

Université LYON I

cours de M2R

**INTRODUCTION TO DE
BRANGES–ROVNYAK SPACES**

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Chapter 1

Hilbert space operators

1.1 Douglas' factorization theorem

Let $A : H_1 \rightarrow H$ and $B : H_2 \rightarrow H$ be bounded operators between Hilbert spaces. In certain applications, we need to know when there is a contraction $C : H_1 \rightarrow H_2$ such that $A = BC$ holds.

Figure 1.1: The factorization $A = BC$

If such a contraction exists, then

$$BB^* - AA^* = BB^* - BCC^*B^* = B(I - CC^*)B^* \geq 0.$$

Douglas showed that the condition $AA^* \leq BB^*$ is also sufficient for the existence of C .

Theorem 1.1 (Douglas). *Let H, H_1 and H_2 be Hilbert spaces and let $A : H_1 \rightarrow H$ and $B : H_2 \rightarrow H$ be bounded operators. Then there is a contraction $C : H_1 \rightarrow H_2$ such that $A = BC$ if and only if $AA^* \leq BB^*$.*

Proof. The necessity was shown above. The other direction is a bit more delicate. Suppose that $AA^* \leq BB^*$. Hence

$$\|A^*x\|_{H_1} \leq \|B^*x\|_{H_2}, \quad (x \in H). \quad (1.1)$$

This inequality enables us to define an operator from the range of B^* to the range of A^* . Let

$$\begin{aligned} D : \mathcal{R}(B^*) &\longrightarrow \mathcal{R}(A^*) \\ B^*x &\longmapsto A^*x. \end{aligned}$$

If an element in $\mathcal{R}(B^*)$ has two representations, i.e. $z = B^*x = B^*y$, then $B^*(x-y) = 0$. Hence, by (1.1), $A^*(x-y) = 0$. In other words, $Dz = A^*x = A^*y$ is well defined. Moreover,

$$\|D(B^*x)\|_{H_1} \leq \|B^*x\|_{H_2}, \quad (x \in H).$$

Therefore, by continuity, D extends to a contraction from the closure of $\mathcal{R}(B^*)$ in H_2 into H_1 . In the last step of extension, we extend D to a contraction from H_2 to H_1 by defining

$$D(z) = 0, \quad (z \in \mathcal{R}(B^*)^\perp).$$

According to our primary definition, the contraction D satisfies $DB^* = A^*$. Hence $A = BD^*$. Take $C = D^*$. \square

Exercises

Exercise 1.1.1. Let H_1, H_2 and H_3 be Hilbert spaces, let $C : H_1 \rightarrow H_2$ be a contraction, and let $A : H_2 \rightarrow H_3$ be a bounded operator. Show that

$$A(I - CC^*)A^* \geq 0,$$

where $I : H_2 \rightarrow H_2$ is the identity operator.

1.2 The square root of a positive operator

If $B \in \mathcal{L}(H)$ and we define $A = BB^*$, then certainly A is a positive operator. Our goal is to show that every positive operator is obtained that way. First we need two preliminary lemmas.

Lemma 1.2. *Let $A, B \in \mathcal{L}(H)$ be positive and $AB = BA$. Then AB is positive.*

Proof. If $TS = ST$, T and S positive, it is clear that TS^2 is positive. Without loss of generality assume that $\|B\| \leq 1$. Let $B_0 = B$, and let

$$B_{n+1} = B_n - B_n^2, \quad (n \geq 0).$$

Then the relations

$$B_{n+1} = B_n^2(I - B_n) + B_n(I - B_n)^2$$

and

$$I - B_{n+1} = (I - B_n) + B_n^2$$

imply that

$$0 \leq B_n \leq I, \quad (n \geq 0).$$

Thus,

$$\sum_{k=0}^n B_k^2 = B - B_{n+1} \leq B, \quad (n \geq 0). \quad (1.2)$$

Therefore, for each $x \in H$ and $n \geq 0$,

$$\begin{aligned} \sum_{k=0}^n \|B_k x\|^2 &= \sum_{k=0}^n \langle B_k x, B_k x \rangle = \sum_{k=0}^n \langle B_k^2 x, x \rangle \\ &= \left\langle \sum_{k=0}^n B_k^2 x, x \right\rangle \leq \langle Bx, x \rangle. \end{aligned}$$

Thus $\sum_{k=0}^{\infty} \|B_k x\|^2 < \infty$. In particular, $\lim_{n \rightarrow \infty} B_n x = 0$ and, by (1.2),

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n B_k^2 \right) x = Bx.$$

Since $A \sum_{k=0}^n B_k^2$ is positive, we conclude that AB is also positive. □

Lemma 1.3. *Let $A_1, A_2, \dots, B \in \mathcal{L}(H)$ be self adjoint. Suppose that*

$$A_n A_m = A_m A_n, \quad A_n B = B A_n, \quad (m, n \geq 1),$$

and that

$$A_n \leq A_{n+1} \leq B, \quad (n \geq 1).$$

Then there is $A \in \mathcal{L}(H)$, A self adjoint, such that

$$Ax = \lim_{n \rightarrow \infty} A_n x, \quad (x \in H).$$

Proof. Let $C_n = B - A_n$, $n \geq 1$. Hence

$$0 \leq C_{n+1} \leq C_n, \quad (n \geq 1). \quad (1.3)$$

Then, by Lemma 1.2,

$$0 \leq C_n^2 \leq C_n C_m \leq C_m^2, \quad (m \leq n).$$

In the first place, for each $x \in H$,

$$\|C_n x\| \leq \|C_m x\|, \quad (m \leq n). \quad (1.4)$$

Since $(\|C_n x\|)_{n \geq 1}$ is a decreasing sequence of positive numbers, we conclude that

$$\lim_{n \rightarrow \infty} \|C_n x\|$$

exists. Secondly,

$$\|C_m x - C_n x\|^2 \leq \|C_m x\|^2 - \|C_n x\|^2, \quad (m \leq n),$$

and thus $(C_n x)_{n \geq 1}$ is a Cauchy sequence. Let

$$Cx = \lim_{n \rightarrow \infty} C_n x, \quad (x \in H).$$

Clearly C is linear. Moreover, by (1.4), we have

$$\|Cx\| = \lim_{n \rightarrow +\infty} \|C_n x\| \leq \|C_1 x\|, \quad x \in H,$$

which proves that $C \in \mathcal{L}(H)$ and $\|C\| \leq \|C_1\|$. Since C_n is self adjoint, C is also self adjoint. Put $A = B - C$. \square

Theorem 1.4. *Let $A \in \mathcal{L}(H)$ be positive. Then there is a unique positive operator $B \in \mathcal{L}(H)$ such that $A = B^2$.*

Proof. Without loss of generality assume that $\|A\| \leq 1$. Put $B_0 = A$ and

$$B_{n+1} = B_n + \frac{A - B_n^2}{2}, \quad (n \geq 0).$$

Clearly $0 \leq B_0 \leq I$. Moreover, by Lemma 1.2, the relations

$$I - B_{n+1} = \frac{(I - B_n)^2 + (I - A)}{2}$$

and

$$B_{n+1} - B_n = (B_n - B_{n-1}) \frac{(I - B_n) + (I - B_{n-1})}{2}$$

imply that

$$0 \leq B_n \leq B_{n+1} \leq I, \quad (n \geq 0).$$

Hence, by Lemma 1.3, there is $B \in \mathcal{L}(H)$, B positive, with

$$Bx = \lim_{n \rightarrow \infty} B_n x, \quad (x \in H).$$

But

$$B_{n+1}x = B_n x + \frac{B_n^2 x - Ax}{2}, \quad (n \geq 0),$$

which immediately gives $B^2 x = Ax$.

It remains to show that B is unique. Suppose that there is $C \in \mathcal{L}(H)$, C positive, such that $C^2 = A$. Then $AC = C^2 C = CC^2 = CA$. Therefore, $p(A)C = Cp(A)$, where p is any polynomial. In particular, $B_n C = CB_n$, $n \geq 0$, and thus $BC = CB$. Fix $x \in H$, and let $y = (B - C)x$. Then

$$\langle (B + C)y, y \rangle = \langle (B^2 - C^2)x, y \rangle = 0.$$

Since B and C are positive operators, we thus have

$$\langle By, y \rangle = \langle Cy, y \rangle = 0.$$

But these assumptions imply $By = Cy = 0$. For example, to verify that $By = 0$, based on the first paragraph, we know that $B^{1/2}$ exists. Hence

$$\|B^{1/2}y\|^2 = \langle B^{1/2}y, B^{1/2}y \rangle = \langle By, y \rangle = 0.$$

Thus $B^{1/2}y = 0$ which implies $By = 0$. Finally,

$$\|(B - C)x\|^2 = \langle (B - C)x, (B - C)x \rangle = \langle (B - C)^2 x, x \rangle = \langle (B - C)y, x \rangle = 0.$$

Therefore, $B = C$. \square

The operator B whose unique existence was guaranteed by the preceding theorem is called the square root of A and is denoted by $A^{1/2}$. The following two results were also implicitly obtained in the proof of Theorem 1.4.

Corollary 1.5. *Let $A \in \mathcal{L}(H)$ be positive. Then there is a sequence of real polynomials $(p_n)_{n \geq 1}$ with $p_n(0) = 0$ such that*

$$A^{1/2}x = \lim_{n \rightarrow \infty} p_n(A)x, \quad (x \in H).$$

Corollary 1.6. *Let $A \in \mathcal{L}(H)$ be positive, and let $x \in H$. Suppose that*

$$\langle Ax, x \rangle = 0.$$

Then $Ax = 0$.

Exercises

Exercise 1.2.1. In Lemma 1.3, can we conclude that

$$\lim_{n \rightarrow \infty} \|A_n - A\| = 0?$$

Hint: Let $(x_n)_{n \geq 1}$ be an orthonormal basis for an infinite dimensional separable Hilbert space H . Let A_n be the orthogonal projection onto the closed space generated by $\{x_1, x_2, \dots, x_n\}$. Then $A_n x \rightarrow x$, but $\|A_n - I\| = 1$.

Exercise 1.2.2. Let H be a real Hilbert space, and let $A \in \mathcal{L}(H)$ be positive. Suppose that $\langle Ax, x \rangle = 0$ for all $x \in H$. Show that $A = 0$.

Hint: Corollaries 1.5 and 1.6 are also valid for real Hilbert spaces.

1.3 Partial isometries and the polar decomposition

An operator $A \in \mathcal{L}(H_1, H_2)$ is called a partial isometry if $\|Ax\|_{H_2} = \|x\|_{H_1}$ for all $x \in (\ker A)^\perp$. In other words, if

$$\|Ax\|_{H_2} = \|P_{(\ker A)^\perp} x\|_{H_1}, \quad (x \in H_1). \quad (1.5)$$

A partial isometry is clearly a contraction. The subspace $(\ker A)^\perp$ is called the initial space of A . Note that the range $\mathcal{R}(A)$ of A is a closed subspace of H_2 and it is called the final space of the partial isometry A .

If $\ker A = \{0\}$, then we have

$$\|Ax\|_{H_2} = \|x\|_{H_1}, \quad (x \in H_1),$$

and in this case A is called an isometry. If H is finite dimensional and $A \in \mathcal{L}(H)$ is an isometry, then A is necessarily a unitary operator. This is no longer true for infinite dimensional Hilbert spaces. For example, by (??), the forward shift operator $S \in \mathcal{L}(\ell^2)$ is an isometry which is not surjective, and thus it is not a unitary operator.

Theorem 1.7. *Let $A : H_1 \rightarrow H_2$ be an operator between Hilbert spaces H_1 and H_2 . Then the following are equivalent.*

- (i) A is a partial isometry;
- (ii) A^* is a partial isometry;
- (iii) AA^* is an orthogonal projection;
- (iv) A^*A is an orthogonal projection.

Proof. (i) \implies (iv) : Since A is a contraction, for all $x \in H_1$,

$$\begin{aligned} \langle (I - A^*A)x, x \rangle_{H_1} &= \langle x, x \rangle_{H_1} - \langle A^*Ax, x \rangle_{H_1} \\ &= \langle x, x \rangle_{H_1} - \langle Ax, Ax \rangle_{H_2} \\ &= \|x\|_{H_1}^2 - \|Ax\|_{H_2}^2 \geq 0. \end{aligned}$$

Hence, $I - A^*A$ is a positive self adjoint operator on H_1 . Moreover,

$$\langle (I - A^*A)x, x \rangle_{H_1} = 0, \quad (x \in (\ker A)^\perp). \quad (1.6)$$

Then, by Corollary 1.6, (1.6) implies

$$(I - A^*A)x = 0, \quad (x \in (\ker A)^\perp).$$

Since

$$A^*Ax = x, \quad (x \in (\ker A)^\perp),$$

and

$$A^*Ax = 0, \quad (x \in \ker A),$$

then A^*A is the orthogonal projection on $(\ker A)^\perp$.

(iv) \implies (i) : Let us remind that $\ker A^*A = \ker A$. Since A^*A is an orthogonal projection, $A^*Ax = x$ for all $x \in (\ker A)^\perp$. Hence $\langle A^*Ax, x \rangle_{H_1} = \langle x, x \rangle_{H_1}$ which is equivalent to

$$\|Ax\|_{H_2} = \|x\|_{H_1}, \quad (x \in (\ker A)^\perp).$$

Hence A is a partial isometry.

Reversing the roles of A and A^* , we see that (ii) \iff (iii).

(iv) \implies (iii) : If A^*A is an orthogonal projection, then

$$A^*A = P_{(\ker A^*A)^\perp} = P_{(\ker A)^\perp}.$$

Thus

$$(AA^*)^2 = A(A^*A)A^* = AP_{(\ker A)^\perp}A^* = AA^*.$$

Therefore, AA^* is also an orthogonal projection. The proof of (iii) \implies (iv) is similar. \square

For each $T \in \mathcal{L}(H)$, clearly $T^*T \geq 0$, and thus by Theorem 1.4, $(T^*T)^{1/2}$ is a well defined self adjoint positive operator which is the unique positive square root of T^*T . We denote this operator by

$$|T| = (T^*T)^{1/2}.$$

For each $x \in H$, we have

$$\| |T|x \|^2 = \langle (T^*T)^{1/2}x, (T^*T)^{1/2}x \rangle = \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2.$$

Therefore

$$\| |T|x \| = \|Tx\|, \quad (x \in H). \quad (1.7)$$

In particular, this identity implies $\ker |T| = \ker T$ and $\| |T| \| = \|T\|$.

If z is a complex number and we put $r = (\bar{z}z)^{1/2}$, then $r \geq 0$ and there is a complex number of modulus one ζ such that we have the polar decomposition $z = \zeta r$. We show that a similar decomposition exists for each $T \in \mathcal{L}(H)$. However, since $\mathcal{L}(H)$ is not commutative, certain technical difficulties will arise.

Theorem 1.8 (Polar Decomposition Theorem). *Let $T \in \mathcal{L}(H)$. Then there is a partial isometry $U \in \mathcal{L}(H)$ such that*

$$T = U |T|.$$

Proof. The proof has the same flavor as the proof of Douglas' factorization theorem (Theorem 1.1). According to the identity (1.7), the mapping

$$\begin{aligned} V : \mathcal{R}(|T|) &\longrightarrow \mathcal{R}(T) \\ |T|x &\longmapsto Tx \end{aligned}$$

is well defined. Since, if $y = |T|x = |T|x'$, then $|T|(x - x') = 0$, and thus, by (1.7), $T(x - x') = 0$, which implies $Vy = Tx = Tx'$. Moreover,

$$\|Vy\| = \|V(|T|x)\| = \|Tx\| = \| |T|x \| = \|y\|, \quad (y \in \mathcal{R}(|T|)).$$

Hence V is actually an isometry on $\mathcal{R}(|T|)$ onto $\mathcal{R}(T)$. By continuity, we extend V to the unitary operator $\mathbf{V} : \overline{\mathcal{R}(|T|)} \longrightarrow \overline{\mathcal{R}(T)}$. Let

$$U = \mathbf{V} P_{\overline{\mathcal{R}(|T|)}}.$$

Hence $\mathcal{R}(U) = \mathcal{R}(\mathbf{V}) = \overline{\mathcal{R}(T)}$ and

$$U^*U = P_{\overline{\mathcal{R}(|T|)}}^* \mathbf{V}^* \mathbf{V} P_{\overline{\mathcal{R}(|T|)}} = P_{\overline{\mathcal{R}(|T|)}} I_{\overline{\mathcal{R}(|T|)}} P_{\overline{\mathcal{R}(|T|)}} = P_{\overline{\mathcal{R}(|T|)}}.$$

Theorem 1.7 ensures that U is a partial isometry. Moreover, for each $x \in H$,

$$U|T|x = \mathbf{V} P_{\overline{\mathcal{R}(|T|)}} |T|x = \mathbf{V}|T|x = V|T|x = Tx.$$

□

Exercises

Exercise 1.3.1. Let M and N be respectively closed subspaces of the Hilbert spaces H_1 and H_2 . Suppose that $\dim M = \dim N$. Show that there is a partial isometry $A \in \mathcal{L}(H_1, H_2)$ with the initial space M and range N .

Hint: Let $(x_\iota)_{\iota \in I}$ and $(y_\iota)_{\iota \in I}$ be respectively orthonormal bases for M and N . Each $x \in H_1$ has the unique representation $x = x' + \sum_\iota \alpha_\iota x_\iota$, where $x' \perp M$. Define $Ax = \sum_\iota \alpha_\iota y_\iota$.

Exercise 1.3.2. Show that if B is a partial isometry, then the range of B is a closed subspace.

Exercise 1.3.3. Let B be a partial isometry. Show that the orthogonal complement of $\ker B$ is exactly the range of B^* .

Exercise 1.3.4. Let $T \in \mathcal{L}(H)$. Show that there is a partial isometry $V \in \mathcal{L}(H)$ such that

- (i) $T = |T^*|V$;
- (ii) $V^*V = P_{(\ker T)^\perp}$;
- (iii) $VV^* = P_{\overline{\mathcal{R}(T)}}$.

Hint: Apply the polar decomposition theorem to T^* gives the following result.

Exercise 1.3.5. Let $T \in \mathcal{L}(H)$. Show that there is a sequence of real polynomial $(p_n)_{n \geq 1}$ with $p_n(0) = 0$ such that

$$p_n(|T|^2) \xrightarrow{s} |T|.$$

Hint: Use Corollary 1.5.

Exercise 1.3.6. Let $T \in \mathcal{L}(H)$, and let $T = U|T|$ be its polar decomposition. Show that $U^*U = P_{\overline{\mathcal{R}(|T|)}}$ and $UU^* = P_{\overline{\mathcal{R}(T)}}$.

Exercise 1.3.7. Let $T \in \mathcal{L}(H)$, and let $T = U|T|$ be its polar decomposition. Show that

$$\ker U = \ker |T| = \ker T$$

and

$$\|Ux\| = \|x\|, \quad (x \in (\ker T)^\perp).$$

Hint: Use (1.7) and Exercise 1.3.6.

Exercise 1.3.8. Let $A \in \mathcal{L}(H_1, H)$ and $B \in \mathcal{L}(H_2, H)$. Show that the following are equivalent:

- (i) there is a partial isometry $C \in \mathcal{L}(H_1, H_2)$, with $\ker(A)^\perp$ as initial space and $\overline{\mathcal{R}(B^*)}$ as final space, such that $A = BC$.
- (ii) $AA^* = BB^*$.

1.4 Reproducing kernel Hilbert spaces

We say that a Hilbert space H of functions on some set Ω is a reproducing kernel Hilbert space on Ω if it satisfies the following properties:

- (i) for each $z \in \Omega$, the mapping

$$\begin{aligned} \Lambda_z : H &\longrightarrow \mathbb{C} \\ f &\longmapsto f(z) \end{aligned}$$

is a continuous linear functional on H .

- (ii) for each $z \in \Omega$, there is $f_z \in H$ such that $f_z(z) \neq 0$.

According to the Riesz representation theorem (Theorem ??) and the first assumption, for each $z \in \Omega$, there is a unique $k_z \in H$, called the reproducing kernel of H , such that

$$f(z) = \langle f, k_z \rangle, \quad (f \in H). \quad (1.8)$$

This relation yields several elementary properties of k_z . First of all, with $f = k_z$, we obtain

$$\|k_z\|^2 = k_z(z), \quad (z \in \Omega). \quad (1.9)$$

This identity implies that $k_z(z) \neq 0$ which is equivalent to

$$k_z \neq 0, \quad (\text{in } H). \quad (1.10)$$

Since otherwise we will have $f(z) = 0$ for all $f \in H$ and this is not true by assumption (ii). By Riesz theorem, the norm of the evaluation functional (1.8) is equal to $\|k_z\|$. Then, for each $z, w \in \Omega$, by (1.8), we have

$$k_w(z) = \langle k_w, k_z \rangle \quad \text{and} \quad k_z(w) = \langle k_z, k_w \rangle.$$

Hence

$$k_z(w) = \overline{k_w(z)}, \quad (z, w \in \Omega). \quad (1.11)$$

Here is another consequence of (1.8).

Lemma 1.9. *Let H be a reproducing kernel Hilbert space on Ω . Then*

$$\{k_z : z \in \Omega\}^\perp = \{0\}.$$

In other words, the linear manifold of all finite linear combinations of k_z , $z \in \Omega$, is dense in H .

Proof. If $f \in \{k_z : z \in \Omega\}^\perp$, then, by (1.8), we immediately have $f(z) = 0$ for all $z \in \Omega$. \square

A function φ on Ω is called a multiplier for H if $\varphi f \in H$ for all $f \in H$. The space of all multipliers of H is denoted by $\mathcal{M}(H)$.

Let $\varphi \in \mathcal{M}(H)$ and define

$$\begin{aligned} M_\varphi : H &\longrightarrow H \\ f &\longmapsto \varphi f. \end{aligned}$$

This mapping is well defined and linear. Moreover, by the closed graph theorem, it is continuous. As a matter of fact, let $f_n \rightarrow f$ and $M_\varphi f_n \rightarrow g$. Since the evaluation functionals are continuous, for each $z \in \Omega$, we have $f_n(z) \rightarrow f(z)$ and $\varphi(z)f_n(z) \rightarrow g(z)$. Hence $g(z) = \varphi(z)f(z)$, $z \in \Omega$, which means $M_\varphi f = g$. Therefore, in short, the mapping

$$\begin{aligned} \mathcal{M}(H) &\longrightarrow \mathcal{L}(H) \\ \varphi &\longmapsto M_\varphi \end{aligned}$$

is well defined.

The next result says that the reproducing kernel k_z is the eigenvector of the conjugate of each multiplication operator. Despite its simple proof, this result has many applications.

Theorem 1.10. *Let H be a reproducing kernel Hilbert space on Ω , and let $\varphi \in \mathcal{M}(H)$. Then*

$$M_\varphi^* k_z = \overline{\varphi(z)} k_z, \quad (z \in \Omega).$$

Proof. Let $f \in H$. By definition

$$\langle f, M_\varphi^* k_z \rangle = \langle M_\varphi f, k_z \rangle = \langle \varphi f, k_z \rangle.$$

Moreover, by (1.8),

$$\varphi(z)f(z) = \langle \varphi f, k_z \rangle \quad \text{and} \quad f(z) = \langle f, k_z \rangle.$$

Hence, for each $f \in H$,

$$\langle f, M_\varphi^* k_z \rangle = \varphi(z)f(z) = \varphi(z)\langle f, k_z \rangle = \langle f, \overline{\varphi(z)} k_z \rangle.$$

Therefore, $M_\varphi^* k_z = \overline{\varphi(z)} k_z$. \square

Corollary 1.11. *Let H be a reproducing kernel Hilbert space on Ω , and let $\varphi \in \mathcal{M}(H)$. Then φ is bounded on Ω and*

$$\sup_{z \in \Omega} |\varphi(z)| \leq \|M_\varphi\|_{\mathcal{L}(H)}.$$

Proof. Fix $z \in \Omega$. By Theorem 1.10,

$$|\varphi(z)| \|k_z\|_H = \|\overline{\varphi(z)} k_z\|_H = \|M_\varphi^* k_z\|_H \leq \|M_\varphi^*\|_{\mathcal{L}(H)} \|k_z\|_H.$$

By (1.10), $\|k_z\|_H \neq 0$, and thus we can divide both sides by $\|k_z\|_H$. Moreover, by Theorem 1.10(v), $\|M_\varphi^*\|_{\mathcal{L}(H)} = \|M_\varphi\|_{\mathcal{L}(H)}$. \square

By Theorem 1.10, each k_z , $z \in \Omega$, is an eigenvector of the conjugate of every multiplication operator. It is rather amazing that no other operator in $\mathcal{L}(H)$ has this property.

Theorem 1.12. *Let H be a reproducing kernel Hilbert space on Ω , and let $A \in \mathcal{L}(H)$. Suppose that, for each $z \in \Omega$, k_z is an eigenvector of A^* . Then there is $\varphi \in \mathcal{M}(H)$ such that*

$$A = M_\varphi.$$

Proof. Let the function φ on Ω be defined by

$$A^* k_z = \overline{\varphi(z)} k_z, \quad (z \in \Omega).$$

For each $f \in H$, we have

$$(Af)(z) = \langle Af, k_z \rangle = \langle f, A^* k_z \rangle = \langle f, \overline{\varphi(z)} k_z \rangle = \varphi(z) \langle f, k_z \rangle = \varphi(z) f(z)$$

for all $z \in \Omega$. Therefore

$$\varphi f = Af, \quad (f \in H),$$

which ensures that $\varphi \in \mathcal{M}(H)$ and that $A = M_\varphi$. \square

We get immediately from Theorem 1.10 and Theorem 1.12 the following characterization of multipliers.

Corollary 1.13. *Let H be a reproducing kernel Hilbert space on Ω and let φ be a function on Ω . Then $\varphi \in \mathcal{M}(H)$ if and only if the map*

$$k_w \longmapsto \overline{\varphi(w)} k_w$$

extends to a continuous linear map on H .

Frequently, the set Ω will be a domain, i.e. an open and connected set, in the complex plane and the functions in H will be holomorphic; in this case, we shall speak of a holomorphic reproducing kernel Hilbert space. We note by $H(\Omega)$ the family of all analytic functions on Ω , and let $H^\infty(\Omega)$ be the subclass consisting of all bounded functions in $H(\Omega)$.

Corollary 1.14. Let H be a holomorphic reproducing kernel Hilbert space on a domain Ω . Assume that there is a function $f_0 \in H$ such that $f_0(z) \neq 0, \forall z \in \Omega$. Then

$$\mathcal{M}(H) \subset H^\infty(\Omega).$$

Proof. Let $\varphi \in \mathcal{M}(H)$. First note that

$$\varphi(z) = \frac{(M_\varphi f_0)(z)}{f_0(z)} \quad (z \in \Omega),$$

and since $H \subset H(\Omega)$, the function φ is analytic on Ω . So it remains to apply Corollary 1.11 to get that $\varphi \in H^\infty(\Omega)$. □

Exercises

Exercise 1.4.1. Let H be a reproducing kernel Hilbert space on Ω and let $(e_i)_{i \in I}$ be any orthonormal basis for H . Show that

$$k_w(z) = \sum_{i \in I} \overline{e_i(w)} e_i(z), \quad z, w \in \Omega.$$

Exercise 1.4.2. Let $H \subset H(\Omega)$ be a reproducing kernel Hilbert space. Suppose that $\mathcal{M}(H) = H^\infty(\Omega)$. Show that

$$\sigma(M_\varphi) = \overline{\mathcal{R}(\varphi)}, \quad (\varphi \in \mathcal{M}(H)),$$

where $\mathcal{R}(\varphi)$ is the range of φ , i.e.

$$\mathcal{R}(\varphi) = \{ \varphi(z) : z \in \Omega \}$$

and $\overline{\mathcal{R}(\varphi)}$ represents the closure of $\mathcal{R}(\varphi)$.

Hint: By Theorem 1.10, $\overline{\varphi(z)} \in \sigma(M_\varphi^*)$, which, by Theorem ??(x), gives $\varphi(z) \in \sigma(M_\varphi)$. If $\lambda \notin \overline{\mathcal{R}(\varphi)}$, consider $\psi = 1/(\varphi - \lambda) \in H^\infty(\Omega)$.

Exercise 1.4.3. Let H be a reproducing kernel Hilbert space on Ω . If $\varphi \in \mathcal{M}(H)$, we denote by

$$\|\varphi\|_{\mathcal{M}(H)} := \|M_\varphi\|_{\mathcal{L}(H)}.$$

- a) Show that $(\mathcal{M}(H), \|\cdot\|_{\mathcal{M}(H)})$ is a Banach algebra.
- b) Let Ω be a set and let $k : \Omega \times \Omega \rightarrow \mathbb{C}$ be a function. We say that k is a kernel if the following hold:

(bi) $k(z, w) = \overline{k(w, z)}$, $z, w \in \Omega$.

(bii) k is positive semi-definite, i.e.

$$\sum_{i,j=1}^N \overline{a_i} a_j k(\lambda_i, \lambda_j) \geq 0,$$

for any finite set $\{\lambda_1, \dots, \lambda_N\}$ of distinct points in Ω and any complex numbers a_1, a_2, \dots, a_N .

(biii) $k(z, z) \neq 0$, $z \in \Omega$.

A weak kernel on Ω is a function $k : \Omega \times \Omega \rightarrow \mathbb{C}$ which satisfies (bi) and (bii).

Show that if k_z is the reproducing kernel of H , then $k(z, w) := k_w(z)$ ($z, w \in \Omega$) is a kernel on Ω .

c) Let H be a reproducing kernel Hilbert space on Ω and let $\rho > 0$. Show that the following are equivalent:

(i) $\varphi \in \mathcal{M}(H)$ with $\|\varphi\|_{\mathcal{M}(H)} \leq \rho$.

(ii) The function

$$K(z, w) := (\rho^2 - \varphi(z)\overline{\varphi(w)})k(z, w) \quad (z, w \in \Omega),$$

is a weak kernel on Ω .

1.5 Fredholm theory

Let X and Y be Banach spaces. An operator $T \in \mathcal{L}(X, Y)$ is called *Fredholm* if $\mathcal{R}(T)$ is a closed subspace of Y , and $\dim \ker T < +\infty$, $\dim \ker T^* < +\infty$. The difference

$$\text{ind } T = \dim \ker T - \dim \ker T^*$$

is called the *index* of T .

We will use in this text the following property of Fredholm operators.

Lemma 1.15. *Let $T \in \mathcal{L}(X)$ and assume that $T = V + K$ where V is an invertible operator and K is a compact operator. Then T is Fredholm and $\text{ind } T = 0$.*

Proof. Recall that Fredholm alternative asserts that if S is a compact operator then $Id + S$ has closed range and

$$\dim \ker(Id + S) = \dim \ker(Id + S)^* < +\infty$$

(see [] for a proof of this result). Therefore, $Id + S$ is a Fredholm operator with $\text{ind}(Id + S) = 0$. Now $V^{-1}T = Id + V^{-1}K$ and $V^{-1}K$ is a compact operator; thus we get that $V^{-1}T$ is a Fredholm operator and $\text{ind}(V^{-1}T) = 0$. Then we easily obtain that $T = VV^{-1}T$ is also a Fredholm operator with $\text{ind } T = 0$. \square

Let H_1, H_2 be Hilbert spaces and $x \in H_1, y \in H_2$. We denote by $x \otimes y$ the rank one operator in $\mathcal{L}(H_1, H_2)$ defined by

$$(x \otimes y)(h) = \langle h, y \rangle_{H_2} x, \quad h \in H_2.$$

Lemma 1.16. *Let $T \in \mathcal{L}(H_1, H_2)$ and $x \in H_1, y \in H_2$. We have*

$$T(x \otimes y) = Tx \otimes y, \quad \text{and} \quad (x \otimes y)T = x \otimes T^*y.$$

Proof. Let $h \in H_2$. Then

$$T(x \otimes y)(h) = T(\langle h, y \rangle_{H_2} x) = \langle h, y \rangle_{H_2} Tx = (Tx \otimes y)(h),$$

which proves the first relation. For the second, let $w \in H_1$. Then

$$((x \otimes y)T)(w) = \langle Tw, y \rangle_{H_2} x = \langle w, T^*y \rangle_{H_1} x = (x \otimes T^*y)(w),$$

which proves the second relation. □

1.6 The operator of multiplication by the independent variable on $L^2(\mu)$.

Let μ be a finite and positive Borel measure on \mathbb{T} and let Z_μ be the operator of multiplication by the independent variable on $L^2(\mu)$,

$$(Z_\mu f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}), \quad f \in L^2(\mu).$$

We recall that the support of the measure μ , denoted by $\text{supp } \mu$, is the largest closed set $C \subset \mathbb{T}$ (with respect to the inclusion) such that

$$U \text{ open subset of } \mathbb{T}, U \cap C \neq \emptyset \implies \mu(U \cap C) > 0,$$

i.e. every open subset of \mathbb{T} that has a non-trivial intersection with the support has a positive measure. It is easy to prove that

$$\text{supp } \mu = \{\zeta \in \mathbb{T} : \zeta \in V_\zeta \text{ open subset of } \mathbb{T} \implies \mu(V_\zeta) > 0\}.$$

Moreover, if $A \in \text{Bor } (\mathbb{T})$ such that $A \subset \mathbb{T} \setminus \text{supp } \mu$, then $\mu(A) = 0$.

