

EXTREMAL RATIONAL FUNCTIONS ON SYMMETRIC DISCRETE SETS AND SUPERLINEAR CONVERGENCE OF THE ADI METHOD

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Abstract. There is a very fruitful interaction between numerical linear algebra and logarithmic potential theory. For instance, we may describe weak asymptotics for a polynomial extremal problem occurring in the convergence analysis of conjugate gradients (CG) or of Ritz values, here the link with a constrained minimal energy problem allows to quantify the effect of super-linear convergence.

In the present paper, we introduce an extremal problem for rational functions on two discrete sets E_N, F_N , the so-called third Zolotarev problem. Roughly speaking, we look for rational functions of a prescribed degree which are as small as possible on E_N and as large as possible on F_N . Such a problem occurs naturally in the convergence analysis for the so-called ADI method for solving Lyapunov equations, but also in the approximation of particular matrix functions, and in the decay rate of singular values of matrices with small displacement rank.

Whereas asymptotics for this extremal problem on continuous sets is well-studied, we require new tools in order to handle the case of discrete sets. The main contribution of this paper is to show that a minimal energy problem for signed and constrained measures allows to describe the asymptotics. This new minimal energy problem is a natural extension of that of Kuijlaars and Beckermann used to describe weak asymptotics for the polynomial extremal problem related to CG.

We discuss the sharpness of our asymptotic findings for general discrete sets in the complex plane, and suggest new formulas for the extremal constant in the particular case where the two discrete sets are real and symmetric with respect to the origin. Finally, by discussing a model example, we show the impact of our findings for analyzing the rate of superlinear convergence of the ADI method applied to a Lyapunov equation.

Key words. Logarithmic potential theory, Minimal energy problems with constraint, ADI

AMS subject classifications. 30E10, 41A20, 65F10, 65E05

1. Introduction. It is already known for a long time that there is an extremely fruitful interaction between the convergence analysis of iterative methods in numerical linear algebra and complex approximation theory [Tre90], and in particular logarithmic potential theory [DTT98, TrBa97]. In the beginning of this century, Kuijlaars [Kui00] for the convergence of Ritz values and Beckermann & Kuijlaars [BeK01a, BeK01b, BeK02] for the superlinear convergence of the conjugate gradients method (CG) showed that a decisive step in the convergence analysis is to understand the asymptotic behavior of the quantity

$$\min \left\{ \frac{\|p\|_{L^\infty(E_N)}}{|p(0)|} : p \text{ polynomial of degree } \leq n \right\},$$

where E_N is a discrete set (the spectrum of the underlying matrix of coefficients). In many applications like the discretization of a differential equation while varying a stepsize parameter, we have to our disposal a sequence of sets E_N , with a joint limit distribution described by some signed measure σ (the weak-star limit of the counting measures of the sets E_N for $N \rightarrow \infty$). Kuijlaars & Beckermann showed using previous results on weak asymptotics of discrete orthogonal polynomials [Rak96, DrSa97, Be00] that the knowledge of σ only allows to describe the n th root asymptotics of the above polynomial extremal problem in the limit case where $n, N \rightarrow \infty$ such that n/N tends to some constant $t \in (0, 1)$. Their expression for the limit involves the solution of an extremal problem in logarithmic potential theory, namely within all measures μ

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with total mass t satisfying the constraint $\mu \leq \sigma$ one looks for the one with minimal logarithmic energy. In terms of (two-dimensional) electrostatics, this means that we look for the equilibrium state of a conductor with a positive charge of t units which is subject to a maximum charge constraint. As a consequence, we may quantify (in a weak sense) the rate of superlinear convergence of CG in terms of the asymptotic eigenvalue distribution σ . We refer the reader to [Be06, Kui06] for different summaries on these findings, and to [LevS06] or the books [Ran95, ST97] for an introduction to logarithmic potential theory.

In the present paper we are interested in deriving weak asymptotics for the extremal constant in the rational extremal problem

$$(1.1) \quad Z_n(E_N, F_N) := \min_{r \in \mathcal{R}_n} \|r\|_{L^\infty(E_N)} \|r^{-1}\|_{L^\infty(F_N)},$$

where E_N, F_N are discrete disjoint subsets of the complex plane, and \mathcal{R}_n denotes the set of rational functions with numerator and denominator degree bounded by n . This problem is usually referred to as the third Zolotarev problem, since it generalizes one of the four extremal problems solved by Zolotarev, a student of Chebyshev, in terms of elliptic functions [Akh90, Zol32]. Roughly speaking, we are looking for a rational function being small on E_N and large on F_N . Obviously, for any closed and disjoint $E, F \subset \overline{\mathbb{C}}$ and T a Moebius transformation, it is easy to see that r is extremal for $Z_n(E, F)$ iff the rational function $r \circ T^{-1}$ is extremal for $Z_n(T(E), T(F))$. Hence, we may suppose without loss of generality that E and F are compact.

Concerning the existence of a minimizer, a KKT reformulation of our problem together with a compactness argument allows us to show that the infimum in $Z_n(E_N, F_N)$ is attained for at least one extremal function. Moreover, under the assumption that the sets E_N and F_N might be strictly separated by a straight line (resp. a circle), the proof of the existence of a minimizer becomes standard, see for instance [Ach56, Section 33] or [Bra80, Lemma V.1.1]. In this case, we may consider rational functions with zeros and poles in two disjoint open half-planes (resp. interior and exterior of a disk).

Also, the example $Z_1(\{0\}, \{-2, -1\}) = 0$ shows that the minimizer in (1.1) is not necessarily unique, since the value 0 is attained for any rational function having a zero in 0.

Let us shortly revisit the well-studied case of closed and disjoint $E, F \subset \mathbb{C}$ of positive capacity, see for instance [ST97, Section VIII] for a summary. Gonchar [Gon78] showed that the quantity $Z_n(E, F)^{1/n}$ has a limit for $n \rightarrow +\infty$ which can be described with help of electrostatics on a condenser with a plate E with positive unit charge and a plate F containing a negative unit charge, see [ST97]. This generalizes the original findings of Zolotarev [Zol32] in the case where $E = -F = [\alpha, \beta] \subset (0, +\infty)$. Zolotarev gave an explicit expression for $Z_n([\alpha, \beta], [-\beta, -\alpha])$ as well as for an underlying extremal function in terms of Jacobi elliptic functions, in particular it follows that

$$\lim_{n \rightarrow +\infty} Z_n([\alpha, \beta], [-\beta, -\alpha])^{1/n} = \rho, \quad \rho := \exp\left(-2\pi \frac{K(k)}{K'(k)}\right), \quad k = \frac{\alpha}{\beta},$$

$K'(k) = K(\sqrt{1-k^2})$, and K is the complete elliptic integral

$$K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

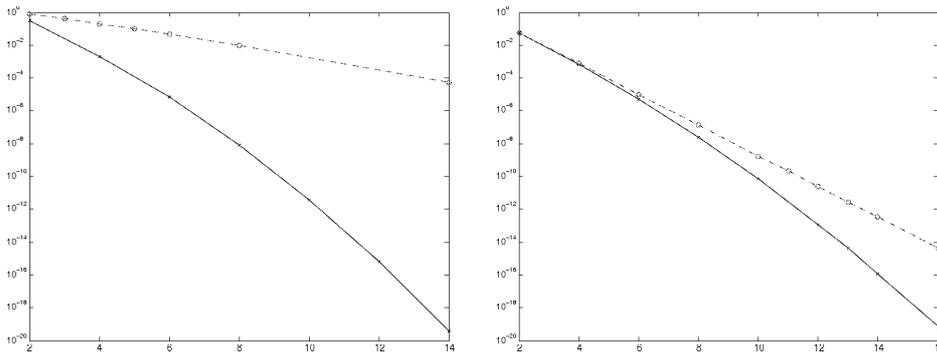


FIG. 1.1. The quantity $Z_n(E_N, -E_N)$ (solid with crosses) versus $Z_n(\text{conv}(E_N), -\text{conv}(E_N))$ (dashed with circles) on a semi-logarithmic scale for increasing degree $n = 2, 3, \dots$ and $E_N = \{1/N^4 + \cos(k\pi/(2N)) : k = 1, \dots, N\}$, $N = 20$, (on the left) and $E_N = \{k/N : k = 1, \dots, N\}$, $N = 25$ (on the right). The distance between the two curves is more important if E_N is closer to $F_N = -E_N$. We obtain in accordance with (1.2) approximate straight lines for the convex hull, whereas the curves for discrete sets seem to be concave.

By the way, in the case of intervals we know from [Bra87, Appendix], [Bra80, Theorem 5.5.D.5] and [ST97, Theorem 8.3.1] the stronger inequalities

$$(1.2) \quad \rho^{n-1} \leq Z_n([\alpha, \beta], [-\beta, -\alpha]) \leq 16\rho^{n-1}.$$

In numerical linear algebra and control theory (see for instance the examples at the end of this section), one has to find upper bounds for $Z_n(E_N, F_N)$ where typically E_N, F_N are (parts) of spectra of given matrices, and their asymptotic distribution for $N \rightarrow \infty$ is often known. To our knowledge, the Zolotarev problem for discrete sets has not been considered before; usually, the authors replace E_N, F_N by some larger disjoint sets of positive capacity (e.g., their convex hull), and then use the trivial inequality $Z_n(E_N, F_N) \leq Z_n(E, F)$ together with the results of Gonchar and Zolotarev for obtaining an upper bound, see, e.g., [EFLSV02]. However, this step may lead to a severe over-estimation, see for instance Figure 1.1. The reason is that a rational function small on a discrete set is not necessarily small on its convex hull, see for instance Figure 1.2 for a simple example. This observation is also the starting point of the work of Beckermann & Kuijlaars for CG: one should take into account the fine structure, that is, the asymptotic distribution of the families of sets E_N and F_N for $N \rightarrow \infty$ in order to get more precise results.

In this paper, we derive asymptotics for $Z_n(E_N, F_N)^{1/N}$ for $n, N \rightarrow +\infty$ such that $n/N \rightarrow t$ in the case where the families of sets $(E_N)_N, (F_N)_N$ have an asymptotic distribution described by some limit measures (see the assumptions (A1), (A2) of §2). Thus the asymptotics will strongly depend on these limit measures and the parameter t . For simplicity, we restrict ourselves to limit measures being sufficiently regular (see the assumption (A3) of §2).

In Theorem 2.1 we establish an upper bound for $\limsup Z_n(E_N, F_N)^{1/N}$. However, it is clear that this bound may only be sharp if the elements of E_N, F_N are sufficiently separated, since clusters of close but distinct elements of E_N or F_N are essentially contributing as one point for $Z_n(E_N, F_N)$, whereas they may change completely the asymptotic distribution. Our separating assumptions (A4), (A5) are shown to be sufficient for sharpness. In our proofs we have been partly inspired by the work

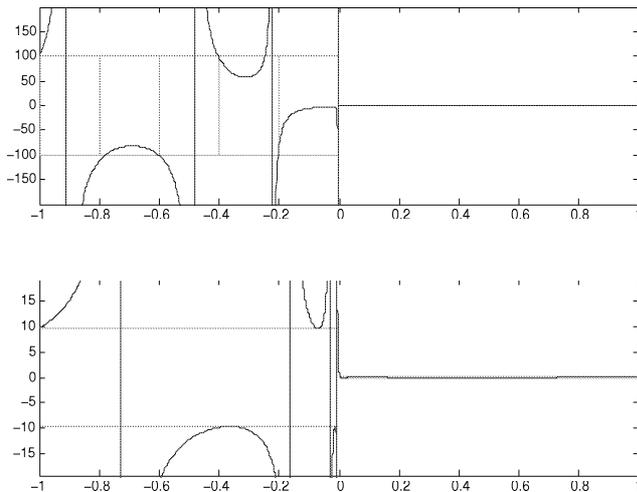


FIG. 1.2. The extremal rational functions r for $Z_4(E_N, -E_N) \approx 10^{-4}$ (top) versus $Z_4(\text{conv}(E_N), -\text{conv}(E_N)) \approx 10^{-2}$ (bottom) for $E_N = 1/200 + \{0/5, 1/5, \dots, 5/5\}$. The long vertical lines correspond to the location of poles, and, for $x > 0$, one may reconstruct r by using the symmetry $r(-x) = 1/r(x)$. Clearly, the extremal function for $Z_4(E_N, -E_N)$ is only large in $-E_N$ (see the small vertical lines) but not uniformly in its convex hull.

[BeK01a] on CG, where one may also find the first four assumptions.

The assertion of Theorem 2.1 involves a quantity which occurs while discussing a condenser in electrostatics with a plate of positive charge t and a plate of negative charge t , where again both charges are subject to a maximum charge constraint. That is, compared to [BeK01a] we no longer work with measures but with signed measures. We establish in Theorem 2.2 existence and uniqueness of an extremal signed measure for this extremal problem in logarithmic potential theory, and give a characterization in terms of equilibrium conditions, involving the constants of Theorem 2.1.

In Sections §3 and §4 we give the proofs of our main Theorem 2.1, and Theorem 2.2, respectively. The aim of Section 5 is to derive in Theorem 5.1 an integral representation for the extremal constant occurring in Theorem 2.1. The corresponding representation of [BeK01a] relies on the Buyarov-Rakhmanov formulas [BuRa99], but in our case we first have to make in Proposition 5.2 a link to vector-valued minimal energy problems with external fields, before being able to use the findings of Lapik [La06] on vector-valued Mhaskar-Saff-Rakhmanov functionals. For the particular case of real symmetric sets $E_N = -F_N$, we describe in Corollary 5.7 and Proposition 5.8 how to make this integral formula more explicit, in terms of complete elliptic integrals. We illustrate our findings in §6 by discussing two (academic) numerical examples.

But before entering into details, let us shortly describe some applications for Zolotarev problems on discrete sets.

