Two-periodic Aztec diamond and matrix valued orthogonal polynomials

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with Maurice Duits (arXiv 1712:05636) and

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Approximation and Matrix Functions

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Outline

1. Matrix Valued Orthogonal Polynomials

MVOP

Matrix valued polynomial of degree j

$$P_j(z) = \sum_{i=0}^{j} C_i z^j, \qquad C_i \text{ is } d \times d \text{ matrix}$$

- **Matrix valued orthogonality**

$$\int_\gamma P_j(x) W(x) P_k^t(x) dx = H_j \delta_{j,k}$$

where $W(x)$ is given matrix valued weight on γ.

Mathematical properties:

- **Three term recurrence with matrix coefficients**
- **Christoffel Darboux formula**
- **Riemann-Hilbert problem**

Connections

MVOP appear in representation theory and spectral theory

We found MVOP in periodic tiling problems

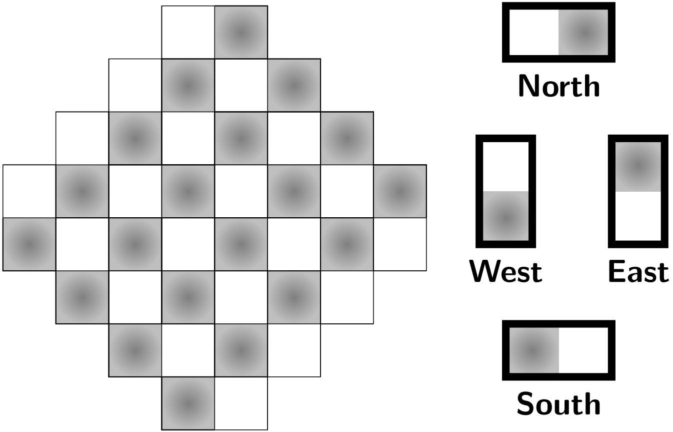
- **Varying weight W^N on closed contour γ around 0**

$$\frac{1}{2\pi i}\int_\gamma P_j(z)W^N(z)P_k^t(z)dz = H_j\delta_{j,k}$$

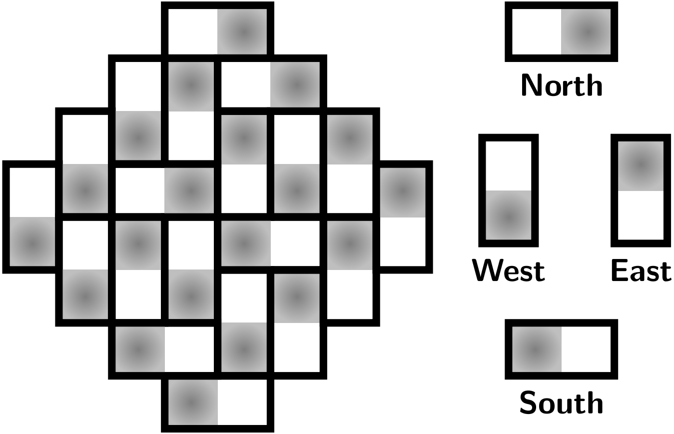
- **Example:** $W(z) = \frac{1}{z^2}\begin{pmatrix} 1+z & 1+\alpha \\ (1+\alpha)z & 1+\alpha^2 z \end{pmatrix}^2$
- **W^N is matrix analogue of a Jacobi weight $(z-1)^{-N}(z+1)^N$ with nonstandard parameters.**
- **Main interest in behavior of the reproducing kernel in the limit $N \to \infty$.**