Lemma 1.17. *We have*

$$\sigma(Z_\mu) = \sigma_a(Z_\mu) = \text{supp } \mu.$$

Proof. First note that since Z_μ is a unitary operator, then $\sigma(Z_\mu) \subset \mathbb{T}$. Now take $\lambda \in \mathbb{T} \setminus \text{supp } \mu$ and let us show that $\lambda \notin \sigma(Z_\mu)$. On one hand, if $f \in \ker(Z_\mu - \lambda Id)$, then $(z - \lambda)f(z) = 0$, a.e. $z \in \mathbb{T}$. That is $f(z) = 0$, a.e. $z \in \mathbb{T} \setminus \{\lambda\}$. But since $\mu(\{\lambda\}) = 0$, that implies that $f = 0$, a.e. with respect

to μ . In others words, $\ker(Z_\mu - \lambda Id) = \{0\}$. On the other hand, let $g \in L^2(\mu)$ and define

$$f(z) := \frac{g(z)}{z - \lambda}, \quad \text{a.e. } z \in \mathbb{T}.$$

Then f is well defined a.e. with respect to μ (again because $\mu(\{\lambda\}) = 0$). Let us check that $f \in L^2(\mu)$. Since $\lambda \in \mathbb{T} \setminus \text{supp } \mu$, we have

$$d := \text{dist}(\lambda, \text{supp } \mu) > 0.$$

Hence

$$\begin{aligned} \int_{\mathbb{T}} |f(z)|^2 d\mu(z) &= \int_{\text{supp } \mu} \frac{|g(z)|^2}{|z - \lambda|^2} d\mu(z) \\ &\leq \frac{1}{d^2} \int_{\text{supp } \mu} |g(z)|^2 d\mu(z) \\ &= \frac{\|g\|_{L^2(\mu)}^2}{d^2} < +\infty. \end{aligned}$$

Thus $f \in L^2(\mu)$ and

$$((Z_\mu - \lambda Id)f)(z) = (z - \lambda)f(z) = g(z),$$

for a.e. $z \in \mathbb{T}$, which proves that

$$(Z_\mu - \lambda Id)f = g.$$

Therefore $Z_\mu - \lambda Id$ is onto and then invertible. That proves that $\mathbb{T} \setminus \text{supp } \mu$ is contained in the complement of $\sigma(Z_\mu)$ or

$$\sigma(Z_\mu) \subset \text{supp } \mu.$$

Now we will prove that $\text{supp } \mu \subset \sigma_a(Z_\mu)$. Take $\lambda \in \text{supp } \mu$ and define for $n \geq 1$

$$f_n(z) = \begin{cases} \frac{1}{n} & \text{if } z \in B(\lambda, 1/n) \cap \mathbb{T} \\ 0 & \text{otherwise} \end{cases}.$$

Then we have

$$\int_{\mathbb{T}} |f_n(z)|^2 d\mu(z) = \int_{B(\lambda, 1/n) \cap \mathbb{T}} \frac{1}{n^2} d\mu(z) = \frac{1}{n^2} \mu(B(\lambda, 1/n) \cap \mathbb{T}), \quad (1.12)$$

which proves that $f_n \in L^2(\mu)$ and since $\mu(B(\lambda, 1/n) \cap \mathbb{T}) > 0$, $f_n \neq 0$ in $L^2(\mu)$. So let us define

$$g_n := \frac{f_n}{\|f_n\|_{L^2(\mu)}}.$$

Then $g_n \in L^2(\mu)$, $\|g_n\|_{L^2(\mu)} = 1$ and

$$\begin{aligned}
\|(Z_\mu - \lambda Id)g_n\|_{L^2(\mu)}^2 &= \int_{\mathbb{T}} |z - \lambda|^2 |g_n(z)|^2 d\mu(z) \\
&= \frac{1}{\|f_n\|_{L^2(\mu)}^2} \int_{\mathbb{T}} |z - \lambda|^2 |f_n(z)|^2 d\mu(z) \\
&= \frac{1}{n^2 \|f_n\|_{L^2(\mu)}^2} \int_{B(\lambda, 1/n) \cap \mathbb{T}} |z - \lambda|^2 d\mu(z) \\
&\leq \frac{1}{n^4 \|f_n\|_{L^2(\mu)}^2} \mu(B(\lambda, 1/n) \cap \mathbb{T}),
\end{aligned}$$

and by (1.12), we get

$$\|(Z_\mu - \lambda Id)g_n\|_{L^2(\mu)}^2 \leq \frac{1}{n^2}.$$

That implies

$$\lim_{n \rightarrow +\infty} \|(Z_\mu - \lambda Id)g_n\|_{L^2(\mu)} = 0.$$

In others words, we have $\lambda \in \sigma_a(Z_\mu)$. Therefore, we have proved that

$$\sigma(Z_\mu) \subset \text{supp } \mu \subset \sigma_a(Z_\mu), \quad (1.13)$$

and since we always have $\sigma_a(T) \subset \sigma(T)$, we get the conclusion that the three sets in (1.13) coincide. □

Chapter 2

Analytic functions on the open unit disc

2.1 The Poisson integral

Let μ be a complex Borel measure on \mathbb{T} and let $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous with respect to the Lebesgue measure m and μ_s is singular with respect to m . The Poisson integral of the measure μ is defined on \mathbb{D} by

$$P[d\mu](z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \zeta|^2} d\mu(\zeta), \quad (z \in \mathbb{D}).$$

In this section, we briefly recall the principal and well-known properties of this function (we refer the reader to [] for the proof of the results).

First we have

$$\lim_{r \rightarrow 1} \int_{\mathbb{T}} \frac{1 - r^2}{|re^{i\theta} - \zeta|^2} d\mu(\zeta) = \frac{d\mu_a}{dm}(e^{i\theta}), \quad (2.1)$$

for almost all $e^{i\theta}$ on \mathbb{T} , with respect to the Lebesgue measure m . If the measure μ is assumed to be positive, then we also have

$$\lim_{r \rightarrow 1} \int_{\mathbb{T}} \frac{1 - r^2}{|re^{i\theta} - \zeta|^2} d\mu(\zeta) = +\infty, \quad (2.2)$$

for almost all $e^{i\theta}$ on \mathbb{T} , with respect to the measure μ_s .

It is clear that if μ is a complex Borel measure on \mathbb{T} , then its Poisson integral $h = P[d\mu]$ is an harmonic function on \mathbb{D} . Moreover using Fubini's theorem and the following trivial equality

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta = 1, \quad (2.3)$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |h(re^{i\theta})| d\theta \leq \int_{\mathbb{T}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{|re^{i\theta} - \zeta|^2} d\theta \right) d|\mu|(\zeta) = \|\mu\|,$$

where $\|\mu\|$ is the total variation of the measure μ on \mathbb{T} . In particular, we get

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} |h(re^{i\theta})| d\theta < +\infty. \quad (2.4)$$

Now if μ is assumed to be positive, then h is positive and $|h| = h$. Thus by the mean value property for harmonic functions, we have

$$\int_{-\pi}^{\pi} |h(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} h(re^{i\theta}) d\theta = 2\pi h(0), \quad 0 \leq r < 1.$$

So the condition (2.4) is automatically satisfied by any positive harmonic function. In fact, Herglotz proved a converse to the preceding observation.

Theorem 2.1 (Herglotz). *Let h be a function defined on \mathbb{D} .*

- a) *The following assertions are equivalent:*
 - (i) *h is an harmonic function on \mathbb{D} which satisfies the condition (2.4);*
 - (ii) *there exists a unique complex Borel measure μ on \mathbb{T} such that $h = P[d\mu]$.*
- b) *The following assertions are equivalent:*
 - (i) *h is a positive harmonic function on \mathbb{D} .*
 - (ii) *there exists a positive Borel measure μ on \mathbb{T} such that $h = P[d\mu]$.*

Proof. we refer the reader to [] for a proof of this classical result. □

2.2 Classical Hardy spaces H^p

Let f be an analytic function on the open unit disc \mathbb{D} . Let

$$\|f\|_p = \sup_{0 \leq r < 1} \|f_r\|_p = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}},$$

if $p \in (0, \infty)$, and

$$\|f\|_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

Then the Hardy space $H^p(\mathbb{D})$ is the family of all analytic functions f where $\|f\|_p < \infty$. We are mainly concerned with H^1 , H^2 and H^{∞} . A simple application of Hölder's inequality shows that

$$H^{\infty}(\mathbb{D}) \subset H^q(\mathbb{D}) \subset H^p(\mathbb{D})$$

if $0 < p < q < \infty$. In particular, we have $H^{\infty} \subset H^2 \subset H^1$.

According to a celebrated theorem of Fatou, for each $f \in H^p(\mathbb{D})$, $0 < p \leq \infty$,

$$f(\zeta) = \lim_{z \underset{\triangleleft}{\rightarrow} \zeta} f(z)$$

exists for almost all $\zeta \in \mathbb{T}$. Moreover, $f|_{\mathbb{T}} \in L^p(\mathbb{T})$ and $\|f\|_{L^p(\mathbb{T})} = \|f\|_{H^p(\mathbb{T})}$, $0 < p \leq \infty$, and

$$\lim_{r \rightarrow 1} \|f_r - f\|_p = 0, \quad (0 < p < \infty). \quad (2.5)$$

This result establishes a norm preserving correspondence between $H^p(\mathbb{D})$ and a closed subspaces of $L^p(\mathbb{T})$ which we denote by $H^p(\mathbb{T})$. In particular, if $1 \leq p \leq \infty$, we also have the equivalent characterization

$$H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, n \leq -1\}$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt, \quad (n \in \mathbb{Z}),$$

is n th Fourier coefficient of f . A special role is played by $L^2(\mathbb{T})$ and its close subspace $H^2(\mathbb{T})$ which are Hilbert spaces endowed with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} dt = \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}.$$

Let $f \in H^p(\mathbb{D})$, $1 \leq p \leq \infty$. Let $z \in \mathbb{D}$, and let $|z| < R < 1$. Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w - z} dw = \frac{R}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{R - e^{-it}z} dt.$$

Let $R \rightarrow 1$. Then, by (2.5), we obtain

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z} dt, \quad (z \in \mathbb{D}). \quad (2.6)$$

Using similar argument, we also have

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{it}|^2} f(e^{it}) dt, \quad (z \in \mathbb{D}). \quad (2.7)$$

Now if $f \in H^2(\mathbb{D})$, the relation (2.6) can be rewritten as

$$f(z) = \langle f, k_z \rangle, \quad (z \in \mathbb{D}), \quad (2.8)$$

where

$$k_z(w) = \frac{1}{1 - \bar{z}w}, \quad (z, w \in \mathbb{D}). \quad (2.9)$$

In particular the Hardy space $H^2(\mathbb{D})$ is an example of analytic reproducing kernel Hilbert space and the function k_z defined by (2.9) is the reproducing kernel of H^2 .

Lemma 2.2. *Let $p \geq 1$ and let $f \in H^p$. Assume that $f|_{\mathbb{T}} \subset \mathbb{R}$. Then f is constant.*

Here in this lemma, the assumption $f|_{\mathbb{T}} \subset \mathbb{R}$ means that $f(\zeta)$ is real for almost all ζ on \mathbb{T} .

Proof. According to (2.7), we get from hypothesis that $f(\mathbb{D}) \subset \mathbb{R}$. But now it is well-known that the only real-valued function which are analytic on a domain are the constants (it is for instance an easy consequence of the Riemann equations). \square

If $f \in H^p(\mathbb{D})$, $p > 0$, and if $(\lambda_n)_{n \geq 1}$ is the zero sequence of f in \mathbb{D} (each zero is repeated according to its multiplicity), then we know that

$$\sum_{n \geq 1} (1 - |\lambda_n|) < +\infty. \quad (2.10)$$

This condition (2.10) is called the Blaschke condition. Then we can consider the infinite product defined by

$$B(z) = \prod_{n \geq 1} \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}, \quad z \in \mathbb{D}.$$

This product converges uniformly on compact subsets of \mathbb{D} . Moreover $|B| \leq 1$ in \mathbb{D} and $|B| = 1$ a.e. on \mathbb{T} . Now if we consider $g := f/B$, then $g \in H^p(\mathbb{D})$ and $\|g\|_p = \|f\|_p$.

Lemma 2.3. *Let $f \in H^1$, $f \not\equiv 0$. The following hold:*

- (a) $\log|f|$ is integrable on \mathbb{T} .
- (b) $f \neq 0$ almost everywhere on \mathbb{T} .

Proof. (a): if B is the Blaschke product formed on the zeros of f and if $g = f/B$, then we have $g \in H^1$ and $|g| = |f|$ a.e. on \mathbb{T} . Hence it is sufficient to prove the result assuming that f has no zeros in \mathbb{D} and $f(0) = 1$. Let us define now $\log^+(x) = \max(0, \log(x))$ and $\log^-(x) = \log^+(x) - \log(x)$, $x > 0$. Since $\log|f|$ is harmonic on \mathbb{D} , with $\log|f(0)| = 0$, the mean value property of harmonic functions implies that

$$\int_{\mathbb{T}} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_{\mathbb{T}} \log^- |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

for $0 < r < 1$. But $\log^+(x) \leq x$, whence

$$\int_{\mathbb{T}} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \int_{\mathbb{T}} |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \|f\|_1.$$

Then we get from Fatou's lemma that $\log^+ |f|$ and $\log^- |f|$ belongs to L^1 . Hence $\log|f|$ is also in L^1 .

- (b): it is an immediate consequence of (a).

\square

Lemma 2.4. Let $p, q, r \geq 1$ and let $f \in H^p$, $g \in H^q$. Assume that f is outer and $g/f \in L^r(\mathbb{T})$. Then we have $g/f \in H^r$.

Proof. Let $g = \lambda_1 BS[g]$ be the canonical factorization of g . Then we get that

$$\frac{g}{f} = \frac{\lambda_1 BS[g]}{\lambda[f]} = \frac{\lambda_1}{\lambda} BS \left[\frac{g}{f} \right].$$

Now since $g/f \in L^r$ and $\log |g/f| \in L^1$, we know that $[g/f] \in H^r$ and thus $g/f \in H^r$. □

Lemma 2.5. Let $f_1, f_2 \in H^p$. Then the product $f_1 f_2$ is an outer function if and only if each function f_1, f_2 is an outer function.

Proof. Assume that $f_1 f_2$ is an outer function and let $f_1 = \lambda_1 B_1 S_1[f_1]$ and $f_2 = \lambda_2 B_2 S_2[f_2]$ be the canonical factorization of f_1, f_2 . Then $f_1 f_2 = \lambda_1 \lambda_2 B_1 B_2 S_1 S_2[f_1 f_2]$. According to the uniqueness of the Riesz–Smirnov factorization, we get $B_1 B_2 S_1 S_2 \equiv 1$ and thus $B_1 \equiv B_2 \equiv S_1 \equiv S_2 \equiv 1$. Finally we have $f_1 = \lambda_1[f_1]$ and $f_2 = \lambda_2[f_2]$, which exactly means that f_1 and f_2 are outer functions.

The reverse implication is obvious because $[f_1][f_2] = [f_1 f_2]$. □

Lemma 2.6. Let f be an analytic function in \mathbb{D} and assume that $\Re f(z) > 0$, for all $z \in \mathbb{D}$. Then $f \in H^p$, $0 < p < 1$, and f is an outer function.

Proof. Let $0 < p < 1$ and let $\log z$ be the determination of the logarithmic defined by $\log z = \log |z| + i \arg_{[-\pi, \pi]}(z)$, which is holomorphic on $\mathbb{C} \setminus \mathbb{R}_-$. Then $z \mapsto f(z)^p = \exp(p \log f(z))$ is analytic on \mathbb{D} . Since $0 < p < 1$ and since $\Re f(z) > 0$, for all $z \in \mathbb{D}$, then there is a constant $c_p > 0$ such that

$$|f(z)|^p \leq c_p \Re(f(z)^p).$$

Then if we apply the mean value theorem to the harmonic function $\Re(f(z)^p)$, we get

$$\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \leq c_p \int_0^{2\pi} \Re(f(re^{i\theta})^p) \frac{d\theta}{2\pi} = c_p \Re(f(0)^p),$$

for $0 \leq r < 1$, which proves that $f \in H^p$. Moreover since $1/f$ is also an analytic function on \mathbb{D} and

$$\Re \left(\frac{1}{f(z)} \right) = \frac{\Re f(z)}{|f(z)|^2} > 0, \quad z \in \mathbb{D},$$

we also have $1/f \in H^p$, for $0 < p < 1$. It remains now to apply Lemma 2.5 to conclude that f is outer. □

2.3 The shift operator on H^2

The mapping

$$U \begin{array}{ccc} \ell^2 & \longrightarrow & H^2(\mathbb{D}) \\ (a_0, a_1, \dots) & \longmapsto & \sum_{n=0}^{\infty} a_n z^n \end{array}$$

is a unitary operator between ℓ^2 and H^2 . As a consequence, the shift operator that was defined in section ?? on ℓ^2 corresponds to an operator on H^2 which we also denote by S . It is not difficult to see that S is indeed given by

$$S : \begin{array}{ccc} H^2 & \longrightarrow & H^2 \\ f & \longmapsto & \chi f, \end{array}$$

where $\chi(z) = \chi_1(z) = z$, $z \in \mathbb{D}$. For obvious reason, S is called the forward shift operator on H^2 . The adjoint of S can be obtained using the adjoint of its cousin on ℓ^2 and the preceding unitary operator. However, we adopt a direct method. First note that

$$\langle \chi, 1 \rangle_{H^2} = 0.$$

Hence, for each $f, g \in H^2$, we have

$$\begin{aligned} \langle Sf, g \rangle_{H^2} &= \langle \chi f, g \rangle_{H^2} \\ &= \langle \chi f, (g - g(0)) \rangle_{H^2} \\ &= \langle f, (g - g(0))\bar{\chi} \rangle_{L^2}. \end{aligned}$$

However, $(g - g(0))\bar{\chi}$ is an element of H^2 . That is indeed why we replaced g by $g - g(0)$. Otherwise the identity

$$\langle Sf, g \rangle_{H^2} = \langle f, \bar{\chi}g \rangle_{L^2}$$

is also perfectly fine. Hence we can write

$$\langle Sf, g \rangle_{H^2} = \langle f, (g - g(0))\bar{\chi} \rangle_{H^2}, \quad (f, g \in H^2).$$

Therefore,

$$S^*g = (g - g(0))\bar{\chi}.$$

If $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then a more explicit formula for S^* is

$$(S^*g)(z) = \frac{g(z) - g(0)}{z} = \sum_{n=0}^{\infty} a_{n+1} z^n, \quad (z \in \mathbb{D}). \quad (2.11)$$

Lemma 2.7. *We have*

$$\sigma_p(S) = \emptyset \quad \text{and} \quad \sigma_p(S^*) = \mathbb{D}.$$

Moreover,

$$\sigma(S) = \sigma(S^*) = \bar{\mathbb{D}}.$$

Proof. To avoid confusion, in the proof of this lemma, we will denote by S_{ℓ^2} the shift operator on ℓ^2 and by S_{H^2} the shift operator on H^2 . Then using the unitary operator U , defined in the beginning of this section, we have $S_{H^2} = US_{\ell^2}U^{-1}$ and $S_{H^2}^* = US_{\ell^2}^*U^{-1}$. Thus it immediately follows that $\sigma_p(S_{H^2}) = \sigma_p(S_{\ell^2})$, $\sigma_p(S_{H^2}^*) = \sigma_p(S_{\ell^2}^*)$ and $\sigma(S_{H^2}) = \sigma(S_{\ell^2})$, $\sigma(S_{H^2}^*) = \sigma(S_{\ell^2}^*)$. Therefore the conclusion is a consequence of Theorem ??.

□

If $w \in \mathbb{D}$, then $1 - wS^*$ is invertible in $\mathcal{L}(H^2)$. Hence we have

$$Q_w = (1 - wS^*)^{-1} S^* \in \mathcal{L}(H^2).$$

This family of operators will enter our discussion many times. Here we study some of their elementary properties.

Theorem 2.8. Fix $w \in \mathbb{D}$, and let $Q_w = (1 - wS^*)^{-1} S^*$. Then, for each $f \in H^2$,

$$(Q_w f)(z) = \frac{f(z) - f(w)}{z - w}, \quad (z \in \mathbb{D}).$$

Proof. Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \quad (z \in \mathbb{D}).$$

Then, for each $n \geq 1$,

$$S^{*n} f(z) = \sum_{m=0}^{\infty} a_{m+n} z^m, \quad (z \in \mathbb{D}).$$

Since

$$Q_w = \sum_{n=1}^{\infty} w^{n-1} S^{*n},$$

we obtain

$$\begin{aligned} Q_w f(z) &= \sum_{n=1}^{\infty} w^{n-1} S^{*n} f(z) \\ &= \sum_{n=1}^{\infty} w^{n-1} \left(\sum_{m=0}^{\infty} a_{m+n} z^m \right) \\ &= \sum_{k=1}^{\infty} a_k \left(\sum_{m+n=k} z^m w^{n-1} \right) \\ &= \sum_{k=1}^{\infty} a_k \frac{z^k - w^k}{z - w} \\ &= \frac{\sum_{k=1}^{\infty} a_k z^k - \sum_{k=1}^{\infty} a_k w^k}{z - w} \\ &= \frac{f(z) - f(w)}{z - w}. \end{aligned}$$

□

Corollary 2.9. *Let $f, g \in H^2$. Then*

$$Q_w(fg) = f Q_w g + g(w) Q_w f.$$

In particular,

$$S^*(fg) = f S^* g + g(0) S^* f.$$

Proof. By Theorem 2.8, for each $z \in \mathbb{D}$,

$$Q_w(fg)(z) = \frac{f(z)g(z) - f(w)g(w)}{z - w}.$$

Hence

$$Q_w(fg)(z) = f(z) \frac{g(z) - g(w)}{z - w} + g(w) \frac{f(z) - f(w)}{z - w}.$$

Again by Theorem 2.8, this is exactly the first identity. Since $Q_0 = S^*$, the second identity is a special case of the first one. □

2.4 The F. M. Riesz Theorem

Theorem 2.10 (F.&M. Riesz). *Let ν be a complex Borel measure on \mathbb{T} . Assume that*

$$\int_{\mathbb{T}} e^{in\theta} d\nu(e^{i\theta}) = 0, \quad n \geq 1. \quad (2.12)$$

Then ν is absolutely continuous with respect to the Lebesgue measure.

Proof. For $z \in \mathbb{D}$, define

$$f(z) = \int_{\mathbb{T}} \frac{d\nu(\zeta)}{1 - z\bar{\zeta}}.$$

It is clear that f is analytic on \mathbb{D} . Let us prove that $f \in H^1$. If $z = re^{i\theta}$, we have

$$\frac{1}{1 - ze^{-it}} = \sum_{n=0}^{+\infty} r^n e^{in(\theta-t)},$$

and since

$$\frac{1 - r^2}{|e^{it} - re^{i\theta}|^2} = \Re \left(\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} \right) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-t)},$$

it follows from (2.12) that

$$f(re^{i\theta}) = \int_{\mathbb{T}} \frac{1 - r^2}{|e^{it} - re^{i\theta}|^2} d\nu(e^{it}).$$

Hence by Fubini's theorem, we get

$$\int_0^{2\pi} |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \int_{\mathbb{T}} \left(\int_0^{2\pi} \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} \frac{d\theta}{2\pi} \right) d|\nu|(e^{it}) = \|\nu\|,$$

which proves that $f \in H^1$. But by (2.7), we have

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|r|^2}{|e^{it}-re^{i\theta}|^2} f(e^{it}) dt, \quad (re^{i\theta} \in \mathbb{D}),$$

which gives

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2} f(\zeta) dm(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2} d\nu(\zeta) \quad (z \in \mathbb{D}).$$

In other words, we have $P[f dm] = P[d\nu]$. Now using the unicity of the Herglotz-representation (Theorem 2.1), we get that $d\nu(\zeta) = f(\zeta) dm(\zeta)$, which gives the conclusion. \square

Corollary 2.11. *Let λ and μ be complex Borel measures on \mathbb{T} such that*

$$\int_{\mathbb{T}} \frac{e^{-i\theta} + z}{e^{-i\theta} - z} d\lambda(e^{i\theta}) = \int_{\mathbb{T}} \frac{e^{-i\theta} + z}{e^{-i\theta} - z} d\mu(e^{i\theta}), \quad (2.13)$$

for all $z \in \mathbb{D}$. Then $\lambda - \mu$ is absolutely continuous with respect to the Lebesgue measure.

Proof. For $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, we have

$$\begin{aligned} \frac{\bar{\zeta} + z}{\bar{\zeta} - z} &= 1 + \frac{2z}{\bar{\zeta} - z} = 1 + \frac{2z\zeta}{1 - 2z\zeta} \\ &= 1 + 2 \sum_{n=1}^{+\infty} z^n \zeta^n, \end{aligned}$$

and the serie is uniformly convergent with respect to $\zeta \in \mathbb{T}$. Therefore using (2.13), we get

$$\lambda(\mathbb{T}) + 2 \sum_{n=1}^{+\infty} z^n \int_{\mathbb{T}} e^{in\theta} d\lambda(e^{i\theta}) = \mu(\mathbb{T}) + 2 \sum_{n=1}^{+\infty} z^n \int_{\mathbb{T}} e^{in\theta} d\mu(e^{i\theta}),$$

for every $z \in \mathbb{D}$. But both functions of z in the preceeding equality are analytic in the unit disc. Thus the Taylor coefficient must be the same and we get that

$$\int_{\mathbb{T}} e^{in\theta} d\lambda(e^{i\theta}) = \int_{\mathbb{T}} e^{in\theta} d\mu(e^{i\theta}), \quad n \geq 1.$$

The conclusion now follows from Theorem 2.10. \square

2.5 Generalized Hardy spaces $H^2(\nu)$

The Hardy space $H^2(\mathbb{T})$ is the closure of analytic polynomials in $L^2(m)$. This interpretation gives us the motivation to define $H^2(\nu)$, where ν is a Borel measure on \mathbb{T} . Since each $\nu \in \mathcal{M}(\mathbb{T})$ has a finite total variation, analytic polynomials are in $L^2(\nu)$. Then $H^2(\nu)$ is defined to be the the closure of analytic polynomials in $L^2(\nu)$.

Each kernel function k_z is bounded. Thus at least $k_z \in L^2(\nu)$. We naturally expect to have $k_z \in H^2(\nu)$. As a matter of fact we can say more. To treat this case, we consider a slightly more generalized concept. Each analytic polynomial is in $\mathcal{C}(\mathbb{T})$, and we remind that $\mathcal{C}(\mathbb{T})$ is endowed with the uniform norm. The closure of analytic polynomials in $\mathcal{C}(\mathbb{T})$ is called the disc algebra and is denoted by \mathcal{A} .

Lemma 2.12. *Let*

$$k_z(w) = \frac{1}{1 - \bar{z}w}, \quad (z \in \mathbb{D}).$$

Then, the following hold:

- (a) *for each $z \in \mathbb{D}$, $k_z \in \mathcal{A}$.*
- (b) *The linear manifold of finite linear combinations of k_z , $z \in \mathbb{D}$, is uniformly dense in \mathcal{A} .*

Proof. (a): fix $z \in \mathbb{D}$. Then we have

$$\left| k_z(w) - \sum_{n=0}^N \bar{z}^n w^n \right| \leq \frac{|z|^{N+1}}{1 - |z|}$$

for all $w \in \mathbb{T}$. In other words, k_z is the uniform limit of analytic polynomials. Thus $k_z \in \mathcal{A}$.

(b): to show that the linear manifold of finite linear combinations of reproducing kernel functions is uniformly dense in \mathcal{A} , it is enough to verify that each monomial $\chi_n(w) = w^n$, $n \geq 0$, can be uniformly approximated by a sequence in the manifold. For $n = 0$, just note that $k_0 = 1$. For $n \geq 1$, let $\zeta = e^{i2\pi/n}$. Then we have

$$\frac{1 + \zeta^\ell + \zeta^{2\ell} + \dots + \zeta^{(n-1)\ell}}{n} = \begin{cases} 0 & \text{if } n \nmid \ell, \\ 1 & \text{if } n \mid \ell. \end{cases}$$

Therefore, for each $0 < r < 1$,

$$\frac{k_r(w) + k_{r\zeta}(w) + \dots + k_{r\zeta^{n-1}}(w)}{n} = \sum_{k=0}^{\infty} r^{kn} w^{kn}.$$

From this formula we obtain

$$\left| \frac{k_r(w) + k_{r\zeta}(w) + \cdots + k_{r\zeta^{n-1}}(w) - nk_0(w)}{nr^n} - \chi_n(w) \right| \leq \frac{r^n}{1-r}$$

for all $w \in \mathbb{T}$. Hence

$$\frac{k_r + k_{r\zeta} + \cdots + k_{r\zeta^{n-1}} - nk_0}{nr^n} \longrightarrow \chi_n$$

uniformly on \mathbb{T} , as $r \rightarrow 0$.

□

The following result immediately follows from Lemma 2.12 and the fact that each element of $\mathcal{M}(\mathbb{T})$ has a finite total variation. In the proof of Lemma 2.12, we hesitated to use the notation $\|\cdot\|_\infty$, and instead we wrote *for all* $w \in \mathbb{T}$ wherever it was necessary. This was to emphasize that those relations are valid at all points of \mathbb{T} with no exception. This fact is implicitly used in the following Theorem.

Theorem 2.13. *Let $\nu \in \mathcal{M}(\mathbb{T})$. Then, for each $z \in \mathbb{D}$, $k_z \in H^2(\nu)$. Moreover, the linear manifold of finite linear combinations of k_z , $z \in \mathbb{D}$, is dense in $H^2(\nu)$.*

Knowing that the family of reproducing kernel functions also generates the Hardy space $H^2(\nu)$ in $L^2(\nu)$, we can give another characterization for the orthogonal complement of $H^2(\nu)$. Of course, the original definition says that

$$\left(H^2(\nu) \right)^\perp = \{ f \in L^2(\nu) : \langle f, \chi_n \rangle_\nu = 0, n \geq 0 \}.$$

Corollary 2.14. *Let $\nu \in \mathcal{M}(\mathbb{T})$. Then*

$$\left(H^2(\nu) \right)^\perp = \{ f \in L^2(\nu) : \langle f, k_z \rangle_\nu = 0, z \in \mathbb{D} \}.$$

2.6 The Cauchy integral

Let $\nu \in \mathcal{M}(\mathbb{T})$. For each $f \in L^1(\nu)$, the formula

$$(K_\nu f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\bar{\zeta}} d\nu(\zeta) \tag{2.14}$$

gives a well defined analytic function at least over $\mathbb{C} \setminus \mathbb{T}$.

Lemma 2.15. *The linear map $K_\nu : L^1(\nu) \longrightarrow \text{Hol}(\mathbb{C} \setminus \mathbb{T})$ is continuous.*

Proof. Let $f \in L^1(\nu)$ and let K be a compact subset of $\mathbb{C} \setminus \mathbb{T}$. For $z \in K$, we have

$$|(K_\nu f)(z)| = \left| \int_{\mathbb{T}} \frac{f(\zeta)}{1 - z\bar{\zeta}} d\nu(\zeta) \right| \leq \frac{\|f\|_{L^1(\nu)}}{|1 - |z||}.$$

Since $K \subset \mathbb{C} \setminus \mathbb{T}$, $\delta := \inf_{z \in K} |1 - |z|| > 0$ and we get that

$$\sup_{z \in K} |(K_\nu f)(z)| \leq \frac{\|f\|_{L^1(\nu)}}{\delta}. \quad (2.15)$$

Now let $(f_n)_{n \geq 1}$ be a sequence in $L^1(\nu)$ which converges to f . It follows from (2.15) that $(K_\nu f_n)_{n \geq 1}$ converges uniformly to $K_\nu f$ on compact subset K , which exactly means that K_ν is continuous from $L^1(\nu)$ into the topological space $\text{Hol}(\mathbb{C} \setminus \mathbb{T})$. □

If $f \in L^2(\nu)$, over the open unit disc, we can write

$$(K_\nu f)(z) = \langle f, k_z \rangle_\nu, \quad (z \in \mathbb{D}), \quad (2.16)$$

where k_z is the reproducing kernel for the Hardy space H^2 . The range of K_ν when f ranges over $L^2(\nu)$ is denoted by $K^2(\nu)$. Hence, for the time being, $K^2(\nu)$ is simply an aggregate of analytic functions on $\mathbb{C} \setminus \mathbb{T}$. In short, we defined the map

$$\begin{array}{ccc} K_\nu : L^2(\nu) & \longrightarrow & K^2(\nu) \\ f & \longmapsto & K_\nu f. \end{array}$$

Theorem 2.16. *Let $\nu \in \mathcal{M}(\mathbb{T})$, and let $f \in L^2(\nu)$. Then*

$$K_\nu f|_{\mathbb{D}} \equiv 0 \iff f \in \left(H^2(\nu) \right)^\perp.$$

Proof. This follows directly from (2.16) and Corollary 2.14. □

We denote the function $K_\nu 1 \in K^2(\nu)$ by $K\nu$. Hence $K\nu$ has the representation

$$K\nu(z) = \int_{\mathbb{T}} \frac{d\nu(\zeta)}{1 - z\bar{\zeta}}, \quad (z \in \mathbb{C} \setminus \mathbb{T}). \quad (2.17)$$

This is called the Cauchy integral of ν .

