On the numerical condition of polynomial bases: Estimates for the Condition Number of Vandermonde, Krylov and Hankel matrices

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Summary

A study is made of the numerical condition of the coordinate map associating to each polynomial its coefficients with respect to a given basis of polynomials. This problem depends on the choice of norms for the sets of polynomials of a given maximal degree, and the corresponding sets of coefficients. In the present work we discuss three different choices of norms for the sets of polynomials, and equip the set of coefficients with a suitable Hölder vector norm.

In the first part we choose for the sets of polynomials the supremum norm with respect to some compact set in the complex plane. Relations to Zolotarov–type and Markov–type extremal problems for polynomials in the complex plane are used to derive approximately tight estimates for various coordinate maps, including the bases of Faber polynomials and Newton polynomials. In particular, we discuss the numerical condition of the basis of monomials on intervals and ellipses, generalizing previous work of Gautschi.

The second part deals with bases of orthogonal polynomials, where the sets of polynomials are equipped with an L_2–norm induced by some other scalar product. Here, equivalently, one has to study the condition number of (modified) moment matrices such as positive definite Toeplitz or Hankel matrices. We propose a lower bound for the condition number in terms of the supports of the underlying measures. Furthermore, asymptotics are given for a particular class of modified moment matrices.

The aim of the third part is to derive approximately tight lower bounds for the condition number of special structured matrices, such like Vandermonde-like, Krylov and Hankel matrices. Here the link to coordinate maps is obtained by taking for the sets of polynomials the supremum norm with respect to some discrete set. In particular, we give lower bounds for the p-condition number of a real Vandermonde matrix of order n growing exponentially in n. These bounds are shown to be attained up to a factor n^2. In addition, we determine explicitly the optimal node configuration which minimizes the 1-condition number.

Krylov matrices consist of colums $B^j \cdot b$, $j = 0, 1, .., n$, where B is a normal matrix of size $m > n$ with eigenvalues being located in some real or complex set G. It is shown that, for estimating the Euclidean condition number, it is sufficient to discuss the case of diagonal B, and in this case a Krylov matrix becomes a rowscaled Vandermonde matrix. We provide an explicit expression for the nth root limit of the optimal lower bound for the condition number of such Krylov matrices. In the case of real G, lower bounds in terms of n, G are given which again are shown to be approximately tight, improving results obtained recently by Tyrtyshnikov. As an application, we discuss the condition number of positive definite Hankel matrices, including for example moment matrices such as the famous Hilbert matrix.

Contents

Chapter 1

Introduction

In this thesis we are concerned with the numerical condition of special structured matrices. Examples of such structured matrices include Vandermonde matrices, Krylov matrices, positive definite Hankel and Toeplitz matrices, and more general moment matrices.

A fundamental operation in numerous branches of mathematics is to solve a linear system of equations $A \cdot x = b$, with A being a nonsingular matrix. Often the given data, A and b, are themselves results of (sometimes extensive) computations, and as such we are faced with the problem of how perturbations δA, δb in the data affect the solution x. A common answer to this problem is to compute the *condition number* $\kappa_p(A) = ||A||_p \cdot ||A^{-1}||_p$ of the matrix of coefficients, where $|| \cdot ||_p$ denotes a Hölder matrix norm, $p \in [1;\infty]$ (for a short summary on vector and matrix norms see Appendix A). In fact, from the formula

$$\kappa_p(A) = \max_{b \neq 0, \delta b \neq 0} \frac{||A^{-1} \cdot (b + \delta b) - A^{-1} \cdot b||_p}{||A^{-1} \cdot b||_p} \cdot \left(\frac{||\delta b||_p}{||b||_p} \right)^{-1}$$

we see that $\kappa_p(A)$ is equal to the maximum magnification of the relative error in the right hand side b (the maximum magnification of absolute errors is measured by $||A^{-1}||_p$). In addition, it is known that $\kappa_p(A)$ serves for measuring the magnification of relative errors in the matrix of coefficients (see [GoVL93, Subsections 2.7.2 and 2.7.4]). Thus, the condition number $\kappa_p(A)$ quantifies the sensitivity of the $A \cdot x = b$ problem. Another characteristic of the condition number is that $1/\kappa_p(A)$ coincides with the relative distance of A to the set of singular matrices. In particular, with $\kappa_p(A)$ we obtain a measure for the linear independence of the columns of A [GoVL93, p.80].

To establish terminology and notation, we call a *Vandermonde matrix* a matrix of the form

$$V_n := \begin{pmatrix} 1 & z_0 & \cdots & z_0^n \\ 1 & z_1 & \cdots & z_1^n \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \cdots & z_n^n \end{pmatrix},$$

where the z_j are distinct real or complex numbers called *nodes*. Systems of linear equations having as a matrix of coefficients V_n (or its transposed) occur naturally in polynomial interpolation. Other applications include the determination of Christoffel numbers for interpolatory quadrature formulas, and the interpolatory approximation of linear functionals (see [BjEf73]). We will also consider the case of *Vandermonde–like* matrices where the successive powers, i.e., the monomials, are replaced by a sequence $(p_k)_{k\geq 0}$ of polynomials, p_k of degree k.

Krylov matrices are obtained by taking as columns the Krylov vectors $A^j \cdot a$, $j = 0, 1, ..., n$. These vectors are basic, for instance, in the context of iterative methods for solving (large) linear systems such as Lanczos type methods (see, e.g., [GoVL93, Chapter 9]). In such cases one is concerned with the nearly linear dependence of Krylov vectors, a problem that requires a knowledge of the numerical condition of such matrices. The third class of matrices included in our work consists of *(modified) moment matrices.* These are Hermitian positive definite matrices which occur in the context of the numerical computation of orthogonal polynomials. In particular, we will be interested in the subclass of positive definite *Hankel* matrices, where we recall that a Hankel matrix has constant entries along all its antidiagonals. Included among such matrices is the *Hilbert* matrix, a famous example of an ill–conditioned matrix.

In this context, let us mention without giving details that there exist several concepts of structured condition numbers for special structured matrices (see, e.g., [GoKo92, HiHi92]).

We denote by $\mathcal{P}$ the space of polynomials with complex coefficients, and by $\mathcal{P}_n$ the subset of polynomials of degree less or equal to n. Clearly, two norms on $\mathcal{P}_n$ are equivalent, and, for given norms $||\cdot||_\alpha$, $||\cdot||_\beta$, it is natural to ask about the size of the equivalence constant

$$\max\{\frac{||P||_\alpha}{||P||_\beta} : P \in \mathcal{P}_n, P \neq 0\}.$$

This question is closely related to the problem of determining the condition number of the special structured matrices mentioned previously. If both norms are induced by (different) scalar products, then the above quantity is shown to coincide with the square root of the norm of the corresponding modified moment matrix. Let us mention a further example: given a sequence of polynomials $(p_k)_k$ as above, we will discuss the numerical condition of the *coordinate map* $\Pi_n : \mathbb{C}^{n+1} \to \mathcal{P}_n$ (see, e.g., [Gau72, Gau79, Rei85]) defined by

$$\Pi_n(a)(z) = \sum_{j=0}^{n} a_j \cdot p_j(z), \quad a = (a_0, .., a_n)^T \in \mathbb{C}^{n+1},$$

where we equip $\mathbb{C}^{n+1}$ with a vector Hölder norm, and $\mathcal{P}_n$ with the maximum norm w.r.t. some compact set G. Again, the induced operator norms of both Π_n and Π_n^{-1} are equivalence constants. They enable us to quantify the change of values of a polynomial in $\mathcal{P}_n$ in G while perturbating its coefficients (with respect to the basis $\{p_0, .., p_n\}$). Here the link to Vandermonde–like matrices is given by taking as G the finite set of nodes. In addition, there are applications to interpolation processes (Newton basis, see, e.g., [FiRe89]) and to the numerical solution of integral and differential equations by collocation or other discretizing methods (basis of Chebyshev polynomials, see, e.g., [ErSt]).

For investigating the above problems, we will stress the connection to extremal problems for polynomials in the complex plane of the following type: under all polynomials in $\mathcal{P}_n$ satisfying a linear constraint, find the one with smallest deviation from zero ρ_n on some compact set G. As a linear constraint we fix the value at some argument $c \notin G$ (the *constrained Chebyshev problem*), fix the leading coefficient (*Chebyshev polynomials* of G), or, more generally, we fix the value of a derivative at some fixed argument (*Markov–type problems*). Here, depending on G, a typical answer is to give the asymptotic of ρ_n (or of its nth root) for n tending to infinity. Also, for particular sets G one may either obtain explicitly ρ_n, or at least give lower and upper bounds for all n being asymptotically tight (e.g., a polynomial in n times γ^n, with the same $\gamma > 1$ for both bounds).

Using the shorthand ρ_n for the condition number of coordinate maps, or for the optimal lower bound $\inf\{\kappa_p(A) : A \in \mathcal{A}_n\}$ for the condition number of a matrix of order $n+1$ having a particular structure as mentioned above, our findings may be classified in the same way. Depending on the given framework, we will describe the nth root behaviour of $(\rho_n)_n$, its asymptotic, or give explicit intervals containing ρ_n for all n. Here the third characterization is of particular practical interest, since it enables us to conclude that matrices of a particular structure are necessarily ill–conditioned, even for not very high orders n.

In what follows we briefly describe the main results of this work, together with a short survey of previously known results.

It is well-known that real Vandermonde matrices are ill-conditioned (see for instance Gautschi's survey on Vandermonde–like matrices [Gau90], where several examples are also given). To give an idea of the size of the condition number, Wilkinson discussed three different configurations of nodes [Wil65, p.372f] for Vandermonde matrices of order 20. For the cases of equidistant nodes on $[0;1]$, on $[-1;1]$, and geometric nodes $z_j = 2^{-j}$, he gave the (rough) lower bounds 2^{42}, 2^{24}, and 2^{171}, respectively. Gautschi and Inglese [GaIn88, Theorem 2.1] later showed that for any configuration of nonnegative nodes $z_0, .., z_n$ it is the case that $\kappa_1(V_n) > 2^n$. Moreover, Tyrtyshnikov [Tyr94a, Theorem 4.1] gives the lower bound $\kappa_2(V_n)) > 2^{n-1}/\sqrt{n+1}$, valid for any configuration of real nodes. This improves a result of Taylor [Tay78, Section 4] who proved that $\kappa_2(V_n)$ for real nodes grows asymptotically at least as 2^n.

In Theorem 5.8 of Section 5.2 we show that, for all $n \geq 0$, $\kappa_p(V_n)$ is at least as large as $(1+\sqrt{2})^n/(4 \cdot (n+1)^{1/p})$ for any configuration of real nodes $z_0, .., z_n$, and $\kappa_p(V_n) \geq (1+\sqrt{2})^{2n}/(2 \cdot (n+1)^{1/p}$ for nodes having one sign. Moreover, both bounds are attained roughly up to a factor $2n^2$. In the case of nodes being located in a real interval $[a;b]$, we point out the relation to the numerical condition of the basis of monomials with respect to the interval $[a;b]$. The latter problem had been studied and partly solved by Gautschi [Gau79]. In Section 2.5 we derive asymptotically tight lower and upper bounds for the condition of the basis of monomials for arbitrary intervals and ellipses.

In the context of real Vandermonde matrices, Gautschi [Gau75b] raised the question of determining a configuration of real (nonnegative) nodes minimizing the p-condition number of Vandermonde matrices. For the $p = 1$ case, some results have been obtained in [Gau75b, GaIn88]. An explicit solution for the ∞-condition number is given in Section 5.2. This section also includes some numerical results for this problem.

To our knowledge, little is known on bounds for the condition number of Krylov matrices, though they are also suspected to be notoriously ill–conditioned (see, e.g., [Wil65, p.374]). Besides Krylov matrices, we also consider more generally *Krylov–like* matrices. These are defined on a column basis by

$$K_n := (p_0(A) \cdot a, p_1(A) \cdot a, ..., p_n(A) \cdot a),$$

where $(p_k)_k$ a sequence of polynomials as above, and where A is assumed to be normal and having eigenvalues lying in some compact set G. In Section 4.2 we show that for norm considerations it is sufficient to discuss the case of diagonal A. In this case K_n may be rewritten as a product of a (in general unknown) diagonal factor, and a Vandermonde–like matrix. Moreover, in order to give lower bounds for the condition number, we may even restrict ourselves to particular diagonal matrices. In Section 4.3 we make some regularity assumptions and are able to give the nth root limit of the optimal lower bound for the condition number of Krylov–like matrices. Our proof makes use of some tools from potential theory.

For the particular case of Hermitian A and ordinary Krylov matrices, we are able to substantially improve the above results. Namely, in Section 5.3 we show that $\kappa_2(K_n) \geq 1.79^n/(n+1)$ for any such Krylov matrix, and this bound is attained roughly up to a factor $n^{3/2}$. We also discuss particular cases such as (scaled) positive definite A.

As an application, we establish, in Section 5.4, that the Euclidean condition number of a positive definite real Hankel matrix of order $n + 1$ has an approximately tight lower bound of $3.21^n/(n + 1)^2$. This improves a recent statement of Tyrtyshnikov [Tyr94a].

In the following we give a brief outline of the organization of the thesis. More detailed summaries are included in the introduction sections of each chapter.

Chapter 2 treats the numerical condition of coordinate maps, with an emphasis on the bases of monomials, orthogonal polynomials, Faber polynomials and Newton polynomials. In particular, we study the connection to extremal problems for polynomials. In Chapter 3 we investigate modified moment matrices. Here, a lower bound for the nth root behaviour is given, and asymptotics are considered for a particular class.

Chapter 4 is devoted to Vandermonde–like and Krylov–like matrices. We make use of several examples to illustrate the important role of the growth of the (weighted) Lebesgue function. The link between these two classes of matrices is discussed, and we provide an explicit formula for the nth root limit of optimal lower bounds for their condition number. Finally, in Chapter 5 we consider real Vandermonde, Krylov and positive definite Hankel matrices, and establish approximately tight lower bounds for their condition numbers.

For the sake of completeness, we have included in Appendix A a brief summary on vector and matrix norms. In Appendix B we present some results from potential theory required for nth root asymptotics, and adapt them to our framework. Finally, in the index we have listed the most important notation and keywords used throughout this work.

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Chapter 2

The maximum norm and coordinate maps

Given a compact set $G \subset \mathbb{C}$, we denote by $\mathcal{C}(G)$ the set of functions being continuous on G, which throughout this chapter will be equipped with the maximum norm

$$||f||_G := \max_{z \in G} |f(z)|, \quad f \in \mathcal{C}(G).$$

Furthermore, for a given sequence of polynomials $(p_n)_n$, p_n of degree n, we introduce the corresponding sequences of coordinate maps $\Pi_n : \mathbb{C}^{n+1} \to \mathcal{P}_n$, $n \geq 0$, being defined by

$$\Pi_n(a)(z) = \sum_{j=0}^{n} a_j \cdot p_j(z), \quad a = (a_0, .., a_n)^T \in \mathbb{C}^{n+1}. \tag{2.1}$$

Obviously, Π_n is bijective, and its inverse is built up by the component maps $\pi_{k,n} : \mathcal{P}_n \to \mathbb{C}$, $0 \leq k \leq n$, where

$$\pi_{k,n}\left(\sum_{j=0}^{n} a_j \cdot p_j\right) = a_k.$$

In the sequel of this chapter, we equip $\mathbb{C}^{n+1}$ with the Hölder infinity vector norm $||\cdot||_\infty$ (see Appendix A), $\mathcal{P}_n$ with the maximum norm $||\cdot||_G$, and denote also by $||\cdot||_G$ the induced operator norms, $\kappa_G(\Pi_n) := ||\Pi_n||_G \cdot ||\Pi_n^{-1}||_G$. Note that, in the particular case of finite $G = \{z_0, .., z_n\}$, the norm of Π_n coincides with the Hölder infinity norm of the *Vandermonde–like matrix* $V = (p_j(z_k))_{k=0,..,n}^{j=0,..,n}$, whereas $||\Pi_n^{-1}||_G$ is equal to the norm of the inverse of V.

For each $a = (a_0, .., a_n)^T \in \mathbb{C}^{n+1}$ and $z \in G$ we have

$$|\Pi_n(a)(z)| \leq ||a||_\infty \cdot \sum_{j=0}^{n} |p_j(z)|,$$

with equality if $a_j = |p_j(z)|/p_j(z)$, $0 \leq j \leq n$. Consequently, we obtain the following explicit expressions for the norm of Π_n and its inverse

$$||\Pi_n||_G = ||K_n||_G, \quad ||\Pi_n^{-1}||_G = \max_{0 \leq k \leq n} ||\pi_{k,n}||_G, \tag{2.2}$$

with the *Kernel function* $K_n := |p_0| + |p_1| + .. + |p_n|$. Due to the recursive structure of the sequence of coordinate maps one immediately verifies

$$||\Pi_0||_G \leq ||\Pi_1||_G \leq ... \leq ||\Pi_n||_G, \quad ||\Pi_0^{-1}||_G \leq ||\Pi_1^{-1}||_G \leq ... \leq ||\Pi_n^{-1}||_G. \tag{2.3}$$

Let us also mention the following simple inequalities

$$||\Pi_n||_{G_1} \leq ||\Pi_n||_{G_2} \text{ and } ||\Pi_n^{-1}||_{G_1} \geq ||\Pi_n^{-1}||_{G_2}, \text{ provided that } G_1 \subset G_2. \tag{2.4}$$

Moreover, there will be approximately equality in (2.4) if G_1 is a discrete set representing the particular characteristics of the continuous set G_2 (such like *Fekete, Leja or Fejer points*, see below). This will enable us to calculate the condition number of particular Vandermonde–like matrices in Chapter 4.

This chapter is organized as follows: The preliminary Section 2.1 is intended to serve as a motivation for the following considerations. We review in Section 2.1.1 Gautschi's results concerning the condition of monomials with respect to particular real intervals. It is shown that the calculation of $||\Pi_n^{-1}||_G$ for monomials requires the solution of Markov–type extremal problems for polynomials, in general a quite difficult task. In contrast, simple estimates for $||\Pi_n^{-1}||_G$ are obtained in Section 2.1.2 for sequences of (bi)orthogonal polynomials. In Section 2.1.3, the influence of scaling of the polynomials on the numerical condition is discussed, where as illustration the case of monomials on real intervals $[-\gamma; \gamma]$ is studied.

Though for instance tight estimates for $\kappa_H(\Pi_n)$ on the unit disk H are easily available in the case of monomials, it seems to be an open problem to give (tight) estimates for arbitrary real intervals H. In Section 2.2 we show that the link between these two quantities is given by a set function $\Delta_n(\cdot, \cdot)$, being related to the constrained Chebyshev problem. From the theory of extremal problems for polynomials in the complex plane it is well–known that $(\Delta_n(\cdot, \cdot)^{1/n})_n$ converges, with limit $\Delta(\cdot, \cdot)$, which may be expressed in terms of Green functions. This allows us to formulate in Section 2.3 necessary conditions for a sequence of polynomials $(p_n)_n$ in order to insure that $(\kappa_G(\Pi_n))_n$ does not grow exponentially. Also, we show that, for given $(p_n)_n$, there is essentially only one set G with this property. The above necessary conditions turn out to be also sufficient for bases of orthogonal polynomials, as well as for bases of Newton–polynomials. As an illustration, we mention the Newton basis of Leja nodes.

In Section 2.4 we restrict ourselves to simply connected compact sets G where explicit intervals for the quantity $\Delta_n(\cdot, G)/\Delta(\cdot, G)^n$ are available. Some results on the size of this interval in dependence of the smoothness of G are reviewed. We also study the numerical condition of the basis of G–Faber polynomials, including for instance (shifted) Chebyshev polynomials.

The numerical condition of the basis of monomials on (scaled) ellipses G is considered in Section 2.5. For this particular case, improved estimates for the quantity $\Delta_n(\cdot, G)/\Delta(\cdot, G)^n$ have been given by several authors. In Section 2.5.1 we provide an explicit expression for ellipses with foci lying on the real axis. In particular, we deduce explicit inequalities for the condition number of the basis of monomials on arbitrary real intervals, improving previous results of Gautschi. These estimates are tight up to the factor $2n+2$. Finally, we discuss in Section 2.5.2 the case of ellipses G being sufficiently far away from the origin. Here we compute explicitly the quantity $\kappa_G(\Pi_n)$, by solving some Markov–type extremal problems which to our knowledge have not been considered before.

2.1 Some preliminary results

2.1.1 The basis of monomials on intervals

As a first example, let us consider the numerical condition of the basis of monomials $p_j(z) = z^j$, where in view of (2.2) it remains to determine $||\pi_{j,n}||_G$, $0 \le j \le n$, with

$$||\pi_{j,n}||_G = \max\{\frac{|P^{(j)}(0)|}{j!} : P \in \mathcal{P}_n, ||P||_G = 1\}.$$

The case of a real interval $G = [\alpha - \beta, \alpha + \beta]$ has been studied by Gautschi [Gau79, Theorem 3.1, Theorem 3.2, p.346] who showed the identities

$$|\alpha| \ge \beta > 0: \qquad ||\Pi_n^{-1}||_{[\alpha-\beta;\alpha+\beta]} = ||\vec{T}_n(\frac{x-\alpha}{\beta})||_\infty, \tag{2.5}$$

$$\beta > 0: \qquad ||\Pi_n^{-1}||_{[-\beta;\beta]} = \max\{||\vec{T}_{n-1}(\frac{x}{\beta})||_\infty, ||\vec{T}_n(\frac{x}{\beta})||_\infty\}. \tag{2.6}$$

Here T_n denotes the classical Chebyshev polynomial being defined by

$$T_n(x) = \cos(n \cdot \arccos(x)), \quad n \ge 0, \quad x \in [-1;1].$$

Moreover, in order to evaluate the Hölder norm of the coefficient vector of a polynomial, we write shorter $\vec{P} = (c_0, .., c_n)^T$ for any $P \in \mathcal{P}_n$, $P(z) = c_0 + c_1 z + .. + c_n z^n$.

Gautschi [Gau79] also proved the asymptotics

$$||\vec{T}_n(\frac{x}{\beta})||_\infty = \frac{1}{\sqrt{2\pi \cdot n}} \cdot \frac{(1+\beta^2)^{3/4}}{\beta} \cdot \left(\frac{1+\sqrt{1+\beta^2}}{\beta}\right)^n \cdot (1 + \mathcal{O}(n^{-1})_{n\to\infty}),$$

$$||\vec{T}_n(\frac{-\beta+x}{\beta})||_\infty = \frac{1}{\sqrt{8\pi \cdot n}} \cdot \frac{(1+2\beta)^{3/4}}{\beta^{1/2}} \cdot \left(\frac{1+\beta+\sqrt{1+2\beta}}{\beta}\right)^n \cdot (1 + \mathcal{O}(n^{-1})_{n\to\infty}).$$

Note that (2.5), (2.6), together with (2.4) implies already that Vandermonde matrices with real nodes necessarily have to be ill–conditioned.

For a proof of (2.5), (2.6) one has to solve the equivalent problem of maximizing the absolute value of the jth derivative at a real argument $c = -\alpha/\beta$ of a polynomial p of degree at most

n, being bounded by 1 on the interval $[-1;1]$. Outside the interval, this quantity is maximal [Riv74, p.93] for the Chebyshev polynomial, namely $|p^{(j)}(c)| \leq |T_n^{(j)}(c)|$ for all $|c| \geq 1$, leading to (2.5). The size of the jth derivative in the interior of the interval (i.e., $|c| < 1$,) is maximal for so-called *Zolotarev polynomials*, for a characterization see Schönhage [Sch71, Section 6.4, p.162ff]. However, the only explicit extremal polynomials are known for the particular case $c = 0$ (as required for (2.6)); by a Theorem due to V.A. Markov [Sch71, Satz 6.12] we have $|p^{(j)}(0)| \leq |T_n^{(j)}(0)|$, if $n - j$ is even, and $|p^{(j)}(0)| \leq |T_{n-1}^{(j)}(0)|$ if $n - j$ is odd. In the case $0 < |c| < 1$, V.A. Markov gave explicitly the norms for $j = 1, n = 2, 3$, and Voronovskaja and Gusev proposed (complicated) techniques for $0 < j < n$ (see [MMR94, p.539f] and the references therein).

In the case $0 < |\alpha| < \beta$, the bounds for $||\Pi_n||_{[\alpha-\beta,\alpha+\beta]}$ obtained by Gautschi [Gau79] are not tight and will be improved in Section 2.5.1, see Corollary 2.19. The Markov–type extremal problems related to $||\Pi_n^{-1}||_G$ may also be explicitly solved for some other sets G (see, e.g., Theorem 2.20 and Lemma 5.5), however, for the general case we will apply different methods in order to obtain asymptotically tight bounds (see Section 2.2).

2.1.2 Biorthogonal functions

In [Gau72], Gautschi discussed the numerical condition of polynomials being orthogonal with respect to a measure with support being a subset of G. His method of proof also applies for families $(p_n)_n$ of biorthogonal functions

THEOREM 2.1 *Let μ be a positive Borel measure with support* $\text{supp}(\mu)$ *being a subset of a compact set $G \subset \mathbb{C}$. Furthermore, suppose that there exist $q_0, .., q_n \in \mathcal{C}(G)$ such that for $j, k = 0, 1, ...n$*

$$\int q_j(z) \cdot p_k(z)\, d\mu(z) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Then for the coordinate map of $(p_n)_n$ we have the estimate

$$||\Pi_n^{-1}||_G \leq \max_{0 \leq j \leq n} \int |q_j(z)|\, d\mu(z).$$

Proof: For each $P = \sum_{j=0}^n a_j \cdot p_j$ there holds

$$|a_j| = \left| \int q_j(z) P(z)\, d\mu(z) \right| \leq \int |q_j(z)| \cdot |P(z)|\, d\mu(z) \leq \int |q_j(z)| \cdot ||P||_G\, d\mu(z),$$

and therefore $||\pi_{j,n}|| \leq \int |q_j(z)|\, d\mu(z)$. □

One also verifies that there holds equality in Theorem 2.1 if the set G consists of $n + 1$ elements. Let us study some simple applications of Theorem 2.1

EXAMPLE 2.2 *If $p_0, .., p_n$ are orthogonal polynomials with respect to the (Hermitian) scalar product induced by μ (see Definition 3.1), then we may take*

$$q_j(z) = \frac{1}{\nu_j}\overline{p_j(z)}, \quad \nu_j = \int |p_j(z)|^2 \, d\mu(z).$$

Moreover, by the Cauchy–Schwarz inequality

$$\int |q_j(z)| d\mu(z) \leq \sqrt{\mu_0 \cdot \int |q_j(z)|^2 \, d\mu(z)} = \sqrt{\frac{\mu_0}{\nu_j}}, \quad \mu_0 = \int d\mu(z),$$

which together with (2.2) leads to the estimate [Gau72]

$$||\Pi_n^{-1}||_G \leq \max_{0 \leq k \leq n} \left(\sqrt{\frac{\mu_0}{\nu_k}} \right), \quad ||\Pi_n||_G = || \sum_{j=0}^{n} |p_j| \, ||_G.$$

Gautschi mentioned that, among all possible normalizations, the upper bound for $\kappa_G(\Pi_n)$ becomes minimal for orthonormal polynomials, i.e., $\nu_j = 1$, $j \geq 0$. He also gave the following two examples: for Chebyshev polynomials $p_n = T_n$ we have $\kappa_{[-1;1]}(\Pi_n) \leq \sqrt{2} \cdot (n+1)$, whereas for Legendre polynomials there holds $\kappa_{[-1;1]}(\Pi_n) \leq \sqrt{2n+1} \cdot (n+1)$. □

EXAMPLE 2.3 *The monomials $p_j(z) = z^j$ are orthonormal with respect to the scalar product*

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \cdot \overline{g(e^{it})} \, dt,$$

i.e., with respect to a measure with support being the unit circle $\partial \mathbb{D}$, the boundary of the closed unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Let H be some compact set satisfying $\partial \mathbb{D} \subset H$. Then $||\Pi_n^{-1}||_H \geq ||\Pi_n^{-1} 1||_\infty = 1$ by construction, and from Example 2.2 we may conclude that $||\Pi_n^{-1}||_H = 1$ for all $n \geq 0$. Furthermore, we get $||\Pi_n||_H = n + 1$ for all $n \geq 0$ provided that H is also a subset of $\mathbb{D}$. □

2.1.3 Scaling of functions

Let us discuss the problem of reducing the condition number $\kappa_G(\Pi_n)$ by scaling the functions $p_0, .., p_n$ with positive scalars $d_0, .., d_n$. In view of (2.2), we obtain for the coordinate map Π_n^{scal} corresponding to the sequence $p_n^{scal} = p_n / d_n$, $n \geq 0$

$$\kappa_G(\Pi_n^{scal}) = \left(\max_{0 \leq j \leq n} d_j \cdot ||\pi_{j,n}||_G \right) \cdot || \sum_{j=0}^{n} \frac{|p_j|}{d_j} ||_G.$$

By applying the Hölder inequality (A.1) one easily verifies that the minimum over all possible choices of the scalars d_j is obtained for $p_j^{opt}(z) = ||\pi_{j,n}||_G \cdot p_j$, $0 \leq j \leq n$, with

$$\kappa_G(\Pi_n^{opt}) = ||\Pi_n^{opt}||_G = || \sum_{j=0}^{n} ||\pi_{j,n}||_G \cdot |p_j| \, ||_G. \tag{2.7}$$

However, this scaling has the important drawback that one might have to rescale the whole family of functions if one wants to add one additional function. Hence, following Reichel [Rei85, Eqn.(2.6)] it seems to be more convenient to consider the scaling $p_j^{dyn}(z) = p_j/||p_j||_G$, $j \geq 0$, referred to as the *dynamical scaling*; here

$$\kappa_G(\Pi_n^{dyn}) \leq (n+1) \cdot ||[\Pi_n^{dyn}]^{-1}||_G = (n+1) \cdot \max_{0 \leq j \leq n} ||\pi_{j,n}||_G \cdot ||p_j||_G. \tag{2.8}$$

Comparing (2.8) and (2.7), one easily verifies that

$$\kappa_G(\Pi_n^{opt}) \leq \kappa_G(\Pi_n^{dyn}) \leq (n+1) \cdot \kappa_G(\Pi_n^{opt}), \tag{2.9}$$

hence the dynamical scaling is optimal up to a factor $(n+1)$. Obviously, applying the dynamic scaling to the family of monomials is equivalent to a rescaling of the set of arguments G: we replace G by $\frac{1}{\gamma} \cdot G = \{z : z \cdot \gamma \in G\}$, where $\gamma = \max_{z \in G} |z|$.

EXAMPLE 2.4 *Let us consider the basis of scaled monomials on the interval $[-\beta; \beta]$, $\beta > 0$. By the Markov Theorem mentioned in Section 2.1.1 we have for the corresponding component maps for $0 \leq k \leq n$*

$$\beta^k \cdot ||\pi_{k,n}||_{[-\beta;\beta]} = \begin{cases} |T_n^{(k)}(0)|/k! & \text{if } n-k \text{ even,} \\ |T_{n-1}^{(k)}(0)|/k! & \text{if } n-k \text{ odd,} \end{cases}$$

and for $n \geq 1$ we obtain from (2.7)

$$\kappa_{[-\beta;\beta]}(\Pi_n^{scal}) \geq \kappa_{[-\beta;\beta]}(\Pi_n^{opt}) = \sum_{k=0}^{n} \frac{|T_n^{(k)}(0)| + |T_{n-1}^{(k)}(0)|}{k!}$$
$$= |T_n(i)| + |T_{n-1}(i)| = \frac{1}{\sqrt{2}} \cdot \left((\sqrt{2}+1)^n - (1-\sqrt{2})^n\right).$$

□

2.2 Connection to the constrained Chebyshev problem

In Section 2.1 we have seen that, for monomials, the quantity $\kappa_G(\Pi_n)$ is exponentially increasing in the case of a real interval G, but only polynomially if $G = \mathbb{D}$. The exact dependence may be described with help of

DEFINITION 2.5 *Let G, H be compact subsets of the complex plane, $n \geq 0$, such that G contains at least $n+1$ elements. Then the* constrained Chebyshev problem *consists in determining the quantity*

$$\Delta_n(H, G) := \max\{\frac{||P||_H}{||P||_G} : P \in \mathcal{P}_n\}.$$

We will also write shorter $\Delta_n(z, G) := \Delta_n(\{z\}, G)$. □

As an illustration, let us mention the following result of Bernstein

$$|z| = R \geq 1 \quad \Longrightarrow \quad \Delta_n(z, \mathbb{D}) = R^n,$$

implying

$$\Delta_n(G, \mathbb{D}) = \max\{\gamma^n, 1\}, \quad \gamma := \max_{z \in G} |z|. \tag{2.10}$$

By definition there holds $\Delta_{n+1}(H, G) \geq \Delta_n(H, G) \geq \Delta_0(H, G) = 1$. The nth root asymptotic of $\Delta_n(H, G)$ is discussed in Theorem B.22 of Appendix B, and will be basic for the considerations of Section 2.3.

For $c \notin G$, $n \geq 1$ we have

$$\frac{1}{\Delta_n(c, G)} = \min_{Q \in \mathcal{P}_{n-1}} \max_{z \in G} |1 - (z - c) \cdot Q(z)|,$$

i.e., we look for the element of best approximation of 1 out of $span\{(z-c), (z-c)^2, .., (z-c)^n\}$, satisfying the Haar condition on G. Thus [Sch71, Satz 6.3, p.155], up to normalisation, there always exists a unique 'extremal' polynomial such that the maximum in the definition of $\Delta_n(c, G)$ is attained. However, these polynomials are explicitly known only for special cases, e.g., $\Delta_n(c, [-1; 1]) = |T_n(c)|$ for $c \in \mathbb{R} \setminus [-1; 1]$.

Given a compact set G, we denote by $\mathcal{D}_\infty(G)$ the unbounded open connected component of $\mathbb{C} \setminus G$, and call $\partial\mathcal{D}_\infty(G)$ the outer boundary of G. With help of the maximum modulus principle for analytic functions one easily verifies that $||P||_G = ||P||_{\partial\mathcal{D}_\infty(G)}$ for each polynomial P. Consequently, the condition number and the norm of a coordinate map will not depend on G but only on its outer boundary. Moreover

$$\begin{aligned} \Delta_n(H, G) &= \Delta_n(H, \partial\mathcal{D}_\infty(G)) = \Delta_n(\partial\mathcal{D}_\infty(H), G), \\ \Delta_n(H, G) &= \Delta_n(H \cap \mathcal{D}_\infty(G), G), \end{aligned} \tag{2.11}$$

where for the second formula $\Delta_n(\emptyset, G) = 1$ (in other words, $\Delta_n(H, G) = 1$ if $\partial\mathcal{D}_\infty(G)$ 'surrounds' the set H).

THEOREM 2.6 *Let G, H be compact sets. We have for each $n \geq 0$*

$$\begin{aligned} \frac{\Delta_n(H, G)}{||\Pi_n||_H} &\leq ||\Pi_n^{-1}||_G \leq \Delta_n(H, G) \cdot ||\Pi_n^{-1}||_H, \\ \frac{\Delta_n(G, H)}{||\Pi_n^{-1}||_H} &\leq ||\Pi_n||_G \leq \Delta_n(G, H) \cdot ||\Pi_n||_H. \end{aligned}$$

Proof: The assertion follows from simple norm manipulations by applying the norm equivalencies

$$\frac{1}{\Delta_n(G, H)} \leq \frac{||P||_H}{||P||_G} \leq \Delta_n(H, G), \quad P \in \mathcal{P}_n,$$

e.g.,

$$||\Pi_n^{-1}||_G = \max_{P \in \mathcal{P}_n} \frac{||\Pi_n^{-1} P||_\infty}{||P||_G} \geq \max_{P \in \mathcal{P}_n} \frac{||P||_H}{||\Pi_n||_H \cdot ||P||_G} \geq \frac{\Delta_n(H, G)}{||\Pi_n||_H}.$$

□

Note that the ratio of the upper and lower bounds of Theorem 2.6 is given by $\kappa_H(\Pi_n)$. Consequently, we obtain 'tight' bounds for $\kappa_G(\Pi_n)$ iff $\kappa_H(\Pi_n)$ is 'small'. For instance, combining Example 2.3 (with $H = \partial\mathbb{D}$), equation (2.10) and Theorem 2.6 we get

COROLLARY 2.7 *For the coordinate map of the basis of monomials there holds*

$$||\Pi_n||_G = \sum_{j=0}^{n} \gamma^j, \qquad \frac{||\Pi_n||_G}{\max\{1,\gamma\}^n} \in [1; n+1],$$

where $\gamma := \max_{z\in G} |z|$, *and*

$$\frac{||\Pi_n^{-1}||_G}{\Delta_n(\mathbb{D}, G)} \in [\frac{1}{n+1}; 1].$$

Moreover, if $\partial\mathbb{D} \subset G$, *then* $||\Pi_n^{-1}||_G = 1$. □

In Section 2.5 we will give bounds for the quantity $\Delta_n(\mathbb{D}, G)$ for G being an ellipse, which enables us to derive explicit estimates for the condition number of the basis of monomials. The case of discrete G and its relation to the Lebesgue function will be studied in Section 4.1.

2.3 The nth root behavour for coordinate maps

For the following considerations we will require the concept of the Green function, where we follow [NiSo88, Section 5, pp.188-191]. Recall that a domain D with compact boundary ∂D is called regular if the Dirichlet problem has a solution for each function being continuous on ∂D.

DEFINITION 2.8 *Let* $G \subset \mathbb{C}$ *be compact, with* $\mathcal{D}_\infty(G)$ *being regular. Then there exists a unique real function* g_G *called* Green function *(with singularity at infinity) with the properties*

- (*a*) g_G *is continuous on the closure of* $\mathcal{D}_\infty(G)$,
- (*b*) g_G *vanishes on* $\partial\mathcal{D}_\infty(G)$,
- (*c*) g_G *is harmonic and positive on* $\mathcal{D}_\infty(G)$
- (*d*) $g(z) := g_G(z) - \log|z|$ *is bounded around infinity.*

The capacity $\mathrm{cap}(G)$ *of* G *is obtained by the formula* $\lim_{z\to\infty} g_G(z) - \log|z| = -\log \mathrm{cap}(G)$. *Furthermore, we define for* $r \geq 1$ *the (compact) level set*

$$G_r := \mathbb{C} \setminus \{z \in \mathcal{D}_\infty(G) : g_G(z) > \log(r)\}.$$

In addition, for any compact set H *let*

$$\Delta(H, G) = \min\{r \geq 1 : H \subset G_r\},$$

where we write shorter $\Delta(z, G) := \Delta(\{z\}, G)$. □

In the case of an arbitrary compact set G with positive capacity, we may still define a unique Green function $g_G : \mathcal{D}_\infty(G) \to \mathbb{R}$ as the limit of Green functions of suitable compact sets (see [NiSo88, p.191]). Here properties (c) and (d) remain valid, moreover, up to a set of capacity zero (the set of irregular points, see [Tsu59, Theorem III.38, p.83]), property (b) remains valid for the limiting values of g_G on the boundary of G. Also, in this more general context, the formulas of Definition 2.8 for $cap(G)$, G_r, and $\Delta(H,G)$ may be applied, and G_r remains compact.

Note that, by definition, the quantities g_G, $cap(G)$, G_r, and $\Delta(H,G)$ only depend on the shape of the outer boundary of G, they remain invariant if we fill any 'holes' of G. Also, $r = \Delta(H,G)$ is the parameter of the smallest levelset G_r such that its outer boundary 'surrounds' the outer boundary of H. In particular, since $G_1 = \mathbb{C} \setminus \mathcal{D}_\infty(G)$, we obtain $\Delta(H,G) = 1$ iff $H \cap \mathcal{D}_\infty(G) = \emptyset$, and otherwise $\Delta(H,G) = \sup\{\exp(g_G(z)) : z \in H \cap \mathcal{D}_\infty(G)\}$.

