s u+\bar t)=v(-\overline{\lambda\zeta}u+\bar\zeta)=\zeta u-\lambda\zeta=s+tu,$$

which implies by (5.4) that $C(g)=g$, with $g:=s+tu$. Since $t\neq 0$, we may apply Theorem 4.3 to conclude that Θ is symmetrizable.

We have thus proved that in the case $0<\|Y\|<1$, Θ_T is symmetrizable if and only if $v=\mu b_\lambda(u)$ with $|\lambda|<1$ and $|\mu|=1$. Now applying Theorem 3.1 ends the proof. □

Using Theorem 5.2, it is easy to construct examples of complex symmetric as well as noncomplex symmetric operators with defect indices 2.

It is not surprising that the condition obtained depends on the norm of $\|Y\|$ (or, rather, its modulus), and not on Y itself. Indeed, with a little effort one can show that all operators T with u, v and $\|Y\|$ fixed are unitarily equivalent.

References

1. Gr. Arsene and A. Gheondea, *Completing matrix contractions*, J. Operator Theory **7** (1982), 179–189. MR0650203 (83i:47010)
2. C. Foias and A.E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Birkhäuser Verlag, Basel, 1990. MR1120546 (92k:47033)
3. S.R. Garcia, *Conjugation, the backward shift, and Toeplitz kernels*, J. Operator Theory **54:2** (2005), 239–250. MR2186351 (2006g:30055)
4. ———, *Conjugation and Clark operators*, Contemporary Mathematics, vol. 393, Amer. Math. Soc., Providence, RI, 2006. MR2198373
5. S.R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358:3** (2006), 1285–1315. MR2187654 (2006j:47036)
6. B. Sz-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, New York, 1970. MR0275190 (43:947)

Institut Camille Jordan, UFR de Mathématiques, Université Claude Bernard Lyon I, 69622 Villeurbanne Cedex, France

E-mail address: chevrot@math.univ-lyon1.fr

Institut Camille Jordan, UFR de Mathématiques, Université Claude Bernard Lyon I, 69622 Villeurbanne Cedex, France

E-mail address: fricain@math.univ-lyon1.fr

Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania

E-mail address: Dan.Timotin@imar.ro

6.4.2 Référence [T11]

Auteurs
I. Chalendar, E. Fricain et D. Timotin

Titre
A note on the stability of linear combinations of algebraic operators.

A Paraître dans
Extracta Mathematicae.

A note on the stability of linear combinations of algebraic operators

I. Chalendar[*], E. Fricain[†] and D. Timotin[‡]

April 8, 2008

Abstract

The aim of this note is to characterize all the algebraic operators S and T having the same minimal polynomial and for which many spectral properties of linear combinations of S and T do not depend on their coefficients.

1 Introduction

Let X be a Banach space, and T, S two idempotent operators on X. Several papers [2, 5] have addressed stability properties of the linear combination $c_1T + c_2S$; it has been proved that a large number of properties (e.g. injectivity, invertibility, Fredholmness) are shared by all such linear combinations, provided $c_1, c_2 \neq 0$ and $c_1 + c_2 \neq 0$.

An idempotent T is defined by the relation $T^2 = T$; in other words, it is an algebraic operator, and its minimal polynomial (except in trivial cases) is $p(z) = z^2 - z$. A natural question is whether the stability results above can be extended to more general situations. Thus, we may consider two

[*]I. C. J., UFR de Mathématiques, Université Lyon 1, 43 bld. du 11/11/1918, 69622 Villeurbanne Cedex, France. chalenda@math.univ-lyon1.fr.

[†]I. C. J., UFR de Mathématiques, Université Lyon 1, 43 bld. du 11/11/1918, 69622 Villeurbanne Cedex, France. fricain@math.univ-lyon1.fr

[‡]Institute of Mathematics of the Romanian Academy, PO Box 1-764, Bucharest 014700, Romania. Dan.Timotin@imar.ro

algebraic operators T, S with the same minimal polynomial p, and look for similar stability results. We will show below that essentially there is no such extension; in other words, these properties of idempotents are rather special. The situation is the same even if we restrict ourselves to matrices instead of operators.

On the positive side, if we assume that the two operators T, S commute, then we can easily obtain stability results of the type discussed, even if their minimal polynomials are different. This is a consequence of (multidimensional) spectral theory.

2 Main result

As in [2], instead of $c_1T + c_2S$ we will rather consider the operator $T - zS$, and thus work with a single parameter z.

Theorem 2.1. *Let p be a unital polynomial of degree $d \geq 1$. The following assertions are equivalent:*

a) $p(z) = z - a$ or $p(z) = z^2 - bz$ where $b \neq 0$;

b) there exists a finite set F such that for all matrices S, T whose minimal polynomial is p, $z \mapsto \dim\ker(T - zS)$ is constant on $\mathbb{C} \setminus F$.

Proof : $a) \Rightarrow b)$: If $p(z) = z - a$, then $T - zS = (1 - z)aI$ and the result is obvious with $F = \{1\}$. If $p(z) = (z^2 - bz)$ with $b \neq 0$, then S/b and T/b are idempotents. Since $\dim\ker(T - zS) = \dim\ker(T/b - zS/b)$, using the main result of [2] or [5], we get the result with $F = \{0, 1\}$.

$b) \Rightarrow a)$: We will discuss the several possible cases.

I. Degree of $p = 2$.

Ia. If $p(z) = (z - a)(z - b)$ with $a, b \in \mathbb{C} \setminus \{0\}$ and $a \neq b$, take $S_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $T_\theta = R_\theta S_{a,b} R_\theta^{-1}$ where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The minimal polynomial of $S_{a,b}$ and T_θ is $p(z) = (z - a)(z - b)$ since $S_{a,b}$ and T_θ are unitarily equivalent. The determinant of $T_\theta - zS_{a,b}$ is equal to

$$d(z) = abz^2 - z(2ab + (a - b)^2 \sin^2\theta) + ab.$$

We have $\dim\ker(T_\theta - zS_{a,b}) = 0$ if z is not a root of $d(z)$; since the set of values of these roots, when $\theta \in [0, 2\pi)$, is infinite, there is no set F as required.

Ib. If $p(z) = (z-a)^2$ with $a \neq 0$, consider $S_a = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ and $T_\theta = R_\theta S R_\theta^{-1}$. The minimal polynomial of S_a and T_θ is $p(z) = (z-a)^2$. The determinant of $T_\theta - zS_a$ is equal to

$$d(z) = a^2 z^2 - z(\sin\theta + 2a^2) + a^2.$$

As above, the set of its roots is infinite when $\theta \in [0, 2\pi)$.

Ic. If $p(z) = z^2$, take $S_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $T_\theta = U_\theta S_0 U_\theta^{-1}$ where U_θ is the unitary matrix defined by $U_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}$. Since we have

$$T_\theta - zS_0 = \begin{pmatrix} 0 & \cos\theta - z & 0 & -\sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & \sin\theta & 0 & \cos\theta - z \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the dimension of $\ker(T_\theta - zS_0)$ is 2 for all $z \in \mathbb{C} \setminus \{e^{i\theta}, e^{-i\theta}\}$ and is 3 or 4 (for $\theta = k\pi$, $k \in \mathbb{Z}$) otherwise. Therefore, in each of the above cases, there is no finite set F such that for all $z \in \mathbb{C} \setminus F$ the dimension of $\ker(T - zS)$ is constant independently of the choice of S and T.

