A lenticular version of a von Neumann inequality

Bernhard Beckermann* and Michel Crouzeix

Abstract

We generalize to lens-shaped domains the classical von Neumann inequality for the disk.

1 Introduction

We will say that L is a convex lens-shaped domain of the complex plane, with vertices σ and σ', if

- either there exist two disks
  $D_1 := \{z \in \mathbb{C}\,; |z-\alpha_1| < r_1\}$ and $D_2 := \{z \in \mathbb{C}\,; |z-\alpha_2| < r_2\}$
  such that $L = D_1 \cap D_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial D_2$,
- either there exist a disk and a half-plane
  $D_1 := \{z \in \mathbb{C}\,; |z-\alpha_1| < r_1\}$ and $\Pi_2 := \{z \in \mathbb{C}\,; \mathrm{Re}\, e^{i\theta}(z-\sigma) < 0\}$
  such that $L = D_1 \cap \Pi_2$, $\sigma \neq \sigma'$ and $\{\sigma, \sigma'\} = \partial D_1 \cap \partial \Pi_2$.

We will denote by $2\alpha \in]0, \pi]$ the angle of the lens L at the vertices. We will consider also as a lens the limit case where $L = D_1 = D_2$ is a disk. Then, any point of the boundary may be considered as a vertex and $\alpha = \frac{\pi}{2}$.

Now let us consider a bounded linear operator $A \in \mathcal{B}(H)$ on a complex Hilbert space H. We will say that the operator A is of the lenticular L-type if we have

- $\|A-\alpha_1 I\| \leq r_1$ and $\|A-\alpha_2 I\| \leq r_2$, if $L = D_1 \cap D_2$,
- $\|A-\alpha_1 I\| \leq r_1$ and $\mathrm{Re}\, e^{i\theta}((A-\sigma)v, v) \leq 0, \forall\, v \in H$, if $L = D_1 \cap \Pi_2$.

*Supported in part by INTAS network NeCCA 03-51-6637, and in part by the Ministry of Science and Technology (MCYT) of Spain and the European Regional Development Fund (ERDF) through the grant BFM2001-3878-C02-02.

In this paper, the norm used for a linear operator on a Hilbert space H (as well as for a matrix) is always the operator norm induced by the hilbertian structure.

The aim of this paper is to prove the following result.

Theorem 1. *Let L be a convex lens-shaped domain of the complex plane with angle 2α. There exists a best constant $C(\alpha) \in \mathbb{R}$ such that the inequality*

$$\|p(A)\| \le C(\alpha) \sup_{z\in L} |p(z)|, \tag{1}$$

holds for all polynomials $p : \mathbb{C} \to \mathbb{C}$, for all linear operators $A \in \mathcal{B}(H)$ of L-type and for all Hilbert spaces H. Furthermore this constant, which is only depending on the angle α, is a continuous decreasing function of $\alpha \in (0, \frac{\pi}{2}]$ and we have the estimate

$$\frac{\pi}{2\alpha} \sin\alpha \le C(\alpha) \le \min(2 + 2/\sqrt{3}, \frac{\pi - \alpha}{\alpha}). \tag{2}$$

Note that for $\alpha = \frac{\pi}{2}$, which corresponds to the case where L is a disk, we have $C(\frac{\pi}{2}) = 1$ and we recover a famous von Neumann inequality [4]. Theorem 1 can be generalized in several directions. For instance, by Mergelyan's Theorem, the inequality (1) remains valid if instead of polynomials we take p holomorphic in L and continuous in $\overline{L}$. The theorem is also valid in a completely bounded form. More precisely, there exists a continuous decreasing function $C_{cb}(\alpha)$ (which satisfies the bounds (2)) such that the inequality

$$\|P(A)\| \le C_{cb}(\alpha) \sup_{z\in L} \|P(z)\|,$$

holds for all polynomials with matrix values $P : \mathbb{C} \to \mathbb{C}^{n,n}$, for all $n \ge 1$, for all linear operators $A \in \mathcal{B}(H)$ of type L and for all Hilbert spaces H.