In Theorem B.22 of Appendix B, we have discussed connections between the quantity $\Delta_n(H,G)$ introduced in Definitions 2.5 and the quantity $\Delta(H,G)$ of Definition 2.8. In particular, it is shown in (B.16), (B.17) that for any compact sets H, G, $cap(G) > 0$ there holds

$$\Delta_n(H,G)^{1/n} \leq \Delta(H,G), \quad \lim_{n\to\infty} \Delta_n(H,G)^{1/n} = \Delta(H,G),$$

the inequality obtained by reformulating the classical Bernstein–Walsh Lemma. This enables us to show the following

THEOREM 2.9 *Let $G, H \subset \mathbb{C}$ be compact sets with positive capacity. Furthermore let $(p_n)_n$ be a sequence of polynomials, p_n of degree n with leading coefficient a_n. Suppose that for the corresponding sequence of coordinate maps there holds*

$$\lim_{n\to\infty} \kappa_G(\Pi_n)^{1/n} = 1.$$

Then necessarily

$$\lim_{n\to\infty} ||p_n||_G^{1/n} = 1, \quad \lim_{n\to\infty} |a_n|^{-1/n} = \mathrm{cap}(G). \tag{2.12}$$

Moreover,

$$\lim_{n\to\infty} ||\Pi_n||_H^{1/n} = \Delta(H,G) \geq 1, \quad \lim_{n\to\infty} ||\Pi_n^{-1}||_H^{1/n} = \Delta(G,H) \geq 1,$$

and $\lim_{n\to\infty} \kappa_H(\Pi_n)^{1/n} = 1$ *if and only if* $\mathcal{D}_\infty(G) = \mathcal{D}_\infty(H)$.

Proof: We first notice that by (2.3)

$$\lim_{n\to\infty} ||\Pi_n||_G^{1/n} = 1, \quad \lim_{n\to\infty} ||\Pi_n^{-1}||_G^{1/n} = 1.$$

In particular, from the explicit formula for $||\Pi_n||_G$ we obtain

$$\limsup_{n\to\infty} ||p_n||_G^{1/n} \leq 1.$$

Moreover, denoting by $LC(P)$ the leading coefficient of a polynomial, $LC(p_n) = a_n$, there holds

$$||\pi_{n,n}||_G = \max_{P\in\mathcal{P}, deg\ P=n} \frac{|\pi_{n,n}(P)|}{||P||_G} = \frac{1}{|a_n| \cdot m_n^*}, \quad m_n^* := \min_{P\in\mathcal{P}, deg\ P=n} \frac{||P||_G}{|LC(P)|}.$$

It is well–known [NiSo88, Proposition 5.4] that the sequence of Chebyshev coefficients $(m_n^*)^{1/n}$ converges, with limit *cap*(G). Using the fact that $|a_n| \cdot m_n^* \le ||p_n||_G$, we get

$$1 = \lim_{n\to\infty} ||\Pi_n^{-1}||_G^{1/n} \ge \limsup_{n\to\infty} ||\pi_{n,n}||_G^{1/n} = \limsup_{n\to\infty} \left(\frac{1}{|a_n| \cdot m_n^*}\right)^{1/n}$$
$$\ge \limsup_{n\to\infty} \frac{1}{||p_n||_G^{1/n}} \ge \liminf_{n\to\infty} \frac{1}{||p_n||_G^{1/n}} \ge 1.$$

This yields the desired nth root asymptotic for $||p_n||_G$, and that for $|a_n|$ is obtained in the same way.

The rest of the assertion now follows from Theorem 2.6 and (B.17), where we notice that $\Delta(G,H) \cdot \Delta(H,G) = 1$ iff $\mathcal{D}_\infty(G) = \mathcal{D}_\infty(H)$. □

Consequently, for a given sequence $(p_n)_n$ of polynomials, there is essentially at most one set G where $\kappa_G(\Pi_n)$ is not exponentially increasing. Let us also mention that, if beside (2.12) also p_n, $n \ge 0$ is supposed to have no zero in $\mathcal{D}_\infty(G)$, then the sequence $(\log|p_n|^{1/n})_n$ necessarily converges to the Green function g_G locally uniformly in $\mathcal{D}_\infty(G)$ (see [NiSo88, Proposition 5.3]). A connection between the kernel function and the Green function g_G will be discussed in Lemma 4.14.

Example 2.10 *Let $(p_n)_n$ be a sequence of polynomials being orthonormal with respect to some measure μ with $S := \mathrm{supp}(\mu)$ being compact. Then*

$$1 = \nu_n := \int |p_n(z)|^2 \, d\mu(z) \le \mu(\mathbb{C}) \cdot ||p_n||_S^2, \quad n \ge 0,$$

and thus

$$\lim_{n\to\infty} ||p_n||_S^{1/n} \ge 1. \tag{2.13}$$

From Example 2.2 we know that $\kappa_S(\Pi_n)$ does not increase exponentially if and only if there holds equality in (2.13). Following Saff [Saf90, Definition 3.1], in such a case the measure μ is said to be completely regular *(see also Definition 3.7). In contrast, the necessary condition*

$$\lim_{n\to\infty} |a_n|^{-1/n} = \mathrm{cap}(S)$$

of Theorem 2.9 was used by Stahl & Totik [StTo92] in order to characterize so–called regular *measures. Notice that a regular measure μ with $\mathcal{D}_\infty(\mathrm{supp}(\mu))$ being regular also is shown to be completely regular (see [StTo92, Theorem 3.2.3]).* □

In view of Theorem 2.9, it remains the question whether, for any compact set G with positive capacity, there exist a sequence of polynomials where $\kappa_G(\Pi_n)$ is not exponentially increasing. For the case of regular $\mathcal{D}_\infty(G)$, a constructive answer has been given by Reichel [Rei90] who considered Newton interpolation at Leja points. Adapting the proof of [Rei90, Theorem 2.4], we may even show that, for a suitable scaled Newton basis, the necessary condition (2.9) already is sufficient. Given a sequence $(z_n)_{n\ge 0}$ of not necessarily distinct elements of G, the Newton polynomials are defined by $\omega_0(z) = 1$, and $\omega_n(z) := (z - z_0) \cdot (z - z_1) \cdot .. \cdot (z - z_{n-1})$ for $n \ge 1$.

THEOREM 2.11 *Let $G \subset \mathbb{C}$ be compact, with $\mathcal{D}_\infty(G)$ being regular. Furthermore, let $z_0, z_1, .. \in G$ such that condition (2.12) holds for the scaled Newton polynomials $p_n = a_n \cdot \omega_n$, $n \geq 0$, with suitable scalars a_n (e.g., $a_n = 1/||\omega_n||_G$), in particular*

$$\lim_{n\to\infty} ||\omega_n||_G^{1/n} = \mathrm{cap}(G). \tag{2.14}$$

Then for the coordinate map corresponding to $(p_n)_n$ we have

$$\lim_{n\to\infty} ||\Pi_n||_G^{1/n} = 1, \quad \lim_{n\to\infty} ||\Pi_n^{-1}||_G^{1/n} = 1.$$

Proof: First from (2.2) together with (2.12) it follows that

$$\lim_{n\to\infty} ||\Pi_n||_G^{1/n} = 1.$$

Also, $||\Pi_n^{-1}||_G \geq ||\Pi_0^{-1}||_G$, and therefore

$$\liminf_{n\to\infty} ||\Pi_n^{-1}||_G^{1/n} \geq 1.$$

For an upper bound, let us first discuss $||\Pi_n^{-1}||_{G_r}$ for a fixed $r > 1$. The component map $\pi_{k,n}$ corresponding to the scaled Newton basis is a scaled divided difference; we will apply its well–known integral representation

$$\pi_{k,n}(P) = \frac{1}{a_k} \cdot [z_0, .., z_k]P = \frac{1}{2 \cdot \pi \cdot i \cdot a_k} \cdot \int_{\partial G_r} \frac{P(z)}{\omega_{k+1}(z)}\, dz,$$

valid for $0 \leq k \leq n$ and $P \in \mathcal{P}$. Consequently,

$$||\pi_{k,n}||_{G_r} = \max_{P\in\mathcal{P}_n} \frac{|\pi_{k,n}(P)|}{||P||_{\partial G_r}} \leq \frac{1}{|a_k|} \cdot ||\frac{1}{\omega_{k+1}}||_{\partial G_r} \cdot \frac{1}{2\pi} \cdot \int_{\partial G_r} |dz|.$$

Notice that the regularity of $\mathcal{D}_\infty(G)$ implies that ∂G_r is a compact subset of $\mathcal{D}_\infty(G)$. By [NiSo88, Proposition 5.3], from the relation (2.14) it follows that

$$\lim_{n\to\infty} |\omega_n(z)|^{1/n} = \mathit{cap}(G) \cdot e^{g_G(z)} = \mathit{cap}(G) \cdot r$$

uniformly for $z \in \partial G_r$. Let now $\epsilon > 0$ be determined by $(1-\epsilon)^2 \cdot r = 1$. Then there exist an $K = K(r)$ such that for all $k \geq K$ and for all $z \in \partial G_r$

$$|\omega_{k+1}(z)|^{1/k} \geq (1-\epsilon) \cdot \mathit{cap}(G) \cdot r, \quad |a_k|^{1/k} \cdot \mathit{cap}(G) \geq (1-\epsilon)^{-1},$$

and therefore

$$\frac{1}{|a_k|} \cdot ||\frac{1}{\omega_{k+1}}||_{\partial G_r} \leq 1, \quad k \geq K(r).$$

Consequently, for each $r > 1$, we may find a constant $C = C(r)$ such that for all $n \geq 0$ there holds $||\Pi_n^{-1}||_{G_r} \leq C(r)$. Applying Theorem 2.6 with $H = G_r$ yields

$$\limsup_{n\to\infty} ||\Pi_n^{-1}||_G^{1/n} \leq \left(\limsup_{n\to\infty} \Delta_n(G_r, G)^{1/n}\right) \cdot \left(\limsup_{n\to\infty} ||\Pi_n^{-1}||_{G_r}^{1/n}\right) \leq r,$$

where we have applied (B.17) together with $\Delta(G_r, G) = r$. Since $r > 1$ may be chosen arbitrarily close to 1, the assertion of the Theorem follows. □

Notice that condition (2.14) is also sufficient for the following assertion: for each f being analytic in $\mathbb{C}\backslash\mathcal{D}_\infty(G)$, the sequence $(Q_n)_n$ of polynomials converges to f uniformly in $\mathbb{C}\backslash\mathcal{D}_\infty(G)$, where Q_n of degree n interpolates f at the nodes $z_0, .., z_n$.

Reichel [Rei90, Lemma 2.1] also mentioned the following result of Leja: let us define a not necessarily unique sequence $(z_n)_n$ of elements of G (called *Leja nodes*) by

$$|z_0| = \max_{z\in G} |z|, \quad \text{and for n=1,2,...:} \quad |\omega_n(z_n)| = ||\omega_n||_G. \tag{2.15}$$

Then condition (2.14) holds, i.e., the condition number of the corresponding coordinate map is not exponentially increasing.

2.4 Simply connected G and Faber polynomials

In order to exploit Theorem 2.6 for the numerical condition of polynomial bases, we require information on $\Delta_n(H, G)$ which is available if we impose some restrictions on G, such as $\mathcal{D}_\infty(G)\cup\{\infty\}$ being simply connected. Due to (2.11) we may assume without loss of generality that there are no holes in G, i.e., $G = \mathbb{C} \setminus \mathcal{D}_\infty(G)$. Hence the sets G which we wish to consider in this section are simply connected compact subsets of $\mathbb{C}$ whose boundary ∂G is a rectifiable Jordan curve. We will shorter denote its complement by $G^c = \mathcal{D}_\infty(G)$, and by $\mathbb{D} := \{w \in \mathbb{C} : |w| \leq 1\}$ the closed unit disk.

2.4.1 The Riemann map

Let $w = \Phi_G(z)$ be the function that maps the exterior G^c conformally and univalently on $\mathbb{D}^c$, with $\Phi_G(\infty) = \infty$, and $\Phi'_G(\infty) > 0$. This function is called the *Riemann map* of G. By the Carathéodory Theorem, Φ_G can be continued continuously to a bijective map from $G^c \cup \partial G$ to $\mathbb{D}^c \cup \partial\mathbb{D}$. Then the Green function of G (see Definition 2.8) is given by $g_G(z) = \log|\Phi_G(z)|$, $z \in G^c \cup \partial G$. In particular, $cap(G) = 1/\Phi'_G(\infty)$, and the level sets G_r are given by $G_r^c = \{z \in G^c : |\Phi_G(z)| > r\}$, $r \geq 1$. We also denote the inverse Riemann map by $z = \Psi_G(w)$.

EXAMPLE 2.12 *The Riemann map for the unit disc trivially is given by* $\Phi_{\mathbb{D}}(z) = z$, $\text{cap}(\mathbb{D}) = 1$. *For an arbitrary disk we may apply a linear transformation as follows: writing* $\alpha + \beta \cdot G = \{\alpha + \beta \cdot z : z \in G\}$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, *we have*

$$\Phi_{(\alpha+\beta\cdot G)}(z) = \frac{\beta}{|\beta|} \cdot \Phi_G(\frac{z-\alpha}{\beta}).$$

Therefore we observe the following invariance under a linear transformation of the complex plane

$$\mathrm{cap}(\alpha+\beta\cdot G)=|\beta|\cdot \mathrm{cap}(G), \quad \Delta(H,\alpha+\beta\cdot G)=\Delta(-\frac{\alpha}{\beta}+\frac{1}{\beta}\cdot H,G), \tag{2.16}$$

the latter relation being obviously valid also for the quantity $\Delta_n(\cdot,\cdot)$. □

EXAMPLE 2.13 *For* $\rho\in[1,+\infty)$, *we define the ellipse*

$$\mathcal{E}_\rho := \{J(\rho\cdot w) : |w|\le 1\}, \quad J(s):=\frac{1}{2}\cdot(s+\frac{1}{s}),$$

where J *is called the* Joukowski map. *Note that, for* $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, *the set* $\alpha+\beta\cdot\mathcal{E}_\rho$ *is an ellipse with foci at* $\alpha-\beta,\alpha+\beta$, *with the sum of the semiaxes given by* $|\beta|\cdot\rho$. *In particular, we describe segments* $\alpha+\beta\cdot\mathcal{E}_1=\{\alpha+\beta\cdot t : -1\le t\le 1\}$, *i.e., for real* α,β *we obtain the interval* $[\alpha-\beta,\alpha+\beta]$. *Moreover,* $\frac{2}{\rho}\cdot\mathcal{E}_\rho$ *becomes for* $\rho\to\infty$ *the unit disk* $\mathbb{D}$.

The reciprocal Joukowski map is defined by

$$J^{-1}(z)=z\cdot(1+\sqrt{1-1/z^2}),$$

where the branch of the square root is chosen such that $\sqrt{1}=1$. *Then for* $G=\mathcal{E}_\rho$ *there holds*

$$\Psi_G(w)=J(\rho\cdot w), \quad \Phi_G(z)=\frac{1}{\rho}\cdot J^{-1}(z), \quad \mathrm{cap}(G)=\frac{\rho}{2}, \quad G_r=\mathcal{E}_{r\cdot\rho},$$

in particular $\mathrm{cap}([-1;1])=1/2$. □

2.4.2 Estimates for the constrained Chebyshev problem

Let $H\subset\mathbb{C}$ be compact, $H\not\subset G$. Inequality (B.16) yields tight upper bounds for the quantity $\Delta_n(H,G)$, $n\ge 0$. In order to derive lower bounds, we introduce the *Faber polynomial* $F_n=F_{G,n}$ as the polynomial part of the Laurent expansion of Φ_G^n at infinity, i.e., $F_{G,n}$ is of precise degree n with leading coefficient $cap(G)^{-n}$, $n\ge 0$, and in particular $F_{G,0}=1$. These polynomials have been introduced in the Thesis of Faber, and are very useful, e.g., for finding a best polynomial approximation of functions continuous on G, and analytic in the interior of G (for further properties see, e.g., [Sue71], [SmLe68]). By the Cauchy integral formula we obtain immediately for each z lying in the interior of G and for each $r\ge 1$

$$F_{G,n}(z)=\frac{1}{2\pi i}\int_{\partial G_r}\frac{\Phi_G(\zeta)^n}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{|w|=r} w^n\cdot\frac{\Psi_G'(w)}{\Psi_G(w)-z}\,dw,$$

leading to the following two alternative definitions of $F_{G,n}$: the Laurent expansion of $F_{G,n}(\Psi_G(w))-w^n$ contains only negative powers of w, and we have the generating function

$$\frac{\Psi_G'(w)}{\Psi_G(w)-z}=\sum_{n=0}^{\infty} w^{-n-1}\cdot F_{G,n}(z), \quad z\in G, \quad |w|>1.$$

As an example, there holds for $n \geq 0$

$$F_{\mathbb{D},n}(z) = z^n, \quad F_{\alpha+\beta\cdot G,n}(z) = \Big(\frac{\beta}{|\beta|}\Big)^n \cdot F_{G,n}(\frac{z-\alpha}{\beta}).$$

Moreover, with help of the Joukowski map we may represent the classical Chebyshev polynomials in the form

$$T_n(J(w)) = J(w^n), \quad n \geq 0, \tag{2.17}$$

implying that for $\rho \geq 1$

$$F_{\mathcal{E}_\rho,0} = 1, \quad F_{\mathcal{E}_\rho,n} = \frac{2}{\rho^n} \cdot T_n, \quad n \geq 1. \tag{2.18}$$

We are now prepared to prove

THEOREM 2.14 *Let G be as described in the introduction of Section 2.4. Then for each compact set H and for each $n \geq 0$ we have*

$$\frac{1}{1+2\cdot\epsilon_{G,n}} \leq \frac{\Delta_n(H,G)}{\Delta(H,G)^n} \leq 1,$$

where $\epsilon_{G,n} := ||F_{G,n} - \Phi_G^n||_{\partial G}$.

Proof: The upper bound is a consequence of the Bernstein–Walsh Lemma, see (B.16). In order to show the lower bound, let $\zeta \in H \setminus G$ (the case $H \subset G$ is trivial) such that $\Delta(H,G) = |\Phi_G(\zeta)|$, and define $p_n \in \mathcal{P}_n$ by $p_n(z) = F_{G,n}(z) + (\Phi_G(\zeta)^n - F_{G,n}(\zeta))$. Notice that $\Phi_G^n - F_{G,n}$ is analytic in $G^c \cup \{\infty\}$, and continuous in $G^c \cup \partial G$. By the maximum modulus principle we get

$$|\Phi_G(\zeta)^n - F_{G,n}(\zeta)| \leq ||\Phi_G^n - F_{G,n}||_{\partial G} = \epsilon_{G,n},$$

and

$$||p_n||_G = ||p_n||_{\partial G} \leq ||\Phi_G^n||_{\partial G} + ||\Phi_G^n - F_{G,n}||_{\partial G} + |\Phi_G(\zeta)^n - f_n(\zeta)| \leq 1 + 2\cdot\epsilon_{G,n}.$$

Consequently,

$$\frac{\Delta(H,G)^n}{1+2\epsilon_{G,n}} = \frac{|p_n(\zeta)|}{1+2\epsilon_{G,n}} \leq \frac{||p_n||_H}{||p_n||_G},$$

and the lower bound becomes immediate. □

Of course, the sharpness of the bounds in Theorem 2.14 depends on the behaviour of the sequence $(\epsilon_{G,n})_n$ which for sufficiently smooth G is shown to tend to zero, and, for general G, increases at most as $\mathcal{O}(\sqrt{n})$. For instance, for ellipses one verifies with help of (2.18) that $\epsilon_{\mathcal{E}_\rho,n} = \rho^{-2n}$, $n \geq 1$. Some classical results are summarized in the following

LEMMA 2.15 *Let $\epsilon_{G,n} := ||F_{G,n} - \Phi_G^n||_{\partial G}$, $n \geq 0$.*

(a) *If G has an analytic boundary, and the function Ψ_G is analytic and univalent in $\{w \in \mathbb{C} : |w| \geq R\}$ with $R \in (0;1)$, then*

$$\epsilon_{G,n} \leq \sqrt{n \cdot \log(\frac{1}{1-R^2})} \cdot R^n, \quad n \geq 0.$$

(b) *We say that G belongs to the class $\mathcal{C}(\ell, \alpha)$, where ℓ is a nonnegative integer, and $0 \leq \alpha < 1$, if the parametrization of the boundary ∂G is $s \to z(s)$, where s is the arc length, and the periodic function $z(\cdot)$ is ℓ times continuously differentiable, with $z^{(\ell)}(\cdot) \in Lip_\alpha$ ($\alpha = 0$ means that we drop this additional condition).*

If $G \in \mathcal{C}(\ell+1, \alpha)$, where $\ell \geq 0$, $\alpha > 0$, then

$$\epsilon_{G,n} = \mathcal{O}(\frac{\log n}{n^{\ell+\alpha}})_{n\to\infty}.$$

(c) *If G is convex but not a segment then $\epsilon_{G,n} < 1$ for all $n \geq 0$.*

(d) *For any G, there exists an $\alpha \in (0;1/2)$ such that*

$$\epsilon_{G,n} = \mathcal{O}(n^\alpha)_{n\to\infty}.$$

Moreover

$$\epsilon_{G,n} \leq 1 + \sqrt{e} + \sqrt{n \cdot \log(n+1)}, \quad n \geq 0.$$

Proof: Part (a) is cited from Smirnov & Lebedev [SmLe68, Eqn.(14), p.136], assertion (b) was proved by Suetin [Sue71, Lemma 1.3], and part (c) has been shown by Kövari & Pommerenke [KöPo67, Theorem 2]. The first part of assertion (d) is an immediate consequence of [KöPo67, Theorem 1(i)], and the final part may be found in [SmLe68, Eqn.(10), p.136]. □

From Lemma 2.15(c) we may conclude that, by the argument principle, the Faber polynomial has all its zeros in the interior of G for all n, provided that G is convex, but not a segment (in the latter case, of course the Faber polynomials are suitable shifted Chebyshev polynomials with zeros on ∂G). Moreover, this property remains true at least for sufficiently large n if the conditions of (a) or (b) are valid.

Following Geronimus [Ger52], we will call G of class Γ if $(\epsilon_{G,n})_n$ tends to zero.

2.4.3 The basis of Faber polynomials

In Example 2.3 we have already seen that the monomials – the Faber polynomials of the unit disk – are well–conditioned with respect to this set. In fact, it is easily seen that this fact remains true in a more general context, leading beside Corollary 2.7 to another application of Theorem 2.6: Since $w \to F_{G,n}(\Psi_G(w)) - w^n$ is analytic in $\mathbb{D}^c \cup \{\infty\}$ including infinity, we obtain for $j, k = 0, 1, ...$

$$\frac{1}{2\pi}\int_{\partial\mathbb{D}} F_{G,j}(\Psi_G(w)) \cdot \overline{w}^k \, |dw| = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{F_{G,j}(\Psi_G(w))}{w^{k+1}} \, dw = \delta_{j,k}.$$

Hence the Faber polynomials $(F_{G,n})_n$ and the successive powers of $\overline{\Phi_G}$ are biorthogonal with respect to some measure with support ∂G. Let us denote the corresponding coordinate map by Π_n, $n \geq 0$. Applying the reasoning of Section 2.1.2 yields

$$||\Pi_n^{-1}||_{\partial G} \leq \max_{0\leq k\leq n} \frac{1}{2\pi} \int_{\partial \mathbb{D}} |\overline{w}^k| \, |dw| = 1,$$

and in addition $||\Pi_n^{-1}||_{\partial G} \geq ||\Pi_0^{-1}||_{\partial G} = 1$. Together with (2.2) and the first formula of Lemma 2.15(d) we obtain

$$||\Pi_n^{-1}||_{\partial G} = 1, \quad ||\Pi_n||_{\partial G} \geq 1, \quad ||\Pi_n||_{\partial G} = \mathcal{O}(n^{3/2}). \tag{2.19}$$

Notice also that the second formula of Lemma 2.15(d) allows us to specify an explicit upper bound for $||\Pi_n||_{\partial G}$, which may be improved if G satisfies in addition one of the initial conditions of Lemma 2.15(a)–(c). For instance, if G is of class Γ, then $||\Pi_n||_{\partial G} = \mathcal{O}(n)$.

As a final example, the Faber polynomials on ellipses (2.18), namely the shifted Chebyshev polynomials

$$F_{G,n}(\Psi_G(w)) = w^n + w^{-n} \cdot \rho^{-2n}, G = \mathcal{E}_\rho, \quad n \geq 1,$$

have been considered by Reichel and Opfer [ReOp91], here we get explicitly for the corresponding coordinate map

$$||\Pi_n^{-1}||_{\mathcal{E}_\rho} = 1, \quad ||\Pi_n||_{\mathcal{E}_\rho} = 1 + \sum_{j=1}^{n} (1 + \rho^{-2j}) = n + 1 + \frac{1 - \rho^{-2n}}{\rho^2 - 1} \leq 2n + 1.$$

2.5 Monomials on Ellipses

In order to obtain the numerical condition of monomials with respect to ellipses of the form $G = \alpha + \beta \cdot \mathcal{E}_\rho$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $\rho \in [1, +\infty)$ (see Example 2.13), we may apply Corollary 2.7, where estimates for the unknown quantity

$$\Delta_n(\mathbb{D}, \alpha + \beta \cdot \mathcal{E}_\rho) = \Delta_n(-\frac{\alpha}{\beta} + \frac{1}{\beta} \cdot \mathbb{D}, \mathcal{E}_\rho)$$

may be obtained in terms of $\Delta(-\frac{\alpha}{\beta} + \frac{1}{\beta} \cdot \mathbb{D}, \mathcal{E}_\rho) = \Delta(\mathbb{D}, \alpha + \beta \cdot \mathcal{E}_\rho)$ by applying Theorem 2.14 with $\epsilon_{\mathcal{E}_\rho,n} = \rho^{-2n}$. By the way, notice that the numerical condition of the monomial basis is invariant under rotation, such that we may (and we will) assume without loss of generality that $\beta > 0$.

In the particular case of ellipses and intervals, however, Theorem 2.14 may be substantially improved; we will report some results below. In Section 2.5.1 we compute explicitly the quantity $\Delta(\mathbb{D}, G)$ for ellipses with real foci, and give a closed form estimate for the numerical condition of monomials on real intervals. Ellipses with real foci being sufficiently far away from the origin are considered in Section 2.5.2.

The constrained Chebyshev problem for the real interval $[-1;1]$ (i.e., $\rho = 1$) was considered by Freund and Ruscheweyh [FrRu86], in particular they proved [FrRu86, Theorem (6.12)] that

$$\Delta_n(J(i \cdot R), [-1;1]) = \Delta_n(\frac{1}{\beta} \cdot \mathrm{ID}, [-1;1]) = \frac{1}{2} \cdot (R^n + R^{n-2}), \quad R := \Delta(\frac{1}{\beta} \cdot \mathrm{ID}, [-1;1]). \quad (2.20)$$

This covers completely the case $\alpha = 0$, $\rho = 1$ of intervals of the form $G = [-\beta; \beta]$. Frappier and Rahman [FrRa82] showed that

$$\frac{1}{2} \cdot R^n + \frac{\sqrt{2}-1}{2} R^{n-2} \leq \Delta_n(\mathcal{E}_R, [-1;1]) \leq \frac{1}{2} \cdot R^n + \frac{5+\sqrt{17}}{4} R^{n-2}, \quad (2.21)$$

and conjectured that $\Delta_n(\mathcal{E}_R, [-1;1]) = \Delta_n(J(i \cdot R), [-1;1])$ (which was given in (2.20)).

In the case $\rho > 1$, we are also able to improve Theorem 2.14. Here, a tighter lower bound was given by Fischer & Freund [FiFr90, Theorem 2, p301]. The upper bound given by Fischer & Freund is only valid if n is sufficiently large, however, we may give the tighter bound (2.23) which in addition is valid for all $n \geq 0$.

LEMMA 2.16 *For $1 < \rho < R$, $t \in [0; 2\pi]$ we have*

$$\frac{R^n + R^{-n}}{\rho^n + \rho^{-n}} \leq \Delta_n(\mathcal{E}_R, \mathcal{E}_\rho) \leq \frac{R^n}{\rho^n}, \quad (2.22)$$

$$\Delta_n(J(R \cdot e^{it}), \mathcal{E}_\rho) \leq \sqrt{\frac{J(R^n)^2 - \sin(n \cdot t)^2}{J(\rho^n)^2 - 1}}. \quad (2.23)$$

Proof: As proved in [FiFr90, Theorem 2], the lower bound of (2.22) is obtained by considering $p_n(z) = T_n(z) + i \cdot \Im T_n(J(R \cdot e^{it}))/(J(R^n)^2 - 1)$, where T_n denotes the classical Chebyshev polynomial of degree n. The upper bound of (2.22) was already stated in Theorem 2.14.

Assertion (2.23) follows by a standard technique based on the Rouché Theorem: suppose that this inequality does not hold. Then there exists a polynomial p with $p(J(R \cdot e^{it})) = T_n(J(R \cdot e^{it}))$, and

$$||p||_{\mathcal{E}_\rho} = \max_{z \in \partial \mathcal{E}_\rho} |p(z)| < \sqrt{J(\rho^n)^2 - 1} = \min_{z \in \partial \mathcal{E}_\rho} |T_n(z)|.$$

Consequently, $|p(z)| < |T_n(z)|$ for all $z \in \partial \mathcal{E}_\rho$. But T_n has n zeros in the interior of $\mathcal{E}_\rho$, whereas $p - T_n$ has the zero $J(R \cdot e^{it}) \notin \mathcal{E}_\rho$, a contradiction to the Rouché Theorem. □

As also shown by Fischer and Freund [FiFr90], the lower bound of (2.22) is attained for sufficiently large R/ρ. We therefore obtain the following complement of Corollary 2.7

COROLLARY 2.17 *Consider the basis of monomials, and let $G = \alpha + \beta \cdot \mathcal{E}_\rho$, $\rho \geq 1$, be an ellipse with foci $\alpha - \beta \neq \alpha + \beta$, with $|\beta| \cdot \rho$ denoting the sum of the semiaxes (in the limiting case $\rho = 1$,* *the support G becomes a line segment). If $\mathbb{D} \subset G$, then $||\Pi_n^{-1}||_G = 1$, $n \geq 0$, and otherwise for all $n \geq 0$*

$$\frac{1}{n+1} \cdot \frac{1+R^{-2n}}{1+\rho^{-2n}} \leq (\frac{R}{\rho})^{-n} \cdot ||\Pi_n^{-1}||_G \leq 1,$$

where $R = \Delta(\mathbb{D}, [\alpha-\beta; \alpha+\beta]) = \rho \cdot \Delta(\mathbb{D}, G)$ is the smallest parameter such that the ellipse $\mathcal{E}_R$ contains the disk centered at $-\alpha/\beta$ with radius $1/|\beta|$ (i.e., $R > \rho$). □

2.5.1 Ellipses with real foci

In the particular case of ellipses $G = \alpha + \beta \cdot \mathcal{E}_\rho$ with foci lying on a straight line through the origin we are able to determine explicitly the quantities γ and $R = \Delta(\mathbb{D}, [\alpha-\beta; \alpha+\beta])$ required in Corollaries 2.7 and 2.17 in terms of α, β and ρ. Since the norms $||\Pi_n||_G$ and $||\Pi_n^{-1}||_G$ are invariant under rotation of G, we may assume without loss of generality that $\alpha, \beta \in \mathbb{R}$, and $\alpha \geq 0$, $\beta > 0$. Then the quantity γ of Corollary 2.7 is given by

$$\gamma = \max_{z \in \alpha+\beta\cdot\mathcal{E}_\rho} |z| = \alpha + \beta \cdot J(\rho) = \alpha + \frac{\beta}{2} \cdot (\rho + 1/\rho).$$

The aim of the following geometrical Lemma is to determine explicitly the quantity $R = \Delta(\mathbb{D}, [\alpha-\beta; \alpha+\beta])$ of Corollary 2.17.

LEMMA 2.18 *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$, $\beta > 0$. Then*

$$\Delta(\mathbb{D}, [\alpha-\beta; \alpha+\beta]) = \begin{cases} \sqrt{\frac{1}{\beta^2-\alpha^2}} + \sqrt{\frac{1}{\beta^2-\alpha^2} + 1}, & \text{if } \beta^2 \geq \alpha^2 + \alpha, \\ \frac{1+\alpha}{\beta} + \sqrt{\left(\frac{1+\alpha}{\beta}\right)^2 - 1} & \text{if } \beta^2 \leq \alpha^2 + \alpha. \end{cases}$$

Proof: Equivalently, we may look for the largest r such that the boundary of the corresponding ellipse $\mathcal{E}_r$ has a non–empty intersection with the circle $-\frac{\alpha}{\beta} + \frac{1}{\beta} \cdot \partial\mathbb{D}$.

The equation

$$\frac{\left(-\frac{\alpha}{\beta} + \frac{1}{\beta} \cdot \cos(t)\right)^2}{\left(\frac{1}{2} \cdot (r + \frac{1}{r})\right)^2} + \frac{\left(\frac{1}{\beta} \cdot \sin(t)\right)^2}{\left(\frac{1}{2} \cdot (r - \frac{1}{r})\right)^2} = 1$$

can be equivalently rewritten as

$$F(r,t) := \left(\cos(t) + \alpha \cdot \left(\frac{1}{2} \cdot (r - \frac{1}{r})\right)^2\right)^2$$
$$- \left((\alpha^2 - \beta^2) \cdot \left(\frac{1}{2} \cdot (r - \frac{1}{r})\right)^2 + 1\right) \cdot \left(\frac{1}{2} \cdot (r + \frac{1}{r})\right)^2 = 0.$$

Now possible extremals t_1 of the function $r = r(t)$ being defined by $F(r(t), t) = 0$ have to satisfy $r'(t_1) = 0$. Therefore, for these extremals there holds

$$\frac{\partial F}{\partial t}(r,t) = 0 = -\sin(t) \cdot \left(\cos(t) + \alpha \cdot \left(\frac{1}{2} \cdot (r - \frac{1}{r}) \right)^2 \right). \tag{2.24}$$

The case $sin(t) = 0$ corresponds to the intersection of the circle with the real axis, here the maximal $r = r_1$ is given by

$$\frac{1}{2}\left(r + \frac{1}{r}\right) = \frac{1+\alpha}{\beta}, \quad \text{or} \quad r_1 = \frac{1+\alpha}{\beta} + \sqrt{\left(\frac{1+\alpha}{\beta}\right)^2 - 1}.$$

On the other hand, the right hand term in (2.24) vanishes if and only if

$$-\cos(t) = \alpha \cdot \left(\frac{1}{2} \cdot (r - \frac{1}{r}) \right)^2 \leq 1$$

and in this case we obtain from $F(r,t) = 0$ the representation

$$\left(\frac{1}{2} \cdot (r - \frac{1}{r}) \right)^2 = \frac{1}{\beta^2 - \alpha^2}, \quad \text{or} \quad r_2 = \sqrt{\frac{1}{\beta^2 - \alpha^2}} + \sqrt{\frac{1}{\beta^2 - \alpha^2} + 1},$$

being a possible candidate for an extremum if $\beta^2 \geq \alpha^2 + \alpha$. The assertion now follows by verifying that $r_2 > r_1$ in the case $\beta^2 > \alpha^2 + \alpha$ (and $r_2 = r_1$ in the case $\beta^2 = \alpha^2 + \alpha$, respectively). □

The condition of the basis of monomials for supports being arbitrary real intervals has been studied by Gautschi [Gau79, Section 4], and some first results have already been reported in Section 2.1.1. In order to compare, we summarize our findings in the following Corollary where we have applied (2.21) and Corollaries 2.7, 2.17

COROLLARY 2.19 *Consider the basis of monomials, and let the support $G = [\alpha - \beta; \alpha + \beta]$ be a real interval, with $\beta > 0$ and without loss of generality $\alpha \geq 0$, furthermore let $R = \Delta(\mathbb{D}, [\alpha - \beta; \alpha + \beta])$ as in Lemma 2.18. Then for each integer $n \geq 0$*

$$\kappa_G(\Pi_n) \geq R^n \cdot (\alpha+\beta)^{n+1} \cdot \frac{1 - (\alpha+\beta)^{-n-1}}{\alpha+\beta-1} \cdot \frac{1}{2n+2} \cdot \max\{1 + R^{-2n}, 1 + (\sqrt{2}-1) \cdot R^{-2}\}$$

and

$$\kappa_G(\Pi_n) \leq R^n \cdot (\alpha+\beta)^{n+1} \cdot \frac{1 - (\alpha+\beta)^{-n-1}}{\alpha+\beta-1} \cdot \min\{1, \frac{1}{2} + \frac{5+\sqrt{17}}{4} R^{-2}\}.$$

In particular, the sequence $(\kappa_G(\Pi_n)^{1/n})_n$ tends to

$$\begin{cases} \sqrt{\frac{\beta+\alpha}{\beta-\alpha}} + \sqrt{\frac{\beta+\alpha}{\beta-\alpha} + (\beta+\alpha)^2}, & \text{if } \beta^2 \geq \alpha^2 + \alpha \text{ and } \alpha + \beta \geq 1, \\ (\alpha+\beta) \cdot [\frac{1+\alpha}{\beta} + \sqrt{\left(\frac{1+\alpha}{\beta}\right)^2 - 1}] & \text{if } \beta^2 \leq \alpha^2 + \alpha \text{ and } \alpha + \beta \geq 1, \\ \sqrt{\frac{1}{\beta^2-\alpha^2}} + \sqrt{\frac{1}{\beta^2-\alpha^2} + 1}, & \text{if } \beta^2 \geq \alpha^2 + \alpha \text{ and } \alpha + \beta \leq 1, \\ \frac{1+\alpha}{\beta} + \sqrt{\left(\frac{1+\alpha}{\beta}\right)^2 - 1} & \text{if } \beta^2 \leq \alpha^2 + \alpha \text{ and } \alpha + \beta \leq 1. \end{cases}$$

□

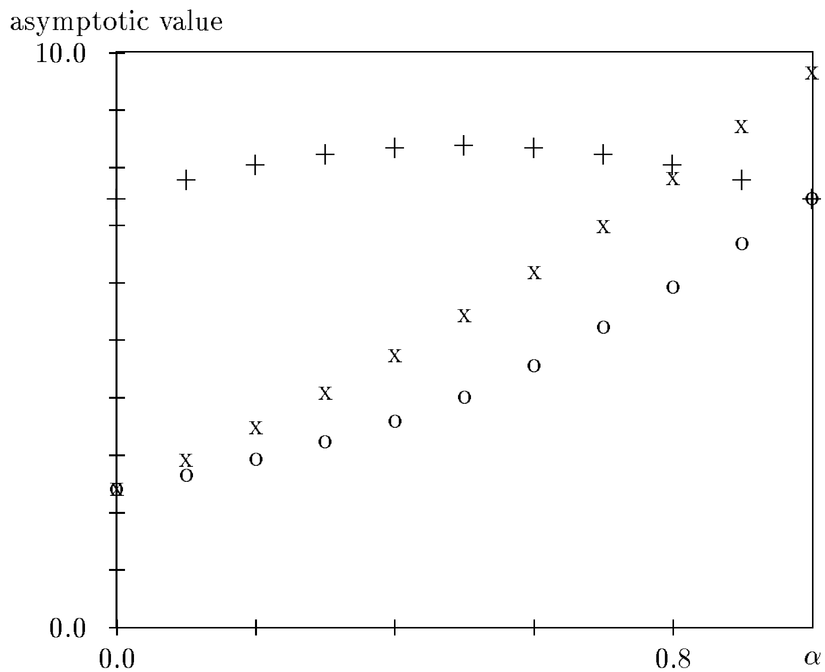

Table 2.1: Asymptotic upper bounds of Gautschi (x,+) and the exact value (o)

Let us first mention that our bounds enable us to determine the condition number up to a factor $2n+2$. Moreover, in the cases $\alpha = 0$ and $\alpha \geq \beta$, Corollary 2.19 is in accordance with Gautschi's (asymptotic) results mentioned in Section 2.1.1. For the asymmetric case $0 < \alpha < \beta$, Gautschi [Gau79, Theorem 4.1, p.350] gave upper bounds for $\kappa_G(\Pi_n)$ being weaker, at least for large n. For instance, for intervals with half-width $\beta = 1$ Gautschi derived the asymptotic upper bound

$$\kappa_{[\alpha-1;\alpha+1]}(\Pi_n)^{1/n} \leq \begin{cases} (1+\alpha)^2(1+\sqrt{2}) & \text{if } 0 \leq \alpha \leq 0.8216.., \\ (1+\alpha)(2-\alpha)(2+\sqrt{3}) & \text{if } 0.8216.. \leq \alpha \leq 1\ . \end{cases}$$

whereas the exact asymptotic grow is given by

$$\lim_{n\to\infty} \kappa_{[\alpha-1;\alpha+1]}(\Pi_n)^{1/n} = \begin{cases} \sqrt{\frac{1+\alpha}{1-\alpha}} + \sqrt{\frac{1+\alpha}{1-\alpha} + (1+\alpha)^2}, & \text{if } 0 \leq \alpha \leq 0.6180.., \\ (\alpha+1)\cdot[(1+\alpha) + \sqrt{(1+\alpha)^2 - 1}] & \text{if } \alpha \geq 0.6180..\ . \end{cases}$$

In Table 2.1, we have plotted both right hand sides for α varying between 0 and 1.

