Sparse representations from moments

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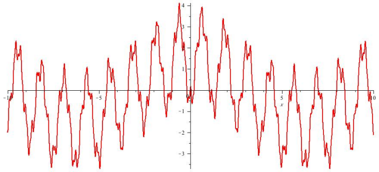
Sparse representation of signals

Given a function or signal $f(t)$:



decompose it as

$$f(t) = \sum_{i=1}^{r'} (a_i cos(\mu_i t) + b_i sin(\mu_i t)) e^{\nu_i t} = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$$

Prony's method (1795)

For the signal $f(t) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

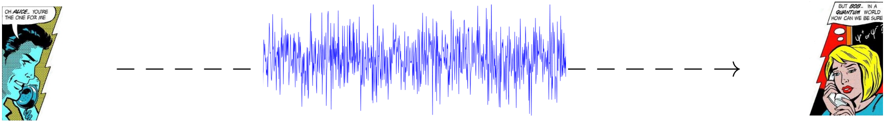
- Evaluate f at $2\,r$ regularly spaced points: $\sigma_0 := f(0), \sigma_1 := f(1), \ldots$
- Compute a non-zero element $\mathbf{p} = [\mathbf{p_0}, \ldots, \mathbf{p_r}]$ in the kernel:

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \mathbf{e}^{\zeta_1}, \ldots, \xi_{\mathbf{r}} = \mathbf{e}^{\zeta_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_{\mathbf{1}} \\ \omega_{\mathbf{2}} \\ \vdots \\ \omega_{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

Decoding



An algebraic code:

$$E = \{c(f) = [f(\xi_1), \ldots, f(\xi_m)] \mid f \in \mathbb{K}[x]; \deg(f) \le d\}.$$

Encoding messages using the dual code:

$$C = E^{\perp} = \{\mathbf{c} \mid \mathbf{c} \cdot [f(\xi_1), \ldots, f(\xi_l)] = 0 \ \forall f \in V = \langle \mathbf{x}^{\mathbf{a}} \rangle \subset \mathbb{F}[\mathbf{x}]\}$$

Message received: $r = m + e$ for $m \in C$ where $e = [\omega_1, \ldots, \omega_m]$ is an error with $\omega_j \neq 0$ for $j = i_1, \ldots, i_r$ and $\omega_j = 0$ otherwise.

☞ **Find the error e.**

Berlekamp-Massey method (1969)

- Compute the syndrome $\sigma_k = c(x^k) \cdot r = c(x^k) \cdot e = \sum_{j=1}^{r} \omega_{i_j} \xi_{i_j}^k$.
- Compute the matrix

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \dots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

  and its kernel $p = [p_0, \dots, p_r]$.
- Compute the roots of the error locator polynomial $p(x) = \sum_{i=0}^{r} p_i \, x^i = p_r \prod_{j=1}^{r} (x - \xi_{i_j})$.
- Deduce the errors ω_{i_j}.

1. **Sequences of moments, series, duality**

2. Algebraic structure

3. Decomposition

4. Optimization

Sequences, series, duality (1D)

Sequences: $\sigma = (\sigma_k)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ indexed by $k \in \mathbb{N}$.

Formal power series:

$$\sigma = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]] \quad (\text{or } \sum_{k=0}^{\infty} \sigma_k z^k \in \mathbb{K}[[z]])$$

Linear functionals: $\mathbb{K}[x]^* = \{\Lambda : \mathbb{K}[x] \to \mathbb{K} \text{ linear}\}$.

Example:

- $p \mapsto$ coefficient of x^i in $p = \frac{1}{i!}\partial^i(p)(0)$
- $\mathbf{e}_\zeta : p \mapsto p(\zeta)$.

Series as linear functionals: For $\sigma = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]]$,

$$\sigma : p = \sum_k p_k x^k \mapsto \langle \sigma | p \rangle = \sum_k \sigma_k p_k$$

$(\frac{y^k}{k!})$ is the dual basis of the monomial basis $(x^k)_{k\in\mathbb{N}}$.

Example: $\mathbf{e}_\zeta = \sum_{k=0}^{\infty} \zeta^k \frac{y^k}{k!} = e^{\zeta y} \in \mathbb{K}[[y]]$.

Structure of $\mathbb{K}[x]$-module: $p(x) \star \Lambda : q \mapsto \Lambda(p\, q)$.

$$\begin{aligned} x \star \sigma &= \sum_{k=1}^{\infty} \sigma_k \frac{y^{k-1}}{(k-1)!} = \partial(\sigma(y)) \\ p(x) \star \sigma &= p(\partial)(\sigma(y)). \end{aligned}$$

Sequences, series, duality (nD)

Multi-index sequences: $\sigma = (\sigma_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

Taylor series:

$$\sigma(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{K}[[y_1, \dots, y_n]] \quad (\text{or } \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \mathbf{z}^\alpha \in \mathbb{K}[[z_1, \dots, z_n]])$$

where $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$.