II. Degree of $p \geq 3$.

IIa. Suppose that the roots of p are all distinct. Then p has at least two nonzero distinct roots a, b. Consider $S = S_{a,b} \oplus A$ and $T_\theta = R_\theta S_{a,b} R_\theta^{-1} \oplus A$, where A is a matrix whose minimal polynomial is p. As for the case $p(z) = (z-a)(z-b)$, considering $T_\theta - zS$, there is no finite set F on which $z \mapsto \dim\ker(T_\theta - zS)$ is constant on $\mathbb{C} \setminus F$.

IIb. If p has a root a of multiplicity at least 2, take A an arbitrary matrix whose minimal polynomial is p. Consider $S = S_a \oplus A$, $T_\theta = R_\theta S_a R_\theta^{-1} \oplus A$ if $a \neq 0$, and $S = S_0 \oplus A$, $T_\theta = U_\theta S_0 U_\theta^{-1}$ if $a = 0$. As above, we obtain that there exists no finite set F, independent of the choice of S and T, on which $z \mapsto \dim\ker(T_\theta - zS)$ is constant on $\mathbb{C} \setminus F$. □

Remark 2.2. The remarkable property of a pair of idempotents cannot be extended to more than two. One might hope for instance that if P, Q, R are three idempotents, then $\dim\ker(P+zQ+wR)$ is constant outside a fixed algebraic variety (not depending on the idempotents). But this is easily seen not to be true. Indeed, denote $P_t = \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix}$. Then the determinant of $P_0 + zP_t + wP_\theta$ is $z\sin^2 t + w\sin^2\theta + zw\sin^2(t-\theta)$.

3 Commuting operators

As opposed to the general case, it is rather simple to obtain stability if the two operators $T, S \in \mathcal{L}(X)$ commute.

Remember that the left spectrum $\sigma^l(T,S)$ is defined as the set of $(z,w) \in \mathbb{C}^2$ for which $T - zI$ and $S - wI$ generate a proper left ideal of $\mathcal{L}(X)$. A similar definition gives the right spectrum $\sigma^r(T,S)$, while the Harte spectrum is $\sigma^H(T,S) = \sigma^l(T,S) \cup \sigma^r(T,S)$. We have then the spectral mapping theorem [3]:

Lemma 3.1. *If $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subset \mathbb{C}^2$ containing $\sigma^H(T,S)$, then $\sigma^l(f(T,S)) = f(\sigma^l(T,S))$, $\sigma^r(f(T,S)) = f(\sigma^r(T,S))$, and $\sigma^H(f(T,S)) = f(\sigma^H(T,S))$.*

Theorem 3.2. *Suppose $T, S \in \mathcal{L}(X)$ are two commuting algebraic operators, with corresponding minimal polynomials p, q. Suppose that the roots of p are λ_i, $i = 1, \ldots, m$ and those of q are μ_j, $j = 1, \ldots, n$. Define the set $F = \{\frac{\lambda_i}{\mu_j} : i = 1, \ldots, m, j = 1, \ldots, n, \mu_j \neq 0\}$. Then, for all $z \notin F$, $T - zS$ is simultaneously left invertible or not.*

Proof : Applying Lemma 3.1 to the function $f(\lambda, \mu) = \lambda - z\mu$, it follows that $T - zS$ is left invertible iff $\lambda - z\mu \neq 0$ for all $(\lambda, \mu) \in \sigma^l(T,S)$. If $(0,0) \in \sigma^l(T,S)$, then this last condition is not satisfied for any z, and thus $T - zS$ is not invertible for all $z \in \mathbb{C}$.

Suppose now $(0,0) \notin \sigma^l(T,S)$. Take $(\lambda,\mu) \in \sigma^l(T,S)$. If $\mu = 0$, then $\lambda \neq 0$, and thus $\lambda - z\mu \neq 0$; therefore $T - zS$ is left invertible. If $\mu \neq 0$, but $\lambda - z\mu = 0$, then $z = \frac{\lambda}{\mu}$. Since $\sigma^l(T,S) \subset \sigma^l(T) \times \sigma^l(S)$, it follows that $z \in F$. Therefore $T - zS$ is left invertible for any $z \notin F$. □

Remark 3.3. Note that if T, S are commuting algebraic operators, then $T - zS$ is also algebraic, since the algebras generated by T and S are finite dimensional, while the algebra generated by $T - zS$ is contained in their product. As the spectrum of an algebraic operator is equal to its point spectrum, injectivity is equivalent to left, right or simple invertibility, or boundedness below (they are all equivalent to the fact that $0 \notin \sigma(T)$). One can therefore reformulate Theorem 3.2 in each of these terms.

An operator $T \in \mathcal{L}(X)$ is called semi-Fredholm if its range $R(T)$ is closed and either $X/R(T)$ or $\ker T$ have finite dimension, and Fredholm if both have finite dimension. More precisely, it is upper semi-Fredholm if $\dim \ker T < \infty$ and lower semi-Fredholm if $\dim X/R(T) < \infty$. Also, T upper semi-Fredholm implies T left essentially invertible, T lower semi-Fredholm implies T right essentially invertible, and T Fredholm implies T essentially invertible ("essentially" meaning modulo compact operators). A procedure introduced in [6, 1, 4] allows us to extend the results above to these classes. Namely, if X is a Banach space, one can define the spaces

$$\begin{aligned} \ell^\infty(X) &= \{\mathbf{x} = (x_n) : x_n \in X, \ \sup \|x_n\| < \infty\}, \\ \tau(X) &= \{\mathbf{x} \in \ell^\infty(X) : \{x_n : n \in \mathbb{N}\} \text{ is totally bounded in } X\}, \\ \tilde{X} &= \ell^\infty(X)/\tau(X), \end{aligned}$$

and one has the following result [6, 1, 4]:

Proposition 3.1. *If $T \in \mathcal{L}(X)$, then T is upper semi-Fredholm if and only if $\tilde{T}$ is injective.*

If T is algebraic then $\tilde{T}$ is also algebraic (with the same minimal polynomial), and Remark 3.3 applies to $\tilde{T}, \tilde{S}$. We obtain thus the following corollary.

Corollary 3.4. *With the above notation, for all $z \notin F$ the operator $T - zS$ is simultaneously lower semi-Fredholm, upper semi-Fredholm, Fredholm, left essentially invertible, right essentially invertible, essentially invertible.*

We may compare Theorem 3.2, Remark 3.3 and Corollary 3.4 with the Main Theorem in [2], or with [5, Theorem 3.1].

ACKNOWLEDGEMENTS. The authors wish to thank Mostafa Mbekhta for drawing their attention to this problem.

References

[1] J. J. Buoni, R. Harte, and T. Wickstead. Upper and lower Fredholm spectra. I. *Proc. Amer. Math. Soc.*, 66(2):309–314, 1977.

[2] H.-K. Du, C.-Y. Deng, M. Mbekhta, and V. Müller. On spectral properties of linear combinations of idempotents. *Studia Math.*, 180(3):211–217, 2007.

[3] R. Harte. A spectral mapping theorem for holomorphic functions. *Math. Z.*, 154(1):67–69, 1977.

[4] R. Harte. *Invertibility and singularity for bounded linear operators*, volume 109 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1988.