2. Aztec diamond

Aztec diamond

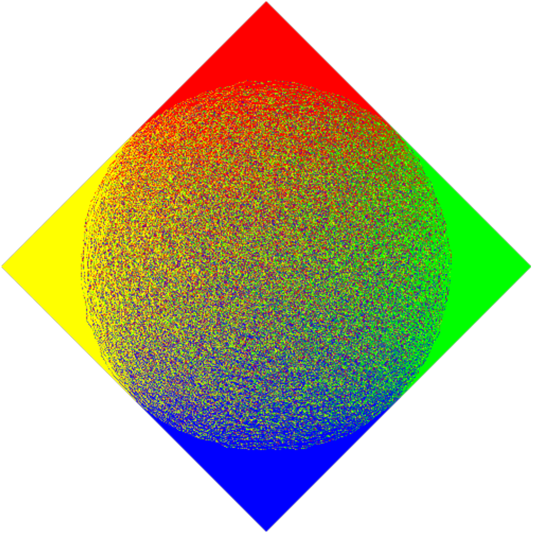


Tiling of an Aztec diamond



- **Tiling with 2×1 and 1×2 rectangles (dominos)**
- **Four types of dominos**

Large random tiling



Deterministic pattern near corners

Solid region

or

Frozen region

Disorder in the middle

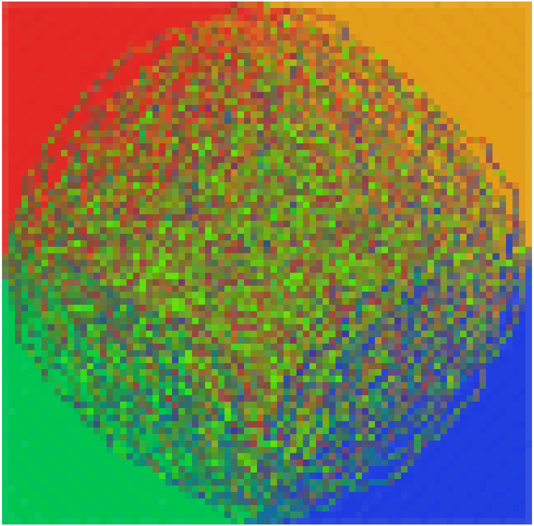
Liquid region

Boundary curve

Arctic circle

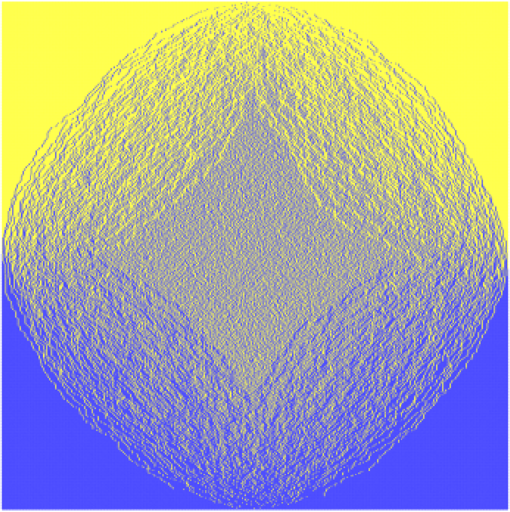
Recent development

- **Two-periodic weighting** **Chhita, Johansson (2016)**
  Beffara, Chhita, Johansson (2018 to appear)