Theorem 2.17. *Let $\nu \in \mathcal{M}(\mathbb{T})$. Then*

$$K\nu(z) = \sum_{n=0}^{\infty} \hat{\nu}(n) z^n, \quad (z \in \mathbb{D}),$$

and

$$K\nu(z) = - \sum_{n=1}^{\infty} \frac{\hat{\nu}(-n)}{z^n}, \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}),$$

where

$$\hat{\nu}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} d\nu(e^{i\theta}), \quad n \in \mathbb{Z}.$$

Proof. If $z \in \mathbb{D}$, then

$$\frac{1}{1 - z\bar{\zeta}} = \sum_{n=0}^{\infty} \bar{\zeta}^n z^n,$$

and if $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$, then

$$\frac{1}{1 - z\bar{\zeta}} = - \sum_{n=1}^{\infty} \frac{\zeta^n}{z^n}.$$

Moreover, for a fixed z , both series are uniformly convergent on \mathbb{T} . Hence we can change the order of summation and integration and the result follows. \square

Lemma 2.18. *Let ν be a complex Borel measure on \mathbb{T} . Then*

$$\int_{\mathbb{T}} \frac{\bar{\zeta}}{1 - \bar{\zeta}z} d\nu(\zeta) = \frac{K\nu(z) - K\nu(0)}{z}, \quad z \in \mathbb{C} \setminus \mathbb{T}, z \neq 0.$$

Proof. For $z \in \mathbb{C} \setminus \mathbb{T}$, $z \neq 0$, we have

$$\frac{\bar{\zeta}}{1 - \bar{\zeta}z} = \frac{1}{z} \frac{\bar{\zeta}z}{1 - \bar{\zeta}z} = \frac{1}{z} \left(\frac{1}{1 - \bar{\zeta}z} - 1 \right).$$

Hence it follows from (2.17) that

$$\int_{\mathbb{T}} \frac{\bar{\zeta}}{1 - \bar{\zeta}z} d\nu(\zeta) = \frac{1}{z} \left(\int_{\mathbb{T}} \frac{d\nu(\zeta)}{1 - \bar{\zeta}z} - \int_{\mathbb{T}} d\nu(\zeta) \right) = \frac{1}{z} (K\nu(z) - K\nu(0)).$$

\square

Let ν be a Borel measure on \mathbb{T} and let Z_ν denote the operator on $H^2(\nu)$ of multiplication by the independent variable,

$$(Z_\nu f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta}), \quad f \in H^2(\nu).$$

We have the following connection between K_ν and Z_ν .

Lemma 2.19. *Let ν be a complex Borel measure on \mathbb{T} and let $g \in H^2(\nu)$. Then*

$$(K_\nu Z_\nu^* g)(z) = \frac{(K_\nu g)(z) - (K_\nu g)(0)}{z}, \quad z \in \mathbb{D}, z \neq 0.$$

Proof. Let $g \in H^2(\nu)$, $z \in \mathbb{D}$, $z \neq 0$. Then according to (2.16), we have

$$(K_\nu Z_\nu^* g)(z) = \langle Z_\nu^* g, k_z \rangle_\nu = \langle g, Z_\nu k_z \rangle_\nu = \int_{\mathbb{T}} \frac{g(e^{i\theta}) e^{-i\theta}}{1 - z e^{-i\theta}} d\nu(e^{i\theta}).$$

Applying Lemma 2.18 to $g(e^{i\theta}) d\nu(e^{i\theta})$, we get the result, because $K_\nu g = K(\nu g)$. \square

Chapter 3

Toeplitz operators

3.1 The multiplication operator M_φ

Let $\varphi \in L^\infty(\mathbb{T})$, and let

$$\begin{aligned} M_\varphi : L^2(\mathbb{T}) &\longrightarrow L^2(\mathbb{T}) \\ f &\longmapsto \varphi f. \end{aligned}$$

Since $\|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2$, M_φ is a bounded operator and we have $\|M_\varphi\| \leq \|\varphi\|_\infty$. As a matter of fact, we show that $\|M_\varphi\| = \|\varphi\|_\infty$.

Theorem 3.1. *Let $\varphi \in L^\infty(\mathbb{T})$. Then*

$$\|M_\varphi\|_{\mathcal{L}(L^2(\mathbb{T}))} = \|\varphi\|_{L^\infty(\mathbb{T})}.$$

Proof. Fix $\varepsilon > 0$, and let

$$E = \{e^{it} : |\varphi(e^{it})| \geq \|\varphi\|_\infty - \varepsilon\}.$$

Consider $f = \chi_E$, the characteristic function of E . The set E has a positive Lebesgue measure, and thus $f \in L^2(\mathbb{T})$ with $\|f\|_2 \neq 0$. We use the inequality

$$\|M_\varphi f\|_2 \leq \|M_\varphi\| \|f\|_2$$

to obtain a lower bound for $\|M_\varphi\|$. First note that

$$\begin{aligned} \|M_\varphi f\|_2^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |(M_\varphi f)(e^{it})|^2 dt \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |\varphi(e^{it})|^2 |f(e^{it})|^2 dt \\ &= \frac{1}{2\pi} \int_E |\varphi(e^{it})|^2 dt \\ &\geq \frac{(\|\varphi\|_\infty - \varepsilon)^2}{2\pi} \int_E dt \\ &= (\|\varphi\|_\infty - \varepsilon)^2 \|f\|_2^2. \end{aligned}$$

Since $\|f\|_2 \neq 0$, we get $\|M_\varphi\| \geq \|\varphi\|_\infty - \varepsilon$. Let $\varepsilon \rightarrow 0$ to obtain $\|M_\varphi\| \geq \|\varphi\|_\infty$. \square

It is easy to determine the adjoint of M_φ . Indeed, for any $f, g \in L^2(\mathbb{T})$, we have

$$\begin{aligned} \langle M_\varphi f, g \rangle_2 &= \frac{1}{2\pi} \int_0^{2\pi} (M_\varphi f)(e^{it}) \overline{g(e^{it})} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) f(e^{it}) \overline{g(e^{it})} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{\varphi(e^{it}) g(e^{it})} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{(M_{\bar{\varphi}} g)(e^{it})} dt \\ &= \langle f, M_{\bar{\varphi}} g \rangle_2. \end{aligned}$$

Therefore, for each $\varphi \in L^\infty(\mathbb{T})$,

$$M_\varphi^* = M_{\bar{\varphi}}. \quad (3.1)$$

3.2 The norm of T_φ

Let $\varphi \in L^\infty(\mathbb{T})$. Then the Toeplitz operator associate with φ is defined by

$$\begin{array}{ccc} T_\varphi : H^2(\mathbb{T}) & \longrightarrow & H^2(\mathbb{T}) \\ f & \longmapsto & P_+(\varphi f). \end{array}$$

In other words,

$$T_\varphi = P_+ \circ M_\varphi \circ i,$$

where P_+ is the Riesz projection, $M_\varphi : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ denotes the multiplication operator introduced in Section 3.1 and $i : H^2(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is the inclusion map. We mention here an easy property of P_+ we will use several times in this course.

Lemma 3.2. P_+ is a bounded operator on $L^2(\mathbb{T})$ and for every $f, g \in L^2$, we have

$$\langle P_+ f, P_+ g \rangle_2 = \langle P_+ f, g \rangle_2.$$

Proof. It is an easy exercise! \square

Hence, T_φ is a bounded operator and, by Theorem 3.1,

$$\|T_\varphi\| \leq \|P_+\| \|M_\varphi\| \|i\| = \|\varphi\|_\infty.$$

However, we indeed will see that $\|T_\varphi\| = \|\varphi\|_\infty$. Throughout this chapter we use the notation

$$\chi_m(e^{it}) = e^{imt}, \quad (m \in \mathbb{Z}).$$

Lemma 3.3. *Let $\varphi \in L^\infty(\mathbb{T})$, and let $f \in L^2(\mathbb{T})$. Then*

$$\lim_{m \rightarrow +\infty} \left\| \chi_{-m} T_\varphi(P_+(\chi_m f)) - \varphi f \right\|_2 = 0.$$

Proof. Let

$$f_m(e^{it}) = (\chi_{-m} P_+(\chi_m f))(e^{it}) = \sum_{n=-m}^{\infty} \hat{f}(n) e^{int}, \quad (m \in \mathbb{Z}).$$

By Parseval's identity,

$$\|f_m - f\|_2^2 = \sum_{n=-\infty}^{-m-1} |\hat{f}(n)|^2.$$

Therefore, for each $f \in L^2(\mathbb{T})$,

$$\lim_{m \rightarrow +\infty} \|f_m - f\|_2 = 0.$$

Since

$$\begin{aligned} \chi_{-m} T_\varphi(P_+(\chi_m f)) &= \chi_{-m} T_\varphi(\chi_m f_m) \\ &= \chi_{-m} P_+(\varphi \chi_m f_m) \\ &= \chi_{-m} P_+(\varphi \chi_m (f_m - f)) + \chi_{-m} P_+(\chi_m \varphi f) \\ &= \chi_{-m} P_+(\varphi \chi_m (f_m - f)) + (\varphi f)_m, \end{aligned}$$

and

$$\|\chi_{-m} P_+(\varphi \chi_m (f_m - f))\|_2 \leq \|\varphi\|_\infty \|f_m - f\|_2,$$

and $\|(\varphi f)_m - \varphi f\|_2 \rightarrow 0$, the result follows. \square

We are now ready to show that $\|T_\varphi\| = \|\varphi\|_\infty$.

Theorem 3.4. *Let $\varphi \in L^\infty(\mathbb{T})$. Then*

$$\|T_\varphi\|_{\mathcal{L}(H^2(\mathbb{T}))} = \|\varphi\|_{L^\infty(\mathbb{T})}.$$

Proof. We already saw that $\|T_\varphi\| \leq \|\varphi\|_\infty$. Lemma 3.3 and the inequality

$$\|T_\varphi g\|_2 \leq \|T_\varphi\| \|g\|_2, \quad (g \in H^2(\mathbb{T})),$$

to obtain a lower bound for $\|T_\varphi\|$. Let $f \in L^2(\mathbb{T})$ and put $g = P_+(\chi_m f)$, $m \in \mathbb{Z}$. Since χ_m is unimodular, the last inequality implies

$$\left\| \chi_{-m} T_\varphi(P_+(\chi_m f)) \right\|_2 \leq \|T_\varphi\| \|f\|_2, \quad (f \in L^2(\mathbb{T})).$$

By Lemma 3.3, we let $m \rightarrow +\infty$ to obtain

$$\|\varphi f\|_2 \leq \|T_\varphi\| \|f\|_2, \quad (f \in L^2(\mathbb{T})).$$

In other words,

$$\|M_\varphi f\|_2 \leq \|T_\varphi\| \|f\|_2, \quad (f \in L^2(\mathbb{T})),$$

where M_φ is the multiplication operator introduced in Section 3.1. The last inequality is equivalent to $\|\varphi\| \leq \|T_\varphi\|$. But, by Theorem 3.1, we know that $\|M_\varphi\| = \|\varphi\|_\infty$. Hence we also have $\|M_\varphi\|_\infty \leq \|T_\varphi\|$. \square

3.3 The adjoint of T_φ

It is not difficult to find the adjoint of T_φ . As a matter of fact, by Lemma 3.2, for each $f, g \in H^2(\mathbb{T})$, we have

$$\begin{aligned} \langle T_\varphi f, g \rangle_{H^2(\mathbb{T})} &= \langle P_+(\varphi f), g \rangle_{H^2(\mathbb{T})} \\ &= \langle \varphi f, g \rangle_{L^2(\mathbb{T})} \\ &= \langle f, \bar{\varphi} g \rangle_{L^2(\mathbb{T})} \\ &= \langle f, P_+(\bar{\varphi} g) \rangle_{H^2(\mathbb{T})} \\ &= \langle f, T_{\bar{\varphi}} g \rangle_{H^2(\mathbb{T})}. \end{aligned}$$

Therefore, for each $\varphi \in L^\infty(\mathbb{T})$,

$$T_\varphi^* = T_{\bar{\varphi}}. \quad (3.2)$$

In the following result, φ being real means that $\varphi(e^{it}) \in \mathbb{R}$ for almost all $e^{it} \in \mathbb{T}$. Other statements should be interpreted similarly.

Theorem 3.5. *Let $\varphi \in L^\infty(\mathbb{T})$. Then the following hold.*

- (i) T_φ is self adjoint if and only if φ is a real function.
- (ii) T_φ is positive if and only if φ is a positive function.
- (iii) T_φ is an orthogonal projection if and only if either $\varphi \equiv 1$ or $\varphi \equiv 0$. In this case, T_φ is the identity operator I or the zero operator 0 .

Proof. (i) This is an immediate consequence of (3.2).

(ii) By Lemma 3.2, for each $f \in H^2(\mathbb{T})$,

$$\langle T_\varphi f, f \rangle_{H^2(\mathbb{T})} = \langle \varphi f, f \rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |f(e^{it})|^2 dt.$$

If φ is positive, the last identity shows that T_φ is a positive operator.

Now suppose that T_φ is positive. This means that

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |f(e^{it})|^2 dt \geq 0, \quad (f \in H^2(\mathbb{T})).$$

But we show that this inequality in fact holds for any $f \in L^2(\mathbb{T})$. Fix $f \in L^2(\mathbb{T})$, and let $m \geq 1$. Then $P_+(\chi_m f) \in H^2(\mathbb{T})$ and thus

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |P_+(\chi_m f)(e^{it})|^2 dt \geq 0.$$

Since $|\chi_{-m}| = 1$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |\chi_{-m}(e^{it}) P_+(\chi_m f)(e^{it})|^2 dt \geq 0.$$

But, by Lemma 3.3, $\chi_{-m} P_+(\chi_m f) \rightarrow f$ in $L^2(\mathbb{T})$ norm. Hence, for each $f \in L^2(\mathbb{T})$,

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |f(e^{it})|^2 dt \geq 0.$$

Therefore, by taking f to be the characteristic function of an arbitrary measurable set, we deduce that $\varphi \geq 0$.

(iii) It is trivial that $T_1 = I$ and $T_0 = 0$ are orthogonal projections. Now suppose that T_φ is an orthogonal projection. By (ii) and Theorem 3.4, we necessarily have $0 \leq \varphi \leq 1$. In the light of Lemma 3.2, the property

$$\langle T_\varphi f, T_\varphi f \rangle_{H^2(\mathbb{T})} = \langle T_\varphi f, f \rangle_{H^2(\mathbb{T})}, \quad (f \in H^2(\mathbb{T})),$$

is written as

$$\langle \varphi f, T_\varphi f \rangle_{L^2(\mathbb{T})} = \langle \varphi f, f \rangle_{L^2(\mathbb{T})}, \quad (f \in H^2(\mathbb{T})).$$

Hence, for all $f \in H^2(\mathbb{T})$,

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) f(e^{it}) \overline{P_+(\varphi f)(e^{it})} dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |f(e^{it})|^2 dt.$$

In particular, for $f = \chi_m$, $m \geq 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) \overline{\chi_{-m}(e^{it}) P_+(\varphi \chi_m)(e^{it})} dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) dt.$$

But, by Lemma 3.3, $\chi_{-m} P_+(\chi_m \varphi) \rightarrow \varphi$ in $L^2(\mathbb{T})$ norm. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi^2(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) dt.$$

Since $0 \leq \varphi \leq 1$, the identity

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) (1 - \varphi(e^{it})) dt = 0$$

holds if and only if either $\varphi = 1$ or $\varphi = 0$, or equivalently, when φ is the characteristic function of a measurable set. Therefore, for each $f \in H^2(\mathbb{T})$, we have

$$\langle T_\varphi f, f \rangle_{H^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{it}) |f(e^{it})|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it}) f(e^{it})|^2 dt$$

and this quantity is equal to

$$\langle T_\varphi f, T_\varphi f \rangle_{H^2(\mathbb{T})} = \frac{1}{2\pi} \int_0^{2\pi} |P_+(\varphi f)(e^{it})|^2 dt.$$

In short

$$\|\varphi f\|_{L^2} = \|P_+(\varphi f)\|_{L^2}, \quad (f \in H^2(\mathbb{T})).$$

But this happens if and only if $\varphi f \in H^2(\mathbb{T})$, for all $f \in H^2(\mathbb{T})$. In particular, we must have $\varphi \in H^2(\mathbb{T})$. The only real functions in $H^2(\mathbb{T})$ are real constant functions. Hence either $\varphi \equiv 1$ or $\varphi \equiv 0$. \square

3.4 Toeplitz operators with (anti-)analytic symbols

If $\varphi \in H^\infty(\mathbb{T})$, then

$$T_\varphi f = \varphi f$$

for each $f \in H^2(\mathbb{T})$. In other words, φ is a multiplier of the reproducing kernel Hilbert space $H^2(\mathbb{T})$ and T_φ is indeed the multiplier operator M_φ introduced in Section 1.4. However, it should not be mixed up with the multiplier operator of Section 3.1 on $L^2(\mathbb{T})$. Therefore, as a special case of Theorem 1.10 and by (3.2), we have

$$T_{\bar{\varphi}} k_z = \overline{\varphi(z)} k_z, \quad (z \in \mathbb{D}), \quad (3.3)$$

where k_z is the kernel of H^2 .

Theorem 3.6. *Let $\varphi \in H^\infty(\mathbb{T})$, $\varphi \neq 0$. Then the following are equivalent.*

- (i) φ is inner.
- (ii) T_φ is an isometry.
- (iii) T_φ is a partial isometry.

Proof. (i) \implies (ii) If φ is inner, then $T_\varphi(f) = \varphi f$, $f \in H^2$, and since φ is unimodular on \mathbb{T} ,

$$\|T_\varphi(f)\|_{H^2} = \|f\|_{H^2}.$$

(ii) \implies (iii) Obvious.

(iii) \implies (i) If T_φ is a partial isometry, then, by Theorem 1.7, $T_{|\varphi|^2} = T_\varphi^* T_\varphi$ is an orthogonal projection. By Theorem 3.5, we must have either $|\varphi| \equiv 1$ or $|\varphi| \equiv 0$. The second possibility is ruled out by assumption. The first one means that φ is inner. \square

Generally speaking, a Toeplitz operator is far away from being one to one. The following result provides a sufficient condition for injectivity.

Theorem 3.7. *Let φ be a function in H^∞ , $\varphi \neq 0$. Then the following hold:*

- (a) T_φ is injective.
- (b) If φ is assumed to be outer, then $T_{\bar{\varphi}}$ is also injective.

Proof. (a) follows immediately from Lemma 2.3. To prove (b), let f be in the kernel of $T_{\bar{\varphi}}$. That means that

$$P_+(\bar{\varphi}f) = 0.$$

Hence we $\bar{\varphi}f \in \overline{H_0^2}$, or equivalently, $\varphi \bar{f} \in H_0^2$. Using Lemma 2.4, we deduce that $\bar{f} \in H_0^2$. Therefore $f \in H^2 \cap \overline{H_0^2} = \{0\}$ and that concludes the proof. \square

3.5 Composition of Toeplitz operators

We saw that the multiplication operators on $L^2(\mathbb{T})$ commute, i.e., for each $\varphi, \psi \in L^\infty(\mathbb{T})$, $M_\varphi M_\psi = M_\psi M_\varphi = M_{\varphi\psi}$. However, the class of Toeplitz operators is not commutative. The following theorem reveals a special result of this type.

Theorem 3.8. *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. Suppose that at least one of them is in $H^\infty(\mathbb{T})$. Then*

$$T_{\bar{\psi}}T_\varphi = T_{\bar{\psi}\varphi}.$$

Proof. If $\varphi \in H^\infty(\mathbb{T})$, then, for each $f \in H^2(\mathbb{T})$,

$$T_{\bar{\psi}}T_\varphi f = T_{\bar{\psi}}P_+(\varphi f) = T_{\bar{\psi}}(\varphi f) = P_+(\bar{\psi}\varphi f) = T_{\bar{\psi}\varphi}f.$$

Hence $T_{\bar{\psi}}T_\varphi = T_{\bar{\psi}\varphi}$. If $\psi \in H^\infty(\mathbb{T})$, then, by what we just proved,

$$T_{\bar{\varphi}}T_\psi = T_{\bar{\varphi}\psi}.$$

Therefore, by (3.2),

$$T_{\bar{\psi}}T_\varphi = T_{\bar{\psi}}^*T_{\bar{\varphi}}^* = (T_{\bar{\varphi}}T_\psi)^* = T_{\bar{\varphi}\psi}^* = T_{\varphi\bar{\psi}}.$$

□

In the preceding theorem, it is important to note that $T_{\bar{\psi}}T_\varphi \neq T_\varphi T_{\bar{\psi}}$. Neglecting this fact is a common source of mistake. However, if $\varphi, \psi \in H^\infty(\mathbb{T})$, we do have

$$T_\varphi T_\psi = T_\psi T_\varphi = T_{\varphi\psi} \quad \text{and} \quad T_{\bar{\varphi}}T_{\bar{\psi}} = T_{\bar{\psi}}T_{\bar{\varphi}} = T_{\bar{\varphi}\bar{\psi}}. \quad (3.4)$$

The first relation is trivial. The second is obtained by taking the conjugate of all sides of the first one.

Theorem 3.9. *Let $\varphi \in H^\infty(\mathbb{T})$. Then*

$$T_\varphi T_{\bar{\varphi}} \leq T_{\bar{\varphi}} T_\varphi.$$

Proof. For each $f \in H^2(\mathbb{T})$, by (??), we have

$$\langle T_\varphi T_{\bar{\varphi}} f, f \rangle_{H^2} = \langle T_{\bar{\varphi}} f, T_{\bar{\varphi}} f \rangle_{H^2} = \|P_+(\bar{\varphi}f)\|_{H^2}^2 \leq \|\bar{\varphi}f\|_{L^2}^2.$$

But, $\|\bar{\varphi}f\|_{L^2} = \|\varphi f\|_{L^2} = \|\varphi f\|_{H^2}$ and

$$\|\varphi f\|_{H^2}^2 = \|T_\varphi f\|_{H^2}^2 = \langle T_\varphi f, T_\varphi f \rangle_{H^2} = \langle T_{\bar{\varphi}} T_\varphi f, f \rangle_{H^2}.$$

Thus

$$\langle T_\varphi T_{\bar{\varphi}} f, f \rangle_{H^2} \leq \langle T_{\bar{\varphi}} T_\varphi f, f \rangle_{H^2}, \quad (f \in H^2(\mathbb{T})),$$

which means $T_\varphi T_{\bar{\varphi}} \leq T_{\bar{\varphi}} T_\varphi$. □

3.6 The compactness

It is clear that $T_0 = 0$ is compact. In this section we show that this is the only compact Toeplitz operator.

The Fourier coefficients of an function $\psi \in L^1(\mathbb{T})$ was defined by

$$\hat{\psi}(n) = \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{it}) e^{-int} dt, \quad (n \in \mathbb{Z}).$$

By the Riemann–Lebesgue lemma,

$$\lim_{n \rightarrow \pm\infty} \hat{\psi}(n) = 0.$$

In particular, if $f, g \in L^2(\mathbb{T})$, then $f\bar{g} \in L^1(\mathbb{T})$, and the Riemann–Lebesgue lemma says

$$\lim_{n \rightarrow \pm\infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} e^{-int} dt = 0.$$

Using the inner product of $L^2(\mathbb{T})$, this fact is rewritten as

$$\lim_{n \rightarrow \pm\infty} \langle \chi_n f, g \rangle = 0. \quad (3.5)$$

In technical language, this means that, for each fixed $f \in L^2(\mathbb{T})$, the sequence $(\chi_n f)_{n \in \mathbb{Z}}$ weakly converges to zero.

Theorem 3.10. *Let $\varphi \in L^\infty(\mathbb{T})$. Then the Toeplitz operator T_φ is compact if and only if $\varphi \equiv 0$.*

Proof. Clearly $T_0 = 0$ is compact. Now suppose that K is any compact operator on $H^2(\mathbb{T})$. By the triangle inequality,

$$\begin{aligned} \|M_\varphi f\|_2 &\leq \|\varphi f - \chi_{-m} T_\varphi(P_+(\chi_m f))\|_2 \\ &\quad + \|T_\varphi(P_+(\chi_m f)) - K(P_+(\chi_m f))\|_2 \\ &\quad + \|K(P_+(\chi_m f))\|_2 \end{aligned}$$

for each $f \in L^2(\mathbb{T})$ and $m \in \mathbb{Z}$. By Lemma 3.3,

$$\lim_{m \rightarrow +\infty} \|\chi_{-m} T_\varphi(P_+(\chi_m f)) - \varphi f\|_2 = 0.$$

For the second term, we have

$$\|T_\varphi(P_+(\chi_m f)) - K(P_+(\chi_m f))\|_2 \leq \|T_\varphi - K\| \|P_+(\chi_m f)\| \leq \|T_\varphi - K\| \|f\|.$$

Since, by (3.5), $\chi_m f$ weakly converges to 0 and K is compact, then

$$\lim_{m \rightarrow +\infty} \|K(P_+(\chi_m f))\|_2 = 0.$$

Therefore, we obtain the estimation

$$\|M_\varphi f\|_2 \leq \|T_\varphi - K\| \|f\|, \quad (f \in L^2(\mathbb{T})).$$

By Theorem 3.1, we conclude that

$$\|\varphi\|_\infty \leq \|T_\varphi - K\|$$

for any compact operator K on $H^2(\mathbb{T})$. Hence if T_φ is compact we must have $\varphi \equiv 0$. □

Corollary 3.11. *Let $\varphi \in L^\infty(\mathbb{T})$, and suppose that $\varphi \not\equiv 0$. Then the linear manifold $\mathcal{R}(T_\varphi)$ of $H^2(\mathbb{T})$ is infinite dimensional.*

3.7 The operator K_φ

The notion of K_ν , $\nu \in \mathcal{M}(\mathbb{T})$, as a mapping on $L^1(\nu)$, was introduced in Section 2.6. In this section, we will be interested by absolutely continuous measures (with respect to the Lebesgue measure), $d\nu(\zeta) = \varphi(\zeta) dm(\zeta)$, $\varphi \in L^1(\mathbb{T})$. In this case, we also write K_ϕ for K_ν , $K\phi$ for $K\nu$ and $L^1(\phi)$ for $L^1(\nu)$. So for $\varphi \in L^1(\mathbb{T})$ and $f \in L^1(\phi)$, we have

$$(K_\varphi f)(z) = \int_{\mathbb{T}} \frac{f(\zeta)\varphi(\zeta)}{1 - z\bar{\zeta}} dm(\zeta),$$

and K_φ is a continuous operator from $L^1(\varphi)$ into $\text{Hol}(\mathbb{C} \setminus \mathbb{T})$.

In this section, we see that under the slightly stronger condition $\varphi \in L^\infty(\mathbb{T})$, K_φ is in fact a bounded operator from $L^2(\varphi)$ into H^2 . We also find its adjoint and explore the connection between K_φ and Toeplitz operator T_φ .

If $\varphi \in L^2(\mathbb{T})$, the first formula in Theorem 2.17 is written as

$$K\varphi(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n) z^n, \quad (z \in \mathbb{D}), \quad (3.6)$$

and thus, by Parseval's identity,

$$\frac{1}{2\pi} \int_0^{2\pi} |K\varphi(re^{it})|^2 dt = \sum_{n=0}^{\infty} |\hat{\varphi}(n)|^2 r^{2n}, \quad (0 < r < 1).$$

Hence

$$K\varphi \in H^2(\mathbb{D}) \quad (3.7)$$

and

$$\|K\varphi\|_2 \leq \|\varphi\|_2. \quad (3.8)$$

As usual we also use $K\varphi$ to denote the corresponding boundary value function on \mathbb{T} which is an element of $H^2(\mathbb{T})$. With this convention, (3.6) shows that we have

$$K\varphi = P_+\varphi, \quad (3.9)$$

where P_+ is the Riesz projection.

If $\varphi \in L^\infty(\mathbb{T})$ and $\psi \in L^2(\varphi)$, then $\varphi\psi \in L^2(\mathbb{T})$ and also

$$K_\varphi\psi = K(\varphi\psi) = P_+(\varphi\psi). \quad (3.10)$$

Hence, by (3.7) and (3.9), $K_\varphi\psi \in H^2$. Moreover, by (3.8),

$$\|K_\varphi\psi\|_2 = \|K(\varphi\psi)\|_2 \leq \|\varphi\psi\|_2 \leq \|\varphi\|_\infty^{1/2} \|\psi\|_{L^2(\varphi)}.$$

Therefore,

$$\begin{array}{ccc} K_\varphi : & L^2(\varphi) & \longrightarrow & H^2 \\ & \psi & \longmapsto & K(\varphi\psi) \end{array} \quad (3.11)$$

is a well defined operator whose norm is at most $\|\varphi\|_\infty^{1/2}$. We clearly have

$$H^2(\mathbb{T}) \subset L^2(\mathbb{T}) \subset L^2(\varphi).$$

Thus the injection mapping $i : H^2 \hookrightarrow L^2(\varphi)$ is well defined. For further reference, we denote this operator by J_φ , i.e.

$$\begin{array}{ccc} J_\varphi : & H^2 & \longrightarrow & L^2(\varphi) \\ & f & \longmapsto & f. \end{array} \quad (3.12)$$

Theorem 3.12. *Let $\varphi \in L^\infty(\mathbb{T})$. Then*

$$K_\varphi^* = J_\varphi.$$

Proof. For each $\psi \in L^2(\varphi)$ and $f \in H^2$, by (3.9) and (3.10),

$$\langle K_\varphi\psi, f \rangle_{H^2} = \langle K_{\varphi\psi}, f \rangle_{H^2} = \langle P_+(\varphi\psi), f \rangle_{H^2}.$$

Hence, by Lemma 3.2,

$$\langle K_\varphi\psi, f \rangle_{H^2} = \langle \varphi\psi, f \rangle_{L^2}.$$

But, directly from the definition,

$$\langle \varphi\psi, f \rangle_{L^2} = \langle \varphi, f \rangle_{L^2(\varphi)}.$$

Hence, we can write

$$\langle K_\varphi\psi, f \rangle_{H^2} = \langle \varphi, J_\varphi f \rangle_{L^2(\varphi)}.$$

This identity shows that $K_\varphi^* = J_\varphi$. □

Corollary 3.13. *Let $\varphi \in L^\infty(\mathbb{T})$. Then*

$$K_\varphi J_\varphi = T_\varphi.$$

Proof. Let $f, g \in H^2$. Then, by Theorem 3.12,

$$\langle K_\varphi J_\varphi f, g \rangle_{H^2} = \langle J_\varphi f, J_\varphi g \rangle_{L^2(\varphi)}.$$

But, right from the definition,

$$\langle J_\varphi f, J_\varphi g \rangle_{L^2(\varphi)} = \langle \varphi f, g \rangle_{L^2}.$$

By Lemma 3.2, we have

$$\langle \varphi f, g \rangle_{L^2} = \langle P_+(\varphi f), g \rangle_{H^2}.$$

Therefore, for each $f, g \in H^2$,

$$\langle K_\varphi J_\varphi f, g \rangle_{H^2} = \langle T_\varphi f, g \rangle_{H^2}.$$

Hence $K_\varphi J_\varphi = T_\varphi$. □

3.8 Toeplitz operators on generalized Hardy spaces $H^2(\nu)$

Recall that for $\phi \in L^\infty$, the Toeplitz operator T_ϕ , with symbol ϕ , is defined on H^2 by the formula $T_\phi f = P_+(\phi f)$, where P_+ is the Riesz projection. We could try to generalize Toeplitz operators to generalized Hardy spaces $H^2(\nu)$, $\nu \in \mathcal{M}(\mathbb{T})$, by using the same formula as in the classical case. In general, however, P_+ is not bounded from $L^2(\nu)$ onto $H^2(\nu)$ so this is not possible. More precisely, the Helson–Szegő Theorem gives the following criterion:

Theorem 3.14 (Helson–Szegő, see []). *Let $\nu \in \mathcal{M}(\mathbb{T})$. The following assertions are equivalent:*

- (i) *The Riesz projection P_+ is bounded on $L^2(\nu)$.*
- (ii) *ν is absolutely continuous with respect to the Lebesgue measure and $d\nu = |h|^2 dm$, where $h \in H^2$ is an outer function such that $\text{dist}(h/\bar{h}, H^\infty) < 1$.*
- (iii) *ν is absolutely continuous with respect to the Lebesgue measure and $d\nu = \omega dm$, where $\omega = e^{u+\bar{v}}$, and u, v are real bounded functions, $\|v\|_\infty < \pi/2$ and \bar{v} is the Hilbert transform of v .*
- (iv) *The family $(z^n)_{n \in \mathbb{Z}}$ is a basis of $L^2(\nu)$.*

A different possibility to generalize Toeplitz operators to $H^2(\nu)$ stems from the observation that when m is in H^∞ , then $T_{\bar{m}}$ maps analytic polynomials to analytic polynomials. Hence we regard $T_{\bar{m}}$ as a densely defined operator on $H^2(\nu)$ and ask the question: when is $T_{\bar{m}}$ bounded on $H^2(\nu)$?

Lemma 3.15. *Let $\varphi \in H^\infty$ and $\omega \in \mathbb{D}$. Then*

$$\overline{\varphi(\omega)} k_\omega = \sum_{n \geq 0} \bar{\omega}^n T_{\bar{\varphi}} \chi_n,$$

where the series converges in $H^2(\nu)$ and $\chi_n(z) = z^n$, $n \geq 0$ and $z \in \mathbb{D}$.

Proof. It is easy to see that

$$(T_{\bar{\varphi}} \chi_n)(z) = \sum_{j=0}^n \overline{\varphi(n-j)} z^j$$

and we get

$$|\bar{\omega}^n T_{\bar{\varphi}} \chi_n(z)| \leq \sum_{j=0}^n |\hat{\varphi}(n-j)| |z|^j |\omega|^n \leq n \|\varphi\|_{\infty} r^n,$$

for all $z \in \mathbb{T}$ and $|\omega| \leq r < 1$. Since $r < 1$, the series $\sum_n n r^n$ is convergent and we obtain that the series

$$\sum_{n \geq 0} \bar{\omega}^n T_{\bar{\varphi}} \chi_n(z)$$

is absolutely and uniformly convergent for $z \in \mathbb{T}$ and $|\omega| \leq r < 1$. Now using Fubini's Theorem, we can write

$$\begin{aligned} \sum_{n \geq 0} \bar{\omega}^n T_{\bar{\varphi}} \chi_n(z) &= \sum_{n \geq 0} \sum_{j=0}^n \overline{\hat{\varphi}(n-j)} \bar{\omega}^n z^j \\ &= \sum_{j \geq 0} \left(\sum_{n \geq j} \overline{\hat{\varphi}(n-j)} \bar{\omega}^n \right) z^j \\ &= \sum_{j \geq 0} \bar{\omega}^j z^j \left(\sum_{n \geq j} \overline{\hat{\varphi}(n-j)} \bar{\omega}^{n-j} \right) \\ &= \sum_{j \geq 0} \bar{\omega}^j z^j \left(\sum_{n \geq 0} \overline{\hat{\varphi}(n)} \bar{\omega}^n \right) \\ &= \sum_{j \geq 0} \bar{\omega}^j z^j \overline{\varphi(\omega)} \\ &= \frac{\overline{\varphi(\omega)}}{1 - \bar{\omega}z} \\ &= \overline{\varphi(\omega)} k_{\omega}(z). \end{aligned}$$

So we get that the series

$$\sum_{n \geq 0} \bar{\omega}^n T_{\bar{\varphi}} \chi_n(z)$$

converges to $\overline{\varphi(\omega)} k_{\omega}(z)$ uniformly for $z \in \mathbb{T}$ and $|\omega| \leq r < 1$. Therefore the convergence is also in $H^2(\nu)$ for each fixed $\omega \in \mathbb{D}$. □

Theorem 3.16. *Suppose that m belongs to H^{∞} . The operator $T_{\bar{m}}$ is bounded on $H^2(\nu)$ if and only if the map*

$$k_w \mapsto \overline{m(w)} k_w$$

extends to a continuous linear operator on $H^2(\nu)$.