1.1. Convergence analysis for the ADI method. The ADI method was designed by Peaceman and Rachford in 1955 [PeRa55] in order to solve elliptic PDEs. This method has been adapted by Wachspress to the case of the general Sylvester equation presented here, see for instance [Wa63], [Wa69] and also the work of Birkhoff and Varga [BiVa59].

A Sylvester equation in the unknown X is given by

$$(1.3) \quad AX - XB = C,$$

where A, B, C are given real matrices of order $N \times N$. We will suppose here that A, B are normal, with disjoint spectra $\Lambda(A), \Lambda(B)$, and thus (1.3) has a unique solution. For the special case $B = -A$ being symmetric positive definite, (1.3) reduces to a Lyapunov equation. An iteration of the ADI method with parameters p_k and q_k is given by the equations

$$(1.4) \quad \begin{aligned} (A - p_k I)X_{k+1/2} &= X_k(B - p_k I) + C, \\ X_{k+1}(B - q_k I) &= (A - q_k I)X_{k+1/2} - C. \end{aligned}$$

Then elementary computations show that

$$\frac{\|X - X_n\|}{\|X - X_0\|} \leq \|r_n(A)\| \|r_n(B)^{-1}\|, \quad r_n(z) := \prod_{k=0}^{n-1} \frac{z - q_k}{z - p_k}.$$

As a consequence, we obtain the following upper bound for the error of the ADI method after n iterations

$$(1.5) \quad \frac{\|X - X_n\|}{\|X - X_0\|} \leq \|r_n\|_{L^\infty(\Lambda(A))} \|r_n^{-1}\|_{L^\infty(\Lambda(B))}.$$

Now, we consider $(A_N)_N, (B_N)_N$ two sequences of normal matrices and stop our ADI iteration at index $n = n(N)$ such that $n/N \rightarrow t > 0$ when $N \rightarrow +\infty$. For optimal parameters p_k, q_k (depending on n and N), we have

$$\frac{\|X - X_n\|}{\|X - X_0\|} \leq Z_n(E_N, F_N), \quad E_N = \Lambda(A_N), \quad F_N = \Lambda(B_N),$$

and hence it is of interest to know $\limsup Z_n(E_N, F_N)^{1/N}$ as a function of t and the limiting distribution of E_N, F_N for $N \rightarrow \infty$. Notice that the occurrence of sequences of Sylvester equations may be natural, e.g., in the context of finite difference discretisation. For instance, in the discretisation of the 2D-Poisson problem with central differences and steplength $1/(N+1)$, the matrices $A_N = -B_N$ represent the discrete 1D-Laplacian, and their asymptotic eigenvalue distribution is described by the arcsin (or Robin equilibrium) distribution on $[0, 4]$. Notice that a similar analysis also applies for Sylvester equations with rectangular X , as occurring, e.g., for the 3D-Laplacian, with A as before, and $-B$ representing the discretized 2D-Laplacian.

In addition, for our analysis we require (at least approximately) optimal parameters, and thus nearly extremal rational functions for $Z_n(E_N, F_N)$. We will discuss our choice of the parameters in §6.

There exist also variants of the ADI method where one takes different degrees for the numerator and the denominator, following so-called ray sequences [LevRei93]. For continuous sets E, F , the corresponding weak asymptotics have been given by [LevS01], and an extension to discrete sets should be possible by combining ideas of this paper with the findings in [LevS01].

1.2. Decay rate of singular values for matrices with small displacement rank. For given normal $A = A_N, B = B_N$, a (possibly rectangular) matrix X has a (A, B) displacement rank ρ if the matrix C defined by (1.3) has rank ρ . For special choices of A, B like diagonal matrices and τ -shifts, many structured matrices like Vandermonde matrices, Krylov matrices, Cauchy matrices, Pick matrices, Loewner matrices, Hankel matrices etc have a displacement rank being equal to 1 or 2.

Denote as usually the singular values of X by $\sigma_1(X) \geq \sigma_2(X) \geq \sigma_3(X) \geq \dots$, then it is shown in [Be04] that for all integers $j, n \geq 1$,

$$(1.6) \quad \sigma_{j+n\rho}(X_N) \leq Z_n(\Lambda(A_N), \Lambda(B_N)) \sigma_j(X_N), \quad \rho = \text{rank}(A_N X_N - X_N B_N),$$

that is, if the spectra of A_N and B_N are well separated then there is an important decay in the singular values of X . Hence our asymptotics help for a better understanding of the singular values of such matrices.

1.3. Approximating the Heaviside function. Given a discrete set E_N , consider the corresponding sign and Heaviside functions defined by

$$\text{sign}_N(z) = 2H_N(z) - 1, \quad H_N(z) = \begin{cases} 1 & \text{if } z \in E_N, \\ 0 & \text{else.} \end{cases}$$

Given an hermitian invertible matrix A_N with $E_N \subset \Lambda(A_N)$ the set of positive eigenvalues, the expression $H_N(A_N)$ gives the eigenprojector onto the eigenspace corresponding to the positive eigenvalues of A_N , whereas $\text{sign}_N(A_N)$ corresponds to the classical sign function. The approximation of $\text{sign}_N(A_N)$ via rational functions is an important task in quantum chromodynamics, see, e.g., [EFLSV02]. Other applications for the sign and the Heaviside function applied to matrices are given in the recent book [Hig08].

In order to see that there is a link between the Zolotarev problem and the approximation of our discrete Heaviside function (and hence of the (discrete) signum function), let

$$E_N = \Lambda(A_N) \cap [0, +\infty[, \quad F_N = \Lambda(A_N) \setminus E_N.$$

We claim that

$$(1.7) \quad \frac{Z_{n,N}}{1 + Z_{n,N}} \leq S_{n,N} := \min_{r^* \in \mathcal{R}_{n,n}} \|H_N - r^*\|_{L^\infty(E_N \cup F_N)} \leq \frac{Z_{n,N}}{1 - Z_{n,N}},$$

where we wrote shorter $Z_{n,N} = \sqrt{Z_n(E_N, F_N)}$. For establishing (1.7), consider $r, r^* \in \mathcal{R}_{n,n}$ with

$$Z = \max\left\{\|r\|_{L^\infty(E_N)}, \|1/r\|_{L^\infty(F_N)}\right\} < 1, \quad S = \|H_N - r^*\|_{L^\infty(E_N \cup F_N)} < 1.$$

Then for $r^* = 1/(1+r)$ we find that

$$S = \max\left\{\left\|\frac{r}{1+r}\right\|_{L^\infty(E_N)}, \left\|\frac{1/r}{1+1/r}\right\|_{L^\infty(F_N)}\right\} \leq \frac{Z}{1-Z}$$

$$Z = \max\left\{\left\|\frac{1-r^*}{r^*}\right\|_{L^\infty(E_N)}, \left\|\frac{r^*}{1-r^*}\right\|_{L^\infty(F_N)}\right\} \leq \frac{S}{1-S}.$$

Since there exists $r \in \mathcal{R}_{n,n}$ normalized such that $Z = \sqrt{Z_n(E_N, F_N)} \leq 1$, and also there exists an extremal function $r^* \in \mathcal{R}_{n,n}$ for $S_{n,N} \leq 1$, the assertion (1.7) follows. We have even shown that an approximately extremal function of one of the two problems allows to compute an approximately extremal function of the other one.

There is an interesting observation in [EFLSV02, End of Section 6]: instead of working with complex hulls of E_N and F_N , it is worthy to deflate first the eigenvalues close to zero. This is exactly what we mentioned before: the rate of approximation $S_{n,N}$ should depend on the fine structure of E_N, F_N (at least of near-by elements).

Similarly one may derive a link between the Zolotarev problem over discrete sets and the minimization of Blaschke products on discrete sets in the particular case $F_N = 1/E_N$, in order to extend the work of [FiSa99] to discrete sets.

2. Regularity assumptions, statement of the main results. We start by stating our main result on the asymptotic behaviour of the Zolotarev quantity on discrete sets, and give subsequently the precise assumptions as well as the theoretical background in logarithmic potential theory. As before, we always consider the limit $n, N \rightarrow \infty$, where $n = n(N)$ in such a way that $n/N \rightarrow t > 0$.

THEOREM 2.1. *Under the assumptions (A1), (A2) and (A3), to be specified below, there is a $T \in (0, 1)$ such that for every $t \in (0, T)$, we have*

$$(2.1) \quad \limsup_{n, N \rightarrow +\infty, n/N \rightarrow t} Z_n(E_N, F_N)^{1/N} \leq e^{-(F_1^t + F_2^t)},$$

where F_1^t, F_2^t are two real non-negative constants defined by the equilibrium conditions of Theorem 2.2 below.

In case we also assume the separation conditions (A4) and (A5), we have the additional result

$$(2.2) \quad \lim_{n, N \rightarrow +\infty, n/N \rightarrow t} Z_n(E_N, F_N)^{1/N} = e^{-(F_1^t + F_2^t)}.$$

In order to obtain the result of Theorem 2.1, we shall impose some technical assumptions which are given here, the first one concerning the sequences of discrete sets $(E_N)_N$ and $(F_N)_N$:

Assumption (A1) : For every N , the sets E_N and F_N are disjoint, of cardinality at most N . Furthermore, $\bigcup_N (E_N \cup F_N)$ is bounded.

The boundedness assumption can be sometimes achieved by using a Moebius transformation, the separation hypothesis is required to assure the existence of a minimizer for the Zolotarev problem as seen in the introduction. To give the second assumption, we have to define the normalized counting measure of a discrete set E

$$\nu_n(E) := \frac{1}{n} \sum_{\lambda \in E} \delta_\lambda.$$

From now on, we denote by *measure* a finite positive Borel measure and by *signed measure* a finite real-valued Borel measure.

Assumption (A2) : We suppose that the normalized counting measures of the sets E_N and F_N weakly tend to some limit measure :

$$\nu_N(E_N) \xrightarrow{*} \sigma_1, \quad \nu_N(F_N) \xrightarrow{*} \sigma_2.$$

In what follows we adopt the convention to write indices corresponding to the different condenser plates E, F as subindices 1 and 2.

Assumption (A1) implies that the total mass $\sigma_1(\mathbb{C})$ (resp. $\sigma_2(\mathbb{C})$) is at most one and it can be strictly less than one (e.g., in the ADI application if the matrices A_N (resp. B_N) have many multiple eigenvalues). Also, it follows from assumption (A1) that σ_j has a support Σ_j being compact for $j = 1, 2$.

If S is a compact subset of the complex plane \mathbb{C} , we denote by $\mathcal{M}(S)$ the set of signed measures on S and $\mathcal{M}^+(S)$ the set of measures on S . We define the logarithmic potential of a measure ρ in $\mathcal{M}^+(S)$

$$U^\rho(z) := \int \log \frac{1}{|z - t|} d\rho(t),$$

This is a superharmonic function on \mathbb{C} taking values in $(-\infty, +\infty]$, see [NS91], [Ran95] or [ST97] for detailed accounts on logarithmic potential theory. If μ belongs to $\mathcal{M}(S)$ with $\mu_1 - \mu_2$ its Jordan decomposition, we define the logarithmic potential of the signed measure μ

$$U^\mu(z) := U^{\mu_1}(z) - U^{\mu_2}(z)$$

defined for z such that $U^{\mu_1}(z) \neq +\infty$ or $U^{\mu_2}(z) \neq +\infty$. It is useful to impose the following regularity assumption on our measures σ_1 and σ_2 .

Assumption (A3) : For $j \in \{1, 2\}$, we assume that σ_j is a measure with continuous potential U^{σ_j} , and with support Σ_j of strictly positive logarithmic capacity such that $\Sigma_1 \cap \Sigma_2$ has logarithmic capacity zero. Furthermore, we write $\sigma = \sigma_1 - \sigma_2$.

The assumption (A3) is not very restrictive. For example, if σ_1 and σ_2 are absolutely continuous with respect to the Lebesgue measure with a density with only logarithmic type or power type singularities at a finite number of points then the continuity of the potentials considered in (A3) is satisfied. On the other hand, the assumption (A3) is not satisfied if σ_1 or σ_2 have point masses.

We define the logarithmic energy of $\rho \in \mathcal{M}^+(S)$ by the integral

$$I(\rho) := \int \int \log \frac{1}{|z-t|} d\rho(t) d\rho(z)$$

and the mutual energy of $(\rho_1, \rho_2) \in \mathcal{M}^+(S)$

$$I(\rho_1, \rho_2) := \int \int \log \frac{1}{|z-t|} d\rho_1(t) d\rho_2(z).$$

The signed measure μ in $\mathcal{M}(S)$ is said of finite energy if $I(\mu_1 + \mu_2) < +\infty$. For $\nu = \nu_1 - \nu_2$ and $\mu = \mu_1 - \mu_2$ in $\mathcal{M}(S)$ two signed measures of finite energy, we define the logarithmic energy of μ

$$I(\mu) := I(\mu_1, \mu_1) - 2I(\mu_1, \mu_2) + I(\mu_2, \mu_2)$$

and the mutual energy of the signed measures μ and ν

$$I(\mu, \nu) := I(\mu_1, \nu_1) - I(\mu_1, \nu_2) - I(\mu_2, \nu_1) + I(\mu_2, \nu_2).$$

Now we give the definition of the regularized mutual energy of the normalized counting measure of two discrete sets E and F :

$$I^*(\nu_N(E), \nu_N(F)) := \frac{1}{N^2} \sum_{x \in E, y \in F, x \neq y} \log \frac{1}{|x-y|}.$$

and the regularized energy of the normalized counting measure of E :

$$I^*(\nu_N(E)) := I^*(\nu_N(E), \nu_N(E)).$$

We might give the following interpretation of these definition : the regularized energy $I^*(\nu_N(E))$ is the energy of a system of N charged particles with equal mass and the same sign in which each particle has an interaction with all other particles, but not with itself, while in the case of the mutual regularized energy, one particle of E interacts with each one of the set F . Let us finally notice that for two disjoint discrete

sets E and F , the regularized mutual energy and the usual logarithmic energy of the measures $\nu_N(E)$ $\nu_N(F)$ coincide.