[5] J. J. Koliha and V. Rakočević. Stability theorems for linear combinations of idempotents. *Integral Equations Operator Theory*, 58(4):597–601, 2007.

[6] B. N. Sadovskiĭ. Limit-compact and condensing operators. *Uspehi Mat. Nauk*, 27(1(163)):81–146, 1972.

6.4.3 Référence [T12]

Auteurs
I. Chalendar, E. Fricain, A. Popov, D. Timotin et V. Troitsky.

Titre
Finitely strictly singular operators between James spaces

A Paraître dans
J. Funct. Anal.

FINITELY STRICTLY SINGULAR OPERATORS BETWEEN JAMES SPACES

ISABELLE CHALENDAR, EMMANUEL FRICAIN, ALEXEY I. POPOV, DAN TIMOTIN, AND VLADIMIR G. TROITSKY

Abstract. An operator $T\colon X \to Y$ between Banach spaces is said to be finitely strictly singular if for every $\varepsilon > 0$ there exists n such that every subspace $E \subseteq X$ with $\dim E \geqslant n$ contains a vector x such that $\|Tx\| < \varepsilon\|x\|$. We show that, for $1 \leqslant p < q < \infty$, the formal inclusion operator from J_p to J_q is finitely strictly singular. As a consequence, we obtain that the strictly singular operator with no invariant subspaces constructed by C. Read is actually finitely strictly singular. These results are deduced from the following fact: if $k \leqslant n$ then every k-dimensional subspace of $\mathbb{R}^n$ contains a vector x with $\|x\|_{\ell_\infty} = 1$ such that $x_{m_i} = (-1)^i$ for some $m_1 < \cdots < m_k$.

1. Introduction

Recall that an operator $T\colon X \to Y$ between Banach spaces is said to be ***strictly singular*** if for every $\varepsilon > 0$ and every infinite dimensional subspace $E \subseteq X$ there is a vector x in the unit sphere of E such that $\|Tx\| < \varepsilon$. Furthermore, T is said to be ***finitely strictly singular*** if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace $E \subseteq X$ with $\dim E \geqslant n$ there exists a vector x in the unit sphere of E such that $\|Tx\| < \varepsilon$. Finitely strictly singular operators are also known in literature as ***superstrictly singular***. Note that

$$\text{compact} \quad \Rightarrow \quad \text{finitely strictly singular} \quad \Rightarrow \quad \text{strictly singular},$$

and that each of these three properties defines a closed subspace in $L(X, Y)$. Actually, each property defines an operator ideal. We refer the reader to [7, 9, 14, 2, 11, 10] for more information about strictly and finitely strictly singular operators. All the Banach spaces in this paper are assumed to be over real scalars.

We say that a subspace $E \subseteq X$ is invariant under an operator $T\colon X \to X$ if $\{0\} \neq E \neq X$ and $T(E) \subseteq E$. Every compact operator has invariant subspaces by [1]. On the other hand, Read constructed in [12] an example of a strictly singular operator without nontrivial closed invariant subspaces (this answered a question of Pełczyński). Read's operator acts on an infinite direct sum which involves James spaces. Recall that James' p-space J_p is a sequence space consisting of all sequences $x = (x_n)_{n=1}^{\infty}$ in

c_0 satisfying $\|x\|_{J_p} < \infty$ where

$$\|x\|_{J_p} = \Bigl(\sup\Bigl\{\sum_{i=1}^{n-1} |x_{k_{i+1}} - x_{k_i}|^p \;:\; 1 \leqslant k_1 < \cdots < k_n,\ n \in \mathbb{N}\Bigr\}\Bigr)^{\frac{1}{p}}$$

is the norm in J_p. For more information on James' spaces we refer the reader to [6, 13, 7, 3, 8].

It was an open question whether every finitely strictly singular operator has invariant subspaces. Some partial results in this direction were obtained in [2, 11]. We answer this question in the negative by showing that the operator in [12] is, in fact, finitely strictly singular. As an intermediate result, we prove that the formal inclusion operator from J_p to J_q with $1 \leqslant p < q < \infty$ is finitely strictly singular. The latter statement in a certain sense refines the result of Milman [9] that the formal inclusion operator from ℓ_p to ℓ_q with $1 \leqslant p < q < \infty$ is finitely strictly singular.

Milman's proof is based on the fact that every k-dimensional subspace E of $\mathbb{R}^n$ contains a vector "with a flat", namely, a vector x with sup-norm one with (at least) k coordinates equal in modulus to 1. For such a vector, one has $\|x\|_{\ell_q} \ll \|x\|_{\ell_p}$. The proofs of our results are based on the following refinement of this observation. We will show that x can be chosen so that these k coordinates have alternating signs. For such a "highly oscillating" vector x one has $\|x\|_{J_q} \ll \|x\|_{J_p}$. More precisely, a finite or infinite sequence of real numbers in $[-1, 1]$ will be called a ***zigzag*** of order k if it has a subsequence of form $(-1, 1, -1, 1, \dots)$ of length k. Our results will be based on the following theorem; two different proofs of it will be presented in Sections 2 and 3.

Theorem 1. *For every $k \leqslant n$, every k-dimensional subspace of $\mathbb{R}^n$ contains a zigzag of order k.*

Corollary 2. *Let $k \in \mathbb{N}$, then every k-dimensional subspace of c_0 contains a zigzag of order k.*

Proof. Let F be a subspace of c_0 with $\dim F = k$. For every $n \in \mathbb{N}$, define $P_n \colon c_0 \to \mathbb{R}^n$ via $P_n \colon (x_i)_{i=1}^{\infty} \mapsto (x_i)_{i=1}^{n}$. Let n_1 be such that $\dim P_{n_1}(F) = k$. There exists n_2 such that every vector in F attains its norm on the first n_2 coordinates. Indeed, define $g \colon F \setminus \{0\} \to \mathbb{N}$ via $g(x) = \max\bigl\{i \;:\; |x_i| = \|x\|_\infty\bigr\}$. Then g is upper semi-continuous, hence bounded on the unit sphere of F, so that we put $n_2 = \max\bigl\{g(x) \;:\; x \in F, \|x\| = 1\bigr\}$.

Put $n = \max\{n_1, n_2\}$. Since $P_n(F)$ is a k-dimensional subspace of $\mathbb{R}^n$, by Theorem 1 there exists $x \in F$ such that $P_n x$ is a zigzag of order k. It follows from our definition of n that x is a zigzag of order k in F. □

Suppose that $1 \leqslant p < q$. Since $\|x\|_{J_p}$ is defined as the supremum of ℓ_p-norms of certain sequences, $\|\cdot\|_{\ell_q} \leqslant \|\cdot\|_{\ell_p}$ implies $\|\cdot\|_{J_q} \leqslant \|\cdot\|_{J_p}$. It follows that $J_p \subseteq J_q$ and the formal inclusion operator $i_{p,q} : J_p \to J_q$ has norm 1. We show next that it is finitely strictly singular. The proof is analogous to that of Proposition 3.3 in [14]. The main difference, though, is that we use Corollary 2 instead of the simpler lemma from [9, 14].