Except for the value $\alpha = \frac{\pi}{2}$, we do not know the exact values of $C(\alpha)$ and $C_{cb}(\alpha)$. Even more, we do not know if $C(\alpha) = C_{cb}(\alpha)$ or not... A small improvement

$$C_{cb}(\alpha) \le \frac{\pi - \alpha}{\pi}\Big(2 - \frac{2}{\pi} \log\tan\big(\frac{\alpha\,\pi}{4(\pi-\alpha)}\big)\Big),$$

of the upper bound in (2) can be deduced from Theorem 4.2 in [1].

We should mention that a preliminary version of this theorem, in the particular case where L has a straight face, has been implicitly used in [2] to study the convergence of the GMRES method.

2 The proof

Our proof of Theorem 1 is heavily based on the result of the paper [3], that we recall now. Let S_α be a convex sector of the complex plane with angle 2α. An operator $B \in \mathcal{B}(H)$ is said S_α-accretive iff $(Bv, v) \in \overline{S_\alpha}$, for all $v \in H$ satisfying $\|v\| = 1$. The result proved in [3] is

there exists a best constant $C_\alpha \in \mathbb{R}$ such that the inequality

$$\|r(B)\| \le C_\alpha \sup_{z\in S_\alpha} |r(z)|, \tag{3}$$

holds for all rational functions bounded in S_α and for all S_α-accretive operators B (it is easily seen that this definition is not depending on the particular choice of the sector). Furthermore this constant C_α is a continuous and decreasing function of α and it satisfies the estimates

$$\frac{\pi}{2\alpha}\sin\alpha \le C_\alpha \le \min(2 + 2/\sqrt{3}, \frac{\pi-\alpha}{\alpha}).$$

Therefore it is sufficient to prove that $C_\alpha = C(\alpha)$ for getting the theorem.

We turn now to the proof of this equality. Without loss of generality, we can assume that the vertices of L are $\sigma = 0$ and $\sigma' = 1$, and that $\operatorname{Im}\alpha_1 < 0$. We introduce the rational function $g(z) := \frac{z}{z-1}$. It is easily seen that g is an involution and that g realizes a bijection of the disk $D_j := \{z \in \mathbb{C}\,; |z-\alpha_j| < |\alpha_j|\}$ onto the half-plane $P_j := \{z \in \mathbb{C}\,; \operatorname{Re}\bar\alpha_j z < 0\}$. In the case where the lens has a straight face $L = D_1 \cap \Pi_2$ with $\Pi_2 := \{z \in \mathbb{C}\,; \operatorname{Re} i\,z < 0\}$, we remark also that g realizes a bijection of the half-plane Π_2 onto the half-plane $P_2 := \{z \in \mathbb{C}\,; \operatorname{Re} i\,z > 0\}$. Therefore g is a bijection of the lens L onto the sector $S_\alpha = P_1 \cap P_2$. Note that the sector and the lens have the same angle 2α and that $1 \notin S_\alpha$.

Let us consider now a linear operator A such that 1 does not belong to its spectrum $\sigma(A)$, and we set $B = g(A) = A(A-I)^{-1}$. It is easily seen that $(B-I)(A-I) = I$, thus $1 \notin \sigma(B)$, and $A = g(B)$.

Using that $\operatorname{Re}\alpha_j = \frac{1}{2}$, we remark by setting $v = (A-I)w$ that

$$\begin{aligned}
& |\alpha_j|^2\,\|w\|^2 - \|(A-\alpha_j I)w\|^2 \ge 0, \forall\, w \in H,\\
\iff\quad & \|Aw\|^2 - 2\operatorname{Re}\bar\alpha_j(Aw, w) \le 0, \forall\, w \in H,\\
\iff\quad & 2\operatorname{Re}\bar\alpha_j(Aw, (A-I)w) \le 0, \forall\, w \in H,\\
\iff\quad & \operatorname{Re}\bar\alpha_j(Bv, v) \le 0, \forall\, v \in H.
\end{aligned}$$

Therefore if the linear operator A is of L-type, then the operator B is S_α-accretive. Conversely if B is S_α-accretive then $1 \notin \sigma(B)$ (since $1 \notin S_\alpha$) and $A = g(B)$ is of L-type.