For the case of signed normalized counting measures of two discrete and disjoint sets E and F , the previous physical interpretation leads to the definition

$$I^*(\nu_N(E) - \nu_N(F)) := I^*(\nu_N(E)) + I^*(\nu_N(F)) - 2I^*(\nu_N(E), \nu_N(F))$$

which might be extended to the definition of mutual regularized energy for signed discrete measures for (E, F) , (E', F') two pairs of disjoint discrete sets

$$(2.3) \quad \begin{aligned} & 2I^*(\nu_N(E) - \nu_N(F), \nu_N(E') - \nu_N(F')) \\ &= I^*(\nu_N(E \cup E') - \nu_N(F \cup F')) \\ & \quad - I^*(\nu_N(E) - \nu_N(F)) - I^*(\nu_N(E') - \nu_N(F')). \end{aligned}$$

Our separation assumption is expressed in terms of the regularized energy of the counting measures of the families of discrete sets (E_N) and (F_N) .

Assumption (A4) : We have

$$\lim_{N \rightarrow +\infty} I^*(\nu_N(E_N) + \nu_N(F_N)) = I(\sigma_1 + \sigma_2).$$

Assumption (A5) : There holds $\text{dist}(E_N, F_N)^{1/N} \rightarrow 1$ for $N \rightarrow \infty$.

Our assumption (A4) is a separation condition for the elements of the set $E_N \cup F_N$. Different other sufficient separation conditions have been considered before, see [Be00] for a comparison. For example, (A4) holds provided that the following condition [Rak96] is true

$$\exists C > 0, \forall N \geq 1, \quad \inf_{x \neq y \in E_N \cup F_N} |x - y| \geq \frac{C}{N}.$$

In order to see that the assumptions (A4) and (A5) are independent, consider first $E_N = -F_N = \{\exp(-N^\alpha)\} \cup \{1 + j/N : 0 \leq j < N\}$. Here (A5) is true iff $\alpha < 1$, and (A4) holds for $\alpha < 2$. In contrary, for $E_N = -F_N = \{1 + \exp(-N^\alpha)\} \cup \{j/N : 1 < j \leq N\}$, condition (A5) holds for any $\alpha > 0$ whereas (A4) remains true only for $\alpha < 2$.

Now we give the main result about our constrained minimal energy problem, which will help us to describe the asymptotic behaviour of the Zolotarev quantity.

THEOREM 2.2. *Suppose that (A3) holds for the limit signed measure $\sigma = \sigma_1 - \sigma_2$, where as before σ_j is supposed to have compact support Σ_j for $j = 1, 2$. Then, for $t \in (0, \min(\sigma_1(\mathbb{C}), \sigma_2(\mathbb{C})))$, the minimization problem*

$$(P) : \text{Find } \mu^t \text{ such that } \quad I(\mu^t) = \inf \{I(\mu), \mu \in \mathcal{M}_\sigma^t\}$$

where $\mathcal{M}_\sigma^t := \{\mu := \mu_1 - \mu_2, \mu_j \text{ measure}, \mu_j(\mathbb{C}) = t, 0 \leq \mu_j \leq \sigma_j \text{ for } j \in \{1, 2\}\}$ has a unique solution μ^t which verifies the following equilibrium conditions: there are two non-negative constants F_1^t and F_2^t such that

$$(2.4) \quad \begin{cases} U^{\mu^t}(z) = F_1^t, & z \in \text{supp}(\sigma_1 - \mu_1^t), \\ U^{\mu^t}(z) \leq F_1^t, & z \in \mathbb{C}, \\ -U^{\mu^t}(z) = F_2^t, & z \in \text{supp}(\sigma_2 - \mu_2^t), \\ -U^{\mu^t}(z) \leq F_2^t, & z \in \mathbb{C}. \end{cases}$$

and moreover, $F_1^t + F_2^t > 0$.

Conversely, a signed measure $\mu \in \mathcal{M}_\sigma^t$ verifying the system (2.4) for a couple of real constants (F_1, F_2) is the equilibrium signed measure μ^t and we have moreover the equalities $F_1 = F_1^t$ and $F_2 = F_2^t$.

3. Proof of Theorem 2.1. In this section we prove Theorem 2.1 with help of Theorem 2.2, the latter being established in §4. We start with a preliminary lemma before showing the upper bound (2.1) for the Zolotarev quantity under the assumptions (A1), (A2) and (A3).

From now on, we assume that the parameter t belongs to $(0, T)$, with $T := \min(\sigma_1(\mathbb{C}), \sigma_2(\mathbb{C}))$.

LEMMA 3.1. Consider the constraint signed measure σ and let $\mu = \mu_1 - \mu_2$ in \mathcal{M}_σ^t . Then, for every $n = n(N) \leq N$ we can choose a sequence of discrete sets $E_N^* \subset E_N$ (resp. $F_N^* \subset F_N$) such that for every N , $\text{card}(E_N^*) = \text{card}(F_N^*) = n$ and that

$$\nu_N(E_N^*) \xrightarrow{*} \mu_1 \quad \text{and} \quad \nu_N(F_N^*) \xrightarrow{*} \mu_2.$$

Moreover, if K_1, K_2 are two closed sets such that $\sigma_i(\partial K_i) = 0$ and $\sigma_i(K_i) = \mu_i(K_i)$, we can choose E_N^*, F_N^* such that for n large enough,

$$E_N \cap K_1 \subset E_N^* \subset E_N \quad \text{and} \quad F_N \cap K_2 \subset F_N^* \subset F_N.$$

Proof. See [Be00, Lemma 2.1] \square

We also recall from [ST97, Theorem 1.6.8] the principle of descent which will play a key-role in the proof of (2.1).

THEOREM 3.2 (Principle of descent). Let $(\mu_n)_n$ be probability measures all having support in a fixed compact subset of \mathbb{C} and converging to some measure μ in the weak* topology. Suppose moreover that for each n a point z_n is given so that $z_n \rightarrow z^*$ for some z^* in \mathbb{C} . Then,

$$U^\mu(z^*) \leq \liminf_{n \rightarrow +\infty} U^{\mu_n}(z_n) \quad \text{and} \quad I(\mu) \leq \liminf_{n \rightarrow +\infty} I(\mu_n).$$

REMARK 3.3. Under the assumptions of the principle of descent, if we suppose furthermore that for every n z_n belongs to a set Σ_1 and $\text{supp}(\mu_n) \subset \Sigma_2$ where Σ_1 and Σ_2 are two disjoint compact sets of the complex plane, then

$$(3.1) \quad U^\mu(z^*) = \lim_{n \rightarrow +\infty} U^{\mu_n}(z_n),$$

which follows from the the uniform convergence of the corresponding integrands.

REMARK 3.4. A consequence of the assumption (A3) called the Rakhmanov Lemma (see [Rak96]) is that for any signed measure $\mu = \mu_1 - \mu_2$ in \mathcal{M}_σ^t , the potentials U^{μ_1} and U^{μ_2} are continuous. Indeed, U^{μ_1} is lower semi-continuous and since $U^{\mu_1} = U^{\sigma_1} - U^{\sigma_1 - \mu_1}$ with U^{σ_1} continuous and $U^{\sigma_1 - \mu_1}$ lower semi-continuous, U^{μ_1} is also upper semi-continuous, hence continuous, and the same proof holds for U^{μ_2} . As seen before (A4), in general the expression for the logarithmic potential and the logarithmic energy of a signed measure has to be defined with care. However, in the case of a measure $\mu \in \mathcal{M}_\sigma^t$, we get by the Fubini Theorem

$$I(\mu) := \int U^\mu(z) d\mu(z)$$

where U^μ is continuous and $\text{supp}(\mu)$ compact, and hence the logarithmic energy of every element of \mathcal{M}_σ^t is finite.

Now we show the upper bound for the Zolotarev quantity (2.1).

Proof. We define the sets

$$K_1^\epsilon := \{\lambda \in \mathbb{C}, U^{\mu^t}(\lambda) \leq F_1^t - \epsilon\}, \quad K_2^\epsilon := \{\lambda \in \mathbb{C}, -U^{\mu^t}(\lambda) \leq F_2^t - \epsilon\}.$$

By the Rakhmanov Lemma, U^{μ^t} is a continuous function, and hence both sets are closed. Because of the equilibrium conditions (2.4) verified by μ^t , K_i^ϵ is disjoint from $\text{supp}(\sigma_i - \mu_i^t)$, hence $\sigma_i(K_i^\epsilon) = \mu_i^t(K_i^\epsilon)$. Moreover, by choosing a smaller ϵ if necessary, we may assume that $\sigma_i(\partial K_i^\epsilon) = 0$ (see [ST97, theorem 0.1.5]). We define

$$r_n(\lambda) := \prod_{\lambda' \in E_N^*} \left(1 - \frac{\lambda}{\lambda'}\right) \prod_{\lambda' \in F_N^*} \left(1 - \frac{\lambda}{\lambda'}\right)^{-1}$$

where the sets E_N^* and F_N^* are chosen as in Lemma 3.1 for the equilibrium signed measure μ^t . Let us derive an upper bound for

$$\|r_n^{-1}\|_{L^\infty(F_N)} \|r_n\|_{L^\infty(E_N)}.$$

We define $(x_1^N, x_2^N) \in E_N \times F_N$ such that

$$\|r_n\|_{L^\infty(E_N)} = |r_n(x_1^N)| \quad \text{and} \quad \|r_n^{-1}\|_{L^\infty(F_N)} = |r_n^{-1}(x_2^N)|.$$

Since r_n (resp. r_n^{-1}) vanishes on $E_N \cap K_1^\epsilon$ (resp. $F_N \cap K_2^\epsilon$) by choice of the sets E_N^* and F_N^* described in Lemma 3.1, we have

$$(3.2) \quad x_1^N \in \mathbb{C} \setminus K_1^\epsilon = \{\lambda \in \mathbb{C}, U^{\mu^t}(\lambda) > F_1^t - \epsilon\}$$

$$(3.3) \quad x_2^N \in \mathbb{C} \setminus K_2^\epsilon = \{\lambda \in \mathbb{C}, -U^{\mu^t}(\lambda) > F_2^t - \epsilon\}.$$

The sequences $(x_j^N)_N$, $j \in \{1, 2\}$, are bounded because of the assumption (A1). Hence passing to a subsequence if necessary, we may assume that $(x_j^N)_N$ converge as $N \rightarrow +\infty$ with the limits

$$x_j^* := \lim_{N \rightarrow +\infty} x_j^N.$$

Moreover, by Lemma 3.1 applied to the energy minimizer μ^t , we have

$$\nu_N(E_N^*), \overset{*}{\xrightarrow{}} \mu_1^t \quad \text{and} \quad \nu_N(F_N^*) \overset{*}{\xrightarrow{}} \mu_2^t.$$

Then, by application of the principle of descent (Theorem 3.2)

$$(3.4) \quad U^{\mu_1^t}(x_1^*) \leq \liminf_N U^{\nu_N(E_N^*)}(x_N) \quad \text{and} \quad U^{\mu_2^t}(x_2^*) \leq \liminf_N U^{\nu_N(F_N^*)}(x_N^2),$$

and, as the closures of the sets $\mathbb{C} \setminus K_1^\epsilon$ and $\mathbb{C} \setminus K_2^\epsilon$ are disjoint with ϵ small enough because of the strict positivity of at least one of the constants F_1^t and F_2^t , we have with (3.1)

$$(3.5) \quad U^{\mu_1^t}(x_2^*) = \lim_N U^{\nu_N(E_N^*)}(x_2^N) \quad \text{and} \quad U^{\mu_2^t}(x_1^*) = \lim_N U^{\nu_N(F_N^*)}(x_1^N).$$

We have moreover

$$\begin{aligned} \frac{1}{N} \log |r_n(x_1^N)| - \frac{1}{N} \log |r_n(x_2^N)| &= \frac{1}{N} \sum_{\lambda \in E_N^*} \log |\lambda - x_1^N| - \frac{1}{N} \sum_{\lambda \in F_N^*} \log |\lambda - x_1^N| \\ &\quad + \frac{1}{N} \sum_{\lambda \in F_N^*} \log |\lambda - x_2^N| - \frac{1}{N} \sum_{\lambda \in E_N^*} \log |\lambda - x_2^N|. \end{aligned}$$

With (3.4) and (3.5), we obtain that, for $N \rightarrow \infty$,

$$-\frac{1}{N} \sum_{\lambda \in E_N^*} \log |\lambda - x_2^N| \rightarrow U^{\mu_1^t}(x_2^*), \quad -\frac{1}{N} \sum_{\lambda \in F_N^*} \log |\lambda - x_1^N| \rightarrow U^{\mu_2^t}(x_1^*)$$

and

$$\limsup_N \frac{1}{N} \sum_{\lambda \in E_N^*} \log |\lambda - x_1^N| \leq -U^{\mu_1^t}(x_1^*), \quad \limsup_N \frac{1}{N} \sum_{\lambda \in F_N^*} \log |\lambda - x_2^N| \leq -U^{\mu_2^t}(x_2^*).$$

Combining these findings with (3.2), we may conclude that

$$\begin{aligned} &\limsup_N \left[\frac{1}{N} \log |r_n(x_1^N)| - \frac{1}{N} \log |r_n(x_2^N)| \right] \\ &\leq -U^{\mu_1^t}(x_1^*) + U^{\mu_2^t}(x_1^*) - U^{\mu_2^t}(x_2^*) + U^{\mu_1^t}(x_2^*) \\ &= -U^{\mu^t}(x_1^*) + U^{\mu^t}(x_2^*) \leq -F_1^t - F_2^t + 2\epsilon. \end{aligned}$$

Since this inequality is true for every $\epsilon > 0$, our claim (2.1) follows. \square

In the second part of this section we establish the lower bound (2.2) for the n -th root asymptotic of the Zolotarev quantity under the additional assumptions (A4) and (A5). We first define rational Fekete points on discrete sets which will be a helpful tool. We refer the reader to [ST97, p. 395] for the definition of (weighted) rational Fekete points on continuous sets.

