K-spectral sets and intersections of disks of the Riemann sphere

Catalin Badea, Bernhard Beckermann and Michel Crouzeix

Abstract

We prove that if two closed disks X_1 and X_2 of the Riemann sphere are spectral sets for a bounded linear operator A on a Hilbert space, then $X_1 \cap X_2$ is a complete $(2+2/\sqrt{3})$-spectral set for A. When the intersection $X_1 \cap X_2$ is an annulus, this result gives a positive answer to a question of A.L. Shields (1974).

1 Introduction and the statement of the main results.

Let X be a closed set in the complex plane and let $R(X)$ denote the algebra of bounded rational functions on X, viewed as a subalgebra of $C(\partial X)$ with the supremum norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\} = \sup\{|f(x)| : x \in \partial X\}.$$

Here ∂X denotes the boundary of the set X.

1.1 Spectral and complete spectral sets.

Let $A \in \mathcal{L}(H)$ be a bounded linear operator acting on a complex Hilbert space H. For a fixed constant $K > 0$, the set X is said to be a K-*spectral* set for A if the spectrum $\sigma(A)$ of A is included in X and the inequality $\|f(A)\| \leq K\|f\|_X$ holds for every $f \in R(X)$. Notice that, for a rational function $f = p/q \in R(X)$, the poles of f are outside of X, and the operator $f(A)$ is naturally defined as $f(A) = p(A)q(A)^{-1}$ or, equivalently, by the Riesz holomorphic functional calculus. The set X is a *spectral* set for A if it is a K-spectral set with $K = 1$. Thus X is spectral for A if and only if $\|\rho\| \leq 1$, where $\rho : R(X) \mapsto \mathcal{L}(H)$ is the homomorphism given by $\rho(f) = f(A)$.

We let $M_n(R(X))$ denote the algebra of n by n matrices with entries from $R(X)$. If we let the n by n matrices have the operator norm that they inherit as linear transformations on the n-dimensional Hilbert space $\mathbb{C}^n$, then we can endow $M_n(R(X))$ with the norm

$$\|\,(f_{ij})\,\|_X = \sup\{\|\,(f_{ij}(x))\,\| : x \in X\} = \sup\{\|\,(f_{ij}(x))\,\| : x \in \partial X\}.$$

In a similar fashion we endow $M_n(\mathcal{L}(H))$ with the norm it inherits by regarding an element (A_{ij}) in $M_n(\mathcal{L}(H))$ as an operator acting on the direct sum of n copies of H. For a fixed constant $K > 0$, the set X is said to be a *complete* K-*spectral* set for A if $\sigma(A) \subset X$ and the inequality $\|(f_{ij}(A))\| \leq K\|(f_{ij})\|_X$ holds for every matrix $(f_{ij}) \in M_n(R(X))$ and every n. In terms of the complete bounded norm ([14]) of the homomorphism ρ, this means that $\|\rho\|_{cb} \leq K$. A *complete spectral* set is a complete K-spectral set with $K = 1$.

Spectral sets were introduced and studied by J. von Neumann [12] in 1951. In the same paper von Neumann proved that a closed disk $\{z \in \mathbb{C} : |z - \alpha| \leq r\}$ is a spectral set for A if and only if $\|A - \alpha I\| \leq r$. Also [12], the closed set $\{z \in \mathbb{C} : |z - \alpha| \geq r\}$ is spectral for $A \in \mathcal{L}(H)$ if and only if $\|(A - \alpha I)^{-1}\| \leq r^{-1}$. We refer to two books [4, 14] for a survey of known properties of spectral and complete spectral sets.

1.2 The annulus as a K-spectral set

Let r and R be two positive constants with $r < R$. Let $A \in \mathcal{L}(H)$ be an invertible operator such that $\|A\| \le R$ and $\|A^{-1}\| \le 1/r$. Then $X_1 = \{z \in \mathbb{C} : |z| \le R\}$ and $X_2 = \{z \in \mathbb{C} : |z| \ge r\}$ are spectral sets for A. The annulus

$$X(r,R) = \{z \in \mathbb{C} : r \le |z| \le R\} = X_1 \cap X_2$$

is not necessarily spectral for a given invertible operator A. Examples can be found in [21, 11, 13]. Given an invertible operator A with $\|A\| \le R$ and $\|A^{-1}\| \le 1/r$, Shields proved in [17] that $X(r,R)$ is a K-spectral set for A with $K = 2 + ((R+r)\,/\,(R-r))^{1/2}$. The following questions were asked by Shields (see [17, Question 7]):

Question 1.1. Find the best constant $K(r,R)$, i.e., the smallest constant C such that $X(r,R)$ is a C-spectral set for all invertible $A \in \mathcal{L}(H)$ with $\|A\| \le R$ and $\|A^{-1}\| \le r^{-1}$.