Note that this theorem is not a tautology because we define a priori the Toeplitz operator $T_{\bar{m}}$ on the set of analytic polynomials and not on the dense set of k_w , $w \in \mathbb{D}$.

Proof. Suppose first that $T_{\bar{m}}$ is bounded on $H^2(\nu)$ and fix $w \in \mathbb{D}$. Since

$$k_w(z) = \frac{1}{1 - \bar{w}z} = \sum_{n=0}^{+\infty} \bar{w}^n z^n$$

converges uniformly on \mathbb{T} , the series converges also in $H^2(\nu)$. Hence

$$T_{\bar{m}}k_w = \sum_{n=0}^{+\infty} \bar{w}^n T_{\bar{m}}z^n$$

with the series converging in $H^2(\nu)$. By Lemma 3.15, we know that this series converges to $\overline{m(w)}k_w$ in $H^2(\nu)$, for each fixed $w \in \mathbb{D}$. Hence $T_{\bar{m}}k_w = \overline{m(w)}k_w$ and using Theorem 2.13, we conclude that $T_{\bar{m}}$ is itself the continuous linear extension to $H^2(\nu)$ of the map $k_w \mapsto \overline{m(w)}k_w$.

Now suppose conversely that the map $k_w \mapsto \overline{m(w)}k_w$ extends to a continuous linear operator T on $H^2(\nu)$. Since $k_w(z) = \sum_{n=0}^{+\infty} \bar{w}^n z^n$ in $H^2(\nu)$, we have

$$Tk_w = \sum_{n=0}^{+\infty} \bar{w}^n T\chi_n.$$

But $Tk_w = \overline{m(w)}k_w$ and by Lemma 3.15, we also have

$$\overline{m(w)}k_w = \sum_{n=0}^{+\infty} \bar{w}^n T_{\bar{m}}\chi_n.$$

Therefore we get

$$\sum_{n=0}^{+\infty} \bar{w}^n T_{\bar{m}}\chi_n = \sum_{n=0}^{+\infty} \bar{w}^n T\chi_n.$$

Remark now that these two power series in \bar{w} represent two conjugate analytic functions in \mathbb{D} and thus their coefficients must be the same. So

$$T\chi_n = T_{\bar{m}}\chi_n, \quad n \geq 0,$$

which proves that T is the continuous linear extension to $H^2(\nu)$ of $T_{\bar{m}}$. □

Chapter 4

The spaces $\mathcal{M}(A)$ and $\mathcal{H}(A)$

4.1 The space $\mathcal{M}(A)$

Suppose that H_1 is a Hilbert space, H_2 a set and $A : H_1 \rightarrow H_2$ a set bijection between H_1 and H_2 . Then the map A can be used to transfer the Hilbert space structure of H_1 to H_2 . It is enough to define

$$\langle Ax, Ay \rangle_{H_2} = \langle x, y \rangle_{H_1} \quad (4.1)$$

for each $x, y \in H_1$. The algebraic operations on H_2 are defined similarly. If H_2 is a linear space and A is an algebraic isomorphism between H_1 and H_2 the latter requirement is already fulfilled.

The notation $\mathcal{R}(A)$ was used to denote the range of an operator. This notation remains valid whenever we look at $\mathcal{R}(A)$ as a set. In the following we are going to define a Hilbert structure on $\mathcal{R}(A)$ which is imposed by A . Hence we will use $\mathcal{M}(A)$ to denote the range of A endowed with that structure. Therefore, the equality $\mathcal{R}(A) = \mathcal{R}(B)$ means that the ranges of A and B as linear manifolds are equal, while $\mathcal{M}(A) = \mathcal{M}(B)$ says that not only $\mathcal{R}(A) = \mathcal{R}(B)$ holds, but also they have the same Hilbert space structure, i.e.

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(B)}$$

for all possible w .

Suppose that H_1 and H_2 are Hilbert spaces and that $A \in \mathcal{L}(H_1, H_2)$. By the first homomorphism theorem, the operator A induces an isomorphism between the quotient space $H_1/\ker A$ and $\mathcal{M}(A)$. Hence, by (4.1), the identity

$$\langle Ax, Ay \rangle_{\mathcal{M}(A)} = \langle x + \ker A, y + \ker A \rangle_{H_1/\ker A}, \quad (x, y \in H_1), \quad (4.2)$$

defines a Hilbert space structure on $\mathcal{R}(A)$. The norm of $x + \ker A$ in $H_1/\ker A$ is originally defined by

$$\|x + \ker A\|_{H_1/\ker A} = \inf_{z \in \ker A} \|x + z\|_{H_1}.$$

But, we easily see that

$$\|x + \ker A\|_{H_1/\ker A} = \|P_{(\ker A)^\perp} x\|_{H_1}, \quad (x \in H_1).$$

Hence, by the polarization identity, we have

$$\langle x + \ker A, y + \ker A \rangle_{H_1/\ker A} = \langle P_{(\ker A)^\perp} x, P_{(\ker A)^\perp} y \rangle_{H_1}, \quad (x, y \in H_1).$$

Moreover, it is also easy to verify that

$$\langle P_{(\ker A)^\perp} x, P_{(\ker A)^\perp} y \rangle_{H_1} = \langle x, P_{(\ker A)^\perp} y \rangle_{H_1} = \langle P_{(\ker A)^\perp} x, y \rangle_{H_1}.$$

Therefore, the definition (4.2) reduces to

$$\begin{aligned} \langle Ax, Ay \rangle_{\mathcal{M}(A)} &= \langle P_{(\ker A)^\perp} x, P_{(\ker A)^\perp} y \rangle_{H_1} \\ &= \langle x, P_{(\ker A)^\perp} y \rangle_{H_1} \\ &= \langle P_{(\ker A)^\perp} x, y \rangle_{H_1} \end{aligned} \quad (4.3)$$

for each $x, y \in H_1$. In particular, for each $x \in H_1$,

$$\|Ax\|_{\mathcal{M}(A)} = \|P_{(\ker A)^\perp} x\|_{H_1}. \quad (4.4)$$

Moreover, if at least one of x or y is orthogonal to $\ker A$, then, by (4.3),

$$\langle Ax, Ay \rangle_{\mathcal{M}(A)} = \langle x, y \rangle_{H_1}. \quad (4.5)$$

The rather trivial inequality

$$\|Ax\|_{\mathcal{M}(A)} \leq \|x\|_{H_1}, \quad (x \in H_1), \quad (4.6)$$

will also be frequently used.

On $\mathcal{M}(A)$ we now have two structures. The old structure inherited from H_2 and the new one imposed by A . In the following, we always assume that $\mathcal{M}(A)$ is endowed with the latter structure. If this is not the case, we will explicitly mention that. To explore the relation between these two structure, note that A is a bounded operator, and thus

$$\|Ax\|_{H_2} = \|AP_{(\ker A)^\perp} x\|_{H_2} \leq \|A\| \|P_{(\ker A)^\perp} x\|_{H_1}, \quad (x \in H_1).$$

Therefore, by (4.4),

$$\|Ax\|_{H_2} \leq \|A\| \|Ax\|_{\mathcal{M}(A)}, \quad (x \in H_1). \quad (4.7)$$

This inequality means that the inclusion map

$$\begin{array}{ccc} i : \mathcal{M}(A) & \hookrightarrow & H_2 \\ & w & \longmapsto w \end{array}$$

is continuous. That is why we say that $\mathcal{M}(A)$ is boundedly contained in H_2 . If A is a contraction, i.e. $\|A\| \leq 1$, then we say that $\mathcal{M}(A)$ is contractively contained in H_2 . If

$$\|w\|_{\mathcal{M}(A)} = \|w\|_{H_2}, \quad (w \in \mathcal{M}(A)),$$

we naturally say that $\mathcal{M}(A)$ is isometrically contained in H_2 .

Let M be a linear manifold in a Hilbert space H . We do not assume that M is closed in H . Suppose that M equipped with an inner product, not necessarily the same inner product as in H , is a Hilbert space. Then we say that M is boundedly contained in H if the inclusion map

$$\begin{array}{ccc} i: M & \hookrightarrow & H \\ w & \longmapsto & w \end{array}$$

is continuous. If i is a contraction, we say that M is contractively contained in H , and finally if i is an isometry we say that M is isometrically contained in H . Note that $M = \mathcal{M}(i)$ and the definitions given in this paragraph for M are consistent with those in the preceding one for $\mathcal{M}(i)$.

Our first result shows that if $A \in \mathcal{L}(H)$ is an orthogonal projection, then in fact we do not obtain a new structure on $\mathcal{M}(A)$. The Hilbert space structure of $\mathcal{M}(A)$ is exactly the one it has in the first place a closed subspace of H .

Lemma 4.1. *Let M be a closed subspace of H , and let P_M denote the orthogonal projection on M . Then $\mathcal{M}(P_M) = M$ and*

$$\|w\|_{\mathcal{M}(P_M)} = \|w\|_H, \quad (w \in M).$$

Proof. The set identity $\mathcal{M}(P_M) = M$ is an immediate consequence of the definition of an orthogonal projection. Since M is closed, $(M^\perp)^\perp = M$. Also remember that $\ker P_M = M^\perp$. Hence, by (4.4),

$$\|P_M x\|_{\mathcal{M}(P_M)} = \|P_{(\ker P_M)^\perp} x\|_H = \|P_M x\|_H, \quad (x \in H_1).$$

□

Exercises

Exercise 4.1.1. Let H be a set endowed with two inner products whose corresponding norms are complete and equivalent, i.e.

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1, \quad (x \in H),$$

where c, C are positive constants. Show that $(H, \langle \cdot, \cdot \rangle_1)$ is boundedly contained in $(H, \langle \cdot, \cdot \rangle_2)$, and vice versa.

Exercise 4.1.2. Let (X, \mathcal{A}) be a measurable space, and let μ and ν be two positive measures on the σ -algebra \mathcal{A} . Suppose that

$$\mu(E) \leq \nu(E) \tag{4.8}$$

for all $E \in \mathcal{A}$. Show that $L^2(\nu)$ is contractively contained in $L^2(\mu)$.

Hint: The assumption (4.8) can be rewritten as

$$\int_X \chi_E d\mu \leq \int_X \chi_E d\nu,$$

where χ_E is the characteristic function of E . Take positive linear combinations, and then apply the monotone convergence theorem to obtain

$$\int_X \varphi d\mu \leq \int_X \varphi d\nu$$

for all positive measurable functions φ . Hence $\|f\|_{L^2(\mu)} \leq \|f\|_{L^2(\nu)}$.

Exercise 4.1.3. Let $\varphi \in L^\infty(\mathbb{T})$, and consider the multiplication operator

$$M_\varphi : \begin{array}{ccc} L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ f & \longmapsto & \varphi f \end{array}$$

which was studied in Section 3.1. Show that

$$\|\varphi f\|_{\mathcal{M}(M_\varphi)} = \left(\frac{1}{2\pi} \int_{\mathbb{T} \setminus E} |f(e^{it})|^2 dt \right)^{\frac{1}{2}}, \quad (f \in L^2(\mathbb{T})),$$

and that

$$\langle \varphi f, \varphi g \rangle_{\mathcal{M}(M_\varphi)} = \frac{1}{2\pi} \int_{\mathbb{T} \setminus E} f(e^{it}) \overline{g(e^{it})} dt, \quad (f, g \in L^2(\mathbb{T})),$$

where $E = \{\zeta \in \mathbb{T} : \varphi(\zeta) = 0\}$.

The first identity implies that $\mathcal{M}(M_\varphi) = \varphi L^2(\mathbb{T})$ is contractively contained in $L^2(\mathbb{T})$. Under what condition $\mathcal{M}(M_\varphi)$ is isometrically contained in $L^2(\mathbb{T})$?

Exercise 4.1.4. Let Θ be an inner function for the open unit disc, and let

$$M_\Theta : \begin{array}{ccc} H^2(\mathbb{D}) & \longrightarrow & H^2(\mathbb{D}) \\ F & \longmapsto & \Theta F. \end{array}$$

Show that

$$\|\Theta F\|_{\mathcal{M}(M_\Theta)} = \|F\|_{H^2(\mathbb{D})} = \|\Theta F\|_{H^2(\mathbb{D})}, \quad (F \in H^2(\mathbb{D})).$$

Thus $\mathcal{M}(M_\Theta) = \Theta H^2$ is isometrically contained in $H^2(\mathbb{D})$.

Hint: M_Θ is injective and $|\Theta| = 1$ almost everywhere on \mathbb{T} .

Remark: The subspaces $\Theta H^2(\mathbb{D})$ are called Beurling subspaces of $H^2(\mathbb{D})$. According to his celebrated theorem, these are the only closed subspaces of $H^2(\mathbb{D})$ which are invariant under the forward shift operator $S = M_z$, i.e.

$$S : \begin{array}{ccc} H^2(\mathbb{D}) & \longrightarrow & H^2(\mathbb{D}) \\ F & \longmapsto & zF. \end{array}$$

4.2 Contractive inclusion of $\mathcal{M}(A)$ in $\mathcal{M}(B)$

Let $A \in \mathcal{L}(H_1, H)$ and $B \in \mathcal{L}(H_2, H)$. Suppose that there is a contraction $C \in \mathcal{L}(H_1, H_2)$ such that $A = BC$. In the first place, since for each $x \in H_1$, $Ax = B(Cx)$, we have the set inclusion

$$\mathcal{R}(A) \subset \mathcal{R}(B).$$

Secondly, by (4.6) and that $\|C\| \leq 1$,

$$\|Ax\|_{\mathcal{M}(B)} = \|BCx\|_{\mathcal{M}(B)} \leq \|Cx\|_{H_2} \leq \|x\|_{H_1}.$$

By (4.4), replacing x by $P_{(\ker A)^\perp}x$ gives us

$$\|Ax\|_{\mathcal{M}(B)} \leq \|Ax\|_{\mathcal{M}(A)}, \quad (x \in H_1).$$

Hence, $\mathcal{M}(A)$ is in fact contractively contained in $\mathcal{M}(B)$. We use Douglas' factorization theorem (Theorem 1.1) to show that the existence of C is also necessary for the contractive inclusion of $\mathcal{M}(A)$ in $\mathcal{M}(B)$.

Theorem 4.2. *Let $A \in \mathcal{L}(H_1, H)$ and $B \in \mathcal{L}(H_2, H)$. Then $\mathcal{R}(A) \subset \mathcal{R}(B)$ and the inclusion*

$$\begin{array}{ccc} i : \mathcal{M}(A) & \hookrightarrow & \mathcal{M}(B) \\ & w & \longmapsto w \end{array}$$

is a contraction if and only if $AA^ \leq BB^*$.*

Proof. If $AA^* \leq BB^*$, then, by Theorem 1.1, there is a contraction $C \in \mathcal{L}(H_1, H_2)$ such that $A = BC$. Hence, as we discussed above, $\mathcal{M}(A)$ is contractively contained in $\mathcal{M}(B)$.

To prove the other direction, suppose that the set inclusion $\mathcal{R}(A) \subset \mathcal{R}(B)$ holds and moreover $\|w\|_{\mathcal{M}(B)} \leq \|w\|_{\mathcal{M}(A)}$ for all $w \in \mathcal{M}(A)$. An element $w \in \mathcal{M}(A)$ is of the form $w = Ax$ with some $x \in H_1$. Hence for each $x \in H_1$ there is $y \in H_2$ such that

$$Ax = By. \quad (4.9)$$

The element y is not necessarily unique. However, if $By = By'$, with $y, y' \in H_2$, then $B(y - y') = 0$ and thus $y - y' \in \ker B$. In other words, $P_{(\ker B)^\perp}y = P_{(\ker B)^\perp}y'$. Therefore, the map

$$\begin{array}{ccc} C : H_1 & \longrightarrow & H_2 \\ & x & \longmapsto P_{(\ker B)^\perp}y, \end{array}$$

where $y \in H_2$ is given by (4.9), is well defined and

$$BCx = BP_{(\ker B)^\perp}y = By = Ax, \quad (x \in H_1).$$

In short, $A = BC$. Moreover, by (4.4) and (4.6),

$$\begin{aligned} \|Cx\|_{H_2} &= \|P_{(\ker B)^\perp}y\|_{H_2} \\ &= \|By\|_{\mathcal{M}(B)} \\ &= \|Ax\|_{\mathcal{M}(B)} \\ &\leq \|Ax\|_{\mathcal{M}(A)} \\ &\leq \|x\|_{H_1}. \end{aligned}$$

Hence C is a contraction. Thus, by Theorem 1.1, $AA^* \leq BB^*$. □

We mention two corollaries. The first one follows immediately from the Theorem 4.2. We remind that $\mathcal{M}(A) = \mathcal{M}(B)$ means $\mathcal{R}(A) = \mathcal{R}(B)$ and they have the same Hilbert space structure.

Corollary 4.3. *Let $A \in \mathcal{L}(H_1, H)$ and $B \in \mathcal{L}(H_2, H)$. Then*

- (i) $\mathcal{M}(A) = \mathcal{M}(B)$ if and only if $AA^* = BB^*$.
- (ii) $\mathcal{M}(A) = \mathcal{M}(|A|)$, where $|A| = (AA^*)^{1/2}$.

Corollary 4.4. *Let $A \in \mathcal{L}(H_1, H)$. Then $\mathcal{M}(A)$ is a closed subspace of H and $\|w\|_{\mathcal{M}(A)} = \|w\|_H$ for each $w \in \mathcal{M}(A)$ if and only if A is a partial isometry. In this case, we have the set identity*

$$\mathcal{M}(A) = \mathcal{R}(AA^*).$$

Proof. If A is a partial isometry then, by Theorem 1.7, $P = AA^*$ is an orthogonal projection and thus $|A| = P$. Hence, by Corollary 4.3(ii), $\mathcal{M}(A) = \mathcal{M}(P)$ and $\|w\|_{\mathcal{M}(A)} = \|w\|_{\mathcal{M}(P)}$ for each $w \in \mathcal{M}(A)$. But, by Lemma 4.1, $\mathcal{M}(P)$ is a closed subspace of H and $\|w\|_{\mathcal{M}(P)} = \|w\|_H$ for each $w \in \mathcal{M}(P)$.

Now suppose that $M = \mathcal{M}(A)$ is a closed subspace of H . Let $P_M \in \mathcal{L}(H)$ denote the orthogonal projection on M . The set identity $\mathcal{M}(A) = \mathcal{M}(P_M)$ is trivial. Then, by Lemma 4.1 and our assumptions, we have $\|w\|_{\mathcal{M}(A)} = \|w\|_H = \|w\|_{\mathcal{M}(P_M)}$ for each $w \in \mathcal{M}(A)$. Hence, by Corollary 4.3(i),

$$AA^* = P_M P_M^* = P_M.$$

Therefore, again by Theorem 1.7, A is a partial isometry. □

Exercises

Exercise 4.2.1. Let $A \in \mathcal{L}(H_1, H)$ and $B \in \mathcal{L}(H_2, H)$. Show that $\mathcal{M}(|A|) = \mathcal{M}(|B|)$ if and only if $|A| = |B|$.

Hint: Use Corollary 4.3(i) and Theorem 1.4.

Exercise 4.2.2.

Hint:

Exercise 4.2.3.

Hint:

4.3 Linear functionals on $\mathcal{M}(A)$

Let $A \in \mathcal{L}(H_1, H_2)$. Suppose that

$$\Lambda : H_2 \longrightarrow \mathbb{C}$$

is a bounded linear functional on H_2 . Then, by Riesz' theorem, there is a unique $w \in H_2$ such that

$$\Lambda z = \langle z, w \rangle_{H_2}, \quad (z \in H_2).$$

According to (4.7), the inclusion map

$$i : \mathcal{M}(A) \hookrightarrow H_2$$

is continuous. Hence

$$\Lambda \circ i : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$. Thus, again by Riesz' theorem, there is a unique $w' \in \mathcal{M}(A)$ such that

$$(\Lambda \circ i)(z) = \langle z, w' \rangle_{\mathcal{M}(A)}, \quad (z \in \mathcal{M}(A)).$$

We want to find the relation between w and w' . Note that $\Lambda \circ i$ is the restriction of Λ to $\mathcal{M}(A)$ and thus, abusing the notation, we may also use Λ instead of $\Lambda \circ i$.

Theorem 4.5. *Let $A \in \mathcal{L}(H_1, H_2)$. Let $w \in H_2$, and let*

$$\Lambda z = \langle z, w \rangle_{H_2}, \quad (z \in H_2),$$

be the corresponding bounded linear functional on H_2 . Then its restriction

$$\Lambda : \mathcal{M}(A) \longrightarrow \mathbb{C}$$

is a bounded linear functional on $\mathcal{M}(A)$ and

$$\Lambda(Ax) = \langle Ax, AA^*w \rangle_{\mathcal{M}(A)}, \quad (x \in H_1).$$

Moreover,

$$\|\Lambda\|_{\mathcal{M}(A)^*} = \|A^*w\|_{H_1}.$$

Remark: By Riesz' theorem

$$\|\Lambda\|_{H_2^*} = \|w\|_{H_2}.$$

Proof. By the definition of adjoint operator,

$$\Lambda(Ax) = \langle Ax, w \rangle_{H_2} = \langle x, A^*w \rangle_{H_1}, \quad (x \in H_1).$$

But, by Theorem ??(vii),

$$A^*w \in \mathcal{R}(A^*) \subset (\ker A)^\perp.$$

Hence, by (4.5),

$$\langle x, A^*w \rangle_{H_1} = \langle Ax, AA^*w \rangle_{\mathcal{M}(A)}, \quad (x \in H_1).$$

Therefore, $\Lambda(Ax) = \langle Ax, AA^*w \rangle_{\mathcal{M}(A)}$. This representation shows that

$$\|\Lambda\|_{\mathcal{M}(A)^*} = \|AA^*w\|_{\mathcal{M}(A)}.$$

However, by (4.4), we have

$$\|AA^*w\|_{\mathcal{M}(A)} = \|A^*w\|_{H_1}.$$

□

Exercises

Exercise 4.3.1.

Hint:

Exercise 4.3.2.

Hint:

Exercise 4.3.3.

Hint:

4.4 The complementary space $\mathcal{H}(A)$

If A is a Hilbert space contraction, then

$$\mathcal{H}(A) = \mathcal{M}((I - AA^*)^{1/2})$$

is called the complementary space of $\mathcal{M}(A)$. The intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ is called the overlapping space. In the rest of this chapter we study $\mathcal{H}(A)$ and its relation to $\mathcal{M}(A)$.

Lemma 4.6. *Let $A \in \mathcal{L}(H_1, H)$ be a contraction. Then $\mathcal{H}(A)$ is a closed subspace of H and $\|w\|_{\mathcal{H}(A)} = \|w\|_H$ for each $w \in \mathcal{H}(A)$ if and only if A is a partial isometry. In this case,*

$$\mathcal{H}(A) = \mathcal{R}(I - AA^*).$$

Proof. By Corollary 4.4, $\mathcal{H}(A)$ is a closed subspace of H and $\|w\|_{\mathcal{M}(A)} = \|w\|_H$ for each $w \in \mathcal{H}(A)$ if and only if $(I - AA^*)^{1/2}$ is a partial isometry. By Theorem 1.7, this happens if and only if $I - AA^*$ is an orthogonal projection. Clearly $I - AA^*$ is an orthogonal projection if and only if AA^* is an orthogonal projection. Finally, again by Theorem 1.7, AA^* is an orthogonal projection if and only if A is a partial isometry. □

Lemma 4.7. *Let $A \in \mathcal{L}(H_1, H_2)$. Then, with respect to the Hilbert space structure of $\mathcal{M}(A)$, the linear manifold $\mathcal{M}(AA^*)$ is dense in $\mathcal{M}(A)$.*

Proof. In the first place, note that

$$\mathcal{M}(AA^*) \subset \mathcal{M}(A) \subset H_2$$

and thus $\mathcal{M}(AA^*)$ is indeed a linear manifold of $\mathcal{M}(A)$.

To show that $\mathcal{M}(AA^*)$ is dense $\mathcal{M}(A)$ we use the standard Hilbert space technic. If 0 is the only vector in $\mathcal{M}(A)$ which is orthogonal to $\mathcal{M}(AA^*)$, then this linear manifold is dense in $\mathcal{M}(A)$. Thus let $w \in \mathcal{M}(A)$ be such that

$$\langle w, z \rangle_{\mathcal{M}(A)} = 0$$

for all $z \in \mathcal{M}(AA^*)$. We need to show that $w = 0$. By definition, $w = Ax$, for some $x \in H_1$, and $z = AA^*y$, where y runs through H_2 . Without loss of generality, assume that $x \perp \ker A$. Hence, by (4.5),

$$\begin{aligned} 0 &= \langle w, z \rangle_{\mathcal{M}(A)} \\ &= \langle Ax, AA^*y \rangle_{\mathcal{M}(A)} \\ &= \langle x, A^*y \rangle_{H_1} \\ &= \langle Ax, y \rangle_{H_2} \\ &= \langle w, y \rangle_{H_2} \end{aligned}$$

for all $y \in H_2$. Therefore, $w = 0$. □

Corollary 4.8. *Let $A \in \mathcal{L}(H_1, H_2)$ be a Hilbert space contraction. Then, with respect to the Hilbert space structure of $\mathcal{H}(A)$, the linear manifold $\mathcal{M}(I - AA^*)$ is dense in $\mathcal{H}(A)$. Moreover, for each $z \in H_2$ and $w \in \mathcal{H}(A)$,*

$$\|(I - AA^*)z\|_{\mathcal{H}(A)}^2 = \|(I - AA^*)^{1/2}z\|_{H_2}^2 = \|z\|_{H_2}^2 - \|A^*z\|_{H_1}^2$$

and

$$\langle w, (I - AA^*)z \rangle_{\mathcal{H}(A)} = \langle w, z \rangle_{H_2}.$$

Proof. It is enough to consider the self adjoint operator $(I - AA^*)^{1/2} \in \mathcal{L}(H_2)$ and apply Lemma 4.7 to see that $\mathcal{M}(I - AA^*)$ is dense in $\mathcal{H}(A)$.

To prove the first relation, note that $(I - AA^*)^{1/2}z \perp \ker(I - AA^*)^{1/2}$. Thus

$$\begin{aligned} \|(I - AA^*)z\|_{\mathcal{H}(A)}^2 &= \|(I - AA^*)^{1/2}z\|_{H_2}^2 \\ &= \langle (I - AA^*)^{1/2}z, (I - AA^*)^{1/2}z \rangle_{H_2} \\ &= \langle (I - AA^*)z, z \rangle_{H_2} \\ &= \|z\|_{H_2}^2 - \|A^*z\|_{H_1}^2. \end{aligned}$$

For the second, we write $w = (I - AA^*)^{1/2}w'$, where $w' \perp \ker(I - AA^*)^{1/2}$. Hence, by (4.5),

$$\begin{aligned} \langle w, (I - AA^*)z \rangle_{\mathcal{H}(A)} &= \langle (I - AA^*)^{1/2}w', (I - AA^*)z \rangle_{\mathcal{H}(A)} \\ &= \langle w', (I - AA^*)^{1/2}z \rangle_{H_2} \\ &= \langle (I - AA^*)^{1/2}w', z \rangle_{H_2} \\ &= \langle w, z \rangle_{H_2} \end{aligned}$$

□

Exercises

Exercise 4.4.1. Let $A \in \mathcal{L}(H_1, H_2)$. Show that

$$\|w\|_{\mathcal{M}(A)} \leq \|A\|_{\mathcal{L}(H_1, H_2)} \|w\|_{\mathcal{M}(AA^*)}, \quad (w \in \mathcal{M}(AA^*)).$$

Exercise 4.4.2. Let $A \in \mathcal{L}(H_1, H_2)$ be a Hilbert space contraction. Show that

$$\|w\|_{\mathcal{H}(A)} \leq \|w\|_{\mathcal{M}(I - AA^*)}, \quad (w \in \mathcal{M}(I - AA^*)).$$

Hint: Apply Exercise 4.4.1.

4.5 The Halmos intertwining theorem

To further study these spaces the following intertwining relation is needed.

Theorem 4.9 (Halmos). *Let $A \in \mathcal{L}(H_1, H_2)$ be a Hilbert space contraction. Then*

$$A(I - A^*A)^{1/2} = (I - AA^*)^{1/2}A.$$

Proof. We obviously have $A(I - A^*A) = (I - AA^*)A$. Hence, by induction, $A(I - A^*A)^n = (I - AA^*)^n A$ holds for any integer $n \geq 0$. Therefore, for any polynomial p ,

$$Ap(I - A^*A) = p(I - AA^*)A.$$

By Corollary 1.5, there is a sequence of polynomials $(p_n)_{n \geq 1}$ such that

$$p_n(I - A^*A)x \longrightarrow (I - A^*A)^{1/2}x \quad \text{and} \quad p_n(I - AA^*)x \longrightarrow (I - AA^*)^{1/2}x,$$

for each $x \in H$. Thus the required identity follows immediately. □

Replacing A by A^* in Halmos' theorem we obtain the equivalent identity

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*. \quad (4.10)$$

4.6 The Lotto–Sarason theorem

In this section we explore the relation between $\mathcal{H}(A)$ versus $\mathcal{H}(A^*)$. In particular, we obtain a frequently used identity which exhibits the bridge between the inner product in $\mathcal{H}(A)$ and $\mathcal{H}(A^*)$.

Theorem 4.10 (Lotto–Sarason). *Let $A \in \mathcal{L}(H_1, H_2)$ be a contraction, and let $w \in H_2$. Then $w \in \mathcal{H}(A)$ if and only if $A^*w \in \mathcal{H}(A^*)$. Moreover, if $w_1, w_2 \in \mathcal{H}(A)$, then*

$$\langle w_1, w_2 \rangle_{\mathcal{H}(A)} = \langle A^*w_1, A^*w_2 \rangle_{\mathcal{H}(A^*)} + \langle w_1, w_2 \rangle_{H_2}.$$

Proof. Replacing A by A^* in the intertwining relation (Theorem 4.9) gives

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*.$$

Hence the set inclusion $A^*\mathcal{H}(A) \subset \mathcal{H}(A^*)$ follows which is equivalent to say that

$$w \in \mathcal{H}(A) \implies A^*w \in \mathcal{H}(A^*).$$

Now, let $w \in H_2$ be such that $A^*w \in \mathcal{H}(A^*)$. Thus, by definition, there is $x \in H_1$ such that

$$A^*w = (I - A^*A)^{1/2}x.$$

By the intertwining relation, the trivial identity

$$w = (I - AA^*)w + AA^*w = (I - AA^*)w + A(I - A^*A)^{1/2}x$$

is rewritten as

$$w = (I - AA^*)^{1/2}((I - AA^*)^{1/2}w + Ax). \quad (4.11)$$

Hence $w \in \mathcal{H}(A)$.

To prove the identity for the inner products, let $w_1, w_2 \in \mathcal{H}(A)$. Hence, A^*w_1 and $A^*w_2 \in \mathcal{H}(A)$, i.e. there are $x_1, x_2 \in H_1$ such that

$$A^*w_k = (I - A^*A)^{1/2}x_k, \quad (k = 1, 2). \quad (4.12)$$

Without loss of generality we assume that $x_k \perp \ker(I - A^*A)^{1/2}$, and remember that $\ker(I - A^*A)^{1/2} = \ker(I - A^*A)$. Therefore, by (4.5),

$$\langle A^*w_1, A^*w_2 \rangle_{\mathcal{H}(A^*)} = \langle x_1, x_2 \rangle_{H_1}. \quad (4.13)$$

Moreover, by (4.11), we have $w_k = (I - AA^*)^{1/2}z_k$, where $z_k \in H_2$ is given by

$$z_k = (I - AA^*)^{1/2}w_k + Ax_k, \quad (k = 1, 2). \quad (4.14)$$

At this point it is important to observe that

$$z_k \perp \ker(I - AA^*), \quad (k = 1, 2).$$

As a matter of fact, for each $w \in \ker(I - AA^*)$, we have

$$(I - A^*A)(A^*w) = A^*(I - AA^*)w = 0$$

which means $A^*w \in \ker(I - A^*A)$. Then

$$\begin{aligned} \langle w, z_k \rangle_{H_2} &= \langle w, (I - AA^*)^{1/2}w_k + Ax_k \rangle_{H_2} \\ &= \langle (I - AA^*)^{1/2}w, w_k \rangle_{H_2} + \langle A^*w, x_k \rangle_{H_1} = 0. \end{aligned}$$

Again we used the fact that $\ker(I - AA^*) = \ker(I - AA^*)^{1/2}$.