DEFINITION 3.5. *The rational Fekete points of order n maximize the quantity*

$$\mathcal{F}((\lambda_1^0, \dots, \lambda_1^n), (\lambda_2^0, \dots, \lambda_2^n)) := \prod_{0 \leq i \neq j \leq n} \frac{|\lambda_1^i - \lambda_1^j| |\lambda_2^i - \lambda_2^j|}{|\lambda_1^i - \lambda_2^j| |\lambda_2^i - \lambda_1^j|}$$

among all the sets $(\lambda_1^i)_{0 \leq i \leq n} \subset E_N$, $(\lambda_2^i)_{0 \leq i \leq n} \subset F_N$.

In what follows, we denote $E_{n,N} := (\lambda_1^i)_{i=0}^n$ (resp. $F_{n,N} := (\lambda_2^i)_{i=0}^n$) the set of the rational Fekete points of order n of the set E_N (resp. F_N). Here we remove the dependence on n, N in the notation for the rational Fekete points of E_N and F_N of order n in order to simplify our notations.

LEMMA 3.6. *Under the additional assumption (A4), for every sequence of sets $(G_{1,N}, G_{2,N}) \subset E_N \cup F_N$ such that for $j \in \{1, 2\}$,*

$$\nu_N(G_{j,N}) \xrightarrow{*} \nu_j, \quad \text{we have} \quad \lim_{N \rightarrow +\infty} I^*(\nu_N(G_{1,N}), \nu_N(G_{2,N})) = I(\nu_1, \nu_2).$$

Proof. Writing $k_M(z) = \min\{M, \log \frac{1}{|z|}\}$ for the truncated logarithmic kernel, we have for every $M > 0$

$$\begin{aligned} I^*(\nu_N(G_{1,N}), \nu_N(G_{2,N})) &= \int \int k_M(x-y) d\nu_N(G_{1,N})(x) d\nu_N(G_{2,N})(y) \\ &\quad + \frac{1}{N^2} \sum_{(x,y) \in U_M(G_{1,N}, G_{2,N})} \left(\log \frac{1}{|x-y|} - M \right) - M \frac{\text{card}(G_{1,N} \cap G_{2,N})}{N^2}, \end{aligned}$$

where $U_M(G_{1,N}, G_{2,N}) := \{(x, y) \in G_{1,N} \times G_{2,N}, x \neq y \text{ and } |x - y| < e^{-M}\}$. Hence, as

$$\sum_{(x,y) \in U_M(G_{1,N}, G_{2,N})} \left(\log \frac{1}{|x-y|} - M \right) \geq 0$$

and by continuity of k_M on $\mathbb{C} \times \mathbb{C}$, we have

$$\lim_{N \rightarrow +\infty} \int \int k_M(x-y) d\nu_N(G_{1,N})(x) d\nu_N(G_{2,N})(y) = \int \int k_M(x-y) d\nu_1(x) d\nu_2(y),$$

and, by the monotone convergence theorem,

$$\lim_{M \rightarrow +\infty} \int \int k_M(x-y) d\nu_1(x) d\nu_2(y) = I(\nu_1, \nu_2).$$

This shows the semi-continuity inequality

$$(3.6) \quad \liminf_{N \rightarrow +\infty} I^*(\nu_N(G_{1,N}), \nu_N(G_{2,N})) \geq I(\nu_1, \nu_2)$$

for every sequence of sets $(G_{1,N}, G_{2,N}) \subset E_N \cup F_N$ which normalized counting measure have a weak- \star limit (ν_1, ν_2) . We now write

$$\begin{aligned} & I^*(\nu_N(G_{1,N}), \nu_N(G_{2,N})) \\ &= I^*(\nu_N(E_N \cup F_N)) - I^*(\nu_N(G_{1,N}), \nu_N(E_N \cup F_N \setminus G_{2,N})) \\ & \quad - I^*(\nu_N(E_N \cup F_N \setminus G_{1,N}), \nu_N(G_{2,N})) \\ & \quad - I^*(\nu_N(E_N \cup F_N \setminus G_{1,N}), \nu_N(E_N \cup F_N \setminus G_{2,N})) \end{aligned}$$

and we conclude with assumption (A4) and (3.6) that

$$\lim_{N \rightarrow +\infty} I^*(\nu_N(G_{1,N}), \nu_N(G_{2,N})) = I(\nu_1, \nu_2).$$

□

REMARK 3.7. *The proof of Lemma 3.6 shows that the inequalities $\liminf_N I^*(\nu_N(E_N)) \geq I(\sigma_1)$, $\liminf_N I^*(\nu_N(F_N)) \geq I(\sigma_2)$, and $\liminf_N I^*(\nu_N(E_N) + \nu_N(F_N)) \geq I(\sigma_1 + \sigma_2)$ are already a consequence of (A2) for every family of discrete set as in (A2). Hence, the assumption (A4) is equivalent to the equalities $\lim_N I^*(\nu_N(E_N)) = I(\sigma_1)$, $\lim_N I^*(\nu_N(F_N)) = I(\sigma_2)$, and $\lim_N I^*(\nu_N(E_N), \nu_N(F_N)) = I(\sigma_1, \sigma_2)$.* ■

REMARK 3.8. *Lemma 3.6 can be extended as well to the case of mutual regularized energies for discrete signed measures with a weak \star limit by writing the discrete mutual regularized energy term of two normalized signed counting measures as in (2.3) and using our lemma for each term.*

The new lemma shows that the solution of the extremal problem (P) describes the asymptotic behaviour of the rational Fekete points.

LEMMA 3.9. *We have the following result for the counting measures of the Fekete points, where $\mu^t = \mu_1^t - \mu_2^t$ is the minimizer for the constrained energy problem (P) with μ_1^t and μ_2^t of mass t*

$$(3.7) \quad \nu_N(E_{n,N}) \xrightarrow{\star} \mu_1^t \quad \text{and} \quad \nu_N(F_{n,N}) \xrightarrow{\star} \mu_2^t$$

Proof. Recall from (A1) that the discrete sets E_N and F_N are contained in two compact sets. Hence we may apply Helly's selection theorem, and we extract two subsequences $(n_k)_k$ and $(N_k)_k$ such that $\nu_{N_k}(E_{n_k, N_k}) \xrightarrow{*} \mu_1$ and $\nu_{N_k}(F_{n_k, N_k}) \xrightarrow{*} \mu_2$ and we define the weak limit signed measure $\mu := \mu_1 - \mu_2$. Then μ belongs to \mathcal{M}_σ^t and with Lemma 3.6 and the previous remark, we have

$$\lim_{k \rightarrow +\infty} I^*(\nu_N(E_{n_k, N_k}) - \nu_N(F_{n_k, N_k})) = I(\mu).$$

Let $E_N^* \subset E_N$, $F_N^* \subset F_N$, both with cardinality $n+1$. Then, using the assumptions (A1) and (A5) we may conclude that

$$\lim_{N \rightarrow \infty} \log \left(\mathcal{F}(E_{n+1}^*, F_{n+1}^*)^{1/N^2} \right) + I^*(\nu_N(E_N^*) - \nu_N(F_N^*)) = 0.$$

Moreover, by Definition 3.5, $\mathcal{F}(E_{n, N}, F_{n, N}) \geq \mathcal{F}(E_{n+1}^*, F_{n+1}^*)$. Hence, for the sets E_N^*, F_N^* as in Lemma 3.1 with weak* limit μ^t , we get applying again Lemma 3.6 that

$$\begin{aligned} I(\mu) &= \lim_{k \rightarrow +\infty} I^*(\nu_N(E_{n_k, N_k}) - \nu_N(F_{n_k, N_k})) \\ &\leq \lim_{k \rightarrow +\infty} I^*(\nu_N(E_{N_k}^*) - \nu_N(F_{N_k}^*)) = I(\mu^t), \end{aligned}$$

and thus $\mu = \mu^t$ by unicity of the minimizer in the theorem 2.2. \square

Now we show that the family of rational function built with zeros and poles chosen among the rational discrete Fekete sets of E_N and F_N is asymptotically optimal for the Zolotarev problem considered here. In particular, we give a relation with the extremal constants occurring in our problem (P).

LEMMA 3.10. *We define for $i, j \in \{0, 1, \dots, n\}$*

$$r_{i,j}(z) := \frac{\prod_{\ell \neq i} (z - \lambda_1^\ell)}{\prod_{\ell \neq j} (z - \lambda_2^\ell)}.$$

Then, we have

$$\liminf_{n, N, n/N \rightarrow t} \min_{0 \leq i, j \leq n} \|r_{i,j}\|_{L^\infty(E_{n, N})}^{1/N} \geq e^{-F_1^t}, \quad \liminf_{n, N, n/N \rightarrow t} \min_{0 \leq i, j \leq n} \|r_{i,j}^{-1}\|_{L^\infty(F_{n, N})}^{1/N} \geq e^{-F_2^t}.$$

Proof. First, we have for all integers $i, j \in \{0, n\}$ and for every $x \in E_N$,

$$|r_{i,j}(x)| = |r_{i,i}(x)| \left| \frac{x - \lambda_2^j}{x - \lambda_2^i} \right|.$$

Hence, we obtain

$$\|r_{i,j}\|_{L^\infty(E_{n, N})} = |r_{i,j}(\lambda_1^i)| \geq \frac{\delta_N}{\Delta} \|r_{i,i}\|_{L^\infty(E_{n, N})} = \frac{\delta_N}{\Delta} |r_{i,i}(\lambda_1^i)|,$$

where $\delta_N = \text{dist}(E_N, F_N)$, and $\Delta := \sup_{N \geq 1} \max_{x \in E_N, y \in F_N} |x - y|$ being finite because of assumption (A1).

Moreover, we have for every x in E_N

$$\begin{aligned} & \sqrt{\frac{\mathcal{F}((\lambda_1^0, \dots, \lambda_1^{i-1}, x, \lambda_1^{i+1}, \dots, \lambda_1^n), (\lambda_2^0, \dots, \lambda_2^n))}{\mathcal{F}((\lambda_1^0, \dots, \lambda_1^n), (\lambda_2^0, \dots, \lambda_2^n))}} \\ &= \prod_{0 \leq \ell \leq n, \ell \neq i} \frac{|x - \lambda_1^\ell|}{|\lambda_1^i - \lambda_1^\ell|} \prod_{0 \leq \ell \leq n, \ell \neq i} \frac{|\lambda_1^i - \lambda_2^\ell|}{|x - \lambda_2^\ell|} = \left| \frac{r_{i,i}(x)}{r_{i,i}(\lambda_1^i)} \right|. \end{aligned}$$

Taking account the definition of Fekete points, we conclude that, for every $i \in \{0, n\}$,

$$\left| \frac{r_{i,i}(x)}{r_{i,i}(\lambda_1^i)} \right| \leq 1,$$

which gives for every $x \in E_N$,

$$|r_{i,j}(\lambda_1^i)| \geq \frac{\delta_N}{\Delta} |r_{i,i}(x)|.$$

Taking the product for $x \in E_N \setminus E_{n,N}$, we deduce that

$$\begin{aligned} \frac{N-n+1}{N^2} \log |r_{i,j}(\lambda_1^i)| &\geq \frac{N-n+1}{N^2} \log \frac{\delta_N}{\Delta} + \frac{1}{N^2} \sum_{x \in E_N \setminus E_{n,N}} \log |r_{i,i}(x)| \\ &= \frac{N-n+1}{N^2} \log \frac{\delta_N}{\Delta} - \frac{1}{N^2} \sum_{x \in E_N \setminus E_{n,N}} \left(\sum_{y \in E_{n,N} \setminus \lambda_1^i} \log \frac{1}{|x-y|} - \sum_{y \in F_{n,N} \setminus \lambda_2^i} \log \frac{1}{|x-y|} \right) \\ &\geq \frac{1}{N} \log \frac{\delta_N}{\Delta} - I^*(\nu_N(E_N \setminus E_{n,N}), \nu_N(E_{n,N} \setminus \{\lambda_1^i\}) - \nu_N(F_{n,N} \setminus \{\lambda_2^i\})). \end{aligned}$$

Now, as the fact to remove one point does not change the weak \star limit of a sequence of normalized counting measures, we know from Lemma 3.9 that

$$\nu_N(E_N \setminus E_{n,N}) \xrightarrow{*} \sigma_1 - \mu_1^t, \quad \nu_N(E_{n,N} \setminus \{\lambda_1^i\}) \xrightarrow{*} \mu_1^t, \quad \nu_N(F_{n,N} \setminus \{\lambda_2^i\}) \xrightarrow{*} \mu_2^t,$$

and Lemma 3.6 gives

$$\begin{aligned} \lim_{n,N \rightarrow +\infty, n/N \rightarrow t} I^*(\nu_N(E_N \setminus E_{n,N}), \nu_N(E_{n,N} \setminus \{\lambda_1^i\})) &= I(\sigma_1 - \mu_1^t, \mu_1^t) \\ \lim_{n,N \rightarrow +\infty, n/N \rightarrow t} I^*(\nu_N(E_N \setminus E_{n,N}), \nu_N(F_{n,N} \setminus \{\lambda_2^i\})) &= I(\sigma_1 - \mu_1^t, \mu_2^t). \end{aligned}$$

In addition, from assumption (A5) we have $\lim_{N \rightarrow +\infty} \delta_N^{1/N} = 1$. Hence, using the equilibrium condition of Theorem 2.2,

$$\begin{aligned} (1-t) \liminf_{n,N,n/N \rightarrow t} \frac{1}{N} \log |r_{i,j}(\lambda_1^i)| &\geq -I(\sigma_1 - \mu_1^t, \mu_1^t) \\ &= - \int U^{\mu_1^t}(z) d(\sigma_1 - \mu_1^t)(z) = -(\sigma_1 - \mu_1^t)(\mathbb{C}) F_1^t = (1-t) F_1^t, \end{aligned}$$

as claimed in the first part of Lemma 3.10. The second follows by reversing the roles played by the sets E_N and F_N . \square

We define for E, F disjoint discrete sets and $n \geq 1$

$$Z_n^*(E, F) := \min_{r \in \mathcal{R}_n(E, F)} \|r\|_{L^\infty(E)} \|r^{-1}\|_{L^\infty(F)}$$

where the set of candidates $\mathcal{R}_n(E, F) := \{r \in \mathcal{R}_n, r \text{ has zeros in } E \text{ and poles in } F\}$ is finite, and thus there is an extremal function for $Z_n^*(E, F)$ as seen in the introduction. Also, there should be a link between $Z_n^*(E, F)$ and $Z_n(E, F)$, since a simple way of getting a rational function small on E is to put its zeros there.