Question 1.2. Fixing (for instance) R, is this best constant bounded (as a function of r) ?

In analogy with Question 1.1, we will denote by $K_{cb}(r,R)$ the smallest constant C such that $X(r,R)$ is a complete C-spectral set. The same proof of Shields (see also [7, 14]) shows that in fact $K_{cb}(r,R) \le 2 + ((R+r)\,/\,(R-r))^{1/2}$.

1.3 Statement of the main results.

The aim of the present note is to study the intersection of two closed disks of the Riemann sphere which are spectral sets for a Hilbert space bounded linear operator. In the case of the annulus we give an estimate for $K(r,R)$ (a partial answer to Question 1.1) which allows to give a positive answer to Question 1.2.

We describe now the main results of this paper. By possibly multiplying the operator by a scalar, we see that $K(r,R) = K(\sqrt{r/R}, \sqrt{R/r})$. This allows to assume, without any loss of generality, that $r = R^{-1}$. We have the following result.

Theorem 1.3. *Let $R > 1$, $X = X(R^{-1}, R) = \{z \in \mathbb{C} : R^{-1} \le |z| \le R\}$, and denote by $K(R) = K(R^{-1}, R)$ (and $K_{cb}(R) = K_{cb}(R^{-1}, R)$, respectively), the smallest constant C such that X is a C-spectral set (and a complete C-spectral set, respectively) for any invertible $A \in \mathcal{L}(H)$ verifying $\|A\| \le R$ and $\|A^{-1}\| \le R$. Then*

$$\begin{aligned} \frac{2}{1+R^{-2}} &< K(R) \le K_{cb}(R) \\ &\le 2 + \min\left(\sqrt{\frac{R^2+2R+1}{R^2+R+1}}, \sqrt{\frac{R^2+1}{R^2-1}}\right) \le 2 + \frac{2}{\sqrt{3}} < 3.2. \end{aligned}$$

In particular $K(R)$ and $K_{cb}(R)$ are bounded functions of R. We obtain the following consequence about normal dilations.

Corollary 1.4. *Let $R > 1$. Let $A \in \mathcal{L}(H)$ be an invertible operator verifying $\|A\| \le R$ and $\|A^{-1}\| \le R$. Let $X = \{z \in \mathbb{C} : R^{-1} \le |z| \le R\}$. Then there exist an invertible operator $L \in \mathcal{L}(H)$ with $\|L\| \cdot \|L^{-1}\| \le 2 + 2/\sqrt{3}$, a larger Hilbert space $\mathcal{H} \supset H$ and an invertible normal operator $N \in \mathcal{L}(\mathcal{H})$ with $\sigma(N) \subset \partial X$ such that*

$$L^{-1} f(A) L = P_H f(N)\,|_H \quad (f \in R(X)).$$

Here P_H is the orthogonal projection of $\mathcal{H}$ onto H.

Besides the annulus, (complete) K-spectral sets which are intersections of spectral disks of the complex plane have been considered in [19, 20, 10, 5, 3] ; we refer to [3] for a discussion of the best possible constant K. In the second part of our paper we consider the more general case of intersection of two closed disks X_1 and X_2 of the Riemann sphere. We prove the following result.

Theorem 1.5. *Let X_1 and X_2 be two closed disks of the Riemann sphere. If X_1 and X_2 are spectral sets for a bounded operator A in a Hilbert space, then $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for A.*

This theorem extends previously known results concerning the intersection of two disks in $\mathbb{C}$ to not necessarily convex or simply connected $X_1 \cap X_2$. Note that the case of finitely connected compact sets has been studied in [7, 14], however, without a uniform control on the constant K.

Note also that, if we consider two distinct convex and closed subsets X_1 and X_2 of the complex plane, and if we assume that X_1 and X_2 are spectral sets for A, then $X_1 \cap X_2$ is a complete 11.08-spectral set for A. Indeed, the fact that X_j is a spectral set for A implies that the numerical range $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$ is included in X_j, $j = 1, 2$, and according to [6] the closure of the numerical range $W(A)$ is a complete 11.08-spectral set for A.

We refer also to [15, 2, 6] for some normal dilation results for the numerical range, in the spirit of Corollary 1.4.

The remainder of the paper is organized as follows: we first show in §2 that Theorem 1.3 together with some results from [5, 3] implies Theorem 1.5. Our proof of Theorem 1.3 is based on a representation formula for $f(A)$ established in §3. Finally, the proofs of Theorem 1.3 and Corollary 1.4 are provided in §4.

2 Proof of Theorem 1.5 using Theorem 1.3

Let X_1 and X_2 be two closed disks of the Riemann sphere, which are spectral sets for a bounded linear operator A in a Hilbert space. Here six different situations have to be considered, see Figure 1.

Case 1: $X_1 \cap X_2 = \{\lambda\}$ is a singleton. Then we have $A = \lambda I$ and $X_1 \cap X_2$ clearly is a complete spectral set for A.