2.5.2 Ellipses not containing the origin

In (2.5) of Section 2.1.1 we have seen that, for monomials, the quantity $||\Pi_n^{-1}||_{[\alpha-\beta;\alpha+\beta]}$ can be computed explicitly in terms of Chebyshev polynomials provided that the origin does not lie in the interior of $[\alpha-\beta;\alpha+\beta]$. This remains true for ellipses $G=\alpha+\beta\cdot\mathcal{E}_\rho$ with real foci (or foci lying on a straight line through the origin) which are sufficiently far away from the origin. For a proof, we first have to solve the corresponding Markov–type extremal problems

THEOREM 2.20 *Let $\rho\geq 1$, and $0\leq j\leq n$. We have for $p\in\mathcal{P}_n$*

$$|p^{(j)}(c)|\leq |T_n^{(j)}(c)|\cdot\frac{||p||_{\mathcal{E}_\rho}}{J(\rho^n)}$$

provided that $c\in\mathbb{R}$, $|c|\geq J(\rho)$, and

$$|c|\geq\frac{n-j}{n}\cdot\frac{\rho^n+\rho^{-n}}{\rho^{n-1}+\rho^{-n+1}}. \tag{2.25}$$

A proof of Theorem 2.20 will be given below. Note that condition (2.25) of Theorem 2.20 holds for $0\leq j\leq n$ if $|c|\geq\rho$. Hence in the particular case of $\mathcal{E}_\rho$ being an interval ($\rho=1$) we obtain the classical results on the grow of (derivatives of) polynomials outside $\mathcal{E}_\rho$ mentioned already in the context of (2.5).

Taking $j=0$ in Theorem 2.20 leads to

$$\Delta_n(c,\mathcal{E}_\rho)=\frac{|T_n(c)|}{J(\rho^n)},\quad c\in\mathbb{R},\quad |c|\geq|\rho|. \tag{2.26}$$

In many textbooks on numerical analysis it is stated without proof that the constrained Chebyshev problem on ellipses has the solution (2.26) for all $c\in\mathbb{R}$ not lying in the interior of $\mathcal{E}_\rho$, i.e., $|c|\geq J(\rho)$. However, as shown by Fischer & Freund [FiFr91, Theorem 1], this is only true for $n\geq 5$ if $c\in\mathbb{R}\setminus[-J(\rho);J(\rho)]$ satisfies a complementary condition $|c|\geq c^*(\rho)>J(\rho)$, at least for ρ being sufficiently large. Some choices for $c^*(\rho)$ different from (2.26) may be found in [FiFr91, Theorem 2].

In the case $j\geq n/2$ one shows that (2.25) is true for all c on the real axis not lying in the interior of $\mathcal{E}_\rho$. Thus the Chebyshev polynomial is optimal if one wants to maximize sufficiently high derivatives outside of ellipses. In particular ($j=n$), the monic counterpart of T_n is, under all monic polynomials of degree n, the one that deviates least from zero on an arbitrary ellipse $\mathcal{E}_\rho$, $\rho\geq 1$ (this property of course is well–known, see, e.g., [SmLe68, Corollary 7, p.360]).

Before giving a proof of Theorem 2.20, we first summarize the consequences for the coordinate map of monomials in the following

THEOREM 2.21 *Let the ellipse $G=\alpha+\beta\cdot\mathcal{E}_\rho$ with real foci be sufficiently far away from the origin such that $|\alpha/\beta|\geq\rho\geq J(\rho)$. Then for the basis of monomials $p_k(z)=z^k$, $k\geq 0$, we have*

$$||\Pi_n^{-1}||_G=\frac{2}{\rho^n+\rho^{-n}}\cdot||\vec{T}_n(\frac{x-\alpha}{\beta})||_\infty,\quad n\geq 0.$$

Moreover, given any basis of scaled monomials $f_k(z) = d_k \cdot z^k$, $d_k \neq 0$, $k \geq 0$, we have for the corresponding coordinate map Φ_n

$$\kappa_G(\Phi_n) \geq \frac{T_n(|\frac{2\alpha}{\beta}| + J(\rho))}{T_n(J(\rho))}, \quad n \geq 0,$$

and the right hand side is attained for the optimal scaling of Section 2.1.3.

Proof: We may assume without loss of generality that $\alpha \geq \beta > 0$. For $0 \leq k \leq n$ we obtain from Theorem 2.20 with help of a linear transformation

$$||\pi_{k,n}||_G = \max_{P \in \mathcal{P}_n} \frac{|P^{(k)}(0)|}{k! \cdot ||P||_{\alpha+\beta\cdot\mathcal{E}_\rho}} = \max_{Q \in \mathcal{P}_n} \frac{|Q^{(k)}(\frac{\alpha}{\beta})|}{|\beta|^k \cdot k! \cdot ||Q||_{\mathcal{E}_\rho}} = \frac{|P_n^{(k)}(0)|}{k! \cdot ||T_n||_{\mathcal{E}_\rho}},$$

where $P_n(z) := T_n((z-\alpha)/\beta)$, $||T_n||_{\mathcal{E}_\rho} = (\rho^n + \rho^{-n})/2 = J(\rho^n)$. This yields the first part of the assertion. Moreover, taking into account the results of Section 2.1.3, for the second part we only have to evaluate

$$\kappa_G(\Phi_n^{opt}) = \sum_{k=0}^{n} \frac{|P_n^{(k)}(0)|}{k! \cdot ||T_n||_{\mathcal{E}_\rho}} \cdot ||z^k||_G$$

$$= \sum_{k=0}^{n} \frac{P_n^{(k)}(0)}{k! \cdot ||T_n||_{\mathcal{E}_\rho}} \cdot (-1)^{n-k} \cdot (\alpha + \beta \cdot J(\rho))^k = \frac{(-1)^n P_n(-\alpha - \beta \cdot J(\rho))}{T_n(J(\rho))} = \frac{T_n(\frac{2\alpha}{\beta} + J(\rho))}{T_n(J(\rho))}.$$

□

For a proof of Theorem 2.20 we require a technical lemma where some sufficient and necessary conditions are given for 'real' functionals $\lambda : \mathcal{P}_n \to \mathbb{C}$ in order to satisfy $||\lambda||_{\mathcal{E}_\rho} = |\lambda(T_n)|/J(\rho^n)$. For the particular case $\rho = 1$ of the interval $[-1;1]$ and polynomials having only real coefficients, equivalent conditions have been obtained by Rivlin [Riv74, Theorems 2.16 and 2.20].

LEMMA 2.22 *Let $\rho \geq 1$, and let the linear functional $\lambda : \mathcal{P}_n \to \mathbb{C}$ be 'real', i.e., $\lambda_k := \lambda(T_k)/J(\rho^k) \in \mathbb{R}$, $k = 0,..,n$. Then $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$ if and only if the cosine polynomial*

$$C(t) = \frac{\lambda_n}{2} + \sum_{k=1}^{n-1} \lambda_{n-k} \cdot \cos(k \cdot t) + \frac{\lambda_0}{2} \cdot \cos(n \cdot t)$$

does not change sign at the arguments $t = \pi \cdot j/n$, $j = 0,..,n$.

Moreover, $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$ if $(\lambda_k)_{0 \leq k \leq n}$ is nonnegative, nondecreasing, and convex, i.e.,

$$\lambda_0 \geq 0, \quad \lambda_1 \geq \lambda_0, \quad \text{and for } k = 1,..,n-1\text{:} \quad \lambda_{k+1} - 2 \cdot \lambda_k + \lambda_{k-1} \geq 0.$$

Proof: By continuity, it is sufficient to prove the assertion for $\rho > 1$. Notice that $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$ (i.e., T_n is extremal for λ) if and only if 0 is the element of best approximation (with respect to $||\cdot||_{\mathcal{E}_\rho}$) to T_n out of the null-space of λ, a linear space of dimension n denoted by $N(\lambda)$. In order to apply the classical characterization of elements of best approximation (see Rivlin, [Riv74, p.63ff], or Schönhage [Sch71, Satz 6.2, p.152]), we first require the arguments where T_n attains its maximum on $\mathcal{E}_\rho$. These are precisely given by $z_j := J(\rho\cdot\exp(\pi\cdot i\cdot j/n))$, $j = 0,..,2n-1$, and for $0\le k\le n$, $0\le j<2n$, we have

$$T_k(z_j) = J(\rho^k)\cdot\cos(\pi\cdot j\cdot k/n) + i\cdot\tilde{J}(\rho^k)\cdot\sin(\pi\cdot j\cdot k/n), \quad \tilde{J}(y) := \frac{1}{2}(y - 1/y).$$

Also, notice that a basis of $N(\lambda)$ is given by $\lambda(T_n)\cdot T_k - \lambda(T_k)\cdot T_n$, $k = 0,..,n-1$.

Consequently, applying the characterization of elements of best approximation we obtain $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$ if and only if there exist a solution $\sigma_j \ge 0$, $j = 0,...,2n-1$, for the following system of equation

$$k = 0,..,n: \qquad \lambda(T_k) = \lambda(T_n)\cdot\sum_{j=0}^{2n-1}\sigma_j\cdot\overline{T_n(z_j)}\cdot T_k(z_j).$$

This system may be rewritten as

$$\begin{aligned}\frac{\lambda_k}{J(\rho^n)^2\cdot\lambda_n} &= \sigma_0\cdot(-1)^0\cdot\cos(k\cdot 0) + \sigma_n\cdot(-1)^n\cdot\cos(k\cdot\pi) \\ &+ \sum_{j=1}^{n-1}\sigma'_j\cdot(-1)^j\cdot\cos(\pi\cdot\frac{j\cdot k}{n}) + i\cdot\frac{\tilde{J}(\rho^k)}{J(\rho^k)}\cdot\sum_{j=1}^{n-1}\sigma''_j\cdot(-1)^j\cdot\sin(\pi\cdot\frac{j\cdot k}{n}),\end{aligned} \tag{2.27}$$

$k = 0,..,n$, where

$$\sigma'_j := \sigma_j + \sigma_{2n-j}, \quad \sigma''_j := \sigma_j - \sigma_{2n-j}.$$

Since the left hand side of (2.27) is real, we see that, for any solution, the sine polynomial

$$s(\phi) = \sum_{j=1}^{n-1}\sigma''_j\cdot(-1)^j\cdot\sin(j\cdot\phi)$$

must have the zeros $\pi\cdot k/n$, $k = 0,..,n$. Now, $s(\phi)/\sin(\phi)$ is a polynomial of degree at most $n-2$ in $\cos(\phi)$, and therefore must vanish identically. Hence, for any solution of (2.27) there necessarily holds $\sigma''_{\rho,j} = 0$, and, by taking into account that $(-1)^j\cdot\cos(\pi\cdot j\cdot k/n) = \cos(\pi\cdot j\cdot (n-k)/n)$, we have $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$ if and only if there exist $\tau_j \ge 0$, $j = 0,...,n$, satisfying

$$\frac{1}{\lambda_n}\cdot(\lambda_n,..,\lambda_1,\lambda_0)^T = V_{n+1}\cdot(\tau_0,..,\tau_n)^T, \quad V_{n+1} := (\cos(\pi\cdot j\cdot k/n))_{k=0,..,n}^{j=0,..,n}.$$

The inverse of V_{n+1} is explicitly known, namely, with $D_{n+1} = \mathit{diag}\,(1/2,1,..,1,1/2)$ there holds $2\cdot D_{n+1}\cdot V_{n+1}\cdot D_{n+1}\cdot V_{n+1} = n\cdot I_{n+1}$. Thus, the equivalent characterization of Lemma 2.22 follows from multiplying the above equation by $\lambda_n\cdot V_{n+1}\cdot D_{n+1}$.

Finally, Fejer showed that if $(\lambda_k)_{0\le k\le n}$ satisfies the conditions of the second part of Lemma 2.22 then the cosine polynomial C of Lemma (2.22) is nonnegative on $[0;2\pi]$ (see [MMR94, Theorem 1.2.8, p.310]). This implies $||\lambda||_{\mathcal{E}_\rho} = |\lambda_n|$. □

Proof of Theorem 2.20: We may assume without loss of generality that $c \geq J(\rho) \geq 1$ (otherwise consider $p(-x)$). We introduce the abbreviation

$$t_n^j = |T_n^{(j)}(c)| = T_n^{(j)}(c), \quad \rho_n := T_n(J(\rho)) = J(\rho^n), \quad n, j \geq 0.$$

In view of Lemma 2.22, it is sufficient to show that the sequences $(t_k^j/\rho_k)_{0 \leq k \leq n}$ are nondecreasing and convex for $j = 0, 1, 2, .., n$.

The aim of the first part is to show that the sequences

$$(t_k^0/\rho_k)_{0 \leq k \leq n} \quad \text{and for } 1 \leq j \leq n: \quad (t_k^j/(k \cdot \rho_k))_{j \leq k \leq n} \tag{2.28}$$

are nondecreasing. Any sequence of polynomials $(p_k)_k$ being orthonormal with respect to a scalar product induced by a weight function on $[-1, 1]$ satisfies a Rodrigues formula

$$\sum_{j=0}^{n} p_j(x) \cdot p_j(y) = \frac{k_n}{k_{n+1}} \cdot \begin{cases} \dfrac{p_{n+1}(y) \cdot p_n(x) - p_n(y) \cdot p_{n+1}(x)}{y - x} & \text{if } x \neq y, \\ p'_{n+1}(x) \cdot p_n(x) - p'_n(x) \cdot p_{n+1}(x) & \text{if } x = y, \end{cases}$$

where $k_n > 0$ denotes the leading coefficient of p_n. Since p_j has all its zeros in $[-1; 1]$, we may conclude that, for all $y \geq x \geq 1$, the sequences $(p_k(y)/p_k(x))_{k \geq 0}$ and $(p'_k(y)/p_k(y))_{k \geq 0}$ are nonnegative and increasing, and this property remains valid if we multiply p_k by a suitable constant. Now, $(T_k^{(j)})_{k \geq j}$ are orthogonal with respect to the weight function $(1 - x^2)^{j-1/2}$ on $[-1; 1]$, $j = 0, 1, ...$ Consequently, for each $c \geq \rho_1$, the sequences $(t_k^j/\rho_k)_{k \geq j}$, as a product of nondecreasing sequences, are nondecreasing, $j = 0, 1, ..., n$. Moreover, (2.28) follows by showing that $(t_k^1/(k \cdot t_k^0))_{0 \leq k \leq n}$ is nondecreasing, the latter being an immediate consequence of $t_k^1/k = U_{k-1}(c)$, where U_k denotes the Chebyshev polynomial of second kind.

Since $t_k^j = 0$ for $k < j$, it remains to show that

$$\delta_k^j := \frac{t_{k+1}^j}{\rho_{k+1}} - 2 \cdot \frac{t_k^j}{\rho_k} + \frac{t_{k-1}^j}{\rho_{k-1}} \geq 0, \quad 1 \leq k < n, \quad 0 \leq j \leq k.$$

Here we need to consider the three term recurrencies for t_k^j and ρ_k. The classical Chebyshev recurrence may be rewritten as

$$\rho_{k+1} = 2 \cdot \rho_1 \cdot \rho_k - \rho_{k-1}, \quad k \geq 1.$$

Moreover, for $j \geq 1$, $T_k^{(j)}/k$ coincides, up to a constant independent of k, with the ultraspherical polynomial of degree $k - j$ and index j (see [Sze67, Eqns.(4.7.8) and (4.7.14)]), and hence the classical three term recurrence for ultraspherical polynomials [Sze67, Eqn.(4.7.17)] may be applied

$$t_{k+1}^j = 2 \cdot c \cdot \frac{k+1}{k+1-j} \cdot t_k^j - \frac{(k+1)(k-1+j)}{(k+1-j)(k-1)} \cdot t_{k-1}^j, \quad k \geq 1, \quad 0 \leq j \leq k.$$

Therefore, for $1 \leq k < n$ and $0 \leq j \leq k$ we have

$$\delta_k^j = \frac{2 \cdot t_k^j \cdot \rho_{k-1} \cdot (c \cdot \frac{k+1}{k+1-j} \cdot \rho_k - \rho_{k+1}) + t_{k-1}^j \cdot \rho_k \cdot (\rho_{k+1} - \frac{(k+1)(k-1+j)}{(k+1-j)(k-1)} \cdot \rho_{k-1})}{\rho_{k+1} \cdot \rho_k \cdot \rho_{k-1}}.$$

Now, by assumption (2.25) on c,

$$c \cdot \frac{k+1}{k+1-j} \cdot \rho_k - \rho_{k+1} \geq \frac{k+1}{k+1-j} \cdot \rho_k \cdot \min_{k \in \{j, j+1, .., n-1\}} (c - \frac{k+1-j}{k+1} \cdot \frac{\rho_{k+1}}{\rho_k})$$
$$= \frac{k+1}{k+1-j} \cdot \rho_k \cdot (c - \frac{n-j}{n} \cdot \frac{\rho_n}{\rho_{n-1}}) \geq 0,$$

and hence we obtain a lower bound for δ_k^j if we replace the quantity $t_k^j \cdot \rho_{k-1}$ by a smaller one.

We study the case $j = 0$ separately. Applying (2.28) gives $t_k^0 \cdot \rho_{k-1} \geq t_{k-1}^0 \cdot \rho_k$, and

$$\delta_k^0 \geq \frac{t_{k-1}^0 \cdot (2 \cdot c \cdot \rho_k - \rho_{k+1} - \rho_{k-1})}{\rho_{k+1} \cdot \rho_{k-1}} = \frac{2 \cdot (c - \rho_1) \cdot \rho_k \cdot t_{k-1}^0}{\rho_{k+1} \cdot \rho_{k-1}} \geq 0.$$

Therefore, the sequence $(t_k^0 / \rho_k)_{0 \leq k \leq n}$ is convex.

The case $j > 0$ is slightly more involved. First notice that $\delta_j^j \geq 0$ according to $t_{j-1}^j = 0$. In the case $n > k > j \geq 1$ we again apply (2.28), leading to $t_k^j \cdot \rho_{k-1} \geq t_{k-1}^j \cdot \rho_k \cdot k/(k-1)$, and

$$\frac{\rho_{k+1} \cdot \rho_{k-1} \cdot \delta_k^j}{t_{k-1}^j}$$
$$\geq 2 \cdot \frac{k}{k-1} \cdot (c \cdot \frac{k+1}{k+1-j} \cdot \rho_k - \rho_{k+1}) + (\rho_{k+1} - \frac{(k+1)(k-1+j)}{(k+1-j)(k-1)} \cdot \rho_{k-1})$$
$$= \frac{2c \cdot k(k+1)}{(k-1)(k+1-j)} \rho_k - \frac{k+1}{k-1} (2\rho_1 \rho_k - \rho_{k-1}) - \frac{(k+1)(k-1+j)}{(k+1-j)(k-1)} \rho_{k-1}$$
$$= (c - \rho_1) \cdot \frac{2k(k+1)}{(k-1)(k+1-j)} \cdot \rho_k + \frac{2(j-1)(k+1)}{(k-1)(k+1-j)} \cdot (\rho_1 \rho_k - \rho_{k-1}).$$

One verifies that, according to the restrictions $c \geq \rho_1 = J(\rho) \geq 1$, and $k > j \geq 1$, each individual factor in the final expression is nonnegative, leading to the conclusion of Theorem 2.20. □

It remains the question whether not only (2.5), but also (2.6) can be generalized to ellipses. In this context it is interesting to observe that a simple extension of the Markov Theorem for ellipses is not valid, i.e., it is not possible to bound a coefficient of a polynomial of degree less or equal to n in terms of its maximum on an ellipse $\mathcal{E}_\rho$, $\rho > 1$, times the corresponding coefficient of $T_n / J(\rho^n)$ or $T_{n-1} / J(\rho^{n-1})$. For a proof note that, e.g., both sequences

$$\Big(\frac{|T_{2n+1}^{(1)}(0)|}{||T_{2n+1}||_{\mathcal{E}_\rho}} \Big)_n, \quad \text{and} \quad \Big(\frac{|T_{2n}^{(2)}(0)|}{||T_{2n}||_{\mathcal{E}_\rho}} \Big)_n,$$

are no longer increasing for $\rho > 1$.

Chapter 3

The Euclidean norm and modified moment matrices

Given a sequence of polynomials $(p_n)_n$, p_n of degree n, let us study the numerical condition of the polynomial coordinate map $\Pi_n : \mathbb{C}^{n+1} \to \mathcal{P}_n$ defined in (2.1), where, in contrast to the considerations of Chapter 2, we equip $\mathcal{P}_n$ with the L^2 norm induced by some scalar product. Also, we will restrict ourselves to sequences $(p_n)_n$ being orthonormal with respect to some other scalar product, and equip $\mathbb{C}^{n+1}$ with the Euclidean vector norm. Then, as we show in Section 3.1, our problem consists in giving tight bounds for the absolute value of so-called *transmission coefficients* which are required if one expresses the orthonormal polynomials of one scalar product in terms of the orthogonal polynomials with respect to the other scalar product. Equivalently, we have to study the condition number of a so–called *modified moment matrix*, a square Hermitian matrix of order $n+1$ being positive definite.

This chapter is organized as follows: in Section 3.1 we recall the concept of orthogonality with respect to some measure μ. Modified moment matrices w.r.t. measures μ, ν are introduced, and different techniques for estimating their condition number are provided. Here we use the basic observation that the inverse of a modified moment matrix is similar to a modified moment matrix. We also mention particular cases such as positive definite Hankel and Toeplitz matrices. The importance of the condition number of modified moment matrices for the numerical calculation of orthogonal polynomials by different methods, such as the E. Schmidt orthogonalization method, or the modified Chebyshev algorithm, is outlined.

Some simple examples of modified moment matrices are considered in Section 3.2, in particular we review classical results on the Hilbert matrix. In Section 3.3 we show that the condition number of modified moment matrices of order $n+1$ w.r.t. measures μ, ν does necessarily increase at least exponentially in n if the supports of the measures are essentially different. The reciprocal of this assertion is established for the subclass of completely regular measures, for which some examples are given.

In the final Section 3.4, we restrict ourselves to measures supported on 'smooth' contours $\partial G, \partial H$, and induced by density functions satisfying the Szegö condition. Norm estimates are given for the case $H \subset G$. As the main result of this section, we determine in the case $H \not\subset G$ the asymptotic of the norm of the corresponding modified moment matrix for large dimensions $n+1$. This generalizes well–established results for the inverse of the Hilbert matrix, as well as results of Wilf and Widom for the case of inverses of ordinary moment matrices. For illustrating our results, we discuss orthogonality on two intervals, and orthogonality on the semi–circle (see Example 3.5, and Example 3.16, respectively).

3.1 Notation

We first summarize the concept of μ–orthogonality in the following

DEFINITION 3.1 *Given a measure μ with compact and non–finite support* $\mathrm{supp}(\mu)$ *(see Section B.1 of Appendix B), we may associate a scalar product on* $\mathcal{C}(\mathrm{supp}(\mu))$ *and the induced norm being defined by*

$$(f,g)_\mu = \int \overline{f(z)} \cdot g(z)\, d\mu(z), \quad ||f||_\mu := \sqrt{(f,f)_\mu}.$$

$(p_n)_n$ is called a sequence of μ–orthogonal polynomials if p_n is a polynomial of degree n, $n \geq 0$, and $(p_j, p_k)_\mu = 0$ for $j \neq k$. Uniqueness is obtained by the additional requirements that $(p_n, p_n)_\mu = 1$, and p_n has a positive leading coefficient, $n = 0, 1, \ldots$. In this case we will speak of μ–orthonormal polynomials, and write shorter $p_n = p_n^\mu$. □

We will also require the reproducing kernel

$$K_n^\mu(x,y) := \sum_{j=0}^{n} \overline{p_j(x)} \cdot p_j(y), \quad K_n^\mu(z) := K_n^\mu(z,z) > 0, \quad n \geq 0, \tag{3.1}$$

having the property that $\overline{K_n^\mu(x,y)} = K_n^\mu(y,x)$, and

$$(K_n^\mu(z,\cdot), P)_\mu = P(z)$$

for $P \in \mathcal{P}_n$, $z \in \mathbb{C}$.

By orthogonality we obtain for a polynomial of degree n

$$P = \sum_{j=0}^{n} a_j p_j^\mu \quad \Longrightarrow \quad ||P||_\mu = ||(a_0, \ldots, a_n)^T||_2 = \sqrt{|a_0|^2 + |a_1|^2 + \ldots + |a_n|^2}.$$

Consequently, if we equip $\mathcal{P}_n$ with the norm $||\cdot||_\nu$, then for the coordinate map Π_n associated to the orthogonal polynomials with respect to some measure μ we have

$$||\Pi_n||_\nu = \max_{P \in \mathcal{P}_n} \frac{||P||_\nu}{||P||_\mu} =: \Delta_n(\nu,\mu), \quad ||\Pi_n^{-1}||_\nu = \Delta_n(\mu,\nu). \tag{3.2}$$

Also, by orthogonality

$$p_k^\mu = \sum_j p_j^\nu \cdot t_{j,k}, \quad t_{j,k} = t_{j,k}(\nu,\mu) = (p_j^\nu, p_k^\mu)_\nu, \quad k \geq 0,$$

where $t_{j,k} = 0$ for $k < j$. Defining the upper triangular non–singular matrix of transmission coefficients

$$T_n(\nu,\mu) = \Big(t_{j,k}(\nu,\mu)\Big)_{j=0,..,n}^{k=0,..,n}, \quad n \geq 0, \tag{3.3}$$

we obtain for a polynomial P of degree at most n

$$P = \sum_{k=0}^n a_k p_k^\mu = \sum_{j=0}^n b_j p_j^\nu, \quad (b_0,..,b_n)^T = T_n(\nu,\mu)\cdot(a_0,..,a_n)^T,$$

and hence $\Delta_n(\nu,\mu) = ||T_n(\nu,\mu)||_2$. Now, for any matrix B, the quantity $||B||_2^2$ is obtained as the largest eigenvalue of the positive semidefinite Hermitian matrix $B^H \cdot B$, where B^H denotes the transposed and adjoint counterpart of B, and $||B||_2 = ||B^H||_2$. The entry on position (j,k) of the matrix $T_n(\nu,\mu)^H \cdot T_n(\nu,\mu)$ has the form

$$m_{j,k}(\nu,\mu) := \sum_{\ell=0}^n \overline{(p_\ell^\nu, p_j^\mu)_\nu} \cdot (p_\ell^\nu, p_k^\mu)_\nu = \int \overline{p_j^\mu(z)} \cdot (K_n^\nu(z,\cdot), p_k^\mu)_\nu \, d\nu(z) = (p_j^\mu, p_k^\mu)_\nu.$$

This gives raise to consider the following Hermitian positive definite matrix, called a *modified moment matrix*

$$M_n(\nu,\mu) = \Big(m_{j,k}(\nu,\mu)\Big)_{j=0,..,n}^{k=0,..,n}, \quad n \geq 0. \tag{3.4}$$

Note that, by definition, $T_n(\mu,\nu)$ is the inverse of $T_n(\nu,\mu)$, yielding the following identities for $n \geq 0$

$$\Delta_n(\nu,\mu) = ||T_n(\nu,\mu)||_2 = \sqrt{||M_n(\nu,\mu)||_2}, \tag{3.5}$$

$$M_n(\nu,\mu) = T_n(\nu,\mu)^H \cdot T_n(\nu,\mu) = T_n(\mu,\nu)^{-H} \cdot T_n(\mu,\nu)^{-1}, \tag{3.6}$$

$$M_n(\nu,\mu)^{-1} = T_n(\nu,\mu)^{-1} \cdot T_n(\nu,\mu)^{-H} = T_n(\mu,\nu) \cdot T_n(\mu,\nu)^H, \tag{3.7}$$

In particular, $M_n(\mu,\nu)^{-1}$ is similar to $M_n(\nu,\mu)$, and thus these matrices have the same 2–norm and the same Froebenius norm. Equations (3.5)–(3.7) together with (3.2) provide several possibilities to compute or estimate the numerical condition of a basis of orthogonal polynomials. Let us also mention the following two formulas: writing shorter $m_{j,k} = m_{j,k}(\nu,\mu)$, we obtain by the Cauchy–Schwarz inequality

$$|m_{j,k}| \leq \sqrt{|m_{j,j}| \cdot |m_{k,k}|} \leq \frac{1}{2} \cdot (m_{j,j} + m_{k,k}).$$

Together with (A.4), (A.5) of Appendix A we arrive at

$$||M_n(\nu,\mu)||_{Turing} = \sqrt{n} \cdot \max_{j=0,..,n} ||p_j^\mu||_\nu^2, \quad ||T_n(\nu,\mu)||_F^2 = \sum_{j=0}^n ||p_j^\mu||_\nu^2 = \int K_n^\mu(z) d\nu(z). \tag{3.8}$$

$M_n = M_n(\nu,\mu)$ is called a (classical) moment matrix, if $p_k^\mu(z) = z^k$, $k \geq 0$, being orthogonal with respect to the equilibrium measure $\mu = \mu_{\mathbb{D}}$ on the unit circle (see Example 2.3). If in addition the support of ν is a subset of the real axis, then the entries of M_n are constant along antidiagonals, and M_n is a (positive definite) *Hankel matrix*. Similarly, if $supp(\nu) \subset \partial\mathbb{D}$, then the entries of M_n are constant along diagonals, in this case M_n is a (positive definite) *Toeplitz matrix*. In the case of other supports like algebraic curves, one may also recover some particular structure in M_n, leading to a small UV–displacement rank (see [Bre95]).

To the end of this section, let us discuss some applications where the matrices of (3.3), (3.4) occur. Suppose that we want to compute the first elements $p_0^\nu, .., p_0^\nu$ of the sequence of orthonormal polynomials with respect to some measure ν, where we want to use the known polynomials $p_0^\mu, ..., p_n^\mu$, being orthonormal with respect to some other measure μ, i.e., we want to determine the transmission coefficients being the entries of the matrix $T_n(\mu, \nu)$.

One attempt could be to determine for $k = 0, .., n$ the coefficients in the linear combination $P_k = b_{k,0} p_0^\mu + ... + b_{k,k} p_k^\mu$ in such a way that P_k is ν-orthogonal to $p_0^\mu, .., p_{k-1}^\mu$ (inverse Choleski decomposition of $M_n(\nu, \mu)$) or orthogonal to the already computed polynomials $p_0^\nu, ..., p_{k-1}^\nu$. In the latter case we successively solve the systems

$$T_{k-1}(\nu, \mu) \cdot (b_{k,0}, .., b_{k,k-1})^T = -b_{k,k} \cdot \left((p_j^\nu, p_k^\mu)_\nu \right)_{j=0,..,k-1}, \quad k = 1, .., n,$$

or, equivalently, we compute $T_n(\nu, \mu)^{-1}$ by a forward substitution.

Let us also study the orthonormalization procedure of E. Schmidt based on the recurrencies

$$\tilde{q}_k = p_k^\mu - \sum_{j=0}^{k-1} (q_j, p_k^\mu)_\nu \cdot q_j, \quad q_k := \frac{\tilde{q}_k}{||\tilde{q}_k||_\nu},$$

$k = 0, .., n$, which for exact arithmetic gives the result $q_k = p_k^\nu$, $k = 0, .., n$. Notice that $\tilde{q}_k$ is a scalar multiple of p_k^ν, and by comparing the leading coefficients one obtains $\tilde{q}_k = p_k^\nu \cdot (p_k^\nu, p_k^\mu)_\nu$. Using the abbreviation $T = T_n(\nu, \mu)$, we may hence rewrite the Schmidt procedure in the form

$$(p_0^\nu, .., p_n^\nu) \cdot T = (p_0^\mu, .., p_n^\mu),$$

where we recall that the coefficients of T are also computed by the procedure. If one wants to implement this method, then of course errors occur due to floating point arithmetic, and due to a numerical evaluation of the scalar products. We will restrict ourselves to the accumulation of errors due to numerical integration. Here it seems to be appropriate to discuss a recurrence of the form

$$(q_0, .., q_n) \cdot (\tilde{T} + R) = (p_0^\mu, .., p_n^\mu), \quad \tilde{T} = \begin{pmatrix} (q_0, p_0^\mu)_\nu & \cdots & \cdots & (q_0, p_n^\mu)_\nu \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (q_n, p_n^\mu)_\nu \end{pmatrix}, \tag{3.9}$$

where the entries in the upper triangular matrix R are errors due to a discretization of the scalar products. We suppose that $||R||_2 \leq \epsilon$, and neglect in the sequel terms of order ϵ^2. Let us show that an ill–conditioned matrix of transmission coefficients T might lead to a significant loss of ν–orthogonality between the resulting polynomials $q_0, .., q_n$.

First, if $\epsilon = 0$, then (3.9) leads to the correct result $q_k = p_k^\nu$, $k = 0, .., n$, hence we may suppose that

$$(q_0, .., q_n) = (p_0^\nu, .., p_n^\nu) \cdot (I + E),$$

with the unknown upper triangular matrix E satisfying $||E|| = \mathcal{O}(\epsilon)$. Notice that

$$\tilde{T} = (I + E)^H \cdot T - S,$$

where the strictly lower triangular matrix S contains the entries $(q_j, p_k^\mu)_\nu$ $(j > k)$ missing in $\tilde{T}$; in particular the elements of the first row of S are identically zero, and $||S|| = \mathcal{O}(\epsilon)$. Consequently,

$$\begin{aligned} &(q_0, .., q_n) \cdot (\tilde{T} + R) = (p_0^\mu, .., p_n^\mu) = (p_0^\nu, .., p_n^\nu) \cdot T \\ &= (p_0^\nu, .., p_n^\nu) \cdot (I + E) \cdot ((I + E)^H \cdot T - S + R), \end{aligned}$$

and we obtain

$$(E + E^H) \cdot T = -R + S + \mathcal{O}(\epsilon^2).$$

Thus, in first order, the first row of $E + E^H$ is given by the first row of $-R \cdot T^{-1}$. Since

$$\begin{pmatrix} (q_0, q_0)_\nu & \cdots & (q_0, q_n)_\nu \\ \vdots & & \vdots \\ (q_n, q_0)_\nu & \cdots & (q_n, q_n)_\nu \end{pmatrix} = I + E + E^H + \mathcal{O}(\epsilon^2),$$

we are therefore only able to insure 'good' orthogonality if $||T^{-1}||$ is sufficiently small. On the other hand, procedure (3.9) is only feasible if the (off–diagonal) entries of $\tilde{T} + R$ are small, or in first order, $||T||$ has to be small. Consequently, also this third method should only be applied if the underlying modified moment matrix is well–conditioned.

In the particular case of a measure ν supported on the real axis, three consecutive orthogonal polynomials $p_{n-1}^\nu, p_n^\nu, p_{n+1}^\nu$ are connected via a recurrence relation. In order to obtain a certain value of p_n^ν, it is numerically more interesting (and more efficient) to compute the coefficients of the recurrence relation instead of computing the coefficients of p_n^ν. The recurrence coefficients may be obtained, e.g., with help of the modified Chebyshev algorithm of Wheeler (see, e.g., [Gau85, Section 3]), where as a starting point one uses the modified moments $m_{0,k}(\nu, \mu)$, $k = 0, .., 2n - 1$, obtained from a known family $(p_n^\mu)_n$ of polynomials being orthogonal with respect to a measure μ which is supposed to be also supported on the real line.

In a number of papers [Gau82, Gau85, Gau86], Gautschi studied the question whether the (nonlinear) map $\mathcal{K}_n = \mathcal{K}_n(\nu, \mu)$ mapping the first $2n$ modified moments to the first $2n$ recurrence coefficients of (monic) ν–orthogonal polynomials is sensitive to perturbations. In particular, it is of interest to know which μ – for given ν – is suitable in order to have a well–conditioned map $\mathcal{K}_n(\nu, \mu)$. Notice that the modified Chebyshev algorithm uses as auxiliary quantities the transmission coefficients which may be found in the matrix $T_{n-1}(\nu, \mu)$. Also, the numerical experiments reported by Gautschi in [Gau85, Examples 4.1–4.3] seem to indicate that, as a necessary condition, the matrix $M_{n-1}(\nu, \mu)$ has to be well–conditioned.

3.2 Some examples of modified moment matrices

Let us first consider the case where the measures μ, ν have the same support, and the measure ν is obtained by a simple modification of the measure μ, i.e.,

$$S = supp(\mu) = supp(\nu), \quad d\nu(z) = \rho(z) \cdot d\mu(z),$$

and $\rho \geq 0$ is bounded on S. Of course for any polynomial P we have $||P||_\nu^2 = \int |P(z)|^2 \cdot \rho(z)\, d\mu(z) \leq ||P||_\mu^2 \cdot \max_{z\in S} \rho(z)$, which together with (3.5) yields

$$||T_n(\nu, \mu)||_2 \leq \max_{z\in S} \sqrt{\rho(z)}, \quad n \geq 0. \tag{3.10}$$

For instance, Gautschi discussed in [Gau85, Example 4.1] the case of an 'elliptic' modification

$$d\mu(z) = \frac{dz}{\sqrt{1-z^2}}, \quad d\nu(z) = \frac{dz}{\sqrt{1-z^2} \cdot \sqrt{1-k^2 z^2}}, \quad z \in [-1; 1],$$

with $k < 1$. Here one verifies using (3.10) that $\kappa_2(M_n(\nu, \mu)) \leq (1-k^2)^{1/2}$ for all $n \geq 0$. Also, (3.10) may be applied in the case of a polynomial modification of a given weight supported on the real line, where (up to a normalization) the transmission coefficients are explicitly known (see [Sze67, Theorem 2.5]).