The next proposition links the quantities $Z_n(E_N, F_N)$ and $Z_n^*(E_{n,N}, F_{n,N})$, which is a capital step in the proof of (2.2), since the N th root asymptotics of the latter quantity has been analyzed already in Lemma 3.10.

PROPOSITION 3.11. *We have*

$$\frac{1}{(n+1)^2} Z_n^*(E_{n,N}, F_{n,N}) \leq Z_n(E_{n,N}, F_{n,N}) \leq Z_n(E_N, F_N).$$

Proof. The inequality $Z_n(E_{n,N}, F_{n,N}) \leq Z_n(E_N, F_N)$ is a trivial consequence of the inclusions $E_{n,N} \subset E_N$, $F_{n,N} \subset F_N$. Hence it remains to prove

$$(3.8) \quad Z_n^*(E_{n,N}, F_{n,N}) \leq (n+1)^2 Z_n(E_{n,N}, F_{n,N}).$$

For a given choice of the square root, we define the matrix

$$X := \left[\frac{1}{\lambda_1^i - \lambda_2^j} \sqrt{\frac{\omega_1(\lambda_2^j) \omega_2(\lambda_1^i)}{\omega_1'(\lambda_1^i) \omega_2'(\lambda_2^j)}} \right]_{0 \leq i, j \leq n}, \quad \omega_j(z) := \prod_{\ell=0}^n (z - \lambda_i^\ell).$$

It follows as an application of (1.6) that

$$(3.9) \quad \frac{1}{\|X\| \|X^{-1}\|} \leq Z_n(E_{n,N}, F_{n,N}),$$

where $\|\cdot\|$ denotes the spectral matrix norm. For completeness, we give elements of proof of this inequality. Indeed, the inequality

$$(3.10) \quad \frac{1}{\|X\| \|X^{-1}\|} \leq \|r\|_{L^\infty(E_{n,N})} \|r^{-1}\|_{L^\infty(F_{n,N})}$$

for any $r \in \mathcal{R}_n$ has been already derived by Penzl in [Pen00, Theorem 1] for symmetric sets $E_{n,N} = -F_{n,N}$. It is not difficult to check that his arguments do not require symmetry : from [GoVLo96, Theorem 2.5.3], we have

$$\inf_{\tilde{X} \in M_{n+1}(\mathbb{R}), \text{rank}(\tilde{X}) \leq n} \frac{\|X - \tilde{X}\|}{\|X\|} = \frac{\sigma_{n+1}(X)}{\sigma_1(X)} = \frac{1}{\|X\| \|X^{-1}\|}.$$

Moreover, one can easily show that with $D_i := \text{diag}(\lambda_0^i, \lambda_1^i, \dots, \lambda_n^i)$ for $i = 1, 2$, we have $\text{rank}(D_1 X - X D_2) = 1$, and consequently that $\text{rank}(X_n) \leq n$ where X_n is the result obtained after n iterations of the ADI method applied to the Sylvester equation $D_1 X - X D_2 = C$ with $X_0 = 0$. Consequently, we have with (1.6)

$$\frac{1}{\|X\| \|X^{-1}\|} \leq \frac{\|X - X_n\|}{\|X\|} \leq Z_n(E_{n,N}, F_{n,N}).$$

Also, observe that

$$\begin{aligned} Z_n^*(E_{n,N}, F_{n,N}) &= \min_{0 \leq i, j \leq n} \left| \frac{r_{i,j}(\lambda_1^i)}{r_{i,j}(\lambda_2^j)} \right| \\ &= \min_{0 \leq i, j \leq n} \left| \left(\lambda_1^i - \lambda_2^j \right)^2 \frac{\omega_1'(\lambda_1^i) \omega_2'(\lambda_2^j)}{\omega_2(\lambda_1^i) \omega_1(\lambda_2^j)} \right| = \min_{0 \leq i, j \leq n} \frac{1}{|X_{ij}|^2}. \end{aligned}$$

One may show that $X^{-1} = -X^t$ using the explicitly known formulae for the inverse of a Cauchy matrix, and reminding the well-known inequality

$$\max_{0 \leq i, j \leq n} |X_{ij}|^2 \geq \frac{1}{(n+1)^2} \|X\|_2^2,$$

we obtain

$$(3.11) \quad \frac{1}{\|X\| \|X^{-1}\|} \geq \frac{1}{(n+1)^2} Z_n^*(E_{n,N}, F_{n,N}).$$

which with (3.9) shows the desired inequality (3.8). \square

We are now prepared to give the proof of (2.2) under the assumptions (A1), (A2), (A3), (A4), and (A5).

Proof. We have seen that

$$Z_n^*(E_{n,N}, F_{n,N}) = \min_{0 \leq i, j \leq n} \left| \frac{r_{ij}(\lambda_1^i)}{r_{ij}(\lambda_2^j)} \right|.$$

Combining Lemma 3.10 and Proposition 3.11, we get

$$\begin{aligned} \liminf_N Z_n(E_N, F_N)^{1/N} &\geq \liminf_N Z_n^*(E_{n,N}, F_{n,N})^{1/N} \\ &= \liminf_N \min_{0 \leq i, j \leq n} \|r_{i,j}\|_{L^\infty(E_{n,N})}^{1/N} \|r_{i,j}^{-1}\|_{L^\infty(F_{n,N})}^{1/N} \\ &\geq e^{-(F_1^t + F_2^t)}, \end{aligned}$$

as claimed in (2.2). \square

4. Proof of Theorem 2.2. As before, let $T = \min(\sigma_1(\mathbb{C}), \sigma_2(C))$. For $t \in (0, T)$, the set of candidates of the constrained energy minimization problem for signed measures of Theorem 2.2 is given by

$$\mathcal{M}_\sigma^t := \{\mu := \mu_1 - \mu_2, \mu_j \text{ measure}, \mu_j(\mathbb{C}) = t, 0 \leq \mu_j \leq \sigma_j \text{ for } j \in \{1, 2\}\}.$$

We assume in this section that the assumptions of Theorem 2.2 hold, in particular we denote by Σ_j the compact support of σ_j for $j = 1, 2$, and suppose that $\Sigma_1 \cap \Sigma_2$ has zero logarithmic capacity. For $\mu \in \mathcal{M}_\sigma^t$, we know from Remark 3.4 that its potential is continuous and its logarithmic energy is finite.

PROPOSITION 4.1. *For $t \in (0, T)$, the quantity $V_\sigma := \inf(I(\mu), \mu \in \mathcal{M}_\sigma^t)$ is finite and there is a unique minimizer μ^t in \mathcal{M}_σ^t such that $I(\mu^t) = V_\sigma$.*

Proof. In our proof we apply standard techniques for minimizing a quadratic form on a convex set (see for example [ST97, Theorem 8.1.4] for the case of signed measures without constraint but with external field). However, we have to pay attention to the fact that the set $\Sigma_1 \cap \Sigma_2$ is not supposed to be empty, which implies a technical difficulty in the proof of the lower semi-continuity of the energy functional I for signed measures. We return to this point at the end of the proof.

As said before in Remark 3.4, for every $\mu = \mu_1 - \mu_2 \in \mathcal{M}_\sigma^t$, the measures μ_1 , μ_2 , and μ have finite logarithmic energy and $\mu(\mathbb{C}) = 0$. Then [ST97, Lemma 1.1.8] implies that $V_\sigma \geq 0$, and obviously V_σ is finite.

In order to show existence of a minimizer, let $(\mu^{(k)})_{k>0}$ be a sequence of signed measures of \mathcal{M}_σ^t such that $\lim_{k \rightarrow +\infty} I(\mu^{(k)}) = V_\sigma$. Then Helly's theorem [ST97,

Theorem 0.1.3] allows us to assume that, by possibly extracting a subsequence, the sequence $(\mu_1^{(k)})_{k>0}$ (resp. $(\mu_2^{(k)})_{k>0}$) weakly tends to a measure μ_1 (resp. μ_2). Assuming for the moment the lower semi-continuity of the energy for signed measures from \mathcal{M}_σ^t , we conclude that

$$V_\sigma = \lim_{k \rightarrow +\infty} I(\mu^{(k)}) \geq I(\mu_1 - \mu_2).$$

Since $\mu_1 - \mu_2$ belongs to \mathcal{M}_σ^t , the infimum in the definition of V_σ is indeed attained.

In order to show uniqueness of the minimizer, assume that we have two signed measures μ and ν in \mathcal{M}_σ^t with $I(\mu) = I(\nu) = V_\sigma$. Then,

$$V_\sigma = \frac{1}{2}(I(\mu) + I(\nu)) = I\left(\frac{\mu + \nu}{2}\right) + I\left(\frac{\mu - \nu}{2}\right),$$

and $(\mu + \nu)/2$ belongs to \mathcal{M}_σ^t by convexity, hence $I\left(\frac{\mu - \nu}{2}\right) \leq 0$ by definition of V_σ . Since $\mu - \nu$ is a signed measure of total mass zero and finite energy, the relation $I(\mu - \nu) \leq 0$ implies $\mu = \nu$ by [ST97, Lemma 1.1.8].

It remains to show that the energy functional is l.s.c in \mathcal{M}_σ^t : this property is well-known for the energy functional restricted to measures, see [Ran95, Lemma 3.3.3]. Let $(\mu^{(k)})_{k>0}$ be a sequence of signed measures of \mathcal{M}_σ^t with Jordan decomposition $\mu^{(k)} = \mu_1^{(k)} - \mu_2^{(k)}$, where we suppose that the sequence $(\mu_1^{(k)})_{k>0}$ (resp. $(\mu_2^{(k)})_{k>0}$) weakly tends to a measure μ_1 (resp. μ_2), and $\mu = \mu_1 - \mu_2 \in \mathcal{M}_\sigma^t$. Using the semi-continuity for measures, we get

$$\liminf_{k \rightarrow \infty} I(\mu^{(k)}) - I(\mu) \geq 2\left(I(\mu_1, \mu_2) - \limsup_{k \rightarrow \infty} I(\mu_1^{(k)}, \mu_2^{(k)})\right).$$

By the theorem of dominated convergence, the right-hand side is zero, provided that

$$(4.1) \quad \lim_{\delta \rightarrow 0} \int \int_{U_\delta} \log \frac{1}{|z - t|} d\sigma_1(t) d\sigma_2(z) = 0,$$

where $U_\delta := \{(z, t) \in \Sigma_1 \times \Sigma_2, |z - t| < \delta\}$. In order to show (4.1), we will make use of our assumption (A3). Notice that the function $(t, z) \rightarrow \log \frac{1}{|z - t|}$ is positive on $U_{1/2}$, and the Fubini theorem insures that the quantity

$$\int \int \log \frac{1}{|z - t|} d\sigma_1(t) d\sigma_2(z) = \int_{\mathbb{C}} U^{\sigma_1}(z) d\sigma_2(z)$$

is finite. In addition, for every $\delta \in (0, \frac{1}{2})$

$$0 \leq \chi_{U_\delta}(t, z) \log \frac{1}{|z - t|} \leq \chi_{U_{1/2}}(t, z) \log \frac{1}{|z - t|},$$

and the family of sets U_δ is decreasing, with

$$U := \bigcap_{\delta > 0} U_\delta = \{(x, x), x \in \text{supp}(\sigma_1) \cap \text{supp}(\sigma_2)\}.$$

By assumption (A3), the set $\Sigma_1 \cap \Sigma_2$ is of logarithmic capacity zero.

Hence the theorem of monotone convergence gives

$$\begin{aligned}
0 \leq \lim_{\delta \rightarrow 0} \int_{U_\delta} \log \frac{1}{|z-t|} d\sigma_1(t) d\sigma_2(z) &= \lim_{\delta \rightarrow 0} \int_{U_{1/2}} \chi_{U_\delta}(t, z) \log \frac{1}{|z-t|} d\sigma_1(t) d\sigma_2(z) \\
&= \int_{U_{1/2}} \chi_U(t, z) \log \frac{1}{|z-t|} d\sigma_1(t) d\sigma_2(z) \\
&\leq \int_{U_{1/2}} \chi_{\Sigma_1 \cap \Sigma_2}(t) \chi_{\Sigma_1 \cap \Sigma_2}(z) \log \frac{1}{|z-t|} d\sigma_1(t) d\sigma_2(z) \\
&= I(\sigma_1|_{\Sigma_1 \cap \Sigma_2}, \sigma_2|_{\Sigma_1 \cap \Sigma_2}) = 0,
\end{aligned}$$

as claimed in (4.1). \square

In the context of the equilibrium conditions of Theorem 2.2, we will require the so-called *principle of domination* quoted from [ST97, Theorem 2.3.2].

THEOREM 4.2 (Principle of domination). *Let μ and ν be two measures with compact support on \mathbb{C} , and suppose that the total mass of ν does not exceed the total mass of μ . Assume further that μ has finite logarithmic energy. If, for some constant c the inequality*

$$U^\mu(z) \leq U^\nu(z) + c$$

holds μ -almost everywhere, then it holds for all z in \mathbb{C} .

Now we are prepared for a proof of the second part of Theorem 2.2.