Theorem 3. *If $1 \le p < q < \infty$ then the formal inclusion operator $i_{p,q} : J_p \to J_q$ is finitely strictly singular.*

Proof. Given any $x \in J_p$, then $|x_{i+1} - x_i|^q \leqslant \big(2\|x\|_\infty\big)^{q-p}|x_{i+1} - x_i|^p$ for every $i \in \mathbb{N}$, so that $\|x\|_{J_q} \leqslant \big(2\|x\|_\infty\big)^{1-\frac{p}{q}}\|x\|_{J_p}^{\frac{p}{q}}$. Fix an arbitrary $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that $(k-1)^{\frac{1}{p}-\frac{1}{q}} > \frac{1}{\varepsilon}$. Suppose that E is a subspace of J_p with $\dim E = k$. By Corollary 2, there is a zigzag $z \in E$ of order k. By the definition of norm in J_p, we have $\|z\|_{J_p} \geqslant 2(k-1)^{\frac{1}{p}}$.

Put $y = \frac{z}{\|z\|_{J_p}}$. Then $y \in E$ with $\|y\|_{J_p} = 1$. Obviously, $\|y\|_\infty \leqslant \frac{1}{2}(k-1)^{-\frac{1}{p}}$, so that

$$\|i_{p,q}(y)\|_{J_q} = \|y\|_{J_q} \leqslant (k-1)^{\frac{1}{q}-\frac{1}{p}}\|y\|_{J_p}^{\frac{p}{q}} < \varepsilon.$$

Hence, $i_{p,q}$ is finitely strictly singular. □

We will now use Theorem 3 to show that the strictly singular operator T constructed by Read in [12] is finitely strictly singular. Let us briefly outline those properties of T that will be relevant for our investigation. The underlying space X for this operator is defined as the ℓ_2-direct sum of ℓ_2 and Y, $X = (\ell_2 \oplus Y)_{\ell_2}$, where Y itself is the ℓ_2-direct sum of an infinite sequence of J_p-spaces $Y = \big(\bigoplus_{i=1}^\infty J_{p_i}\big)_{\ell_2}$, with (p_i) a certain strictly increasing sequence in $(2, +\infty)$. The operator T is a compact perturbation of $0 \oplus W_1$, where $W_1 \colon Y \to Y$ acts as a weighted right shift, that is,

$$W_1(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \dots), \quad x_i \in J_{p_i}$$

with $\beta_i \to 0$. Note that one should rather write $\beta_i i_{p_i,p_{i+1}} x_i$ instead of $\beta_i x_i$. Clearly, it suffices to show that W_1 is finitely strictly singular.

For $n \in \mathbb{N}$, define $V_n \colon Y \to Y$ via

$$V_n(x_1, x_2, x_3, \dots) = (0, \beta_1 x_1, \dots, \beta_n x_n, 0, 0 \dots), \quad x_i \in J_{p_i}.$$

It follows from $\beta_i \to 0$ that $\|V_n - W_1\| \to 0$. Since finitely strictly singular operators from Y to Y form a closed subspace of $L(Y)$, it suffices to show that V_n is finitely strictly singular for every n. Given $n \in \mathbb{N}$, one can write

$$V_n = \sum_{i=1}^n \beta_i j_{i+1} i_{p_i,p_{i+1}} P_i,$$

where $P_i\colon Y \to J_{p_i}$ is the canonical projection and $j_i\colon J_{p_i} \to Y$ is the canonical inclusion. Thus, V_n is finitely strictly singular because finitely strictly singular operators form an operator ideal. This yields the following result.

Theorem 4. *Read's operator T is finitely strictly singular.*

In the remaining two sections, we present two different proofs of Theorem 1, one based on combinatorial properties of polytopes and the other based on the geometry of the set of all zigzags and algebraic topology.

2. Proof of Theorem 1 via combinatorial properties of polytopes

By a ***polytope*** in $\mathbb{R}^k$ we mean a convex set which is the convex hull of a finite set. A set is a polytope iff it is bounded and can be constructed as the intersection of finitely many closed half-spaces. A ***facet*** of P is a face of (affine) dimension $k-1$. We refer the reader to [5, 15] for more details on properties of polytopes.

A polytope P is centrally symmetric iff it can be represented as the absolutely convex hull of its vertices, that is, $P = \mathrm{conv}\{\pm\bar{u}_1, \dots, \pm\bar{u}_n\}$ where $\pm\bar{u}_1, \dots, \pm\bar{u}_n$ are the vertices of P. Clearly, P is centrally symmetric iff it can be represented as the intersection of finitely many centrally symmetric "bands". More precisely, there are vectors $\bar{a}_1, \dots, \bar{a}_m \in \mathbb{R}^k$ such that $\bar{u} \in P$ iff $-1 \leqslant \langle \bar{u}, \bar{a}_i \rangle \leqslant 1$ for all $i = 1, \dots, m$, and the facets of P are described by $\big\{u \in P \,:\, \langle \bar{u}, \bar{a}_i \rangle = 1\big\}$ or $\big\{u \in P \,:\, \langle \bar{u}, -\bar{a}_i \rangle = 1\big\}$ as $i = 1, \dots, m$.

A ***simplex*** in $\mathbb{R}^k$ is the convex hull of $k+1$ points with non-empty interior. A polytope P in $\mathbb{R}^k$ is ***simplicial*** if all its faces are simplexes (equivalently, if all the facets of P are simplexes). Every polytope can be perturbed into a simplicial polytope by an iterated "pulling" procedure, see e.g., [5, Section 5.2] for details. We will outline a slight modification of the procedure such that it preserves the property of being centrally symmetric. Suppose that P is a centrally symmetric polytope with vertices, say $\pm\bar{u}_1, \dots, \pm\bar{u}_n$. Pull $\bar{u}_1$ "away from" the origin, but not too far, so that it does not reach any affine hyperplane spanned by the facets of P not containing $\bar{u}_1$; denote the resulting point $\bar{u}_1'$. Let $Q = \mathrm{conv}\{\bar{u}_1', -\bar{u}_1, \pm\bar{u}_2, \dots, \pm\bar{u}_n\}$. By [5, 5.2.2, 5.2.3] this procedure does not affect the facets of P not containing $\bar{u}_1$, while all the facets of Q containing $\bar{u}_1'$ become pyramids having apex at $\bar{u}_1'$. Note that no facet of P contains both $\bar{u}_1$ and $-\bar{u}_1$. Hence, if we put $R = \mathrm{conv}\{\pm\bar{u}_1', \pm\bar{u}_2, \dots, \pm\bar{u}_n\}$, then, by symmetry, all the facets of R containing $-\bar{u}_1'$ become pyramids with apex at $-\bar{u}_1'$, while the rest of the facets (in particular, the facets containing $\bar{u}_1'$) are not affected.

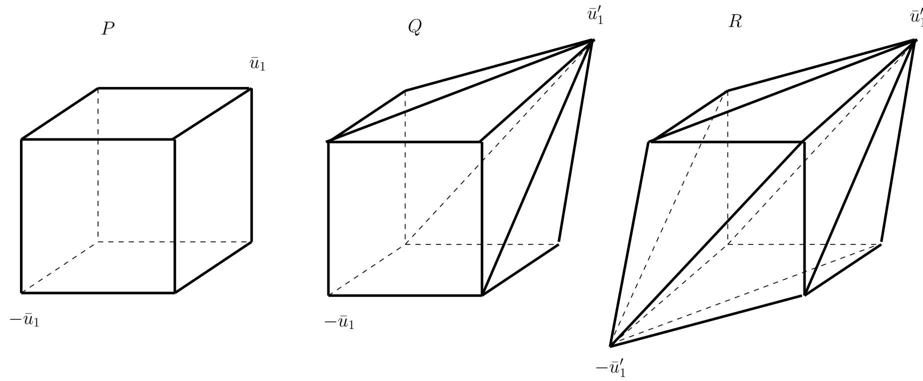

FIGURE 1. Pulling out the first pair of vertices.