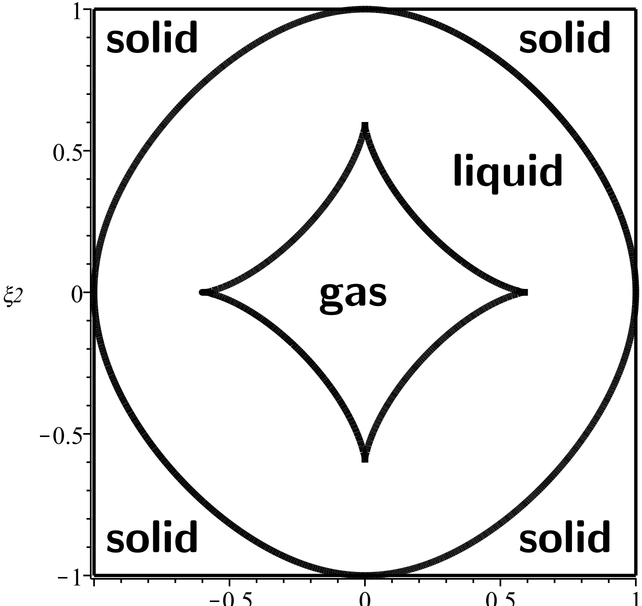


Two-periodic weights

- **A new phase within the liquid region: gas region**

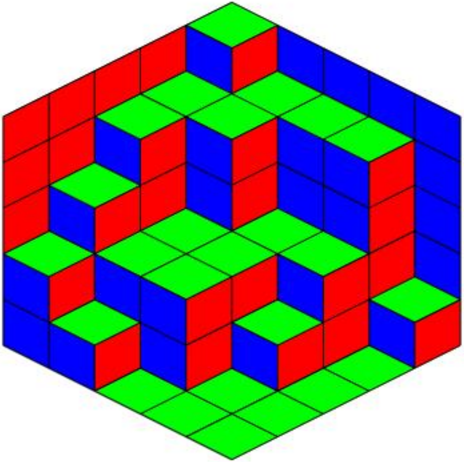


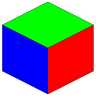
Phase diagram



3. Hexagon tilings

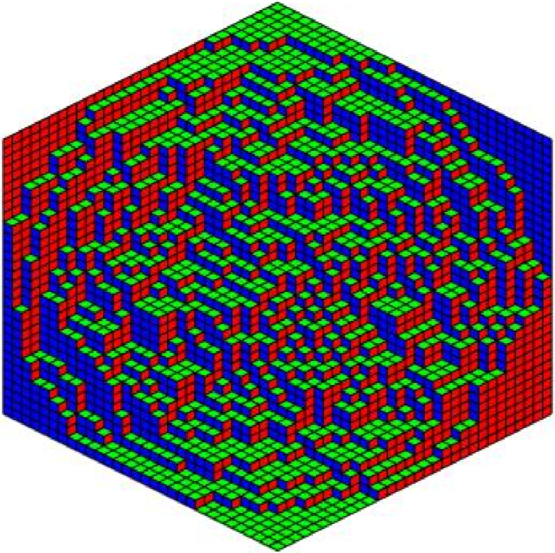
Lozenge tiling of a hexagon



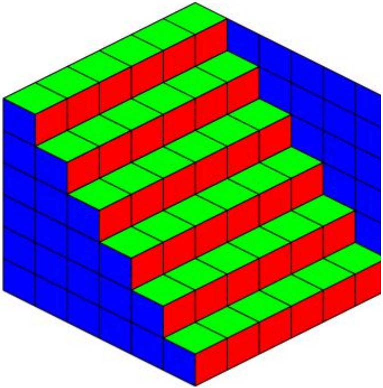


three types of lozenges

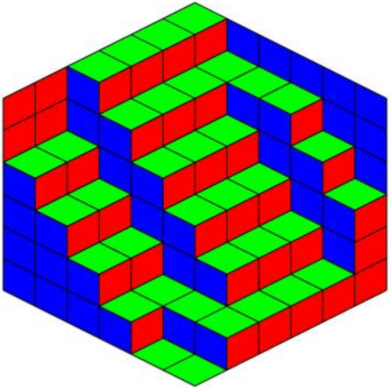
Arctic circle phenomenon



Two periodic hexagon (size 6)

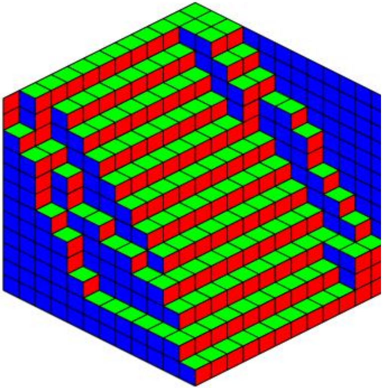


$\alpha = 0$

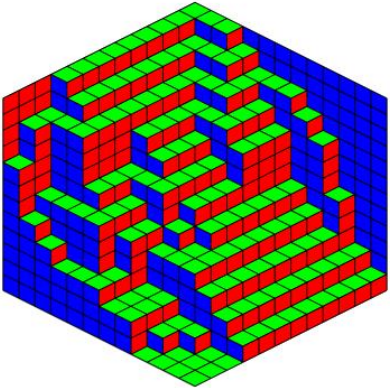


$\alpha = 0.1$

Two periodic hexagon (size 30)