PROPOSITION 4.3. *Under the assumptions of Theorem 2.2, the minimizer $\mu^t \in \mathcal{M}_\sigma^t$ satisfies the equilibrium conditions (2.4) with non-negative F_1^t, F_2^t such that $F_1^t + F_2^t > 0$. Conversely, a signed measure $\mu \in \mathcal{M}_\sigma^t$ satisfying the system (2.4) for some constants F_1, F_2 necessarily coincides with the equilibrium signed measure μ^t , and $F_1 = F_1^t$ and $F_2 = F_2^t$.*

Proof. Let us denote $\mu^t = \mu_1^t - \mu_2^t$ the solution of the extremal problem (P) of Theorem 2.2. For some continuous external field Q , the weighted energy is given by the expression

$$I_Q(\nu) = \int \int \log \frac{1}{|z-t|} d\nu(t) d\nu(z) + 2 \int Q(z) d\nu(z).$$

Notice also that $\mathcal{M}_{\sigma_1}^t = \{\nu_1 \text{ measure} : \nu_1(\mathbb{C}) = t, 0 \leq \nu_1 \leq \sigma_1\}$. We denote by ν_Q^t the solution of the extremal problem

$$\text{find } \nu_Q^t \in \mathcal{M}_{\sigma_1}^t \text{ such that } (\widehat{P}) \quad I_Q(\nu_Q^t) = \inf (I_Q(\nu), \nu \in \mathcal{M}_{\sigma_1}^t),$$

with the continuous external field $Q(z) = -U^{\mu_2^t}(z)$. This minimization problem (\widehat{P}) admits a unique minimizer, see, e.g., [Be06, Theorem 2.2.1]. We claim that $\nu_Q^t = \mu_1^t$. Indeed, for any $\nu \in \mathcal{M}_{\sigma_1}^t$

$$\begin{aligned}
I_Q(\nu) &= \int \int \log \frac{1}{|z-t|} d\nu(t) d\nu(z) + 2 \int Q(z) d\nu(z) \\
&= \int \int \log \frac{1}{|z-t|} d\nu(t) d\nu(z) - 2 \int U^{\mu_2^t}(z) d\nu(z) \\
&= \int \int \log \frac{1}{|z-t|} d\nu(t) d\nu(z) - 2 \int \int \log \frac{1}{|z-t|} d\mu_2^t d\nu(z) \\
&= I(\nu - \mu_2^t) - I(\mu_2^t),
\end{aligned}$$

which becomes minimal for $\nu = \mu_1^t$, and hence $\nu_Q^t = \mu_1^t$. In other words, the global minimizer of the energy functional must also be a coordinatewise minimum for this functional. Consequently, from [Rak96] we get the following equilibrium conditions

$$\begin{cases} U^{\mu_1^t}(z) - U^{\mu_2^t}(z) \geq F_1^t, & z \in \text{supp}(\sigma_1 - \mu_1^t), \\ U^{\mu_1^t}(z) - U^{\mu_2^t}(z) \leq F_1^t, & z \in \text{supp}(\mu_1^t), \end{cases}$$

for some real constant F_1^t . Recalling that μ_1^t, μ_2^t are two measures with compact support, equal mass t and finite logarithmic energy, the second equilibrium condition together with the domination principle stated in Theorem 4.2 allows to conclude that

$$U^{\mu_1^t}(z) - U^{\mu_2^t}(z) \leq F_1^t$$

for all $z \in \mathbb{C}$. We apply the same idea to analyze the energy minimization problem with respect to the constraint σ_2 with external field $Q = -U^{\mu_1^t}$, which leads to the equilibrium conditions (2.4) for the signed measure μ^t .

Taking into account that $\lim_{|z| \rightarrow \infty} U^{\mu^t}(z) = 0$, we have from (2.4) the inequality $-F_2^t \leq 0 \leq F_1^t$, and $F_1^t + F_2^t > 0$ follows from [ST97, Lemma 1.1.8] since the measure μ^t is non-trivial.

In order to show the second part of Proposition 4.3, let $\mu \in \mathcal{M}_\sigma^t$ satisfying the equilibrium conditions (2.4) for two real constants F_1 and F_2 . As previously, we see that both F_1 and F_2 non-negative and that $F_1 + F_2 > 0$.

First, we prove that the minimizer μ^t for the problem (P) is characterized by the following inequality

$$(4.2) \quad I(\mu^t) \leq I(\nu, \mu^t) \quad \forall \nu \in \mathcal{M}_\sigma^t.$$

This characterization of the minimizer is a usual property of the element of smallest norm in a convex subset of a real Hilbert space, see [NS91, Lemma 5.4.1]. To prove (4.2), we choose $s \in (0, 1]$. Then $s\nu + (1-s)\mu^t$ belongs to \mathcal{M}_σ^t by convexity, and

$$0 \leq I(s\nu + (1-s)\mu^t) - I(\mu^t) \leq s^2 I(\nu) + (s^2 - 2s)I(\mu^t) + 2s(1-s)I(\nu, \mu^t),$$

which leads to (4.2) if we divide by $2s$ and we let s tend to 0.

Conversely, if $\mu \in \mathcal{M}_\sigma^t$ verifies $I(\mu) \leq I(\nu, \mu)$ for every ν in \mathcal{M}_σ^t , then by taking into account that

$$I(\nu, \mu) = \frac{1}{2} (I(\nu) + I(\mu) - I(\mu - \nu)),$$

we obtain by choosing $\nu = \mu^t$, the solution of (P),

$$I(\mu) \leq I(\mu^t) - I(\mu - \mu^t).$$

Consequently, $I(\mu - \mu^t) = 0$ and $\mu = \mu^t$ by [ST97, Lemma 1.1.8].

In view of (4.2), it will be therefore sufficient to show that the equilibrium conditions (2.4) for μ imply that $I(\mu) \leq I(\mu, \nu)$ for every ν in \mathcal{M}_σ^t . We have

$$I(\nu, \mu) - I(\mu) = \int U^\mu(z) d(\nu - \mu)(z) = \int U^\mu(z) d(\nu_1 - \mu_1)(z) - \int U^\mu(z) d(\nu_2 - \mu_2)(z)$$

and, as $\mu_1(\mathbb{C}) = \mu_2(\mathbb{C}) = \nu_1(\mathbb{C}) = \nu_2(\mathbb{C})$,

$$\int U^\mu(z) d(\nu_1 - \mu_1)(z) = \int (U^\mu(z) - F_1^t) d(\nu_1 - \mu_1)(z).$$

Consider the sets

$$\Sigma_1^- := \{z : U^\mu(z) < F_1^t\}, \quad \Sigma_2^- := \{z : -U^\mu(z) < F_2^t\}.$$

From (2.4) we know that $U^\mu(z) \leq F_1^t$ for all $z \in \mathbb{C}$, and $(\sigma_1 - \mu_1)(\Sigma_1^-) = 0$. Thus

$$\begin{aligned} \int (U^\mu(z) - F_1^t) d(\nu_1 - \mu_1)(z) &= \int_{\Sigma_1^-} (U^\mu(z) - F_1^t) d(\nu_1 - \mu_1)(z) \\ &= \int_{\Sigma_1^-} (U^\mu(z) - F_1^t) d(\nu_1 - \mu_1 - (\sigma_1 - \mu_1))(z) \\ &= \int_{\Sigma_1^-} (U^\mu(z) - F_1^t) d(\nu_1 - \sigma_1)(z) \geq 0, \end{aligned}$$

where for the last inequality we used the definition of the set Σ_1^- and the inequality $\nu_1 \leq \sigma_1$. In the same way one shows the inequality

$$\int_{\Sigma_1 \cup \Sigma_2} (-U^\mu(z) + F_2^t) d(\nu_2 - \mu_2)(z) \geq 0,$$

implying that

$$I(\nu, \mu) - I(\mu) = \int U^\mu(z) d(\nu - \mu)(z) \geq 0,$$

which allows us to conclude that μ is the minimizer of (P) .

We finally notice that the uniqueness of the constants F_1^t and F_2^t immediately follows from the equality conditions in (2.4) which are non-trivial since t is strictly less than both $\sigma_1(\mathbb{C})$ and $\sigma_2(\mathbb{C})$. \square

REMARK 4.4. *We could prove the following statement by application of the maximum principle for subharmonic functions : if we add the additional requirement on the support of the constraint that $\text{supp}(\sigma_j)$ have empty interior and connected complement for $j \in \{1, 2\}$, then $\text{supp}(\sigma_j) = \text{supp}(\mu_j^t)$ for $j = 1, 2$ and moreover both extremal constants F_1^t and F_2^t are strictly positive.*

5. Integral formulation for the extremal constant. The main aim of this section is to give the following integral formula of the extremal quantity $F^t = F_1^t + F_2^t$ which occurs in Theorem 2.1 and Theorem 2.2.

THEOREM 5.1. *Suppose that $\text{supp}(\sigma_j)$ have empty interior and connected complement for $j = 1, 2$, and that $\sigma_1(\mathbb{C}) = \sigma_2(\mathbb{C})$. We have for $t \in (0, \sigma_1(\mathbb{C}))$ the integral formulation*

$$(5.1) \quad F^t = \int_0^t \frac{d\tau}{\text{cap}(\text{supp}(\sigma_1 - \mu_1^\tau), \text{supp}(\sigma_2 - \mu_2^\tau))},$$

where

$$(5.2) \quad \frac{1}{\text{cap}(K_1, K_2)} = \min\{I(\mu_1 - \mu_2) : \mu_j \text{ is a probability measure supported on } K_j \text{ for } j = 1, 2\}$$

denotes the logarithmic capacity of a condenser with plates K_1 with positive unit charge, and K_2 with negative unit charge.

Theorem 5.1 does apply for example for a constraint σ supported on the real line. Since the sets $\text{supp}(\sigma_j - \mu_j^t)$ are shown in Proposition 5.4 to be decreasing in t , it follows that the function $t \mapsto -F^t$ is concave, showing some super-linear convergence behavior for the Zolotarev quantity $Z_n(E_N, F_N)$.

Our integral representation becomes an explicit expression in terms of complete elliptic integrals (see, e.g., Corollary 5.7) once we know that the sets $\text{supp}(\sigma_j - \mu_j^t)$ are real intervals. We will therefore discuss in Proposition 5.8 some sufficient conditions on the constraint σ insuring that this interval case does occur.

Our findings described above rely on recent results of Lapik on vector-valued minimal energy problems with external field [La06]. Hence, in a first step, we have to establish in § 5.1 a duality relation between two minimal energy problems on condensers, the first one with external field but without constraint, and the second one with constraint but without external field. Again we are inspired by similar results for measures: here the duality is known already from [DrSa97] and was used in [BeK01a] to derive an integral representation from preceding work of Buyarov & Rakhmanov [BuRa99]. Our sufficient condition for the interval case is inspired by work of Kuijlaars & Dragnev [KuDr99].

In all results of Section § 5 we suppose in addition that $\text{supp}(\sigma_j)$ for $j = 1, 2$ have empty interior and connected complements. This holds for instance if σ is supported on the real line, or on some open Jordan arc in the complex plane. Also, for convenience we will suppose that $T = \sigma_1(\mathbb{C}) = \sigma_2(\mathbb{C})$. This latter condition can be achieved for instance without changing the extremal signed measure μ^t by augmenting σ_j on its unconstrained part $\text{supp}(\sigma_j - \mu_j^t)$.

5.1. Duality constraint/external field. The next result which we will refer to as *duality constraint/external field* allows us to show that our constrained energy minimization problem (P) for signed measures may in fact be described by an unconstrained minimization problem for signed measures with external field Q . The latter problem has been analyzed in detail in [ST97, Chapter VIII].

We suppose from now on that the constraint σ has total mass 0, in other words, $\sigma_1(\mathbb{C}) = \sigma_2(\mathbb{C})$. Let us consider the set

$$\mathcal{Q}^s := \{\tilde{\mu} = \tilde{\mu}_1 - \tilde{\mu}_2, \tilde{\mu}_i \text{ is a measure on } \Sigma_i : \tilde{\mu}_i(\mathbb{C}) = s\},$$

and the following minimal energy problem

$$\text{Find } \tilde{\mu}^s \in \mathcal{Q}^s \text{ such that } (\tilde{P}) \quad I_Q(\tilde{\mu}^s) = \inf(I_Q(\tilde{\mu}), \tilde{\mu} \in \mathcal{Q}^s),$$

with the weighted energy $I_Q(\tilde{\mu}) = I(\tilde{\mu}) + 2 \int Q(z) d\tilde{\mu}(z)$ as defined before. We will consider the continuous external field $Q = -U^\sigma$. From [ST97, Theorem VIII.1.4] we know that (\tilde{P}) has a unique solution.

PROPOSITION 5.2. *Let $s, t > 0$ such that $s + t = \sigma_1(\mathbb{C}) = \sigma_2(\mathbb{C})$. Then the solution $\mu^t = \mu_1^t - \mu_2^t$ for the extremal problem (P) and the solution $\tilde{\mu}^s = \tilde{\mu}_{Q,1}^s - \tilde{\mu}_{Q,2}^s$ for the problem (\tilde{P}) with external field $Q = -U^\sigma$ are related by $\tilde{\mu}^s = \sigma - \mu^t$.*

Proof. We use the following result quoted from [ST97, Theorem VIII.2.2]: if

$\tilde{\mu} \in \mathcal{Q}^s$ of finite energy satisfies the equilibrium conditions

$$(5.3) \quad \begin{cases} U^{\tilde{\mu}}(z) + Q(z) \geq \tilde{F}_{1,Q}^s, & z \in \Sigma_1, \\ U^{\tilde{\mu}}(z) + Q(z) \leq \tilde{F}_{1,Q}^s, & z \in \text{supp}(\tilde{\mu}_1), \\ -U^{\tilde{\mu}}(z) - Q(z) \geq \tilde{F}_{2,Q}^s, & z \in \Sigma_2, \\ -U^{\tilde{\mu}}(z) - Q(z) \leq \tilde{F}_{2,Q}^s, & z \in \text{supp}(\tilde{\mu}_2), \end{cases}$$

for some constants $\tilde{F}_{1,Q}^s, \tilde{F}_{2,Q}^s$ then $\tilde{\mu}$ is the unique extremal solution of (\tilde{P}) .