Therefore, by (4.5),

$$\langle w_1, w_2 \rangle_{\mathcal{H}(A)} = \langle z_1, z_2 \rangle_{H_2}. \quad (4.15)$$

To obtain the required result, we expand $\langle z_1, z_2 \rangle_{H_2}$. Hence, by (4.14),

$$\begin{aligned} \langle w_1, w_2 \rangle_{\mathcal{H}(A)} &= \langle (I - AA^*)^{1/2}w_1 + Ax_1, (I - AA^*)^{1/2}w_2 + Ax_2 \rangle_{H_2} \\ &= \langle (I - AA^*)^{1/2}w_1, (I - AA^*)^{1/2}w_2 \rangle_{H_2} \\ &\quad + \langle Ax_1, (I - AA^*)^{1/2}w_2 \rangle_{H_2} \\ &\quad + \langle (I - AA^*)^{1/2}w_1, Ax_2 \rangle_{H_2} \\ &\quad + \langle Ax_1, Ax_2 \rangle_{H_2}. \end{aligned}$$

Let us simplify the right side. Thus

$$\begin{aligned} \langle (I - AA^*)^{1/2}w_1, (I - AA^*)^{1/2}w_2 \rangle_{H_2} &= \langle (I - AA^*)w_1, w_2 \rangle_{H_2} \\ &= \langle w_1, w_2 \rangle_{H_2} - \langle A^*w_1, A^*w_2 \rangle_{H_1}, \end{aligned}$$

and, by intertwining relation and (4.12),

$$\begin{aligned} \langle Ax_1, (I - AA^*)^{1/2}w_2 \rangle_{H_2} &= \langle (I - AA^*)^{1/2}Ax_1, w_2 \rangle_{H_2} \\ &= \langle A(I - A^*A)^{1/2}x_1, w_2 \rangle_{H_2} \\ &= \langle A^*w_1, A^*w_2 \rangle_{H_1}. \end{aligned}$$

Similarly,

$$\langle (I - AA^*)^{1/2}w_1, Ax_2 \rangle_{H_2} = \langle A^*w_1, A^*w_2 \rangle_{H_1}.$$

Finally, by (4.12),

$$\begin{aligned} \langle A^*w_1, A^*w_2 \rangle_{H_1} &= \langle (I - A^*A)^{1/2}x_1, (I - A^*A)^{1/2}x_2 \rangle_{H_1} \\ &= \langle (I - A^*A)x_1, x_2 \rangle_{H_1} \\ &= \langle x_1, x_2 \rangle_{H_1} - \langle Ax_1, Ax_2 \rangle_{H_2}. \end{aligned}$$

Therefore, the preceding identities imply

$$\langle w_1, w_2 \rangle_{\mathcal{H}(A)} = \langle w_1, w_2 \rangle_{H_2} + \langle x_1, x_2 \rangle_{H_1}.$$

By (4.13), this is the required identity. \square

Corollary 4.11. *Let $A \in \mathcal{L}(H_1, H_2)$ be a contraction. Then a vector $x \in H_1$ belongs to $\mathcal{H}(A^*)$ if and only if $Ax \in \mathcal{H}(A)$. Moreover, if $x_1, x_2 \in \mathcal{H}(A^*)$, then*

$$\langle x_1, x_2 \rangle_{\mathcal{H}(A^*)} = \langle Ax_1, Ax_2 \rangle_{\mathcal{H}(A)} + \langle x_1, x_2 \rangle_{H_1}.$$

Exercises

Exercise 4.6.1. Let $A \in \mathcal{L}(H_1, H_2)$ be a contraction, and let $w \in \mathcal{H}(A)$. Show that

$$\|w\|_{\mathcal{H}(A)}^2 = \|A^*w\|_{\mathcal{H}(A^*)}^2 + \|w\|_{H_2}^2.$$

Hint: This immediately follows from Theorem 4.10.

4.7 The overlapping space

Let $A \in \mathcal{L}(H_1, H_2)$ be a Hilbert space contraction. In Section 4.4, the intersection $\mathcal{M}(A) \cap \mathcal{H}(A)$ was called the overlapping space. We first we find a formula for the overlapping space.

Lemma 4.12. *Let $A \in \mathcal{L}(H_1, H_2)$ be a contraction. Then*

$$\mathcal{M}(A) \cap \mathcal{H}(A) = A\mathcal{H}(A^*).$$

Proof. By Corollary 4.11, $A\mathcal{H}(A^*) \subset \mathcal{H}(A)$. Moreover, by definition, $A\mathcal{H}(A^*) \subset \mathcal{M}(A)$. Hence $A\mathcal{H}(A^*) \subset \mathcal{M}(A) \cap \mathcal{H}(A)$.

To prove the other inclusion, let $w \in \mathcal{M}(A) \cap \mathcal{H}(A)$. Therefore, $w = Ax$, for some $x \in H_1$, and $Ax \in \mathcal{H}(A)$. Thus, again by Corollary 4.11, we necessarily have $x \in \mathcal{H}(A^*)$, and this means $w = Ax \in A\mathcal{H}(A^*)$. \square

We naturally wonder when the overlapping space is trivial. We are now able to fully characterize this case.

Theorem 4.13. *Let $A \in \mathcal{L}(H_1, H_2)$ be contraction. Then the following are equivalent.*

- (i) A is a partial isometry;
- (ii) $\mathcal{M}(A)$ is a closed subspace of H and inherits its Hilbert space structure;
- (iii) $\mathcal{H}(A)$ is a closed subspace of H and inherits its Hilbert space structure;
- (iv) $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other;
- (v) $\mathcal{M}(A) \cap \mathcal{H}(A) = \{0\}$;
- (vi) $\mathcal{H}(A^*) \subset \ker A$.

Moreover, under the preceding equivalent conditions, we have the orthogonal decomposition

$$H = \mathcal{M}(A) + \mathcal{H}(A).$$

Proof. (i) \iff (ii) Proved in Corollary 4.4.

(i) \iff (iii) Proved in Lemma 4.6.

(i) \implies (iv) If A is a partial isometry, then $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are the range of complementary orthogonal projections AA^* and $I - AA^*$. Hence $\mathcal{M}(A)$ and $\mathcal{H}(A)$ are orthogonal complements of each other.

(iv) \implies (v) Trivial.

(v) \iff (vi) This is an immediate consequence of Lemma 4.12.

(vi) \implies (i) By assumption $A(I - A^*A)^{1/2} = 0$. If so, then also $A(I - A^*A) = 0$. Hence $A = AA^*A$ which implies $(AA^*)^2 = AA^*$. In other words, AA^* is an orthogonal projection. Therefore, by Theorem 1.7, A is a partial isometry. \square

4.8 Decomposition of $\mathcal{H}(A)$

If an operator decomposes as $A = A_2A_1$ we naturally ask about the relation between $\mathcal{H}(A)$ on one hand, and $\mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ on the other hand. In this section we address this question.

Theorem 4.14. *Let $A_1 \in \mathcal{L}(H_3, H_1)$ and $A_2 \in \mathcal{L}(H_1, H_2)$ be contractions, and let $A = A_2A_1$. Then the following hold.*

(i) $\mathcal{H}(A)$ decomposes as

$$\mathcal{H}(A) = A_2\mathcal{H}(A_1) + \mathcal{H}(A_2).$$

(ii) If $w \in \mathcal{H}(A)$ has the representation

$$w = A_2w_1 + w_2,$$

where $w_i \in \mathcal{H}(A_i)$, then

$$\|w\|_{\mathcal{H}(A)}^2 \leq \|w_1\|_{\mathcal{H}(A_1)}^2 + \|w_2\|_{\mathcal{H}(A_2)}^2.$$

(iii) For each $w \in \mathcal{H}(A)$ there is a unique pair of $w_1 \in \mathcal{H}(A_1)$ and $w_2 \in \mathcal{H}(A_2)$ such that $w = A_2w_1 + w_2$ and

$$\|w\|_{\mathcal{H}(A)}^2 = \|w_1\|_{\mathcal{H}(A_1)}^2 + \|w_2\|_{\mathcal{H}(A_2)}^2.$$

(iv) $\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.

(v) The operator A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.

Proof. (i) Let $B_1 = A_2(I - A_1A_1^*)^{1/2} \in \mathcal{L}(H_1, H_2)$ and $B_2 = (I - A_2A_2^*)^{1/2} \in \mathcal{L}(H_2)$.

Figure 3.1: The factorization $A = A_2A_1$

Then

$$\begin{aligned}
 I - AA^* &= I - (A_2A_1)(A_2A_1)^* \\
 &= A_2(I - A_1A_1^*)A_2^* + (I - A_2A_2^*) \\
 &= B_1B_1^* + B_2B_2^* \\
 &= BB^*,
 \end{aligned}$$

where $B = [B_1 \ B_2] \in \mathcal{L}(H_1 \oplus H_2, H_2)$. Therefore, by Corollary 4.3(ii),

$$\begin{aligned}
 \mathcal{H}(A) &= \mathcal{M}((I - AA^*)^{1/2}) \\
 &= \mathcal{M}(B) \\
 &= \mathcal{M}(B_1) + \mathcal{M}(B_2) \\
 &= A_2\mathcal{M}((I - A_1A_1^*)^{1/2}) + \mathcal{M}((I - A_2A_2^*)^{1/2}) \\
 &= A_2\mathcal{H}(A_1) + \mathcal{H}(A_2).
 \end{aligned}$$

Note that $\mathcal{H}(A)$ and $\mathcal{M}(B)$ have the same Hilbert structure.

(ii) If $w = A_2w_1 + w_2$ with $w_i \in \mathcal{H}(A_i)$, then we can write $w_i = (I - A_iA_i^*)^{1/2}x_i$ with $x_i \perp \ker(I - A_iA_i^*)$. Then have

$$\begin{aligned}
 w &= A_2w_1 + w_2 \\
 &= A_2(I - A_1A_1^*)^{1/2}x_1 + (I - A_2A_2^*)^{1/2}x_2 \\
 &= B_1x_1 + B_2x_2 \\
 &= B(x_1 \oplus x_2).
 \end{aligned}$$

Therefore, by Corollary 4.3(ii) and (4.4),

$$\begin{aligned}
 \|w\|_{\mathcal{H}(A)}^2 &= \|w\|_{\mathcal{M}(B)}^2 \\
 &= \|B(x_1 \oplus x_2)\|_{\mathcal{M}(B)}^2 \\
 &\leq \|x_1 \oplus x_2\|_{H_1 \oplus H_2}^2 \\
 &= \|x_1\|_{H_1}^2 + \|x_2\|_{H_2}^2 \\
 &= \|w_1\|_{\mathcal{H}(A_1)}^2 + \|w_2\|_{\mathcal{H}(A_2)}^2.
 \end{aligned}$$

(iii) Among all possible representations

$$w = A_2w_1 + w_2 = B(x_1 \oplus x_2),$$

if we choose x_1 and x_2 such that $x_1 \oplus x_2 \perp \ker B$, then, in the light of (??), we certainly have $x_i \perp \ker(I - A_iA_i^*)$. Hence in the last paragraph of (ii) equality holds everywhere. Thus this choice of x_1 and x_2 gives at least a suitable pair w_1 and w_2 for which $\|w\|_{\mathcal{H}(A)}^2 = \|w_1\|_{\mathcal{H}(A_1)}^2 + \|w_2\|_{\mathcal{H}(A_2)}^2$ holds. But, to have this equality, we certainly need $x_1 \oplus x_2 \perp \ker B$ and this choice of $x_1 \oplus x_2$ is unique. Hence, in return, w_1 and w_2 are unique.

(iv) By (i), $\mathcal{H}(A_2) \subset \mathcal{H}(A)$. For each $w_2 \in \mathcal{H}(A_2)$, consider $w = A_2 0 + w_2$. Hence, by (ii),

$$\|w_2\|_{\mathcal{H}(A)} \leq \|w_2\|_{\mathcal{H}(A_2)}.$$

This means that $\mathcal{H}(A_2)$ is contractively contained in $\mathcal{H}(A)$.

(v) By (i), $A_2 \mathcal{H}(A_1) \subset \mathcal{H}(A)$. For each $w_1 \in \mathcal{H}(A_1)$, consider $w = A_2 w_1 + 0$. Hence, by (ii),

$$\|A_2 w_1\|_{\mathcal{H}(A)} \leq \|w_1\|_{\mathcal{H}(A_1)}.$$

This means that A_2 acts as a contraction from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$. \square

In part (iii) of the preceding theorem the existence of a unique pair of w_1 and w_2 was established. However we did not present a procedure or formula to find them. Indeed we are able to do this just in a special case. In Corollary 4.8, we saw that $\mathcal{M}(I - AA^*)$ is dense in $\mathcal{H}(A)$. Let $w \in \mathcal{M}(I - AA^*)$. Hence there is $y \in H_2$ such that

$$w = (I - AA^*)y. \quad (4.16)$$

Let

$$x_1 = B_1^* y \quad \text{and} \quad x_2 = B_2^* y.$$

Then, by Theorem ??(xi),

$$x_1 \oplus x_2 = B_1^* y \oplus B_2^* y = B^* y \in \mathcal{R}(B^*) \subset (\ker B)^\perp.$$

Moreover,

$$B(x_1 \oplus x_2) = B_1 x_1 + B_2 x_2 = (B_1 B_1^* + B_2 B_2^*) y = BB^* y = (I - AA^*) y = w.$$

Therefore, the unique pair is obtained by

$$w_1 = (I - A_1 A_1^*)^{1/2} x_1 = (I - A_1 A_1^*) A_2^* y \quad (4.17)$$

and

$$w_2 = (I - A_2 A_2^*)^{1/2} x_2 = (I - A_2 A_2^*) y. \quad (4.18)$$

The relation

$$\mathcal{H}(A) = \mathcal{M}(B) = \mathcal{M}(B_1) + \mathcal{M}(B_2) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$$

shows that the decomposition

$$\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$$

is an algebraic direct sum of $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ if and only if

$$\ker B = \ker B_1 \oplus \ker B_2.$$

Assuming the decomposition is an algebraic direct sum, if

$$w = A_2 w_1 + w_2 = A_2 w'_1 + w'_2$$

then we necessarily have $A_2 w_1 = A_2 w'_1$ and $w_2 = w'_2$. Hence the choice of w_2 in the representation $w = A_2 w_1 + w_2$ is unique. However, still there is some liberty for w_1 .

Corollary 4.15. *Let $A_1 \in \mathcal{L}(H_3, H_1)$ and $A_2 \in \mathcal{L}(H_1, H_2)$ be contractions, and let $A = A_2A_1$. Suppose that the decomposition*

$$\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$$

is an algebraic direct sum. Then the following hold.

- (i) $\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.
- (ii) Relative to the Hilbert space structure of $\mathcal{H}(A)$, $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$. In other words, the decomposition $\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$ is fact an orthogonal direct sum.
- (iii) The operator A_2 acts as a partial isometry from $\mathcal{H}(A_1)$ into $\mathcal{H}(A)$.

Proof. We use the notations appeared in Theorem 4.14. The main ingredient of the proof is the relation (??).

(i) Let $w_2 \in \mathcal{H}(A_2)$. Hence $w_2 = B_2x_2 = (I - A_2A_2^*)^{1/2}x_2$ with $x_2 \in H_2$. Therefore, by (??),

$$\begin{aligned} \|w_2\|_{\mathcal{H}(A)} &= \|w_2\|_{\mathcal{M}(B)} \\ &= \|B_2x_2\|_{\mathcal{M}(B)} \\ &= \|B(0 \oplus x_2)\|_{\mathcal{M}(B)} \\ &= \|P_{(\ker B)^\perp}(0 \oplus x_2)\|_{H_1 \oplus H_2} \\ &= \|0 \oplus P_{(\ker B_2)^\perp}x_2\|_{H_1 \oplus H_2} \\ &= \|P_{(\ker B_2)^\perp}x_2\|_{H_2} \\ &= \|B_2x_2\|_{\mathcal{M}(B_2)} \\ &= \|w_2\|_{\mathcal{H}(A_2)}. \end{aligned}$$

Hence $\mathcal{H}(A_2)$ is contained isometrically in $\mathcal{H}(A)$.

(ii) By part (i), with respect to the structure of $\mathcal{H}(A)$, $\mathcal{H}(A_2)$ is a closed subspace of $\mathcal{H}(A)$. By Theorem 4.14(ii), $\|A_2w_1\|_{\mathcal{H}(A)} \leq \|w_1\|_{\mathcal{H}(A_1)}$. In the light of part (iii) of that theorem and our assumption that the decomposition is an algebraic direct sum, it is possible to choose w_1 such that $\|A_2w_1\|_{\mathcal{H}(A)} = \|w_1\|_{\mathcal{H}(A_1)}$. This observation shows that $A_2\mathcal{H}(A_1)$ is also a closed subspace of $\mathcal{H}(A)$.

Let $w_i \in \mathcal{H}(A_i)$, $i = 1, 2$. Hence $w_i = (I - AA^*)^{1/2}x_i$ with $x_i \in H_i$. Therefore, by (??),

$$\begin{aligned} \langle A_2w_1, w_2 \rangle_{\mathcal{H}(A)} &= \langle A_2w_1, w_2 \rangle_{\mathcal{M}(B)} \\ &= \langle B_1x_1, B_2x_2 \rangle_{\mathcal{M}(B)} \\ &= \langle B(x_1 \oplus 0), B(0 \oplus x_2) \rangle_{\mathcal{M}(B)} \\ &= \langle P_{(\ker B)^\perp}(x_1 \oplus 0), P_{(\ker B)^\perp}(0 \oplus x_2) \rangle_{H_1 \oplus H_2} \\ &= \langle P_{(\ker B_1)^\perp}x_1 \oplus 0, 0 \oplus P_{(\ker B_2)^\perp}x_2 \rangle_{H_1 \oplus H_2} \\ &= \langle P_{(\ker B_1)^\perp}x_1, 0 \rangle_{H_1} + \langle 0, P_{(\ker B_2)^\perp}x_2 \rangle_{H_2} = 0. \end{aligned}$$

Hence $A_2 \mathcal{H}(A_1)$ and $\mathcal{H}(A_2)$ are complementary orthogonal subspaces of $\mathcal{H}(A)$.

(iii) Let us use temporarily the notation $\mathbb{A}_2 : \mathcal{H}(A_1) \rightarrow \mathcal{H}(A)$ for the restriction of $A_2 : H_1 \rightarrow H_2$ to $\mathcal{H}(A_1)$. Hence $\ker \mathbb{A}_2 = (\ker A_2) \cap \mathcal{H}(A_1)$. To show that \mathbb{A}_2 is a partial isometry, we need to verify that if w_1 is orthogonal to $\ker \mathbb{A}_2$ with respect to the inner product of $\mathcal{H}(A_1)$, then $\|w_1\|_{\mathcal{H}(A_1)} = \|A_2 w_1\|_{\mathcal{H}(A)}$.

Fix $w_1 \perp \ker \mathbb{A}_2$, and let $x \in \ker B_1$. Then $(1 - A_1 A_1^*)^{1/2} x \in \ker \mathbb{A}_2$. Hence

$$\langle w_1, (1 - A_1 A_1^*)^{1/2} x \rangle_{\mathcal{H}(A_1)} = 0. \quad (4.19)$$

Write $w_1 = (I - A_1 A_1^*)^{1/2} x_1$ with $x_1 \in H_1$ and, without loss of generality, let x_1 be orthogonal to the $\ker (I - A_1 A_1^*)$ with respect to the inner product of H_1 . Therefore, by (4.5) and (4.19), we have $\langle x_1, x \rangle_{H_1} = 0$. By Theorem 4.14(iii), we thus obtain $\|A_2 w_1\|_{\mathcal{H}(A)} = \|w_1\|_{\mathcal{H}(A_1)}$. □

4.9 Decomposition of H

Let H be a Hilbert space and let M be a closed subspace of H . By Corollary ??, we have $H = M \oplus M^\perp$. On the other hand, by Lemma 4.1, we have $\mathcal{M}(P_M) = M$ and $\mathcal{H}(P_M) = M^\perp$, and thus we can write $H = \mathcal{M}(P_M) \oplus \mathcal{H}(P_M)$. In this section, we generalize this result.

Theorem 4.16. *Let $A \in \mathcal{L}(H_1, H)$ be a Hilbert space contraction. Then*

$$H = \mathcal{M}(A) + \mathcal{H}(A).$$

For each decomposition $w = w_1 + w_2$, with $w \in H$, $w_1 \in \mathcal{M}(A)$ and $w_2 \in \mathcal{H}(A)$, we have

$$\|w\|_H^2 \leq \|w_1\|_{\mathcal{M}(A)}^2 + \|w_2\|_{\mathcal{H}(A)}^2.$$

Moreover,

$$\|w\|_H^2 = \|w_1\|_{\mathcal{M}(A)}^2 + \|w_2\|_{\mathcal{H}(A)}^2$$

if and only if

$$w_1 = AA^* w \quad \text{and} \quad w_2 = (I - AA^*) w.$$

Proof. In this proof we write T instead of A . This is because we want to apply Theorem 4.14 and use the notations there, but the operator A which appears in that Theorem is not the same as one introduced here.

Figure 3.2: The factorization $0 = A0$

Consider the decomposition

$$0 = T0,$$

where on the left side we have $0 \in \mathcal{L}(H_1, H)$ and on the right side $0 \in \mathcal{L}(H_1)$. Hence we have $A = A_2 A_1$ with $A = 0 \in \mathcal{L}(H_1, H)$, $A_2 = T$ and $A_1 = 0 \in \mathcal{L}(H_1)$. Also note that

$$\mathcal{H}(A) = H \quad \text{and} \quad \mathcal{H}(A_1) = H_1.$$

The diagram in Figure 3.1 simplifies as follows. Thus the decomposition $\mathcal{H}(A) = A_2 \mathcal{H}(A_1) + \mathcal{H}(A_2)$ obtained in Theorem 4.14(i) is written as

$$H = TH_1 + \mathcal{H}(T) = \mathcal{M}(T) + \mathcal{H}(T)$$

and if $z = Tz_1 + z_2$ with $z_1 \in H_1$ and $z_2 \in \mathcal{H}(T)$, then, by Theorem 4.14(ii),

$$\|z\|_H^2 \leq \|z_1\|_{H_1}^2 + \|z_2\|_{\mathcal{H}(T)}^2.$$

In particular, if we take $z_1 \perp \ker T$, we obtain

$$\|z\|_H^2 \leq \|Tz_1\|_{\mathcal{M}(T)}^2 + \|z_2\|_{\mathcal{H}(T)}^2.$$

Finally, by (4.16), (4.17) and (4.18), the unique pair z_1 and z_2 for which

$$\|z\|_H^2 = \|z_1\|_{H_1}^2 + \|z_2\|_{\mathcal{H}(T)}^2$$

holds are given by

$$z_1 = (I - A_1 A_1^*) A_2^* z = T^* z$$

and

$$z_2 = (I - A_2 A_2^*) z = (I - TT^*) z.$$

But

$$z_1 \in \mathcal{R}(T^*) \subset (\ker T)^\perp$$

implies $\|z_1\|_{H_1} = \|Tz_1\|_{\mathcal{M}(T)}$. To be consistent with the notation of theorem, just take $w = z$, $w_1 = Tz_1$ and $w_2 = z_2$. \square

We are now able to add one more item to the equivalent conditions mentioned in Theorem 4.13.

Corollary 4.17. *Let $A \in \mathcal{L}(H_1, H)$ be a contraction. Then the decomposition*

$$H = \mathcal{M}(A) + \mathcal{H}(A)$$

is an algebraic direct sum if and only if A is a partial isometry.

Proof. It suffices to apply Theorem 4.13! \square

Chapter 5

Hilbert spaces in H^2

In this chapter, our ambient Hilbert space is H^2 , the Hardy space of analytic functions on the open unit disc \mathbb{D} , or equivalently their boundary values on the unit circle \mathbb{T} . Using contractive Toeplitz operators on H^2 , we apply the theory developed in Chapter 4 to obtain some Hilbert spaces of analytic functions which live in H^2 . Understanding the structure of these spaces is the principal goal of this text.

5.1 The space $\mathcal{H}(b)$

Let $\varphi \in L^\infty(\mathbb{T})$ with $\|\varphi\|_\infty \leq 1$. Then, by Theorem 3.4, the corresponding Toeplitz operator T_φ is a contraction on the Hilbert space H^2 . Hence the Hilbert spaces $\mathcal{M}(T_\varphi)$ and $\mathcal{H}(T_\varphi)$ are well defined. For simplicity, we denote the complementary space $\mathcal{H}(T_\varphi)$ by $\mathcal{H}(\varphi)$. By the same token, the norm and inner product in $\mathcal{H}(\varphi)$ will be denoted by $\|\cdot\|_\varphi$ and $\langle \cdot, \cdot \rangle_\varphi$.

Our main concern is when φ is a nonconstant analytic function in the unit ball of H^∞ . In this case, by tradition, we use b instead of φ . Therefore, from now on, we assume that

- (i) $b \in H^\infty$,
- (ii) b is nonconstant,
- (iii) and $\|b\|_\infty \leq 1$,

and the corresponding Hilbert spaces created by T_b are denoted by $\mathcal{M}(b)$ and $\mathcal{H}(b)$, i.e.

$$\mathcal{M}(b) = \mathcal{M}(T_b)$$

and

$$\mathcal{H}(b) = \mathcal{M}((I - T_b T_{\bar{b}})^{1/2}).$$

To better understand the structure of $\mathcal{H}(b)$ we will also naturally face with

$$\mathcal{H}(\bar{b}) = \mathcal{M}((I - T_{\bar{b}} T_b)^{1/2}).$$

The structure of $\mathcal{M}(b)$ is simple. Indeed, directly from the definition, we obtain the set identity

$$\mathcal{M}(b) = bH^2.$$

Moreover, since T_b is injective, for each $f \in H^2$, we have

$$\|bf\|_{\mathcal{M}(b)} = \|f\|_{H^2}. \quad (5.1)$$

Lemma 4.12 also implies

$$\mathcal{M}(b) \cap \mathcal{H}(b) = T_{\bar{b}} \mathcal{H}(\bar{b}). \quad (5.2)$$

Theorem 5.1. *The space $\mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{H}(b)$. Moreover, $f \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$ and*

$$\langle f, g \rangle_b = \langle f, g \rangle + \langle T_{\bar{b}}f, T_{\bar{b}}g \rangle_{\bar{b}}, \quad (f, g \in \mathcal{H}(b)).$$

Proof. By Theorem 3.9,

$$I - T_{\bar{b}}T_b \leq I - T_bT_{\bar{b}}.$$

Hence, by Theorem 4.2, $\mathcal{M}((I - T_{\bar{b}}T_b)^{1/2}) = \mathcal{H}(\bar{b})$ is contractively contained in $\mathcal{M}((I - T_bT_{\bar{b}})^{1/2}) = \mathcal{H}(b)$.

The relation between the inner products of $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ is a special case of Lotto–Sarason theorem (Theorem 4.10). \square

5.2 Model subspaces K_{Θ}

For each b in the unit ball of H^∞ , we have

$$\mathcal{H}(\bar{b}) = \mathcal{M}((I - T_{\bar{b}}T_b)^{1/2}) = \mathcal{M}(T_{1-|b|^2}^{1/2}).$$

Hence, if b is inner, then $\mathcal{H}(\bar{b}) = \{0\}$. But, in this case, $\mathcal{M}(b)$ and $\mathcal{H}(b)$ have rich structures. If b is not inner, in the light of $\mathcal{M}(T_{1-|b|^2}) \subset \mathcal{M}(T_{1-|b|^2}^{1/2})$, the Corollary 3.11 ensures that $\mathcal{H}(\bar{b})$ is infinite dimensional.

Theorem 5.2. *The complementary space $\mathcal{H}(b)$ is a closed subspace of H^2 and inherits its Hilbert space structure if and only if b is inner. Moreover, in this case,*

$$\mathcal{M}(b) = bH^2 \quad \text{and} \quad \mathcal{H}(b) = (bH^2)^\perp.$$

Proof. By Lemma 4.6, $\mathcal{H}(b)$ is a closed subspace of H^2 and inherits its Hilbert space structure if and only if T_b is a partial isometry. But, by Theorem 3.6, this happens if and only if b is inner.

The relation $\mathcal{M}(b) = bH^2$ is indeed valid for any b . But, for an inner b , T_b is a partial isometry and thus, by Theorem 4.13(iv), $\mathcal{M}(b)$ and $\mathcal{H}(b)$ are orthogonal complement of each other. \square

Again by tradition, we write Θ instead of b when it is an inner function. The space $\mathcal{M}(\Theta)$ is the subset ΘH^2 of the Hardy space H^2 . Moreover, since Θ is unimodular, by (5.1), we have

$$\|\Theta f\|_{\mathcal{M}(\Theta)} = \|f\|_{H^2} = \|\Theta f\|_{H^2}, \quad (f \in H^2).$$

We can equivalently say

$$\|g\|_{\mathcal{M}(\Theta)} = \|g\|_{H^2}, \quad (g \in \mathcal{M}(\Theta)).$$

That is why it is legitimate to say that $\mathcal{M}(\Theta)$ is the closed subspace ΘH^2 of H^2 .

Indeed, they are also called Beurling subspaces of H^2 . A. Beurling showed ΘH^2 , Θ inner, are the only closed invariant subspaces of H^2 under the forward shift operator. Hence, their orthogonal complements, i.e. $\mathcal{H}(\Theta)$, are the only closed subspaces of H^2 which are invariant under the backward shift operator. We emphasize that we only considered closed subspaces of H^2 . Theorem 5.7 below shows that H^2 has other invariant linear manifolds too. These subspaces are usually denoted by K_Θ and called the model subspaces of H^2 .

Here we give another useful description of space K_Θ .

Lemma 5.3. *Let Θ be an inner function. Then*

$$K_\Theta = H^2 \cap \Theta H_-^2.$$

Proof. Let f be a function in H^2 . Then by definition, f belongs to K_Θ if and only if $\langle f, \Theta h \rangle = 0$, for every $h \in H^2$. Using the fact that $|\Theta| = 1$ almost everywhere on \mathbb{T} , this is equivalent to $\langle \Theta, h \rangle = 0$, for every $h \in H^2$. That means that $\Theta f \in H_-^2$. In other words, $f \in \Theta H_-^2$ and we get the conclusion. \square

We now give the relation between model spaces and Toeplitz operators.

Corollary 5.4. *Let φ be a function in H^∞ and let φ_i be the inner part of φ . Then $\ker T_{\bar{\varphi}} = K_{\varphi_i}$.*

Proof. Let φ_e be the outer part of φ . Then according to Theorem 3.8 and Theorem 3.7, we have

$$\begin{aligned} \ker T_{\bar{\varphi}} &= \{f \in H^2 : T_{\bar{\varphi}} f = 0\} \\ &= \{f \in H^2 : T_{\bar{\varphi}_e} T_{\bar{\varphi}_i} f = 0\} \\ &= \{f \in H^2 : T_{\bar{\varphi}_i} f = 0\} \\ &= \{f \in H^2 : \bar{\varphi}_i f \in H_-^2\} \\ &= H^2 \cap \varphi_i H_-^2 \end{aligned}$$

It remains to apply Lemma 5.3. \square

Theorem 5.5. *The complementary space $\mathcal{H}(\Theta b)$ is the orthogonal direct sum of K_Θ and $\Theta \mathcal{H}(b)$. The model space K_Θ is contained isometrically in $\mathcal{H}(\Theta b)$. The operator T_Θ act as an isometry from $\mathcal{H}(b)$ into $\mathcal{H}(\Theta b)$.*

Proof. \square

5.3 The reproducing kernel of $\mathcal{H}(b)$

The reproducing kernel for the Hardy space H^2 is

$$k_z(w) = \frac{1}{1 - \bar{z}w}, \quad (z, w \in \mathbb{D}).$$

In other words, for each $f \in H^2$,

$$f(z) = \langle f, k_z \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{1 - e^{-it}z} dt, \quad (z \in \mathbb{D}). \quad (5.3)$$

Theorem 4.5 enables us to find the reproducing kernel of $\mathcal{H}(b)$.

Theorem 5.6. *The reproducing kernel of $\mathcal{H}(b)$ is*

$$k_z^b = (1 - \overline{b(z)}b) k_z, \quad (z \in \mathbb{D}),$$

or equivalently

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \bar{z}w}, \quad (z, w \in \mathbb{D}).$$

Moreover, the norm of the evaluation functional

$$f(z) = \langle f, k_z^b \rangle_b, \quad (f \in \mathcal{H}(b), z \in \mathbb{D}),$$

is equal to

$$\|k_z^b\|_b = (k_z^b(z))^{1/2} = \left(\frac{1 - |b(z)|^2}{1 - |z|^2} \right)^{1/2}.$$

Proof. Since $\mathcal{H}(b) = \mathcal{M}((I - T_b T_{\bar{b}})^{1/2})$, by Theorem 4.5 and (5.3), we have

$$k_z^b = (I - T_b T_{\bar{b}}) k_z.$$

But, by (3.3),

$$T_{\bar{b}} k_z = \overline{b(z)} k_z.$$

Clearly $T_b k_z = b k_z$. Hence

$$k_z^b = (1 - \overline{b(z)}b) k_z.$$

The last two identities of theorem were proved in the general context of reproducing kernel Hilbert spaces in Section 1.4. \square

Exercises

Exercise 5.3.1. Show that the reproducing kernel of $\mathcal{M}(b)$ is

$$k_z(w) = \frac{\overline{b(z)}b(w)}{1 - \bar{z}w}, \quad (z, w \in \mathbb{D}).$$

Moreover, the norm of the evaluation functional

$$f(z) = \langle f, k_z \rangle_{\mathcal{M}(b)}, \quad (f \in \mathcal{M}(b), z \in \mathbb{D}),$$

is given by

$$\|k_z\|_{\mathcal{M}(b)} = (k_z(z))^{1/2} = \left(\frac{|b(z)|^2}{1 - |z|^2} \right)^{1/2}.$$

5.4 $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ as invariant subspaces

In Section 5.2, we saw that $K_\Theta = \mathcal{H}(\Theta)$ is a closed subspace of H^2 which is invariant under the backward shift operator $S^* = T_{\bar{z}}$. In this section we further explore this property and show that each $\mathcal{H}(b)$ is invariant under a large family of operators.