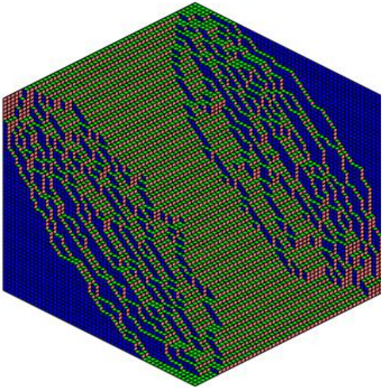


$\alpha = 0.1$

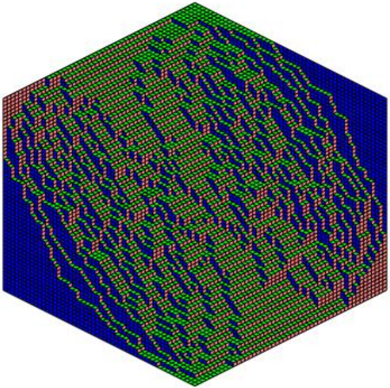


$\alpha = 0.18$

Two periodic hexagon (size 50)

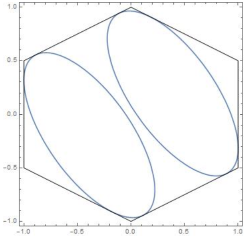


$\alpha = 0.1$

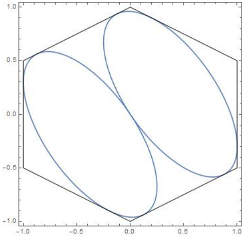


$\alpha = 0.15$

Phase Diagrams



$\alpha < 1/9$,



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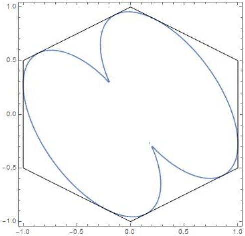
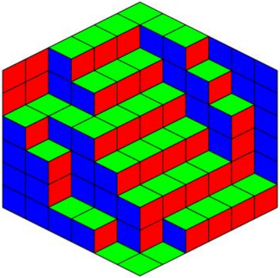
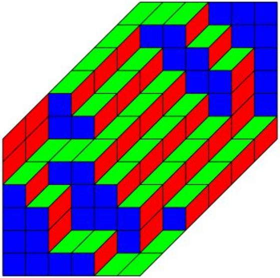


$\alpha > 1/9$

4. The two periodic model

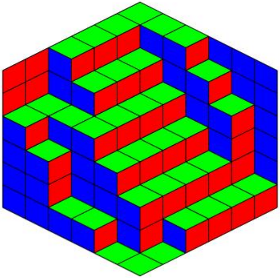
Oblique hexagon and weights

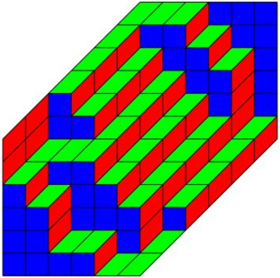




- **Vertices are on the integer lattice** $\mathbb{Z}^2$

Oblique hexagon and weights





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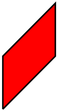
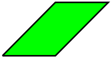


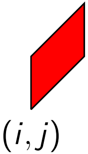
(i, j)

has weight $\begin{cases} \alpha < 1, & \textbf{if } i+j \textbf{ is even}, \\ 1, & \textbf{if } i+j \textbf{ is odd}, \end{cases}$

have weight 1

Weight



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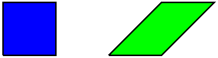


have weight 1

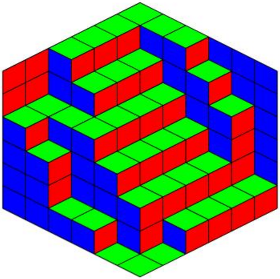
- Weight of a tiling T is the product of the weights of the lozenges in the tiling.
- Probability is proportional to the weight

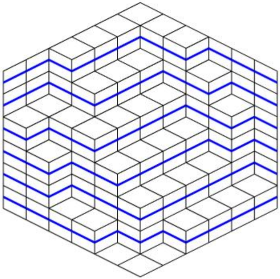
$$\text{Prob}(T) = \frac{w(T)}{Z_N}$$

where $Z_N = \sum_T w(T)$ is the normalizing constant (partition function)