Let $\tilde{\mu} = \sigma - \mu^t = \tilde{\mu}_1 - \tilde{\mu}_2$ with $\tilde{\mu}_j = \sigma_j - \mu_j^t$ for $j = 1, 2$. Notice that $\mu^t \in \mathcal{M}_\sigma^t$ implies that $\tilde{\mu}$ is indeed an element of \mathcal{Q}^s , in particular it has finite energy. With $Q(z) = -U^\sigma(z)$ as before, we observe that $U^{\tilde{\mu}}(z) + Q(z) = -U^{\mu^t}(z)$. Thus (5.3) is an immediate consequence of (2.4) by setting $\tilde{F}_{j,Q}^s = -F_j^t$ for $j = 1, 2$. Hence, $\tilde{\mu}^s = \sigma - \mu^t$. \square

REMARK 5.3. *Concerning the first part of the preceding proof, we should mention that, by [ST97, Theorem VIII.2.2], also the reversed statement is true: the solution $\tilde{\mu} = \tilde{\mu}^s$ of the extremal problem (\tilde{P}) with external field $Q = -U^\sigma$ does always satisfy (5.3), with unique extremal constants $\tilde{F}_{j,-U^\sigma}^s$ (which because of Proposition 5.2 coincide with $-F_j^t$ for $j \in \{1, 2\}$). Our Proposition 5.2 gives us further a priori unexpected properties for our special external field: for instance we have that the first and the third inequality of (5.3) are true for all $z \in \mathbb{C}$. Also, $\tilde{\mu}_j^s \leq \sigma_j$ for $j = 1, 2$ (which is also a consequence of [ST97, Theorem IV.4.5]).*

5.2. Proof of Theorem 5.1. As said before, our proof of Theorem 5.1 depend heavily on recent work of Lapik [La06] who discussed a generalization of the Mhaskar-Saff-Rakhmanov functional for unconstrained vector-valued minimal energy problems with external field. Applying the duality constraint/external field of Proposition 5.2, we may state her main findings directly for our constrained problem for signed measures.

PROPOSITION 5.4. *Under the assumptions of Theorem 5.1, we define the functional F_t acting on a couple of disjoint compact sets K_1, K_2 of positive capacity by*

$$F_t(K_1, K_2) := (\sigma_1(\mathbb{C}) - t) \frac{1}{\text{cap}(K_1, K_2)} - \int U^\sigma(z) d\omega_{K_1, K_2}(z)$$

where ω_{K_1, K_2} is the minimizer in (5.2), that is, the equilibrium signed measure for the condenser (K_1, K_2) .

Then, we have with $S_j^t := \text{supp}(\sigma_j - \mu_j^t)$ for $j \in \{1, 2\}$

$$F_t(K_1, K_2) \geq F_t(S_1^t, S_2^t) = -F^t,$$

and equality holds if and only if, for $j = 1, 2$,

$$S_j^t \subset K_j \subset \{z \in \mathbb{C} : U^{\mu^t}(z) = (-1)^{j-1} F_j^t\}.$$

Moreover, we have for every $t, \delta > 0$ with $t + \delta < \sigma_1(\mathbb{C})$

$$S_j^{t+\delta} \subset \{z \in \mathbb{C} : U^{\mu^{t+\delta}}(z) = (-1)^{j-1} F_j^t\} \subset S_j^t \subset \{z \in \mathbb{C} : U^{\mu^t}(z) = (-1)^{j-1} F_j^t\}.$$

We should mention that the original result in [La06] is stated and proved for the case of σ supported on the real axis, but the proof is the same for constraints σ as in Theorem 5.1.

Before entering in the proof of Theorem 5.1, let us quickly return to (5.2), that is, to the unconstrained condenser (S_1^t, S_2^t) without external field. We know from [ST97, Theorem VIII.1.4] that for the extremal signed measure denoted by $\omega^t = \omega_1^t - \omega_2^t$ there holds

$$(5.4) \quad \begin{cases} U^{\omega^t}(z) = c_1^t & \text{quasi-everywhere on } S_1^t, \\ U^{\omega^t}(z) = -c_2^t & \text{quasi-everywhere on } S_2^t. \end{cases}$$

From the inclusion properties of the last part of Proposition 5.4 we see that $\text{supp}(\omega_j^{t+\epsilon}) \subset S_j^t$, and thus $I(\omega^{t+\epsilon}) \geq I(\omega^\epsilon)$. Consequently,

$$t \mapsto I(\omega^t) = \int U^{\omega^t} d\omega^t = c_1^t + c_2^t = \frac{1}{\text{cap}(S_1^t, S_2^t)}$$

is increasing in t , and its set N of discontinuities in $(0, \sigma_1(\mathbb{C}))$ is at most countable.

We are now prepared to proceed with the proof of Theorem 5.1.

Proof. Let t, ϵ, η such that $0 < t - \eta < t < t + \epsilon < \sigma_1(\mathbb{C})$. We claim that

$$(5.5) \quad \frac{F^t - F^{t-\eta}}{\eta} \leq \frac{1}{\text{cap}(S_1^t, S_2^t)} \leq \frac{F^{t+\epsilon} - F^t}{\epsilon},$$

showing the convexity of the function $\phi : (0, \sigma_1(\mathbb{C})) \ni t \mapsto F^t$. Moreover, for $t \notin N$ we may also conclude from (5.5) that ϕ is differentiable in t , with

$$\phi'(t) = \frac{\partial F^t}{\partial t} = \frac{1}{\text{cap}(S_1^t, S_2^t)}.$$

If we are able to show that $\phi(t) \rightarrow 0$ for $t \rightarrow 0+$, then the claimed identity (5.1) of Theorem 5.1 follows since convexity implies that ϕ is absolutely continuous.

In order to establish (5.5), notice that, by Proposition 5.4,

$$\begin{aligned} -F^{t+\epsilon} &= F_{t+\epsilon}(S_1^{t+\epsilon}, S_2^{t+\epsilon}) \leq F_{t+\epsilon}(S_1^t, S_2^t) \\ &= F_t(S_1^t, S_2^t) - \frac{\epsilon}{\text{cap}(S_1^t, S_1^t)} = -F^t - \frac{\epsilon}{\text{cap}(S_1^t, S_2^t)} \end{aligned}$$

and similarly

$$-F^{t-\eta} \geq F_{t-\eta}(S_1^t, S_2^t) = -F^t + \frac{\eta}{\text{cap}(S_1^t, S_2^t)}.$$

Thus (5.5) follows.

For the limit $\phi(0+)$ we observe that, by (2.4),

$$I(\mu^t, \sigma - \mu^t) = \int U^{\mu^t} d(\sigma_1 - \mu_1^t) - \int U^{\mu^t} d(\sigma_1 - \mu_1^t) = (\sigma_1(\mathbb{C}) - t) F^t,$$

and, by (4.2),

$$\begin{aligned} 0 &\leq (\sigma_1(\mathbb{C}) - t) F^t \leq |I(\mu^t, \sigma - \mu^t)| \\ &\leq |I(\mu^t, \mu^t)| + |I(\mu^t, \sigma)| \leq \left| I\left(\mu^t, t \frac{\sigma}{\sigma_1(\mathbb{C})}\right) \right| + |I(\mu^t, \sigma)| \\ &\leq \left(1 + \frac{t}{\sigma_1(\mathbb{C})}\right) |I(\mu^t, \sigma)| \leq 4t \sup_{z \in \mathbb{C}} |U^\sigma(z)|, \end{aligned}$$

which yields the required relation $F^t \rightarrow 0$ for $t \rightarrow 0+$. \square

REMARK 5.5. *The sets $\text{supp}(\sigma_j - \mu_j^t)$ occurring in Theorem 5.1 are usually very difficult to describe and may even have a Cantor structure. To see that such examples may be constructed, let $S_j(t)$ for $0 < t < T$ and $j = 1, 2$ be arbitrary real compact sets (say, regular with respect to the Dirichlet problem), decreasing in t , with $S_1(t) \cap S_2(t)$ empty, and continuous in the sense that the closure of $\cup_{\tau > t} S_j(\tau)$ coincides with $S_j(t)$. We claim that for the constraint*

$$\sigma = \int_0^T \omega_{S_1(t), S_2(t)} dt$$

there holds $\text{supp}(\sigma_j - \mu_j^t) = S_j(t)$ for all $0 < t < T$ and $j = 1, 2$. To see this, define $\mu = \int_0^t \omega_{S_1(\tau), S_2(\tau)} d\tau$, then $\mu \in \mathcal{M}_\sigma^t$, with $\text{supp}(\sigma_j - \mu_j) = S_j(t)$. For its potential we find using Fubini

$$U^\mu(z) = \int \log \frac{1}{|z-x|} \int_0^t d\omega_{S_1(\tau), S_2(\tau)}(x) d\tau = \int_0^t U^{\omega_{S_1(\tau), S_2(\tau)}}(z) d\tau,$$

and the relations (5.4) together with the nested structure of the $S_j(\tau)$ allows to verify the equilibrium conditions (2.4). Hence, by Theorem 2.2, we have $\mu = \mu^t$.

5.3. The special case of a symmetric real constraint. In this subsection we will restrict ourselves to constraints σ being supported on the real line and being symmetric, that is, $\sigma_1(x) = \sigma_2(-x)$, or $\sigma(x) = -\sigma(-x)$. Such a situation is for instance true if $E_N = -F_N \subset (0, \infty)$ in assumption (A2). Notice also that, with σ , also the extremal signed measure μ^t has this symmetry property by the uniqueness result of Theorem 2.2. Here we concentrate on what we call the *symmetric interval case*.

DEFINITION 5.6. *We say that the symmetric interval case holds for a constraint σ if there are increasing $t \mapsto a_t$ and decreasing $t \mapsto b_t$ such that, for $0 < t < \sigma_1(t)$*

$$\text{supp}(\sigma_1 - \mu_1^t) = -\text{supp}(\sigma_2 - \mu_2^t) = [a_t, b_t].$$

We know from [ST97, Example 2.5.14] or [LevLub01, §5] that, in the symmetric interval case,

$$(5.6) \quad \text{cap}(\text{supp}(\sigma_1 - \mu_1^t), \text{supp}(\sigma_2 - \mu_2^t)) = \frac{K'(a_t/b_t)}{2\pi K(a_t/b_t)}.$$

It follows from Remark 5.5 that the constraint σ is necessarily supported on the real line and symmetric, and thus verifies the assumptions of Theorem 5.1. Hence a combination of Theorem 2.1 and Theorem 5.1 yields the following.

COROLLARY 5.7. *Suppose that the assumptions (A1), (A2), (A3), (A4), and (A5) are verified with $E_N = -F_N \subset (0, +\infty)$, such that the symmetric interval case holds. Then we have the following explicit formula for the asymptotic behaviour of the Zolotarev quantity*

$$\lim_{n, N \rightarrow +\infty, n/N \rightarrow t} \frac{1}{N} \log Z_n(E_N, -E_N) = -2\pi \int_0^t \frac{K(a_\tau/b_\tau)}{K'(a_\tau/b_\tau)} d\tau.$$

Compared to the formula for $Z_n([a, b], [-b, -a])$ given in the introduction, we see that the term a_τ/b_τ increasing in τ has to be replaced by the constant term a/b . Hence Corollary 5.7 which takes into account the asymptotic fine structure of the sets E_N leads to a sharper result as long as a_τ/b_τ is not constant.

In order to be able to apply this corollary, we still need to know whether for a symmetric real constraint the symmetric interval case is true. Here we suggest the following sufficient condition being inspired by [BeK01a, Lemma 3.1].

PROPOSITION 5.8. *Consider a symmetric constraint $\sigma(x) = -\sigma(-x)$ with support $\text{supp}(\sigma_1) = [A, B] \subset [0, +\infty)$ and density σ'_1 .*

Provided that $x \rightarrow \sqrt{(x^2 - A^2)(B^2 - x^2)}\sigma'_1(x)$ is an increasing function on $[A, B]$, the symmetric interval case holds with $b_\tau = B$.

Similarly, provided that $x \rightarrow \sqrt{(x^2 - A^2)(B^2 - x^2)}\sigma'_1(x)$ is a decreasing function on $[A, B]$, the symmetric interval case holds with $a_\tau = A$.