Linear functionals: $\sigma \in R^* = \{\sigma : R \rightarrow \mathbb{K}, \textbf{linear}\}$

$$\sigma : p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mapsto \langle \sigma | p \rangle = \sum_\alpha \sigma_\alpha p_\alpha$$

The coefficients $\langle \sigma | \mathbf{x}^\alpha \rangle = \sigma_\alpha \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are called the **moments** of σ.

Structure of R-module: $\forall p \in R, \sigma \in R^*$, $p \star \sigma : q \mapsto \langle \sigma | p\, q \rangle$.

☞ $$p \star \sigma = p(\partial_1, \dots, \partial_n)(\sigma)(\mathbf{y})$$

Hankel operators

Hankel operator: For $\sigma \in \mathbb{K}[[\mathbf{y}]]$ (**symbol** of the Hankel operator),

$$\begin{array}{rcl} H_\sigma : \mathbb{K}[\mathbf{x}] & \to & \mathbb{K}[[\mathbf{y}]] \\ p & \mapsto & p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y}) \end{array}$$

Truncated Hankel operator: $V, V' \subset \mathbb{K}[\mathbf{x}]$,

$$H_\sigma^{V',V} : p \in V \to p \star \sigma_{|V'} \in V'^*$$

Example: $V = \langle \mathbf{x}^A \rangle, V' = \langle \mathbf{x}^B \rangle \subset \mathbb{K}[\mathbf{x}]$

$$H_\sigma^{A,B} = [\sigma_{\alpha+\beta}]_{\alpha \in A, \beta \in B}.$$

Ideal: $I_\sigma = \ker H_\sigma = \{p \in \mathbb{K}[\mathbf{x}] \mid p \star \sigma = 0\},$

$$= \{p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mid \forall \beta \in \mathbb{N}^n \sum_\alpha p_\alpha \sigma_{\alpha+\beta} = 0\}$$

Linear recurrence relations on the sequence $\sigma = (\sigma_\alpha)_{\alpha \in \mathbb{N}^n}$.

Quotient algebra: $\mathcal{A}_\sigma = \mathbb{K}[x_1, \ldots, x_n]/I_\sigma$

Studied case: $\dim \mathcal{A}_\sigma < \infty$ **i.e.** $\mathcal{A}_\sigma$ **Artinian.**

1. Sequences of moments, series, duality
2. **Algebraic structure**
3. Decomposition
4. Optimization

Structure of an Artinian algebra $\mathcal{A}$

Definition: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ is **Artinian** if $\dim_{\mathbb{K}} \mathcal{A} < \infty$.

Hilbert nullstellensatz: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian $\Leftrightarrow \mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$ is finite.

Assuming $\mathbb{K} = \overline{\mathbb{K}}$ is algebraically closed, we have

- $I = Q_1 \cap \cdots \cap Q_r$ where Q_i is m_{ξ_i}-primary where $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$.
- $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r,$ with
  - $\mathcal{A}_i = \mathbf{u}_i\, \mathcal{A} \sim \mathbb{K}[x_1, \ldots, x_n]/Q_i$,
  - $\mathbf{u}_i^2 = \mathbf{u}_i$, $\mathbf{u}_i\, \mathbf{u}_j = 0$ if $i \neq j$, $\mathbf{u}_1 + \cdots + \mathbf{u}_r = 1$.
- $\dim R/Q_i = \mu_i$ is the multiplicity of ξ_i.

Structure of the dual $\mathcal{A}^*$

Sparse series:

$$\mathcal{P}ol\mathcal{E}xp = \left\{ \sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i(\mathbf{y})\, \mathbf{e}_{\xi_i}(\mathbf{y}) \mid \omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}], \right\}$$

where $\mathbf{e}_{\xi_i}(\mathbf{y}) = e^{\mathbf{y}\cdot\xi_i} = e^{y_1\xi_{1,i}+\cdots+y_n\xi_{n,i}}$ with $\xi_{i,j} \in \mathbb{K}$.

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\mathcal{D}(\omega_1, \ldots, \omega_r) = \langle \partial_{\mathbf{y}}^{\alpha}(\omega_i), \alpha \in \mathbb{N}^n \rangle$$

Theorem

For $\mathbb{K} = \overline{\mathbb{K}}$ algebraically closed,

$$\mathcal{A}^* = \oplus_{i=1}^{r} \mathcal{D}_i\, \mathbf{e}_{\xi_i}(\mathbf{y}) \quad \subset \quad \mathcal{P}ol\mathcal{E}xp$$

- $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$
- $\mathcal{D}_i = \mathcal{D}(\omega_{i,1}, \ldots, \omega_{i,l_i}) = Q_i^{\perp}$ *with* $\omega_{i,j} \in \mathbb{K}[\mathbf{y}]$ *and* $I = Q_1 \cap \cdots \cap Q_r$
- $\mu(\omega_{i,1}, \ldots, \omega_{i,l_i}) := \dim_{\mathbb{K}}(\mathcal{D}_i) = \mu_i$ *multiplicity of* ξ_i.

The roots by eigencomputation

Hypothesis: $\mathcal{V}_{\mathbb{K}}(I) = \{\xi_1, \ldots, \xi_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian.