Let us consider now a polynomial p and set $r(z) = p(g(z))$, then we have $p(A) = r(B)$ and $\sup_{z\in S_\alpha} |r(z)| = \sup_{\zeta\in L} |p(\zeta)|$. We deduce from (3) that

$$\|p(A)\| \leq C_\alpha \sup_{\zeta\in L} |p(\zeta)|.$$

Note that, if $1 \in \sigma(A)$, then for $0 < \varepsilon < 1$, the operator $A_\varepsilon := (1-\varepsilon)A$ is of L-type and $1 \notin \sigma(A_\varepsilon)$, which shows that the previous inequality is still valid by using a limit argument. Therefore we have $C(\alpha) \leq C_\alpha$.

Conversely if we consider a rational function r bounded in S_α, $p(z) = r(g(z))$ is a rational function bounded in L, therefore we deduce from (1) (which is valid for such rational functions) that

$$\|r(B)\| \leq C(\alpha) \sup_{z\in S_\alpha} |r(z)|,$$

which implies $C(\alpha) \geq C_\alpha$, and thus finally $C(\alpha) = C_\alpha$.

The proofs would be the same for the completely bounded form of our estimates.

References

[1] C. Badea, M. Crouzeix, B. Delyon, Convex domains and K-spectral sets, preprint

[2] B. Beckermann, S.A. Goreinov, E.E. Tyrtyshnikov, Some remarks on the Elman estimate for GMRES, preprint.

[3] M. Crouzeix, B. Delyon, Some estimates for analytic functions of strip or sectorial operators, *Arch. Math.*, **81**, 553-566, 2003.

[4] J. von Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachrichten* **4** (1951) 258–281.

e-mail : Bernhard.Beckermann@univ-lille1.fr,

Address : Laboratoire de Mathématiques Paul Painlevé, UMR CNRS 8524,
Bâtiment M2 - Cité Scientifique,
Université des Sciences et Technologies de Lille,
F-59655 Villeneuve d'Ascq cedex, France

e-mail : michel.crouzeix@univ-rennes1.fr,

Address : IRMAR, UMR CNRS 6625, Université de Rennes 1,
Campus de Beaulieu, F-35042 Rennes cedex, France