Example 3.2 *For $\alpha, \beta > -1$, let us denote by $\mu^{(\alpha,\beta)}$ the Jacobi measure $d\mu^{(\alpha,\beta)}(x) = (1-x)^\alpha \cdot (1+x)^\beta dx$ on $[-1; 1]$. Here the corresponding orthonormal polynomials are known to be (suitably scaled) Jacobi polynomials such as Legendre polynomials ($\alpha = \beta = 0$) or Chebyshev polynomials ($\alpha = \beta = -1/2$). Since*

$$\max_{x\in[-1;1]} (1-x)^\alpha \cdot (1+x)^\beta = \frac{(2\alpha)^\alpha \cdot (2\beta)^\beta}{(\alpha+\beta)^{\alpha+\beta}}, \quad \alpha, \beta \geq 0,$$

we may apply (3.10) in order to obtain absolute bounds for $||T_n(\mu^{(\alpha,\beta)}, \mu^{(0,0)})||_2$ ($\alpha, \beta \geq 0$) and for $||T_n(\mu^{(\alpha,\beta)}, \mu^{(-1/2,-1/2)})||_2$ ($\alpha, \beta \geq -1/2$). In order to obtain bounds for the norm of its inverse, we apply (3.8). By combining [Sze67, Equation (4.3.3), (7.32.2)] we get writing shorter $\nu := \mu^{(\alpha,\beta)}$

$$\max_{x\in[-1;1]} |p_n^\nu(x)| = \mathcal{O}(n^{\max\{\alpha+1/2, \beta+1/2, 0\}})_{n\to\infty},$$

leading to

$$||K_n^\nu||_{[-1;1]} = \mathcal{O}(n^{\max\{2\alpha+2, 2\beta+2, 1\}})_{n\to\infty}.$$

It follows that both quantities $||T_n(\mu^{(0,0)}, \mu^{(\alpha,\beta)})||_2$ ($\alpha, \beta \geq 0$) and $||T_n(\mu^{(-1/2,-1/2)}, \mu^{(\alpha,\beta)})||_2$ ($\alpha, \beta \geq -1/2$) grow at most as $\mathcal{O}(n^{\max\{\alpha+1, \beta+1\}})_{n\to\infty}$. □

Let us also discuss a classical moment matrix, namely the Hilbert matrix being a well-known example for an extremely ill-conditioned matrix.

EXAMPLE 3.3 *The perhaps most famous positive definite Hankel matrix is the Hilbert matrix*

$$H_{n-1} = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix},$$

$H_n = M_n(\nu, \mu_{\mathbb{D}})$, $d\nu(x) = dx$ on $[0;1]$, where $\mu = \mu_{\mathbb{D}}$ denotes the equilibrium distribution of the unit circle (i.e., $p_n^\mu(z) = z^n$, $n \geq 0$). Notice that Hilbert originally introduced the infinite matrix $H = (1/(j+k-1))_{j,k=0,1,2,..}$ as an example of a bounded linear operator on ℓ^2 (namely $||H|| \leq \pi$, see, e.g., [Pow82, p.58]) whose row and column sums are divergent. Thus the norm of H_n is an element of the interval $[1, \pi]$. It seems to be Todd [Tod54] who gave a first asymptotic expression for the Turing condition number of the Hilbert matrix H_n. Let us here mention a result of Wilf [Wif70, Equation (3.35)] who showed that

$$||H_n^{-1}||_2 = ||M_n(\mu_{\mathbb{D}}, \nu)||_2 = \frac{1}{\sqrt{\pi} \cdot 2^{15/4}} \cdot \frac{(1+\sqrt{2})^{4n+4}}{\sqrt{n}} \cdot (1 + o(1)_{n \to \infty}).$$

□

3.3 Behavour for large dimension

The main result of this section (see Corollary 3.6) is to show that a modified moment matrix $\mathcal{M}_n(\nu, \mu)$ can only be asymptotically well–conditioned if the supports of the two underlying measures essentially coincide.

LEMMA 3.4 *Let* $\mathrm{cap}(\mathrm{supp}(\mu)) > 0$, *then we have*

$$\liminf_{n \to \infty} ||M_n(\nu, \mu)||_2^{1/n} \geq \Delta(\mathrm{supp}(\nu), \mathrm{supp}(\mu))^2 \geq 1.$$

Proof: As in [Sze67, Theorem 3.1.3] one shows that the constrained Chebyshev problem with respect to the $L_2(\mu)$ norm is explicitly solvable

$$\max_{P \in \mathcal{P}_n} \frac{|P(z)|}{||P||_\mu} = \sqrt{K_n^\mu(z)}, \quad z \in \mathbb{C}, \quad n \geq 0. \tag{3.11}$$

Since $||P||_\mu \leq \sqrt{\mu(\mathbb{C})} \cdot ||P||_{supp(\mu)}$ for any polynomial P, we may conclude that

$$K_n^\mu(z) \geq \frac{\Delta_n(z, supp(\mu))^2}{\mu(\mathbb{C})}, \quad z \in \mathbb{C}, \quad n \geq 0.$$

Taking into account (A.6) of Appendix A we get

$$\frac{||M_n(\nu, \mu)||_2}{||T_n(\nu, \mu)||_F^2} = \frac{||T_n(\nu, \mu)||_2^2}{||T_n(\nu, \mu)||_F^2} \in [\frac{1}{n+1}, 1].$$

Thus, by (3.8), the sequences $(||M_n(\nu,\mu)||_2^{1/n})_n$ and $([\int K_n^\mu(z)\,d\nu(z)]^{1/n})_n$ have the same accumulation points.

In particular, we have

$$\liminf_{n\to\infty} ||M_n(\nu,\mu)||_2^{1/n} \geq \liminf_{n\to\infty} c_n, \quad c_n := \left[\int \Delta_n(z, supp(\mu))^2\, d\nu(z)\right]^{1/n}.$$

Let us show that $(c_n)_n$ converges, with limit R^2, where $R := \Delta(supp(\nu), S)$, $S := supp(\mu)$. First notice that $\Delta_n(z,S) \leq R^n$ for all $z \in supp(\nu)$ by (B.16) and by definition of R, and thus $\limsup_{n\to\infty} c_n \leq R^2$. Moreover, if $supp(\nu) \cap \mathcal{D}_\infty(S) = \emptyset$, then $\Delta_n(\cdot, S)$ is identical 1 on $supp(\nu)$, and the assertion is trivially true. Suppose now that $supp(\nu) \cap \mathcal{D}_\infty(S)$ is nonempty, and let $\rho, r \in (1; R)$, $\rho < r$. Recall that $R = \sup\{\exp(g_S(z)) : z \in supp(\nu) \cap \mathcal{D}_\infty(S)\}$. By continuity of g_S in $\mathcal{D}_\infty(S)$ and by definition of $supp(\nu)$, we may find some compact set $V \subset \mathcal{D}_\infty(S)$ satisfying $\Delta(z,S) \geq r$ for all $z \in V$, and $\nu(V) > 0$. From Theorem B.22 in Appendix B we know that $(\Delta_n(\cdot,S)^{1/n})_n$ converges to $\Delta(\cdot,S)$ locally uniformly in $\mathcal{D}_\infty(S)$. Thus there exists an $N \geq 0$ such that $\Delta_n(z,S) \geq \rho^n$ for all $z \in V$ and for all $n \geq N$. Consequently,

$$\liminf_{n\to\infty} c_n \geq \liminf_{n\to\infty} \left[\int_V \Delta_n(z, supp(\mu))^2\, d\nu(z)\right]^{1/n} \geq \rho^2.$$

Since $\rho < R$ may be chosen arbitrarily close to R, the assertion follows. □

Example 3.5 *As in [Gau85, Example 4.3], consider a measure μ being supported on a union of two intervals* $\mathrm{supp}(\mu) = [0; 1/3] \cup [2/3; 1]$. *Here the quantity $\Delta(\cdot, \mathrm{supp}(\mu))$ may be explicitly computed using the fact that, for $0 < c < 1$, the Green function of the set $[-1; -c] \cup [c; 1]$ is given by (see, e.g., [Wid69, p.225])*

$$g_{[-1;-c]\cup[c;1]}(z) = \frac{1}{2}\log\left|J^{-1}\left(\frac{1+c^2-2z^2}{1-c^2}\right)\right|.$$

For instance, for the inverse of the underlying positive definite Hankel matrix we obtain by Lemma 3.4

$$\liminf_{n\to\infty} ||M_n(\nu,\mu)||_2^{1/n} \geq \Delta(\mathbb{D}, \mathrm{supp}(\mu))^2 = \Delta(-1+2\cdot\mathbb{D}, [-1; -\frac{1}{3}]\cup[\frac{1}{3}; 1])^2 = |J^{-1}(-19)| \approx 37.97.$$

If in contrast we use as p_n^ν the Chebyshev polynomials shifted on $[0; 1]$, *then*

$$\liminf_{n\to\infty} ||M_n(\nu,\mu)||_2^{1/n} \geq \Delta([0;1], \mathrm{supp}(\mu))^2 = \Delta([-1;1], [-1; -\frac{1}{3}] \cup [\frac{1}{3}; 1])^2 = |J^{-1}(\frac{5}{4})| = 2.$$

This confirms an observation of Gautschi [Gau85] who reported about numerical difficulties if one tries to compute $(p_n^\mu)_n$ from the shifted Chebyshev polynomials by applying the modified Chebyshev algorithm. □

Corollary 3.6 *Let μ, ν be measures with supports having positive capacity. A necessary condition for*

$$\lim_{n\to\infty} \kappa_2(M_n(\nu,\mu))^{1/n} = 1 \tag{3.12}$$

to hold is that $\mathcal{D}_\infty(\mathrm{supp}(\mu)) = \mathcal{D}_\infty(\mathrm{supp}(\nu))$.

Proof: Notice that

$$||M_n(\nu,\mu)^{-1}||_2 = ||T_n(\nu,\mu)^{-1}||_2^2 = ||T_n(\mu,\nu)||_2^2 = ||M_n(\mu,\nu)||_2.$$

Applying Lemma 3.4 gives

$$\liminf_{n\to\infty} \kappa_2(M_n(\nu,\mu))^{1/n} \geq \Delta(supp(\nu), supp(\mu))^2 \cdot \Delta(supp(\mu), supp(\nu))^2.$$

Consequently, (3.12) implies $\mathcal{D}_\infty(supp(\mu)) = \mathcal{D}_\infty(supp(\nu))$. □

For a subclass of measures we may even be more precise (compare [Saf90, Definition 3.1] and Example 2.10).

Definition 3.7 *A measure μ with compact support is called* completely regular *if*

$$\mathrm{cap}(\mathrm{supp}(\mu)) > 0, \quad \textit{and} \quad \lim_{n\to\infty} ||p_n^\mu||_{\mathrm{supp}(\mu)}^{1/n} = 1.$$

□

Corollary 3.8 *Let μ,ν be completely regular measures. Then (3.12) holds if and only if $\mathcal{D}_\infty(\mathrm{supp}(\mu)) = \mathcal{D}_\infty(\mathrm{supp}(\nu))$.*

Proof: One implication of the assertion has been established in Corollary 3.6. In order to show the other one, we denote by S the complement of $\mathcal{D}_\infty(supp(\mu)) = \mathcal{D}_\infty(supp(\nu))$. First notice that

$$||p_n^\mu||_{supp(\mu)} = ||p_n^\mu||_S, \quad ||p_n^\nu||_{supp(\nu)} = ||p_n^\nu||_S, \quad n \geq 0,$$

by the maximum principle for analytic functions. Consequently,

$$\limsup_{n\to\infty} ||K_n^\mu||_S^{1/n} \leq \limsup_{n\to\infty} \Big[\sum_{j=0}^n ||p_j^\mu||_S^2\Big]^{1/n} = \max\Big\{1, \limsup_{n\to\infty}[||p_n^\mu||_S]^{2/n}\Big\} = 1.$$

On the other hand, we have $K_n^\mu(z) \geq K_0^\mu(z) = 1/\mu(\mathbb{C})$ for all $z \in S$. Therefore, the sequence $([K_n^\mu]^{1/n})_{n\geq 0}$ tends to 1 uniformly in $supp(\nu) \subset S$. Similarly, $([K_n^\nu]^{1/n})_{n\geq 0}$ tends to 1 uniformly in $supp(\mu)$, and the assertion follows from (3.8). □

In order to illustrate Corollary 3.8, we give some examples of completely regular measures (part (a)-(c) are cited from Saff [Saf90, Example 3.A], and part (d) follows from results of Widom [Wid69, Theorem 9.1, Theorem 12.3]).

Example 3.9 *The following measures μ are completely regular*

(a) *$d\mu = w\, dx\, dy$ over a bounded Jordan region S, where the weight $w \geq 0$ and some negative powers of w are integrable with respect to area over S.*

(b) *$d\mu = w\, ds$, where ds is arclength over a rectifiable curve S and the weight $w \geq 0$ and some negative powers of w are integrable with respect to ds.*

(c) *$d\mu(x) = w(x)\, dx$ on a real interval $[a, b]$, where $w > 0$ almost everywhere with respect to the Lebesgue measure on $[a, b]$.*

(d) *$S = \mathrm{supp}(\mu)$ consists of finitely many $\mathcal{C}(2, \alpha)$–curves or arcs, $\alpha > 0$ (e.g., several real intervals, or the unit circle), and $d\mu = w\,|dz|$ satisfies the Szegö–type condition*

$$\int_S w(z)\,|dz| < \infty, \quad \int_S \log(w(z))\,|dz| > -\infty, \tag{3.13}$$

where here and in the sequel $|dz|$ denotes the differential of the arc length of S.

□

3.4 Asymptotics for particular modified moment matrices

The aim of this section is to discuss a class of measures (μ, ν) where we may establish the asymptotic behaviour

$$||M_n(\nu, \mu)||_2 = \frac{\Delta(supp(\nu), supp(\mu))^{2n}}{n^{1/m}} \cdot (C + o(1)_{n\to\infty}) \tag{3.14}$$

with m an even integer, and C a suitable explicit constant. Notice that for the inverse of the Hilbert matrix studied in Example 3.3 formula (3.14) holds with $m = 2$, since $(1 + \sqrt{2})^2 = 3 + 2 \cdot \sqrt{2} = \Delta(\mathbb{D}, [0; 1])$, see Lemma 2.18. The case of inverses of ordinary moment matrices, i.e., ν being the equilibrium distribution on the unit circle, was already discussed by Wilf [Wif70] and Widom [Wid69]. Wilf considered measures of the form $d\mu(x) = w(x)\, dx$ on a real interval $[a; b]$ with the weight function $w \geq 0$ satisfying the Szegö condition (see (3.16) below), and established (3.14) with $m \in \{2, 4\}$ for the resulting class of inverses of particular positive definite Hankel matrices (see [Wif70, Theorem 3.2]). In [Wid69, Theorem 10.1, Theorem 10.4, Theorem 10.5, and Theorem 13.1], Widom was able to show that (3.14) remains valid with m an even positive integer including infinity, if $d\mu(x) = w(x)\, dx$ is supported on a system of (smooth) curves or arcs in the complex plane, and $w \geq 0$ satisfies the Szegö–type condition (3.13).

As discussed by Widom, the case of a measure with support consisting of several curves or arcs requires the solution of some rather complicated extremal problems in the complex plane. Here we will restrict ourselves to the case of measures being supported on just one arc or curve, namely

$$d\nu(z) = w_H(z)\,|dz| \text{ on } supp(\nu) = \partial H, \quad d\mu(z) = w_G(z)\,|dz| \text{ on } supp(\mu) = \partial G, \tag{3.15}$$

where G, H are simply connected compact subsets of the complex plane with rectifiable boundary, and $w_H \geq 0$, $w_G \geq 0$. We will specify further conditions on μ, ν below. As in Section 2.4.1, denote by Φ_G, Φ_H the corresponding Riemann maps, and by Ψ_G, Ψ_H their inverses.

In the sequel of this section we will also consider the case where w_G satisfies the Szegö condition

$$\int_{\partial G} \log(w_G(z)) \cdot |\Phi_G'(z)||dz| = \int_{|w|=1} \log(w_G(\Psi_G(w)))\,|dw| > -\infty. \tag{3.16}$$

Define the Szegö function

$$D_G(\Psi_G(w)) := \exp\Big\{-\frac{1}{4\pi} \cdot \int_0^{2\pi} \frac{w+e^{it}}{w-e^{it}} \cdot \log(w_G(\Psi_G(e^{it})))\,dt\Big\}, \quad |w| > 1,$$

then D_G has limit values almost everywhere on ∂G satisfying $|D_G(z)|^{-2} \cdot |\Phi_G'(z)| = w_G(z)$ (see, e.g., [SmLe68, Section 4.4.1]). We are now prepared to study the case $supp(\nu) \subset G$, which in fact is not covered by (3.14).

THEOREM 3.10

(a) *Let μ be as in (3.15) satisfying (3.16). For any measure ν with support being a proper subset of the interior of G we have $||M_n(\nu,\mu)||_2 = \mathcal{O}(1)_{n\to\infty}$.*

(b) *Let μ be as in (3.15), with weight function w_G being bounded away from zero, that is, $w_G(z) > w_0 > 0$ for all $z \in \partial G$. Then for any measure ν with* $\mathrm{supp}(\nu) \subset G$ *there holds $||M_n(\nu,\mu)||_2 = \mathcal{O}(n)_{n\to\infty}$.*

Proof: **(a)** It is well–known that, under the above assumptions, the sequence $(K_n^\mu)_n$ converges uniformly on compact subsets of the interior of G (see, e.g., [Sze67, Theorem 16.3], [SmLe68, Section 4.2.4]), and in particular is bounded there. Hence the first part of the assertion follows from (3.8).

(b) The aim of the following considerations will be to establish bounds for K_n^μ on the boundary of G, or more generally on G^c. Notice that w_G trivially satisfies (3.16). Also,

$$K_n^\mu(z) \le \max_{P\in\mathcal{P}_n} \frac{2 \cdot |P(z)|^2}{\int_{\partial G} |P(\zeta)|^2 \cdot w_G(\zeta)\,|d\zeta|}, \quad z \in \mathbb{C},$$

where we added the factor 2 in order to include also the case of $supp(\mu)$ being an arc. For any $P \in \mathcal{P}_n$, define the function

$$f_P(w) := \frac{P(\Psi_G(w))}{D_G(\Psi_G(w)) \cdot w^n}.$$

f_P is analytic in $\mathbb{D}^c$ including infinity, hence by the Cauchy integral formula we have for $|w| = r > \rho > 1$

$$|f_P(w)|^2 \le \frac{1}{2\pi}\left|\int_{|t|=\rho} \frac{f_P(t)^2}{t-w}\,dt\right| \le \frac{1}{2\pi} \cdot \frac{1}{r-\rho} \cdot \int_{\partial G_\rho} |f_P(\Phi_G(\zeta))|^2 \cdot |\Phi_G'(\zeta)|\,|d\zeta|.$$

By definition of the Szegö function D_G we know that, for almost all $\zeta_0 \in \partial G$, $|f_P(\Phi_G(\zeta))|^2 \cdot |\Phi_G'(\zeta)|$ tends to $|P(\zeta_0)|^2 \cdot w_G(\zeta_0)$ for $\zeta \to \zeta_0$. Consequently, for $|\Phi_G(z)| = r > 1$,

$$|P(z)|^2 \le \frac{r^{2n}}{r-1} \cdot |D_G(z)|^2 \cdot \int_{\partial G} |P(\zeta)|^2 \cdot w_G(\zeta)\,|d\zeta|$$

(the passage to the limit in the integral is justified by the theory of Hardy spaces, for details see, e.g., [SmLe68, Section 4.4.1]). On the other hand, for $w = r \cdot e^{is}$, $r > 1$, there holds

$$\Re \frac{w + e^{it}}{w - e^{it}} = \frac{r^2 - 1}{r^2 - 2r\cos(s-t) + 1} > 0,$$

hence the Szegö function may be estimated as follows

$$\log |D_G(\Psi_G(w))| = -\frac{1}{4\pi} \cdot \int_0^{2\pi} \left(\Re \frac{w + e^{it}}{w - e^{it}} \right) \cdot \log(w_G(\Psi_G(e^{it})))\, dt$$
$$\leq -\frac{\log(w_0)}{4\pi} \cdot \int_0^{2\pi} \frac{r^2 - 1}{r^2 - 2r\cos(s-t) + 1}\, dt = -\frac{\log(w_0)}{2}.$$

Summarizing we get for any polynomial $P \in \mathcal{P}_n$ by applying the maximum modulus principle for analytic functions

$$\max_{z \in G} |P(z)|^2 \leq \max_{|\Phi_G(z)| = 1 + 1/n} |P(z)|^2 \leq n \cdot (1 + 1/n)^{2n} \cdot \frac{1}{w_0} \cdot \int_{\partial G} |P(\zeta)|^2 \cdot w_G(\zeta)\, |d\zeta|,$$

or $K_n^\mu(z) \leq n \cdot 2e^2/w_0$ for all $z \in G$. Thus, part (b) again is a consequence of (3.8). □

For the rest of this section we will study the remaining case where $supp(\mu)$ does not 'surround' $supp(\nu)$, or $R := \Delta(supp(\nu), supp(\mu)) > 1$. Our aim is to establish (3.14) for a class of measures (μ, ν) satisfying (3.15). Here we require power asymptotics for $(p_n^\mu)_n$, and suppose (3.18) below. Furthermore, we will need some further information about the points of intersection of ∂G_R and ∂H. To simplify the presentation, we will restrict ourselves to the case of piecewise analytic ∂H, assume that w_H does not vanish on $\partial H \cap G_R$, and w_H is of class Lip_α in a neighborhood of this set for some $\alpha > 0$.

In order to describe the geometrical form of $\partial G_R \cap \partial H$, define on $I := \{t \in [0; 2\pi] : \Psi_H(e^{it}) \notin G\}$ the function

$$h(t) := |\Phi_G(\Psi_H(e^{it}))|. \tag{3.17}$$

By assumption, h is continuous, and $\log(h)$ is the real part of a function being piecewise analytic in an open neighborhood of I. Note that we have a point of intersection between ∂G_R and ∂H at $z_0 = \Psi_H(e^{i \cdot t_0})$ if and only if h has a maximum at t_0, with $h(t_0) = R$. Thus, $\{t \in I : h(t) = R\}$ must consist of a finite number of intervals $[a_j; b_j]$ and a finite number of isolated points c_j, and in any of the points a_j, b_j, c_j we may define a (right or left handed) multiplicity of the equation $h(t) = R$, denoted by $m_-(t)$ or $m_+(t)$. Notice that $m_+(a_j) = m_-(b_j) = \infty$, whereas the numbers $m_-(a_j)$, $m_+(b_j)$, $m_-(c_j)$ and $m_+(c_j)$ are even, positive integers.

We say that the sequence of orthonormal polynomials $(p_n^\mu)_n$ has a power asymptotic if there exists a function g, being analytic in $G^c \cup \{\infty\}$, and having no zeros there, such that

$$\frac{p_n^\mu(z)}{g(z) \cdot \Phi_G(z)^n} \to 1, \tag{3.18}$$

locally uniformly in $G^c = \mathbb{C} \setminus G$. Szegö established power asymptotic for ∂G being a real interval [Sze67, Theorem 12.1.2] and w_G satisfying (3.16), here $g(z) := D_G(z)/\sqrt{2\pi}$ (see also

[Sze67, Theorem 16.4]). As shown by Geronimus [Ger52], instead of intervals one may also allow curves ∂G where G is of class Γ (see the remark at the end of Section 2.4.2). Power asymptotics for ∂G being a $\mathcal{C}(2,\alpha)$–curve or arc are contained in [Wid69, Theorem 12.3]. Widom showed (in fact his assertion is much more general) that if also (3.13) holds then (3.18) is valid with $g(z) = \sqrt{\Phi_G'(z)/2\pi} \cdot \exp(-\tilde{g}(z)/2)$. Here, $\Re\tilde{g}$ is harmonic in $G^c \cup \{\infty\}$, with (limiting) boundary values $\Re\tilde{g}(\zeta) = \log w_G(\zeta)$ for $\zeta \in \partial G$, and $\Im\tilde{g}$ is its harmonic conjugate satisfying $\Im\tilde{g}(\infty) = 0$ (compare [Wid69, p.155, p.160]).

As a consequence of (3.18), we get

$$\frac{K_n^\mu(z)}{|\Phi_G(z)|^{2n+2}} \to \frac{|g(z)|^2}{|\Phi_G(z)|^2 - 1},$$

locally uniformly in G^c. Hence if we denote by ν_r, $r \geq 1$, the restriction of ν on the set $\{z \in G^c : |\Phi_G(z)| \geq r\}$, then using (3.8) for the measure $\nu - \nu_r$ we obtain

$$||M_n(\nu,\mu)||_2 = ||M_n(\nu_r,\mu)||_2 + \mathcal{O}(r^{2n})_{n\to\infty}, \quad r > 1, \tag{3.19}$$

which leads to the first estimate $||M_n(\nu,\mu)||_2 = \mathcal{O}(R^{2n})_{n\to\infty}$. To be more precise, we have examine the behaviour of ν_r for $r \to R-$.

Lemma 3.11 *Let Γ denote a subarc of ∂H with endpoints $z_j := \Psi_H(e^{it_j})$, $j = 1, 2$, $t_1 < t_2$, and $\Gamma \cap \partial G_R = \{z_1\}$, $m := m_+(t_1)$. Then*

$$\int_\Gamma |P(z)|^2 \cdot w_H(z)\,|dz| = \frac{|P(z_1)|^2}{(2n)^{1/m}} \cdot \Gamma(1+\frac{1}{m}) \cdot w_H(z_1) \cdot |\Psi_H'(e^{it_1+})| \cdot \left(\frac{R \cdot m!}{-h^{(m)}(t_1+)}\right)^{1/m} + o(\frac{R^{2n}}{n^{1/m}}),$$

uniformly for $P \in \mathcal{P}_n$ satisfying $\int |P(z)|^2\,d\mu(z) = 1$.

Proof: Let $P \in \mathcal{P}_n$ satisfying $\int |P(z)|^2\,d\mu(z) = 1$, and define $F(z) := P(z)/\Phi_G(z)^n$. We want to show that F^2 satisfies a Lipschitz condition on Γ with a Lipschitz constant independent of P, more precisely, we may give an upper bound for the derivative of F^2 on Γ which does not depend on P. First according to (3.19) we may assume without loss of generality that $\Gamma \subset \mathbb{C}\setminus G_r$ with an $r \in (1; R)$. Since F^2 is analytic in G^c including infinity, by the Cauchy integral formula it is sufficient to give a uniform bound for F^2 on ∂G_r. By (3.18), there exists a constant C such that $|p_k^\mu(z)| \leq C \cdot |\Phi_G(z)|^k = r^k$ for all $k \geq 0$ and for all $z \in \partial G_r$. Let $P = a_0 p_0^\mu + .. + a_n p_n^\mu$, then by the scaling of P we have $a_0^2 + .. + a_n^2 = 1$, and for $z \in \partial G_r$

$$|F(z)|^2 \leq r^{-2n} \cdot \sum_{j=0}^n |a_j|^2 \cdot \sum_{k=0}^n |p_k^\mu(z)|^2 \leq C \cdot \sum_{k=0}^n r^{2k-2n} \leq \frac{C}{1-r^{-2}}.$$

Thus it remains to determine the asymptotic of an integral of the form

$$I_n := \int_\Gamma |\Phi_G(z)|^{2n} \cdot d(z)\,|dz|, \quad d(z) = |F(z)|^2 \cdot w_H(z),$$

where $d \in \mathrm{Lip}_\alpha(\Gamma)$ for some $\alpha > 0$. Here we will apply the Laplace method. First, by eventually making Γ smaller, we may assume without loss of generality that $\Psi_H(e^{it})$ is differentiable on $(t_1; t_2)$, and h defined in (3.17) is strictly decreasing. Thus

$$I_n = R^{2n} \cdot \int_{t_1}^{t_2} \frac{h(t)^{2n}}{h(t_1)^{2n}} \cdot d_1(t)\, dt, \quad d_1(t) = d(\Psi_H(e^{it})) \cdot |\Psi_H'(e^{it})|,$$

with $d_1 \in \mathrm{Lip}_\alpha([t_1; t_2])$. We now introduce the new variable

$$s = s(t) = \left[\log \frac{h(t_1)}{h(t)}\right]^{1/m}, \quad d_2(s(t)) = \frac{d_1(t)}{s'(t)},$$

and notice that $s([t_1; t_2]) = [0; s_2]$, and $s'(t_1) = c^{1/m}$, where $c > 0$ is determined from

$$\log(\frac{h(t_1)}{h(t)}) \approx \log(1 + c \cdot (t - t_1)^m) \approx c \cdot (t - t_1)^m, \quad t \to t_1+, \quad c = -\frac{h^{(m)}(t_1+)}{R \cdot m!}.$$

It remains to discuss the integral

$$I_n = R^{2n} \cdot \int_0^{s_2} \exp(-2n \cdot s^m) \cdot d_2(s)\, ds$$

with $d_2 \in \mathrm{Lip}_\alpha([0; s_2])$, for which by elementary computations one establishes the asymptotics

$$I_n = \frac{R^{2n}}{(2n)^{1/m}} \cdot \Gamma(1 + \frac{1}{m}) \cdot (d_2(0) + o(1)_{n\to\infty}), \quad d_2(0) = \frac{d_1(t_1+)}{c^{1/m}} = \frac{d(z_1) \cdot |\Psi_H'(e^{it_1+})|}{c^{1/m}}.$$

□

Of course, Lemma 3.11 remains valid if $t_1 > t_2$, here we have to take $m = m_-(t_1)$, and the derivatives of h, Ψ_H are evaluated at $t = t_1-$. Let us denote by $\nu|_\Gamma$ the restriction of ν on Γ. Then by taking $P = p_n^\mu$ in Lemma 3.11 and using (3.18) we obtain

$$C_1 \cdot \frac{R^{2n}}{n^{1/m}} \le ||M_n(\nu|_\Gamma, \mu)||_2 + o(\frac{R^{2n}}{n^{1/m}}),$$

with a suitable constant C_1. Also, with a suitable constant C_2 we get

$$||M_n(\nu|_\Gamma, \mu)||_2 \le ||T_n(\nu|_\Gamma, \mu)||_F^2 \le C_2 \cdot \frac{R^{2n}}{n^{1/m}} + o(\frac{R^{2n}}{n^{1/m}})$$

by a superposition of Lemma 3.11, $P = p_k^\mu$, $k = 0, ., , n$. In particular, both estimates remain valid with $m = \infty$ if Γ is a subarc of $\partial H \cap \partial G_R$. We therefore have shown the following

LEMMA 3.12 *Let $H_0 := \partial H \cap \partial G_R$. If H_0 contains a proper arc, then*

$$\int_{\partial H} |P(z)|^2 \cdot w_H(z)\, |dz| = \int_{H_0} |P(z)|^2 \cdot w_H(z)\, |dz| + o(R^{2n}),$$

uniformly for $P \in \mathcal{P}_n$ satisfying $\int |P(z)|^2\, d\mu(z) = 1$.

Otherwise, denote by m the highest multiplicity of intersection, and let $z_1, .., z_s \in H_0$ be the corresponding intersection points. Furthermore, define for $j = 1, .., s$

$$v_j := \Gamma(1 + \frac{1}{m}) \cdot \left(\frac{R \cdot m!}{2}\right)^{1/m} \cdot w_H(z_j) \cdot \begin{cases} \tilde{h}(t_j+) & \text{if } m = m_+(t_j) \neq m_-(t_j), \\ \tilde{h}(t_j-) & \text{if } m = m_-(t_j) \neq m_+(t_j), \\ (\tilde{h}(t_j-) + \tilde{h}(t_j+)) & \text{if } m = m_-(t_j) = m_+(t_j), \end{cases}$$

where $z_j = \Psi_H(e^{it_j})$, and $\tilde{h}(t) = |\Psi_H'(t)| \cdot [-h^{(m)}(t)]^{-1/m}$. Then

$$\int_{\partial H} |P(z)|^2 \cdot w_H(z)\, |dz| = \sum_{j=1}^{s} \frac{|P(z_j)|^2}{n^{1/m}} \cdot v_j + o(\frac{R^{2n}}{n^{1/m}}),$$

uniformly for $P \in \mathcal{P}_n$ satisfying $\int |P(z)|^2\, d\mu(z) = 1$. □

Taking into account (3.8) and (3.18), the following assertion becomes immediate

COROLLARY 3.13 *With the notations of Lemma 3.11, if H_0 contains a proper arc, then*

$$||M_n(\nu,\mu)||_{Turing} = \sqrt{n} \cdot R^{2n} \cdot \Big(\int_{H_0} |g(z)|^2 \cdot w_H(z)\, |dz| + o(1)\Big),$$

$$||T_n(\nu,\mu)||_F^2 = \frac{R^{2n+2}}{R^2 - 1} \cdot \Big(\int_{H_0} |g(z)|^2 \cdot w_H(z)\, |dz| + o(1)\Big).$$

Otherwise, we have

$$||M_n(\nu,\mu)||_{Turing} = \sqrt{n} \cdot \frac{R^{2n}}{n^{1/m}} \cdot \Big(\sum_{j=1}^{s} |g(z_j)|^2 \cdot v_j + o(1)\Big),$$

$$||T_n(\nu,\mu)||_F^2 = \frac{R^{2n+2}}{R^2 - 1} \cdot \frac{1}{n^{1/m}} \cdot \Big(\sum_{j=1}^{s} |g(z_j)|^2 \cdot v_j + o(1)\Big).$$

□

Recall from (A.6), (A.7) of Appendix A that $||M_n(\nu,\mu)||_{Turing}/\sqrt{n} \leq ||M_n(\nu,\mu)||_2 \leq ||T_n(\nu,\mu)||_F^2$. Hence Corollary 3.13 gives already inclusions for $||M_n(\nu,\mu)||_2$ with bounds having the same asymptotic behavour as the right hand side of (3.14). In the case of finitely many points of intersection, we may even determine the corresponding constant C

THEOREM 3.14 *Consider measures μ, ν satisfying (3.15) and the assumptions formulated after Theorem 3.10, in particular $R = \Delta(H, G) > 1$. If $\partial G_R \cap \partial H$ does not contain a proper arc, then with m, v_j, z_j as in Lemma 3.12 there holds*

$$||M_n(\nu,\mu)||_2 = \frac{R^{2n+2}}{n^{1/m}} \cdot \Big(\sigma + o(1)_{n\to\infty}\Big),$$

where σ is the largest eigenvalue of the positive definite matrix

$$B := \Big(\frac{\sqrt{v_j \cdot v_k} \cdot \overline{g(z_j)} \cdot g(z_k)}{\overline{\Phi_G(z_j)} \cdot \Phi_G(z_k) - 1}\Big)_{j=1,..,s}^{k=1,..,s}.$$

Proof: Let us first show that

$$||M_n(\nu,\mu)||_2 = \frac{c_n + o(R^{-2n})}{n^{1/m}}, \quad c_n := \max_{a\in\mathbb{C}^{n+1}} \frac{1}{||a||_2^2} \cdot \sum_{j=1}^{s} v_j \cdot \left|\sum_{k=0}^{n} a_k g(z_j)\Phi_G(z_j)^k\right|^2 \tag{3.20}$$

In fact, for determining $||M_n(\nu,\mu)||_2$, we have to take in the second part of Lemma 3.12 the maximum over all polynomials of the form

$$P(z) = \sum_{k=0}^{n} a_k \cdot p_k^{\mu}(z), \quad |a_0|^2 + ... + |a_n|^2 = 1.$$

Writing $Q(z) := \sum_{k=0}^{n} a_k \cdot g(z) \cdot \Phi_G(z)^k$, we obtain

$$\begin{aligned} &\left| n^{1/m} \cdot \int_{\partial H} |P(z)|^2 \cdot w_H(z)\,|dz| - \sum_{j=1}^{s} v_j \cdot \left|\sum_{k=0}^{n} a_k g(z_j)\Phi_G(z_j)^k\right|^2 \right| \\ &\leq \sum_{j=1}^{s} v_j \cdot \left||P(z_j)|^2 - |Q(z_j)|^2\right| + o(R^{2n}), \end{aligned}$$

uniformly for $a \in \mathbb{C}^{n+1}$ satisfying $||a||_2 = 1$. Due to (3.18), there exists a constant C such that $|p_k^{\mu}(z)| \leq C \cdot |\Phi_G(z)|^k = C \cdot R^k$ and $|g(z)| < C$ for all $k \geq 0$ and for all $z \in H_0 := \partial H \cap \partial G_R$. Moreover, for all $\epsilon > 0$ there exists a $K > 0$ such that for all $k \geq K$ and for all $z \in H_0$ we have $|p_k^{\mu}(z) - g(z) \cdot \Phi_G(z)^k| \leq \epsilon \cdot R^k$. Using the Cauchy–Schwarz inequality, one verifies that, for $n' := [n/2] > K$,

$$\begin{aligned} &\left||P(z_j)|^2 - |Q(z_j)|^2\right| \leq (|P(z_j)| + |Q(z_j)|) \cdot |P(z_j) - Q(z_j)| \\ &\leq \frac{2 \cdot C \cdot R^{n+1} \cdot (2 \cdot C \cdot R^{n'+1} + \epsilon \cdot R^{n+1})}{R^2 - 1}. \end{aligned}$$

Since $\epsilon > 0$ may be chosen arbitrarily close to zero, it follows that $\left||P(z_j)|^2 - |Q(z_j)|^2\right| = o(R^{2n})$, uniformly for $a \in \mathbb{C}^{n+1}$ satisfying $||a||_2 = 1$. This proves (3.20).

By definition, c_n is the largest eigenvalue (namely the largest Rayleigh quotient) of the matrix $A^H \cdot D^H \cdot D \cdot A$ with

$$A = \left(\Phi_G(z_j)^k\right)_{j=1,..,s}^{k=0,..,n}, \quad D = diag\left(\sqrt{v_j} \cdot g(z_j)\right)_{j=1,..,s},$$

coinciding with the largest eigenvalue of

$$D \cdot A \cdot A^H \cdot D^H = \left(\sqrt{v_j \cdot v_k} \cdot \overline{g(z_j)} \cdot g(z_k) \cdot \frac{\overline{\Phi_G(z_j)}^{n+1} \cdot \Phi_G(z_k)^{n+1} - 1}{\overline{\Phi_G(z_j)} \cdot \Phi_G(z_k) - 1}\right)_{k=1,..,s}^{j=1,..,s}.$$

Hence with help of a similarity transformation with the matrix $diag\,(\Phi_G(z_j)^{n+1} \cdot R^{-n-1})_{j=1,..,s}$ we may conclude that $c_n = R^{2n+2} \cdot \sigma + \mathcal{O}(1)$, yielding the assertion. □

If (as in the case $H = \mathbb{D}$, $G = [0;1]$ of the Hilbert matrix) both sets G, H are symmetric with respect to the real axis, then in general there will be two complex conjugate points of intersection

$z_1 = \overline{z_2}$. If in addition the two weight functions w_G, w_H are invariant under conjugation of the arguments, then $v_1 = v_2$, and also $g(z_1) = \overline{g(z_2)}$. Consequently, the quantity σ of Theorem 3.14 is equal to

$$\sigma = v_1 \cdot |g(z_1)|^2 \cdot \left(\frac{1}{1-R^2} + \frac{1}{|1-\Phi_G(z_1)^2|} \right),$$

and we obtain as a particular case the result [Wif70, Theorem 3.2] of Wilf.

We summarize a weaker form of our findings in the following

Corollary 3.15 *Let μ, ν be measures of the form*

$$d\nu(z) = w_H(z)\,|dz| \text{ on } \operatorname{supp}(\nu) = \partial H, \quad d\mu(z) = w_G(z)\,|dz| \text{ on } \operatorname{supp}(\mu) = \partial G,$$

where G, H are simply connected compact subsets of the complex plane, with boundaries being an analytic arc or curve. Furthermore, suppose that the density functions w_H, w_G are strictly positive, and of class Lip_α for some $\alpha > 0$. Then there exist constants $C_1, C_2 > 0$ such that for all $n \geq 0$

$$C_1 \cdot n^{\ell_1} \leq \frac{\kappa_2(M_n(\mu,\nu))}{\Delta(G,H)^{2n} \cdot \Delta(H,G)^{2n}} \leq C_2 \cdot n^{\ell_2}.$$

Here $\ell_1 = \ell_2 = 0$ in the case $H = G$, $\ell_1 = -1$, $\ell_2 = 0$ in the case $G \not\subset H \not\subset G$, and $\ell_1 = -1/2$, $\ell_2 = 1$ otherwise.