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \mathcal{A}^* & \to & \mathcal{A}^* \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are $\{a(\xi_1), \ldots, a(\xi_r)\}$.*
- *The eigenvectors of all $(\mathcal{M}_a^{\mathsf{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathbf{e}_{\xi_i} : p \mapsto p(\xi_i)$.*

Proposition

If the roots are simple, the operators $\mathcal{M}_a$ are diagonalizable. Their common eigenvectors are, up to a scalar, interpolation polynomials $\mathbf{u}_i$ at the roots and idempotent in $\mathcal{A}$.

Example

Roots of polynomial systems

$$\left\{\begin{array}{lcl} f_1 & = & x_1^2x_2 - x_1^2 \\ f_2 & = & x_1x_2 - x_2 \end{array}\right. \qquad I = (f_1, f_2) \subset \mathbb{C}[\mathbf{x}]$$

$$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I \equiv \langle 1, x_1, x_2 \rangle \quad I = (x_1^2 - x_2, x_1x_2 - x_2, x_2^2 - x_2)$$

$$M_1 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right), \quad M_2 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right) \quad \begin{array}{c} \text{common} \\ \text{eigvecs of} \\ M_1^t, M_2^t \end{array} = \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right)$$

$$I = Q_1 \cap Q_2 \quad \text{where} \quad Q_1 = (x_1^2, x_2), \quad Q_2 = \mathbf{m}_{(1,1)} = (x_1 - 1, x_2 - 1)$$

$$I^\perp = Q_1^\perp \oplus Q_2^\perp \quad Q_1^\perp = \langle 1, y_1 \rangle = \langle 1, y_1 \rangle\, \mathbf{e}_{(0,0)}(\mathbf{y}) \quad Q_2^\perp = \langle 1 \rangle\, \mathbf{e}_{(1,1)}(\mathbf{y}) = \langle e^{y_1+y_2} \rangle$$

Solution of partial differential equations (with constant coeff.)

$$\left\{\begin{array}{lcl} \partial_{y_1}^2\partial_{y_2}\sigma - \partial_{y_1}^2\sigma & = & 0 \\ \partial_{y_1}\partial_{y_2}\sigma - \partial_{y_2}\sigma & = & 0 \end{array}\right. \quad \begin{array}{l} f_1 \star \sigma = 0 \\ f_2 \star \sigma = 0 \end{array} \quad \Rightarrow \quad \sigma \in I^\perp = Q_1^\perp \oplus Q_2^\perp$$

$$\sigma = a + b\, y_1 + c\, e^{y_1+y_2} \quad a, b, c \in \mathbb{C}$$

Hankel operators and Gorenstein algebra

Hankel operator: For $\sigma \in \mathbb{K}[[\mathbf{y}]]$,

$$\begin{array}{rcl} H_\sigma : \mathbb{K}[\mathbf{x}] & \rightarrow & \mathbb{K}[[\mathbf{y}]] \\ p & \mapsto & p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y}) \end{array}$$

Quotient algebra: $\mathcal{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma$ where $I_\sigma = \ker H_\sigma$.

Definition: $\mathcal{A}$ **Gorenstein** iff $\exists\, \sigma \in \mathcal{A}^*$ such that $\mathcal{A}^* = \mathcal{A} \star \sigma$ is a free $\mathcal{A}$.

☞ **Isomorphism between $\mathcal{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma$ and its dual space $\mathcal{A}_\sigma^*$:**

$$\begin{tikzcd} 0 \arrow[r] & I_\sigma \arrow[r] & \mathbb{K}[\mathbf{x}] \arrow[r, "H_\sigma"] & \mathcal{A}_\sigma^* \arrow[r] & 0 \\ & & p \arrow[r, mapsto] & p \star \sigma & \end{tikzcd}$$

Correspondence between **sequences** $\sigma \in \mathbb{K}^{\mathbb{N}^n}$ **with** $\mathrm{rank} H_\sigma < \infty$ and **Artinian Gorenstein algebras** $\mathcal{A}_\sigma := \mathbb{K}[\mathbf{x}]/I_\sigma$.

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Univariate series:

Kronecker (1881)

The Hankel operator $H_\sigma : (p_m) \in \mathbb{C}^{\mathbb{N},finite} \mapsto (\sum_m \sigma_{m+n} p_m)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ *is of* **finite rank** r *iff* $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[y]$ *and* $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ *distincts s.t.*

$$\sigma(y) = \sum_{n\in\mathbb{N}} \sigma_n \frac{y^n}{n!} = \sum_{i=1}^{r'} \omega_i(y) \mathbf{e}_{\xi_i}(y) \text{ with } \sum_{i=1}^{r'} (\deg(\omega_i)+1) = r.$$

Multivariate series:

Theorem (Generalized Kronecker Theorem)

H_σ *is of rank* r *iff* $\sigma = \sum_{i=1}^{r'} \omega_i(\mathbf{y})\, \mathbf{e}_{\xi_i}(\mathbf{y}) \in \mathcal{P}ol\mathcal{E}xp$ *with* $r = \sum_{i=1}^{r'} \mu(\omega_i)$. *In this case, we have*

- $\mathcal{V}_{\mathbb{C}}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$ *with* $Q_i^\perp = \mathcal{D}(\omega_i)\, \mathbf{e}_{\xi_i}(\mathbf{y})$.
- $\mathcal{A}_\sigma^* = \mathcal{A}_\sigma \star \sigma$ *(free* $\mathcal{A}_\sigma$*-module of rank 1).*
- $(a, b) \mapsto \langle \sigma | ab \rangle$ *is non-degenerate in* $\mathcal{A}_\sigma$.