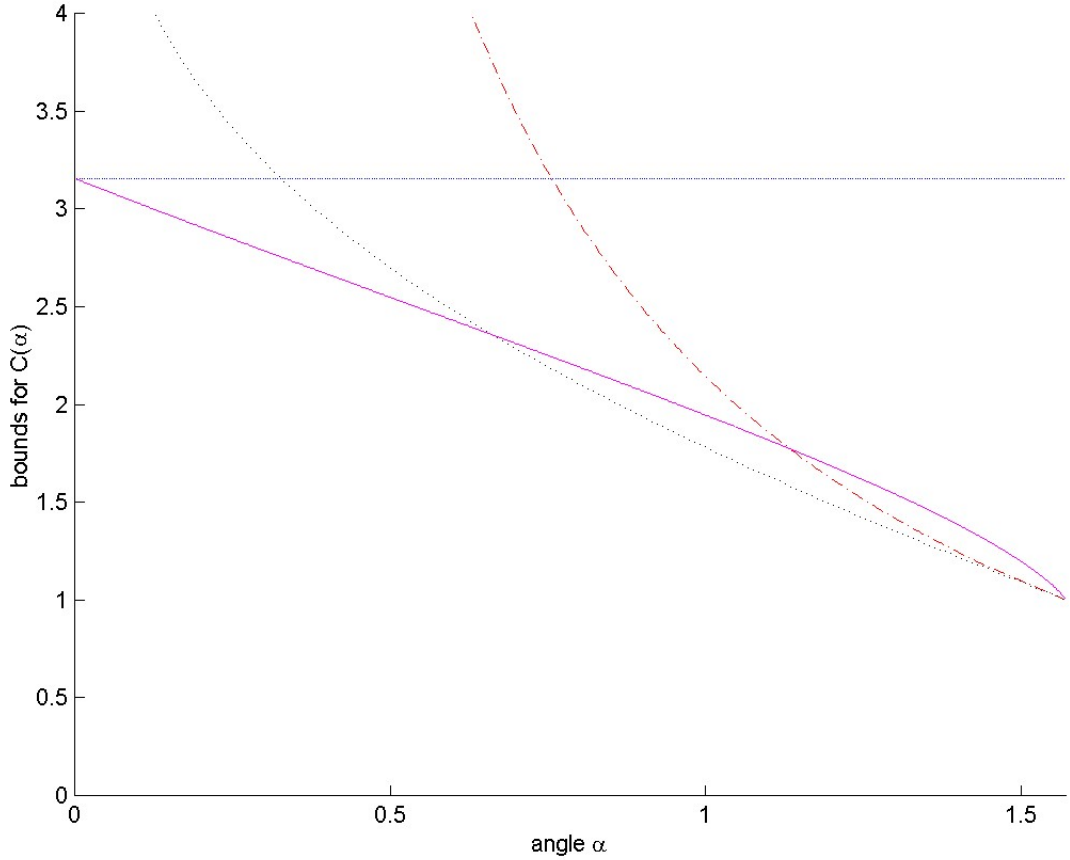

Figure 1: *Comparision of the different upper bounds for $C(\alpha)$, in magenta/solid the new bound, in blue/dashed the Crouzeix/Delyon bound (constant $2 + 2/\sqrt{3}$), in black/dashed the Crouzeix/Delyon bound (Fourier), in red/dashdot the Badea/Crouzeix/Delyon bound ($(\pi - \alpha)/\alpha$)*

A Slight improvement for a sector

Theorem 2. *Let A be an operator acting on a Hilbert space H, with numerical range $W(A) \subset \overline{S_\alpha}$, with the sector $S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$, $0 < \alpha \leq \pi/2$. Then for any rational function r being analytic in $\overline{S_\alpha}$ we have that*

$$||r(A)|| \leq (2 - \frac{2\alpha}{\pi} + \mu(\alpha))\, ||r||_{L_\infty(S_\alpha)},$$

or, more precisely,

$$||r(A)|| \leq (2 - \frac{2\alpha}{\pi})\, ||r||_{L_\infty(S_\alpha)} + \mu(\alpha)\, ||r||_{L_\infty(\mathbb{R})},$$

where

$$\mu(\alpha) = \frac{\sin(2\alpha)}{\pi} \int_0^\infty \frac{du}{u^2 - 2\cos 2\alpha u + \cos^2 \alpha},$$

or

$$\mu(\alpha) = \frac{2\cos(\alpha)}{\pi\sqrt{|4\cos^2(\alpha)-1|}}\beta(\alpha),$$

with $\beta(\alpha) = \arccos(\frac{\cos(\pi-2\alpha)}{\cos(\alpha)})$ *if* $\alpha \leq \frac{\pi}{3}$, *and* $\beta(\alpha) = \arg\cosh(\frac{\cos(\pi-2\alpha)}{\cos(\alpha)})$ *otherwise.*

Proof. We may suppose without loss of generality that $r(\infty) = 0$ (else consider $r_\epsilon(z) = r(z)/(1 - \epsilon e^{-\alpha i} z)$ for $\epsilon > 0$ and let tend $\epsilon \to 0$). Also, we may suppose without loss of generality that A is a bounded invertible operator...

As in [1, 3] we write

$$r(A) = \int_{S_\alpha} r(\sigma)\mu(\sigma, A)|d\sigma| + \frac{1}{2\pi}\int_{S_\alpha} r(\sigma)(\overline{\sigma} - A^*)^{-1} d\overline{\sigma}$$

and recall from [1, Section 3] that

$$|| \int_{S_\alpha} r(\sigma)\mu(\sigma, A)|d\sigma||| \leq ||r||_{L_\infty(S_\alpha)} \int_{S_\alpha} \mu(\sigma, A)|d\sigma| = (2 - \frac{2\alpha}{\pi})\, ||r||_{L_\infty(S_\alpha)}.$$

Hence we only have to bound the norm of

$$\begin{aligned} M &:= \frac{1}{2\pi i}\int_{S_\alpha} r(\sigma)(\overline{\sigma} - A^*)^{-1} d\overline{\sigma} \\ &= \frac{1}{2\pi i}\int_0^\infty [r(e^{-i\alpha}x)(x - e^{-i\alpha}A^*)^{-1} - r(e^{i\alpha}x)(x - e^{+i\alpha}A^*)^{-1}]dx. \end{aligned}$$