Now iterate this procedure with every other pair of opposite vertices. Let P' be the resulting polytope, $P' = \operatorname{conv}\{\pm\bar{u}'_1, \dots, \pm\bar{u}'_n\}$. Clearly, P' is centrally symmetric and simplicial as in [5, 5.2.4]. It also follows from the construction that if F is a facet of P' then all the vertices of P corresponding to the vertices of F belong to the same facet of P.

We will call a polytope P ***marked*** if the following assumptions are satisfied:

(i) P is simplicial, centrally symmetric, and has a non-empty interior.

(ii) Every vertex is assigned a natural number, called its ***index***, such that two vertices have the same index iff they are opposite to each other.

(iii) All the vertices of P are painted in two colors, say, black and white, so that opposite vertices have opposite colors.

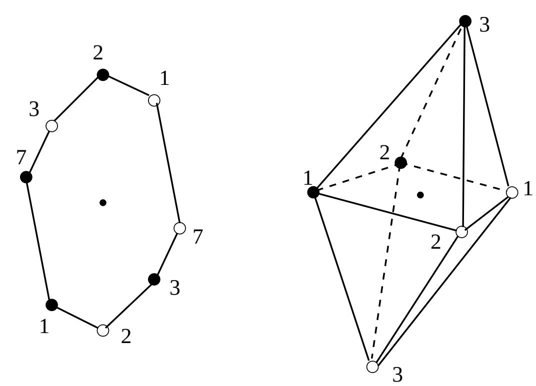

FIGURE 2. Examples of marked polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

See Figure 2 for examples of marked polytopes. A face of a marked polytope is said to be ***happy*** if, when one lists its vertices in the order of increasing indices, the colors of the vertices alternate. For example, the front top facet of the marked polytope in the right hand side of Figure 2 is happy. See Figure 3 for more examples of happy faces.

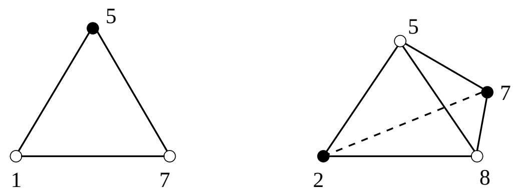

FIGURE 3. Examples of happy simplexes in $\mathbb{R}^2$ and $\mathbb{R}^3$.

We will reduce Theorem 1 to the claim that every marked polytope has a happy facet, which we will prove afterwards. Suppose that $k \leqslant n$ and E is a subspace of $\mathbb{R}^n$ with $\dim E = k$. Let $\{\bar{b}_1, \dots, \bar{b}_k\}$ be a basis of E. We need to find a linear combination of these vectors $\bar{x} := a_1\bar{b}_1 + \dots + a_k\bar{b}_k$ such that $\bar{x}$ is a zigzag. Let B be the $n \times k$ matrix with columns $\bar{b}_1, \dots, \bar{b}_k$, and let $\bar{u}_1, \dots, \bar{u}_n$ be the rows of B. If $\bar{a} = (a_1, \dots, a_k)$, then $x_i = \langle \bar{u}_i, \bar{a} \rangle$ as $i = 1, \dots, n$. Thus, it suffices to find $\bar{a} \in \mathbb{R}^k$ such that the vector $\big(\langle \bar{u}_i, \bar{a} \rangle\big)_{i=1}^n$ is a zigzag of order k.

Let P be the centrally symmetric convex polytope spanned by $\bar{u}_1, \dots, \bar{u}_n$, i.e., $P = \operatorname{conv}\{\pm\bar{u}_1, \dots, \pm\bar{u}_n\}$. Then some of the $\pm\bar{u}_i$'s will be the vertices of P, while the others might end up inside P. Suppose that $\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}$ are the vertices of P, so that $P = \operatorname{conv}\{\pm\bar{u}_{m_1}, \dots, \pm\bar{u}_{m_r}\}$. Following the "pulling" procedure that was described before, construct a simplicial centrally symmetric polytope $P' = \operatorname{conv}\{\pm\bar{u}'_{m_1}, \dots, \pm\bar{u}'_{m_r}\}$. Every vertex of P' is either $\bar{u}'_{m_i}$ or $-\bar{u}'_{m_i}$ for some i. Paint the vertex white in the former case and black in the latter case; assign index i to this vertex. This way we make P' into a marked polytope.

We claim that happy facets of P' correspond to zigzags. Indeed, suppose that P' has a happy facet. Then this facet (or the facet opposite to it) is spanned by some $-\bar{u}'_{m_{i_1}}$, $\bar{u}'_{m_{i_2}}$, $-\bar{u}'_{m_{i_3}}$, $\bar{u}'_{m_{i_4}}$, etc, for some $1 \leqslant i_1 < \dots < i_k \leqslant r$. It follows that $-\bar{u}_{m_{i_1}}$, $\bar{u}_{m_{i_2}}$, $-\bar{u}_{m_{i_3}}$, $\bar{u}_{m_{i_4}}$, etc, are all contained in the same facet of P. Hence, they are contained in an affine hyperplane, say L, such that P "sits" between L and $-L$. Let $\bar{a}$ be the vector defining L, that is, $L = \big\{\bar{u} \;:\; \langle \bar{u}, \bar{a} \rangle = 1\big\}$. Since P is between L and $-L$, we have $-1 \leqslant \langle \bar{u}, \bar{a} \rangle \leqslant 1$ for every $\bar{u}$ in P. In particular, $-1 \leqslant x_i = \langle \bar{u}_i, \bar{a} \rangle \leqslant 1$ for $i = 1, \dots, n$. On the other hand, it follows from $-\bar{u}_{m_{i_1}}, \bar{u}_{m_{i_2}}, -\bar{u}_{m_{i_3}}, \bar{u}_{m_{i_4}}, \dots \in L$ that $x_{m_{i_1}} = -1$, $x_{m_{i_2}} = 1$, $x_{m_{i_3}} = -1$, $x_{m_{i_4}} = 1$, etc. Hence, $\bar{x}$ is a zigzag of order k.

Thus, to complete the proof, it suffices to show that *every marked polytope has a happy facet*. Throughout the rest of this section, P will be a marked polytope in $\mathbb{R}^k$; $\mathcal{F}_j$ stands for the set of all j-dimensional faces of P for $j = 0, \dots, k-1$. In particular, $\mathcal{F}_{k-1}$ is the set of all facets of P, while $\mathcal{F}_0$ is the set of all vertices of P.

By [5, 3.1.6], every $(k-2)$-dimensional face E of P is contained in exactly two facets, say F and G; in this case $E = F \cap G$. Suppose that $R \subseteq \mathcal{F}_{k-1}$. For $E \in \mathcal{F}_{k-2}$, we say that E is a boundary face of R if $E = F \cap G$ for some facets F and G such that $F \in R$ and $G \notin R$. The set of all boundary faces of R will be referred to as the ***face boundary*** of R and denoted $\tilde{\partial}R$. Clearly, $\tilde{\partial}R \subset \mathcal{F}_{k-2}$. If F is a single facet, we put $\tilde{\partial}F = \tilde{\partial}\{F\}$. Clearly, $\tilde{\partial}F$ is the set of all the facets of F.