Case 2: $X_1 \cap X_2$ is a circle or a straight line. Then A is a normal operator with spectrum $\sigma(A)$ contained in $X_1 \cap X_2$. This yields that $X_1 \cap X_2$ is a complete spectral set for A.

Case 3: $X_1 \cap X_2$ is a convex sector or a strip of the complex plane. In this case, both X_1 and X_2 are half-planes, and a closed half-plane Π is a spectral set for A if and only if the numerical range $W(A)$ is a subset of Π. Thus $W(A) \subset X_1 \cap X_2$. It follows from [5] that $X_1 \cap X_2$ is a complete K-spectral set, with $K \leq 2 + 2/\sqrt{3}$.

Case 4: $\partial X_1 \cap \partial X_2 = \{\lambda_1, \lambda_2\}$ is a set consisting of two distinct points of $\mathbb{C}$. Here $X_1 \cap X_2$ is lens-shaped. If it is in addition convex, then from [3] we know that $X_1 \cap X_2$ is a complete K-spectral set, with $K \leq 2 + 2/\sqrt{3}$. The proof for not convex lenses is the same, we repeat here the main idea for the sake of completeness. Let us first assume that $\lambda_1 \notin \sigma(A)$ and set $B = \varphi(A)$ with $\varphi(z) = (\lambda_1 - z)^{-1}$ and $Y_j = \varphi(X_j)$, $j = 1, 2$. Then both Y_j are closed half-planes. The von Neumann inequality for disks shows that Y_j are spectral sets for B, see also [16, § 154, Lemma 2]. It follows from the previous case that $Y_1 \cap Y_2$ is a complete K-spectral set for B and thus $X_1 \cap X_2$ is a complete K-spectral set for A, with the same constant K. Finally, if $\lambda_1 \in \sigma(A)$, we can replace the disk X_1 of the Riemann sphere, of radius R_1, by a concentric disk $X_1' \supset X_1$, of radius $R_1 \pm \varepsilon$. Then, for $\varepsilon > 0$ small enough, $\partial X_1' \cap \partial X_2 = \{\lambda_1', \lambda_2'\}$ is still a set with two distinct points of $\mathbb{C}$, the set X_1' is a spectral set for A and $\lambda_1' \notin \sigma(A)$. We conclude that $X_1 \cap X_2$ is a complete K-spectral set for A by letting $\varepsilon \to 0$.

Case 5: $\partial X_1 \cap \partial X_2 = \emptyset$, but $X_1 \cap X_2$ is not a strip. For the special case $X_1 \cap X_2 = \{z \in \mathbb{C} ; R^{-1} \leq |z| \leq R\}$, $R > 1$, Theorem 1.3 implies that $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for A. In the general case, we may find $R > 1$ and a linear fractional transformation φ such that $\varphi(X_1) = \{z \in \mathbb{C} ; |z| \leq R\}$ and $\varphi(X_2) = \{z \in \mathbb{C} ; |z| \geq R^{-1}\}$. Then, setting $B = \varphi(A)$ and