Taking into account that $||r(e^{-i\alpha}z)(z - e^{i\alpha}A^*)^{-1}|| = \mathcal{O}(|z|^{-2})_{|z|\to\infty}$, following the approach of [3] for a stripe, we may deform the above two paths of integration, leading to the formula

$$\begin{aligned} M &= \frac{1}{2\pi i}\int_0^\infty r(x)[(x - e^{-2i\alpha}A^*)^{-1} - (x - e^{+2i\alpha}A^*)^{-1}]dx \\ &= \frac{\sin(2\alpha)}{\pi}\int_0^\infty r(x)[A^* - 2x\cos(2\alpha) + x^2(A^*)^{-1}]^{-1}dx. \end{aligned}$$

We now use the fact that, for some bounded matrix $A(x)$ with positive definite $B(x) := \mathrm{Re}(A(x) \geq 0$, and $B(x)^{-1/2}A(x)B(x)^{-1/2}$ being the identity plus i times some symmetric operator, and hence having an inverse bounded in norm by 1,

$$\begin{aligned} \|\int A(x)^{-1}dx\| &= \sup_{u,v\in H,\|u\|=\|v\|=1} |\int (A(x)^{-1}u,v)dx| \\ &\le \sup_{u,v\in H,\|u\|=\|v\|=1} \int \|B(x)^{-1/2}u\|\,\|B(x)^{-1/2}v\|dx \\ &\le \sup_{u\in H,\|u\|=1} \int (B(x)^{-1}u,u)dx = \|\int B(x)^{-1}dx\|. \end{aligned}$$

In our case, with $B = \mathrm{Re}(A)$ we have by assumption $W(A) \subset S_\alpha$ that B and $\mathrm{Re}(A^{-1})$ are positive definite, and thus

$$\sup_{u\in H,\|u\|=1} (\mathrm{Re}(A^{-1})B^{-1/2}u, B^{-1/2}u) = \sup_{u\in H,\|u\|=1} ((I + (B^{-1/2}\,\mathrm{Im}(A)B^{-1/2})^2)^{-1}u,u)$$

$$= \sup_{u\in H} \frac{1}{1+\frac{((B^{-1/2}\,\mathrm{Im}(A)B^{-1/2})^2u,u)}{(u,u)}} = \sup_{u\in H} \frac{1}{1+|\frac{((B^{-1/2}\,\mathrm{Im}(A)B^{-1/2})u,u)}{(u,u)}|^2}$$

$$= \sup_{u\in H} \frac{1}{1+|\frac{(\mathrm{Im}(A)u,u)}{(\mathrm{Re}(A)u,u)}|^2} \ge \frac{1}{1+\tan^2(\alpha)} = \cos^2(\alpha),$$

implying that

$$\begin{aligned} &\mathrm{Re}(A^* - 2x\cos(2\alpha) + x^2(A^*)^{-1}) \ge B - 2x\cos(2\alpha) + x^2\cos^2(\alpha)\,B^{-1} \\ &B^{-1/2}(B - x\cos(\alpha))^2 B^{-1/2} + 2x(\cos(\alpha) - \cos(2\alpha)) > 0, \end{aligned}$$

since $\alpha \in (0, \pi/2)$. The above estimate allows us to conclude that

$$\|M\| \le \frac{\sin(2\alpha)}{\pi}\, \|\widetilde{M}\|\; \|r\|_{L_\infty([0,+\infty))},$$

with the self-adjoint and positive definite operator

$$\begin{aligned} \widetilde{M} &= \int_0^\infty [B - 2x\cos(2\alpha) + x^2\cos^2(\alpha)\,B^{-1}]^{-1}\,dx \\ &= \frac{1}{\cos(\alpha)} \int_0^\infty [B - 2x\frac{\cos(2\alpha)}{\cos(\alpha)} + x^2\,B^{-1}]^{-1}\,dx. \end{aligned}$$

If now $\frac{\cos(2\alpha)}{\cos(\alpha)} = \cos(\beta)$ for some $\beta \in (0,\pi]$ (which is true for $\alpha \le \pi/3$), then

$$\frac{\cos(\alpha)\sin(\beta)}{\pi}\widetilde{M} = \frac{1}{2\pi i}\int_0^\infty [(x - e^{i\beta}B)^{-1} - (x - e^{-i\beta}B)^{-1}]dx = \frac{1}{2}\int_{S_\beta} \mu(x,B) = 1 - \beta/\pi,$$