Theorem 5.7. *Let $\varphi \in H^\infty$. Then $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are both invariant under $T_{\bar{\varphi}}$. Moreover the norm of $T_{\bar{\varphi}}$, as an operator on $\mathcal{L}(\mathcal{H}(b))$, or on $\mathcal{L}(\mathcal{H}(\bar{b}))$, does not exceed $\|\varphi\|_\infty$.*

Proof. Without loss of generality assume that $\|\varphi\|_\infty = 1$. To show that

$$T_{\bar{\varphi}} \mathcal{H}(\bar{b}) \subset \mathcal{H}(\bar{b})$$

note that the linear manifold on the left side is $\mathcal{M}(T_{\bar{\varphi}}(I - T_{\bar{b}}T_b)^{1/2})$, and the one of the right side is $\mathcal{M}((I - T_{\bar{b}}T_b)^{1/2})$. By Theorem 4.2, the inclusion holds and $T_{\bar{\varphi}}$ acts a contraction on $\mathcal{H}(\bar{b})$ if and only if

$$T_{\bar{\varphi}}(I - T_{\bar{b}}T_b)T_{\varphi} \leq I - T_{\bar{b}}T_b.$$

But, by Theorem 3.8, this inequality is equivalent to

$$T_{(1-|b|^2)(1-|\varphi|^2)} \geq 0$$

which is indeed true by Theorem 3.5(ii).

To prove the statement for $\mathcal{H}(b)$, we apply the Lotto–Sarason theorem (Theorem 4.10) several times to go back and forth between $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. Let $f \in \mathcal{H}(b)$. Hence $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$. According to the preceding paragraph, we thus have $T_{\bar{\varphi}}T_{\bar{b}}f \in \mathcal{H}(\bar{b})$. But, by (3.4), $T_{\bar{\varphi}}T_{\bar{b}} = T_{\bar{b}}T_{\bar{\varphi}}$. Hence $T_{\bar{b}}T_{\bar{\varphi}}f \in \mathcal{H}(\bar{b})$. Another application of Lotto–Sarason’s theorem implies that $T_{\bar{\varphi}}f \in \mathcal{H}(\bar{b})$. Moreover,

$$\begin{aligned} \|T_{\bar{\varphi}}f\|_{\bar{b}}^2 &= \|T_{\bar{\varphi}}f\|_2^2 + \|T_{\bar{b}}T_{\bar{\varphi}}f\|_{\bar{b}}^2 \\ &= \|T_{\bar{\varphi}}f\|_2^2 + \|T_{\bar{\varphi}}T_{\bar{b}}f\|_{\bar{b}}^2 \\ &\leq \|T_{\bar{\varphi}}f\|_2^2 + \|T_{\bar{b}}f\|_{\bar{b}}^2 \\ &= \|f\|_{\bar{b}}^2. \end{aligned}$$

□

By Theorem 5.6, the only explicit elements in $\mathcal{H}(b)$ that we know up to now are functions k_z^b , $z \in \mathbb{D}$, where

$$k_z^b(w) = \frac{1 - \overline{b(z)}b(w)}{1 - \bar{z}w}, \quad (w \in \mathbb{D}).$$

However, Theorem 5.7 enables us to distinguish more inhabitants of $\mathcal{H}(b)$. We remind that $Q_w = (1 - wS^*)^{-1}S^* \in \mathcal{L}(H^2)$, $w \in \mathbb{D}$.

Corollary 5.8. *Let $w \in \mathbb{D}$. Then $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are both invariant under Q_w . Moreover the norm of Q_w , as an operator on $\mathcal{L}(\mathcal{H}(b))$, or on $\mathcal{L}(\mathcal{H}(\bar{b}))$, does not exceed $1/(1 - |w|)$.*

Proof. Since $S^* = T_{\bar{z}}$, and

$$Q_w = \sum_{n=1}^{\infty} w^{n-1} S^{*n}$$

the result immediately follows from Theorem 5.7. Moreover, this theorem also says,

$$\|S^*\| \leq \|z\|_{\infty} = 1,$$

where we considered S^* as an operator on $\mathcal{L}(\mathcal{H}(b))$, or on $\mathcal{L}(\mathcal{H}(\bar{b}))$. Hence

$$\|Q_w\| \leq \sum_{n=1}^{\infty} |w|^{n-1} \|S^*\|^n \leq \sum_{n=1}^{\infty} |w|^{n-1} = \frac{1}{1 - |w|}.$$

□

Let $f = (I - T_{\bar{b}}T_b)1$. Clearly $f \in \mathcal{H}(\bar{b})$. Hence, by Theorem 5.7,

$$-S^*f \in \mathcal{H}(\bar{b}).$$

A simple calculation shows

$$-S^*f = S^*T_{\bar{b}}b = T_{\bar{b}}S^*b.$$

Since $T_{\bar{b}}S^*b \in \mathcal{H}(\bar{b})$, the Lotto–Sarason theorem (Theorem 4.10) ensures that

$$S^*b \in \mathcal{H}(b). \quad (5.4)$$

If we knew that $b \in \mathcal{H}(b)$, then (5.4) was an immediate consequence of Theorem 5.7. But $b \in \mathcal{H}(b)$ is not always true. We will explore the possibility of $b \in \mathcal{H}(b)$ in Section ???. Since

$$(S^*b)(z) = \frac{b(z) - b(0)}{z}, \quad (z \in \mathbb{D}),$$

in the light of (5.4), we may wonder if the function

$$\frac{b(z) - b(w)}{z - w}, \quad (z \in \mathbb{D}),$$

where $w \in \mathbb{D}$ is fixed, also lives $\mathcal{H}(b)$. By Theorem 2.8,

$$(Q_w b)(z) = \frac{b(z) - b(w)}{z - w}, \quad (z \in \mathbb{D}),$$

and

$$Q_w b = \sum_{n=0}^{\infty} w^n S^{*n}(S^* b).$$

Therefore, by (5.4) and Theorem 5.7,

$$Q_w b \in \mathcal{H}(b), \quad (w \in \mathbb{D}). \quad (5.5)$$

This gives an affirmative answer to our question.

5.5 The operator X_b

As a special case of Theorem 5.7, the space $\mathcal{H}(b)$ is invariant under the backward shift operator $S^* = T_{\bar{z}}$, and the restriction of S^* is a contraction. Since we have restricted the domain of S^* and moreover the Hilbert space structure of $\mathcal{H}(b)$ is not necessarily inherited from H^2 , the adjoint of S^* is not S . To avoid confusion, we use the notation

$$X_b = S^*|_{\mathcal{H}(b)}$$

and emphasize that

$$X_b \in \mathcal{L}(\mathcal{H}(b)).$$

If furthermore during a discussion b is fixed, we exploit X instead of X_b .

Theorem 5.9. *The adjoint of X_b is given by*

$$X_b^* f = S f - \langle f, S^* b \rangle_b b.$$

Proof. By the definition of reproducing kernel,

$$(X_b^* f)(z) = \langle X_b^* f, k_z^b \rangle_b, \quad (z \in \mathbb{D}),$$

and, by the defining property of the adjoint,

$$\langle X_b^* f, k_z^b \rangle_b = \langle f, X_b k_z^b \rangle_b.$$

According to Theorem 5.6, $k_z^b = (1 - \overline{b(z)}b) k_z$. Hence, by Corollary 2.9, and that $k_z(0) = 1$,

$$\begin{aligned} X_b k_z^b &= S^* ((1 - \overline{b(z)}b) k_z) \\ &= (1 - \overline{b(z)}b) S^* k_z + S^* (1 - \overline{b(z)}b) \\ &= (1 - \overline{b(z)}b) \bar{z} k_z - \overline{b(z)} S^* b \\ &= \bar{z} k_z^b - \overline{b(z)} S^* b. \end{aligned}$$

Hence

$$\begin{aligned}
(X_b^* f)(z) &= \langle f, \bar{z} k_z^b - \overline{b(z)} S^* b \rangle_b \\
&= z \langle f, k_z^b \rangle_b - b(z) \langle f, S^* b \rangle_b \\
&= z f(z) - \langle f, S^* b \rangle_b b(z) \\
&= (Sf)(z) - \langle f, S^* b \rangle_b b(z).
\end{aligned}$$

□

5.6 Integral representation of $\mathcal{H}(\bar{b})$

We denote by ρ the L^∞ function defined on \mathbb{T} by

$$\rho(\zeta) := 1 - |b(\zeta)|^2, \quad \zeta \in \mathbb{T}.$$

Since $\rho \in L^\infty$, we know from Section 3.7 that

$$\begin{array}{ccc}
K_\rho : L^2(\rho) & \longrightarrow & H^2 \\
g & \longmapsto & K(\rho g)
\end{array}$$

is a bounded operator whose norm is at most $\|\rho\|_\infty^{1/2}$ and $K_\rho(g) = P_+(\rho g)$. Moreover, Theorem 3.12 implies that

$$K_\rho^* = J_\rho, \tag{5.6}$$

where $J_\rho : H^2 \longrightarrow L^2(\rho)$ is the canonical injection and it follows from Corollary 3.13 that

$$K_\rho J_\rho = T_\rho. \tag{5.7}$$

The following result gives an integral representation for functions of $\mathcal{H}(\bar{b})$.

Theorem 5.10. *The operator K_ρ is an isometry from $H^2(\rho)$ onto $\mathcal{H}(\bar{b})$.*

Proof. Using (5.6), (5.7) and Theorem 3.8, we have

$$Id - T_{\bar{b}} T_b = T_{1-|b|^2} = T_\rho = K_\rho K_\rho^*. \tag{5.8}$$

Moreover, Theorem 2.16 implies that $K_\rho|_{H^2(\rho)}$ is injective. Hence by Lemma ?? and (5.8), K_ρ is an isometry from $H^2(\rho)$ onto $\mathcal{M}((Id - T_{\bar{b}} T_b)^{1/2}) = \mathcal{H}(\bar{b})$. □

It follows from the definition of K_ρ and Theorem 5.10 that for $f \in \mathcal{H}(\bar{b})$ there is a unique $g \in H^2(\rho)$ such that

$$f(z) = K_\rho(g)(z) = \int_{\mathbb{T}} \frac{\rho(\zeta) g(\zeta)}{1 - z\bar{\zeta}} dm(\zeta), \quad z \in \mathbb{D}. \tag{5.9}$$

Moreover, we have $\|f\|_{\bar{b}} = \|g\|_{L^2(\rho)}$. Using (5.9), we can write for $f \in \mathcal{H}(\bar{b})$ and $z \in \mathbb{D}$

$$f(z) = \langle g, k_z \rangle_{H^2(\rho)} = \langle K_\rho g, K_\rho k_z \rangle_{\bar{b}} = \langle f, K_\rho k_z \rangle_{\bar{b}}.$$

Thus the reproducing kernel of $\mathcal{H}(\bar{b})$ satisfies

$$k_z^{\bar{b}} = K_\rho k_z. \quad (5.10)$$

Note that relation (5.10) follows also from Theorem 4.5.

Lemma 5.11. *We have*

$$k_w^b = (Id - \bar{w}X_b^*)^{-1}k_0^b,$$

for every $w \in \mathbb{D}$.

5.7 Integral representation of $\mathcal{H}(b)$

let b be a function in the unit ball of H^∞ . Since the function

$$\Re \left(\frac{1 + b(z)}{1 - b(z)} \right) = \frac{1 - |b(z)|^2}{|1 - b(z)|^2}, \quad (z \in \mathbb{D})$$

is a positive and harmonic function on \mathbb{D} , we know from Herglotz theorem that there is a unique positive Borel measure μ on \mathbb{T} such that

$$\frac{1 - |b(z)|^2}{|1 - b(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(e^{i\theta}). \quad (5.11)$$

In other words, we have

$$\frac{1 + b(z)}{1 - b(z)} = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) + i\Im \frac{1 + b(0)}{1 - b(0)}. \quad (5.12)$$

Moreover, according to (2.1), for almost all $e^{i\theta}$ on \mathbb{T} (with respect to the Lebesgue measure m), we have

$$\lim_{r \rightarrow 1} \frac{1 - |b(re^{i\theta})|^2}{|1 - b(re^{i\theta})|^2} = \frac{1 - |b(e^{i\theta})|^2}{|1 - b(e^{i\theta})|^2} = \frac{d\mu_a}{dm}(e^{i\theta}), \quad (5.13)$$

where μ_a is the continuous part of the measure μ . In particular, if the function b is an inner function (that is $|b(e^{i\theta})| = 1$ a.e. on \mathbb{T}), then the measure μ is singular with respect to the Lebesgue measure.

Note that if we start with a positive Borel measure μ on \mathbb{T} , we can define b by the formula (5.12) (by setting for instance $b(0) = 0$) and b will be in the unit ball of H^∞ .

In this section, we will use this measure μ to give an integral representation for functions in $\mathcal{H}(b)$. We first begin by an easy computation.

Lemma 5.12. *Let $z, w \in \mathbb{D}$. Then*

$$\frac{1}{(1 - e^{i\theta}\bar{w})(1 - e^{-i\theta}z)} = \frac{1}{2(1 - \bar{w}z)} \left(\frac{e^{-i\theta} + \bar{w}}{e^{-i\theta} - \bar{w}} + \frac{e^{i\theta} + z}{e^{i\theta} - z} \right)$$

Proof. An elementary decomposition of rational functions gives us that

$$\frac{1}{(1 - e^{i\theta}\bar{w})(1 - e^{-i\theta}z)} = \frac{e^{i\theta}}{(1 - e^{i\theta}\bar{w})(e^{i\theta} - z)} = \frac{1}{1 - z\bar{w}} \left(\frac{1}{1 - e^{i\theta}\bar{w}} + \frac{z}{e^{i\theta} - z} \right). \quad (5.14)$$

Then notice that

$$\begin{aligned} \frac{1}{1 - e^{i\theta}\bar{w}} + \frac{z}{e^{i\theta} - z} &= \frac{e^{-i\theta}}{e^{-i\theta} - \bar{w}} + \frac{z}{e^{i\theta} - z} \\ &= \frac{e^{-i\theta}}{e^{-i\theta} - \bar{w}} - 1 + \frac{z}{e^{i\theta} - z} + 1 \\ &= \frac{\bar{w}}{e^{-i\theta} - \bar{w}} + \frac{e^{i\theta}}{e^{i\theta} - z}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{1 - e^{i\theta}\bar{w}} + \frac{z}{e^{i\theta} - z} &= \frac{1}{2} \left(\frac{e^{-i\theta}}{e^{-i\theta} - \bar{w}} + \frac{\bar{w}}{e^{-i\theta} - \bar{w}} + \frac{z}{e^{i\theta} - z} + \frac{e^{i\theta}}{e^{i\theta} - z} \right) \\ &= \frac{1}{2} \left(\frac{e^{-i\theta} + \bar{w}}{e^{-i\theta} - \bar{w}} + \frac{e^{i\theta} + z}{e^{i\theta} - z} \right), \end{aligned}$$

and we get the result by (5.14). \square

The key point of the integral representation for functions in $\mathcal{H}(b)$ is the following result:

Lemma 5.13. *Let $z, w \in \mathbb{D}$. Then*

$$\langle k_w, k_z \rangle_\mu = (1 - \overline{b(w)})^{-1} (1 - b(z))^{-1} k_w^b(z).$$

Proof. Using Lemma 5.12, we have

$$\begin{aligned} \langle k_w, k_z \rangle_\mu &= \int_{\mathbb{T}} \frac{1}{(1 - e^{i\theta}\bar{w})(1 - e^{-i\theta}z)} d\mu(e^{i\theta}) \\ &= \frac{1}{2(1 - \bar{w}z)} \left(\int_{\mathbb{T}} \frac{e^{-i\theta} + \bar{w}}{e^{-i\theta} - \bar{w}} d\mu(e^{i\theta}) + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) \right) \\ &= \frac{1}{2(1 - \bar{w}z)} \left(\int_{\mathbb{T}} \frac{e^{i\theta} + w}{e^{i\theta} - w} d\mu(e^{i\theta}) + \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) \right). \end{aligned}$$

Hence we get from (5.12) that

$$\langle k_w, k_z \rangle_\mu = \frac{1}{2(1 - \bar{w}z)} \left(\frac{1 + \overline{b(w)}}{1 - \overline{b(w)}} + \frac{1 + b(z)}{1 - b(z)} \right).$$

But applying once more Lemma 5.12 with $\theta = 0$, $b(w)$ and $b(z)$, we can write

$$\frac{1 + \overline{b(w)}}{1 - \overline{b(w)}} + \frac{1 + b(z)}{1 - b(z)} = \frac{2(1 - \overline{b(w)}b(z))}{(1 - \overline{b(w)})(1 - b(z))}.$$

Thus, using Theorem 5.6, we obtain

$$\begin{aligned} \langle k_w, k_z \rangle_\mu &= \frac{1 - \overline{b(w)}b(z)}{1 - \bar{w}z} \frac{1}{(1 - \overline{b(w)})(1 - b(z))} \\ &= (1 - \overline{b(w)})^{-1} (1 - b(z))^{-1} k_w^b(z). \end{aligned}$$

□

Let $q \in L^2(\mu)$ and define

$$V_b q(z) = (1 - b(z))K_\mu q(z), \quad z \in \mathbb{D}. \quad (5.15)$$

Then it follows from Lemma 2.15 that V_b is continuous as operator from $L^2(\mu)$ into $\mathcal{H}ol(\mathbb{D})$. In fact, we have a stronger result.

Theorem 5.14. *The map V_b is a partial isometry of $L^2(\mu)$ onto $\mathcal{H}(b)$ and $\ker V_b = (H^2(\mu))^\perp$.*

Proof. Let $z, w \in \mathbb{D}$. Using the definition of K_μ and Lemma 5.13, we have

$$\begin{aligned} (V_b k_w)(z) &= (1 - b(z))(K_\mu k_w)(z) = (1 - b(z)) \int_{\mathbb{T}} \frac{1}{(1 - \bar{w}e^{i\theta})(1 - e^{-i\theta}z)} d\mu(e^{i\theta}) \\ &= (1 - b(z)) \langle k_w, k_z \rangle_\mu \\ &= (1 - \overline{b(w)})^{-1} k_w^b(z), \end{aligned}$$

which means that

$$V_b k_w = (1 - \overline{b(w)})^{-1} k_w^b. \quad (5.16)$$

Hence it follows that

$$V_b (\mathcal{L}in(k_w : w \in \mathbb{D})) = \mathcal{L}in(k_w^b : w \in \mathbb{D}). \quad (5.17)$$

Moreover, using once more Lemma 5.13, we have

$$\langle k_w, k_z \rangle_\mu = \langle (1 - \overline{b(w)})^{-1} k_w^b, (1 - \overline{b(w)})^{-1} k_w^b \rangle_b = \langle V_b k_w, V_b k_z \rangle_b,$$

which implies that

$$\|V_b g\|_b = \|g\|_{L^2(\mu)}, \quad (5.18)$$

for all $g \in \mathcal{L}in(k_w : w \in \mathbb{D})$.

Now let $g \in H^2(\mu)$. Then, by Theorem 2.13, there exists a sequence $(g_n)_{n \geq 1}$, $g_n \in \mathcal{L}in(k_w : w \in \mathbb{D})$, $n \geq 1$, which converges to g in $H^2(\mu)$. Thus we get from (5.18) that $(V_b g_n)_{n \geq 1}$ is a Cauchy sequence in $\mathcal{H}(b)$ and then converges to a function $f \in \mathcal{H}(b)$. On the other hand, we have already noticed that according

to Lemma 2.15, V_b is continuous as operator from $L^2(\mu)$ into $\mathcal{H}ol(\mathbb{D})$. Therefore, $(V_b g_n)_{n \geq 1}$ converges to $V_b g$ in the topological space $\mathcal{H}ol(\mathbb{D})$. In particular,

$$\lim_{n \rightarrow +\infty} (V_b g_n)(z) = (V_b g)(z),$$

for every $z \in \mathbb{D}$ and since by Theorem 5.6, we also have

$$\lim_{n \rightarrow +\infty} (V_b g_n)(z) = f(z),$$

we obtain $V_b g = f$. Moreover, using once more (5.18), we have

$$\|V_b g\|_b = \|f\|_b = \lim_{n \rightarrow +\infty} \|V_b g_n\|_b = \lim_{n \rightarrow +\infty} \|g_n\|_{L^2(\mu)} = \|g\|_{L^2(\mu)}.$$

Finally we have proved that V_b is an isometry from $H^2(\mu)$ into $\mathcal{H}(b)$. But it follows from (5.17) that $\mathcal{L}in(k_w : w \in \mathbb{D}) \subset V_b H^2(\mu)$ and $V_b H^2(\mu)$ is closed in the norm of $\mathcal{H}(b)$. Hence Lemma 1.9 implies that $\mathcal{H}(b) \subset V_b H^2(\mu)$ and we obtain that $V_b H^2(\mu) = \mathcal{H}(b)$. Thus it remains to prove that $\ker V_b = (H^2(\mu))^\perp$. But this equation follows immediately from Theorem 2.16. \square

It follows from the definition of K_μ and Theorem 5.14 that given $f \in \mathcal{H}(b)$, there is a unique $g \in H^2(\mu)$ such that

$$f(z) = (V_b g)(z) = (1 - b(z)) \int_{\mathbb{T}} \frac{g(e^{i\theta})}{1 - z e^{-i\theta}} d\mu(e^{i\theta}).$$

Moreover, we have $\|f\|_b = \|g\|_{L^2(\mu)}$.

5.8 Multipliers of $\mathcal{H}(b)$

Reproducing kernel Hilbert spaces and the space of their multipliers were studied in Section 1.4. The multipliers for a reproducing kernel Hilbert space $H \subset H^2$ can be interpreted slightly differently. An analytic function $\varphi \in H^\infty$ is a multiplier of H , i.e. $\varphi \in \mathcal{M}(H)$, if and only if H is invariant under the Toeplitz operator T_φ . If φ is a multiplier, the multiplication operator M_φ which was introduced in Section 1.4 is exactly the restriction of T_φ to H .

The reproducing kernel Hilbert spaces which are in the center of our discussion are $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. According to Theorem 1.10, if $\varphi \in \mathcal{M}(\mathcal{H}(b))$, then

$$M_\varphi^* k_z^b = \overline{\varphi(z)} k_z^b, \quad (z \in \mathbb{D}).$$

Moreover, by Theorem 1.12, these are the only operators with such a property.

Theorem 5.15. *If $A \in \mathcal{L}(\mathcal{H}(b))$ is such that each kernel function k_z^b , $z \in \mathbb{D}$, is an eigenvector of A^* , then there is a $\varphi \in \mathcal{M}(\mathcal{H}(b))$ such that $A = M_\varphi$.*

The next result shows that $\mathcal{M}(\mathcal{H}(\bar{b}))$ contains $\mathcal{M}(\mathcal{H}(b))$.

Theorem 5.16. *Every multiplier of $\mathcal{H}(b)$ is also a multiplier of $\mathcal{H}(\bar{b})$.*

Proof. Let $\varphi \in \mathcal{M}(\mathcal{H}(b))$. Hence

$$T_\varphi \mathcal{H}(b) \subset \mathcal{H}(b).$$

The set inclusion

$$T_\varphi \mathcal{M}(b) \subset \mathcal{M}(b).$$

is trivial. Hence

$$T_\varphi(\mathcal{H}(b) \cap \mathcal{M}(b)) \subset (\mathcal{H}(b) \cap \mathcal{M}(b)).$$

But, by (5.2),

$$\mathcal{H}(b) \cap \mathcal{M}(b) = T_b \mathcal{H}(\bar{b}).$$

Thus we have

$$T_\varphi T_b \mathcal{H}(\bar{b}) \subset T_b \mathcal{H}(\bar{b}),$$

which is equivalent to

$$T_\varphi \mathcal{H}(\bar{b}) \subset \mathcal{H}(\bar{b}).$$

□

Theorem 5.17. *Let $\varphi \in \mathcal{M}(\mathcal{H}(b))$. Then, for each $w \in \mathbb{D}$,*

$$Q_w \varphi \in \mathcal{M}(\mathcal{H}(b)).$$

Proof. By Corollary 2.9, for each $f, g \in \mathcal{H}(b)$,

$$(Q_w \varphi) f = Q_w(\varphi f) - \varphi(w) Q_w f.$$

By assumption, $\varphi f \in \mathcal{H}(b)$. By Corollary 5.8, $Q_w(\varphi f)$ and $Q_w f$ stay in $\mathcal{H}(b)$. Hence $(Q_w \varphi) f \in \mathcal{H}(b)$. This means that $Q_w \varphi$ is a multiplier of $\mathcal{H}(b)$. □

In Section 3.8, we introduced the notion of Toeplitz operators on generalized Hardy spaces $H^2(\nu)$, $\nu \in \mathcal{M}(\mathbb{T})$. We use this notion to give a criterion for a function m in H^∞ to be a multiplier of $\mathcal{H}(b)$.

Theorem 5.18. *Let $m \in H^\infty$. Then m is a multiplier of $\mathcal{H}(b)$ if and only if $T_{\bar{m}}$ is bounded on $H^2(\mu)$.*

Here μ is the measure associated to b by (5.12).

Proof. According to Corollary 1.13, we have that m is a multiplier of $\mathcal{H}(b)$ if and only if the map

$$k_w^b \longmapsto \overline{\varphi(w)} k_w^b$$

extends to a continuous linear operator on $\mathcal{H}(b)$. Since by Theorem 5.14, the map V_b is a unitary operator from $H^2(\mu)$ onto $\mathcal{H}(b)$ which takes k_w to a constant multiple of k_w^b in $\mathcal{H}(b)$, we get that m is a multiplier of $\mathcal{H}(b)$ if and only if the map

$$k_w \longmapsto \overline{m(w)} k_w$$

extends to a continuous linear operator on $H^2(\mu)$. Now it remains to apply Theorem 3.16 to conclude the proof. □

Corollary 5.19. *Let $\chi(z) = z$. Then χ is a multiplier of $\mathcal{H}(b)$ if and only if*

$$\int_{\mathbb{T}} \log(1 - |b|^2) dm > -\infty.$$

Proof. According to Theorem 5.18, the function χ is a multiplier of $\mathcal{H}(b)$ if and only if $T_{\bar{\chi}}$ is bounded on $H^2(\mu)$. But if p is an analytic polynomial, we have

$$T_{\bar{\chi}}p(z) = \frac{p(z) - p(0)}{z} = \bar{z}(p(z) - p(0)), \quad z \in \mathbb{T}.$$

Since multiplication by \bar{z} is a unitary operator on $L^2(\mu)$, it follows that $T_{\bar{\chi}}$ is bounded on $H^2(\mu)$ if and only if the functional $p \mapsto p(0)$ of evaluation at 0 is bounded on $H^2(\mu)$. By Riesz Theorem, this is equivalent to the existence of $g \in H^2(\mu)$ such that

$$\begin{cases} \langle \chi_n, g \rangle_{H^2(\mu)} = 0, & \text{if } n \geq 1 \\ \langle \chi_0, g \rangle_{H^2(\mu)} = 1, \end{cases}$$

that is $\chi_0 \notin \text{span}(\chi_n : n \geq 1) = H_0^2(\mu)$. According to Theorem 8.4, this is equivalent to b be a non extreme point of the unit ball of H^∞ . □

Corollary 5.20. *Let μ be a positive Borel measure on \mathbb{T} and let m be an analytic polynomial. Assume that $H^2(\mu) = L^2(\mu)$. If $T_{\bar{m}}$ is bounded on $H^2(\mu)$, then m is constant.*

Proof. Let b be the function associated to μ by (5.12) (defining for instance $b(0) = 0$). Then according to Theorem 5.18, we see that m is a multiplier of $\mathcal{H}(b)$. We argue by absurd assuming that m is not constant and let $d \geq 1$ be the degree of m . By Theorem 5.17, we deduce that $S^{*d-1}m = Q_0^{d-1}m$ is also a multiplier of $\mathcal{H}(b)$. But $S^{*d-1}m$ is a polynomial of degree 1 and since $\mathcal{M}(\mathcal{H}(b))$ is an algebra with contains $\chi_0 = 1$, we get that χ_1 is a multiplier of $\mathcal{H}(b)$. Therefore it follows Corollary 5.19 that b is not an extreme point of the unit ball of H^∞ . Then Theorem 8.4 implies that $H^2(\mu) \neq L^2(\mu)$, which is absurd. Thus m is constant. □

Chapter 6

Applications of $\mathcal{H}(b)$ spaces

6.1 Comparison of measures

Theorem 6.1. *Let b be a point in the unit ball of H^∞ , let u be a non-constant inner function, and let μ (respectively ν) be the positive finite Borel measure on \mathbb{T} associated to b (respectively to u) by (5.12). The following assertions are equivalent:*

- (i) ν is absolutely continuous with respect to μ and $\frac{d\nu}{d\mu}$ is in $L^2(\mu)$.
- (ii) The function $\frac{1-b}{1-u}k_0^u$ is in $\mathcal{H}(b)$.

Proof. (i) \longrightarrow (ii): by Theorem 5.14, we know that $V_b\left(\frac{d\nu}{d\mu}\right) \in \mathcal{H}(b)$. But for $z \in \mathbb{D}$, we have

$$\begin{aligned} V_b\left(\frac{d\nu}{d\mu}\right) \in \mathcal{H}(b) &= (1-b(z))K_\mu\left(\frac{d\nu}{d\mu}\right)(z) \\ &= (1-b(z)) \int_{\mathbb{T}} \frac{1}{1-e^{-i\theta}z} d\nu(e^{i\theta}) \\ &= \frac{1-b(z)}{1-u(z)}(1-u(z))(K_\nu 1)(z) \\ &= \frac{1-b(z)}{1-u(z)}(V_u 1)(z). \end{aligned}$$

By (5.16), we have

$$(V_u 1)(z) = V_u k_0 = (1-\overline{u(0)})^{-1}k_0^u,$$

which implies that

$$V_b\left(\frac{d\nu}{d\mu}\right) = \frac{1-b}{1-u}(1-\overline{u(0)})^{-1}k_0^u,$$

and we get that $\frac{1-b}{1-u}k_0^u \in \mathcal{H}(b)$.

(ii) \longrightarrow (i): using once more Theorem 5.14, we know that there is $q \in L^2(\mu)$ such that

$$\frac{1-b}{1-u}k_0^u = V_b q,$$

which gives that

$$\frac{1-\overline{u(0)}u}{1-u} = K_\mu q.$$

Therefore for $z \in \mathbb{D}$, we can write

$$\begin{aligned} \frac{1-\overline{u(0)}u(z)}{1-u(z)} &= \int_{\mathbb{T}} \frac{q(e^{i\theta})}{1-e^{-i\theta}z} d\mu(e^{i\theta}) \\ &= \int_{\mathbb{T}} \frac{e^{i\theta}}{e^{i\theta}-z} q(e^{i\theta}) d\mu(e^{i\theta}) \\ &= \frac{1}{2} \int_{\mathbb{T}} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z} + 1 \right) q(e^{i\theta}) d\mu(e^{i\theta}) \\ &= \frac{1}{2} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} q(e^{i\theta}) d\mu(e^{i\theta}) + \frac{1}{2} \int_{\mathbb{T}} q(e^{i\theta}) d\mu(e^{i\theta}). \end{aligned} \tag{6.1}$$

Now if we take $z = 0$ in (6.1), we get

$$\frac{1-|u(0)|^2}{1-u(0)} = \int_{\mathbb{T}} q(e^{i\theta}) d\mu(e^{i\theta}),$$

and that implies

$$\frac{1-\overline{u(0)}u(z)}{1-u(z)} = \frac{1}{2} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} q(e^{i\theta}) d\mu(e^{i\theta}) + \frac{1}{2} \frac{1-|u(0)|^2}{1-u(0)}.$$

Since

$$\frac{1-\overline{u(0)}u(z)}{1-u(z)} = \overline{u(0)} + \frac{1-\overline{u(0)}}{1-u(z)},$$

we get

$$\frac{1-\overline{u(0)}}{1-u(z)} = \frac{1}{2} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} q(e^{i\theta}) d\mu(e^{i\theta}) + \frac{1}{2} \frac{1-|u(0)|^2}{1-u(0)} - \overline{u(0)},$$

whence

$$\frac{1}{1-u(z)} = \frac{1}{2} (1-\overline{u(0)})^{-1} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} q(e^{i\theta}) d\mu(e^{i\theta}) + \frac{1}{2} \frac{1-|u(0)|^2}{|1-u(0)|^2} - \frac{\overline{u(0)}}{1-\overline{u(0)}}. \tag{6.2}$$

Using straightforward computations, it is easy to check that

$$\frac{1}{2} \frac{1-|u(0)|^2}{|1-u(0)|^2} - \frac{\overline{u(0)}}{1-\overline{u(0)}} = \frac{1}{2} \left(1 + i\Im \left(\frac{1+u(0)}{1-u(0)} \right) \right).$$

Therefore we deduce from (6.2) that

$$\frac{1+u(z)}{1-u(z)} = \frac{2}{1-u(z)} - 1 = (1-\overline{u(0)})^{-1} \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} q(e^{i\theta}) d\mu(e^{i\theta}) + i\Im \left(\frac{1+u(0)}{1-u(0)} \right).$$

By definition of ν , we have

$$\frac{1+u(z)}{1-u(z)} = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(e^{i\theta}) + i\Im \left(\frac{1+u(0)}{1-u(0)} \right).$$

and thus we obtain

$$(1-\overline{u(0)})^{-1} \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} q(e^{i\theta}) d\mu(e^{i\theta}) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\nu(e^{i\theta}).$$

Taking the conjugate, we rewrite this equality as

$$(1-u(0))^{-1} \int_{\mathbb{T}} \frac{e^{-i\theta} + z}{e^{-i\theta} - z} \overline{q(e^{i\theta})} d\mu(e^{i\theta}) = \int_{\mathbb{T}} \frac{e^{-i\theta} + z}{e^{-i\theta} - z} d\nu(e^{i\theta}).$$

It thus follows from Corollary 2.11 that the measure $(1-u(0))^{-1} \overline{q} d\mu - d\nu$ is absolutely continuous with respect to m . That means that there is $h \in L^1$ such that

$$(1-u(0))^{-1} \overline{q} d\mu - d\nu = h dm.$$

Since ν is singular with respect to m , we get $(1-u(0))^{-1} \overline{q} d\mu_s = d\nu$, where μ_s is the singular part of the measure μ with respect to m . In other words, ν is absolutely continuous with respect to μ_s and

$$\frac{d\nu}{d\mu_s} = (1-u(0))^{-1} \overline{q} \in L^2(\mu).$$

Now it is easy to see that implies that ν is absolutely continuous with respect to μ and $\frac{d\nu}{d\mu} \in L^2(\mu)$. □

Corollary 6.2. *Let b be a point in the unit ball of H^∞ and let $z_0 \in \mathbb{T}$. The following assertions are equivalent:*

- (i) $\mu(\{z_0\}) > 0$.
- (ii) $\frac{b(z) - 1}{z - z_0} \in \mathcal{H}(b)$.