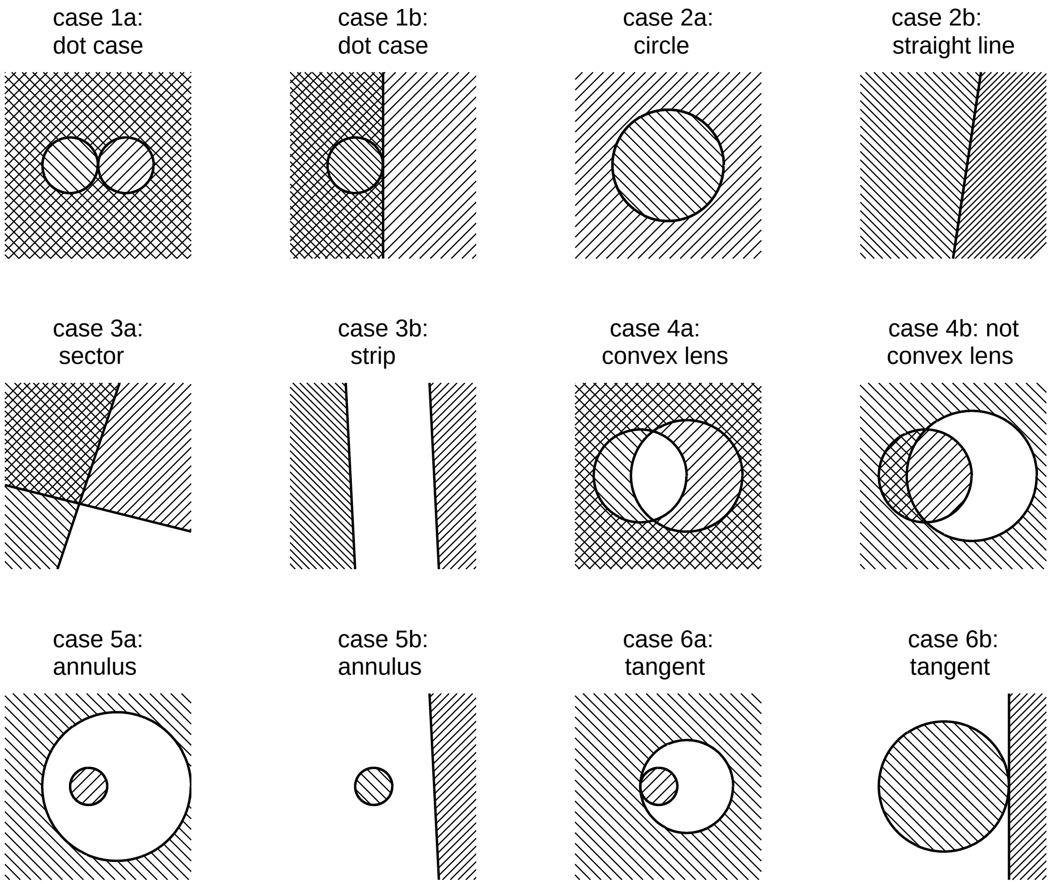

Figure 1: *The six different cases occurring by considering intersections of closed disks on the Riemann sphere.*

$Y_j = \varphi(X_j)$, $j = 1, 2$, we have that Y_j is a spectral set for B, see also [16, § 154, Lemma 2]. Thus $\{z \in \mathbb{C}\,; R^{-1} \le |z| \le R\} = \varphi(X_1 \cap X_2)$ is a complete $(2 + 2/\sqrt{3})$-spectral set for B, which is equivalent to $X_1 \cap X_2$ is a complete $(2 + 2/\sqrt{3})$-spectral set for A.

Case 6: $\partial X_1 \cap \partial X_2 = \{\lambda\}$ is reduced to a single point, but $X_1 \cap X_2$ is neither a singleton, nor a sector nor a strip. In this case at least one of the sets X_j, $j = 1, 2$, is the interior or the exterior of a disk and the boundaries of the sets X_j are tangent in one point. We can replace the disk, say X_1, of radius R_1, by a concentric disk $X_1' \supset X_1$, of radius $R_1 \pm \varepsilon$. Then, for $\varepsilon > 0$ small enough, $\partial X_1' \cap \partial X_2 = \emptyset$, and we obtain from the previous case that $X_1 \cap X_2$ is a complete K-spectral set for A by letting $\varepsilon \to 0$.

3 A decomposition lemma for annuli

In order to give a proof of the upper bound of Theorem 1.3 we need the following representation formula for $f(A)$.

Lemma 3.1. *Let $A \in \mathcal{L}(H)$ be an operator satisfying $\|A\| < R$ and $\|A^{-1}\| < R$. We set $r = 1/R$ and denote by X the annulus $X = X(R^{-1}, R) = \{z \in \mathbb{C}\,; r \le |z| \le R\}$. For any bounded rational function f on X, we have the representation formula*

$$f(A) = \int_0^{2\pi} f(Re^{i\theta})\,\mu(\theta, A)\,d\theta + \int_0^{2\pi} f(re^{i\theta})\,\mu(-\theta, A^{-1})\,d\theta + \int_0^{2\pi} f(e^{i\theta})\,M(\theta, A^*)^{-1}\,d\theta,$$

where

$$\mu(\theta,A) = \frac{1}{4\pi}\big((1+e^{-i\theta}rA)(1-e^{-i\theta}rA)^{-1} + (1+e^{i\theta}rA^*)(1-e^{i\theta}rA^*)^{-1}\big), \quad \textit{and}$$
$$M(\theta,A^*) = \frac{2\pi}{R^2-r^2}(R^2+r^2-(e^{i\theta}A^*)^{-1} - e^{i\theta}A^*).$$

Proof. We get from the Cauchy formula

$$f(A) = \tfrac{1}{2\pi i}\int_{\partial X} f(\sigma)\,((\sigma-A)^{-1}\,d\sigma - (\bar\sigma-A^*)^{-1}\,d\bar\sigma) + \tfrac{1}{2\pi i}\int_{\partial X} f(\sigma)\,(\bar\sigma-A^*)^{-1}\,d\bar\sigma = F_1+F_2.$$