Decomposition from the structure of $\mathcal{A}_\sigma$

For $\sigma = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}}\, \mathbf{e}_{\xi_{\mathbf{i}}}$, with $\omega_{\mathbf{i}} \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct.

- rank $H_\sigma = \mathbf{r}$ and the multiplicity of the points $\xi_{\mathbf{1}}, \ldots, \xi_{\mathbf{r}}$ in $\mathcal{V}(I_\sigma)$ is 1.
- For B, B' be of size r, $H_\sigma^{B',B}$ invertible iff B and B' are bases of $\mathcal{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma$.
- The matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}_\sigma$ is such that

$$\mathbf{H}_\sigma^{\mathbf{B}',\mathbf{x_i B}} = \mathbf{H}_{\mathbf{x_i}\star\sigma}^{\mathbf{B}',\mathbf{B}} = \mathbf{H}_\sigma^{\mathbf{B}',\mathbf{B}}\, \mathbf{M_i}$$

- The common **eigenvectors** of M_i are (up to a scalar) the Lagrange **interpolation polynomials** $\mathbf{u}_{\xi_{\mathbf{i}}}$ at the points $\xi_{\mathbf{i}}$, $i = 1, \ldots, r$.

$$\mathbf{u}_{\xi_{\mathbf{i}}}(\xi_{\mathbf{j}}) = \begin{cases} 1 & \text{if}\, i = j, \\ 0 & \text{otherwise} \end{cases} \quad {\mathbf{u}_{\xi_{\mathbf{i}}}}^2 \equiv \mathbf{u}_{\xi_{\mathbf{i}}},\ \sum_{i=1}^{\mathbf{r}} \mathbf{u}_{\xi_{\mathbf{i}}} \equiv 1.$$

- The common **eigenvectors** of M_i^t are (up to a scalar) the vectors $[B(\xi_{\mathbf{i}})]$, $i = 1, \ldots, r$.

Decomposition algorithm

Input: The first coefficients $(\sigma_\alpha)_{\alpha \in A}$ of the series

$$\sigma = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$$

1. Compute bases $B, B' \subset \langle \mathbf{x}^A \rangle$ s.t. that $H^{B',B}$ invertible and $|B| = |B'| = r = \dim \mathcal{A}_\sigma$;
2. Deduce the tables of multiplications $M_i := (H_\sigma^{B',B})^{-1} H_\sigma^{B',x_i B}$
3. Compute the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $\sum_i l_i M_i$ for a generic $\mathbf{l} = l_1 x_1 + \cdots + l_n x_n$;
4. Deduce the points $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$ s.t. $M_j \mathbf{v}_i - \xi_{i,j} \mathbf{v}_i = 0$ and the weights $\omega_i = \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle$.

Output: The decomposition $\sigma = \sum_{i=1}^{r} \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle \mathbf{e}_{\xi_i}(\mathbf{y})$.

Multivariate Prony method

Let $h(t_1, t_2) = 2 + 3\, 2^{t_1}\, 2^{t_2} - 3^{t_1}$, $\sigma = \sum_{\alpha \in \mathbb{N}^2} h(\alpha) \frac{\mathbf{y}^\alpha}{\alpha!} = 2\mathbf{e}_{(1,1)}(\mathbf{y}) + 3\mathbf{e}_{(2,2)}(\mathbf{y}) - \mathbf{e}_{(3,1)}(\mathbf{y})$.

- Take $B = \{1, x_1, x_2\}$ and compute

$$H_0 := H_\sigma^{B,B} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) \\ h(1,0) & h(2,0) & h(1,1) \\ h(0,1) & h(1,1) & h(0,2) \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 5 & 11 \\ 7 & 11 & 13 \end{bmatrix},$$

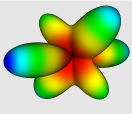
$$H_1 := H_\sigma^{B,x_1 B} = \begin{bmatrix} 5 & 5 & 7 \\ 5 & -1 & 17 \\ 811 & 178 & 23 \end{bmatrix}, \; H_2 := H_\sigma^{B,x_2 B} = \begin{bmatrix} 7 & 11 & 13 \\ 11 & 17 & 23 \\ 13 & 23 & 25 \end{bmatrix}.$$

- Compute the generalized eigenvectors of $(aH_1 + bH_2, H_0)$:

$$U = \begin{bmatrix} 2 & -1 & 0 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} \text{ and } H_0\, U = \begin{bmatrix} 2 & 3 & -1 \\ 2 \times 1 & 3 \times 2 & -1 \times 3 \\ 2 \times 1 & 3 \times 2 & -1 \times 1 \end{bmatrix}.$$

- This yields the weights $2, 3, -1$ and the roots $(1,1), (2,2), (3,1)$.