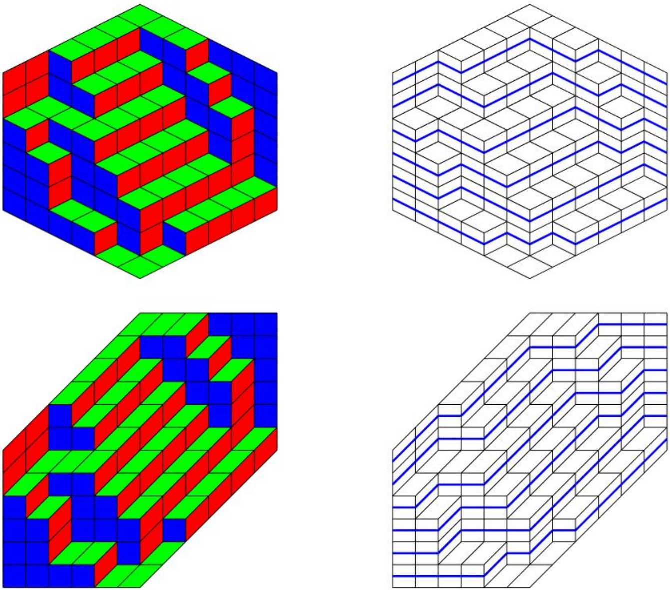
4. Non-intersecting paths

Non-intersecting paths



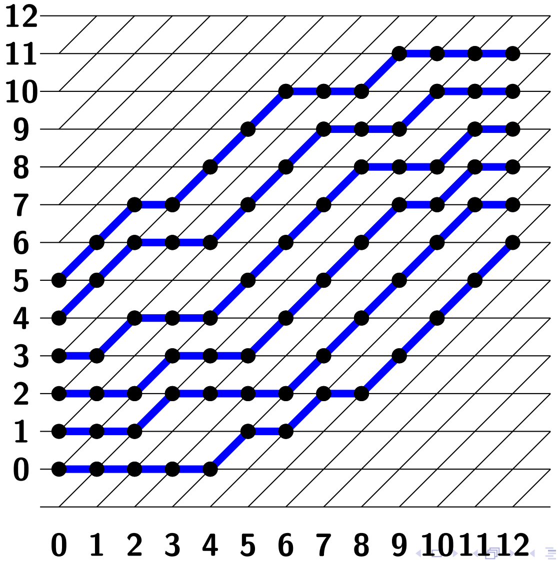


Non-intersecting paths



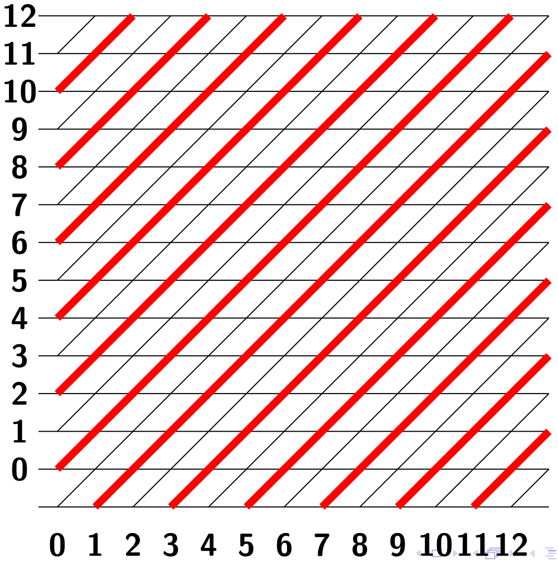
Non-intersecting paths on a graph

Paths fit on a graph

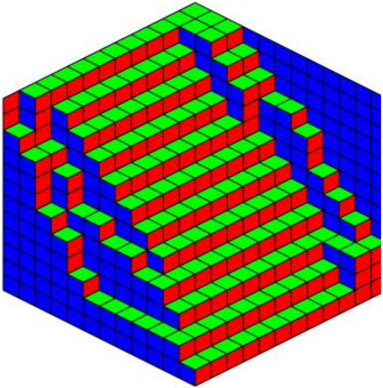


Weights on the graph

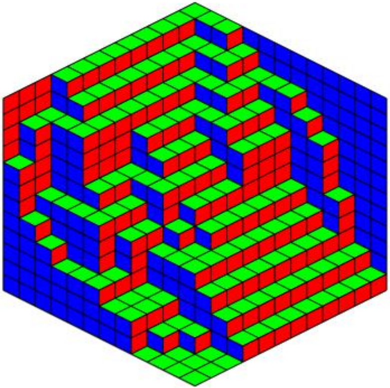
Red edges carry weight $\alpha < 1$. **Other edges weight 1**