For a face F of P we define its ***color code*** to be the list of the colors of its vertices in the order of increasing indices. For example, the color codes of the simplexes in Figure 3 are (wbw) and $(bwbw)$. Here b and w correspond to "black" and "white" respectively. A face in P will be said to be a ***b-face*** if its color code starts with b and a ***w-face*** otherwise.

Lemma 5. *Suppose that F is a facet of P. The following are equivalent:*

(i) *F is happy;*
(ii) *$\tilde{\partial}F$ contains exactly one happy b-face;*
(iii) *$\tilde{\partial}F$ contains an odd number of happy b-faces;*

Proof. Note that since F is a simplex, every face of F can be obtained by dropping one vertex of F and taking the convex hull of the remaining vertices. Hence, the color code of the face is obtained by dropping one symbol from the color code of F.

(i)$\Rightarrow$(ii) Suppose that F is happy, then its color code is either $(bwbw\dots)$ or $(wbwb\dots)$. In the former case, the only happy b-face of F is obtained by dropping the last vertex, while in the latter case the only happy b-face of F is obtained by dropping the first vertex.

(ii)$\Rightarrow$(iii) Trivial.

(iii)$\Rightarrow$(i) Suppose that F has an odd number of happy b-faces. Let E be a happy b-face of F. Then the color code of E is the sequence $(bwbw\dots)$ of length $k-1$. Then the color code of F is obtained by inserting one extra symbol into this sequence. Note that inserting the extra symbol should not result in two consecutive b's or w's, as in this case F would have exactly two happy b-faces (corresponding to removing each of the two consecutive symbols), which would contradict the assumption. Hence, the color code of F should be an alternating sequence, so that F is happy. □

Lemma 6. *For every $R \subseteq \mathcal{F}_{k-1}$, the number of happy facets in R and the number of happy b-faces in $\tilde{\partial} R$ have the same parity.*

Proof. For $R \subseteq \mathcal{F}_{k-1}$, define the ***parity*** of R to be the parity of the number of happy b-faces in $\tilde{\partial} R$. Observe that if R and S are two disjoints subsets of $\mathcal{F}_{k-1}$, then the parity of $R \cup S$ is the sum of the parities of R and S (mod 2). It follows that the parity of R is the sum of the parities of all of the facets that make up R (mod 2). But this is exactly the parity of the number of happy facets in R by Lemma 5. ☐

For every face F of P we write $-F$ for the opposite face. If R is a set of facets, we write $-R = \{-F \,:\, F \in R\}$. Also, we write $\bigcup R$ for the set theoretic union of all the facets in R.

Theorem 7. *Every marked polytope has a happy facet.*

Proof. We will prove a stronger statement: *every marked polytope in $\mathbb{R}^k$ has an odd number of happy b-facets.* The proof is by induction on k. For $k = 1$, the statement is trivial. Let $k > 1$ and let P be a marked polytope in $\mathbb{R}^k$.

For every facet F, let $\bar{n}_F$ be the normal vector of F, directed outwards of P. Fix a vector $\bar{v}$ of length one such that $\bar{v}$ is not parallel to any of the facets of P (equivalently, not orthogonal to $\bar{n}_F$ for any facet F); it is easy to see that such a vector exists. By rotating P we may assume without loss of generality that $\bar{v} = (0, \dots, 0, 1)$. Let T be the projection from $\mathbb{R}^k$ to $\mathbb{R}^{k-1}$ such that $T\colon (x_1, \dots, x_{k-1}, x_k) \mapsto (x_1, \dots, x_{k-1})$. We can think of T as the orthogonal projection onto the "horizontal" hyperplane $\{\bar{x} \in \mathbb{R}^k \,:\, x_k = 0\}$ in $\mathbb{R}^k$. Let $Q = T(P)$. Since T is linear and surjective, Q is again a centrally symmetric convex polytope in $\mathbb{R}^{k-1}$ with a non-empty interior.

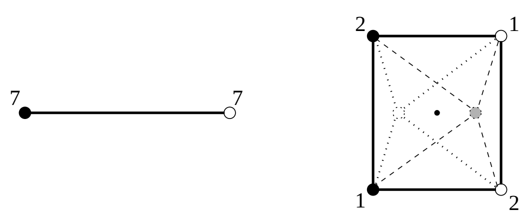

FIGURE 4. The images $T(P)$ of the polytopes in Figure 2.

It follows from our choice of $\bar{v}$ that the k-th coordinate of $\bar{n}_F$ is non-zero for every facet F. Let R be the set of all the facets of P that "face upward", that is,

$$R = \bigl\{\, F \in \mathcal{F}_{k-1} \,:\, \text{the } k\text{-th coordinate of } \bar{n}_F \text{ is positive} \,\bigr\}.$$

Clearly, a facet F is in $-R$ iff the k-th coordinate of $\bar{n}_F$ is negative. Hence, $-R \cap R = \varnothing$ and $-R \cup R = \mathcal{F}_{k-1}$. Observe that $\tilde{\partial} R = \tilde{\partial}(-R)$; hence $\tilde{\partial} R$ is centrally symmetric. Clearly, every vertical line (i.e., a line parallel to $\bar{v}$) that intersects the interior of P meets the boundary of P at exactly two points and corresponds to a point in the interior of Q. It follows that the restriction of T to $\bigcup R$ is a bijection between $\bigcup R$ and Q. The same is also true for $-R$. Therefore, the restriction of T to $\tilde{\partial} R$ is a face-preserving bijection between $\tilde{\partial} R$ and the boundary of Q. Under this bijection, the faces in $\tilde{\partial} R$ correspond to the facets of Q. Hence, this bijection induces a structure of a marked polytope on the boundary of Q, making Q into a marked polytope. It follows, by the induction hypothesis, that the boundary of Q has an odd number of happy b-facets. Hence, $\tilde{\partial} R$ has an odd number of happy b-faces. It follows from Lemma 6 that R has an odd number of happy facets.

Let m and ℓ be the numbers of all happy b-facets and w-facets in R, respectively. Then $m + \ell$ is odd. Observe that F is a happy b-facet iff $-F$ is a happy w-facet. It follows that $-R$ contains ℓ happy b-facets and m happy w-facets. Thus, the total number of happy b-facets of P is $m + \ell$, which we proved to be odd. □

3. Proof of Theorem 1 via algebraic topology

Fix a natural number n and let B^n_∞ and S^{n-1}_∞ be, respectively, the unit ball and the unit sphere of ℓ^n_∞, i.e., $B^n_\infty = \{x \in \mathbb{R}^n : \max|x_i| \leqslant 1\}$ and $S^{n-1}_\infty = \{x \in \mathbb{R}^n : \max|x_i| = 1\}$. For $k \geqslant 1$ we define

$$\begin{aligned}
\Gamma_k &= \{x \in B^n_\infty : x \text{ has at least } k \text{ alternating coordinates } \pm 1\}, \\
A^+_k &= \{x \in B^n_\infty : x \text{ has at least } k \text{ alternating coordinates } \pm 1, \text{ starting with } 1\}, \\
A^-_k &= -A^+_k.
\end{aligned}$$

Note that A^-_k is exactly the set of all zigzags of order k in $\mathbb{R}^n$. Put also $A^+_0 = A^-_0 = \Gamma_0 = B^n_\infty$. For $k \geqslant 1$, $\Gamma_k, A^\pm_k \subset S^{n-1}_\infty$ and we have

$$\begin{aligned}
A^+_k \cup A^-_k &= \Gamma_k, \\
A^+_k \cap A^-_k &= \Gamma_{k+1}.
\end{aligned}$$

Note that the first relation above is true also for $k = 0$.