Symmetric tensor decomposition



$$
\begin{aligned}
\psi &= \mathbf{(x_0 - x_1 + 3\,x_2)^4 + (x_0 + x_1 + x_2)^4 - 3\,(x_0 + 2\,x_1 + 2\,x_2)^4} \\
&= -x_0{}^4 - 24\,x_0{}^3x_1 - 8\,x_0{}^3x_2 - 60\,x_0{}^2x_1{}^2 - 168\,x_0{}^2x_1x_2 - 12\,x_0{}^2x_2{}^2 \\
&\quad -96\,x_0x_1{}^3 - 240\,x_0x_1{}^2x_2 - 384\,x_0x_1x_2{}^2 + 16\,x_0x_2{}^3 - 46\,x_1{}^4 - 200\,x_1{}^3x_2 \\
&\quad -228\,x_1{}^2x_2{}^2 - 296\,x_1x_2{}^3 + 34\,x_2{}^4 \\
\psi^* &\equiv \mathbf{e_{(-1,3)}(y) + e_{(1,1)}(y) - 3e_{(2,2)}(y)} \qquad (\mathit{by\ apolarity\ for}\ \psi^* : p \mapsto \langle \psi, p\rangle_d)
\end{aligned}
$$

For $B = \{1, x_2, x_1\}$,

$$
H^{2,2}_{\psi^*} := \begin{bmatrix}
\boxed{-1} & \boxed{\boxed{-2}} & \boxed{\boxed{-6}} & \boxed{-2} & \boxed{\boxed{-14}} & \boxed{-10} \\
\boxed{-2} & \boxed{\boxed{-2}} & \boxed{\boxed{-14}} & \boxed{4} & \boxed{\boxed{-32}} & \boxed{-20} \\
\boxed{-6} & \boxed{\boxed{-14}} & \boxed{\boxed{-10}} & \boxed{-32} & \boxed{\boxed{-20}} & \boxed{-24} \\
-2 & 4 & -32 & 34 & -74 & -38 \\
-14 & -32 & -20 & -74 & -38 & -50 \\
-10 & -20 & -24 & -38 & -50 & -46
\end{bmatrix}
$$

$$
H^{B,B}_{\psi^*} = \boxed{\begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix}}
$$

$$
H^{B,x_2B}_{\psi^*} = \boxed{\begin{bmatrix} -6 & -14 & -10 \\ -14 & -32 & -20 \\ -10 & -20 & -24 \end{bmatrix}}
$$

$$
H^{B,x_1B}_{\psi^*} = \boxed{\begin{bmatrix} -2 & -2 & -14 \\ -2 & 4 & -32 \\ -14 & -32 & -20 \end{bmatrix}}
$$

- The matrix of multiplication by x_1 in $B = \{1, x_2, x_1\}$ is

$$M_1 = (H_{\psi^*}^{B,B})^{-1} H_{\psi^*}^{B,x_1 B} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials

$$U(x) = \left[\ \frac{1}{2}\,x_2 - \frac{1}{2}\,x_1 \quad -2 + \frac{3}{4}\,x_2 + \frac{1}{4}\,x_1 \quad -1 + \frac{1}{2}\,x_2 + \frac{1}{2}\,x_1\ \right].$$

- We deduce the weights and the frequencies:

$$H_{\psi^*}^{[1,x_1,x_2],U} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} \times \mathbf{-1} & \mathbf{1} \times \mathbf{1} & \mathbf{-3} \times \mathbf{2} \\ \mathbf{1} \times \mathbf{3} & \mathbf{1} \times \mathbf{1} & \mathbf{-3} \times \mathbf{2} \end{bmatrix}.$$

Weights: $1, 1, -3$; frequencies: $(-1, 3), (1, 1), (2, 2)$.

Decomposition: $\psi^*(\mathbf{y}) = \mathbf{e}_{(-1,3)}(\mathbf{y}) + \mathbf{e}_{(1,1)}(\mathbf{y}) - 3\,\mathbf{e}_{(2,2)}(\mathbf{y}) + (\mathbf{y})^4$

Sparse interpolation

$$f(\mathbf{x}) = \sum_{i=1}^{r} \omega_i\, \mathbf{x}^{\alpha_i} \quad \Rightarrow \quad \sigma = \sum_{\gamma} f(\varphi^{\gamma})\, \frac{\mathbf{y}^{\gamma}}{\gamma!} = \sum_{i=1}^{r} \omega_i\, \mathbf{e}_{\varphi^{\alpha_i}}(\mathbf{y})$$