Two periodic hexagon (size 30)



$\alpha = 0.1$



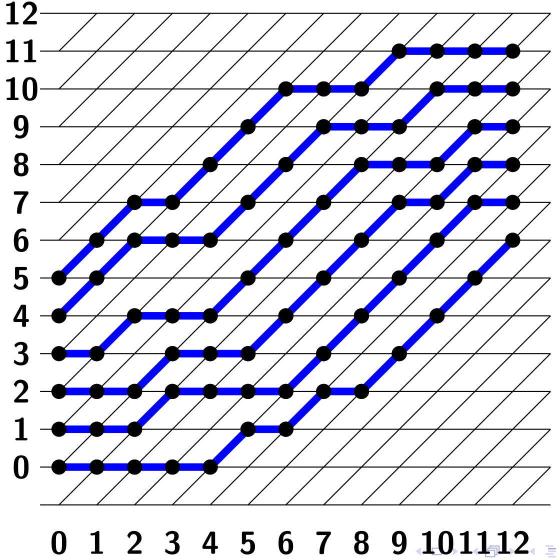
$\alpha = 0.18$

- **For $0 < \alpha < 1$: punishment to cover the red edges.**
- **Staircase region in the middle avoids all red edges.**

6. Determinantal point process : known results

Particle configuration

Consider positions of particles along the paths.



Transitions and LGV theorem

Particles at level m: $x_j^{(m)}$, $j = 0, \dots, N-1$.

Proposition

$$\textbf{Prob}\left((x_j^{(m)})_{j=0,m=1}^{N-1,2N-1} \right) = \frac{1}{Z_n} \prod_{m=0}^{2N-1} \det \left[T_m(x_j^{(m)}, x_k^{(m+1)}) \right]_{j,k=0}^{N-1}$$

with $x_j^{(0)} = j$, $x_j^{(2N)} = N + j$ and transition matrices

$$T_m(x,x) = 1$$

$$T_m(x, x+1) = \begin{cases} \alpha, & \text{if } m+x \text{ is even}, \\ 1, & \text{if } m+x \text{ is odd}, \end{cases}$$

$$T_m(x,y) = 0 \quad \text{otherwise}, \qquad x, y \in \mathbb{Z}$$

This follows from Lindström-Gessel-Viennot lemma.

Lindström (1973) **Gessel-Viennot (1985)**

Determinantal point process

Such a product of determinants defines a determinantal point process on $\mathcal{X} = \{0, \ldots, 2N\} \times \mathbb{Z}$.

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Corollary

There is a correlation kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for every finite $\mathcal{A} \subset \mathcal{X}$

$$\textbf{Prob}\,[\exists \text{ particle at each } (m,x) \in \mathcal{A}]$$

$$= \det\left[K((m,x),(m',x'))\right]_{(m,x),(m',x') \in \mathcal{A}}$$

Eynard Mehta formula

Notation for $m < m'$

$$T_{m,m'} = T_{m'-1} \cdot \cdots \cdot T_{m+1} \cdot T_m$$

is transition matrix from level m **to level** m'**, and**

$$G = [T_{0,2N}(i,j)]_{i,j=0}^{2N-1}$$

is finite section of $T_{0,2N}$**.**

Eynard-Mehta (1998) formula for correlation kernel

$$K((m,x),(m',x')) = -\chi_{m>m'} T_{m',m}(x',x) + \sum_{i,j=0}^{2N-1} T_{0,m}(i,x) \left[G^{-1}\right]_{j,i} T_{m',2N}(x',j)$$