Proof: One verifies that the assumptions of either Theorem 3.10(b) or Theorem 3.14 hold for (μ, ν) and also for (ν, μ) (the case $H = G$ is covered by (3.10) and (3.5)). Thus the assertion follows by recalling that $\kappa_2(M_n(\mu,\nu)) = ||M_n(\mu,\nu)||_2 \cdot ||M_n(\nu,\mu)||_2$. □

Example 3.16 *Suppose that we want to compute the polynomials being orthonormal with respect to the measure*

$$d\mu(z) = |dz| \text{ on } G = \{z \in \mathbb{C} : |z| = 1, \Im z \geq 0\},$$

i.e., we look for polynomials p_n^μ satisfying

$$\int_0^\pi \overline{p_k^\mu(e^{it})} \cdot p_n^\mu(e^{it})\,dt = \delta_{k,n}.$$

For the case of orthogonalization of the monomials we have to consider the moment matrix $M_n(\mu,\nu)$, where

$$d\nu(z) = \frac{1}{2\pi}|dz| \text{ on } \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

Notice that $M_n(\mu,\nu)$ is a positive definite Hermitian Toeplitz matrix, namely

$$M_n(\mu,\nu) = \begin{pmatrix} \pi & 2i/1 & 0 & 2i/3 & 0 & 2i/5 & \cdots \\ -2i/1 & \pi & 2i/1 & 0 & 2i/3 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

First $\mathrm{supp}(\mu) \subset \mathrm{supp}(\nu)$, *hence from Theorem 3.10 (interchange the role of* μ *and* ν*) we may conclude that* $||M_n(\mu,\nu)||_2 = \mathcal{O}(n)$. *In fact, we have* $K_n^\nu(z) = 1 + |z|^2 + ... + |z|^{2n}$, *and with help of (3.8) we obtain*

$$||T_n(\mu,\nu)||_F^2 = \int_0^\pi K_n^\nu(e^{it})\,dt = \pi \cdot (n+1).$$

We now consider the asymptotics of $||M_n(\nu,\mu)||_2$ *where we want to apply Theorem 3.14. In accordance with (3.15), we have* $H = \mathbb{D}$ *being sufficiently smooth,* $w_H(z) = 1/2\pi$, $w_G(z) = 1$. *Also,* $\Psi_H(z) = z$, $\Phi_H(w) = w$, *whereas for the set* G *one obtains the inverse Riemann map*

$$z = \Psi_G(w) = -i + \frac{2}{J(s) - i}, \quad s = \frac{1 + (1+\sqrt{2}) \cdot i \cdot w}{w - (1+\sqrt{2}) \cdot i},$$

where J *denotes the Joukowski map (we may neglect the additional condition* $\Psi_G'(\infty) > 0$ *leading only to an additional factor of modulus 1). Now* $R = \Delta(\mathbb{D}, G)$ *is given as the smallest parameter satisfying* $|\Psi_G(R \cdot e^{it})| \geq 1$ *for all* $t \in [0; 2\pi]$. *One verifies that* $R = 1 + \sqrt{2}$, *and that there is only one point of intersection between* ∂H *and* ∂G_R, *namely*

$$z_1 = -i = \Psi_G(R \cdot e^{i \cdot \pi/2}) = \Psi_H(e^{i \cdot 3\pi/2}),$$

and therefore $t_1 = 3\pi/2$. *The expansion at* t_1 *of the function* h *defined in (3.17) is given by*

$$h(t) = R - \frac{2+\sqrt{2}}{8} \cdot (t-t_1)^2 - \frac{4-\sqrt{2}}{384} \cdot (t-t_1)^4 + \mathcal{O}((t-t_1)^6).$$

Hence we obtain the maximal multiplicity of intersection $m = 2$, *and the quantity* v_1 *of Lemma 3.12 is given by*

$$v_1 = \Gamma(\frac{3}{2}) \cdot \sqrt{R} \cdot \frac{1}{2\pi} \cdot \frac{2}{\sqrt{-h^{(2)}(t_1)}} = \Gamma(\frac{3}{2}) \cdot \frac{1}{2\pi} \cdot \frac{4}{2^{1/4}} = \frac{1}{\pi^{1/2} \cdot 2^{1/4}}.$$

Notice also that (3.13) holds and G *is a sufficiently smooth arc. Thus we obtain (3.18) with* $g(z) = \sqrt{\Phi_G'(z)/2\pi}$, *and*

$$|g(z_1)|^2 = \frac{|\Phi_G'(-i)|}{2\pi} = \frac{1}{2\pi|\Psi_G'(R \cdot i)|} = \frac{R}{4\pi}.$$

Together, we have

$$\sigma = \frac{v_1 \cdot |g(z_1)|^2}{R^2 - 1} = \frac{v_1}{8\pi} = \frac{1}{\pi^{3/2} \cdot 2^{13/4}},$$

and Theorem 3.14 leads to

$$||M_n(\nu,\mu)||_2 = \frac{R^{2n+2}}{\sqrt{n}} \cdot \left(\pi^{-3/2} \cdot 2^{-13/4} + o(1)_{n\to\infty}\right), \quad R = 1 + \sqrt{2}.$$

Thus the underlying moment matrix is a positive definite Toeplitz matrix, with the square of its condition number growing roughly as fast as the condition number of the Hilbert matrix (see Example 3.3). □

Chapter 4

Vandermonde–like and Krylov–like matrices

It is well–known that perturbations in the coefficients $a_0, .., a_n$ of a polynomial $P(z) = a_0 + a_1 z + .. + a_{n-1} z^n$ might lead to significantly magnified perturbations of values of P at some arguments $z_0, .., z_m \in \mathbb{C}$. To be more precise, one has to study the condition number of the *Vandermonde matrix*

$$\begin{pmatrix} 1 & z_0 & z_0^2 & \cdots & z_0^n \\ 1 & z_1 & z_1^2 & \cdots & z_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & z_m & z_m^2 & \cdots & z_m^n \end{pmatrix}.$$

This matrix (or its transposed) occurs beside polynomial interpolation also in other applications, e.g., the determination of Christoffel numbers for a Gaussian quadrature rule, the interpolatory approximation of linear functionals, or the discretization of differential equations. Instead of the monomial, one often chooses other more suitable bases of the space of polynomials. The aim of this chapter is to study the condition number of matrices of the form

$$V_{m,n}(d) = V_n(d, z_0, .., z_m) = \begin{pmatrix} \frac{p_0(z_0)}{d(z_0)} & \frac{p_1(z_0)}{d(z_0)} & \frac{p_2(z_0)}{d(z_0)} & \cdots & \frac{p_n(z_0)}{d(z_0)} \\ \frac{p_0(z_1)}{d(z_1)} & \frac{p_1(z_1)}{d(z_1)} & \frac{p_2(z_1)}{d(z_1)} & \cdots & \frac{p_n(z_1)}{d(z_1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{p_0(z_m)}{d(z_m)} & \frac{p_1(z_m)}{d(z_m)} & \frac{p_2(z_m)}{d(z_m)} & \cdots & \frac{p_n(z_m)}{d(z_m)} \end{pmatrix}, \qquad (4.1)$$

where $(p_k)_k$ is a sequence of polynomials, p_k of degree k, $z_0, .., z_m \in G \subset \mathbb{C}$, the density function $d \in \mathcal{C}(G)$ takes only positive values, and $m \geq n$. Following Gautschi [Gau90], we will refer to $V_{m,n}(d)$ as a *weighted Vandermonde–like matrix*, reducing to a *weighted Vandermonde matrix* in the case $p_k(z) = z^k$, $k \geq 0$, and reducing to a *Vandermonde–like matrix* in the case $d(z) = 1$ (for convenience, we prefer to take the transposed of the matrices introduced by Gautschi).

The occurrence of a weight function d will be natural in the context of Krylov–like matrices, see Section 4.2. Since

$$V_{m,n}(d) = diag\ (d(z_0), .., d(z_m))^{-1} \cdot V_{m,n}(1),$$

we may understand a weighted Vandermonde–like matrix as a (non–weighted) Vandermonde-like matrix being preconditioned by multiplication on the left with a diagonal matrix.

Our main interest in this chapter will be to derive estimates and asymptotic lower bounds for the condition number of $V_{m,n}(d)$ induced by the Hölder vector p–norm

$$\kappa_p(V_{m,n}(d)) = ||V_{m,n}(d)||_p \cdot ||V_{m,n}(d)^+||_p,$$

where $V_{m,n}(d)^+$ is the pseudo–inverse of $V_{m,n}(d)$. However, in order to be able to apply results from Chapter 2, we want to make use of formulas (A.8), (A.10) and (A.11) of Appendix A. Hence we will restrict our attention to either the case of square matrices (i.e., $m = n$ and hence $V_{m,n}(d)^+ = V_{n,n}(d)^{-1}$, see Section 4.1) or to the Euclidean norm (i.e., $p = 2$, see Section 4.2).

In the case $m = n$ of square matrices, equations (A.8) and (A.10) may be summarized in the form

$$(n+1)^{-1/p} \cdot \kappa_{p,\infty}(V_{n,n}(d)) \leq \kappa_p(V_{n,n}(d)) \leq (n+1)^{1/p} \cdot \kappa_{p,\infty}(V_{n,n}(d)), \tag{4.2}$$

where $\kappa_{p,\infty}(V_{n,n}(d)) = ||V_{n,n}(d)^{-1}||_{\infty,p} \cdot ||V_{n,n}(d)||_{p,\infty}$. We prefer to use polynomial language in order to evaluate the latter quantity. As in (2.1), let $\Pi_n : \mathbb{C}^{n+1} \to \mathcal{P}_n$ denote the coordinate map corresponding to the sequence $(p_k)_k$. Here we equip the domain of Π_n with the p–Hölder norm, whereas for the range $\mathcal{P}_n$ we take the weighted maximum norm

$$||f||_{d,G} := \max_{z\in G} \frac{|f(z)|}{d(z)}, \quad f \in \mathcal{C}(G),$$

with $G \subset \mathbb{C}$ being compact. Recall from Appendix A that the matrix norm $||B||_{p,\infty}$ is obtained as the maximum of the q–norms of the rows of B, where $q \in [1;\infty]$ is the Hölder index being complementary to p, i.e., $1/p + 1/q = 1$. Defining the Kernel function $K_{p,n}(z) = ||(p_0(z), .., p_n(z))^T||_q = (\sum_{k=0}^{n} |p_k(z)|^q)^{1/q}$, we get similarly as in (2.2)

$$||V_{n,n}(d)||_{p,\infty} = ||K_{p,n}||_{d,G_0}, \quad ||V_{n,n}(d)^{-1}||_{\infty,p} = \max_{P\in\mathcal{P}_n} \frac{||\Pi_n^{-1} P||_p}{||P||_{d,G_0}}, \tag{4.3}$$

with $G_0 = \{z_0, .., z_n\}$.

For weighted Vandermonde–like matrices, the following problems seem to be of particular interest

Problem (A) Given d, $(p_k)_k$, and a family of nodes $(z_{j,n})_{0\leq j\leq n}$, (e.g., equidistant nodes on the interval $[-1;1]$), find the asymptotics of $(\kappa_p(V_n(d, z_{0,n}, .., z_{n,n})))_n$.

Problem (B) Given d, $(p_k)_k$, $n \geq 0$, find nodes $z_0, .., z_n$ such that $\kappa_p(V_n(d, z_0, .., z_n))$ is 'small'.

Problem (C) Given d, $(p_k)_k$, $n \geq 0$, and a compact set $G \subset \mathbb{C}$, find a lower bound for $\kappa_p(V_n(d, z_0, .., z_n))$ being valid for each $z_0, .., z_n \in G$, which is approximately attained for particular nodes $z_0, .., z_n \in G$.

Problem (D) Given $(p_k)_k$, $m \geq n \geq 0$, and a compact set $G \subset \mathbb{C}$, find a lower bound for $\kappa_2(V_n(d, z_0, .., z_m))$ being valid for each $z_0, .., z_m \in G$ and for each density function d, which is approximately attained for particular nodes $z_0, .., z_m \in G$ and for a particular density function.

This chapter is organized as follows: In Section 4.1 we review classical results concerning Problems (A) and (B) obtained by Gautschi, Taylor, Tyrtyshnikov, Cordova, Ruscheweyh, Reichel, Fischer, and Opfer. By specifying results of Section 2.2, it is shown that properties of the Lebesgue function corresponding to the considered nodes $z_0, .., z_n$ are basic for a solution of Problems (A)–(C) in the case $d = 1$. In particular, we point out the connections between Problem (B) and the problem of finding nodes minimizing the Lebesgue number. In order to treat the weighted case $d \neq 1$, we introduce a weighted analogue of the Lebesgue function, and provide inequalities for particular sequences of polynomials such as Faber polynomials. Finally, we outline how potential theory may be applied for investigating the nth root behaviour of solutions of problems (A)–(C). However, the aim of this section is not to give a complete account of Problems (A)–(C), but instead to motivate the techniques used in the following sections.

In Section 4.2 we study the problem of giving approximately tight lower bounds for the Euclidean condition number of Krylov–like matrices, such as a matrix built up with the columns $B^j \cdot b$, $j = 0, .., n$, where B is a normal matrix with spectrum contained in some compact set G, and b is an arbitrary vector of suitable size. These matrices occur for instance while solving systems of linear equations by some iterative methods. We discuss the connections with Problem (D), which is shown to be closely related to Problem (C) for a particular 'optimal' density function. In addition, we derive approximately tight lower bounds in terms of the solution of Problem (D) for the condition number of particular modified moment matrices $M_n(\mu, \nu)$, namely so–called Hankel–like matrices where one of the measures is fixed, and the other one is known to be supported on some subset of the real line.

Finally, in Section 4.3 we provide a new explicit formula for the limit of the nth root of the solution of Problem (D). For a proof, we require some elements of Potential theory as described in Appendix B. Our result is illustrated by considering several classes of Krylov and Krylov–Chebyshev matrices. In this context let us mention that the main assertion of Section 4.3 will be substantially improved for some particular cases in Section 5.3 and Section 5.4.

4.1 Some Examples of square Vandermonde–like matrices

In the first part of this section, let us study (non–weighted) Vandermonde matrices, i.e., $p_k(z) = z^k$, $k \geq 0$, and $d = 1$. An answer to Problem (B) is immediate, at least for the Euclidean norm

EXAMPLE 4.1 *Let* ω_n *denote the primitive* $n+1$*th root of unity,* $V_n := V_n(1, \omega_n^0, \omega_n^1, .., \omega_n^n)$. *One verifies without difficulty that* $V_n^H \cdot V_n = (n+1) \cdot I_n$. *In particular, all singular values of* V_n *(square root of eigenvalues of* $V_n^H \cdot V_n$*) are equal to* $\sqrt{n+1}$*, and therefore* $\kappa_2(V_n) = 1$*, the best possible value.* □

In contrast, Taylor [Tay78], Gautschi [GaIn88] and Tyrtyshnikov [Tyr94a] (see the remarks after Theorem 5.8) showed that, for any family $(z_{j,n})_{0 \le j \le n}$ of real nodes, the sequence $(\kappa_p(V_n(d, z_{0,n}, .., z_{n,n})))_n$ is at least exponentially increasing. We will study Problems (C), (D) for $G \subset \mathbb{R}$ in detail in Chapter 5. With regard to Problem (A), let us mention the following examples studied by Gautschi [Gau90, Examples 3.1–3.4]

EXAMPLE 4.2 *In the case of harmonic nodes, equidistant nodes on* $[0;1]$*, equidistant nodes on* $[-1;1]$*, and of nodes being the zeros of the Chebyshev polynomial* T_{n+1}*, respectively, we obtain for the* 1*–condition number of the Vandermonde matrix*

$$\kappa_1(V_n(1; z_{0,n}, .., z_{n,n})) \begin{cases} > n^{n+1} & \text{if } z_{j,n} = 1/(j+1) \\ = \frac{2\cdot\sqrt{2}}{\pi} \cdot 8^n \cdot (1 + o(1)_{n\to\infty}) & \text{if } z_{j,n} = j/n, \\ = \frac{\sqrt{2}}{\pi} \cdot (\sqrt{2} \cdot \exp(\pi/4))^n \cdot (1 + o(1)_{n\to\infty}) & \text{if } z_{j,n} = -1 + 2j/n, \\ = \frac{3^{3/4}}{4} \cdot (1+\sqrt{2})^{n+1} \cdot (1 + o(1)_{n\to\infty}) & \text{if } z_{j,n} = \cos(\pi \cdot \frac{2j+1}{2n+2}). \end{cases}$$

□

Let us also mention that the above expressions are obtained by exploiting explicit expressions for $||V_n(1, z_0, .., z_n)^{-1}||_1$ in terms of Lagrange polynomials

$$\ell_{j,n}(z) = \prod_{k=0, k\neq j}^{n} \frac{z - z_{k,n}}{z_{j,n} - z_{k,n}}. \tag{4.4}$$

Such formulas have been given by Gautschi for the case where all nodes are real and non–negative, and for the case of real nodes located symmetrically with respect to the origin, i.e., $z_{j,n} = -z_{n-j,n} \in \mathbb{R}$, $j = 0, .., n$ [Gau75a].

In order to reduce the number of function evaluations, it is of particular interest for applications to have well–conditioned Vandermonde matrices for a linear array of nodes $z_{j,n} = z_j$, $j \ge 0$, instead of a triangular array as discussed in Example 4.1. Such an array was given by Cordova, Gautschi and Ruscheweyh [CGR90],

EXAMPLE 4.3 *We consider the Van der Corput enumeration of the set of* 2^r*th roots of unity,* $r \ge 0$*: for an integer* $j \ge 0$*, let us define the angle* $\phi_j \in [0; 2\pi)$ *by bit reversal of the binary representation of* j

$$\phi_j := 2\pi \cdot \sum_{k=0}^{\infty} 2^{-k-1} \cdot j_k, \quad \text{where } j = \sum_{k=0}^{\infty} j_k \cdot 2^k, j_k \in \{0,1\}.$$

It may be verified that $(z_j)_{j\geq 0}$, $z_j := \exp(i\cdot\phi_j)$, forms a sequence of Leja points for the unit disk (see (2.15), [Rei90, Example 1.3]). Note that if n is a power of 2, then $z_0,..,z_{n-1}$ are the nth roots of unity, and $\kappa_2(V_{n-1}(1,z_0,..,z_{n-1})) = 1$. More generally, Cordova, Gautschi and Ruscheweyh determined the singular values of $V_n(1,z_0,..,z_n)$ for all $n\geq 0$, and showed in particular [CGR90, Corollary 3] that for all $n\geq 0$

$$\kappa_2(V_n(1,z_0,..,z_n)) \leq \sqrt{2(n+1)}.$$

□

Let us now turn to Vandermonde–like matrices built up with more general bases of polynomials, where in a first step we restrict ourselves to the non–weighted case $d=1$. We will suppose here and in the sequel of this chapter that the corresponding sequence of coordinate maps Π_n is well–conditioned with respect to some compact set H, i.e., the sequence $(\kappa_H(\Pi_n))_n$ of condition numbers has modest growth. Thus, by (2.3), both quantities $||\Pi_n||_H$ and $||\Pi_n^{-1}||_H$ grow at most with the same rate. This assumption holds for instance for H–Faber polynomials such as monomials ($H=\mathbb{D}$), and shifted Chebyshev polynomials ($H=\alpha+\beta\cdot\mathcal{E}_\rho$), see Example 2.3, and Section 2.4.3, respectively. A further example is given by polynomials being orthonormal with respect to some completely regular measure with support H, see Example 2.10 and Example 3.9.

From Theorem 2.6 we may conclude that

$$\begin{aligned}\frac{1}{\Delta_n(\{z_0,..,z_n\},H)} &\leq \frac{||V_n(1;z_0,..,z_n)^{-1}||_\infty}{||\Pi_n^{-1}||_H} \leq \Delta_n(H,\{z_0,..,z_n\}),\\ \frac{1}{\Delta_n(H,\{z_0,..,z_n\})} &\leq \frac{||V_n(1;z_0,..,z_n)||_\infty}{||\Pi_n||_H} \leq \Delta_n(\{z_0,..,z_n\},H).\end{aligned} \tag{4.5}$$

This enables us to propose a solution of Problem (B): In order to keep $||V_n(1;z_0,..,z_n)||_\infty$ small, one should take $z_0,..,z_n\in H$ and thus $\Delta_n(\{z_0,..,z_n\},H)=1$. On the other hand, the problem of choosing $z_0,..,z_n\in H$ with small $\Delta_n(H,\{z_0,..,z_n\})$ is a classical problem of interpolation theory: for each polynomial $P\in\mathcal{P}_n$ satisfying $|P(z_j)|\leq 1$, $0\leq j\leq n$ we obtain with help of the Lagrange polynomials (4.4)

$$|P(z)| = \left|\sum_{j=0}^n P(z_j)\cdot\ell_{j,n}(z)\right| \leq \sum_{j=0}^n |\ell_{j,n}(z)|$$

with equality if $|P(z_j)|=1$, $P(z_j)\cdot\ell_{j,n}(z)\geq 0$. Consequently,

$$\Delta_n(z,\{z_0,..,z_n\}) = \sum_{j=0}^n |\ell_{j,n}(z)|,$$

the classical *Lebesgue function*, and $\Delta_n(H,\{z_0,..,z_n\})$ equals the corresponding *Lebesgue constant*. Hence we are left with the problem of finding nodes with a small Lebesgue constant, and some possible choices are given in Example 4.4. Recall that, even in the case of a real interval H, a configuration minimizing the Lebesgue constant is not explicitly known.

EXAMPLE 4.4 **(a)** *For the real interval $H = [-1; 1]$ we have [Sch71, Satz 5.1 and Eqn.(5.60)]*

$$\Delta_n(H, \{z_0, .., z_n\}) \geq \frac{2\log(n)}{\pi^2}, \quad \Delta_n(H, \{z_{0,n}, .., z_{n,n}\}) \leq \frac{2\log(n)}{\pi} + \mathcal{O}(1)_{n\to\infty},$$

where $z_0, .., z_n$ are arbitrary distinct elements of H, and $z_{j,n} = \cos(\pi \cdot j/n)$, $j = 0, .., n$, are extremal points of T_n (these properties are easily adapted to arbitrary segments by taking into account that $\Delta_n(H, \{z_0, .., z_n\})$ is invariant under linear transformations of the complex plane).

(b) *As a generalization of Example 4.1, the nth Fejer nodes of a simply connected compact set H with inverse Riemann map Ψ_H are defined (up to a real constant α) by $z_{j,n-1} := \Psi_H(\exp(i \cdot (2\pi j/n + \alpha)))$, $0 \leq j < n$. Reichel showed [Rei85, Theorem 2.1] that the Lebesgue constant corresponding to Fejer nodes grows at most as $(2/\pi) \cdot \log(n) + \mathcal{O}(1)$, provided that H has an analytic boundary (the growth $\mathcal{O}(\log(n))$ may be also established under weaker assumptions on ∂H).*

(c) *Fischer and Reichel [FiRe89] considered Newton interpolation at nodes obtained by a Van der Corput ordering of the 2^rth Fejer nodes, $r \geq 0$. More precisely, let $z_{j,n} = z_j = \Psi_H(\exp(i \cdot \phi_j))$, $j \geq 0$, with ϕ_j as in Example 4.3. By slightly extending the considerations in the proof of [FiRe89, Lemma 2.5], one shows that $\Delta_n(H, \{z_0, .., z_n\}) = \mathcal{O}(n^s)_{n\to\infty}$ for some $s > 0$, provided that Φ_H' is nonvanishing and of bounded variation on ∂H.*

(d) *For an arbitrary compact set H, the* Fekete nodes *(or* Vandermonde points*) $z_{0,n}, .., z_{n,n} \in H$ are obtained as the arguments where* det $V_n(1, z_0, .., z_n)$ *as a function of $z_0, .., z_n \in H$ attains its maximum. For the corresponding Lebesgue constant there holds $\Delta_n(H, \{z_0, .., z_n\}) \leq n + 1$ (see [SmLe68, Section 1.3.2] or Corollary B.18).*

□

As an illustration of the above remarks, let us mention the following result of Reichel and Opfer [ReOp91, Theorem 3.4] dealing again with a linear array of nodes: Let $p_0 = 1$, and $p_n = 2 \cdot \rho^{-n} \cdot T_n$ for $n \geq 1$, the Faber polynomials of the ellipse $\mathcal{E}_\rho$, $\rho > 1$. Furthermore, let $z_j = J(\rho \cdot \exp(i \cdot \phi_j)$, $j \geq 0$, as described in Example 4.4(c). Then $\kappa_\infty(V_n(1, z_0, .., z_n)) = \mathcal{O}(n^s)$ for some $s > 0$ (this assertion follows from a combination of Example 4.4(c) and (4.5) with the results of Section 2.4.3, whereas the original proof given in [ReOp91] makes use of the corresponding Newton basis).

The following example was considered by Gautschi [Gau90, Section V]

EXAMPLE 4.5 *Let μ be a measure supported on the real line. We consider the corresponding sequence of orthonormal polynomials $p_n = p_n^\mu$, and nodes $z_{0,n}, .., z_{n,n}$ being the (distinct and real) zeros of p_{n+1}^μ. Then*

$$\kappa_2(V_n(1, z_{0,n}, .., z_{n,n})) = \frac{\max_j \sqrt{K_n^\mu(z_{j,n})}}{\min_j \sqrt{K_n^\mu(z_{j,n})}}.$$

This follows at once by observing that the Christoffel numbers of the corresponding Gaussian quadrature rule (see, e.g., [Sze67, Theorem 3.4.2]) are given by $\lambda_{j,n} = 1/K_n^\mu(z_{j,n})$, $0 \leq j \leq n$, *and hence* $V_n(\sqrt{K_n^\mu}, z_{0,n}, .., z_{n,n})$ *is an unitary matrix.*

A particular role is played by the Chebyshev measure $d\mu(z) = (1 - z^2)^{-1/2}$ *on* $[-1; 1]$ *since here all Christoffel numbers coincide, and thus* $\kappa_2(V_n(1, z_{0,n}, .., z_{n,n})) = 1$. □

We conclude this section with a discussion of problems (A) and (C) for the two particular cases of polynomials mentioned above, namely H–Faber polynomials, and orthonormal polynomials w.r.t. a measure μ with support H. Recall that (4.3) gives a quite handy expression for the norm of a weighted Vandermonde matrix, namely

$$||V_n(d, x_0, .., x_n)||_{p,\infty} = \max_{0 \leq j \leq n} \frac{K_{p,n}(z_j)}{d(z_j)} = \max_{0 \leq j \leq n} \frac{||(p_0(z_j), .., p_n(z_j))^T||_q}{d(z_j)}, \tag{4.6}$$

$1/p + 1/q = 1$, which is easily exploitable if the growth of the sequence of polynomials is known. However, there is some need to express the norm of the inverse in terms of quantities for which inequalities or asymptotics are available. Adapting the above reasoning for the case of a non–trivial weight function, we have to switch a weighted maximum norm. Thus — in generalization of Definition 2.5 — we introduce for compact sets H, G and for density functions $h \in \mathcal{C}(H)$, $g \in \mathcal{C}(G)$

$$\Delta_n(h, H; g, G) := \max\{\frac{||P||_{h,H}}{||P||_{g,G}} : P \in \mathcal{P}_n\},$$

in particular $\Delta_n(H, G) = \Delta_n(1, H; 1, G)$. Also, we have a weighted analogue of the Lebesgue function

$$\Delta_n(1, \{z\}; d, \{z_{0,n}, .., z_{n,n}\}) = \sum_{j=0}^{n} |d(z_{j,n})| \cdot |\ell_{j,n}(z)|. \tag{4.7}$$

The nth root behavour of this quantity is studied in Sections B.2.2 and B.3.3 of Appendix B, and in Section B.3.2 we give a weighted analogue of Example 4.4(d).

Example 4.6 *Consider the basis of Faber polynomials* $p_n = F_{H,n}$, $n \geq 0$, *with respect to some simply connected compact set* H. *Let* $P \in \mathcal{P}_n$, *then* $||P||_H \leq ||K_{p,n}||_H \cdot ||\Pi_n^{-1} P||_p$ *by the Hölder inequality. Furthermore, one shows as in Section 2.4.3 using biorthogonality that* $||\Pi_n^{-1} P||_p \leq (n+1)^{1/p} \cdot ||P||_H$. *Combining these estimates with (4.3), we obtain for* $n \geq 0$

$$\frac{1}{||K_{p,n}||_H} \leq \frac{||V_n(d, z_0, .., z_n)^{-1}||_{\infty,p}}{\Delta_n(1, H; d, \{z_0, .., z_n\})} \leq (n+1)^{1/p}.$$

Moreover, by Lemma 2.15,

$$||K_{p,n}||_H \leq [\sum_{j=0}^{n} ||F_{H,j}||_{\partial H}^q]^{1/q} \leq [\sum_{j=0}^{n} (\epsilon_{H,j} + 1)^q]^{1/q} = \mathcal{O}((n+1)^{3/2-1/p})_{n \to \infty},$$

showing that the norm of the inverse of such a weighted Vandermonde–like matrix behaves essentially like a 'weighted Lebesgue constant'.

Estimates for the condition number of such matrices may now be obtained by applying (4.6), where we possibly require the asymptotic behavior of $K_{p,n}$ outside of H. Here bounds may be given by applying the maximum principle (compare Theorem 2.14): for $z \notin H$ there holds $\Delta(z,H) = |\Phi_H(z)|$, and thus

$$|K_{p,n}(z) - ||(1,\Delta(z,H),...,\Delta(z,H)^n)^T||_q| \leq [\sum_{j=0}^{n} |F_{H,j}(z) - \Phi_H(z)^j|^q]^{1/q}$$

$$\leq [\sum_{j=0}^{n} ||F_{H,j} - \Phi_H^j||_{\partial H}^q]^{1/q} = \mathcal{O}((n+1)^{3/2-1/p})_{n\to\infty}.$$

□

With regard to Problem (A), let us mention one application

Example 4.7 *We consider Vandermonde–like matrices obtained by the Faber polynomials of the interval $[-1;1]$ together with equidistant nodes on $[-1;1]$, i.e., $p_0 = 1$, $p_n(z) = 2T_n(z)$ for $n \geq 1$, and $z_{j,n} = -1 + 2j/n$, $0 \leq j \leq n$. It follows from (4.3) that*

$$||V_n(1,z_{0,n},..,z_{n,n})||_{p,\infty} = \max_{0\leq j\leq n} K_{p,n}(x_{j,n}) = \max_{0\leq j\leq n}[1 + \sum_{k=1}^{n} |2\cdot T_k(x_{j,n})|^q]^{1/q} = (1 + 2^q\cdot n)^{1/q}.$$

Therefore, the growth of $\kappa_{p,\infty}(V_n(1,z_{0,n},..,z_{n,n}))$ essentially coincides with the grow of the Lebesgue constant of equidistant nodes on $[-1;1]$. This quantity (being for instance basic for the so–called Runge phenomenon in polynomial interpolation) has been discussed by several authors, e.g., (Turetskii 1940)

$$\Delta_n([-1;1],\{z_{0,n},..,z_{n,n}\}) = \frac{2^{n+1}}{e\cdot n\cdot \log n}\cdot(1 + o(1)_{n\to\infty}).$$

□

Example 4.8 *Let $p_n = p_n^\mu$, $n \geq 0$, be orthonormal with respect to some measure μ with compact $H := \mathrm{supp}(\mu)$. We have seen in Examples 2.2 and 2.10 that the corresponding coordinate map is not necessarily well–conditioned with respect to the maximum norm on H, here a weighted maximum norm is more appropriate. In fact, with help of the Cauchy–Schwarz inequality one obtains $||\Pi_n(a)||_{K_{2,n},H} \leq ||a||_2$ for all $a \in \mathbb{C}^{n+1}$ (recall that $K_{2,n} = \sqrt{K_n^\mu}$, with the Szegö kernel K_n^μ). Moreover, for each $P \in \mathcal{P}_n$,*

$$||\Pi_n^{-1}P||_2^2 = \int |P(z)|^2\, d\mu(z) \leq ||P||_{K_{2,n},H}^2 \cdot \int K_n^\mu(z)\, d\mu(z) = ||P||_{K_{2,n},H}^2\cdot(n+1).$$

Thus we obtain using (4.3)

$$1 \leq \frac{||V_n(d,z_0,..,z_n)^{-1}||_{\infty,2}}{\Delta_n(K_{2,n},H;d,\{z_0,..,z_n\})} \leq (n+1)^{1/2}, \quad n \geq 0.$$

□

Let us suppose for the case of Example 4.8 in addition that the nth root of the Szegö kernel converges locally uniformly in $\mathbb{C}$ (the locally uniform convergence of the nth root of the Faber kernel $K_{p,n}$ of Example 4.6 to the function $\Delta(\cdot, H)$ is easily established). Then, with regard to Problem (A), we may determine the nth root behaviour of $\kappa_p(V_n(d_n, z_{0,n}, .., z_{n,n}))$ for a given family of distinct nodes $(z_{j,n})_{0\leq j\leq n} \subset G$, and a sequence of density functions $(d_n)_n \subset \mathcal{C}(G)$ with $d_n^{1/n} \to d_\infty$ uniformly in G. Here we use (4.2), (4.6), and apply the results on the nth root behaviour of the weighted Lebesgue function as stated in Section B.2.2 of Appendix B.

For instance, in Theorem B.5 together with Lemma B.6 we have given a lower bound in terms of the limit distribution of the family of nodes. In particular, for a linear array $z_{j,n} = z_j$, with $(z_j)_j$ having only a finite number of accumulation points, such as harmonic nodes (see Example 4.2) or geometric nodes $z_j = 2^{-j}$, we know from Corollary B.7 and Example B.9 that the condition number grows faster than exponentially. Also, if the family $(z_{j,n})_{0\leq j\leq n}$ is obtained as the image of equidistant nodes on $[0;1]$ under some conformal map ϕ, then the sequence $(\kappa_p(V_n(d_n, z_{0,n}, .., z_{n,n}))^{1/n})_n$ has a limit, which may be calculated using Theorem B.19.

EXAMPLE 4.9 *We consider weighted Vandermonde–like matrices with Chebyshev polynomials* $p_n = T_n$, *and with the nodes* $z_{j,n} = \exp(2\cdot\pi\cdot i\cdot j/(n+1))$, *i.e.,* $z_{0,n}, .., z_{n,n}$ *are the* $(n+1)$*th roots of unity. From (4.2) and (4.6) together with (2.18) we may conclude that*

$$\lim_{n\to\infty} ||V_n(d_n, z_{0,n}, .., z_{n,n})||_p^{1/n} = \max_{|z|=1} \frac{|J^{-1}(z)|}{d_\infty(z)},$$

whereas by (a slight extension of) Theorem B.8 and Example 4.6

$$\lim_{n\to\infty} ||V_n(d_n, z_{0,n}, .., z_{n,n})^{-1}||_p^{1/n} = \max_{|z|=1} d_\infty(z).$$

Here we have used the fact that the limit distribution of our nodes is given by the equilibrium measure on the unit circle, with potential $\max\{0, \log(1/|z|)\}$. *We observe that the condition number is asymptotically minimal if one chooses* $d_\infty = J^{-1}$, *the limit of the* n*th root of the Kernel function* $K_{2,n}$. *In fact, we will show in Section 4.2 that this choice is also suitable in a more general context, which will enable us in Section 4.3 to give the* n*th root asymptotic of the solution of Problem (D).* □

To the end of this section, some remarks concerning Problem (C): In Theorem B.21 of Appendix B we have given a lower bound for the smallest accumulation point of the sequence of weighted Lebesgue numbers $(\Delta_n(h_n, H; d_n, z_{0,n}, .., z_{n,n})^{1/n})_n$, being valid for each family of nodes $(z_{j,n})_{0\leq j\leq n} \subset G$, and being attained for a particular family of nodes. By Examples 4.6 and 4.8, this leads to a tight lower bound for accumulation points of $(||V_n(d_n, z_{0,n}, .., z_{n,n})^{-1}||_p^{1/n})_n$. This bound is easily seen to increase if one only allows nodes in a compact subset G_1 of G. On the other hand, by (4.6), the smallest accumulation point of $(||V_n(d_n, z_{0,n}, .., z_{n,n})||_p^{1/n})_n$ in general decreases if all nodes lie in a compact subset G_1 of G. Thus, a solution of Problem (C) might be obtained by scanning a suitable family of subsets of G, as done in Section 5.2 for the particular case of real Vandermonde matrices.

4.2 Krylov–like and Hankel–like matrices

For solving a large linear system $B \cdot x = c$ with the square matrix B being sparse, one usually prefers iterative methods such as the method of conjugate gradients, or a Lanczos–type method (see, e.g., [GoOL89, BrSa93]). Also, a Lanczos–type method may be applied in order to obtain the approximate spectrum of B (see, e.g., [GoVL93, Chapter 9]). Here, with a suitable vector b, one successively determines an orthogonal basis of the so–called nth *Krylov space* spanned by the Krylov vectors $b, B \cdot b, B^2 \cdot b, .., B^n \cdot b$. If one wants to analyse the numerical condition of these methods, one is concerned with the problem of nearly linear dependence of Krylov vectors. To be more precise, we have to study the condition number of the *Krylov matrix*

$$K_n(B;b) := (b, B \cdot b, B^2 \cdot b, ..., B^n \cdot b).$$

Depending on the spectrum on B, one observes quite often that the condition number grows at least exponentially in n, see for instance the numerical results reported in [Car94, Chapitre 4.6]. It is the aim of this and the following section to establish asymptotic lower bounds, which will be refined for particular cases in Chapter 5.

In the case of non–Hermitian B, the Lanczos algorithm may suffer from breakdown or near–breakdown. In order to remedy this drawback, several authors have proposed look–ahead strategies in order to 'jump' over numerically unstable subproblems (see, e.g., [BRZS92, Gut92, Nac91]). The matrix of coefficients of the resulting intermediate linear systems is obtained by forming successively vectors of the form $p_k(B) \cdot b$, $k \geq 0$, where p_k is a polynomial of degree k (Chebyshev polynomials in [Nac91, p.55], monomials in [BRZS92]). This motivates the problem of studying more generally the condition number of *Krylov–like matrices*

$$K_n(B;b) := (p_0(B) \cdot b, p_1(B) \cdot b, p_2(B) \cdot b, ..., p_n(B) \cdot b).$$

In the particular case $p_n = T_n$, $n \geq 0$, we will speak of *Krylov–Chebyshev matrices.*

Using the Jordan decomposition $B = U \cdot X \cdot U^{-1}$, one easily verifies that $p_k(B) = U \cdot p_k(X) \cdot U^{-1}$ for each polynomial p_k, and hence

$$K_n(B;b) = U \cdot K_n(X; U^{-1} \cdot b).$$

If the matrix B is supposed to be nondefective [GoVL93, p.338], i.e., X is diagonal, then we may recover a weighted Vandermonde–like matrix: let $X = \textit{diag}\,(z_0, .., z_m)$, $U^{-1} \cdot b =: E \cdot (d(z_0)^{-1}, ..., d(z_m)^{-1})^T$ with E being diagonal and containing elements of modulus 1, then

$$K_n(B;b) = U \cdot E \cdot V_n(d, z_0, .., z_m)$$

(in the case of a defective eigenvalue z_j we will find also derivatives of the polynomials p_k evaluated at z_j).