Let us set $\Gamma_\rho = \{\rho\, e^{i\theta};\theta\in[0,2\pi]\}$. The part Γ_R of ∂X is counterclockwised oriented and, with $\sigma = Re^{i\theta}$, we have

$$\begin{aligned}\tfrac{1}{2\pi i}((\sigma-A)^{-1}\,d\sigma - (\bar\sigma-A^*)^{-1}\,d\bar\sigma) &= \tfrac{1}{2\pi}((Re^{i\theta}-A)^{-1}\,Re^{i\theta} + (Re^{-i\theta}-A^*)^{-1}\,Re^{-i\theta})\,d\theta\\ &= \tfrac{1}{2\pi}((1-e^{-i\theta}rA)^{-1} + (1-e^{i\theta}rA^*)^{-1})\,d\theta\\ &= \tfrac{1}{2\pi}\,d\theta + \mu(\theta,A)\,d\theta.\end{aligned}$$

The other component Γ_r is clockwised oriented and, with $\sigma = re^{i\theta}$, we have

$$\begin{aligned}\tfrac{1}{2\pi i}((\sigma-A)^{-1}\,d\sigma - (\bar\sigma-A^*)^{-1}\,d\bar\sigma) &= \tfrac{1}{2\pi}((re^{i\theta}-A)^{-1}\,re^{i\theta} + (re^{-i\theta}-A^*)^{-1}\,re^{-i\theta})\,d\theta\\ &= \tfrac{1}{2\pi}\,d\theta - \mu(-\theta,A^{-1})\,d\theta.\end{aligned}$$

Noticing that $\int_0^{2\pi} f(Re^{i\theta})\,d\theta = \int_0^{2\pi} f(re^{i\theta})\,d\theta$, we obtain that

$$F_1 = \int_0^{2\pi} f(Re^{i\theta})\,\mu(\theta,A)\,d\theta + \int_0^{2\pi} f(re^{i\theta})\,\mu(-\theta,A^{-1})\,d\theta.$$

We consider now the second term F_2. On the component Γ_R we have $\bar\sigma = R^2/\sigma$, and thus

$$\begin{aligned}\tfrac{1}{2\pi i}\int_{\Gamma_R} f(\sigma)\,(\bar\sigma-A^*)^{-1}\,d\bar\sigma &= -\tfrac{1}{2\pi i}\int_{\Gamma_R} f(\sigma)\,(R^2-\sigma A^*)^{-1}\,\frac{R^2}{\sigma}\,d\sigma\\ &= -\tfrac{1}{2\pi i}\int_{\Gamma_1} f(\sigma)\,(R^2-\sigma A^*)^{-1}\,\frac{R^2}{\sigma}\,d\sigma.\end{aligned}$$

Indeed, the last integrand is holomorphic in σ. Hence we can replace the integration path Γ_R by Γ_1 (counterclockwised oriented). We similarly have for the second component

$$\tfrac{1}{2\pi i}\int_{\Gamma_r} f(\sigma)\,(\bar\sigma-A^*)^{-1}\,d\bar\sigma = \tfrac{1}{2\pi i}\int_{\Gamma_1} f(\sigma)\,(r^2-\sigma A^*)^{-1}\,\frac{r^2}{\sigma}\,d\sigma$$

by taking into account the opposite orientation of Γ_r. Therefore

$$\begin{aligned}F_2 &= \tfrac{1}{2\pi i}\int_{\Gamma_1} f(\sigma)\,\Big((r^2-\sigma A^*)^{-1}\,\frac{r^2}{\sigma} - (R^2-\sigma A^*)^{-1}\,\frac{R^2}{\sigma}\Big)\,d\sigma\\ &= \int_0^{2\pi} f(e^{i\theta})\,M(\theta,A^*)^{-1}\,d\theta,\end{aligned}$$

which completes the proof of the lemma. ☐

4 The complete bound in an annulus

We keep the notation from the previous section. The following lemma shows that $\operatorname{Re} M(\theta, A^*)$ is a positive operator.

Lemma 4.1. *Assume that $\|A\| < R$ and $\|A^{-1}\| < R$. Let $r = R^{-1}$. Then we have the lower bound*

$$\operatorname{Re} M(\theta, A^*) \geq N(\theta) := \frac{2\pi}{R^2 - r^2}\left((R^2 + r^2 - R - r) + \frac{R + r + 2}{4}\left(2 - e^{i\theta}U^* - e^{-i\theta}U\right)\right),$$

where U denotes the unitary operator such that $A = UG$, with G self-adjoint positive definite. Also, $N(\theta)$ is a positive invertible operator.