Example: $f(x_1, x_2) = x_1^{33} x_2^{12} - 5\, x_1 x_2^{45} + 101.$

- Compute $\sigma_\alpha = f(\varphi_1^{\alpha_1}, \varphi_2^{\alpha_2})$ for $\alpha_1 + \alpha_2 \leq 3$ and $\varphi_1 = \varphi_2 = e^{\frac{2\mathbf{i}\pi}{50}}$.
- Compute the Hankel matrix $H_\sigma^{1,2}$:

$$\left[\begin{array}{ccccc} 97.00000 & 97.01771 + 3.93695\,\mathbf{i} & 95.50360 - 1.47099\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} \\ 97.01771 + 3.93695\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 102.35770 + 3.77300\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} \\ 95.50360 - 1.47099\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 95.73130 - .33862\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} & 95.42134 + 1.47250\,\mathbf{i} \end{array}\right.$$

- Deduce the decomposition of $\sigma = \sum_{i=1}^{3} \omega_i \mathbf{e}_{\xi_i}$:

$$\Xi = \begin{bmatrix} 0.99211 + 0.12533\mathbf{i} & 0.80902 - 0.58779\mathbf{i} \\ 1.00000 + 4.86234e^{-11}\mathbf{i} & 1.00000 - 6.91726e^{-10}\mathbf{i} \\ -0.53583 - 0.84433\mathbf{i} & 0.06279 + 0.99803\mathbf{i} \end{bmatrix} \omega = \begin{bmatrix} -5.00000 - 4.43772e^{-7}\mathbf{i} \\ 101.00000 + 4.65640e^{-7}\mathbf{i} \\ 1.00000 - 1.92279e^{-8}\mathbf{i} \end{bmatrix}$$

- and the exponents $\frac{50\,\Xi}{2\pi i}$ mod 50 of the terms of f:

$$\begin{bmatrix} 1.00000 - 0.414119e^{-7}\,\mathbf{i} & -5.00000 + 0.270858e^{-6}\,\mathbf{i}, \\ 0.386933e^{-9} + 0.137963e^{-8}\,\mathbf{i} & -0.550458e^{-8} - 0.38761e^{-8}\,\mathbf{i} \\ -17.00000 - 0.100085e^{-6}\,\mathbf{i} & 12.00000 + 0.700984e^{-6}\,\mathbf{i} \end{bmatrix}$$

A general framework

- $\mathfrak{F}$ the functional space, in which the "signal" lives.
- $S_1, \ldots, S_n : \mathfrak{F} \to \mathfrak{F}$ commuting linear operators: $S_i \circ S_j = S_j \circ S_i$.
- $\Delta : h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ a linear functional on $\mathfrak{F}$.

Generating series associated to $h \in \mathfrak{F}$:

$$\sigma_h(\mathbf{y}) \quad = \quad \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}.$$

- Eigenfunctions:
$$S_j(E) = \xi_j E, j = 1, \ldots, n \Rightarrow \sigma_E = \mathbf{e}_\xi(\mathbf{y}).$$
- Generalized eigenfunctions:
$$S_j(E_k) = \xi_j E_k + \sum_{k' < k} m_{j,k'} E_{k'}) \Rightarrow \sigma_{E_k} = \omega_i(\mathbf{y}) \mathbf{e}_\xi(\mathbf{y}).$$

If $h \mapsto \sigma_h$ is injective $\Rightarrow$ unique decomposition of f as a linear combination of generalized eigenfunctions.

Extends [Peter, Plonka 2013] to the multivariate case.

Sparse interpolation of PolyLog functions

- $\mathcal{F} = \mathcal{POLYLOG}(\mathbf{x}) = \{\sum_{(\beta,\gamma)\in A} h_{\beta,\gamma} \log^{\beta}(\mathbf{x})\mathbf{x}^{\gamma}, A \text{ finite}\}$,
- $S_j : h(x_1, \ldots, x_n) \mapsto h(\ldots, x_{j-1}, \varphi_j x_j, x_{j+1}, \ldots)$ for $\varphi_j \in \mathbb{C} - \{1\}$,
- $\Delta : h(x_1, \ldots, x_n) \mapsto \Delta[h] = h(1, \ldots, 1)$.

Generating series of h: $\sigma_h(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} h(\varphi_1^{\alpha_1}, \ldots, \varphi_n^{\alpha_n}) \frac{\mathbf{y}^{\alpha}}{\alpha!}$.
Eigenfunctions: $\mathbf{x}^{\gamma}$; **Generalized eigenfunctions**: $\log^{\beta}(\mathbf{x})\mathbf{x}^{\gamma}$.

$h = \sum_{i=1}^{r'} \sum_{\beta\in B_i} \omega_{i,\beta} \log^{\beta}(\mathbf{x})\, \mathbf{x}^{\gamma_i}$ iff the generating series σ_h is of the form

$$\sigma_h(\mathbf{y}) = \sum_{i=1}^{r'} \omega_i(\mathbf{y}) \mathbf{e}_{\xi_i}(\mathbf{y})$$

with $\xi_i = (\varphi_1^{\gamma_{i,1}}, \ldots, \varphi_n^{\gamma_{i,n}}) \in \mathbb{C}^n$ and $\omega_i(\mathbf{y}) = \sum_{\beta\in B_i} \omega_{i,\beta} \mathbf{y}^{\beta} \in \mathbb{C}[\mathbf{y}]$.