In our considerations we will restrict ourselves to normal matrices B, i.e., $B^H \cdot B = B \cdot B^H$, with spectrum located in some closed set G. Then B is diagonalizable, and has an orthonormal system of eigenvectors. Consequently, $U \cdot E$ may be chosen to be unitary, and $\kappa_2(K_n(B;b)) =$

$\kappa_2(V_n(d, z_0, .., z_m))$. Our aim is to find a lower bound for $\kappa_2(K_n(B; b))$, being valid for each normal matrix B with eigenvalues in G, and for each b, and which is approximately attained for a particular pair (B, b). Hence we are left with Problem (D).

Let us also mention another application of Problem (D), namely the problem of finding optimal lower bounds for the condition number of particular moment matrices: given a fixed measure μ with compact support, we look for

$$\inf\{\kappa_2(M_n(\nu, \mu)) : \nu \in \mathcal{M}(G)\}. \tag{4.8}$$

E.g., if μ is the equilibrium measure of the unit circle, and $G = \mathbb{R}$, then we look for the optimal lower bound for the condition number of a positive definite Hankel matrix (see Corollary 5.14). We are only able to treat problem (4.8) for the case of so–called *Hankel–like matrices* where G is a real interval. In fact, for any measure ν being supported on the real axis, the link between (4.8) and problem (D) is obtained by the corresponding Gaussian quadrature formula.

Before specifying our findings in Theorem 4.11, let us first turn to Problem (D). One step towards the solution of this problem consists in finding for fixed nodes the weight function leading to a minimal 2–condition number. In other words, we look for an optimal diagonal preconditioning on the left (or an optimal row scaling) of a Vandermonde–like matrix. The problem of optimal diagonal preconditioning of an arbitrary square matrix was studied in detail by Bauer, who gave explicit solutions for the ∞– and the 1–Hölder norm [Bau63, Theorem IIa]. Instead of using Bauer's estimates for the Euclidean norm, for the purpose of Section 4.3 and Chapter 5 we propose the following

THEOREM 4.10 *Consider weighted Vandermonde–like matrices corresponding to a fixed family of polynomials $(p_k)_k$, with kernel $K_{2,n}$ as defined in (4.3). Let $G \subset \mathbb{C}$ be compact, and define for integers $m \geq n \geq 0$*

$$\Theta_n(G) := \max\{\frac{||\Pi_n^{-1} P||_2}{||P||_{K_{2,n},G}} : P \in \mathcal{P}_n, P \neq 0\},$$
$$\Theta_{m,n}(G) := \min\{\Theta_n(G_0) : G_0 \subset G, card(G_0) \leq m+1\}.$$

(a) *For each density function d and for each $z_0, .., z_m \in G$ we have*

$$\kappa_2(V_n(d, z_0, .., z_m)) \geq \frac{1}{\sqrt{m+1}} \cdot \Theta_{m,n}(G),$$

and the lower bound is attained up to a factor $(m+1)$.

(b) *If in addition G is a real interval, then for each density function d and for each $z_0, .., z_m \in G$ we have*

$$\kappa_2(V_n(d, z_0, .., z_m)) \geq \frac{1}{\sqrt{n+1}} \cdot \Theta_{n,n}(G),$$

and the lower bound is attained up to a factor $(n+1)$.

(c) *There holds for $m \geq n$*

$$\Theta_{n,n}(G) \geq \Theta_{m,n}(G) \geq \Theta_n(G) \geq \frac{1}{n+1} \cdot \Theta_{n,n}(G). \tag{4.9}$$

Proof: **(a)** We first require an analogue of (4.2) for the case of rectangular matrices $V_{m,n}(d)$ with $m \geq n$ and for Euclidean matrix norms. For $G_0 := \{z_0, .., z_m\}$, let

$$\kappa_n(d, G_0) = \Theta_{m,n}(d, G_0) \cdot ||K_{2,n}||_{d,G_0}, \quad \Theta_{m,n}(d, G_0) := \max_{P \in \mathcal{P}_n} \frac{||\Pi_n^{-1} P||_2}{||P||_{d,G_0}}.$$

Similar to (4.3), one verifies using (A.8), (A.11) that

$$(m+1)^{-1/2} \cdot \kappa_n(d, G_0) \leq \kappa_2(V_n(d, z_0, .., z_m)) \leq (m+1)^{1/2} \cdot \kappa_n(d, G_0). \tag{4.10}$$

Hence, for a fixed configuration of nodes $G_0 = \{z_0, .., z_m\}$, we obtain approximately the weight function $d : G_0 \to (0; +\infty)$ minimizing $\kappa_2(V_n(d, z_0, .., z_m))$ if we minimize $\kappa_n(d, G_0)$ as a function of $d(z_j) \in (0; +\infty)$, $j = 0, .., m$.

We want to show that, for each density function d on G_0, there holds $\kappa_n(d, G_0) \geq \kappa_n(K_{2,n}, G_0)$. In fact, since $\kappa_n(d, G_0)$ is invariant under multiplication of d by a positive constant, we may assume without loss of generality that $||K_{2,n}||_{d,G_0} = 1$; in particular

$$K_{2,n}(z_j) \leq d(z_j), \quad j = 0, .., m.$$

Consequently, for each polynomial $P \in \mathcal{P}_n$ we get $||P||_{K_{2,n},G_0} \geq ||P||_{d,G_0}$, or

$$\kappa_n(d, G_0) = \Theta_{m,n}(d, G_0) \geq \Theta_{m,n}(K_{2,n}, G_0) = \kappa_n(K_{2,n}, G_0).$$

Thus, for each configuration of nodes, the 'optimal' density function is given by $d = K_{2,n}$. Using in addition (4.10), we obtain for weighted Vandermonde–like matrices with a fixed configuration of nodes $z_0, .., z_m$

$$\min_d \kappa_2(V_n(d, z_0, .., z_m)) = \epsilon \cdot \kappa_n(K_{2,n}, \{z_0, .., z_m\})$$

with a suitable $\epsilon \in [(m+1)^{-1/2}; (m+1)^{1/2}]$. We now take the minimum over all nodes $z_0, .., z_m \in G$, and recall that each continuous function $F : \mathbb{C}^{m+1} \to [0; +\infty]$ attains its minimum on compact subsets of $\mathbb{C}^{m+1}$. This yields part **(a)** of the assertion.

(b) In view of the first part, it is sufficient to show that, for each density function d and for each distinct $z_0, .., z_m \in G$, there exist a density function d^* and distinct $z_0^*, ..., z_n^* \in G$ with $\kappa_2(V_n(d, z_0, .., z_m)) = \kappa_2(V_n(d^*, z_0^*, ..., z_n^*))$, and vice versa. By the particular form of $V_n(d, z_0, .., z_m)^H \cdot V_n(d, z_0, .., z_m)$, we obtain the sufficient condition that for all $Q \in \mathcal{P}_{2n}$ there holds

$$\sum_{j=0}^{m} \frac{Q(z_j)}{d(z_j)^2} = \sum_{j=0}^{n} \frac{Q(z_j^*)}{d^*(z_j^*)^2}. \tag{4.11}$$

Notice that the left hand side of (4.11) may be rewritten as $\int Q(z)\, d\mu(z)$ with μ a (discrete) measure being supported on G. Hence (4.11) holds by taking as nodes $z_0^*, .., z_n^*$ the zeros of the $(n+1)$th μ–orthogonal polynomial, and $d^*(z) = \sqrt{K_n^\mu(z)}$, the data of the corresponding Gaussian quadrature formula (see, e.g., [Sze67, Theorem 3.4.2]). On the other hand, given the data $z_0^*, .., z_n^*$, and d^*, equation (4.11) holds with $z_j = z_j^*$, $d(z_j) = d(z_j^*)$ for $j = 0, .., n-1$, and $z_j = z_n^*$, $d(z_j) = d(z_n^*) \cdot \sqrt{m+1-n}$ for $j = n, n+1, .., m$.

(c) The first two estimates of (4.9) are immediate. In order to show the third one, notice that for any nodes $z_0, .., z_n \in G$ we have

$$\Theta_{n,n}(G) \leq \Theta_n(G) \cdot \Delta_n(K_{2,n}, G; K_{2,n}, \{z_0, .., z_n\}).$$

As shown in Corollary B.18, there holds $\Delta_n(K_{2,n}, G; K_{2,n}, \{z_0, .., z_n\}) \leq n+1$ for the weighted Fekete nodes (i.e., the arguments maximizing $det\, V_n(K_{2,n}, z_0, .., z_n)$ in G), leading to (4.9). □

THEOREM 4.11 *Let $G \subset \mathbb{C}$ be compact.*

(a) *For each integers $m \geq n \geq 0$ for each normal matrix B of size $(m+1)$ with spectrum in G and for each $b \in \mathbb{C}^{m+1}$, the Euclidean condition number of the Krylov–like matrix $K_n(B,b)$ is bounded below by $\kappa_2(K_n(B,b)) \geq \Theta_{m,n}(G)/\sqrt{m+1}$, and the lower bound is attained up to a factor $(m+1)$. If in addition G is a real interval, and therefore B is Hermitian, then we have the lower bound $\Theta_{n,n}(G)/\sqrt{n+1}$, being attained up to a factor $(n+1)$.*

(b) *If G is a real interval, then for each measure μ with compact support*

$$\frac{1}{n+1} \cdot \Theta_{n,n}(G)^2 \leq \inf\{\kappa_2(M_n(\nu,\mu)) : \nu \in \mathcal{M}(G)\} \leq (n+1) \cdot \Theta_{n,n}(G)^2,$$

where for $\Theta_{n,n}(G)$ we consider the sequence of the μ–orthonormal polynomials.

Proof: Part (a) follows from the remarks at the beginning of this section together with Theorem 4.10(a), (b). In order to show part (b), denote the zeros of the $(n+1)$th ν–orthogonal polynomial p_{n+1}^ν by $z_{0,n}, .., z_{n,n} \in G$, then we have the Gaussian quadrature formula

$$\int P(z)\, d\nu(z) = \sum_{j=0}^{n} \frac{1}{\sqrt{K_n^\nu(z_{j,n})}} \cdot P(z_{j,n}), \qquad P \in \mathcal{P}_{2n+1}.$$

Consequently,

$$M_n(\nu,\mu) = \Big(\int \overline{p_j^\mu(z)} \cdot p_k^\mu(z)\, d\nu(z)\Big)_{j=0,..,n}^{k=0,..,n} = V_n(\sqrt{K_n^\nu}, z_{0,n}, .., z_{n,n})^H \cdot V_n(\sqrt{K_n^\nu}, z_{0,n}, .., z_{n,n})$$

with Vandermonde–like matrices corresponding to μ–orthogonal polynomials, and the assertion follows from Theorem 4.10(a). □

As a consequence of Theorem 4.11 and Theorem 4.10(c), we may establish approximately tight lower bounds for the condition number of Krylov–like matrices and Hankel–like matrices in terms of $\Theta_{n,n}(G)$ or $\Theta_n(G)$. In order to obtain explicit expressions for the latter quantities, we have to restrict ourselves to particular sequences $(p_n)_n$ and particular sets G. The nth root behaviour of $\Theta_n(G)$ will be established in Section 4.3 for sequences $(p_n)_n$ where $(\kappa_H(\Pi_n)^{1/n})_n$ tends to 1 for some set H. In the particular case of monomials $p_k(z) = z^k$, $k \geq 0$, and G a real interval, we will give explicit intervals for $\Theta_{n,n}(G)$ in Section 5.3, which enables us to derive approximately tight lower bounds for the condition number of positive definite Hankel matrices (see Corollary 5.14), as well as for Krylov matrices built up with Hermitian B (see Theorem 5.11).

4.3 The nth root behaviour

The aim of this section is to give a proof and some examples for the following

THEOREM 4.12 *Given a sequence of polynomials $(p_n)_n$ with a corresponding sequence of coordinate maps $(\Pi_n)_n$, suppose that there exists a compact set H having the K–property (see Definition B.2) such that*

$$\lim_{n\to\infty} \kappa_H(\Pi_n) = 1.$$

Then for each compact set G with K–property there holds

$$\lim_{n\to\infty} \Theta_n(G)^{1/n} = \lim_{n\to\infty} \Theta_{n,n}(G)^{1/n} = \Delta^w(H,G).$$

Here

$$\log \Delta^w(H,G) := \max_{z\in H} \int_{\mathbb{C}\setminus G} g_G(z;t)\, d\mu_H(t),$$

where μ_H denotes the equilibrium measure of the set H (see Example B.14), and $g_G(\cdot,t)$ is the Green function of G with singularity at $t \notin G$ (see Definition B.1).

The quantity $\Delta^w(H,G)$ may be understood as a weighted counterpart of the quantity $\Delta(H,G)$ introduced in Definition 2.8. In the examples mentioned at the end of this section we will restrict ourselves to simply connected G; here, the quantity $\log \Delta^w(H,G)$ may be evaluated by using the fact that, for $z, t, z_0 \notin G$, $t \neq z_0$,

$$g_G(z;t) = g_G(t,z) = \log|\phi_t(z)|, \quad g_G(z;z_0) = \log\left|\frac{1-\overline{\phi_t(z_0)}\cdot\phi_t(z)}{\phi_t(z)-\phi_t(z_0)}\right|, \tag{4.12}$$

where ϕ_t maps the exterior of G conformally on the exterior of the unit disk, with $\phi_t(t) = \infty$.

Before giving a proof of Theorem 4.12, let us summarize some further properties in the following

THEOREM 4.13 **(a)** *The assertion of Theorem 4.12 remains valid for closed (not necessarily bounded) sets G with boundary consisting of segments, straight lines and circular arcs, provided that $G \cup H \neq \mathbb{C}$.*

(b) *Let* $H_0 := \partial\mathcal{D}_\infty(H)\setminus G$*. We have* $\Delta^w(H,G) \geq 1$*, and* $\Delta^w(H,G) = 1$ *if and only if* $H_0 = \emptyset$*, i.e.,* G *contains the outer boundary of* H*. Moreover, in the case* $H_0 \neq \emptyset$*, the maximum in the definition of* $\Delta^w(H,G)$ *is attained for* $z \in H_0$*.*

(c) *If* $G \setminus \mathcal{D}_\infty(H) = \emptyset$*, then* $\Delta^w(H,G) = \Delta(H,G)$*.*

(d) *In the case of ordinary Krylov and Hankel matrices, i.e.,* $p_n(z) = z^n$*,* $n \geq 0$*, the assumptions of Theorem 4.12 are true with* $H = \partial\mathbb{D}$*. Here we have the simpler formula*

$$\log \Delta^w(\partial\mathbb{D}, G) := \max_{s\in[0;2\pi]} \frac{1}{2\pi} \cdot \int_{t\in[0;2\pi], e^{it}\notin G} g_G(e^{is}, e^{it})\, dt.$$

In the case of Krylov–Chebyshev *matrices, i.e., Krylov-like matrices with respect to Chebyshev polynomials* $p_n = T_n$*,* $n \geq 0$*, the assumptions of Theorem 4.12 are true with* $H = [-1;1]$*. Here we have the simpler formula*

$$\log \Delta^w([-1;1], G) := \max_{s\in[-1;1]} \frac{1}{\pi} \cdot \int_{[-1;1]\setminus G} \frac{g_G(s,z)}{\sqrt{1-z^2}}\, dz.$$

For a proof of Theorem 4.12 we require the following complement of Theorem 2.9

Lemma 4.14 *Under the assumptions of Theorem 4.12, the sequence* $(g_n)_n$ *being defined by*

$$g_n(z) := \log K_{2,n}(z)^{1/n} = \log\Big(\sum_{j=0}^{n} |p_j(z)|^2\Big)^{1/2n}, \quad z \in \mathbb{C}, \quad n \geq 0,$$

converges to the Green function $g_H(\cdot,\infty)$*, locally uniformly in* $\mathbb{C}$*.*

Proof: First by definition of the kernel $K_{2,n}$ we obtain for each $z \in \mathbb{C}$ and for each $n \geq 0$ using (A.2)

$$\begin{aligned} e^{n\cdot g_n(z)} &= \max_{a\in\mathbb{C}^{n+1}} \frac{|(\Pi_n(a))(z)|}{||a||_2} \leq \max_{a\in\mathbb{C}^{n+1}} \frac{|(\Pi_n(a))(z)|}{||a||_\infty} \\ &\leq ||\Pi_n||_H \cdot \max_{a\in\mathbb{C}^{n+1}} \frac{|(\Pi_n(a))(z)|}{||\Pi_n(a)||_H} = ||\Pi_n||_H \cdot \Delta_n(z,H), \end{aligned}$$

and similarly

$$e^{n\cdot g_n(z)} \geq \frac{1}{\sqrt{n+1}} \cdot \max_{a\in\mathbb{C}^{n+1}} \frac{|(\Pi_n(a))(z)|}{||a||_\infty} \geq \frac{1}{\sqrt{n+1}\cdot ||\Pi_n^{-1}||_H} \cdot \Delta_n(z,H).$$

Due to (2.3), with $(\kappa_H(\Pi_n)^{1/n})_n$, also the sequences $(||\Pi_n||_H^{1/n})_n$ and $(||\Pi_n^{-1}||_H^{1/n})_n$ tend to 1. Consequently, the sequences $(g_n)_n$ and $(d_n)_n$, $d_n(z) := \log \Delta_n(z,H)^{1/n}$, have the same limit behavour.

Now H has the K–property, and therefore in particular H is regular with respect to the Dirichlet problem. By Definition 2.8 and Definition B.1 we have

$$\log \Delta(z, H) = g_H(z, \infty), \quad z \in \mathbb{C},$$

and the assertion follows from the last part of Theorem B.22. □

Let us mention that, by (A.2), the assertion of Lemma 4.14 remains valid if one replaces the kernel $K_{2,n}$ by the kernel $K_n = K_{\infty,n}$ of Chapter 2.

Proof of Theorem 4.12: Using (A.2) we obtain

$$\frac{1}{||\Pi_n||_H} \cdot \max_{P \in \mathcal{P}_n} \frac{||P||_{1,H}}{||P||_{K_{2,n},G}} \leq \Theta_n(G) = \max_{P \in \mathcal{P}_n} \frac{||\Pi_n^{-1} P||_2}{||P||_{K_{2,n},G}} \leq \sqrt{n+1} \cdot ||\Pi_n^{-1}||_H \cdot \max_{P \in \mathcal{P}_n} \frac{||P||_{1,H}}{||P||_{K_{2,n},G}}.$$

Hence the sequences

$$(\log \Theta_n(G)^{1/n})_n \quad \text{and} \quad (\log \Delta_n(1, H; K_{2,n}, G)^{1/n})_n$$

have the same accumulation points, the latter sequence being discussed in Appendix B, where we have used the abbreviation $(\delta_n)_n$. By Lemma 4.14, the assumptions of Section B.2.1 hold with $\Omega = \mathbb{C}$,

$$\text{for } z \in H\text{: } f_H(z) = 0, \quad \text{and for } z \in G\text{: } f_G(z) = g_H(z, \infty).$$

Thus the assertion of Theorem 4.12 follows from Theorem B.21. □

Proof of Theorem 4.13:

(a) Let us first mention that the quantities $\Theta_n(G)$ and $\Delta_n(1, H; K_{2,n}, G)$ also are well–defined for not bounded G, since, for each $P \in \mathcal{P}_n$, the function $P/K_{2,n}$ is continuous around infinity. To be more precise, we consider the following linear transformation of the extended complex plane

$$\tilde{z} = \frac{1}{z - c}, \quad \tilde{G} = \{\frac{1}{z - c} : z \in G \cup \{\infty\}\}, \quad \tilde{H} = \{\frac{1}{z - c} : z \in H\},$$

with a fixed $c \in \mathbb{C} \setminus (H \cup G)$. Any polynomial P of degree less or equal to n may be rewritten as

$$P(z) = P(c + 1/\tilde{z}) = \tilde{P}(\tilde{z})/\tilde{z}^n$$

with $\tilde{P} \in \mathcal{P}_n$, and thus

$$\Delta_n(1, H; K_{2,n}, G) = \Delta_n(\tilde{h}_n, \tilde{H}; \tilde{g}_n, \tilde{G}), \quad \tilde{h}_n(\tilde{z}) = \tilde{z}^n, \quad \tilde{g}_n(\tilde{z}) = \tilde{z}^n \cdot K_{2,n}(c + 1/\tilde{z}).$$

Taking into account the arguments of the proof of Theorem 4.12, we see that the limit of the sequence $(\Theta_n(G)^{1/n})_n$ equals the limit of $(\Delta_n(\tilde{h}_n, \tilde{H}; \tilde{g}_n, \tilde{G})^{1/n})_n$. The latter may be determined using Theorem B.21 since $\tilde{H}, \tilde{G}$ are compact sets with K–property, and

$$\log \tilde{h}_n(\tilde{z})^{1/n} \to \log |\tilde{z}|, \quad \log \tilde{g}_n(\tilde{z})^{1/n} \to \log |\tilde{z}| + g_H(c + 1/\tilde{z}, \infty),$$

uniformly in $\tilde{H}$, and $\tilde{G}$, respectively.

(b) Let

$$h(z) := \int_{\mathbb{C}\setminus G} g_G(z;t)\, d\mu_H(t).$$

Recall from Example B.14 that μ_H is supported on the outer boundary $\partial\mathcal{D}_\infty(H)$ of H. Consequently, if $H_0 = \emptyset$, then h is identically zero. Suppose now that $H_0 \neq \emptyset$. Then, by Definition B.1, h is identically zero on G, and strictly positive on H_0. Let $H_1 := \mathbb{C} \setminus (\mathcal{D}_\infty(H) \cup G)$. Notice that $H_0 \subset H \setminus G \subset H_1$ and $\partial H_1 \subset H_0 \cup \partial G$. Moreover, h is harmonic in the interior of H_1, and continuous in $\mathbb{C}$. Thus by the maximum principle for harmonic functions we obtain

$$0 < \max_{z\in H_0} h(z) \leq \max_{z\in H\setminus G} h(z) \leq \max_{z\in H_1} h(z) \leq \max_{z\in\partial H_0\cup\partial G} h(z) = \max_{z\in\partial H_0} h(z),$$

yielding assertion (b).

(c) By Lemma 4.14, the identity $G \setminus \mathcal{D}_\infty(H) = \emptyset$ implies that $(\log K_{2,n}^{1/n})_n$ converges uniformly on G to the zero function. Hence $(\Theta_n(G)^{1/n})_n$ behaves asymptotically like $(\Delta_n(H,G)^{1/n})_n$, the latter having the limit $\Delta(H,G)$ (see Theorem B.22).

(d) This assertion follows from Example 2.3, and Example 2.2, respectively, by noticing that $d\mu_{\mathbb{D}}(z) = |dz|/(2\pi)$ for $z \in \partial\mathbb{D}$, and $d\mu_{[-1;1]}(z) = |dz|/(\pi \cdot \sqrt{1-z^2})$ for $z \in [-1;1]$, respectively. □

Before studying some examples, let us mention that the quantity $\Delta^w(H,G)$ is invariant under a linear transformation, i.e., for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ there holds

$$\Delta^w(\alpha + \beta \cdot H, \alpha + \beta \cdot G) = \Delta^w(H,G). \tag{4.13}$$

Hence the quantity $\Delta^w(\mathbb{D}, G)$ is invariant under a rotation of G, and $\Delta^w([-1;1], G) = \Delta^w([-1;1], -G)$. We first study the case of G being a halfplane

Example 4.15 *Let*

$$G = \mathbb{C}_+ := \{z \in \mathbb{C} : \Re z \geq 0\},$$

the right halfplane. The function $\phi_{(-1)}(z) = (z-1)/(z+1)$ *maps* G^c *conformally on* $\mathbb{D}^c$*, with* $\phi_{(-1)}(-1) = +\infty$*. One verifies using (4.12) that*

$$g_G(z,t) = \log|\frac{z+t}{z-t}|, \quad z,t \in [-1;0] = [-1;1] \setminus G,$$

and by means of elementary computations

$$\log \Delta^w([-1,1], \mathbb{C}_+) = \max_{z\in[-1;0]} \frac{1}{\pi} \cdot \int_{-1}^{0} \log|\frac{z+t}{z-t}| \frac{dt}{\sqrt{1-t^2}}$$
$$= \max_{\beta\in[0;\pi/2]} \frac{-1}{\pi} \int_{-\beta}^{\pi-\beta} \log|\tan(\frac{\alpha}{2})|\, d\alpha = \max_{\beta\in[0;\pi/2]} \frac{-1}{\pi} \int_{-\beta}^{\beta} \log|\tan(\frac{\alpha}{2})|\, d\alpha.$$

Consequently, the maximum is attained for $\beta = \pi/2$ *(or* $z = -1$*), with the value*

$$\log \Delta^w([-1,1], \mathbb{C}_+) = \frac{2}{\pi} \cdot \int_0^{\pi/2} \log |\frac{1}{\tan(\alpha/2)}| \, d\alpha =: 2 \cdot \delta^*.$$

For the final integral there exists no close form expression. The quantity δ^* *will be basic also for the following considerations, let us mention its relation with the* Catalan constant

$$\delta^* = \frac{2}{\pi} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.583, \quad e^{\delta^*} \approx 1.792, \quad e^{2\cdot\delta^*} \approx 3.210 \tag{4.14}$$

□

Consequently, for large n, the nth root of the condition number of a Krylov–Chebyshev matrix built up with a normal matrix B having eigenvalues with nonnegative real part is approximately greater or equal to 3.210, and this bound is best possible. We will discuss some particular cases below. Let us first show that for ordinary Krylov matrices we obtain in the same context the lower bound $1.792 \approx \sqrt{3.210}$.

Example 4.16 *Let again* $G = \mathbb{C}_+ = \{z \in \mathbb{C} : \Re z \geq 0\}$*, the right halfplane. With help of (4.12) and Example 4.15 we obtain for* $e^{is}, e^{it} \in \partial \mathbb{D} \setminus G$

$$\phi_{(-1)}(e^{it}) = i \cdot \tan(t/2), \quad g_G(e^{is}, e^{it}) = \log |\frac{1}{\tan((s-t)/2)}|,$$

with $s, t \in (\pi/2, 3\pi/2)$*. Thus*

$$\log \Delta^w(\mathbb{D}, \mathbb{C}_+) = \max_{s \in [\pi/2; 3\pi/2]} \frac{-1}{2\pi} \cdot \int_{\pi/2}^{3\pi/2} \log | \tan(\frac{t-s}{2})| \, dt = \delta^*,$$

where for the final equality we have taken into account the computations of Example 4.15. □

In the next example we consider Krylov(–Chebyshev) matrices built up with Hermitian and anti–Hermitian matrices, i.e., $B = B^H$ and $B^H = -B$, respectively. These matrices are trivially normal, and their spectrum is contained in $G = \mathbb{R}$, and $G = i \cdot \mathbb{R}$, respectively.

Example 4.17 *In the case* $G = i \cdot \mathbb{R}$*, we have a Green function for each connected component of* $\mathbb{C} \setminus G$*, namely*

$$g_{i\mathbb{R}}(z,t) = \begin{cases} g_{\mathbb{C}_+}(z,t) & \text{if } t, z \in \mathbb{C}_+ \\ g_{\mathbb{C}_+}(-z,-t) & \text{if } t, z \in -\mathbb{C}_+, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for $H \in \{\partial \mathbb{D}, [-1;1]\}$ *one gets* $\Delta^w(H, i \cdot \mathbb{R}) = \Delta^w(H, \mathbb{C}_+)$*. In particular, by (4.13) we obtain for Krylov matrices the lower bounds*

$$\Delta^w(\partial \mathbb{D}, i \cdot \mathbb{R}) = \Delta^w(\partial \mathbb{D}, \mathbb{R}) = e^{\delta^*},$$

whereas for Krylov–Chebyshev matrices

$$\Delta^w([-1;1], i\cdot \mathbb{R}) = e^{2\cdot\delta^*}, \quad \Delta^w([-1;1], \mathbb{R}) = 1.$$

In fact, we know from Example 4.5 that a Krylov–Chebyshev matrix based on the (real) zeros of Chebyshev polynomials is perfectly conditioned. Note however that Krylov–Chebyshev matrices become ill-conditioned for anti-Hermitian matrices B, at least for large dimension. □

The aim of the next example is to study Krylov(–Chebyshev) matrices built up with Hermitian positive definite matrices B. We will even allow more generally sectors.

EXAMPLE 4.18 *With $\lambda \in [0,2)$, we consider the sector*

$$S_\lambda := \{z \in \mathbb{C} : -\lambda\cdot\frac{\pi}{2} \le \arg z \le \lambda\cdot\frac{\pi}{2}\},$$

i.e., $S_1 = \mathbb{C}_+$, and $S_0 = [0;+\infty)$. The function $f_\lambda(z) := z^{1/(2-\lambda)}\cdot\exp(\frac{2-2\lambda}{2-\lambda}\frac{i\pi}{2})$ maps the exterior of S_λ conformally to the exterior of $\mathbb{C}_+$ (we take the branch of the logarithm on $\mathbb{C}\setminus[0;+\infty)$ with $\log(-1) = i\pi$*). Consequently, $g_{S_\lambda}(z,t) = g_{C_+}(f_\lambda(z), f_\lambda(t))$ for $z,t \notin S_\lambda$. In particular,*

$$\log\Delta^w(\partial\mathbb{D}, S_\lambda) = \max_{s\in[\lambda\pi/2;2\pi-\lambda\pi/2]} \frac{-1}{2\pi}\cdot\int_{\lambda\pi/2}^{2\pi-\lambda\pi/2} \log|\tan(\frac{t-s}{2\cdot(2-\lambda)})|\,dt = (2-\lambda)\cdot\delta^*.$$

Therefore, we obtain for Krylov matrices

$$\Delta^w(\partial\mathbb{D}, S_\lambda) = e^{(2-\lambda)\cdot\delta^*}, \quad \Delta^w(\partial\mathbb{D}, [0;+\infty)) = \Delta^w(\partial\mathbb{D}, \mathbb{R})^2 = e^{2\cdot\delta^*}.$$

In fact, if $\lambda \to 2$, we get the limiting value $\Delta^w(\partial\mathbb{D}, \mathbb{C}) = 1$. With regard to Krylov–Chebyshev matrices, we only mention the case $\lambda = 0$ of positive definite Hermitian matrices B. Evaluating the resulting integral numerically gives the lower bound $\Delta^w([-1;1],[0;+\infty)) \approx 4.422$. □

In order to prevent overflow, in applications one often scales the matrix B such that $||B|| \le 1$ with a suitable (subordinate) norm. Then the spectrum of B is contained in the unit disk, and we may determine asymptotically lower bounds by applying Theorem 4.13(c).

EXAMPLE 4.19 *For Krylov(–Chebyshev) matrices built up with Hermitian positive definite scaled matrices B we have the lower bound*

$$\Delta^w(\partial\mathbb{D}, [0;1]) = \Delta^w([-1;1],[0;1]) = J^{-1}(3) = (1+\sqrt{2})^2 \approx 5.828.$$

In the case of Hermitian scaled matrices B we obtain

$$\Delta^w(\partial\mathbb{D}, [-1;1]) = |J^{-1}(i)| = (1+\sqrt{2}) \approx 2.414, \quad \Delta^w([-1;1],[-1;1]) = 1.$$

□

Thus, in general, a scaling of the matrix B might increase the condition number of the corresponding Krylov(–Chebyshev) matrix.

As a final remark, we believe that Theorem 4.12 also holds for more general closed sets G, H, namely, H being compact, and G, H being regular with respect to the Dirichlet problem.

Chapter 5

Tight bounds for particular matrices

Beside the asymptotic lower bounds of Section 4.3 for the condition number of special structured matrices, it is of particular practical interest to have lower bounds as a function of the dimension n which are approximately attained for a particular choice of parameters. Here we will restrict ourselves to a subclass of the matrices discussed in Chapter 4, namely square weighted Vandermonde matrices with real nodes

$$V_n(d) := V_n(d, x_0, .., x_n) = \begin{pmatrix} \frac{1}{d(x_0)} & \frac{x_0}{d(x_0)} & \frac{x_0^2}{d(x_0)} & \cdots & \frac{x_0^n}{d(x_0)} \\ \frac{1}{d(x_1)} & \frac{x_1}{d(x_1)} & \frac{x_1^2}{d(x_1)} & \cdots & \frac{x_1^n}{d(x_1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{1}{d(x_n)} & \frac{x_n}{d(x_n)} & \frac{x_n^2}{d(x_n)} & \cdots & \frac{x_n^n}{d(x_n)} \end{pmatrix},$$

where $d(x_j) > 0$, $0 \leq j \leq n$, and $x_0, .., x_n \in M \subset \mathbb{R}$. Instead of using the coordinate map Π_n of the basis of monomials, we will use the shorthand notation $\vec{P} = (c_0, .., c_n)^T = \Pi_n^{-1}(P)$ for any $P \in \mathcal{P}_n$, $P(z) = c_0 + c_1 z + .. + c_n z^n$.

In Section 5.1 we deal with the problem of finding

$$\min_{x_0,..,x_n \in M} \kappa_{p,\infty}(V_n(d; x_0, .., x_n))$$

for fixed p, n, d and a set $M \subset \mathbb{R}$. Nodes where the minimum is attained will be referred to as *optimal nodes*. Recall that $\kappa_{p,\infty}(V_n(d)) = ||V_n(d)^{-1}||_{\infty,p} \cdot ||V_n(d)||_{p,\infty}$. In order to be able to monitor the size of $||V_n(d)||_{p,\infty}$ while calculating the minimum of $||V_n(d)^{-1}||_{\infty,p}$, we will restrict ourselves to a class of admissible density functions described in Definition 5.2. Then a solution of the above problem may be given in terms of alternants of (weighted) Chebyshev polynomials with respect to (d, M).

In Section 5.2 we find the configuration of nodes out of $[0; +\infty)$, and of $\mathbb{R}$, respectively, minimizing the condition number $\kappa_\infty(V_n(1))$ of ordinary Vandermonde matrices. The problem of minimizing $\kappa_1(V_n(1))$ was studied before by Gautschi [Gau75b]. We also give lower bounds for the condition number $\kappa_p(V_n(1))$ of (column–scaled) real Vandermonde matrices improving results obtained by Gautschi, Taylor, Inglese, and Tyrtyshnikov.

Krylov matrices built up with Hermitian matrices B are studied in Section 5.3, and applications to positive definite Hankel matrices, and to the rational interpolation problem, respectively, are discussed in Section 5.4.

5.1 Real weighted Vandermonde matrices

For the following considerations, we will require (generalized) Chebyshev polynomials with respect to some real compact set M and some density function $d \in \mathcal{C}(M)$ being defined as follows (see, e.g., [SmLe68, p.351])

DEFINITION 5.1 *Let $M \subset \mathbb{C}$ be compact containing at least $n+1$ elements, and let $d : M \to (0; +\infty)$ be continuous on M. Then there exists a polynomial $T_n[d, M]$ of degree n with positive leading coefficient called nth Chebyshev polynomial which maximizes*

$$\{|c_n| : P \in \mathcal{P}_n, ||P||_{d,M} \leq 1, P(z) = c_0 + c_1 z + .. + c_n z^n\}.$$

□

Note that, under the above assumptions, $T_n[d, M]$ is unique (see [Sch71, Satz 6.3, p.155]), and $||T_n[d, M]||_{d,M} = 1$. As an example we have $T_n[1, [-1; 1]] = T_n$, the classical Chebyshev polynomial. Moreover, by a linear transformation,

$$T_n[d, \alpha + \beta \cdot M](z) = T_n[d(\alpha + \beta \cdot z), M](\frac{z - \alpha}{\beta}). \tag{5.1}$$

In addition, $T_n[d, M]$ inherits symmetry properties of (d, M) (see Meinardus [Mei67, Theorem 27, p.26]). For instance, suppose that $M = -M$, and $d(x) = d(-x)$ for all $x \in M$. Then with $M' := \{z^2 : z \in M\}$ we obtain

$$T_{2n}[d, M](z) = T_n[d(\sqrt{z}), M'](z^2), \quad T_{2n+1}[d, M](z) = z \cdot T_n[\tfrac{d(\sqrt{z})}{\sqrt{z}}, M' \setminus \{0\}](z^2). \tag{5.2}$$

Also, Chebyshev polynomials are (partly) explicitly known for polynomial densities d and M being real intervals; these results will be important for our study of Krylov matrices.

In the following considerations, we will restrict ourselves to real sets M. Given a polynomial $P \in \mathcal{P}_n$, $x_0 < .. < x_m$ is called an *alternant* of P in M (with respect to a fixed density function d) if for $j = 0, .., m$

$$x_j \in M \quad \text{and} \quad \frac{P(x_j)}{d(x_j)} = (-1)^{m-j} \cdot ||P||_{d,M} \tag{5.3}$$

(note that $m \leq n$ for each nontrivial P). Chebyshev showed that the solution of the problem of best approximation on real compact sets M with respect to the maximum norm may be characterized with help of alternants. Since the set of functions $1/d, x/d, .., x^{n-1}/d$ satisfies the Haar condition, by the Chebyshev criterion [Sch71, Theorem 6.6] we know that, for each $M \subset \mathbb{R}$ being compact, and for each density function d being continuous on a interval containing M, we have $T = T_n[d, M]$ if and only if T has an alternant in M consisting of exactly $(n+1)$ elements.

The set of admissible density functions is given by

Definition 5.2 *Let $n \geq 1$ be an integer, $p \in [1; \infty]$ a Hölder index. We say that a pair (d, M) is p–admissible if $M \subset [0; \infty)$, and with $\gamma := \max\{z : z \in M\}$ there holds*

(a) *$M \subset M_1 \subset \mathbb{R}$, where M is compact containing at least $n+1$ elements, and M_1 is a compact interval. Furthermore, $d : M_1 \to (0; +\infty)$ is continuous on M_1.*

(b) *We have*

$$||K_{p,n}||_{d,M} = \frac{K_{p,n}(\gamma)}{d(\gamma)},$$

with the kernel function $K_{p,n}(z) = (1+|z|^q+..+|z|^{n\cdot q})^{1/q}$ introduced in (4.3), $1/p+1/q = 1$. In addition, γ is an element of the alternant of $T_n[d, M]$.

We say that a pair (d, M) is s–admissible if conditions (a), (b) hold with $\gamma := \max\{z : z \in M\}$, and if in addition M and d are symmetric with respect to the origin, i.e., $M = -M$ and $d(x) = d(-x)$ for all $x \in M$. □

For instance, with help of (5.1) we verify that $(1, M)$ is p–admissible for each real compact interval $M \subset [0; +\infty)$, and s–admissible for each real compact interval of the form $M = [-\gamma; \gamma]$.

Notice that most of the conditions of Definition 5.2 are easily checked, perhaps up to the condition on the alternant of $T_n[d, M]$. Here, the following observation might be helpful

Lemma 5.3 *Let (d, M) be as in Definition 5.2(a), $\gamma = \max\{z : z \in M\}$. If there exists a polynomial $q \in \mathcal{P}_{2n}$ being positive on M_1 and satisfying*

$$\max_{z \in M_1} \frac{\sqrt{q(z)}}{d(z)} = \frac{\sqrt{q(\gamma)}}{d(\gamma)},$$

then γ is an element of the alternant of $T_n[d, M]$.

Proof: By a suitable scaling of q, we may assume without loss of generality that $q(\gamma) = d(\gamma)^2$, and therefore $q(x) \leq d(x)^2$ for all $x \in M_1$.