Proof. We have

$$\begin{aligned}
\frac{R^2 - r^2}{2\pi}\operatorname{Re} M(\theta, A^*) &= R^2 + r^2 - \operatorname{Re}((e^{-i\theta}A)^{-1} + e^{i\theta}A^*) \\
&= R^2 + r^2 - \operatorname{Re}\left(e^{i\theta}(G^{-1} + G)U^*\right) \\
&= R^2 + r^2 - \tfrac{R+r+2}{2}\operatorname{Re}\left(e^{i\theta}U^*\right) - \operatorname{Re}\left(e^{i\theta}(G^{-1} + G - \tfrac{R+r+2}{2})U^*\right)
\end{aligned}$$

We note that the assumptions $\|A\| \leq R$ and $\|A^{-1}\| \leq R$ are equivalent to $\|G\| \leq R$ and $\|G^{-1}\| \leq R$. Since G is self-adjoint, this means that $r \leq G \leq R$, and hence

$$\|G^{-1} + G - \tfrac{R+r+2}{2}\| \leq \sup_{r \leq x \leq R} |x^{-1} + x - \tfrac{R+r+2}{2}| = \tfrac{R+r-2}{2}.$$

It follows that

$$\begin{aligned}
\frac{R^2 - r^2}{2\pi}\operatorname{Re} M(\theta, A^*) &\geq R^2 + r^2 - \tfrac{R+r+2}{2}\operatorname{Re}\left(e^{i\theta}U^*\right) - \tfrac{R+r-2}{2} \\
&= R^2 + r^2 - R - r + \tfrac{R+r+2}{2}\operatorname{Re}\left(1 - e^{i\theta}U^*\right),
\end{aligned}$$

which completes the proof of the lemma. ∎

Proof of the upper bound of Theorem 1.3. We can suppose that $\|A\| < R$ and $\|A^{-1}\| < R$. Using the notation of Lemma 3.1, it follows from the condition $\|A\| < R$ that $\mu(\theta, A) \geq 0$ for all $\theta \in \mathbb{R}$. Therefore we have

$$\left\|\int_0^{2\pi} f(Re^{i\theta})\, \mu(\theta, A)\, d\theta\right\| \leq \left\|\int_0^{2\pi} \mu(\theta, A)\, d\theta\right\| \, \|f\|_X = \|f\|_X.$$

Here we have used that $\int_0^{2\pi}\mu(\theta, A)\, d\theta = 1$, which follows from the residue formula. Similarly we have $\mu(-\theta, A^{-1}) \geq 0$ and we get the estimate

$$\left\|\int_0^{2\pi} f(re^{i\theta})\, \mu(-\theta, A^{-1})\, d\theta\right\| \leq \|f\|_X.$$

Using Lemma 3.1 and the positivity of $\operatorname{Re} M(\theta, A^*)$ for all $\theta \in \mathbb{R}$ (Lemma 4.1) we obtain the estimate

$$\|f(A)\| \leq K\, \|f\|_X, \quad \text{with} \quad K = 2 + \left\|\int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1}\, d\theta\right\|.$$

Let $\rho : R(X) \mapsto \mathcal{L}(H)$ be the homomorphism given by $\rho(f) = f(A)$. Therefore the norm of ρ is bounded by K. Furthermore, since we only have used arguments based on positivity of operators, it is easily seen that the complete bounded norm $\|\rho\|_{cb}$ is also bounded by K.

Taking into account the bound of Shields [17], for establishing the upper bound of Theorem 1.3 it suffices now to show that

$$\Big\| \int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1}\, d\theta \Big\| \le \sqrt{\frac{R^2+2R+1}{R^2+R+1}} \le \frac{2}{\sqrt{3}}. \tag{1}$$

Consider the function

$$J(z) := \frac{R^2-r^2}{2\pi} \int_0^{2\pi} \left((R^2+r^2-R-r) + \frac{R+r+2}{4} \left(2 - e^{i\theta} z^{-1} - e^{-i\theta} z\right) \right)^{-1} d\theta.$$

Since U is a unitary operator, it follows from Lemma 4.1 that

$$\Big\| \int_0^{2\pi} (\operatorname{Re} M(\theta, A^*))^{-1}\, d\theta \Big\| \le \Big\| \int_0^{2\pi} (N(\theta))^{-1}\, d\theta \Big\| = \|J(U)\| = \sup \left\{ |J(e^{i\phi})| : e^{i\phi} \in \sigma(U) \right\}.$$

On the other hand, we have

$$\begin{aligned} J(e^{i\varphi}) &= \frac{R^2-r^2}{2\pi} \int_0^{2\pi} \frac{1}{(R^2+r^2-R-r) + \frac{R+r+2}{4}(2-2\cos(\theta-\varphi))}\, d\theta \\ &= \frac{R^2-r^2}{2\pi} \int_{-\infty}^{\infty} \frac{2}{(R^2+r^2-R-r)(1+s^2) + (R+r+2)s^2}\, ds \\ &= \frac{R^2-r^2}{2\pi} \int_{-\infty}^{\infty} \frac{2}{(R^2+r^2-R-r) + (R^2+r^2+2)s^2}\, ds \\ &= \sqrt{\frac{R^2+2R+1}{R^2+R+1}} = \sqrt{\frac{1}{1-\frac{1}{(\sqrt{R}+1/\sqrt{R})^2}}} \le \frac{2}{\sqrt{3}}, \end{aligned}$$

which implies (1). This gives a proof of the upper bound of Theorem 1.3 for $K_{cb}(R)$. □