Proof. First let us prove that $\mu(\{z_0\}) > 0$ if and only if $\delta_{z_0} \ll \mu$ and $\frac{\delta_{z_0}}{d\mu} \in L^2(\mu)$, where δ_{z_0} is the Dirac measure associated to z_0 .

Assume that $\mu(\{z_0\}) > 0$ and let E be a measurable subset of \mathbb{T} such that $\mu(\mathbb{T}) = 0$. It follows that $z_0 \notin E$ and thus $\delta_{z_0}(E) = 0$, which proves that

$\delta_{z_0} \ll \mu$. Let $h := \frac{\delta_{z_0}}{d\mu} \in L^1(\mu)$. We have to check that indeed $h \in L^2(\mu)$. Since $h \in L^1(\mu)$, we necessarily have $h(z_0) < +\infty$ and thus

$$\int_{\mathbb{T}} |h|^2 d\mu = \int_{\mathbb{T}} h \delta_{z_0} = h(z_0) < +\infty,$$

which implies that $h \in L^2(\mu)$. Reciprocally assume that $\delta_{z_0} \ll \mu$ and $h := \frac{\delta_{z_0}}{d\mu} \in L^2(\mu)$. Assume that $\mu(\{z_0\}) = 0$. Then since δ_{z_0} is absolutely continuous with respect to μ , we necessarily have $\delta_{z_0}(\{z_0\}) = 0$, which is absurd. Therefore $\mu(\{z_0\}) > 0$ and we get the desired equivalence.

Now let u be the inner function defined by $u(z) = \frac{z}{z_0}$, $z \in \mathbb{D}$. Then for $z \in \mathbb{D}$, we have

$$\frac{1+u(z)}{1-u(z)} = \frac{1+\frac{z}{z_0}}{1-\frac{z}{z_0}} = \frac{z_0+z}{z_0-z} = \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\delta_{z_0}(e^{i\theta}).$$

In other words, u is the inner function associated to δ_{z_0} by (5.12). We can apply Theorem 6.1 and we get that (i) is equivalent to the fact that

$$\frac{1-b}{1-u} k_0^u \in \mathcal{H}(b).$$

But

$$\frac{1-b(z)}{1-u(z)} k_0^u(z) = \frac{1-b(z)}{1-\frac{z}{z_0}} = z_0 \frac{b(z)-b(z_0)}{z-z_0},$$

and the desired equivalence is proved. □

6.2 Angular derivatives

Let ζ_0 be a point on the unit circle \mathbb{T} . A region of the form

$$S_C(\zeta_0) = \{z \in \mathbb{D} : |z - \zeta_0| \leq C(1 - |z|)\}$$

is called a Stoltz's domain anchored at the point ζ_0 . Since $|z - \zeta_0| \geq (1 - |z|)$, we need to assume that $C > 1$. Near the point ζ_0 , the boundaries of $S_C(\zeta_0)$ are tangent to a triangular shaped region with angle

$$2\alpha = 2 \arccos(1/C).$$

Let f be a function on the open unit disc \mathbb{D} . We say that f has the nontangential limit L at ζ_0 provided that

$$\lim_{\substack{z \rightarrow \zeta_0 \\ z \in S_C(\zeta_0)}} f(z) = L$$

for every fixed value of the parameter C . If so, we define $f(z_0) = L$ and write

$$\lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} f(z) = f(z_0).$$

We will use the following simple geometrical property of the Stoltz domains. Let $z \in S_C(\zeta_0)$ and consider the circle

$$\Gamma_z = \left\{ w : |w - z| = \frac{1 - |z|}{2} \right\}.$$

Then, for any $w \in \Gamma_z$, we have

$$1 - |z| \leq 2(1 - |w|),$$

and thus

$$\begin{aligned} |w - \zeta_0| &\leq |w - z| + |z - \zeta_0| \\ &\leq \frac{1 - |z|}{2} + C(1 - |z|) \\ &\leq (2C + 1)(1 - |w|). \end{aligned} \tag{6.3}$$

In other words,

$$\Gamma_z \subset S_{2C+1}(\zeta_0). \tag{6.4}$$

Theorem 6.3. *Let f be analytic on the open unit disc \mathbb{D} , and let $\zeta_0 \in \mathbb{T}$. Then the following are equivalent.*

- (i) *The function f has a nontangential limit at ζ_0 , and so does the quotient $(f(z) - f(\zeta_0))/(z - \zeta_0)$.*
- (ii) *There is a complex number λ such that the quotient $(f(z) - \lambda)/(z - \zeta_0)$ has a nontangential limit at ζ_0 .*
- (iii) *The function f' has a nontangential limit at ζ_0 .*

Moreover, under the preceding conditions,

$$\lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} \frac{f(z) - f(\zeta_0)}{z - \zeta_0} = \lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} f'(z).$$

Proof. (i) \implies (ii) Trivial.

(ii) \implies (iii) Let

$$\lim_{z \underset{\triangleleft}{\rightarrow} \zeta_0} \frac{f(z) - \lambda}{z - \zeta_0} = L$$

and define

$$g(z) = \frac{f(z) - \lambda}{z - \zeta_0} - L, \quad (z \in \mathbb{D}).$$

Fix a Stoltz's domain $S_C(\zeta_0)$. Then, given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, C)$ such that

$$|g(z)| < \varepsilon \quad (6.5)$$

provided that $z \in S_{2C+1}$ and $|z - \zeta_0| < \delta$.

Let $z \in S_C(\zeta_0)$, and let Γ_z denote the circle of radius $(1 - |z|)/2$ and center z . Hence, by Cauchy's integral formula,

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(w)}{(w-z)^2} dw \\ &= L + \frac{1}{2\pi i} \int_{\Gamma_z} \frac{g(w)(w-\zeta_0)}{(w-z)^2} dw. \end{aligned}$$

If we further assume that

$$1 - |z| < \delta' = \frac{2\delta}{1 + 2C}$$

then, by (6.3),

$$|w - \zeta_0| < \delta$$

and thus, by (6.4) and (6.5), we have

$$\left| \frac{g(w)(w-\zeta_0)}{(w-z)^2} \right| \leq \frac{\varepsilon(1+2C)}{2(1-|z|)}$$

for each $w \in \Gamma_z$. Since the length of Γ_z is $\pi(1 - |z|)$, we obtain the estimate

$$|f'(z) - L| \leq \frac{\varepsilon(1+2C)}{4}$$

for each $z \in S_C(\zeta_0)$ with $1 - |z| < 2\delta/(1 + 2C)$. This means that

$$\lim_{z \xrightarrow[\triangleleft]{\zeta_0} \zeta_0} f'(z) = L$$

(iii) \implies (i) The assumption enables us to define

$$\lambda = f(0) + \int_{[0, \zeta_0]} f'(w) dw.$$

The integral is well defined and, by Cauchy's theorem,

$$\lambda = f(z) + \int_{[z, \zeta_0]} f'(w) dw$$

for any $z \in \mathbb{D}$. On the Stoltz domain $S_C(\zeta_0)$,

$$\left| \int_{[z, \zeta_0]} f'(w) dw \right| \leq \left(\sup_{w \in S_C(\zeta_0)} |f'(w)| \right) |z - \zeta_0|,$$

and thus

$$\lim_{\substack{z \rightarrow \zeta_0 \\ \triangleleft}} f(z) = \lambda.$$

As usual, write $\lambda = f(\zeta_0)$. Hence

$$\begin{aligned} \frac{f(z) - f(\zeta_0)}{z - \zeta_0} &= -\frac{1}{z - \zeta_0} \int_{[z, \zeta_0]} f'(w) dw \\ &= f'(\zeta_0) + \frac{1}{z - \zeta_0} \int_{[z, \zeta_0]} (f'(\zeta_0) - f'(w)) dw. \end{aligned}$$

In each Stoltz's domain, the last integral tends to zero as $z \rightarrow \zeta_0$. Hence

$$\lim_{\substack{z \rightarrow \zeta_0 \\ \triangleleft}} \frac{f(z) - f(\zeta_0)}{z - \zeta_0} = f'(\zeta_0).$$

□

6.3 Carathéodory's theorem

In Section 6.2, we studied the angular derivative of analytic functions on the open unit disc \mathbb{D} . In this section we consider the smaller class of analytic functions $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$, i.e. the elements of the unit sphere of $H^\infty(\mathbb{D})$. We say that such a function has an angular derivative in the sense of Carathéodory at $\zeta_0 \in \mathbb{T}$ if it has an angular derivative at ζ_0 and moreover $|f(\zeta_0)| = 1$. By the maximum principle, $f(z) \in \mathbb{T}$, for some $z \in \mathbb{D}$, happens only if f is a constant function of modulus one. Hence from now on, we consider function with values inside \mathbb{D} .

Theorem 6.4. *Let $b : \mathbb{D} \rightarrow \mathbb{D}$ be analytic, and let $\zeta \in \mathbb{T}$. Then the following are equivalent.*

(i)

$$c = \liminf_{z \rightarrow \zeta} \frac{1 - |b(z)|}{1 - |z|} < \infty.$$

(ii) *There is $\lambda \in \mathbb{T}$ such that*

$$\frac{b(z) - \lambda}{z - \zeta} \in \mathcal{H}(b).$$

(iii) *For each function $f \in \mathcal{H}(b)$,*

$$\lim_{\substack{z \rightarrow \zeta \\ \triangleleft}} f(z)$$

exists.

(iv) b has angular derivative in the sense of Carathéodory at ζ .

Moreover, under the preceding conditions,

(a)

$$c = \lim_{z \rightarrow \zeta} \frac{1 - |b(z)|}{1 - |z|} > 0,$$

(b) $|b(\zeta)| = 1$ and

$$b'(\zeta) = \frac{cb(\zeta)}{\zeta},$$

(c) for each $f \in \mathcal{H}(b)$,

$$f(\zeta) = \langle f, k_\zeta^b \rangle_b,$$

where

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)}b(z)}{1 - \bar{\zeta}z} \in \mathcal{H}(b),$$

(d)

$$\lim_{z \rightarrow \zeta} \|k_z^b - k_\zeta^b\|_b = 0,$$

(e) and finally

$$|b'(\zeta)| = k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2 = c.$$

Proof. (i) \implies (ii) If $c < \infty$, then there is a sequence $(z_n)_{n \geq 1} \subset \mathbb{D}$ with $\lim_{n \rightarrow \infty} z_n = \zeta$ such that

$$c = \lim_{n \rightarrow \infty} \frac{1 - |b(z_n)|}{1 - |z_n|} < \infty.$$

Hence we necessarily have $\lim_{n \rightarrow \infty} |b(z_n)| = 1$. Therefore, we can write

$$c = \lim_{n \rightarrow \infty} \frac{1 - |b(z_n)|^2}{1 - |z_n|^2}.$$

In the light of Theorem 5.6, this means that

$$c = \lim_{n \rightarrow \infty} \|k_{z_n}^b\|_b^2.$$

By Lemma ??, $(k_{z_n}^b)_{n \geq 1}$ has a weakly convergent subsequence. Since $(b(z_n))_{n \geq 1}$ is bounded, it also has a convergent subsequence in the complex plane. Hence, replacing $(z_n)_{n \geq 1}$ by a subsequence if needed, we assume that $b(z_n) \rightarrow \lambda \in \overline{\mathbb{D}}$

and that $k_{z_n}^b \xrightarrow{w} k \in \mathcal{H}(b)$. Therefore, for each $z \in \mathbb{D}$,

$$\begin{aligned} k(z) &= \langle k, k_z^b \rangle_b \\ &= \lim_{n \rightarrow \infty} \langle k_{z_n}^b, k_z^b \rangle_b \\ &= \lim_{n \rightarrow \infty} k_{z_n}^b(z) \\ &= \lim_{n \rightarrow \infty} \frac{1 - \overline{b(z_n)} b(z)}{1 - \bar{z}_n z} \\ &= \frac{1 - \bar{\lambda} b(z)}{1 - \bar{\zeta} z}. \end{aligned}$$

Since $k \in H^2(\mathbb{D})$ and $1/(1 - \bar{\zeta} z) \notin H^2(\mathbb{D})$, we thus have $|\lambda| = 1$ and

$$\lambda \bar{\zeta} k(z) = \frac{b(z) - \lambda}{z - \zeta} \in \mathcal{H}(b).$$

Clearly $k \neq 0$, and, by (??), the condition $k_{z_n}^b \xrightarrow{w} k$ implies

$$0 < \|k\|_b^2 \leq \liminf_{n \rightarrow \infty} \|k_{z_n}^b\|_b^2 = c. \quad (6.6)$$

(ii) \implies (iii) By assumption $k \in \mathcal{H}(b)$. Hence

$$b(z) = \lambda + \lambda \bar{\zeta} (z - \zeta) k(z),$$

which, by (??), implies

$$\lim_{z \rightarrow \zeta} b(z) = \lambda.$$

Let us write $b(\zeta)$ for λ and k_ζ^b for k , i.e.,

$$k_\zeta^b(z) = \frac{1 - \overline{b(\zeta)} b(z)}{1 - \bar{\zeta} z}.$$

Since $k_\zeta^b \in \mathcal{H}(b)$, for each $z \in \mathbb{D}$, we have

$$k_\zeta^b(z) = \langle k_\zeta^b, k_z^b \rangle_b.$$

Thus, by Cauchy-Schwarz inequality,

$$|k_\zeta^b(z)| \leq \|k_\zeta^b\|_b \|k_z^b\|_b.$$

Since

$$|k_\zeta^b(z)| = \frac{|1 - \overline{b(\zeta)} b(z)|}{|1 - \bar{\zeta} z|} \geq \frac{1 - |b(z)|}{|z - \zeta|} = \frac{(1 - |z|^2) \|k_z^b\|_b^2}{(1 + |b(z)|)|z - \zeta|},$$

the preceding inequality implies

$$\|k_z^b\|_b \leq \|k_\zeta^b\|_b \frac{1 + |b(z)|}{1 + |z|} \frac{|z - \zeta|}{1 - |z|}.$$

Hence, for each Stoltz's domain $S_C(\zeta)$,

$$\|k_z^b\|_b \leq 2C \|k_\zeta^b\|_b, \quad (z \in S_C(\zeta)). \quad (6.7)$$

This inequality means that $\|k_z^b\|_b$ stays bounded as z tends nontangentially to ζ . For each fixed $w \in \mathbb{D}$,

$$\lim_{z \xrightarrow{\triangleleft} \zeta} k_z^b(w) = \lim_{z \xrightarrow{\triangleleft} \zeta} \frac{1 - \overline{b(z)} b(w)}{1 - \overline{z} w} = \frac{1 - \overline{b(\zeta)} b(w)}{1 - \overline{\zeta} w} = k_\zeta^b(w).$$

We rewrite this as

$$\lim_{z \xrightarrow{\triangleleft} \zeta} \langle k_z^b, k_w^b \rangle_b = \langle k_\zeta^b, k_w^b \rangle_b.$$

Therefore,

$$\lim_{z \xrightarrow{\triangleleft} \zeta} \langle f, k_z^b \rangle_b = \langle f, k_\zeta^b \rangle_b, \quad (6.8)$$

where $f \in \mathcal{H}(b)$ is any element of the form $f = \alpha_1 k_{w_1}^b + \cdots + \alpha_n k_{w_n}^b$. But such elements are dense in $\mathcal{H}(b)$, and thus, by (6.7), the last identity holds for all $f \in \mathcal{H}(b)$. At the same time, (6.8) shows

$$f(\zeta) = \lim_{z \xrightarrow{\triangleleft} \zeta} f(z) = \langle f, k_\zeta^b \rangle_b, \quad (f \in \mathcal{H}(b)).$$

In particular, with $f = k_\zeta^b$, we obtain $k_\zeta^b(\zeta) = \|k_\zeta^b\|_b^2$. The relation (6.8) also implies that $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ .

(iii) \implies (i) Fix any Stoltz's domain $S_C(\zeta)$. Consider k_z^b as an element of the dual space. Then the relation $f(z) = \langle f, k_z^b \rangle_b$ along with our assumption imply that

$$\sup_{z \in S_C(\zeta)} |\langle f, k_z^b \rangle_b| = C(f) < \infty.$$

Thus, by the uniform boundedness principle,

$$C' = \sup_{z \in S_C(\zeta)} \|k_z^b\|_b < \infty.$$

Take $z_n = (1 - 1/n)\zeta$, $n \geq 1$. Then we necessarily have $\lim_{n \rightarrow \infty} |b(z_n)| = 1$ and

$$c \leq \liminf_{n \rightarrow \infty} \frac{1 - |b(z_n)|^2}{1 - |z_n|^2} = \liminf_{n \rightarrow \infty} \|k_{z_n}^b\|_b^2 \leq C'.$$

(i), (ii), (iii) \implies (iv) Since $k_\zeta^b \in \mathcal{H}(b)$ we have

$$\frac{b(z) - b(\zeta)}{z - \zeta} = k_\zeta^b(z) b(\zeta) / \zeta = \langle k_\zeta^b, k_z^b \rangle_b b(\zeta) / \zeta, \quad (z \in \mathbb{D}).$$

On the other hand, $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ . Hence

$$\lim_{z \xrightarrow{\triangleleft} \zeta} \frac{b(z) - b(\zeta)}{z - \zeta} = \|k_\zeta^b\|_b^2 b(\zeta) / \zeta,$$

which, by Theorem 6.3, means

$$b'(\zeta) = \|k_\zeta^b\|_b^2 b(\zeta)/\zeta \quad (6.9)$$

By (6.6), $c \geq \|k_\zeta^b\|_b^2$. To show the reverse inequality, we prove that $\|k_z^b\|_b \rightarrow \|k_\zeta^b\|_b$ as z tends nontangentially to ζ . This result has three consequences. Firstly, $c \leq \|k_\zeta^b\|_b^2$, and thus equality indeed holds. Secondly, since $k_z^b \xrightarrow{w} k_\zeta^b$ as z tends nontangentially to ζ , we have $\|k_z^b - k_\zeta^b\|_b \rightarrow 0$. Thirdly,

$$\lim_{z \rightarrow \zeta} \frac{1 - |b(z)|}{1 - |z|} = \lim_{z \rightarrow \zeta} \frac{1 - |b(z)|^2}{1 - |z|^2} = \lim_{z \rightarrow \zeta} \|k_z^b\|_b^2 = \|k_\zeta^b\|_b^2 = c.$$

To prove that $\|k_z^b\|_b \rightarrow \|k_\zeta^b\|_b$ as z tends nontangentially to ζ , let

$$g(z) = \frac{b(z) - b(\zeta)}{z - \zeta} - b'(\zeta), \quad (z \in \mathbb{D}).$$

Thus

$$b(z) = b(\zeta) + b'(\zeta)(z - \zeta) + (z - \zeta)g(z), \quad (z \in \mathbb{D}),$$

and, by (6.9),

$$|b(z)|^2 = 1 - 2\|k_\zeta^b\|_b^2 \Re(1 - \bar{\zeta}z) + h(z), \quad (z \in \mathbb{D}),$$

where

$$h(z) = (|b'(\zeta)|^2 + |g(z)|^2) |z - \zeta|^2 + 2\Re\left(g(z)(z - \zeta) \overline{(b(\zeta) + b'(\zeta)(z - \zeta))}\right).$$

The only important fact about h is that

$$\lim_{z \rightarrow \zeta} \frac{h(z)}{1 - |z|} = 0.$$

It is also elementary to verify that

$$\lim_{z \rightarrow \zeta} \frac{\Re(1 - \bar{\zeta}z)}{1 - |z|^2} = \frac{1}{2}.$$

Therefore,

$$\lim_{z \rightarrow \zeta} \|k_z^b\|_b^2 = \lim_{z \rightarrow \zeta} \frac{1 - |b(z)|^2}{1 - |z|^2} = \|k_\zeta^b\|_b^2.$$

(iv) \implies (i) If b has angular derivative in the sense of Carathéodory at ζ , then the inequality

$$\frac{1 - |b(r\zeta)|}{1 - r} \leq \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right|$$

implies that

$$\liminf_{z \rightarrow \zeta} \frac{1 - |b(z)|}{1 - |z|} \leq \lim_{r \rightarrow 1} \left| \frac{b(r\zeta) - b(\zeta)}{r\zeta - \zeta} \right| = |b'(\zeta)| < \infty.$$

□

Corollary 6.5 (Julia). *Let $b : \mathbb{D} \rightarrow \mathbb{D}$ be analytic, and let $\zeta \in \mathbb{T}$. Suppose that b has angular derivative in the sense of Carathéodory at ζ . Then*

$$\frac{|b(z) - b(\zeta)|^2}{1 - |b(z)|^2} \leq |b'(\zeta)| \frac{|z - \zeta|^2}{1 - |z|^2}.$$

Moreover, the equality holds if and only if b is a Möbius transformation.

Proof. By Cauchy–Schwarz inequality,

$$|\langle k_\zeta^b, k_z^b \rangle_b|^2 \leq \|k_\zeta^b\|_b^2 \|k_z^b\|_b^2.$$

But, by Theorem 6.4, this is exactly the required inequality. The inequality can be rewritten as

$$\Re \left(\frac{z + \zeta}{z - \zeta} - c \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} \right) \geq 0,$$

where $c = |b'(\zeta)|$. A positive harmonic function either identically vanishes or it has no zeros. Hence, if equality holds even at one point inside \mathbb{D} , then

$$\Re \left(\frac{z + \zeta}{z - \zeta} - c \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} \right) = 0, \quad (z \in \mathbb{D}).$$

Therefore, we have

$$\frac{z + \zeta}{z - \zeta} - c \frac{b(z) + b(\zeta)}{b(z) - b(\zeta)} = i\gamma, \quad (z \in \mathbb{D}),$$

where $\gamma \in \mathbb{R}$. This identity shows that b is a Möbius transformation.

That equality holds for a Möbius transformation is easy to verify directly. \square

Julia's inequality has a geometrical interpretation. The set

$$\left\{ z \in \mathbb{C} : \frac{|z - \zeta|^2}{1 - |z|^2} \leq r \right\}$$

is a disc of radius $r/(1+r)$ in \mathbb{D} whose center is on the ray $[0, \zeta]$ and is tangent to the unit circle \mathbb{T} at the point ζ . Julia's inequality says that this disc is mapped into a similar disc of radius $rc/(1+rc)$ which is tangent to \mathbb{T} at the point $b(\zeta)$.

Chapter 7

The nonextreme case of $\mathcal{H}(b)$ spaces

In this chapter, we will study the specific properties of the space $\mathcal{H}(b)$ when b is not an extreme point of the unit ball of H^∞ . Recall that b is not an extreme point of the unit ball of H^∞ if and only if $\log(1 - |b|^2)$ is integrable. In this case, we let a denote (throughout this chapter) the outer function that has modulus $(1 - |b|^2)^{1/2}$ on \mathbb{T} and that is positive at the origin. More precisely, we have

$$a(z) = \exp\left(\frac{1}{2} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(1 - |b(\zeta)|^2) dm(\zeta)\right), \quad |z| < 1, \quad (7.1)$$

and $a \in H^\infty$, $\|a\|_\infty \leq 1$.

7.1 First properties of $\mathcal{H}(b)$

Theorem 7.1. *Let b be a non extreme point of the unit ball of H^∞ . Then we have*

$$\mathcal{M}(a) \hookrightarrow \mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b),$$

where both inclusions are contractive.

Proof. For the first inclusion, note that for all $f \in H^2$, we have

$$\|T_{\bar{a}}f\|_2^2 = \|P_+(\bar{a}f)\|_2^2 \leq \|\bar{a}f\|_2^2 = \|af\|_2^2 = \|T_a f\|_2^2,$$

which implies, using (3.2) and (??), that

$$T_a T_a^* = T_{\bar{a}}^* T_{\bar{a}} \leq T_a^* T_a = T_{\bar{a}} T_{\bar{a}}^*.$$

By Theorem 4.2, we get that $\mathcal{M}(T_a) \hookrightarrow \mathcal{M}(T_{\bar{a}})$, which exactly means that $\mathcal{M}(a)$ is contractively included in $\mathcal{M}(\bar{a})$.

Using Theorem 3.8, we see that

$$T_{\bar{a}}T_a = T_{|a|^2} = T_{1-|b|^2} = I - T_{\bar{b}}T_b.$$

Hence Corollary 4.3 implies that $\mathcal{M}(\bar{a}) = \mathcal{M}(T_{\bar{a}}) = \mathcal{M}((Id - T_{\bar{b}}T_b)^{1/2}) = \mathcal{H}(\bar{b})$.

The third inclusion is already proved. \square

Let $h \in \mathcal{H}(b)$. Then, using ?? and Theorem 7.1, we know that $T_{\bar{b}}h \in \mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. Therefore Lemma ?? implies that there is a unique $h^+ \in H^2$ such that

$$T_{\bar{b}}h = T_{\bar{a}}h^+. \quad (7.2)$$

Lemma 7.2. *Let $h_1, h_2 \in \mathcal{H}(b)$ and let h_1^+, h_2^+ be the functions of H^2 associated to h_1, h_2 by (7.2). Then we have*

$$\langle h_1, h_2 \rangle_b = \langle h_1, h_2 \rangle_2 + \langle h_1^+, h_2^+ \rangle_2$$

Proof. Using (??), we can write

$$\langle h_1, h_2 \rangle_b = \langle h_1, h_2 \rangle_2 + \langle T_{\bar{b}}h_1, T_{\bar{b}}h_2 \rangle_{\bar{b}} = \langle h_1, h_2 \rangle_2 + \langle T_{\bar{a}}h_1^+, T_{\bar{a}}h_2^+ \rangle_{\bar{b}}.$$

Since $\mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$ (as Hilbert spaces), we have

$$\langle T_{\bar{a}}h_1^+, T_{\bar{a}}h_2^+ \rangle_{\bar{b}} = \langle T_{\bar{a}}h_1^+, T_{\bar{a}}h_2^+ \rangle_{\mathcal{M}(\bar{a})}.$$

Now, according to Lemma ??, it follows that

$$\langle T_{\bar{a}}h_1^+, T_{\bar{a}}h_2^+ \rangle_{\mathcal{M}(\bar{a})} = \langle h_1^+, h_2^+ \rangle_2,$$

which implies that

$$\langle h_1, h_2 \rangle_b = \langle h_1, h_2 \rangle_2 + \langle h_1^+, h_2^+ \rangle_2.$$

\square

The above lemma is very useful to compute the norm of elements in $\mathcal{H}(b)$.

Corollary 7.3. *Let b be a non extreme point of the unit ball of H^∞ . Then $b \in \mathcal{H}(b)$ and we have*

$$\|S^*b\|_b^2 = 1 - |b(0)|^2 - |a(0)|^2 \quad \text{and} \quad \|b\|_b^2 = |a(0)|^{-2} - 1.$$

Proof. We know from ?? that $S^*b \in \mathcal{H}(b)$. We will compute $(S^*b)^+$. Using Theorem 3.8, we have

$$T_{\bar{b}}S^*b = T_{\bar{b}}T_{\bar{z}}b = T_{\bar{z}}T_{\bar{b}}b = S^*P_+|b|^2.$$

Since $|a| = (1 - |b|^2)^{1/2}$, we get

$$T_{\bar{b}}S^*b = S^*P_+(1 - |a|^2) = -S^*P_+(|a|^2) = -S^*T_{\bar{a}}a,$$

and using once more Theorem 3.8, it follows that

$$T_{\bar{b}}S^*b = -T_{\bar{a}}S^*a.$$

Therefore $(S^*b)^+ = -S^*a$ and according to Lemma 7.2, we have

$$\|S^*b\|_b^2 = \|S^*b\|_2^2 + \|S^*a\|_2^2.$$

Now remark that for $g \in H^2$, we have $g = S^*g + g(0)$, with $S^*g \perp g(0)$. Thus

$$\|g\|_2^2 = \|S^*g\|_2^2 + |g(0)|^2,$$

and we get

$$\|S^*b\|_b^2 = \|b\|_2^2 + \|a\|_2^2 - |b(0)|^2 - |a(0)|^2.$$

But

$$\|a\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |a(e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - |b(e^{i\theta})|^2) d\theta = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} |b(e^{i\theta})|^2 d\theta = 1 - \|b\|_2^2,$$

and it follows that

$$\|S^*b\|_b^2 = 1 - |b(0)|^2 - |a(0)|^2.$$

It remains to prove that $b \in \mathcal{H}(b)$ and to compute $\|b\|_b$. Recall that according to ??? and Theorem 7.1, we have $b \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}b \in \mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. But

$$T_{\bar{b}}b = P_+|b|^2 = P_+(1 - |a|^2) = 1 - T_{\bar{a}}a,$$

and we can write $1 = P_+(\bar{a}/\overline{a(0)}) = T_{\bar{a}}(1/\overline{a(0)})$. Therefore we get

$$T_{\bar{b}}b = T_{\bar{a}}\left(\frac{1}{\overline{a(0)}} - a\right) \in \mathcal{M}(\bar{a}),$$

which proves that $b \in \mathcal{H}(b)$. Moreover the last equation says also that $b^+ = 1/\overline{a(0)} - a$. Lemma 7.2 implies that

$$\begin{aligned} \|b\|_b^2 &= \|b\|_2^2 + \|b^+\|_2^2 \\ &= \|b\|_2^2 + \|a\|_2^2 + |a(0)|^{-2} - 2\Re\langle a, \overline{a(0)}^{-1} \rangle_2. \end{aligned}$$

Using the fact that $\|a\|_2^2 + \|b\|_2^2 = 1$ and that $\langle a, 1 \rangle_2 = a(0)$, we conclude that

$$\|b\|_b^2 = |a(0)|^{-2} - 1.$$

□

Lemma 7.4. *Let b be a non extreme point of the unit ball of H^∞ , let $h \in \mathcal{H}(b)$ and let $\varphi \in H^\infty$. Then*

$$(T_{\bar{\varphi}}h)^+ = T_{\bar{\varphi}}h^+.$$

Proof. We know from ?? that $\mathcal{H}(b)$ is invariant under $T_{\bar{\varphi}}$. Consequently, we have $T_{\bar{\varphi}}h \in \mathcal{H}(b)$. Now according to Theorem 3.8, we have

$$T_{\bar{b}}T_{\bar{\varphi}}h = T_{\bar{\varphi}}T_{\bar{b}}h = T_{\bar{\varphi}}T_{\bar{a}}h^+ = T_{\bar{a}}T_{\bar{\varphi}}h^+,$$

which proves by definition of $(T_{\bar{\varphi}}h)^+$ that $(T_{\bar{\varphi}}h)^+ = T_{\bar{\varphi}}h^+$. \square

Lemma 7.5. *Let b be a non extreme point of the unit ball of H^∞ . Then the space $\mathcal{M}(\bar{a})$ is dense in $\mathcal{H}(b)$.*

Proof. Recall that since b is not an extreme point in the unit ball of H^∞ , then we have $\mathcal{M}(\bar{a}) = \mathcal{H}(\bar{b}) \hookrightarrow \mathcal{H}(b)$. Now let $h \in \mathcal{H}(b)$ and assume that h is orthogonal to $\mathcal{M}(\bar{a})$ (of course relatively to the scalar product of $\mathcal{H}(b)$). Then in particular, we have

$$\langle h, T_{\bar{a}}S^{*n}h \rangle_b = 0, \quad (7.3)$$

for all $n \geq 0$. Using Theorem 3.8, we can write

$$T_{\bar{a}}S^{*n}h = T_{\bar{a}}T_{\bar{z}^n}h = T_{\bar{a}\bar{z}^n}h,$$

and since $az^n \in H^\infty$, we get from Lemma 7.4

$$(T_{\bar{a}}S^{*n}h)^+ = T_{\bar{a}\bar{z}^n}h^+.$$

Therefore, according to Lemma 7.2 and the fact that $h, h^+ \in H^2$, we have

$$\begin{aligned} \langle h, T_{\bar{a}}S^{*n}h \rangle_b &= \langle h, T_{\bar{a}}S^{*n}h \rangle_2 + \langle h^+, T_{\bar{a}\bar{z}^n}h^+ \rangle_2 \\ &= \langle h, P_+(\bar{a}\bar{z}^nh) \rangle_2 + \langle h^+, P_+(\bar{a}\bar{z}^nh^+) \rangle_2 \\ &= \langle h, \bar{a}\bar{z}^nh \rangle_2 + \langle h^+, \bar{a}\bar{z}^nh^+ \rangle_2 \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta})(|h(e^{i\theta})|^2 + |h^+(e^{i\theta})|^2)e^{in\theta} d\theta \\ &= \hat{\varphi}(-n), \end{aligned}$$

where φ denotes the function in $L^1(\mathbb{T})$ defined by $\varphi := a(|h|^2 + |h^+|^2)$ (φ belongs to $L^1(\mathbb{T})$ because it is the product of the H^∞ function a and the $L^1(\mathbb{T})$ function $(|h|^2 + |h^+|^2)$). Then equation (7.3) and the previous computation imply that $\hat{\varphi}(n) = 0$, $n \leq 0$. Hence we get that $\varphi \in H_0^1$. Since a is an outer function and $|h|^2 + |h^+|^2 \in L^1(\mathbb{T})$, we deduce from Lemma 2.4 that in fact $|h|^2 + |h^+|^2 \in H_0^1$. Since this function is real-valued, we get from Lemma 2.2 that $|h|^2 + |h^+|^2 \equiv 0$, which implies that $h \equiv 0$. Therefore we can conclude that $\mathcal{M}(\bar{a})$ is dense in $\mathcal{H}(b)$. \square

7.2 The polynomials are dense in $\mathcal{H}(b)$

Theorem 7.6. *Let b be a non extreme point of the unit ball of H^∞ and let \mathfrak{P} denote the space of (analytic) polynomials. Then we have*

- (a) $\mathfrak{P} \subset \mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$;
- (b) \mathfrak{P} is dense in $\mathcal{M}(\bar{a})$;
- (c) \mathfrak{P} is dense in $\mathcal{H}(b)$.