We start with a simple lemma.

Lemma 8. *Suppose p is a real polynomial of degree m, and there are $m+2$ real numbers $t_1 < t_2 < \cdots < t_{m+2}$, such that $p(t_i) \geqslant 0$ for i odd and $p(t_i) \leqslant 0$ for i even. Then $p \equiv 0$.*

Proof. We do induction with respect to m. If $m = 0$, the result is obvious. If the lemma has been proved up to $m - 1$, and p is a polynomial of degree m, then p has at least one real root s. We write $p(t) = (t - s)q(t)$, and q (or $-q$) has a similar property, with respect to at least $m - 1$ values t_i—so we can apply induction. □

Lemma 9. *There exists a sequence of subspaces $\pi_k \subset \mathbb{R}^n$, $\pi_k \supset \pi_{k+1}$, $\dim \pi_k = n - k$, such that, if P_k is the orthogonal projection onto π_k, then $P_k|A_k^+$ is injective.*

Proof. For $1 \leqslant j \leqslant n$ we define the vectors $\zeta^j \in \mathbb{R}^n$ by the formula $\zeta^j_i = i^{j-1}$. One checks easily that the ζ^j's are linearly independent. Define $\pi_0 = \mathbb{R}^n$, and, for $k \geqslant 1$, $\pi_k = (\operatorname{span}\{\zeta^1, \dots, \zeta^k\})^\perp$.

Suppose that $x, y \in A_k^+$, and $P_k x = P_k y$. There exist scalars $\alpha_1, \dots, \alpha_k$, such that $x - y = \sum_{j=1}^k \alpha_j \zeta^j$. We have indices $1 \leqslant r_1 < \dots < r_k \leqslant n$ and $1 \leqslant s_1 < \dots < s_k \leqslant n$, such that $x_{r_l} = y_{s_l} = (-1)^{l-1}$. It follows that $x_{r_l} - y_{r_l} \geqslant 0$ for l odd and $\leqslant 0$ for l even, while $x_{s_l} - y_{s_l} \leqslant 0$ for l odd and $\geqslant 0$ for l even.

Let the polynomial p of degree $k - 1$ be given by $p(t) = \sum_{j=1}^k \alpha_j t^{j-1}$. If $r_l = s_l$ for all l, we obtain

$$\sum_j \alpha_j \zeta^j_{r_l} = \sum_j \alpha_j {r_l}^{j-1} = 0$$

for all $l = 1, \dots k$. Thus p has k distinct zeros; it must be identically 0, whence $x = y$.

Suppose now that we have $r_l \neq s_l$ for at least one index l. We claim then that among the union of the indices r_l and s_l we can find $\iota_1 < \iota_2 < \dots < \iota_{k+1}$, such that $x_{\iota_l} - y_{\iota_l}$ have alternating signs. This can be achieved by induction with respect to k. For $k = 1$ we must have $r_1 \neq s_1$, so we may take $\iota_1 = \min\{r_1, s_1\}$, $\iota_2 = \max\{r_1, s_1\}$. For $k > 1$, there are two cases. If $r_1 = s_1$, we take $\iota_1 = r_1 = s_1$ and apply the induction hypothesis to obtain the rest. If $r_1 \neq s_1$, we take ι_1 as the lesser of the two and ι_2 as the other one, and then we continue "accordingly" to ι_2 (that is, taking as ι's the rest of r's if $\iota_2 = r_1$ and the rest of s's if $\iota_2 = s_1$).

Now, the way ι_l have been chosen implies that $p(t)$ defined above satisfies the hypotheses of Lemma 8: it has degree $k - 1$ and the values it takes in $\iota_1, \dots, \iota_{k+1}$ have alternating signs. It must then be identically 0, which implies $x = y$. □

Since $A_k^- = -A_k^+$, it follows that $P_k|A_k^-$ is also injective.

Lemma 10. *If π_k, P_k are obtained in Lemma 9, then*

$$\Delta_k := P_k(\Gamma_k)$$

is a balanced, convex subset of π_k*, with* 0 *as an interior point (in* π_k*). Moreover,* $\Delta_k = P_k(A_k^-) = P_k(A_k^+)$ *and* $\partial\Delta_k = P_k(\Gamma_{k+1})$ *(the boundary in the relative topology of* π_k*).*

Proof. We will use induction with respect to k. The statement is immediately checked for $k = 0$ (note that $P_0 = I_{\mathbb{R}^n}$ and $\partial\Delta_0 = S_\infty^{n-1} = \Gamma_1$).

Assume the statement true for k; we will prove its validity for $k+1$. By the induction hypothesis, we have

$$\Delta_{k+1} = P_{k+1}P_k(\Gamma_{k+1}) = P_{k+1}\partial\Delta_k = P_{k+1}\Delta_k$$

and is therefore a balanced, convex subset of π_{k+1}, with 0 as an interior point.

Take then $y \in \overset{\circ}{\Delta}_{k+1}$. Suppose $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains a single point. Then $P_{k+1}^{-1}(y) \cap \Delta_k$ also contains a single point, and therefore $P_{k+1}^{-1}(y) \cap \pi_k$ is a support line for the convex set Δ_k. This line is contained in a support hyperplane (in π_k); but then the whole of Δ_k projects onto π_{k+1} on one side of this hyperplane, and thus y belongs to the boundary of this projection. Therefore y cannot be in $\overset{\circ}{\Delta}_{k+1}$.

The contradiction obtained shows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points. But

$$\partial\Delta_k = P_k(\Gamma_{k+1}) = P_k(A_{k+1}^+) \cup P_k(A_{k+1}^-)$$

whence

$$P_{k+1}(\partial\Delta_k) = P_{k+1}(A_{k+1}^+) \cup P_{k+1}(A_{k+1}^-).$$

Since P_{k+1} restricted to each of the two terms in the right hand side is injective by Lemma 9, there exists a unique $z_+ \in A_{k+1}^+$ such that $y = P_{k+1}z_+$ and a unique $z_- \in A_{k+1}^-$ such that $y = P_{k+1}z_-$.

Take $x \in P_{k+1}^{-1}(y) \cap \partial\Delta_k$. Then either $x \in P_k(A_{k+1}^+)$ or $x \in P_k(A_{k+1}^-)$. If $x \in P_k(A_{k+1}^+)$ then $x = P_k z$ for some $z \in A_{k+1}^+$, so that $y = P_{k+1}x = P_{k+1}z$, which yields $z = z_+$; hence $x = P_k z_+$. Similarly, if $x \in P_k(A_{k+1}^-)$ then $x = P_k z_-$. It follows that $P_{k+1}^{-1}(y) \cap \partial\Delta_k \subseteq \{P_k z_+, P_k z_-\}$. Since $P_{k+1}^{-1}(y) \cap \partial\Delta_k$ contains at least two points, we conclude that $P_{k+1}^{-1}(y) \cap \partial\Delta_k = \{P_k z_+, P_k z_-\}$ and $P_k z_+ \neq P_k z_-$. It follows from $y = P_{k+1}z_\pm$ that $\overset{\circ}{\Delta}_{k+1} \subset P_{k+1}(A_{k+1}^\pm)$. But, Δ_{k+1} being a closed convex set with a nonempty interior, it is the closure of its interior $\overset{\circ}{\Delta}_{k+1}$; since the two sets on the right are closed, we have actually $\Delta_{k+1} = P_{k+1}(A_{k+1}^\pm)$.