Let $x_0 < x_1 < .. < x_n$ denote the alternant of $T := T_n[d, M]$ in M. By Rolle's Theorem, T has the zeros $y_1, .., y_n \in M_1$ with $x_{j-1} < y_j < x_j$, $j = 1, .., n$. Let $Q(x) := q(x) - T(x)^2$. Then Q is a real polynomial of degree at most $2n$, satisfying $Q(y_j) = q(y_j) > 0$, $j = 1, .., n$, and $Q(x_j) = q(x_j) - d(x_j)^2 \leq 0$, $j = 0, .., n$. One verifies that, for $j = 1, .., n-1$, Q has at least two zeros in each open interval (y_j, y_{j+1}) (counting multiplicities). Moreover, Q must have at least one zero in $[x_0; y_1)$, and in $(y_n; x_n]$, respectively. By counting the number of zeros being obviously bounded by $2n$, we may conclude that Q is strictly negative on $(x_n; \gamma]$. Assuming now that $x_n \neq \gamma$, we obtain $Q(\gamma) < 0$, or $T(\gamma)^2 > q(\gamma) = d(\gamma)^2$, a contradiction to the definition (5.3) of the alternant. □

As a consequence, $(1, M)$ is p–admissible for each compact set $M \subset [0; \infty)$, and s–admissible for each symmetric compact set (take $q = 1$ in Lemma 5.3). Moreover, in the case $p = 2$, the pair $(K_{2,n}, M)$ is p–admissible for each compact set $M \subset [0; \infty)$, and s–admissible for each symmetric compact set, since $q(z) := K_{2,n}(z)^2$ is a polynomial of degree $2n$.

We are now prepared to prove the following Theorem dealing with nonnegative nodes.

THEOREM 5.4 *Let (d, M) be p–admissible, and $\gamma := \max\{x : x \in M\}$. Then the minimum of*

$$\{\kappa_{p,\infty}(V_n(d; x_0, .., x_n)) : x_0, .., x_n \in M, \max_j x_j = \gamma\}$$

is given by

$$\frac{K_{p,n}(\gamma)}{d(\gamma)} \cdot ||\vec{T}_n[d, M]||_p = \epsilon \cdot \frac{K_{p,n}(\gamma)}{d(\gamma)} \cdot |\vec{T}_n[d, M](-1)|$$

with an $\epsilon \in [(n+1)^{1/p-1}, 1]$. This minimum is attained for $x_0, .., x_n$ being the alternant of $T_n[d, M]$.

Proof: From (4.3) we know that for all $x_0, .., x_n \in M$ satisfying $\max_j x_j = \gamma$ there holds

$$||V_n(d; x_0, .., x_n)||_{p,\infty} = \max_j \frac{|K_{p,n}(x_j)|}{d(x_j)} = \frac{|K_{p,n}(\gamma)|}{d(\gamma)},$$

the final equality following from the assumptions of Definition 5.2(b). Hence the first lower bound for the condition number is an immediate consequence of (4.3) by taking $P = T_n[d, M]$ and noticing that $||T_n[d, M]||_{d,\{x_0,..,x_n\}} \leq ||T_n[d, M]||_{d,M} = 1$.

In order to show that this bound may be attained, denote by $x_0 < .. < x_n$ the alternant (5.3) of $T_n[d, M]$ in M. By Definition 5.2(b) we have $x_n = \gamma$. Recall that the inverse of $V_n(d; x_0, .., x_n)$ has a checkerboard sign distribution: denote by $a_{j,k}$, $j, k = 0, .., n$ the element of $V_n(d; x_0, .., x_n)^{-1}$ in position (j, k), then with the Lagrange polynomials of (4.4) there holds

$$d(x_k) \cdot \ell_{k,n}(x) = \sum_{j=0}^{n} a_{j,k} \cdot x^j, \quad k = 0, .., n. \tag{5.4}$$

Consequently, $a_{j,k} \cdot (-1)^{j+k} \geq 0$, and

$$\begin{aligned} & ||V_n(d; x_0, .., x_n)^{-1}||_{\infty,p} \\ & = ||V_n(d; x_0, .., x_n)^{-1} \cdot \Big((-1)^n, (-1)^{n-1}, .., (-1)^0\Big)^T ||_p = ||\vec{T}_n[d, M]||_p. \end{aligned} \tag{5.5}$$

It remains to show the second representation of the lower bound. Due to (5.3), $T := T_n[d, M]$ has n positive real zeros, and therefore its coefficients have oscillating signs. Using (A.2) we get

$$\frac{||\vec{T}||_1}{(n+1)^{1-1/p}} \leq ||\vec{T}||_p \leq ||\vec{T}||_1 = |T(-1)|,$$

leading to the assertion. □

In the second part of this section, we want to give an analogue of Theorem 5.4 for the case of arbitrary real nodes. However, here the reasoning is more involved. We first formulate some tools in the following lemma. Part (a) is a weighted analogue of the V.A. Markov Theorem mentioned in Section 2.1.1 (see [Sch71, Satz 6.12]), and part (c) reminds of a result due to De la Vallée–Poussin (see [Sch71, Satz 6.7]). Finally, in part (d) we give a relationship between the weighted constrained Chebyshev problem and the $(\infty, 1)$ norm of an inverse weighted Vandermonde matrix (compare Example 4.6 where instead of the monomials more generally Faber polynomials are discussed).

Lemma 5.5 *Let (d, M) satisfy the conditions of Definition 5.2(a), and suppose that (d, M) is symmetric, i.e., $M = -M$ and $d(x) = d(-x)$ for all $x \in M$.*

(a) *The maximum in*

$$c_{j,n}(d, M) := \max\{\frac{|P^{(j)}(0)|}{j! \cdot ||P||_{d,M}} : \deg P \leq n\}$$

is attained for $P = T_n[d, M]$ if $n - j$ is even, and for $P = T_{n-1}[d, M]$ if $n - j$ is odd.

(b) *Let $x_0 < x_1 < .. < x_n$, $x_j \in M$ be symmetric with respect to the origin. Then for $n = 2k$ or for $n = 2k - 1$ we have*

$$||V_n(d; x_0, .., x_n)^{-1}||_{\infty,p} = \epsilon \cdot \max\{C_{0,p}, C_{1,p}\}, \quad \epsilon \in [1, 2^{1/p}],$$

where

$$C_{0,p} := ||V_{n-k}(d(\sqrt{x}); x_k^2, .., x_n^2)^{-1}||_{\infty,p}, \quad C_{1,p} := ||V_{k-1}(\frac{d(\sqrt{x})}{\sqrt{x}}; x_{n-k+1}^2, .., x_n^2)^{-1}||_{\infty,p}.$$

(c) *Given a density function h, $0 \leq y_0 < y_1 < ... < y_\ell$, $y_j \in M$, and a polynomial $T \in \mathcal{P}_\ell$ with $T(y_j) \cdot T(y_{j+1}) < 0$, $j = 0, .., \ell - 1$, there holds*

$$||\vec{T}_\ell[h; \{y_0, .., y_\ell\}]||_p \leq \max_{j=0,..,\ell} \frac{h(y_j)}{|T(y_j)|} \cdot ||\vec{T}||_p.$$

(d) *Let $x_0 < x_1 < .. < x_n$, $x_j \in M$ be symmetric with respect to the origin. Then*

$$||V_n(d; x_0, .., x_n)^{-1}||_{\infty,1} = \epsilon \cdot \Delta_n(1, \{i\}; d, \{x_0, .., x_n\})$$

with $\epsilon \in [1, \sqrt{2}]$. Moreover, we have $\Delta_n(1, \{i\}; d, \{x_0, .., x_n\}) = |Q(i)|$, where $Q \in \mathcal{P}_n$ satisfies

$$Q(x_j) = \sigma \cdot (-1)^{n-j} \cdot \frac{x_j - i}{|x_j - i|} \cdot d(x_j), \quad j = 0, ..., n,$$

with a constant $\sigma \in \mathbb{C}$, $|\sigma| = 1$.

Proof: **(a)** Let the maximum in $c_{j,n}(d, M)$ be attained for some polynomial $P \in \mathcal{P}_n$. We first show that P may be supposed to be either even or odd. In fact, with $Q(z) := (P(z) + (-1)^j \cdot P(-z))/2$, we obtain $Q^{(j)}(0) = P^{(j)}(0)$, and $||P||_{d,M} \leq ||Q||_{d,M}$ according to the symmetry of (d, M). Defining $M' := \{z^2 : z \in M\}$, and the integer k by $n = 2k$ or $n = 2k - 1$, it follows that

$$c_{2j,n}(d, M) = c_{j,n-k}(d(\sqrt{x}), M'), \quad c_{2j+1,n}(d, M) = c_{j,k-1}(\frac{d(\sqrt{x})}{\sqrt{x}}, M').$$

On the other hand, the convex hull of M' is a subset of $[0; +\infty)$, and we have shown implicitly in the proof of Theorem 5.4 that the maximum in $c_{j,\ell}(d', M')$ is attained for $T_\ell[d', M']$. Hence the assertion follows from (5.2).

(b) Taking $M = \{x_0, .., x_n\}$ in part (a) and using (4.3) we obtain

$$||V_n(d, x_0, .., x_n)^{-1}||_{\infty,p} \leq ||(c_{0,n}(d, M), .., c_{n,n}(d, M))^T||_p = ||\vec{T}_n[d, M] + \vec{T}_{n-1}[d, M]||_p,$$

whereas from (4.3)

$$||V_n(d, x_0, .., x_n)^{-1}||_{\infty,p} \geq \max\{||\vec{T}_n[d, M]||_p, ||\vec{T}_{n-1}[d, M]||_p\}.$$

The polynomials $T_n[d, M]$ and $T_{n-1}[d, M]$ have different parity, and by (A.2) we get

$$||\vec{T}_n[d, M] + \vec{T}_{n-1}[d, M]||_p \leq 2^{1/p} \cdot ||(||\vec{T}_n[d, M]||_p, ||\vec{T}_{n-1}[d, M]||_p)^T||_\infty.$$

Therefore the assertion follows by observing that

$$C_{2k-n,p} = ||\vec{T}_n[d, \{x_0, .., x_n\}]||_p, \quad C_{1+n-2k,p} = ||\vec{T}_{n-1}[d, \{x_0, .., x_n\}]||_p, \tag{5.6}$$

being a consequence of (5.2) and (5.5).

(c) We have shown in (5.5) that $||\vec{T}_\ell[h; \{y_0, .., y_\ell\}]||_p = ||V_\ell(h; y_0, .., y_\ell)^{-1}||_{\infty,p}$. Thus the stated inequality is a consequence of the checkerboard sign distribution of $V_\ell(h; y_0, .., y_\ell)^{-1}$.

(d) Recall from Appendix A that $||A||_{\infty,1}$ is evaluated by taking the sum of the 1–norm of the columns of A. Denoting by $\ell_{j,n}$, $0 \leq j \leq n$, the Lagrange polynomials corresponding to $x_0, .., x_n$, we obtain according to (5.4)

$$||V_n(d, x_0, .., x_n)^{-1}||_{\infty,1} = \sum_{j=0}^{n} d(x_j) \cdot ||\vec{\ell}_{j,n}||_1.$$

Since the nodes $x_0 < ... < x_n$ are symmetric with respect to the origin, one verifies that there exist positive $c_{j,k}$ such that

$$\ell_{j,n}(x) = (-1)^{n-j} \cdot (x + x_j) \cdot q_j(z), \quad q_j(z) := c_{j,0} \cdot x^{n-2} - c_{j,2} \cdot x^{n-4} + c_{j,4} \cdot x^{n-6} \mp ...,$$

and consequently

$$\frac{|\ell_{j,n}(i)|}{||\vec{\ell}_{j,n}||_1} = \frac{|q_j(i)| \cdot |i + x_j|}{||\vec{q}_j||_1 \cdot (1 + |x_j|)} = \frac{|i + x_j|}{1 + |x_j|} \in [\frac{1}{\sqrt{2}}, 1]$$

(here we have tacitly excluded the case $z_j = 0$, i.e., $n = 2j$, where one verifies directly that $||\vec{\ell}_{j,n}||_1 = |\ell_{j,n}(i)|$). Thus the first part of assertion (d) follows from recalling that

$$\Delta_n(1, \{i\}; d, \{x_0, .., x_n\}) = \sum_{j=0}^{n} d(x_j) \cdot |\ell_{j,n}(i)|,$$

see (4.7) of Section 4.1. Also, for each polynomial $Q \in \mathcal{P}_n$ we have

$$Q(i) = \sum_{j=0}^{n} Q(x_j) \cdot \ell_{j,n}(i).$$

Therefore, $Q(i) = \sigma_1 \cdot \Delta_n(1, \{i\}; d, M)$ if (and only if) for $j = 0, .., n$

$$Q(x_j) = \sigma_1 \cdot \frac{|\ell_{j,n}(i)|}{\ell_{j,n}(i)} \cdot d(x_j) = \sigma_2 \cdot (-1)^{n-j} \cdot \frac{x_j - i}{|x_j - i|} \cdot d(x_j),$$

with $|\sigma_2/\sigma_1| = 1$, σ_2 being independent of j. □

Proceeding as in the proof of Theorem 5.4, one could imagine that the optimal configuration of nodes for a weighted Vandermonde matrix with respect to symmetric M and d is given by the (symmetric) alternant of $T_n[d, M]$, which by the Chebyshev criterion consists exactly of $n + 1$ elements $x_0, .., x_n$. However, in view of Lemma 5.5(b) and (5.6) we would require that $C_{2k-n,p} \geq C_{1+n-2k,p}$. Let us show in the following example that such an inequality in general is not valid.

EXAMPLE 5.6 *Consider* $n = 3$, $d = 1$, *and the symmetric nodes* $(x_0, x_1, x_2, x_3) = (-\gamma, -\gamma/2, \gamma/2, \gamma)$, $\gamma > 0$, *i.e., the alternant of the Chebyshev polynomial* $T_3[1, [-\gamma, \gamma]]$. *We have*

$$V_1(1; x_2^2, x_3^2)^{-1} = \begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{4}{3\gamma^2} & \frac{4}{3\gamma^2} \end{pmatrix}$$

$$V_1(\frac{1}{\sqrt{x}}; x_2^2, x_3^2)^{-1} = V_1(1; x_2^2, x_3^2)^{-1} \cdot \text{diag }(x_2, x_3)^{-1} = \begin{pmatrix} \frac{8}{3\gamma} & -\frac{1}{3\gamma} \\ -\frac{8}{3\gamma^3} & \frac{4}{3\gamma^3} \end{pmatrix}.$$

Thus

$$C_{0,\infty} = \max\{\frac{5}{3}, \frac{8}{3\gamma^2}\}, \quad C_{1,\infty} = \max\{\frac{3}{\gamma}, \frac{4}{\gamma^3}\},$$

and

$$||V_3(1; x_0, .., x_3)^{-1}||_\infty = \max\{C_{0,\infty}, C_{1,\infty}\} = \begin{cases} 4/\gamma^3 = C_{1,\infty} & \text{if } \gamma \leq \sqrt{4/3}, \\ 3/\gamma = C_{1,\infty} & \text{if } \sqrt{4/3} \leq \gamma \leq 9/5, \\ 5/3 = C_{0,\infty} & \text{if } \gamma \geq 9/5. \end{cases}$$

We see that both cases $\max\{C_{0,\infty}, C_{1,\infty}\} = C_{0,\infty}$ *or* $\max\{C_{0,\infty}, C_{1,\infty}\} = C_{1,\infty}$ *may occur.* □

In fact we are only able to show for sufficiently small γ that the lower bound given in Theorem 5.7 below is attained. For instance, from Example 5.6 it follows that, for $n = 3$, $M = [-\gamma, \gamma]$, and $d = 1$, the lower bound is only attained for the proposed Chebyshev nodes if $\gamma \leq 9/5$. However, in Lemma 5.10 we will show that the range for γ proposed below is sufficient to determine optimally conditioned Vandermonde matrices.

THEOREM 5.7 *Let (d, M) be s–admissible. Furthermore, let $\gamma := \max\{x : x \in M\}$, and*

$$S_n(x) := x \cdot T_n[d, M](x) - t_n \cdot \prod_{j=0}^{n} (x - x_j), \tag{5.7}$$

t_n denoting the leading coefficient of $T_n[d, M]$, and $x_0, .., x_n$ the alternant of $T_n[d, M]$.

A lower bound for

$$\{\kappa_{p,\infty}(V_n(d; x_0, .., x_n)) : x_0, .., x_n \in M, \max_j |x_j| = \gamma\}$$

is given by

$$\frac{K_{p,n}(\gamma)}{d(\gamma)} \cdot ||\vec{T}_n[d, M]||_p.$$

Up to a factor being less or equal to $2^{1/p}$, this bound is attained for $x_0, .., x_n$ being the alternant of $T_n[d, M]$, provided that the condition $\gamma \leq 1$ or the condition $||\vec{S}_n||_p \leq ||\vec{T}_n[d, M]||_p$ holds.

Proof: The lower bound given in the assertion is established in the same way as in the proof of Theorem 5.4. It remains to show that the given bound is approximately attained (exactly attained for $p = \infty$) for the nodes $x_0 < .. < x_n$ constituting the alternant of $T_n[d, M] = T_n[d, \{x_0, .., x_n\}]$. First, by Definition 5.2(b) there holds $\max_j |x_j| = x_n = \gamma$. Consequently, the resulting weighted Vandermonde matrix is included in the class discussed in Theorem 5.7, and it is sufficient to prove that

$$||V_n(d; x_0, .., x_n)^{-1}||_{\infty,p} \leq 2^{1/p} \cdot ||\vec{T}_n[d, M]||_p.$$

Note that $T_n[d, M]$ is either even or odd, depending on the parity of n. In particular, its alternant is symmetric with respect to the origin, and the above matrix norm may be computed by applying Lemma 5.5(b). The aim of the following considerations is to establish the inequality $C_{2k-n,p} \geq C_{1+n-2k,p}$ which by (5.6) and Lemma 5.5(b) implies the above estimate, and therefore the assertion of the theorem.

We first discuss the case $\gamma \leq 1$. The polynomial $P(z) := (z/\gamma) \cdot T_{n-1}[d, M]$ has the same parity as $T_n[d, M]$, and $||P||_{d,M} \leq 1$. Hence by Lemma 5.5(a) we obtain $||\vec{P}||_p \leq ||\vec{T}_n[d, M]||_p$, or $||\vec{T}_{n-1}[d, M]||_p \leq \gamma \cdot ||\vec{T}_n[d, M]||_p \leq ||\vec{T}_n[d, M]||_p$, yielding the desired inequality.

If $n = 2k - 1$, then $T_n[d, M]$ is odd. Consequently, S_n is even, and has a degree bounded by $n - 1$. Define the polynomial T by

$$x \cdot T(x^2) = \frac{1}{2} \cdot \Big(x \cdot S_n(x) + T_n[d, M](x)\Big).$$

Then T has a degree bounded by $k-1$,

$$T(x_j^2) = \frac{x_j^2+1}{2\cdot x_j}\cdot T_n[d,M](x_j) = \frac{x_j^2+1}{2\cdot x_j}\cdot (-1)^{n-j}\cdot d(x_j),$$

and in particular $|T(x_j^2)| \geq d(x_j)$. By applying Lemma 5.5(c) we get

$$\begin{aligned} C_{0,p} &= ||\vec{T}_{n-k}[d(\sqrt{x}), \{x_k^2, .., x_n^2\}]||_p \\ &\leq ||\vec{T}||_p \leq \frac{1}{2}\cdot\Big(||\vec{S}_n||_p + ||\vec{T}_n[d,M]||_p\Big) \leq ||\vec{T}_n[d,M]||_p = C_{1,p}. \end{aligned}$$

In the case $n=2k$ we have $x_k=0$. Therefore, the matrix $V_k(d(\sqrt{x}); x_k^2, .., x_n^2)$ is block lower triangular, containing the matrix $V_{k-1}(d(\sqrt{x})/x; x_{k+1}^2, .., x_n^2)$ as its lower right block. For any block triangular matrix, the (∞, p) norm of the inverse is greater than or equal to the norm of the inverse of any of its diagonal blocks. With help of (5.5) we obtain using polynomial language

$$||\vec{T}_{k-1}[\frac{d(\sqrt{x})}{x}, \{x_{k+1}^2, .., x_n^2\}]||_p \leq ||\vec{T}_k[d(\sqrt{x}), \{x_k^2, .., x_n^2\}]||_p = C_{0,p}.$$

One verifies as above that S_n introduced in (5.7) is odd. Let the polynomial T be defined by

$$T := \frac{1}{2}\cdot\Big(T_{k-1}[\frac{d(\sqrt{x})}{x}, \{x_{k+1}^2, .., x_n^2\}] + T_1\Big), \quad x\cdot T_1(x^2) = S_n(x).$$

Then T has a degree bounded by $k-1$,

$$T(x_j^2) = \frac{(-1)^{n-j}\cdot d(x_j)}{2}\cdot\Big(\frac{1}{x_j^2}+1\Big),$$

and in particular $|T(x_j^2)| \geq d(x_j)/x_j$. With help of Lemma 5.5(c) we arrive at

$$C_{1,p} = ||\vec{T}_{k-1}[\frac{d(\sqrt{x})}{\sqrt{x}}, \{x_{k+1}^2, .., x_n^2\}]||_p \leq ||\vec{T}||_p \leq \frac{1}{2}\cdot\Big(||\vec{S}_n||_p + C_{0,p}\Big) \leq C_{0,p}.$$

□

Let us mention at the end of this section that S_n defined in (5.7) may be rewritten for particular (d,M) in terms of $T_n[d,M]$ and $T_{n-1}[d,M]$. This is for instance true for the cases $(d,M) = (1, [-\gamma;\gamma])$, and $(d,M) = (K_{2,n}, [-\gamma;\gamma])$. Also, for particular (d,M) we may give an approximately tight lower bound for $\kappa_{1,\infty}(V_n(d, x_0, .., x_n))$ in terms of $\Delta_n(1, \{i\}; d, M)$ by exploiting Lemma 5.5(d). These results are studied more detailed in the following two sections.

5.2 Optimal nodes for real Vandermonde matrices

Let us exploit in a first step the assertions of Theorems 5.4 and 5.7 in order to derive approximately tight lower bounds for the p–condition number of a Vandermonde with real nodes.

THEOREM 5.8 *Let $V_n := V_n(1, x_0, .., x_n)$ denote a Vandermonde matrix of order $n+1$, $n \geq 0$, $x_0, .., x_n \in \mathbb{R}$, and let $p \in [1; \infty]$ be a Hölder index. Then*

$$\kappa_p(V_n) \geq \frac{1}{2 \cdot (n+1)^{1/p}} \cdot \Big((1+\sqrt{2})^n + (1-\sqrt{2})^n\Big),$$

and this bound is attained for the nodes $x_j = \cos(j \cdot \pi/n)$ up to the factor $(n+1) \cdot (2n+2)^{1/p}$. If in addition all nodes are non–negative or non–positive then

$$\kappa_p(V_n) \geq \frac{1}{2 \cdot (n+1)^{1/p}} \cdot \Big((1+\sqrt{2})^{2n} + (1+\sqrt{2})^{-2n}\Big),$$

and this bound is attained for the nodes $x_j = (1+\cos(j \cdot \pi/n))/2$ up to the factor $(n+1)^{1+1/p}$.

In particular, both lower bounds remain valid if we scale the columns of V_n, i.e., if we multiply V_n on the right by a diagonal matrix.

Proof: Let $\gamma \in \mathbb{R}$ be the smallest number such that $x_0, .., x_n \in [-\gamma; \gamma] =: M$. Since $(1, M)$ is s–admissible, we may apply Theorem 5.7 which together with (A.8), (A.9) leads to

$$\kappa_p(V_n) \geq (n+1)^{-1/p} \cdot \kappa_{p,\infty}(V_n) \geq (n+1)^{-1/p} \cdot ||\vec{T}_n(\frac{x}{\gamma})||_p \cdot ||(1, \gamma, ..., \gamma^n)^T||_q,$$

where q is the Hölder index being complementary to p. Denoting the coefficients of T_n by $c_0, .., c_n$, we obtain using the Hölder inequality (A.2)

$$||\vec{T}_n(\frac{x}{\gamma})||_p \cdot ||(1, \gamma, ..., \gamma^n)^T||_q \geq \sum_{j=0}^{n} |c_j| = |T_n(i)| = \frac{1}{2} \cdot \Big((1+\sqrt{2})^n + (1-\sqrt{2})^n\Big),$$

yielding the stated lower bound. On the other hand, by taking the nodes constituting the alternant of T_n, Theorem 5.7 together with (A.8), (A.9) gives

$$\begin{aligned} \kappa_p(V_n) &\leq (n+1)^{1/p} \cdot \kappa_{p,\infty}(V_n) \leq (2n+2)^{1/p} \cdot ||\vec{T}_n||_p \cdot K_{p,n}(1) \\ &\leq (2n+2)^{1/p} \cdot ||\vec{T}_n||_1 \cdot (n+1)^{1-1/p} = (2n+2)^{1/p} \cdot |T_n(i)| \cdot (n+1)^{1-1/p}, \end{aligned}$$

showing that this lower bound is attained up to the factor stated in the assertion. The reasoning for the case of non–negative nodes is similar, here we take into account that $(1, [0; \gamma])$ is p–admissible, apply Theorem 5.4, and use the formula

$$||\vec{T}_n[1, [0; 1]]||_1 = ||\vec{T}_n(-1+2x)||_1 = |T_n(-3)| = \frac{1}{2} \cdot \Big((1+\sqrt{2})^{2n} + (1+\sqrt{2})^{-2n}\Big). \tag{5.8}$$

It remains to discuss $\kappa_p(V_n \cdot D)$ with D being diagonal. As in Section 2.1.3 one shows using Lemma 5.5(a) that, in the case of nodes in $[-\gamma; \gamma]$,

$$(n+1)^{1/p} \cdot \kappa_p(V_n \cdot D) \geq ||D^{-1} \cdot \vec{T}_n(\frac{x}{\gamma})||_p \cdot ||D \cdot (1, \gamma, ..., \gamma^n)^T||_q,$$

and an application of the Hölder inequality leads to the same conclusion. A proof for nodes in $[0;\gamma]$ is similar, we omit the details. □

Let us compare the above inequalities with some classical ones obtained in [GaIn88, Tyr94a]. Gautschi and Inglese gave lower bounds for the quantity $\kappa_\infty(V_n^T) = \kappa_1(V_n)$, in particular that $\kappa_1(V_n) > 2^n$ for nonnegative nodes [GaIn88, Theorem 2.1] and $\kappa_1(V_n) > 2^{(n+1)/2}$ for symmetric nodes, i.e., $x_j = -x_{n-j}$ for $j = 0, 1, .., n$ [GaIn88, Theorem 3.1]. Our corresponding estimates of Theorem 5.8 for $p = 1$ are tighter for $n \geq 2$, and $n \geq 6$, respectively. Also, from Example 4.2 we may conclude that our bounds for $p = 1$ and arbitrary real nodes are asymptotically tight at least up to a factor $(3^{3/4}/2) \cdot (1 + \sqrt{2}) \cdot (n + 1)$.

Lower bounds for $\kappa_2(V_n)$ have been derived by Tyrtyshnikov [Tyr94a, Theorem 4.1], namely $2^{n-1}/\sqrt{n+1}$ in the case of arbitrary nodes and 2^{n-1} for $x_j \in [-1; 1]$. Again, our corresponding estimates are tighter for $n \geq 2$.

Let us also mention a result of Taylor [Tay78, p.54] who proved that for any family of nodes $\{x_{j,n}\}_{0\leq j\leq n}$ lying in the real interval $[\alpha - \beta; \alpha + \beta]$, $\beta > 0$, there holds

$$\liminf_{n\to\infty} \kappa_2(V_n(1, x_{0,n}, ..., x_{n,n}))^{1/n} \geq \frac{4 \cdot (\alpha + \beta)}{2\beta},$$

where we have assumed (without loss of generality) that $\alpha \geq 0$. Notice that

$$\liminf_{n\to\infty} \kappa_2(V_n(1, x_{0,n}, ..., x_{n,n}))^{1/n} = \liminf_{n\to\infty} \kappa_\infty(V_n(1, x_{0,n}, ..., x_{n,n}))^{1/n},$$

and for the latter quantity we have according to (4.5) and Example 4.4 the optimal lower bound $\lim_{n\to\infty} \kappa_{[\alpha-\beta;\alpha+\beta]}(\Pi_n)$ being determined in Corollary 2.19 of Section 2.5.

We now turn to the problem of finding the 'optimal' real nodes minimizing the condition number of Vandermonde matrices. We will restrict ourselves to particular real intervals and to the infinity Hölder norm, since here the assertions of Theorem 5.4 and Theorem 5.7 simplify. The configuration of nodes minimizing the 1–condition number of Vandermonde matrices has been investigated by Gautschi [Gau75b]. He conjectured that an optimal real configuration is symmetric with respect to the origin.

Theorem 5.9 *Let $\gamma > 0$, and $n \geq 2$.*

(a) *The minimum of*

$$\{\kappa_\infty(V_n(1; x_0, .., x_n)) : x_0, .., x_n \in [0, \gamma], \max_j x_j = \gamma\}$$

is given by

$$\Gamma_n^+(\gamma) := (1 + \gamma + ... + \gamma^n) \cdot n \cdot \max_{j=0,1,..,n} \frac{(4/\gamma)^{n-j}}{2n - j} \cdot \binom{2n - j}{j}.$$

This minimum is attained for the optimal configuration of nodes

$$x_j = \frac{\gamma}{2} \cdot (1 + \cos(\frac{n - j}{n} \cdot \pi)), \quad j = 0, .., n.$$

(b) *A lower bound for*

$$\{\kappa_\infty(V_n(1; x_0, .., x_n)) : x_0, .., x_n \in [-\gamma; \gamma], \max_j |x_j| = \gamma\}$$

is given by

$$\Gamma_n(\gamma) := (1 + \gamma + ... + \gamma^n) \cdot \frac{n}{2} \cdot \max_{j=0,..,[n/2]} \frac{(2/\gamma)^{n-2j}}{n-j} \cdot \binom{n-j}{j}.$$

This bound is attained for the optimal configuration of nodes

$$x_j = \gamma \cdot \cos(\frac{n-j}{n} \cdot \pi)), \quad j = 0, .., n,$$

provided that $\gamma \leq \sqrt{2}$ *and* $n \geq 2$, *or* $\gamma < \sqrt{3}$ *and* n *being sufficiently large.*

Proof: In order to show (a), we notice that $(1, [0; \gamma])$ is p–admissible, and apply Theorem 5.4 by taking into account that

$$K_{\infty,n}(\gamma) = \sum_{j=0}^{n} \gamma^j, \quad T_n[1, [0; \gamma]](x) = T_n(-1 + \frac{2x}{\gamma}).$$

Now $T_n(-1 + 2y^2) = T_{2n}(y)$, and thus part (a) follows by using the formula (see, e.g., [Riv74])

$$T_n(y) = \sum_{j=0}^{[n/2]} \frac{n}{2} \cdot \frac{(-1)^j}{n-j} \cdot \binom{n-j}{j} \cdot (2y)^{n-2j}, \quad n \geq 1. \tag{5.9}$$

The lower bound of part (b) follows by Theorem 5.7 and (5.9) since $(1, [-\gamma; \gamma])$ is s–admissible, and $T_n[1, [-\gamma; \gamma]](x) = T_n(x/\gamma)$. Also we have shown in Theorem 5.7 that $\Gamma_n(\gamma)$ is attained if $\gamma \leq 1$. In the case $\gamma \geq 1$, we have to examine the polynomial S_n being defined in (5.7). One verifies with help of the Chebyshev polynomials of the second kind U_n that

$$S_n(x) = x \cdot T_n(x/\gamma) - (x^2 - \gamma^2) \cdot \frac{1}{\gamma} \cdot U_{n-1}(x/\gamma) = \gamma \cdot T_{n-1}(x/\gamma).$$

By using (5.9) it is not difficult to show that $||\vec{S}_n||_\infty \leq ||\vec{T}_n(x/\gamma)||_\infty$, provided $n \geq 2$ and $\gamma \leq \sqrt{2}$, implying that in this case the lower bound $\Gamma_n(\gamma)$ is attained. Moreover, by using Gautschi's asymptotic formula for $||\vec{T}_n(\frac{x}{\beta})||_\infty$ mentioned in Section 2.1.1 one verifies that

$$\gamma < \sqrt{3}: \quad \lim_{n\to\infty} \frac{||\vec{T}_n[1, [-\gamma; \gamma]]||_\infty}{||\vec{S}_n||_\infty} = \frac{1 + \sqrt{1 + \gamma^2}}{\gamma^2} < 1,$$

yielding the second part of assertion (b). □

In order to be able to drop the constrain $\max x_j = \gamma$ in Theorem 5.9(b), we require

n	γ_n	$\Gamma_n(\gamma_n)$	γ_n^+	$\Gamma_n^+(\gamma_n)$
1	1	2.0e0	2	3.0000e0
2	$\sqrt{2}$	4.4142e0	1	2.4000e1
3	1.154700538	1.3062e1	1.521379707	1.7331e2
4	1	4.0000e1	1.599999997	9.8810e2
5	1.190804264	1.1492e2	1.190804264	6.4355e3
6	1.302411005	2.9550e2	1.302411006	4.2552e4
7	1.370544100	7.1563e2	1.370544100	2.5190e5
8	1.264911064	1.7199e3	1.181818233	1.5037e6
9	1.154700544	4.3721e3	1.212792543	9.7174e6
10	1.107495548	1.1713e4	1.244471545	5.8498e7
11	1.139975402	3.0690e4	1.200000000	3.3843e8
12	1.164110336	7.7552e4	1.164110336	2.1375e9
13	1.182344594	1.9064e5	1.182344594	1.2982e10
14	1.196310819	4.5863e5	1.196310819	7.5612e10
15	1.144155277	1.1082e6	1.133544020	4.5954e11
16	1.090119774	2.8147e6	1.145377677	2.8062e12
17	1.102485418	7.2331e6	1.154867318	1.6552e13
18	1.112571589	1.8198e7	1.112571589	9.7275e13
19	1.120866477	4.4988e7	1.120866477	5.9623e14

Table 5.1: Condition Numbers and corresponding largest nodes for optimally conditioned Vandermonde matrices

LEMMA 5.10 *Let $n \geq 2$. The function $\Gamma_n : (0; +\infty) \longrightarrow (0; \infty)$ defined in Theorem 5.9(b) is strictly convex, and attains its unique minimum at $\gamma_n \leq \sqrt{2}$.*

Proof: Since Γ_n is a maximum of strictly convex functions, Γ_n is also strictly convex. Moreover,

$$\lim_{\gamma \to 0} \Gamma_n(\gamma) = +\infty, \quad \lim_{\gamma \to +\infty} \Gamma_n(\gamma) = +\infty,$$

so that in fact Γ_n has a unique minimum. It remains to localize this minimum. For $j = 0, 1, .., [n/2]$ let

$$a_{j,n}(x) := (1 + x + .. + x^n) \cdot x^{-n+2j} \cdot \frac{n}{2} \cdot \frac{2^{n-2j}}{n-j} \cdot \binom{n-j}{j}$$

so that

$$\frac{a_{1,n}(x)}{a_{0,n}(x)} = \frac{n}{4} \cdot x^2 \geq 1 \quad \text{for } x \geq \sqrt{2} \text{ and } n \geq 2,$$

$$\frac{a_{2,n}(x)}{a_{1,n}(x)} = \frac{n-3}{8} \cdot x^2 \geq 1 \quad \text{for } x \geq \sqrt{2} \text{ and } n \geq 7.$$

Consequently, if $x > \sqrt{2}$ then $\Gamma_n(x) \neq a_{0,n}(x)$ for all $n \geq 2$ and $\Gamma_n(x) \neq a_{1,n}(x)$ for all $n \geq 7$. Thus, that the minimum of Γ_n is attained at $\gamma_n \leq \sqrt{2}$ if we are able to show that the convex functions $a_{j,n}$ satisfy

$$a'_{1,n}(\sqrt{2}) > 0 \text{ for all } n \leq 6$$
$$a'_{j,n}(\sqrt{2}) > 0 \text{ for all } n \geq 4 \text{ for all } j = 2, .., [n/2].$$

A verification of these inequalities is straight forward by examining the derivatives given by

$$a'_{j,n}(x) = \frac{n \cdot 2^{n-2j-1} \cdot \binom{n-j}{j}}{n-j} \cdot \frac{x^{n+1} \cdot (2j \cdot (x-1) - 1) + (n-2j+1) \cdot (x-1) + 1}{(1-x)^2 \cdot x^{n+1-2j}}.$$

□

In order to determine the configuration of real nodes minimizing the ∞–condition of a Vandermonde matrix of order $(n+1)$, in view of Theorem 5.9(b) and Lemma 5.10 it remains to determine the unique argument $\gamma_n \in]0; \sqrt{2}]$ where the function Γ_n becomes minimal. This is a numerically easy task since each function $\gamma^{2j-n} \cdot (1 + \gamma + ... + \gamma^n)$ is strictly convex.

Similarly, for a nonnegative optimal configuration, we have to find the argument γ_n^+ where the function Γ_n^+ of Theorem 5.9(a) takes its minimum. Some of these optimal arguments together with the minimal condition number are given in Table 5.1 (these quantities have been obtained by using the computer algebra system MAPLE). In fact, the magnitude for the condition number is the same as in the case of $\kappa_1(V_n)$ discussed by Gautschi [Gau75b, Table 4.1] and Gautschi and Inglese [GaIn88, Table 1 and Table 2].

The arguments leading to the assertion of Theorem 5.9 also apply for more general sets M such as the union of disjoint compact intervals. However, in order to give explicit tight lower bounds we require some information about the corresponding Chebyshev polynomials $T_n[1, M]$, as obtained for instance by Peherstorfer in the case of several intervals (see for instance the survey paper [Peh93]).

5.3 Sharp lower bounds for particular Krylov matrices

The results of Section 5.1 allow us also to deduce approximately tight lower bounds for the Euclidean condition numbers of Krylov matrices $K_n(B, b) = (b, B \cdot b, B^2 \cdot b, ..., B^n \cdot b)$ with Hermitian B. For $M \subset \mathbb{R}$ being a real interval, and $n \geq 0$ an integer, we denote by $\Gamma_n^w(M)$ the minimum over all $\kappa_2(K_n(B, b))$, where B is any Hermitian matrix of order $(m+1) \geq (n+1)$ with eigenvalues in M, and b is any element of $\mathbb{C}^{m+1}$. Recall that by a combination of Theorem 4.11(a) and Theorem 4.12 we have

$$\lim_{n \to \infty} \Gamma_n^w(M)^{1/n} = \Delta^w(\mathbb{D}, M),$$

the latter quantity being computed for particular intervals M in Example 4.17, Example 4.18 and Example 4.19.

The aim of this section is to give a proof for the following

THEOREM 5.11 *Let M be a real interval, $n \geq 1$ an integer, and $\Gamma_n^w(M)$ as above.*

(a) *In the case of scaled Hermitian matrices B we have*

$$\frac{1}{n+1} \leq \frac{\Gamma_n^w([-1;1])}{\Big((1+\sqrt{2})^n + (1-\sqrt{2})^n\Big)/2} \leq \sqrt{2} \cdot (n+1).$$

(b) *In the case of scaled positive semidefinite Hermitian matrices B we have*

$$\frac{1}{n+1} \leq \frac{\Gamma_n^w([0;1])}{\Big((1+\sqrt{2})^{2n} + (1+\sqrt{2})^{-2n}\Big)/2} \leq n+1.$$

(c) *In the case of positive semidefinite Hermitian matrices B we have*

$$\frac{9}{10} \cdot \frac{1}{n+1} \leq \frac{\Gamma_n^w([0;+\infty))}{\exp(\delta^*)^{2n+2}/(4 \cdot \sqrt{2})} \leq \frac{6}{5} \cdot \sqrt{n+1},$$

where $\exp(\delta^*)^2 \approx 3.210$, *confer Example 4.15 of Section 4.3.*

(d) *In the case of Hermitian matrices B we have*

$$\frac{9}{10} \cdot \frac{1}{n+1} \leq \frac{\Gamma_n^w(\mathbb{R})}{\exp(\delta^*)^{n+1}/\sqrt{2}} \leq \sqrt{2} \cdot \sqrt{n+1},$$

where $\exp(\delta^*) \approx 1.792$.