☞ Decomposition from the moments $\sigma_{\alpha} = h(\varphi_1^{\alpha_1}, \ldots, \varphi_n^{\alpha_n})$.

Sparse Fourier analysis

- $\mathcal{F} = L^2(\Omega)$,
- $S_j : h(\mathbf{x}) \mapsto e^{\frac{2\pi x_i}{T_i}} h(\mathbf{x}) \in L^2(\Omega)$,
- $\Delta : h(\mathbf{x}) \mapsto \int h(\mathbf{x}) d\mathbf{x}$.

Generating series of h: $\sigma(h) = \frac{1}{\prod_{j=1}^n T_j} \mathscr{F}(h) \left(2\pi \frac{\gamma_1}{T_1}, \ldots, 2\pi \frac{\gamma_n}{T_n}\right)$

Eigenfunctions: δ_ξ; **Generalized eigenfunctions**: $\delta_\xi^{(\beta)}$.

Let $h \in L^2(\Omega)$ and let $\sigma(h) = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n}$ its Fourier coefficients.

$$\Gamma_{\sigma(h)} : (\rho_\beta)_{\beta \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n) \mapsto \left(\sum_\beta \sigma_{\alpha+\beta} \rho_\beta\right)_{\alpha \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n).$$

$\Gamma_{\sigma(h)}$ is of finite rank r iff

$\sigma(h) = \sum_{i=1}^{r'} \sum_{\alpha \in A_i} \omega_{i,\alpha} \mathbf{y}^\alpha \mathbf{e}_{\mathbf{i}\xi_i}(\mathbf{y}) \quad \Leftrightarrow \quad h = \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \omega_{i,\alpha} \mathbf{i}^{|\alpha|} \delta_{\xi_i}^{(\alpha)}$

with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \Omega$, $\omega_{i,\alpha} \in \mathbb{C}$ and $r = \sum_{i=1}^{r'} \mu(\sum_{\alpha \in A_i} \omega_{i,\alpha} \mathbf{y}^\alpha)$

1. Sequences of moments, series, duality
2. Algebraic structure
3. Decomposition
4. **Optimization**

(joint work with C. Josz, J.B. Lasserre)

For noisy data, transform the decomposition problem into an optimization problem which gives the **closest sparse decomposition**.

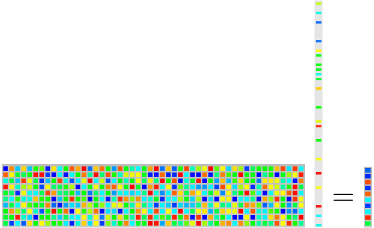
Compress sensing

Problem: For $f(\mathbf{x}) = \sum_{\alpha\in A} \omega_\alpha \mathbf{x}^\alpha$, find A and $(\omega_\alpha)_{\alpha\in A}$ from values $f(\zeta_k)$ for $\zeta_k \in \mathbb{R}^n$, $k = 1, \ldots, s$.

Find a solution of

$$V\,\Omega = F$$

where $V = (\zeta_k^\alpha)_{k,\alpha}$, $\Omega = [\omega_\alpha]_\alpha$ and $F = [f(\zeta_k)]_{k=1,\ldots,m}$, with $\|\Omega\|_0 = \#\{\omega_\alpha \neq 0\}$ minimal.

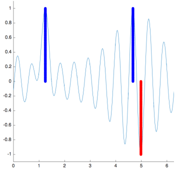


Linear Programming

$$\begin{aligned} \inf \quad & \sum_{\alpha\in A} \omega_\alpha^+ + \omega_\alpha^- \\ s.t. \quad & \omega_\alpha^+ \geq 0, \omega_\alpha^- \geq 0, \sum_\alpha (\omega_\alpha^+ - \omega_\alpha^-)\zeta_k^\alpha - f(\zeta_k) = 0, k = 1, \ldots, s \end{aligned}$$

- ☞ Sparse solution with r terms when $s = r$.
- ☞ Exact recovery for enough points well-separated.

Super-resolution



For $f(\mathbf{x}) = \sum_{\alpha\in A} \omega_\alpha \mathbf{x}^\alpha$, $\zeta \in \mathbb{C}^n$

$$f(\zeta^\beta) = \sum_{\alpha\in A} \omega_\alpha\, \zeta^{\alpha\cdot\beta} = \int \mathbf{x}^\beta d\mu$$

where $\mu = \sum_{\alpha\in A} \omega_\alpha \delta_{\zeta^\alpha}$ is a weighted sum of Dirac measures at ζ^α.

Find a measure $\mu = \sum_{\alpha\in A} \omega_\alpha \delta_\xi$, which is a minimizer of

$$\begin{array}{lll} (*) & \inf & \|\mu\|_{TV} \\ & s.t. & \int \mathbf{x}^\beta d\mu = f(\zeta^\beta), \beta \in B \end{array}$$

$(\|\mu\|_{TV} = \sup_\pi \sum_{A\in\pi} |\mu(A)|$; for $\mu = \sum_{\alpha\in A} \omega_\alpha \delta_\xi$ $\|\mu\|_{TV} = \sum_{\alpha\in A} |\omega_\alpha|.)$

Theorem (Candès, Fernandez-Granda'14)

Let $\zeta \in \mathbb{T}^n$, $\delta = \min\{|\zeta^\alpha - \zeta^{\alpha'}| \mid \alpha \neq \alpha' \in A\}$ with $A \subset [-a, a]^n$. If $\delta > C_n/a$, then $(*)$ has a unique optimal solution, which is a weighted sum of Dirac measures at ζ^α for $\alpha \in A$.