We want to show now that $\partial\Delta_{k+1} = P_{k+1}(\Gamma_{k+2})$. Suppose first that $y \in P_{k+1}(\Gamma_{k+2}) = P_{k+1}(A_{k+1}^+ \cap A_{k+1}^-)$; that is, $y = P_{k+1}z$ with $z \in A_{k+1}^+ \cap A_{k+1}^-$. Clearly, $y \in \Delta_{k+1}$. If

$y \in \mathring{\Delta}_{k+1}$, then, defining z_+ and z_- as before, the injectivity of P_{k+1} on $A^{\pm}_{k+1}$ implies $z = z_- = z_+$. This contradicts $P_k z_+ \neq P_k z_-$; consequently, $y \in \partial\Delta_{k+1}$.

Conversely, take $y \in \partial\Delta_{k+1} = \partial\big(P_{k+1}(\Delta_k)\big)$. Again, take $z_+ \in A^+_{k+1}$, $z_- \in A^-_{k+1}$, such that $P_{k+1}z_+ = P_{k+1}z_- = y$. We have then $P_k z_+ \in \partial\Delta_k$ (if $P_k z_+ \in \mathring{\Delta}_k$, then $P_{k+1}z_+ = P_{k+1}P_k z_+$ must be in the interior of $P_{k+1}\Delta_k$, which is $\mathring{\Delta}_{k+1}$). Similarly, $P_k z_- \in \partial\Delta_k$.

If $P_k z_+ \neq P_k z_-$, then P_{k+1} applied to the whole segment $\big[P_k z_+, P_k z_-\big]$ is equal to y. Therefore the segment belongs to $\partial\Delta_k$. Since $\partial\Delta_k = P_k(A^+_{k+1} \cup A^-_{k+1})$, there exist two values x_1, x_2 either both in A^+_{k+1} or both in A^-_{k+1}, such that $P_k x_1, P_k x_2 \in \big[P_k z_+, P_k z_-\big]$, and thus $P_{k+1}x_1 = P_{k+1}x_2 = y$. This contradicts the injectivity of P_{k+1} on $A^{\pm}_{k+1}$.

Therefore $P_k z_+ = P_k z_-$. But z_+ and z_- both belong to A^+_k, on which P_k is injective. It follows that $z_+ = z_- \in A^+_{k+1} \cap A^-_{k+1} = \Gamma_{k+2}$, and $P_{k+1}z_+ = y$. This ends the proof. □

The main consequence of Lemma 10, in combination with Lemma 9, is the fact that the linear map P_{k-1} maps homeomorphically Γ_k into $\partial\Delta_{k-1}$, which is the boundary of a convex, balanced set, containing 0 in its interior.

Proof of Theorem 1. As noted above, P_{k-1} maps homeomorphically Γ_k onto the boundary of a convex, balanced set, containing 0 in its interior. Composing it with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a homeomorphic map ϕ from Γ_k to S^{n-k}, which satisfies the relation $\phi(-x) = -\phi(x)$.

Suppose that E is a k-dimensional subspace of $\mathbb{R}^n$ with no zigzags. Then $E \cap \Gamma_k = \varnothing$, so that the projection of Γ_k onto $E^\perp$ does not contain 0. Composing this projection with the map $x \mapsto \frac{x}{\|x\|}$, we obtain a continuous map from $\psi : \Gamma_k \to S^{n-k-1}$, that satisfies $\psi(-x) = -\psi(x)$. Then the map $\Phi := \psi \circ \phi^{-1} : S^{n-k} \to S^{n-k-1}$ is continuous and satisfies $\Phi(-x) = -\Phi(x)$. This is however impossible: it is known that such a map does not exist (see, for instance, [4]). □

Acknowledgements. The authors would like to thank Charles J. Read and Jonathan R. Partington for useful discussions.

References

[1] N. Aronszajn and K. T. Smith, Invariant subspaces of completely continuous operators. *Ann. of Math. (2)*, 60:345–350, 1954.

[2] G. Androulakis, P. Dodos, G. Sirotkin, and V.G. Troitsky, Classes of strictly singular operators and their products. *Israel J. Math*, to appear.

[3] P. Casazza and R. Lohman, A general construction of spaces of the type of R.C. James, *Canad. J. Math.*, 27(1975), no. 6, 1263–1270.
[4] W. Fulton, *Algebraic topology*, Springer–Verlag, New York, 1995.
[5] B. Grünbaum, *Convex polytopes.* 2nd edition. Graduate Texts in Mathematics, 221. Springer–Verlag, New York, 2003.
[6] R.C. James, A non-reflexive Banach space isometric with its second conjugate, *Proc. Nat. Acad. Sci. U.S.A.* 37(1951), 174–177.
[7] J. Lindenstrauss, L. Tsafriri, *Classical Banach spaces. I. Sequence spaces.* Springer-Verlag, Berlin-New York, 1977.
[8] V. Maslyuchenko and A. Plichko, Quasireflexive locally convex spaces without Banach subspaces. (Russian) *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.* No. 44 (1985), 78–84; translation in *J. Soviet Math.*, 48 (1990), no. 3, 307–312
[9] V.D. Milman, Operators of class C_0 and C_0^*. *Teor. Funktsiĭ Funkcional. Anal. i Priložen.* no. 10, 1970, 15-26.
[10] A. Plichko, Superstrictly singular and superstrictly cosingular operators. *Functional analysis and its applications*, North-Holland Math. Stud., 197, 2004, Elsevier, Amsterdam, 239–255,
[11] A. Popov, Schreier singular operators. *Houston J. Math.*, (to appear).
[12] C.J. Read, Strictly singular operators and the invariant subspace problem, *Studia Math.*, no. 3(132), 1999, 203–226.
[13] I. Singer, *Bases in Banach Spaces I.* Springer-Verlag, New York-Berlin, 1970.
[14] B. Sari, Th. Schlumprecht, N. Tomczak-Jaegermann, V.G. Troitsky, On norm closed ideals in $L(\ell_p \oplus \ell_q)$. *Studia Math.*, 179 (2007), no. 3, 239–262.
[15] G.M. Ziegler, *Lectures on polytopes.* Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1994.

(I. Chalendar and E. Fricain) Université de Lyon; Université Lyon 1; INSA de Lyon; Ecole Centrale de Lyon; CNRS, UMR5208, Institut Camille Jordan; 43 bld. du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

E-mail address: chalenda@math.univ-lyon1.fr, fricain@math.univ-lyon1.fr

(A. I. Popov and V. G. Troitsky) Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada

E-mail address: apopov@math.ualberta.ca, vtroitsky@math.ualberta.ca

(D. Timotin) Institute of Mathematics of the Romanian Academy, PO Box 1-764, Bucharest 014700, Romania

E-mail address: Dan.Timotin@imar.ro