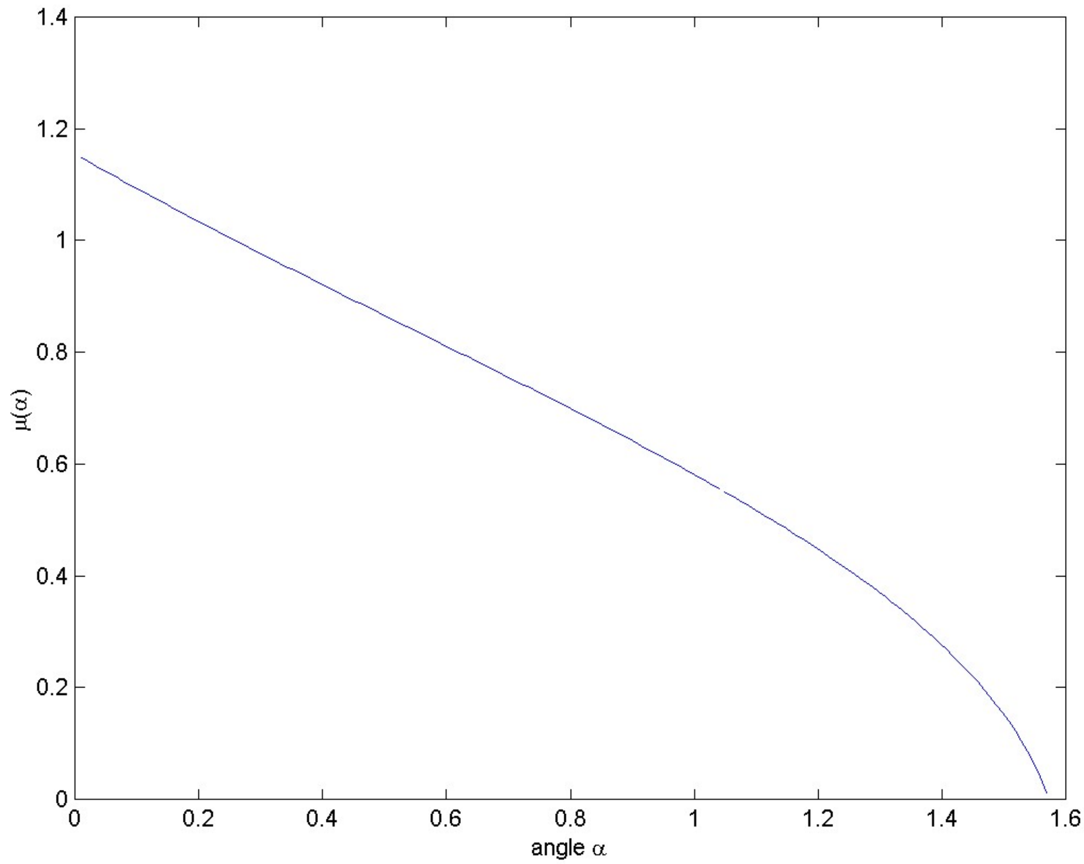

Figure 2: *Plot for $\mu(\alpha)$. Graphically it is decreasing.*

as claimed in the assertion. If on the other hand we have that $\frac{\cos(2\alpha)}{\cos(\alpha)} = -\cosh(2\beta)$ for some $\beta > 0$, then

$$\begin{aligned}\cos(\alpha)\sinh(2\beta)\widetilde{M} &= \frac{1}{2}\int_0^\infty [(x+e^{-\beta}B)^{-1} - (x+e^{\beta}B)^{-1}]dx \\ &= \frac{1}{2}[\log((x+e^{-\beta}B)(x+e^{\beta}B)^{-1}]_0^\infty = \beta,\end{aligned}$$

again as claimed in the assertion. □

The different estimates for $C(\alpha)$ are graphically displayed in Figure 1. The new bound seems to be best for $\alpha \in (0, 0.22\pi]$.

It seems to be true but a bit tricky to prove that μ decreases. We have $\mu(0) = 2/\sqrt{3}$ as before, $\mu(\pi/4) = 1/\sqrt{2}$, $\mu(\pi/3) = \sqrt{3}/\pi$, $\mu(\pi/2) = 0$, compare with Figure 2.