- **How to invert the matrix** G**?**

7. Determinantal point process: new result for periodic T_m

Periodic transition matrices

T_m **is** 2**-periodic:** $T_m(x+2, y+2) = T_m(x, y)$ **for** $x, y \in \mathbb{Z}$

Block Toeplitz matrix $$T_m = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & B_0 & B_1 & \ddots & \\ \ddots & B_{-1} & B_0 & B_1 & \ddots \\ & \ddots & B_{-1} & B_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

Periodic transition matrices

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with **block symbol**

$$A_m(z) = \sum_{j=-\infty}^{\infty} B_j z^j = B_0 + B_1 z = \begin{cases} \begin{pmatrix} 1 & \alpha \\ z & 1 \end{pmatrix} & \text{if } m \text{ is even}, \\ \begin{pmatrix} 1 & 1 \\ \alpha z & 1 \end{pmatrix} & \text{if } m \text{ is odd}. \end{cases}$$

- **Notation** $A(z) = A_1(z) A_0(z)$

Double contour integral formula

Theorem (Duits + K for this special case)

Suppose hexagon of size $2N$. Then

$$\begin{pmatrix} K(2m,2x;2m',2y) & K(2m,2x+1;2m',2y) \\ K(2m,2x;2m',2y+1) & K(2m,2x+1,2m',2y+1) \end{pmatrix}$$
$$= -\frac{\chi_{m>m'}}{2\pi i} \oint_\gamma A^{m-m'}(z) z^{y-x} \frac{dz}{z}$$
$$+ \frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma A^{2N-m'}(w) R_N(w,z) A^m(z) \frac{w^y}{z^{x+1} w^{2N}} dz dw$$

where $R_N(w,z)$ is a reproducing kernel for matrix valued polynomials with respect to weight matrix

$$W_N(z) = \frac{A^{2N}(z)}{z^{2N}} = \frac{1}{z^{2N}} \begin{pmatrix} 1+z & 1+\alpha \\ (1+\alpha)z & 1+\alpha^2 z \end{pmatrix}^{2N}$$

8. Matrix Valued Orthogonal Polynomials (again)

MVOP

- **Matrix valued orthogonality**

$$\frac{1}{2\pi i} \oint_\gamma P_j(z) W_N(z) P_k^t(z)\, dz = H_j \delta_{j,k}$$

Definition

Reproducing kernel for matrix polynomials

$$R_N(w,z) = \sum_{j=0}^{N-1} P_j^t(w) H_j^{-1} P_j(z)$$

- **If Q has degree $\leq N-1$, then**

$$\frac{1}{2\pi i} \oint_\gamma Q(w) W_N(w) R_N(w,z) dw = Q(z)$$

Riemann-Hilbert problem

- **There is a Christoffel-Darboux formula for R_N and a Riemann Hilbert problem for MVOP**

$Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{4\times 4}$ **satisfies**

- Y **is analytic,**
- $Y_+ = Y_- \begin{pmatrix} I_2 & W_N \\ 0_2 & I_2 \end{pmatrix}$ **on** γ**,**
- $Y(z) = (I_4 + O(z^{-1})) \begin{pmatrix} z^N I_2 & 0_2 \\ 0_2 & z^{-N} I_2 \end{pmatrix}$ **as** $z \to \infty$**.**

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Christoffel Darboux formula

$$R_N(w,z) = \frac{1}{z-w} \begin{pmatrix} 0_2 & I_2 \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}$$

Delvaux (2010)

Matrix weights and genus

Lozenge tiling of hexagon

- $A(z) = \begin{pmatrix} 1+z & 1+\alpha \\ (1+\alpha)z & 1+\alpha^2 z \end{pmatrix}$ **has eigenvalues**

$$1 + \frac{1+\alpha^2}{2} z \pm \frac{1-\alpha^2}{2} \sqrt{z\left(z + \frac{4}{(1-\alpha)^2}\right)}$$

that "live" on $y^2 = z\left(z + \frac{4}{(1-\alpha)^2}\right)$ $\rightarrow$ **genus zero**