Proof. We only show the first part of this statement, the second is similar. Define the measure γ supported on $[A^2, B^2]$ with density $\gamma'(x) := \sigma'_1(\sqrt{x})$ and mass

$$T := \int \gamma'(x) dx = \int_{A^2}^{B^2} \sigma'_1(\sqrt{x}) dx = \int_A^B 2x\sigma'_1(x) dx.$$

Then $x \mapsto \sqrt{(x - A^2)(B^2 - x)}\gamma'(x)$ is increasing on $[A^2, B^2]$. For the problem of minimal energy for measures and constraint γ , we learn from [BeK01a, Lemma 3.1] and the modified Buyarov-Rakhmanov formula of [BeK01a, p. 9] that there exists $t \mapsto \alpha(t) \in [A^2, B^2]$ increasing such that

$$\gamma'(x) = \int_0^t \frac{1}{\pi} \frac{\chi_{[\alpha(t), B^2]}(x)}{\sqrt{(B^2 - x)(x - \alpha(t))}} dt.$$

Consequently,

$$\sigma'_1(x) = \gamma'(x^2) = \int_0^t \frac{1}{\pi} \frac{\chi_{[\alpha(t), B^2]}(x)}{\sqrt{(B^2 - x^2)(x^2 - \alpha(t))}} dt.$$

Denote by $\omega_{a,b}$ the equilibrium signed measure of the symmetric condenser with positive plate $[a, b]$ and negative plate $[-b, -a]$. For $x \in [a, b]$, there is an explicit formula for its density, see [ST97, p 413]

$$(5.7) \quad \omega'_{a,b}(x) = \frac{1}{\sqrt{(x^2 - a^2)(b^2 - x^2)}} \frac{bd\tau}{K'(a/b)}.$$

Notice that the function

$$\phi(u) = \int_0^u \frac{K'(\sqrt{\alpha(t)}/B)}{\pi B} dt$$

is clearly continuous and strictly increasing. A simple substitution shows that σ'_1 may be rewritten as

$$\sigma'_1(x) = \sigma'_2(-x) = \int_0^{\sigma_1(\mathbb{C})} \omega_{a_t, B}(x) dt, \quad a_t = \alpha(\phi^{-1}(t)),$$

and, as in Remark 5.5, we conclude that $\text{supp}(\sigma_1 - \mu_1^t) = [a_t, B]$, at least for those t where $t \mapsto a_t$ is continuous, otherwise we take limits from the right. \square

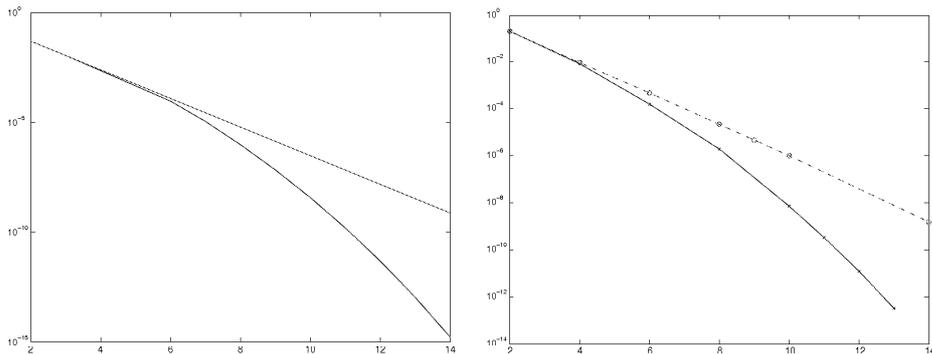


FIG. 6.1. On the left, asymptotic behavior of $n \mapsto Z_n(E_N, -E_N)$ versus $n \mapsto Z_n(\text{conv}(E_N), \text{conv}(-E_N))$: for $N = 20$ and the Laplace example $E_N = \{2 - 2 \cos(\pi k / (N + 1)) : k = 1, \dots, N\}$, we have drawn on a semi-logarithmic scale the curves $n \mapsto \exp(-NF^{n/N})$ resulting from Corollary 5.7 (lower curve), and $n \mapsto \exp(-n \text{cap}(\text{conv}(E_N), \text{conv}(-E_N))^{-1})$ resulting from (1.2) (upper curve). On the right, the quantity $Z_n(E_N, -E_N)$ (solid with crosses) versus $Z_n(\text{conv}(E_N), -\text{conv}(E_N))$ (dashed with circles) on a semi-logarithmic scale for increasing degree $n = 2, 3, \dots$

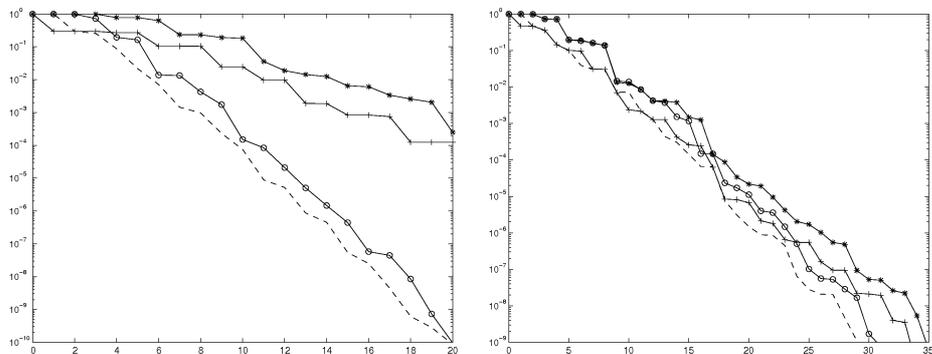


FIG. 6.2. The ADI error (dashed line) and its upper bound (1.5) (solid line with circles) for ADI parameters adapted to the discrete set E_N . For the sake of comparison, we have also drawn the ADI error for Bagby points on the interval given by the convex hull of E_N (solid line with crosses), and its upper bound (1.5) (solid line with stars). On the left we display the case of cosine points, on the right the Laplace case, both for $N = 1000$.

6. Two numerical examples. In what follows we present two numerical examples, in order to illustrate the theoretical findings on asymptotics of the Zolotarev quantity $Z_n(E_N, F_N)$. In particular, we want to illustrate the impact of our work to the choice of parameters in the ADI method, though the examples studied here are still of academic nature. We will consider two families of sets $E_N = -F_N \subset (0, +\infty)$, namely

$$E_N = \left\{ \cos\left(\frac{\pi k}{2N}\right) + \frac{1}{N^4} : 1 \leq k \leq N \right\} \quad (\text{cosine case}),$$

$$E_N = \left\{ 2 - 2 \cos\left(\frac{\pi k}{N+1}\right) : 1 \leq k \leq N \right\} \quad (\text{Laplace case}),$$

the latter giving the eigenvalues of a central finite difference discretization with step-size $1/(N+1)$ of the 1D Laplacian on $[0, 1]$. It is not difficult to verify that all

five assumptions of §2 do hold, with constraint being symmetric, having the densities $\sigma'_1(x)$ being given by

$$\frac{2}{\pi\sqrt{1-x^2}} \text{ on } [A, B] = [0, 1], \text{ and } \frac{1}{\pi\sqrt{x(4-x)}} \text{ on } [A, B] = [0, 4].$$

Hence Proposition 5.8 tells us that, in both cases, $\text{supp}(\sigma_1 - \mu_1^t) = [a_t, B]$ for $0 < t < \sigma_1(\mathbb{C}) = 1$. For the cosine case and $N = 20$, the curves $n \rightarrow Z_n(E_N, -E_N)$ and $n \rightarrow Z_n(E, -E)$ with $E = \text{conv}(E_N)$ have already been given in Figure 1.1.

We have drawn in Figure 6.1 for the Laplace example and $N = 20$ and $1 \leq n \leq N$ on a semi-logarithmic scale the two expressions for the asymptotics of $Z_n(E_N, -E_N)$ and of $Z_n(E, -E)$, with $E = \text{conv}(E_N)$. That is, we put $t = n/N$, and draw the two functions

$$n \mapsto \exp\left(-N \int_0^{n/N} \frac{d\tau}{\text{cap}([-B, a_\tau], [a_\tau, B])}\right), \quad n \mapsto \exp\left(-n \text{cap}(-E, E)^{-1}\right),$$

the first coming from Corollary 5.7, and the second from (1.2). Clearly, the two expressions differ, especially for larger n , which is natural since the first takes into account the fine structure of the sets E_N . Notice also that the first curve is concave (in a semi-logarithmic scale), as predicted by our theory. We should also mention that we have only established N th root asymptotics, whereas our plot suggests indeed strong asymptotics, at least for these particular sets E_N , compare for instance Figure 1.1 with Figure 6.1.

We should also explain how we found numerical values for the constants a_t . According to Proposition 5.4, we may find a_t for fixed t by looking for stationary points of the function

$$a \mapsto F_t([-B, -a], [a, B]),$$

that is, points where the derivative with respect to a becomes zero. Notice that where the right-hand side may be made more explicit using the explicit formulas (5.6) and (5.7). However, since the resulting expression for the derivative is linear in t but complicated in a , it is easier to proceed in the opposite way, by fixing $a \in (A, B)$, and then find the $t > 0$ such that $a = a_t$.

In our particular symmetric interval case, we could also determine a_t by following the lines described in the proof of Proposition 5.8: in our two cases, the function $x \mapsto \sigma'_1(x)\sqrt{(x^2 - A^2)(B^2 - x^2)}$ vanishes for $x = A$, and hence, by [BeK01a, Lemma 3.1], the quantity $\alpha(t)$ is determined by the integral equation

$$t = \int_{A^2}^{\alpha(t)} \sqrt{\frac{B^2 - x}{\alpha(t) - x}} \sigma'_1(\sqrt{x}) dx.$$

This yields for the cosine example $\alpha(t) = (4t/\pi)^2$, whereas for the Laplace example we get more complicated expressions.

Let us now turn to the application of our results to the ADI method. As mentioned in §1.1, our starting point is the Lyapunov equation $A_N X + X A_N = B_N$ in the unknown X , where here A_N is symmetric positive definite with spectrum given by the above sets E_N . In the cosine case, we simply use a diagonal matrix, whereas in the Laplace case we use for A_N the classical discretized 1D Laplacian, such that X represents the solution of a discretized 2D Poisson problem on the unit square.

In our examples, we have always chosen $N = 1000$ and fixed a random solution X , which enables us to compute the corresponding right-hand side B_N . Also, for the ADI iteration we have chosen the starting matrix $X_0 = 0$.

For two sets of parameters $p_j = -q_j$ to be specified below, the first one adapted to the discrete set E_N , and the second one to its convex hull, we have drawn in Figure 6.2 the ADI error $\|X_n - X\|/\|X\|$, as well as its upper bound given in terms of an explicit rational function r_n in (1.5). First it is interesting to observe that in our cases the estimate (1.5) appears to be quite sharp. Also, in our example, the convergence rate for the $\text{conv}(E_N)$ -parameters (upper curves) appears to be linear, as predicted for optimal parameters by (1.2). The rate for the E_N -parameters (lower curves) is better (which is natural), in particular for the cosine case (which is also related to the presence of one "isolated" eigenvalue $1/N^4$ close to 0). In addition, it seems that the curves for the E_N -parameters are globally slightly concave, as predicted by our theory (though the concavity is more pronounced for larger values of n/N , see Figure 6.1).

We still have to explain our choice of parameters, which for the moment is partly heuristic. It follows from our work that rational Fekete points are of theoretical use, but they are difficult to compute. Given a set E in the right half-plane, Bagby points (or rational Leja points) $p_j = -q_j$ for the couple of sets $(E, -E)$ with weight w are found via the recursive procedure [ST97, Definition before Theorem VIII.3.5]

$$q_{m+1} = \arg \min_{z \in E} |w(z)r_m(z)|, \quad r_m(z) = \prod_{j=1}^m \frac{z - q_j}{z + q_j}, \quad r_0(z) = 1.$$

For parameters adapted for E being the convex hull of E_N , we take the weight $w(z) = 1$, and choose the parameters q_m for $m = 1, 2, \dots, [tN]$, with the parameter $t \in (0, 1)$ chosen in advance. Bagby points have been suggested, for instance, in [LevRei93]. Of course, in practice we do not choose the real interval $E = \text{conv}(E_N)$, but take as E the discrete set formed by shifted Chebyshev points of degree, say, $10N$ for this interval which should give about the same asymptotic behavior. It is known that such a choice of parameters gives asymptotically optimal results not only for $n = [tN]$ but for the whole interval $1 \leq n \leq [tN]$, see, e.g., Figure 6.2.

From our theory we know that only the fine structure of E_N outside $\text{supp}(\sigma_1 - \mu_1^t)$ plays a role. Thus, in order to mimic this fine structure, we fix some $a \in (A, B)$ sufficiently large (in order to allow for $[tN]$ iterations with $a = a_t$), and compute explicitly all eigenvalues of our matrix A_N in $[0, a]$, say, q_1, \dots, q_m . We then add $[tN] - m$ further parameters $q_{m+1}, \dots, q_{[tN]}$ by starting the Bagby procedure for $E = [a, B]$ at index $m+1$, or, in other words, we use the weight $w = r_m$. Again, in practice one takes for E shifted Chebyshev points of degree, say, $10N$ for $[a, B]$.

However, the resulting order of our points is not suitable to give good errors for all ADI iterations for $n = 1, 2, \dots, [tN]$. We therefore apply the Bagby procedure a second time to the weight $w = 1$ and the set $E = \{q_1, \dots, q_{[tN]}\}$, and these permuted parameters turn out to give small errors for all ADI iterations. For the moment we may show that the resulting rational functions are asymptotically optimal for $n = [tN]$ (compare with [ST97, Theorem VIII.3.5]), but we have only numerical but no theoretical evidence to show similar results for smaller iteration indices.

The motivating idea behind this approach (also for ADI applied to a general Sylvester equations with normal matrices A_N, B_N) is that in the computation of optimal parameters for $n = [tN]$ one should explicitly know all eigenvalues of A_N outside $\text{supp}(\sigma_1 - \mu_1^t)$ and all eigenvalues of B_N outside $\text{supp}(\sigma_1 - \mu_2^t)$. Thus in our approach we make implicitly a kind of deflation of the corresponding eigenspaces

of these eigenvalues in a suitable order, while keeping the error small for all other eigenspaces.

7. Conclusion. A minimal energy problem in logarithmic potential theory for signed and constrained measures has allowed us to describe in Theorem 2.1 the asymptotics of the third Zolotarev problem on discrete sets under some suitable technical assumptions.

We have found in Theorem 5.1 and in a more explicit way in Corollary 5.7 an integral formulation for the extremal constant involved in the asymptotics of the Zolotarev problem over discrete sets for the particular case where the two discrete sets are real and symmetric with respect to the origin. This quantifies the rate of superlinear convergence of the ADI method applied to a Lyapunov equation.

However, several open problems are still under consideration: first, we would like to drop the symmetry assumptions in §5.3 in order to generalize our work to more general Sylvester equations with symmetric matrices $B \neq -A$. Here we require theoretical results insuring that the interval case holds for the two sets $\text{supp}(\sigma_j - \mu_j^t)$ for $j = 1, 2$. In addition, we would like to provide efficient methods for numerically computing the endpoints of such intervals, e.g., by solving a system on non-linear equations stemming from the generalized Mhaskar-Saff-Rakhmanov functional.

Finally, we have seen in §6 that the choice of Leja-Bagby points on discrete seem to give very promising parameters for the ADI method, but we need a better understanding of their asymptotic properties.

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