Proof. (a) For $n \geq 0$, denote by \mathfrak{P}_n the space of polynomials of degree less or equal to n and let $p \in \mathfrak{P}_n$. In particular, p is orthogonal (in H^2) to the (closed) subspace $z^{n+1}H^2$. Then for all $j \geq n+1$, we have

$$\langle T_{\bar{a}}p, z^j \rangle_2 = \langle \bar{a}p, z^j \rangle_2 = \langle p, az^j \rangle_2.$$

Since $az^j \in z^jH^2 \subset z^{n+1}H^2$, we get that

$$\langle T_{\bar{a}}p, z^j \rangle_2 = 0,$$

for all $j \geq n+1$. That means that the H^2 function $T_{\bar{a}}p$ is in fact a polynomial of degree less or equal to n . Therefore we have proved that $T_{\bar{a}}\mathfrak{P}_n \subset \mathfrak{P}_n$. Since by Lemma ?? $T_{\bar{a}}$ is injective, it follows that $T_{\bar{a}}\mathfrak{P}_n = \mathfrak{P}_n$. In particular, we have $\mathfrak{P}_n \subset \mathcal{M}(\bar{a})$, for all $n \geq 0$, which proves that the set of polynomials is contained in $\mathcal{M}(\bar{a})$. The inclusion $\mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$ is already proved in Theorem 7.1.

(b) Let $f \in \mathcal{M}(\bar{a})$ and let $\varepsilon > 0$. By definition, there exists $g \in H^2$ such that $f = T_{\bar{a}}g$. Since \mathfrak{P} is dense in H^2 , there exists $p \in \mathfrak{P}$ such that $\|g - p\|_2 \leq \varepsilon$. Therefore we have

$$\|f - T_{\bar{a}}p\|_{\mathcal{M}(\bar{a})} = \|T_{\bar{a}}(g - p)\|_{\mathcal{M}(\bar{a})} = \|g - p\|_2 \leq \varepsilon,$$

and since $T_{\bar{a}}p \in \mathfrak{P}$, we get the result.

(c) Let $f \in \mathcal{H}(b)$ and let $\varepsilon > 0$. According to Lemma 7.5, there exists $g \in \mathcal{M}(\bar{a})$ such that $\|f - g\|_b \leq \frac{\varepsilon}{2}$ and thanks to (b), there also exists $p \in \mathfrak{P}$ such that $\|g - p\|_{\mathcal{M}(\bar{a})} \leq \frac{\varepsilon}{2}$. Now it follows from Theorem 7.1 that $\|g - p\|_b \leq \|g - p\|_{\mathcal{M}(\bar{a})} \leq \frac{\varepsilon}{2}$ and triangle inequality implies that $\|f - p\|_b \leq \varepsilon$. □

7.3 The shift on $\mathcal{H}(b)$

Theorem 7.7. *Let b be a non extreme point of the unit ball of H^∞ . Then the space $\mathcal{H}(b)$ is invariant under the unilateral shift S . Moreover, we have*

$$\sigma(S|\mathcal{H}(b)) = \bar{\mathbb{D}}.$$

Proof. Recall that $\mathcal{H}(b)$ is invariant under S^* and denote by X the operator on $\mathcal{H}(b)$ defined by $X := S^*|\mathcal{H}(b)$. Then we know from ??? that X is a contraction and

$$X^* = S - b \otimes S^*b.$$

Therefore for all $f \in \mathcal{H}(b)$, we have

$$Sf = X^*f + \langle f, S^*b \rangle_b b. \tag{7.4}$$

It follows from (7.4) and Corollary 7.3 that $Sf \in \mathcal{H}(b)$. That means that $\mathcal{H}(b)$ is invariant under S . We denote by $Y := S|_{\mathcal{H}(b)}$.

Now let $\lambda \in \sigma(Y)$. We argue by contradiction assuming that $|\lambda| > 1$. Since X^* is a contraction, we have $\sigma(X^*) \subset \overline{\mathbb{D}}$ and thus $X^* - \lambda I$ is invertible (as operator on $\mathcal{H}(b)$). Since

$$Y - \lambda I = (X^* - \lambda I) + b \otimes S^*b,$$

the Lemma 1.15 implies that $Y - \lambda I$ is a Fredholm operator with $\text{ind}(Y - \lambda I) = 0$. In others words, $Y - \lambda I$ has a closed range and

$$\dim \ker(Y - \lambda I) = \dim \ker(Y^* - \bar{\lambda}I).$$

But we know from Lemma 2.7 that $\ker(Y - \lambda I) = \{0\}$, which implies that $\ker(Y^* - \bar{\lambda}I) = \{0\}$ and it follows from Theorem ?? that the range of $Y - \lambda I$ is dense in $\mathcal{H}(b)$. But it is already closed, thus we have $(Y - \lambda I)\mathcal{H}(b) = \mathcal{H}(b)$. Finally we get that $(Y - \lambda I)$ is invertible, which is absurd. Therefore, we have proved the first inclusion, that is $\sigma(Y) \subset \overline{\mathbb{D}}$.

To prove the converse, let $\lambda \in \mathbb{D}$. It is easy to see that

$$\bigcap_{n \geq 0} (Y - \lambda I)^n \mathcal{H}(b) = \{0\}. \quad (7.5)$$

Indeed let h be in the left set. That means that for each $n \geq 0$ there exists $h_n \in \mathcal{H}(b)$ such that $h = (z - \lambda)^n h_n$. In particular, it implies that $h^{(n)}(\lambda) = 0$, ($n \geq 0$). Since h is analytic on \mathbb{D} , it is only possible if $h \equiv 0$.

Assume that $(Y - \lambda I)$ is invertible. Then of course $(Y - \lambda I)^n$ is also invertible for each $n \geq 0$. In particular, we get $(Y - \lambda I)^n \mathcal{H}(b) = \mathcal{H}(b)$, which is absurd by (7.5). Thus $(Y - \lambda I)$ is not invertible and $\lambda \in \sigma(Y)$. We have proved that

$$\mathbb{D} \subset \sigma(Y) \subset \overline{\mathbb{D}},$$

which implies the result since $\sigma(Y)$ is a closed set. □

7.4 The multipliers of $\mathcal{H}(b)$

Theorem 7.8. *Let b be a non extreme point of the unit ball of H^∞ and let f be an holomorphic function on an open set Ω containing $\overline{\mathbb{D}}$. Then the function f is a multiplier of $\mathcal{H}(b)$ and of $\mathcal{M}(\bar{a})$. In particular, we have $f \in \mathcal{M}(\bar{a}) \subset \mathcal{H}(b)$.*

Proof. As in the previous section, we will denote by Y the operator on $\mathcal{H}(b)$ defined by $Y := S|_{\mathcal{H}(b)}$. Since Ω is an open set containing the compact set $\overline{\mathbb{D}}$, there exists $r > 1$ such that $\overline{\mathbb{D}} \subset D(0, r) \subset \overline{D(0, r)} \subset \Omega$. We know from Theorem 7.7 that $\sigma(Y) = \overline{\mathbb{D}} \subset D(0, r)$. By the Riesz-Dunford calculus, we can write

$$f(Y) = \frac{1}{2i\pi} \int_{|\lambda|=r} f(\lambda)(Y - \lambda I)^{-1} d\lambda,$$

and $f(Y)$ defines a bounded operator on $\mathcal{H}(b)$. In particular, for every $h \in \mathcal{H}(b)$, we have $f(Y)h \in \mathcal{H}(b)$. We will show that $f(Y)h = fh$. Indeed since f is analytic on Ω , we can write

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n,$$

with the radius of convergence of the series $R > r$. In particular, the series is normalement convergent on $\partial D(0, r)$. Thus

$$\begin{aligned} f(Y)h &= \frac{1}{2i\pi} \int_{|\lambda|=r} \sum_{n=0}^{+\infty} a_n \lambda^n (S - \lambda I)^{-1} h \, d\lambda \\ &= \sum_{n=0}^{+\infty} a_n \frac{1}{2i\pi} \int_{|\lambda|=r} \lambda^n (S - \lambda I)^{-1} h \, d\lambda. \end{aligned}$$

Now if we denote by $p(z) := z^n$ ($z \in \mathbb{C}$), we have $p(Y) = Y^n$ and

$$\frac{1}{2i\pi} \int_{|z|=r} \lambda^n (Y - \lambda I)^{-1} h \, d\lambda = p(Y)h = S^n h = z^n h,$$

which implies that

$$f(Y)h = \sum_{n=0}^{+\infty} a_n z^n h = fh.$$

We have proved that for every $h \in \mathcal{H}(b)$, we have $fh = f(Y)h \in \mathcal{H}(b)$. That exactly means that f is a multiplier of $\mathcal{H}(b)$. Using ??? and Theorem 7.1, we get that f is also a multiplier of $\mathcal{H}(\bar{b}) = \mathcal{M}(\bar{a})$. It remains to note that thanks to Theorem 7.6, the function identically equals to 1 belongs to $\mathcal{M}(\bar{a})$ and thus we have $f = f1 \in \mathcal{H}(b)$ (since f is a multiplier of $\mathcal{H}(b)$). □

7.5 A result of completeness

For b in the unit ball of H^∞ and $\lambda \in \mathbb{D}$, we denote by \hat{k}_λ^b the function defined by

$$\hat{k}_\lambda^b(z) := \frac{b(z) - b(\lambda)}{z - \lambda}, \quad (z \in \mathbb{D}).$$

We are interested in the following question: does the family $(\hat{k}_\lambda^b)_{\lambda \in \mathbb{D}}$ be complete in $\mathcal{H}(b)$?

This family is called the “difference quotients” or the “cokernels”.

It is easy to see that we can reformulate this question in terms of the completeness of another family.

Lemma 7.9. *Let $b \in H^\infty$. Then the following two conditions are equivalent:*

1. $\text{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b)$.
2. $\text{span}\{S^{*n+1}b : n \geq 0\} = \mathcal{H}(b)$.

Proof. It is easily seen that, for $\lambda \in \mathbb{D}$ and $f \in H^2$, we have

$$\frac{f(z) - f(\lambda)}{z - \lambda} = (Id - \lambda S^*)^{-1} S^* f.$$

In particular, applying this formula to $f = b$, we obtain

$$\frac{b(z) - b(\lambda)}{z - \lambda} = (Id - \lambda S^*)^{-1} S^* b = \sum_{n=0}^{\infty} \lambda^n S^{*n+1} b. \quad (7.6)$$

Now according to (7.6), we have $f \in \mathcal{H}(b) \ominus \text{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\}$ if and only

$$\sum_{n=0}^{\infty} \lambda^n \langle S^{*n+1} b, f \rangle_b = 0, \quad (\lambda \in \mathbb{D},),$$

and, since the function $\lambda \mapsto \sum_{n=0}^{\infty} \lambda^n \langle S^{*n+1} b, f \rangle_b$ is analytic in a neighbourhood of 0, this is equivalent to

$$\langle S^{*n+1} b e, f \rangle_b = 0, \quad (n \geq 0),$$

which gives the result. \square

To give the criterion in the nonextreme case, we have to recall that a function f in the Nevanlinna class of the unit disc \mathbb{D} is said to be *pseudocontinuable* (across \mathbb{T}) if there exist $g, h \in \bigcup_{p>0} H^p$ such that

$$f = \bar{h}/\bar{g}$$

a.e. on \mathbb{T} . The function $\tilde{f} := \bar{h}/\bar{g}$ is the (nontangential) boundary function of the meromorphic function $\tilde{f}(z) := \bar{h}(\frac{1}{\bar{z}})/\bar{g}(\frac{1}{\bar{z}})$ defined for $|z| > 1$, which is called a pseudocontinuation of f . R. Douglas, H. Shapiro and A. Shields have obtained [12] the following characterization: a function $f \in H^2$ is pseudocontinuable if and only if it is not S^* -cyclic, that is $\text{span}(S^{*n} f : n \geq 0) \neq H^2$.

Theorem 7.10. *Suppose b is not an extreme point in the unit ball of H^∞ . Then*

$$\text{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b) \iff b \text{ is not pseudocontinuable.}$$

Proof. Assume that $\text{span}\{\hat{k}_\lambda^b : \lambda \in \mathbb{D}\} = \mathcal{H}(b)$ but b is pseudocontinuable. Then there exists a nonconstant inner function u such that $b \in \mathcal{H}(u)$. Since $\mathcal{H}(u)$ is S^* -invariant, $S^{*n+1}b \in \mathcal{H}(u)$ for all $n \geq 0$. As $\mathcal{H}(b)$ is contained continuously in H^2 , we deduce that

$$\text{span}_{\mathcal{H}(b)}(S^{*n+1}b : n \geq 0) \subset \text{clos}_{\mathcal{H}(b)} \mathcal{H}(u) \subset \mathcal{H}(u),$$

and it follows from Lemma 7.9 that $\mathcal{H}(b) \subset \mathcal{H}(u)$. Now since b is not an extreme point in the unit ball of H^∞ , we know that the polynomials belong to $\mathcal{H}(b)$ and consequently to $\mathcal{H}(u)$. Hence $H^2 \subset \mathcal{H}(u)$, which is absurd. Thus, if the difference quotients are complete in $\mathcal{H}(b)$, then b is not pseudocontinuable.

Conversely, assume b is not pseudocontinuable. Note that $\text{span}_{\mathcal{H}(b)}(S^{*n+1}b : n \geq 0)$ is a closed S^* -invariant subspace of $\mathcal{H}(b)$. But we know from [27] the description of these subspaces when b is not an extreme point: they are just the intersection of $\mathcal{H}(b)$ with the invariant subspaces of S^* . Hence there is an inner function u such that

$$\text{span}_{\mathcal{H}(b)}(S^{*n+1}b : n \geq 0) = \mathcal{H}(b) \cap \mathcal{H}(u).$$

But $S^*b \in \mathcal{H}(u)$ implies that $b \in \mathcal{H}(uz)$, which is absurd unless $u \equiv 0$ (because b is not pseudocontinuable). Hence

$$\text{span}_{\mathcal{H}(b)}(S^{*n+1}b : n \geq 0) = \mathcal{H}(b),$$

and applying once more Lemma 7.9, we obtain that the difference quotients are complete in $\mathcal{H}(b)$. \square

Example: As a consequence of Theorem 7.10, it is simple to give two examples of de Branges-Rovnyak spaces (both corresponding to nonextreme functions b), with the completeness of the difference quotients false for the first and true for the second. Note first that, if $\sup_{z \in \mathbb{T}} |b(z)| < 1$, then $\log(1 - |b|)$ is integrable, and thus b is not extreme. This condition is satisfied by both functions $b_1(z) := 1/(z - 3)$ and $b_2(z) := \exp((z - 2)^{-1})$. The first is pseudocontinuable, and thus the difference quotients are not complete in $\mathcal{H}(b_1)$, while the second is not, whence the difference quotients are complete in $\mathcal{H}(b_2)$.

Chapter 8

Appendix

8.1 Extreme points of a convex set in a Banach space

Let E be a \mathbb{K} -linear space ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}), let Ω be a convex subset of E and let $p \in \Omega$. We say that p is an extreme point of Ω if

$$p \in [a, b], a, b \in \Omega \implies p = a \text{ or } p = b.$$

In the following, if X is a normed linear space, then we denote by B_X^1 the closed unit ball of X and by $Ext(X)$ the set of extreme points of B_X^1 . We recall some easy properties concerning extreme points of the closed unit ball.

Lemma 8.1. *Let X be a normed linear space and let $x \in B_X^1$. The following hold:*

a) $x \in Ext(X)$ if and only if

$$x = \frac{1}{2}(y + z), y, z \in B_X^1 \implies x = y = z. \quad (8.1)$$

b) $Ext(X) \subset \{x \in X : \|x\| = 1\}$.

c) $x \in Ext(X)$ if and only if

$$\|y - x\| \leq 1, \|y + x\| \leq 1, y \in X \implies y = 0. \quad (8.2)$$

Proof. a) : first let $x \in Ext(X)$ and assume that there is two points $y, z \in B_X^1$ such that $x = \frac{1}{2}(y + z)$. Then in particular $x \in [y, z]$ and thus by definition of the extreme points, either $x = y$ or $x = z$. If $x = y$, then $\frac{1}{2}(y + z) = y$, which implies $y = z = x$. If $x = z$, then $\frac{1}{2}(y + z) = z$, which also implies $y = z = x$. Therefore, we have proved that x satisfy the property (8.1).

Now on the contrary, let $x \notin Ext(X)$. That means that there is two points $a, b \in B_X^1$ such that $x \in [a, b]$ and $x \neq a, x \neq b$. Of course $a \neq b$ (otherwise

$x = a = b$). Then there is $\lambda \in]0, 1[$ such that $x = \lambda a + (1 - \lambda)b$. We will construct $y, z \in B_X^1$ such that $x = \frac{1}{2}(y + z)$ and $y \neq z$. For this purpose choose a real r satisfying

$$0 < r < \min((1 - \lambda)\|a - b\|, \lambda\|a - b\|)$$

and define

$$t_1 = \lambda + \frac{r}{\|a - b\|}, \quad t_2 = \lambda - \frac{r}{\|a - b\|}$$

and

$$y = t_1 a + (1 - t_1)b, \quad z = t_2 a + (1 - t_2)b.$$

Then it is easy to check that $t_1, t_2 \in]0, 1[$; thus $y, z \in [a, b]$. In particular $y, z \in B_X^1$ (because B_X^1 is convex). Moreover

$$\begin{aligned} y + z &= (t_1 + t_2)a + (1 - t_1 + 1 - t_2)b = 2\lambda a + (2 - 2\lambda)b \\ &= 2(\lambda a + (1 - \lambda)b) = 2x. \end{aligned}$$

It remains to note that $y \neq z$ otherwise $t_1 = t_2$ which is impossible. Thus we have proved that x does not satisfy the property (8.1), which ends the proof of the point a).

b) : let $x \in X$, $\|x\| < 1$. Then there is $r > 0$ such that

$$\{z \in X : \|z - x\| \leq r\} \subset B_X^1, \quad (8.3)$$

and choose $z \in X$ such that $\|z - x\| = r$. Now put $y := 2x - z$. Of course we have $\|y - x\| = \|x - z\| = r$, whence by (8.3), we get that $y, z \in B_X^1$. Moreover $x = \frac{1}{2}(y + z)$. Finally $y \neq z$ because otherwise $x = z$ and then $r = 0$ which is absurd. Therefore we conclude that x is not an extreme point of B_X^1 . Hence

$$\text{Ext}(X) \subset B_X^1 \setminus \{x \in X : \|x\| < 1\} = \{x \in X : \|x\| = 1\}.$$

c) : first assume that $x \in \text{Ext}(X)$ and let $y \in X$ such that $\|y - x\| \leq 1$, $\|y + x\| \leq 1$. Note that $x = \frac{1}{2}(x + y + x - y)$ and $x + y, x - y \in B_X^1$. Then we get by a) that $x = x - y = x + y$, in other words $y = 0$.

Reciprocally assume that x satisfies the property (8.2) and assume that there is two points $a, b \in B_X^1$ such that $x = \frac{1}{2}(a + b)$. Put $y = x - a$. Then $y - x = -a \in B_X^1$, $y + x = 2x - a = b \in B_X^1$. Therefore by (8.2), we get that $y = 0$, whence $x = a$ and then $x = b$ also. Once again by a), we conclude that $x \in \text{Ext}(X)$. □

Exercises

Exercise 8.1.1. Let X be an Hilbert space. Show that

$$\text{Ext}(X) = \{x \in X : \|x\| = 1\}.$$

Exercise 8.1.2. Let $1 < p < +\infty$. Show that

$$\text{Ext}(L^p) = \{f \in L^p : \|f\|_p = 1\}.$$

Hint: use Lemma 8.1 and the fact that if $f, g \in L^p$, then

$$\|f + g\|_p = \|f\|_p + \|g\|_p \implies \exists \lambda \in \mathbb{R}_+ : f \equiv \lambda g \text{ a.e.}$$

Exercise 8.1.3. Show that

$$\text{Ext}(L^1) = \emptyset.$$

Hint: argue by absurd assuming that there is $f \in \text{Ext}(L^1)$; then by Lemma 8.1 $\|f\|_1 = 1$ and consider

$$\begin{aligned} \varphi : [0, 2\pi] &\longrightarrow \mathbb{R}_+ \\ x &\longmapsto \varphi(x) = \frac{1}{2\pi} \int_0^x |f(e^{i\theta})| d\theta. \end{aligned}$$

Using Vitali's Lemma and mean value theorem, show that there is $x_0 \in]0, 2\pi[$ such that $\varphi(x_0) = \frac{1}{2}$. Now find a contradiction by considering

$$g(e^{i\theta}) = \begin{cases} 2f(e^{i\theta}), & 0 \leq \theta \leq x_0 \\ 0, & x_0 < \theta < 2\pi \end{cases}$$

and

$$h(e^{i\theta}) = \begin{cases} 0, & 0 \leq \theta \leq x_0 \\ 2f(e^{i\theta}), & x_0 < \theta < 2\pi \end{cases}.$$

Exercise 8.1.4. Show that

$$\text{Ext}(L^\infty) = \{f \in L^\infty : |f| = 1 \text{ a.e. } \}.$$

Hint: first let $f \in \text{Ext}(L^\infty)$ and put $E = \{\zeta \in \mathbb{T} : |f(\zeta)| < 1\}$. Assume that $m(E) > 0$ and show a contradiction by considering

$$g := \begin{cases} f + \frac{1-|f|}{2}, & \text{on } E \\ f, & \text{on } \mathbb{T} \setminus E \end{cases}$$

and

$$h := \begin{cases} f - \frac{1-|f|}{2}, & \text{on } E \\ f, & \text{on } \mathbb{T} \setminus E \end{cases}.$$

Conclude that $|f| = 1$ a.e.

Reciprocally let $f \in L^\infty$, $|f| = 1$ a.e. and let $g, h \in B_{L^\infty}^1$, $f = \frac{1}{2}(g + h)$. Show that $|h + g| = |h| + |g|$ a.e. Then consider $E = \{\zeta \in \mathbb{T} : |g(\zeta)| < 1\}$ and $F = \{\zeta \in \mathbb{T} : |h(\zeta)| < 1\}$. Show that $m(E) = m(F) = 0$. Conclude that $g = h$ a.e. and therefore $f \in \text{Ext}(L^\infty)$.

8.2 Extreme points of the unit ball of H^∞ and H^1

In Exercice ??, we describe the extreme point of the unit ball of H^p for $1 < p < +\infty$. The situation was pretty easy because it was more or less the same than in L^p . What can we say about the case $p = 1$ and $p = +\infty$?

We have ever seen that $\text{Ext}(L^1) = \emptyset$ (see Exercice 8.1.3). But H^1 is very different from L^1 in the sense that H^1 is the conjugate of a Banach space. More precisely, we have

$$H^1 \simeq (C(\mathbb{T})/A_0)^*,$$

where A_0 is the closed linear subspace in $C(\mathbb{T})$ generated by χ_n , $n \geq 1$. Now according to Krein–Milman’s Theorem, if a Banach space X is (isometrically isomorphe) to the conjugate of a Banach space Y , then the unit ball of X not only has extreme points, but it has a lot to span this unit ball (in the sense that the closed convex hull of its extreme points coincide with the unit ball; here the closure is relatively to the weak star topology). Therefore we see that the situation for H^1 is dramastically different from the situation for L^1 . The closed unit ball of L^1 has no extreme points; the closed unit ball of H^1 has a lot of extreme points and the closed convex hull of its extreme points coincide with the unit ball.

Independently of the result of Krein–Milman, we will describe in this section exactly the extreme points of H^1 and H^∞ .

First note that if b is a point in the unit ball of H^∞ such that $|b| = 1$ a.e. on \mathbb{T} (that is b is an inner function), then we know from Exercice 8.1.4 that b is an extreme point of the unit ball of L^∞ and then an extreme point of the unit ball of H^∞ . But we will see in the next result that the unit ball of H^∞ has a lot of other extreme points.

Theorem 8.2. *Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. Then b is an extreme point of the unit ball of H^∞ if and only if*

$$\int_0^{2\pi} \log(1 - |b(e^{it})|) dt = -\infty. \quad (8.4)$$

Proof. First assume that (8.4) is satisfied and let $a \in H^\infty$ such that $\|b+a\|_\infty \leq 1$ and $\|b-a\|_\infty \leq 1$. We will show that $a \equiv 0$. Using identity parallelogram, we have, for every $z \in \mathbb{D}$,

$$|b(z)|^2 + |a(z)|^2 = \frac{1}{2} (|b(z) + a(z)|^2 + |b(z) - a(z)|^2) \leq 1,$$

which implies that for almost all $e^{i\theta}$ on \mathbb{T} , we have

$$|a(e^{i\theta})|^2 \leq 1 - |b(e^{i\theta})|^2.$$

Hence

$$|a(e^{i\theta})|^2 \leq (1 - |b(e^{i\theta})|)(1 + |b(e^{i\theta})|) \leq 2(1 - |b(e^{i\theta})|),$$

which gives

$$2 \int_{-\pi}^{\pi} \log |a(e^{i\theta})| d\theta \leq 2\pi \log 2 + \int_{-\pi}^{\pi} \log(1 - |b(e^{i\theta})|) d\theta.$$

Thus we get from (8.4) that

$$\int_{-\pi}^{\pi} \log |a(e^{i\theta})| d\theta = -\infty,$$

and Lemma 2.3 implies that $a \equiv 0$. According to Lemma 8.1, we obtain that b is an extreme point of the unit ball of H^∞ .

For the converse implication, assume that

$$\int_0^{2\pi} \log(1 - |b(e^{it})|) dt \neq -\infty. \quad (8.5)$$

Since $\log(1 - |b(e^{i\theta})|) \leq 0$ a.e. on \mathbb{T} , the condition (8.5) means that $\log(1 - |b(e^{i\theta})|) \in L^1$. But $1 - |b(e^{i\theta})| \in L^\infty$ and thus the function a , defined by

$$a(z) = [1 - |b|](z) = \exp \left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log(1 - |b(\zeta)|) dm(\zeta) \right), \quad z \in \mathbb{D},$$

is an outer function which is in H^∞ . Moreover, we have $|a| = 1 - |b|$ a.e. on \mathbb{T} . Then $a + b \in H^\infty$, $a - b \in H^\infty$ and

$$\|a + b\|_\infty = \sup_{\theta \in [0, 2\pi[} |a(e^{i\theta}) + b(e^{i\theta})| \leq \sup_{\theta \in [0, 2\pi[} (|a(e^{i\theta})| + |b(e^{i\theta})|) = 1.$$

Similarly we also have $\|a - b\|_\infty \leq 1$. Since $a \not\equiv 0$, it follows from Lemma 8.1 that b is not an extreme point of the unit ball of H^∞ . □

Contrary to the case of L^1 , the following result shows that the unit ball of H^1 has a lot of extreme points.

Theorem 8.3. *Let $f \in H^1$. The following are equivalent:*

- (i) f is an extreme point of the unit ball of H^1 .
- (ii) f is an outer function and $\|f\|_1 = 1$.

Proof. (ii) \implies (i) : let f be an outer function in H^1 of unit norm and assume that there exists $g \in H^1$, $\|f \pm g\|_1 \leq 1$. We will show that $g \equiv 0$. It follows from Lemma 2.3 that $f \neq 0$ a.e. on \mathbb{T} . Moreover using the fact that $f = \frac{1}{2}((f + g) + (f - g))$, it is easy to see that $\|f \pm g\|_1 = 1$. Define now ϕ the holomorphic function on \mathbb{D} by

$$\phi(z) = \frac{g(z)}{f(z)}, \quad (z \in \mathbb{D}),$$

and let $\phi(e^{i\theta})$ denote the boundary value of ϕ , which exists a.e. on \mathbb{T} because $f, g \in H^1$ and $f \neq 0$ a.e. on \mathbb{T} . Since $\|f \pm g\|_1 = \|f\|_1 = 1$, we have

$$\int_{-\pi}^{\pi} (|1 + \phi(e^{i\theta})| + |1 - \phi(e^{i\theta})| - 2) |f(e^{i\theta})| d\theta = 0. \quad (8.6)$$

But note that

$$|1 + \phi(e^{i\theta})| + |1 - \phi(e^{i\theta})| \geq |1 + \phi(e^{i\theta}) + 1 - \phi(e^{i\theta})| = 2.$$

Therefore since $f(e^{i\theta}) \neq 0$ a.e. on \mathbb{T} , the equation 8.6 implies that

$$|1 + \phi(e^{i\theta})| + |1 - \phi(e^{i\theta})| = 2,$$

for almost all $e^{i\theta}$ on \mathbb{T} . Now it is easy to check that this relation gives $1 - \Re\phi = |1 - \phi|$ a.e. on \mathbb{T} and since $|1 - \phi|^2 = (1 - \Re\phi)^2 + (\Im\phi)^2$, we get that ϕ is real a.e. on \mathbb{T} . Moreover we have $1 - \phi = 1 - \Re\phi = |1 - \phi| \geq 0$. Changing ϕ by $-\phi$, we get from similar arguments that $1 + \phi = |1 + \phi| \geq 0$. Thus ϕ is real a.e. on \mathbb{T} and $-1 \leq \phi \leq 1$. Since f is outer and $\phi = g/f \in L^\infty$, it follows from Lemma 2.4 that $\phi \in H^\infty$. But we have seen that ϕ is real-valued on \mathbb{T} and then we get from Lemma 2.2 that ϕ is constant. Hence

$$\begin{aligned} (1 - \phi)\|f\|_1 &= \|(1 - \phi)f\|_1 = \left\| \left(1 - \frac{g}{f}\right) f \right\|_1 \\ &= \|f - g\|_1 = 1, \end{aligned}$$

whence $1 - \phi = 1$ because $\|f\|_1 = 1$. In other words, $\phi = 0$ and then $g = 0$. Now it remains to apply Lemma 8.1 to deduce that f is an extreme point of the unit ball of H^1 .

(i) \implies (ii) : let $f \in H^1$ and assume that f is an extreme point of the unit ball of H^1 . We already know from Lemma 8.1 that $\|f\|_1 = 1$. So it remains to show that f is an outer function. We argue by absurd. Then according to Theorem ??, we have $f = IF$, where F is the outer part of f , $F \in H^1$ and I is the inner part of f and I is not constant (because f is assumed to be not outer). Consider

$$\varphi(\alpha) = \int_{-\pi}^{\pi} |f(e^{i\theta})| \Re(e^{i\alpha} I(e^{i\theta})) d\theta, \quad \alpha \in (0, \pi).$$

Since f is in L^1 , it is easy to see that φ is continuous on $(0, \pi)$. Moreover, we have

$$\varphi(0) = \int_{-\pi}^{\pi} |f(e^{i\theta})| \Re(I(e^{i\theta})) d\theta,$$

and

$$\varphi(\pi) = \int_{-\pi}^{\pi} |f(e^{i\theta})| \Re(-I(e^{i\theta})) d\theta = -\varphi(0).$$

Hence by the mean value theorem, there is $\alpha \in [0, \pi]$ such that $\varphi(\alpha) = 0$. Put

$$u := e^{i\alpha} I \quad \text{and} \quad g := \frac{1}{2} e^{-i\alpha} F(1 + u^2).$$

Of course $u \in H^\infty$, $g \in H^1$ and $g \not\equiv 0$ (because u is not constant). Moreover, we have $|u(e^{i\theta})| = 1$ for almost all $e^{i\theta}$ on \mathbb{T} . Hence

$$\begin{aligned} 2\Re(u(e^{i\theta})) &= u(e^{i\theta}) + \overline{u(e^{i\theta})} \\ &= \frac{1 + u^2(e^{i\theta})}{u(e^{i\theta})}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} g(e^{i\theta}) &= \frac{1}{2}e^{-i\alpha}F(e^{i\theta})(1 + u^2(e^{i\theta})) \\ &= e^{-i\alpha}u(e^{i\theta})F(e^{i\theta})\Re(u(e^{i\theta})) \\ &= I(e^{i\theta})F(e^{i\theta})\Re(u(e^{i\theta})) \\ &= f(e^{i\theta})\Re(u(e^{i\theta})). \end{aligned}$$

Then we can write

$$|f(e^{i\theta}) \pm g(e^{i\theta})| = |f(e^{i\theta})| (1 \pm \Re(u(e^{i\theta}))) = |f(e^{i\theta})| \pm |f(e^{i\theta})|\Re(u(e^{i\theta})).$$

But since

$$\int_{-\pi}^{\pi} |f(e^{i\theta})|\Re(u(e^{i\theta})) d\theta = \int_{-\pi}^{\pi} |f(e^{i\theta})|\Re(e^{i\alpha}I(e^{i\theta})) d\theta = \varphi(\alpha) = 0,$$

we get that

$$\|f \pm g\|_1 = \|f\|_1 = 1.$$

Since $g \not\equiv 0$, we obtain a contradiction by Lemma 8.1. \square

8.3 A theorem of Helson–Szegö

We will use the following deep result. For the proof, we refer to [22, Chap. 1].

Theorem 8.4. *Let ν be a finite and positive Borel measure on \mathbb{T} and let b be the function in the unit ball of H^∞ associated to ν by (5.12). The following assertions are equivalent:*

- (i) $H^2(\nu) = L^2(\nu)$.
- (ii) $\overline{\chi_1} \in H^2(\nu)$.
- (iii) $\chi_0 \in H_0^2(\nu)$.
- (iv) b is an extreme point in the unit ball of H^∞ .

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