Matrix weights and genus

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Two periodic Aztec diamond

- **Similar analysis leads to** $\begin{pmatrix} 2\alpha z & \alpha(z+1) \\ \alpha^{-1} z(z+1) & 2\alpha^{-1} z \end{pmatrix}$

  with eigenvalues

$$(\alpha + \alpha^{-1})z \pm \sqrt{z(z+\alpha^2)(z+\alpha^{-2})}$$

$\rightarrow$ **genus one and this leads to gas phase**

9. Results for Aztec diamond

Explicit formulas

- **MVOP of degree N is explicit for N even**

$$P_N(z) = (z-1)^N z^{N/2} A^{-N}(z)$$

- **Explicit formula for correlation kernel (double contour part only)**

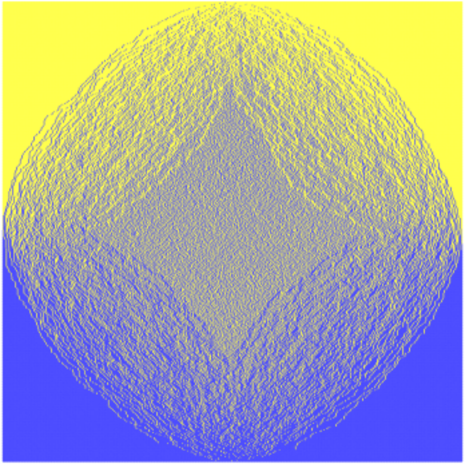
$$\frac{1}{(2\pi i)^2} \oint_{\gamma_{0,1}} \frac{dz}{z} \oint_{\gamma_1} \frac{dw}{z-w} A^{N-m'}(w) F(w) A^{-N+m}(z) \times \frac{z^{N/2}(z-1)^N}{w^{N/2}(w-1)^N} \frac{w^{(m'+n')/2}}{z^{(m+n)/2}}$$

with $F(w) = \frac{1}{2} I_2$

$$+ \frac{1}{2\sqrt{w(w+\alpha^2)(w+\alpha^{-2})}} \begin{pmatrix} (\alpha - \alpha^{-1})w & \alpha(w+1) \\ \alpha^{-1} w(w+1) & -(\alpha - \alpha^{-1})w \end{pmatrix}$$

Steepest descent

- **Classical steepest descent for integrals on the Riemann surface explains the phases and transitions between phases**



10. Results for hexagon

Scalar orthogonality

MVOP for two periodic hexagon are expressed in terms of scalar OP of degree $2N$

$$\frac{1}{2\pi i}\oint_{\gamma_1} P_{2N}(\zeta)\left(\frac{(\zeta-\alpha)^2}{\zeta(\zeta-1)^2}\right)^{2N}\zeta^k d\zeta = 0, \qquad k = 0, 1, \ldots, 2N-1.$$

- **Non-hermitian orthogonality with respect to varying weight**

Scalar orthogonality

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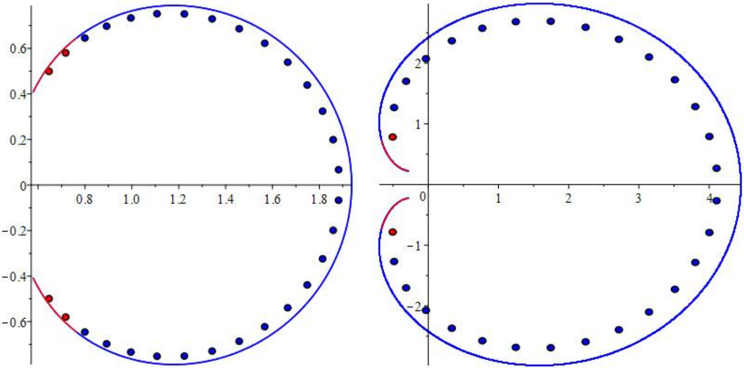
$$k = 0, 1, \ldots, 2N-1.$$

- **Non-hermitian orthogonality with respect to varying weight**
- **We can see the phase transition at $\alpha = 1/9$ in the behavior of the zeros of P_{2N} as $N \to \infty$.**

Zeros



$\alpha = 1/2$ $\alpha = 1/8$

- **Curve closes for $\alpha = 1/9$.**
- **Analysis uses logarithmic potential theory, *S*-curves in external field, and the Riemann-Hilbert problem**

Thank you for your attention