Semi-Definite Programming

Relaxation in convex optimization problems on moment matrices.

$$\begin{array}{lll} \mu \text{ positive measure} & \Rightarrow & \forall p \in \mathbb{R}[\mathbf{x}],\ \int p^2 d\mu \geq 0 \\ & \Rightarrow & H_\mu \succcurlyeq 0 \text{ where } \langle p \mid H_\mu p\rangle = \int p^2 d\mu \end{array}$$

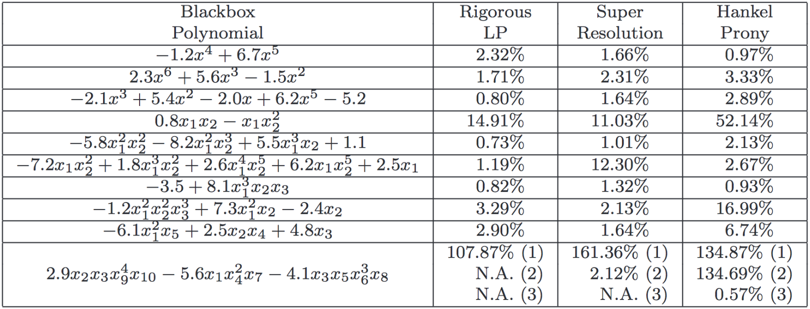
If $\zeta \in \mathbb{R}^n$, $\mu = \sum_{\alpha\in A} \omega_\alpha \delta_{\zeta^\alpha} = \mu^+ - \mu^-$ with μ^+, μ^- positive measures.

Find moment matrices H^+, H^- indexed by $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha| \leq d, |\beta| \leq d$, which is a minimizer of

$$\begin{array}{lll} (\star) & \inf & h^+_{0,0} + h^-_{0,0} \\ & s.t. & H^+ = (h^+_{\alpha,\beta}) \succcurlyeq 0, H^- = (h^-_{\alpha,\beta}) \succcurlyeq 0, \\ & & H^+, H^- \text{ Hankel : } h_{\alpha,\beta} = h_{\alpha',\beta'} \text{if } \alpha+\beta = \alpha'+\beta' \\ & & h^+_{\beta,0} - h^-_{\beta,0} = f(\zeta^\beta) \text{ for } \beta \in B \end{array}$$

If a **flat extension** property holds, decompose the series of moments $h_{\alpha,0}$ as a weighted sum of exponentials to get the support and weights of the optimal measure.

Some numerical results



Blackbox Polynomial	Rigorous LP	Super Resolution	Hankel Prony
$-1.2x^4 + 6.7x^5$	2.32%	1.66%	0.97%
$2.3x^6 + 5.6x^3 - 1.5x^2$	1.71%	2.31%	3.33%
$-2.1x^3 + 5.4x^2 - 2.0x + 6.2x^5 - 5.2$	0.80%	1.64%	2.89%
$0.8x_1x_2 - x_1x_2^2$	14.91%	11.03%	52.14%
$-5.8x_1^2x_2^2 - 8.2x_1^2x_2^3 + 5.5x_1^3x_2 + 1.1$	0.73%	1.01%	2.13%
$-7.2x_1x_2^2 + 1.8x_1^3x_2^2 + 2.6x_1^4x_2^5 + 6.2x_1x_2^5 + 2.5x_1$	1.19%	12.30%	2.67%
$-3.5 + 8.1x_1^3x_2x_3$	0.82%	1.32%	0.93%
$-1.2x_1^2x_2^2x_3^3 + 7.3x_1^2x_2 - 2.4x_2$	3.29%	2.13%	16.99%
$-6.1x_1^2x_5 + 2.5x_2x_4 + 4.8x_3$	2.90%	1.64%	6.74%
$2.9x_2x_3x_9^4x_{10} - 5.6x_1x_4^2x_7 - 4.1x_3x_5x_6^3x_8$	107.87% (1) N.A. (2) N.A. (3)	161.36% (1) 2.12% (2) N.A. (3)	134.87% (1) 134.69% (2) 0.57% (3)

Relative error with uniform noise between -0.1 and 0.1 for the real and imaginary parts on the measurements; evaluations in the points $(e^{i\alpha_1}, \ldots, e^{i\alpha_n})$ with $\alpha_i \in \mathbb{N}$, $\sum_i \alpha_i \leq d = deg(f)$.

Challenges, open questions

- Numerical stability, correction of errors,
- Efficient construction of basis, complexity,
- Super-resolution, collision of points,
- Super-extrapolation,
- Best low rank approximation,
- ...

Thanks for your attention

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