Proof of the lower bound of Theorem 1.3. For $t \in \mathbb{C}$, let $A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ with inverse $A(t)^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ acting on the Hilbert space $\mathbb{C}^2$. For $t_0 = R - R^{-1}$ we have $\|A(t_0)\| = \|A(t_0)^{-1}\| = R$ (compare with [14, p. 152]). We will make use of the following result from geometric function theory about the infinitesimal Carathéodory metric: it is shown by Simha in [18, Example (5.3)] that

$$\sup \left\{ \frac{|f'(1)|}{\|f\|_X} : f \text{ analytic in } X \text{ and } f(1) = 0 \right\} = \frac{2}{R} \prod_{n=1}^{\infty} \Big(\frac{1-R^{-8n}}{1-R^{4-8n}} \Big)^2,$$

with the supremum being attained for some function f_0 analytic in X, with $\|f_0\|_X = 1$ and $f_0(1) = 0$. Therefore

$$K(R) \ \ge\ \frac{1}{\|f_0\|_X} \|f_0(A(t_0))\| = \left\| \begin{pmatrix} f_0(1) & t_0 f_0'(1) \\ 0 & f_0(1) \end{pmatrix} \right\| = t_0\, |f_0'(1)| = \gamma(R)$$

with

$$\begin{aligned} \gamma(R) \ &:=\ 2(1-R^{-2}) \prod_{n=1}^{\infty} \Big(\frac{1-R^{-8n}}{1-R^{4-8n}} \Big)^2 = \frac{2}{1+R^{-2}} \prod_{n=1}^{\infty} \frac{(R^{4n}-R^{-4n})^2}{(R^{4n}-R^{4-4n})(R^{4n}-R^{-4-4n})} \\ &=\ \frac{2}{1+R^{-2}} \prod_{n=1}^{\infty} \Big(1 - \frac{(R^2-R^{-2})^2}{(R^{4n}-R^{-4n})^2} \Big)^{-1}. \end{aligned}$$

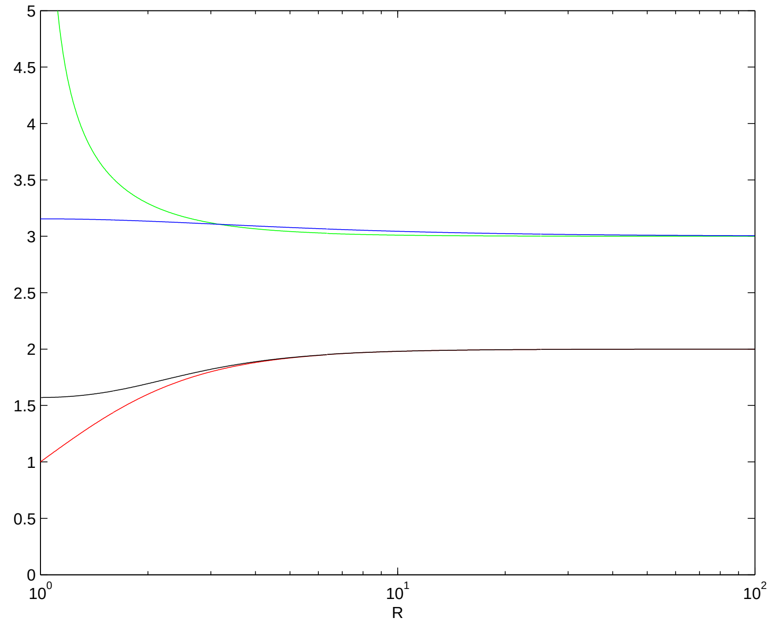

Figure 2: *The two upper bounds and the lower bound for $K(R)$ from Theorem 1.3, and the lower bound $\gamma(R)$ from the proof of Theorem 1.3.*

This yields the estimate

$$K(R) > \frac{2}{1 + R^{-2}}, \tag{2}$$

as claimed in Theorem 1.3. It remains to justify why we are allowed to take for a lower bound of $K(R)$ the function f_0 which is not a rational function. Indeed, by using instead of f_0 partial sums of the Laurent expansion of an extremal function for the infinitesimal Carathéodory metric on the annulus $1/R' < |z| < R'$ for some $R' > R$ we obtain the same conclusion after taking the limit $R' \to R$. □

Remark 4.2. The final estimate (2) of the preceding proof is not very sharp for R close to one (see Figure 2), and $\gamma(R)$ is a sharper but less readable lower bound for $K(R)$. For instance, for $R \to 1$ the lower bound $2/(1 + R^{-2})$ of Theorem 1.3 tends to 1 but

$$\lim_{R\to 1} \gamma(R) = \lim_{R\to 1} \prod_{n=1}^{\infty} \Big(1 - \frac{(R^2 - R^{-2})^2}{(R^{4n} - R^{-4n})^2}\Big)^{-1} = \prod_{n=1}^{\infty} \Big(1 - \frac{1}{4n^2}\Big)^{-1} = \frac{\pi}{2}.$$

In contrast, for our fixed matrix $A(t_0)$, it follows from [9, Theorem 1] and [18] that the function f_0 is extremal within the class of functions analytic in X.

Proof of Corollary 1.4. We use the terminology of Paulsen's book [14]. Let $\rho : R(X) \mapsto \mathcal{L}(H)$ be the homomorphism given by $\rho(f) = f(A)$. Theorem 1.3 implies that the complete bounded norm $\|\rho\|_{cb}$ of ρ is bounded by $2 + 2/\sqrt{3}$. Using a theorem of Paulsen [14, Theorem 9.1], there exists an invertible operator L with $\|L\| \cdot \|L^{-1}\| = \|\rho\|_{cb} \leq 2 + 2/\sqrt{3}$ such that $L^{-1}\rho(\cdot)L$ is a unital contractive homomorphism. Thus X is a spectral set for $L^{-1}AL$. According to a deep result due to Agler [1], X is in fact a complete spectral set for $L^{-1}AL$. Therefore, as a consequence of Arveson's extension theorem (see [14, Corollary 7.8]), $L^{-1}AL$ has a normal dilation with spectrum included in ∂X, as claimed in Corollary 1.4. □

Remark 4.3. The analogue of Agler's theorem is not true for triply connected domains (see [8]).

References

[1] J. Agler, Rational dilation on an annulus, *Ann. of Math.* (2) **121** (1985), 537–563.

[2] C. Badea, M. Crouzeix and B. Delyon, Convex domains and K-spectral sets, *Math. Z.* **252** (2006), 345–365.

[3] B. Beckermann and M. Crouzeix, A lenticular version of a von Neumann inequality, *Arch. Math.*(Basel), **86** (2006), 352–355.

[4] J.B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, 36. American Mathematical Society, Providence, RI, 1991.

[5] M. Crouzeix, B. Delyon, Some estimates for analytic functions of strip or sectorial operators, *Arch. Math.*(Basel), **81** (2003), 553-566.

[6] M. Crouzeix, Numerical range and functional calculus in Hilbert space, *J. Funct. Anal.*, **244** (2007), 668–690.

[7] R.G. Douglas and V.I. Paulsen, Completely bounded maps and hypo-Dirichlet algebras, *Acta Sci. Math.*(Szeged), **50** (1986), 143–157.

[8] M.A. Dritschel and S. McCullough, The failure of rational dilation on a triply connected domain, *J. Amer. Math. Soc.* **18** (2005), 873–918.

[9] S. Fu and B. Russo, Spectral domains in several complex variables, *Rocky Mountain J. Math.* **27** (1997), 1095–1116.

[10] K. A. Lewis, Intersections of K-spectral sets, *J. Operator Theory* **24** (1990), 129–135.

[11] G. Misra, Curvature inequalities and extremal properties of bundle shifts, *J. Operator Theory* **11** (1984), 305–317.

[12] J. von Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.* **4** (1951), 258–281.

[13] V.I. Paulsen, *Toward a theory of K-spectral sets*, in : Surveys of some recent results in operator theory, Vol. I, 221–240, Pitman Res. Notes Math. Ser., 171, Longman Sci. Tech., Harlow, 1988.

[14] V.I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

[15] M. Putinar and S. Sandberg, A skew normal dilation on the numerical range of an operator, *Math. Ann.* **331** (2005), no. 2, 345–357.

[16] F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Books on Advanced Mathematics. Dover Publications, New York, 1990.

[17] A.L. Shields, *Weighted shift operators and analytic function theory*, in : Topics in operator theory, pp. 49–128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.

[18] R.R. Simha, The Carathéodory metric of the annulus, *Proc. Amer. Math. Soc.* **50** (1975), 162-166.

[19] J.G. Stampfli, Surgery on spectral sets, *J. Operator Theory*, **16** (1986), no 2, 235–243.

[20] J.G. Stampfli, Surgery on spectral sets. II. The multiply connected case, *Integral Equations Operator Theory* **13** (1990), 421–432.

[21] J.P. Williams, Minimal spectral sets of compact operators, *Acta Sci. Math.*(Szeged) **28** (1967) 93–106.

Laboratoire Paul Painlevé, UMR CNRS no. 8524,
Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France
Catalin.Badea@math.univ-lille1.fr

Laboratoire Paul Painlevé, UMR CNRS no. 8524,
Université de Lille 1, 59655 Villeneuve d'Ascq Cedex, France
Bernhard.Beckermann@math.univ-lille1.fr

Institut de Recherche Mathématique de Rennes, UMR 6625 au CNRS,
Université de Rennes 1, Campus de Beaulieu, 35042 RENNES Cedex, France
michel.crouzeix@univ-rennes1.fr