Let us compare our findings of Theorem 5.11 with some classical ones obtained in [Tay78, Tyr94a]. Taylor claimed that $\Gamma_n^w(M)$ grows asymptotically for any compact interval M at least as 2^n [Tay78, p.56]. His reasoning however seems to be incomplete since one requires a lower bound for the Rayleigh quotient $b^T \cdot B^{2n-2} \cdot b / b^T \cdot b$ which without any additional assumptions on b is not available. Tyrtyshnikov [Tyr94b] gives for $\Gamma_n^w(\mathbb{R})$ a lower bound of the form $c \cdot \gamma^n$ with $\gamma \leq 1.14$ (the proof of [Tyr94a, Lemma 3.4] is incomplete [Tyr94b]).

Notice that $\kappa_2(K_n(B,b))$ is invariant under replacing B by $-B$, moreover, for invertible B, the matrix $K_n(B,b)$ is obtained by reversing the order of the columns of $K_n(B^{-1}, b')$, $b' = B^n \cdot b$. Consequently, we have $\Gamma_n^w(M) = \Gamma_n^w(-M)$, and $\Gamma_n^w(M) = \Gamma_n^w(M^{-1})$, $M^{-1} := \{1/z : z \in M\}$, showing that, for instance, the inequalities of Theorem 5.11(b) remain valid for the sets $M = [-1;0]$, $M = [1;\infty)$, and $M = (-\infty;-1]$.

The proof of Theorem 5.11 is separated into several parts. Let us first recall from Theorem 4.11(a) that

$$\frac{\Theta_{n,n}(M)}{\sqrt{n+1}} \leq \Gamma_n^w(M) \leq \sqrt{n+1} \cdot \Theta_{n,n}(M), \tag{5.10}$$

where by Theorem 4.10 together with (4.3)

$$\Theta_{n,n}(M) = \inf\{||V_n(K_{2,n}, x_0, .., x_n)^{-1}||_{\infty,2} : x_0, .., x_n \in M\}.$$

Thus for determining the quantity $\Theta_{n,n}(M)$ we may apply Theorem 5.4 and Theorem 5.7 where we require a close form expression for the nth Chebyshev polynomial with respect to a polynomial weight $d = \sqrt{q}$ with q a polynomial of degree $2n$ being positive on M. These polynomials are explicitly known due to Chebyshev, Markov and Bernstein; we will give a suitable representation in Lemma 5.12 below.

The resulting expressions for compact intervals $M = [-\gamma; \gamma]$ or $M = [0; \gamma]$ are not very handy, hence for a proof of parts (a), (b) we prefer to apply partly different arguments leading to slightly weaker estimates

Proof of Theorem 5.11(a),(b): We have $K_{2,n}(x) \in [1; \sqrt{n+1}]$ for $x \in M := [-1; 1]$, and hence for all nodes $x_0, ..., x_n \in M$

$$||V_n(K_{2,n}, x_0, .., x_n)^{-1}||_{\infty,2} = ||V_n(1, x_0, .., x_n)^{-1} \cdot diag\ (K_{2,n}(x_0), .., K_{2,n}(x_n))||_{\infty,2}$$
$$\geq ||V_n(1, x_0, .., x_n)^{-1}||_{\infty,2}.$$

Using in addition (5.10) and (4.3) we obtain

$$\Gamma_n^w(M) \geq \frac{\Theta_{n,n}(M)}{\sqrt{n+1}} \geq \frac{||\vec{T}_n||_2}{\sqrt{n+1}} \geq \frac{||\vec{T}_n||_1}{n+1},$$

which according to $||\vec{T}_n||_1 = |T_n(i)|$ yields the lower bound of Theorem 5.11(a). On the other hand, a Vandermonde matrix also is a Krylov matrix, and in Theorem 5.8 we have given a Vandermonde matrix V_n with $\kappa_2(V_n) \leq \sqrt{2} \cdot (n+1) \cdot |T_n(i)|$, implying the stated upper bound. Finally, Theorem 5.11(b) is proved using similar arguments, where instead of T_n we take the polynomial $T_n[1, [0; 1]]$, and apply (5.8). □

For a proof of part (c) we require a particular representation of $T_n[\sqrt{q}, [a; b]]$ with q a polynomial of degree $2n$.

LEMMA 5.12 *Let $[a; b] \subset \mathbb{R}$, and let q be a real polynomial of degree $2n$ being nonnegative on $[a; +\infty)$, $q(z) = q_{2n} \cdot (z - z_1) \cdot (z - z_2) \cdot .. \cdot (z - z_{2n})$. Then for $z \in \mathbb{C} \setminus [a; b]$*

$$T_n[\sqrt{q}, [a, b]](z) = \sqrt{q(b)} \cdot \left(\frac{z-b}{b-a}\right)^n \cdot$$
$$\cdot \frac{1}{2} \cdot \left(\prod_{j=1}^{2n} (\sqrt{\frac{z-a}{z-b}} - \sqrt{\frac{z_j-a}{z_j-b}}) + \prod_{j=1}^{2n} (\sqrt{\frac{z-a}{z-b}} + \sqrt{\frac{z_j-a}{z_j-b}}) \right).$$

Furthermore, for $z \in \mathbb{C} \setminus [a; +\infty)$

$$T_n[\sqrt{q}, [a; +\infty)](z) := \lim_{b \to +\infty} T_n[\sqrt{q}, [a; b]](z)$$
$$= \frac{\sqrt{q_{2n}} \cdot (-1)^n}{2} \cdot \left(\prod_{j=1}^{2n} (\sqrt{a-z} - \sqrt{a-z_j}) + \prod_{j=1}^{2n} (\sqrt{a-z} + \sqrt{a-z_j}) \right).$$

Here the branch of the square root is chosen such that $\sqrt{e^{i\phi}} = e^{i\phi/2}$, $-\pi < \phi < \pi$.

Proof: By using the argument transformation

$$z' = \frac{2 \cdot z - a - b}{b - a}, \quad z' \in [-1;1] \text{ iff } z \in [a;b],$$

one obtains

$$\frac{z'+1}{z'-1} = \frac{z-a}{z-b}, \quad \frac{z'-1}{2} = \frac{z-b}{b-a},$$

and thus it sufficient to prove the assertion for the interval $= [a;b] = [-1;1]$. Bernstein gave the representation (see, e.g., [Mei67, Section 4.4] or [MMR94, Theorem 1.2.12, p.394])

$$T_n[\sqrt{q}, [-1;1]](J(y)) = \frac{1}{2} \cdot (y^n \cdot Q(1/y) + y^{-n} \cdot Q(y)), \tag{5.11}$$

provided that

$$q(J(y)) = Q(y) \cdot Q(\frac{1}{y}), \quad Q(y) \neq 0 \text{ for } |y| \leq 1, \tag{5.12}$$

where Q is a real polynomial of degree $2n$, having all its zeros outside the unit disk, and $Q(0) > 0$. Here J denotes the Joukowski map introduced in Example 2.13 of Section 2.4.1, mapping the exterior of the interval $[-1;1]$ conformally on the exterior of the unit disk. In particular, the zeros of $q(J(y))$ are given by $J^{-1}(z_j)$ and $1/J^{-1}(z_j)$, $1 \leq j \leq 2n$. Thus

$$q(J(y)) = q(1) \cdot \prod_{j=1}^{2n} \frac{J(y) - z_j}{1 - z_j} = q(1) \cdot y^{-2n} \prod_{j=1}^{2n} \frac{(y - J^{-1}(z_j)) \cdot (y - 1/J^{-1}(z_j))}{(1 - J^{-1}(z_j)) \cdot (1 - 1/J^{-1}(z_j))}$$

$$= q(1) \prod_{j=1}^{2n} \frac{(y - J^{-1}(z_j)) \cdot (1/y - J^{-1}(z_j))}{(1 - J^{-1}(z_j))^2}.$$

Consequently, the polynomial Q in (5.12) is given by

$$Q(y) = \sqrt{q(1)} \cdot \prod_{j=1}^{2n} \frac{y - J^{-1}(z_j)}{1 - J^{-1}(z_j)}. \tag{5.13}$$

The inverse Joukowski map may be rewritten as

$$J^{-1}(z) = S(R(z)), \quad R(z) := \sqrt{\frac{z+1}{z-1}}, \quad S(w) = \frac{w+1}{w-1}.$$

Due to (5.11), for the first part of Lemma 5.12 it remains to show that

$$\frac{1}{\sqrt{q(1)}} \cdot y^{-n} \cdot Q(y) = \frac{(z-1)^n}{2^n} \cdot \prod_{j=1}^{2n} (R(z) - R(z_j)), \quad z = J(y). \tag{5.14}$$

In fact, using (5.13), the left hand side of (5.14) may be rewritten as

$$S(R(z))^{-n} \cdot \prod_{j=1}^{2n} \frac{S(R(z)) - S(R(z_j))}{S(\infty) - S(R(z_j))} = S(R(z))^{-n} \cdot \prod_{j=1}^{2n} \frac{R(z) - R(z_j)}{R(z) - 1}$$

$$= (R(z)^2 - 1)^{-n} \cdot \prod_{j=1}^{2n} (R(z) - R(z_j)) = \frac{(z-1)^n}{2^n} \cdot \prod_{j=1}^{2n} (R(z) - R(z_j)),$$

implying (5.14). The second part of Lemma 5.12 now follows by taking the limit $b \to \infty$. □

We will also require the following property having a nice geometric interpretation

LEMMA 5.13 *Denote by I_n, $n \geq 1$, the product of the distances between i and the $(2n+2)$th roots of unity having negative imaginary part, i.e.,*

$$I_n = \prod_{j=1}^{n} \Big| 1 + \exp(\pi \cdot i \cdot \frac{2j-n-1}{2n+2}) \Big|.$$

Then

$$9/10 \leq \sqrt{2} \cdot I_n \cdot \exp(\delta^*)^{-n-1} \leq 1,$$

and the upper bound is attained for $n \to \infty$.

Proof: We first require a different representation of the constant δ^* being defined in Example 4.15

$$\delta^* = \frac{1}{\pi} \cdot \int_0^{\pi/2} \log | \frac{1}{\tan(\alpha/2)} | \, d\alpha = \log(2) + \int_{-1/2}^{1/2} f(t) \, dt, \quad f(t) = \log(\cos(\frac{\pi \cdot t}{2})).$$

Also, $\log |1 + \exp(\pi \cdot i \cdot t)| = \log 2 + f(t)$, and thus

$$\frac{1}{n+1} \cdot \log(\sqrt{2} \cdot I_n) = \log 2 + \frac{1}{n+1} \cdot \Big(- \log(\sqrt{2}) + \sum_{j=1}^{n} f(-\frac{1}{2} + \frac{j}{n+1}) \Big).$$

Since $f(\pm 1/2) = -\log(\sqrt{2})$, the same sum is obtained by using the composite trapezoidal rule for approximating the above integral. We now apply the well–known error formula

$$\int_{-1/2}^{1/2} f(t) \, dt - \frac{1}{n+1} \cdot \Big(\frac{f(-1/2) + f(1/2)}{2} + \sum_{j=1}^{n} f(-\frac{1}{2} + \frac{j}{n+1}) \Big) = - \frac{f''(\xi)}{12 \cdot (n+1)^3}$$

with a suitable $\xi \in (-1/2, 1/2)$, and notice that $f''(\xi) = -\pi^2/(2 \cdot \cos(\pi \cdot \xi/2))^2 \in [-\pi^2/4; -\pi^2/2]$. Consequently,

$$\frac{\sqrt{2} \cdot I_n}{e^{(n+1) \cdot \delta^*}} \in [\exp(-\frac{\pi^2}{24 \cdot (n+1)^2}); \exp(-\frac{\pi^2}{48 \cdot (n+1)^2})],$$

and the assertion follows by taking into account that $\exp(-\frac{\pi^2}{24 \cdot (n+1)^2}) \geq 9/10$ for $n \geq 1$. □

Proof of Theorem 5.11(c): We know from Lemma 5.3 that $(K_{2,n}, [0; \gamma])$ is p–admissible for each $\gamma \geq 0$. Furthermore, $q := (K_{2,n})^2$ is a polynomial of degree $2n$, and hence the quantity

$$t_n := \lim_{\gamma \to \infty} |\vec{T}_n[\sqrt{q}, [0; \gamma]](-1)|$$

is well–defined according to Lemma 5.12. Using (5.10) and Theorem 5.4 we get

$$\Gamma_n^w([0; +\infty)) \geq \lim_{\gamma \to \infty} \frac{\Theta_{n,n}([0; \gamma))}{\sqrt{n+1}} \geq \frac{t_n}{n+1}, \quad \Gamma_n^w([0; +\infty)) \leq t_n \cdot \sqrt{n+1}.$$

It remains to give estimates for t_n. By Lemma 5.12 we have

$$t_n = \frac{1}{2} \cdot (t'_n + \frac{q(-1)}{t'_n}) = \frac{1}{2} \cdot (t'_n + \frac{n+1}{t'_n}), \quad t'_n = \prod_{j=1}^{2n} 1 + \sqrt{-z_j},$$

where z_j, $j = 1, .., 2n$ are the zeros of q, i.e., those $(2n+2)$th roots of unity being different from ± 1. Since our branch of the square root maps on the right half plane, we get $2 \cdot t'_n = I_{2n+1}$, and by applying Lemma 5.13 we arrive at

$$9/10 \cdot \exp(\delta^*)^{2n+2} \leq 2 \cdot \sqrt{2} \cdot t'_n \leq \exp(\delta^*)^{2n+2}.$$

In particular,

$$0 \leq \frac{n+1}{t'^2_n} \leq \frac{800}{81} \cdot (n+1) \cdot \exp(\delta^*)^{-4n-4} \leq \frac{1}{5},$$

and hence $t_n = \epsilon \cdot \exp(\delta^*)^{2n+2} / (4 \cdot \sqrt{2})$ with an $\epsilon \in [9/10; 6/5]$, leading to the assertion. □

Proof of Theorem 5.11(d): The reasoning of the proof of part (c) may not be applied since the initial condition $||S_n||_2 \leq ||T_n[K_{2,n}, [-\gamma; \gamma]]||_2$ of Theorem 5.7 may be shown to fail for large γ. Instead, define the polynomial

$$Q_n(z) := \prod_{\Im z_j < 0} (z - z_j),$$

where $z_1, .., z_{2n}$ are the zeros of the real polynomial $q := (K_{2,n})^2$, i.e., those $(2n+2)$th roots of unity being different from ± 1. Let us first summarize some properties of Q_n: obviously Q_n is of exact degree n, and

$$|Q_n(x)| = \sqrt{Q_n(x) \cdot \overline{Q_n(\overline{x})}} = \sqrt{q(x)} = K_{2,n}(x), \quad x \in \mathbb{R}. \tag{5.15}$$

Moreover, with z_j, also $-\overline{z_j}$ is a zero of Q_n. Thus Q_n may be factored using terms of the form $z^2 - 2i \cdot z \cdot \Im(z_j) - 1$, $z_j \neq -i$, and eventually $z + i$, showing that $Q_n(i \cdot z) = i^n \cdot \tilde{Q}_n(z)$ with $\tilde{Q}_n$ having only positive coefficients. In particular,

$$||\vec{Q}_n||_1 = |Q_n(i)| = I_n \tag{5.16}$$

with I_n being estimated in Lemma 5.13.

Due to (5.15), (5.16), equation (4.3) together with (5.10) allows us to prove the lower bound of Theorem 5.11

$$\Gamma_n^w(\mathbb{R}) \geq \frac{\Theta_{n,n}(\mathbb{R})}{\sqrt{n+1}} \geq \frac{||\vec{Q}_n||_2}{\sqrt{n+1}} \geq \frac{||\vec{Q}_n||_1}{n+1} = \frac{I_n}{n+1},$$

where it remains to apply Lemma 5.13. On the other hand, for any nodes $x_0, .., x_n \in \mathbb{R}$ we have using (5.10) and (A.2)

$$||V_n(K_{2,n}, x_0, .., x_n)^{-1}||_{\infty,1} \geq \Theta_{n,n}(\mathbb{R}) \geq \Gamma_n^w(\mathbb{R}) / \sqrt{n+1}.$$

The construction of these nodes is motivated by Lemma 5.5(d). In fact, if we are able to find real nodes $x_0 < x_1 < .. < x_n$, being symmetric with respect to the origin, and satisfying

$$Q_n(x_j) = i \cdot (-1)^{n-j} \cdot \frac{x_j - i}{|x_j - i|} \cdot |Q_n(x_j)| = i \cdot (-1)^{n-j} \cdot \frac{|x_j + i|}{x_j + i} \cdot |Q_n(x_j)| \tag{5.17}$$

for $j = 0, ..., n$, then by Lemma (5.5)(d), (5.15), and (5.16) we obtain

$$||V_n(K_{2,n}, x_0, .., x_n)^{-1}||_{\infty,1} \leq \sqrt{2} \cdot |Q_n(i)| \leq \sqrt{2} \cdot I_n,$$

and again Lemma 5.13 leads to the upper bound of Theorem 5.11(d).

In order to prove (5.17), suppose that $z_1, .., z_n$ are those zeros of q with negative imaginary part, define $z_0 := -i$, and let

$$P(z) := \prod_{k=0}^{n} (z - z_k), \quad a(z) := \sum_{k=0}^{n} a_k(z), \quad a_k(z) := \arg(z - z_k).$$

Since $\Im(z_k) < 0$, the function a_k, restricted on the real axis, is continuous and strictly decreasing, with $a_k(-\infty) = \pi$, and $a_k(+\infty) = 0$. In particular, the function a, restricted on the real axis, is continuous and strictly decreasing, with $a(-\infty) = (n+1) \cdot \pi$, and $a(+\infty) = 0$. Notice also the symmetry $a(-z) = (n+1) \cdot \pi - a(z)$, following from $\{z_0, .., z_n\} = \{-\overline{z_0}, ..., -\overline{z_n}\}$. Consequently, we may find unique solutions x_j of the equations $a(x_j) = \pi/2 + (n-j) \cdot \pi$, $0 \leq j \leq n$. Obviously $x_0 < x_1 < ... < x_n$, and these nodes are symmetric with respect to the origin according to the symmetry of a. Finally, we obtain

$$P(x_j) = i \cdot (-1)^{n-j} \cdot |P(x_j)|, \quad j = 0, .., n,$$

being equivalent to (5.17). Thus we have shown the assertion of Theorem 5.11. □

By proceeding as in Lemma 5.12 one shows that

$$T_n[K_{2,n}, \mathbb{R}](z) = \lim_{\gamma \to \infty} T_n[K_{2,n}, [-\gamma; \gamma]](z) = \frac{1}{2} \cdot (Q_n(z) + \overline{Q_n(\overline{z})})$$

with Q_n as in the proof of Theorem 5.11(d), i.e., on the real axis we take the real part of Q_n. This illustrates that the lower bound stated in Theorem 5.7 may not be tight. In particular, the nodes proposed in Theorem 5.7, namely the alternant of $T_n[K_{2,n}, [-\gamma; \gamma]]$ for large γ, are not suitable for showing that the given lower bound is approximately attained. This follows at once by observing that the underlying weighted Vandermonde matrix becomes singular for $\gamma \to \infty$. In contrast, the more suitable nodes chosen in the proof of Theorem 5.11(d) may be shown to coincide with the zeros of $T_{n+1}[\sqrt{p}, (-\infty; \infty)]$, where $p(z) := (1 + z^2) \cdot K_{2,n}(z)^2$.

In all four cases mentioned in Theorem 5.11, we have obtained approximately tight lower bounds $\Gamma_n^w(M)$ for the Euclidean condition number of Krylov matrices which may be shown to be of the same magnitude as the weighted Lebesgue constants $\Delta_n(1, \mathbb{D}; K_{2,n}, M)$. For instance, by adapting the reasoning of Freund and Ruscheweyh in the proofs of [FrRu86, Theorem (3.5) and Theorem (6.12)], one verifies using (5.15), (5.16), and (5.17) that

$$\Delta_n(1, \mathbb{D}; K_{2,n}, \mathbb{R}) = \Delta_n(1, \{i\}; K_{2,n}, \mathbb{R}) = I_n,$$

with I_n being defined in Lemma 5.13.

Let us finally notice that the reasoning leading to the conclusion of Theorem 5.11 may also be applied in order to determine explicitly

$$\min_{x_0,..,x_n \in [0;\gamma]} \min_{d} \kappa_\infty(V_n(d, x_0, .., x_n)), \quad \gamma \in (0; +\infty]$$

(compare Problem (D) of Section 4). In fact, as a consequence of a result of Bauer [Bau63, Theorem II(a)], for any fixed configuration of nodes, the minimum over d is attained for the optimal density function $d = K_{\infty,n}$, which, restricted on $[0;\infty)$, is a polynomial of degree n. Hence it is sufficient to find the quantity

$$\min_{x_0,..,x_n \in [0;\gamma]} \kappa_\infty(V_n(K_{\infty,n}, x_0, .., x_n))$$

which by Theorem 5.4 is equal to $||\vec{T}_n[K_{\infty,n}; [0;\gamma]]||_\infty$. The corresponding optimal nodes are given by the alternant of $\vec{T}_n[K_{\infty,n}; [0;\gamma]]$, a polynomial being explicitly given in Lemma 5.12.

5.4 An application to positive definite Hankel matrices

To the end of this work, let us discuss some consequences of Theorem 5.11.

As already noticed in [Tay78, Tyr94a], a lower bound for the condition number of Krylov matrices also implies a lower bound for the condition number of positive definite Hankel matrices, where we recall that a Hankel matrix has constant entries along antidiagonals. Notice that there exist well–conditioned Hankel matrices, for instance the anti–identity J containing 1 on the main antidiagonal and otherwise zero for which obviously $\kappa_2(J) = 1$. It is widely believed that the condition number of any Hankel matrix can be bounded below by a (fast growing) function which takes as argument the number of non–singular principal submatrices (compare the concluding remarks in [Tyr94a]). However, we may construct well–conditioned dense Hankel matrices H_n by reversing the order of columns of positive definite Toeplitz matrices $M_n(\nu,\mu)$ (see Section 3.1), where μ,ν are measures supported on the unit circle. For instance, let $d\mu(z) = |dz|$, and $d\nu(z) = w(z)\,|dz|$ with a non–constant density function w satisfying $w_1 \le w(z) \le w_2$ for all $z \in \partial\mathbb{D}$ with positive constants w_1, w_2. Then from (3.10) of Section 3.2 it follows that $\kappa_2(H_n) = \kappa_2(M_n(\nu,\mu)) \le w_2/w_1$ for all $n \ge 0$, though all principal submatrices of H_n might be non–singular

The situation is different if we restrict ourselves to positive definite Hankel matrices

Corollary 5.14 *For each real positive definite Hankel matrix $H_n = (h_{j+k})_{j,k=0,..,n}$ of order $n+1$, $n \ge 1$, we have*

$$\kappa_2(H_n) \ge \frac{2}{5(n+1)^2} \cdot \exp(\delta^*)^{2n+2}, \quad \exp(\delta^*)^2 \approx 3.210,$$

and this bound is tight up to the factor $5 \cdot (n+1)^3$.

If in addition $(H_n)_n$ is the family of moment matrices associated to a scalar product on the real line induced by some measure μ, i.e.,

$$h_k = \int_{\mathbb{R}} x^k \, d\mu(x), \quad 0 \le k \le 2n,$$

then $\kappa_2(H_n)$ *is greater or equal to*

$$\begin{cases} \exp(\delta^*)^{4n+4}/(40\cdot(n+1)^2) \approx 10.30^n/(2n+2)^2 & \textit{if}\ \mathrm{supp}(\mu)\subset[0;+\infty), \\ ((1+\sqrt{2})^{2n}+(1+\sqrt{2})^{-2n})^2/(2n+2)^2 \approx 33.97^n/(2n+2)^2 & \textit{if}\ \mathrm{supp}(\mu)\subset[0;1], \\ ((1+\sqrt{2})^{n}+(1-\sqrt{2})^{n})^2/(2n+2)^2 \approx 5.83^n/(2n+2)^2 & \textit{if}\ \mathrm{supp}(\mu)\subset[-1;1], \end{cases}$$

and these bounds are tight, at least up to the factors $2\cdot(n+1)^3$, $(n+1)^4$, *and* $2\cdot(n+1)^4$, *respectively.*

Proof: Let us first show that each real positive definite Hankel matrix $H_n = (h_{j+k})_{j,k=0,..,n}$ may be rewritten in the form

$$H_n = V_n(d, x_0, .., x_n)^H \cdot V_n(d, x_0, .., x_n) \tag{5.18}$$

with real $d, x_0, .., x_n$ (for a linear algebra proof, see [Tyr94a, Lemma 2.1]). In fact, by choosing $h_{2n+1} = 0$ and h_{2n+2} sufficiently large, we may insure that the extended Hankel matrix $H_{n+1} := (h_{j+k})_{j,k=0,..,n+1}$ also is positive definite. As a consequence, the linear functional c defined on $\mathcal{P}_{2n+2}$ by $c(x^k) := h_k$, $0 \leq k \leq 2n+2$, is positive definite. Then assertion (5.18) is equivalent to the existence of a Gaussian quadrature formula

$$Q(f) := \sum_{j=0}^{n} f(x_j)/d(x_j)^2 \tag{5.19}$$

satisfying $Q(x^k) = c(x^k) = h_k$, $k = 0, 1, .., 2n$. Such a formula however can be given explicitly, take as x_j the zeros of the $n+1$th orthogonal polynomial and as $1/d(x_j)^2$ the corresponding Christoffel numbers, e.g., d^2 equals the corresponding nth Szegö kernel.

Notice also that, for each real weighted Vandermonde matrix, the right hand side of (5.18) is a positive definite Hankel matrix. Thus the first part of the assertion follows from Theorem 5.11(d), since for each matrix B we have $\kappa_2(B^H \cdot B) = \kappa_2(B)^2$.

If now the moments $h_0, ..., h_{2n}$ are generated by a measure μ supported on the real line, then (5.19) remains valid, where we recall that the numbers x_j lie in the smallest interval containing the support of μ. Therefore, (5.18) still holds, and the lower bounds of the second part follow from Theorem 5.11(a)–(c). Also, defining a discrete measure by the quadrature formula (5.19) with x_j, d as in the proofs of the different parts of Theorem 5.11, we see that these bounds are approximately attained up to the factors resulting from Theorem 5.11. □

Let us also mention that the lower bounds of Corollary 5.14 are approximately attained for the measure

$$d\mu_n(x) = \frac{dx}{(1+x^2+x^4+...+x^{2n})\cdot(1+x^2)\cdot\sqrt{(x/a-1)\cdot(1-x/b)}}, \quad x\in[a;b],$$

with $[a;b]$ being an appropriately chosen interval. Here one uses well–established connections between the polynomials being orthogonal with respect to the measure μ_n, and the Chebyshev polynomials of Lemma 5.12 (see, e.g., [Sze67, Section 2.6]), we omit the details.

Corollary 5.14 states that positive definite Hankel matrices are ill–conditioned even for not very high orders. To our knowledge, bounds of this type have not been established before (the proof of [Tyr94a, Theorem 5.1] is incomplete [Tyr94b]). It is interesting to compare our findings with the asymptotic of the condition number of the Hilbert matrix discussed in Example 3.3 of Section 3.1. Recall that the Hilbert matrix is the moment matrix $M_n(\mu, \mu_{\mathbb{D}})$ of the measure $d\mu(x) = dx$ on $[0;1]$, and its condition number grows asymptotically as $(1+\sqrt{2})^{4n}/\sqrt{n}$. Hence, though obviously ill–conditioned, the Hilbert matrix is still 'relatively well–conditioned' as a member of the class of positive definite Hankel matrices studied in Corollary 5.14.

In order to compare Corollary 5.14 with our findings of Section 3.3, notice that in general the two quantities

$$\inf_{supp(\mu)\subset I} \liminf_{n\to\infty} \kappa_2(M_n(\mu,\mu_{\mathbb{D}}))^{1/n}, \quad \text{and} \quad \liminf_{n\to\infty} \inf_{supp(\mu)\subset I} \kappa_2(M_n(\mu,\mu_{\mathbb{D}}))^{1/n}$$

for $I \subset \mathbb{R}$ are different. In fact, the second one is discussed in Corollary 5.14, for instance we obtain $\exp(\delta^*)^2 \approx 3.210$ in the case $I = \mathbb{R}$. For the first one, Taylor [Tay78, Corollary to Theorem 3] gave the lower bound

$$I = [a;b]: \quad \max\{a^2, b^2\} \cdot [4/(b-a)]^2 \geq 4,$$

whereas from Lemma 3.4 together with Corollary 3.15 we deduce the exact value

$$I = [a;b]: \quad \Delta(\mathbb{D}, [a;b])^2 \cdot \Delta([a;b], \mathbb{D})^2$$

being valid for the subclass of measures μ with $cap(supp(\mu)) > 0$ satisfying $a, b \in supp(\mu)$. Note that $\Delta([a;b], \mathbb{D})^2 = \max\{1, a^2, b^2\}$, and $\Delta(\mathbb{D}, [a;b])$ has been determined in Lemma 2.18. In particular it follows that

$$\min_{a<b} \Delta(\mathbb{D}, [a;b])^2 \cdot \Delta([a;b], \mathbb{D})^2 = \min_{b>0} [(b^{-1} + \sqrt{1+b^{-2}}) \cdot \max\{1, b\}]^2 = (1+\sqrt{2})^2.$$

Thus we have shown

COROLLARY 5.15 *Denote by $\mathcal{M}_1$ the set of measures with support being a compact subset of $\mathbb{R}$, then*

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} \kappa_2(M_n(\mu,\mu_{\mathbb{D}}))^{1/n} = (1+\sqrt{2})^2 \approx 5.828,$$
$$\liminf_{n\to\infty} \inf_{\mu\in\mathcal{M}_1} \kappa_2(M_n(\mu,\mu_{\mathbb{D}}))^{1/n} = \exp(\delta^*)^2 \approx 3.210.$$

□

Let us finally mention another application of Theorem 5.11, namely the problem of the numerical condition of the problem of rational interpolation at real distinct nodes.

First recall from Appendix A that for each (rectangular) matrix B

$$\kappa_2(B) = \max_{x,y} \frac{||y||_2 \cdot ||B\cdot x||_2}{||B\cdot y||_2 \cdot ||x||_2},$$

hence we obtain a lower bound $\kappa_2(B) \geq \kappa_2(B')$, if B' results from B by dropping some columns. In particular, if B contains $n+1$ columns building up a Krylov matrix $K_n(B,b)$ with Hermitian B , then the lower bounds of Theorem 5.11 are still valid.

'Striped' Krylov matrices containing two column blocks of Krylov matrices occur naturally in the context of the (linearized) rational interpolation problem (see, e.g., [BGM81]). Given two positive integers μ, ν, real nodes $x_0, .., x_{\mu+\nu}$ located in the interval $[\alpha;\beta]$, and real data $(f_0, g_0), .., (f_{\mu+\nu}, g_{\mu+\nu})$, we look for polynomials P and Q with degrees bounded by μ, and ν, respectively, such that

$$f_j \cdot P(x_j) - g_j \cdot Q(x_j) = 0, \quad j = 0, .., \mu+\nu.$$

With $f := (f_0, .., f_{\mu+\nu})^T$, $g = (g_0, .., g_{\mu+\nu})$, and $X = \mathit{diag}\ (x_0, .., x_{\mu+\nu})$, this problem leads to a homogeneous system of $(\mu+\nu+1)$ linear equations and $(\mu+\nu+2)$ unknowns, where the matrix of coefficients is given by

$$(\, K_\mu(X;f) |\, K_\nu(X;g))\,.$$

Due to freedom in scaling, one usually fixes either the leading coefficient of Q or the value of Q at zero, leading to the square matrices of coefficients

$$A = (\, K_\mu(X;f) |\, K_{\nu-1}(X;g)) \text{ or } A = (\, K_\mu(X;f) |\, K_{\nu-1}(X;X\cdot g))\,.$$

One could imagine that in the case of clustered data, i.e., of nearly coinciding nodes, this matrix is ill-conditioned. However, the condition number of A already is very large independent of the location of the (real) nodes, namely at least $3 \cdot 1.79^{n+1}/(5n+5)$ with $n = \min\{\mu, \nu-1\}$ according to Theorem 5.11, and this lower bound certainly may be improved.

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Appendix A

Matrix and Vector Norms

The aim of this part is to give a short summary of well-known results on vector or matrix Hölder norms which are required elsewhere in our work. We will omit proofs (for a short summary, see, e.g., [GoVL93, pp.53-58]). Given an integer $n \geq 1$, and $p \in [1; +\infty]$, the p–Hölder norm of a vector $x = (x_1, .., x_n)^T \in \mathbb{C}^n$ is defined by

$$||x||_p = \Big(\sum_{j=1} |x_j|^p\Big)^{1/p} \quad \text{or} \quad ||x||_\infty = \max_{1 \leq j \leq n} |x_j|.$$

The number $q \in [1; +\infty]$ with $1/p + 1/q = 1$ is called the complementary Hölder index of p. We have the Hölder inequality

$$\sum_{j=1}^{n} |x_j \cdot y_j| \leq ||x||_p \cdot ||y||_q, \quad x, y \in \mathbb{C}^n. \tag{A.1}$$

Of course all norms on $\mathbb{C}^n$ are equivalent. For Hölder vector norms, we may give explicitly the corresponding equivalence constants using the Hölder inequality

$$||x||_{p_2} \leq ||x||_{p_1} \leq n^{1/p_1 - 1/p_2} \cdot ||x||_{p_2}, \quad p_1 \leq p_2, \quad x \in \mathbb{C}^n \tag{A.2}$$

(both bounds are sharp, consider $x = (1, 0, .., 0)^H$, and $x = (1, 1, .., 1)^H$, respectively). The Hölder matrix norm being subordinate to the Hölder vector norms $|| \cdot ||_{p_1}$ and $|| \cdot ||_{p_2}$ is given by

$$||A||_{p_1,p_2} := \max_{x \neq 0} \frac{||A \cdot x||_{p_2}}{||x||_{p_1}} = \max_{x,y \neq 0} \frac{|y^H \cdot A \cdot x|}{||x||_{p_1} \cdot ||y||_{q_2}}, \quad A \in \mathbb{C}^{m \times n}, \tag{A.3}$$

where we write shorter $||A||_p := ||A||_{p,p}$. Note that $||A||_{p_1,p_2} = ||A^H||_{q_2,q_1}$, where A^H denotes the Hermitian counterpart of A.

There are well-know formulas for evaluating $||A||_1$ and $||A||_\infty$. More generally, one verifies that $||A||_{p,\infty}$ is obtained by taking the maximum of the q–norms of the rows of A (and hence

$||A||_{1,p}$ equals the maximum of the p–norms of the columns of A). In particular, $||A||_{1,\infty}$ is the in modulus largest element of A, and one defines for square matrices

$$||A||_{Turing} := \sqrt{n} \cdot ||A||_{1,\infty}, \quad A \in \mathbb{C}^{n\times n}. \tag{A.4}$$

It is not difficult to verify that, for positive definite Hermitian matrices A (such as modified moment matrices discussed in Chapter 3), the in modulus largest element of A lies on the diagonal.

The Euclidean or 2–norm $||\cdot||_2$ of a matrix $A \in \mathbb{C}^{m\times n}$ is obtained as the square root of the largest eigenvalue of the positive semidefinite matrix $A^H \cdot A$, in particular $||A^H \cdot A||_2 = ||A||_2^2 = ||A^H||_2^2$. The Froebenius norm $||\cdot||_F$ is defined by

$$||A||_F^2 = \sum_j \sum_k |a_{j,k}|^2 = \text{trace}(A^H \cdot A), \quad A = (a_{j,k}) \in \mathbb{C}^{m\times n}. \tag{A.5}$$

Recall that both norms are invariant under multiplication on the left or right by unitary factors. Moreover, if A is Hermitian, then both $||A||_2$ and $||A||_F$ are invariant under a similarity transformation of A. We have the estimate

$$\frac{1}{\sqrt{n}} \cdot ||A||_F \le ||A||_2 \le ||A||_F, \quad A \in \mathbb{C}^{n\times n}. \tag{A.6}$$

Inequality (A.2) enables us also to derive various inequalities between matrix Hölder norms, let us mention the examples

$$n^{-1/2} \cdot ||A||_{Turing} \le ||A||_{p_1,p_2}, \quad A \in \mathbb{C}^{n\times n}, \tag{A.7}$$

$$||A||_{p_1,\infty} \le ||A||_{p_1,p_2} \le m^{1/p_2} \cdot ||A||_{p_1,\infty}, \quad A \in \mathbb{C}^{m\times n}, \tag{A.8}$$

$$n^{-1/p_1} \cdot ||A||_{\infty,p_2} \le ||A||_{p_1,p_2} \le ||A||_{\infty,p_2}, \quad A \in \mathbb{C}^{m\times n}. \tag{A.9}$$

The pseudo–inverse of a matrix $A \in \mathbb{C}^{m\times n}$ having full column rank n (and thus $m \ge n$) is given by $A^+ = (A^H \cdot A)^{-1} \cdot A^H$; if $m = n$ then we obtain the classical inverse $A^+ = A^{-1}$. It will be basic for the considerations of Section 4 that, in the cases $p = 2$ or $m = n$, we may estimate the p–norm of A^+ in terms of A

$$n^{-1/p} \cdot ||A^{-1}||_{\infty,p} \le ||A^{-1}||_p \le ||A^{-1}||_{\infty,p} = \max_{x\neq 0} \frac{||x||_p}{||A\cdot x||_\infty}, \quad A \in \mathbb{C}^{n\times n}, \tag{A.10}$$

$$m^{-1/p} \cdot \max_{x\neq 0} \frac{||x||_2}{||A\cdot x||_\infty} \le ||A^+||_2 \le \max_{x\neq 0} \frac{||x||_2}{||A\cdot x||_\infty}, \quad A \in \mathbb{C}^{m\times n}. \tag{A.11}$$

Here, (A.10) is an immediate consequence of (A.9), whereas for (A.11) we require beside (A.2) also the representation of eigenvalues of $A^H \cdot A$ in terms of its Rayleigh quotient. Note that (A.11) in general does not remain valid for Hölder norms with $p \neq 2$.

We finally define for $A \in \mathbb{C}^{m\times n}$ the Hölder condition numbers $\kappa_{p_1,p_2}(A) := ||A^+||_{p_2,p_1} \cdot ||A||_{p_1,p_2}$, and $\kappa_p(A) := ||A^+||_p \cdot ||A||_p$.

Appendix B

Some elements of Potential Theory

Let $\Omega \subset \mathbb{C}$ be open. For a compact set $G \subset \Omega$, we introduce a weighted maximum norm on the space $\mathcal{C}(G)$ of functions f being continuous on G by

$$||f||_{g,G} := \max_{z \in G} \left| \frac{f(z)}{g(z)} \right|,$$

where $g \in \mathcal{C}(G)$ is strictly positive, referred to as a *weight function* on G. On $\mathcal{P}_n$, the space of polynomials of degree at most n, any two norms are equivalent. In order to compare two weighted maximum norms, one has to study the quantity

$$\Delta_n(h, H; g, G) := \max\{\frac{||P||_{h,H}}{||P||_{g,G}} : P \in \mathcal{P}_n, P \neq 0\}. \tag{B.1}$$

Given compact sets $G, H \subset \Omega$, a triangular array $Z := (z_{j,n})_{0 \leq j \leq n}$ of nodes in G, $z_{0,n}, .., z_{n,n}$ being distinct, and sequences of weight functions $(g_n)_n$ on G, and $(h_n)_n$ on H, respectively, in the sequel of this chapter we will be interested in establishing nth root asymptotics for the quantities

$$\Delta_n(h_n, H; g_n, G), \quad \text{and} \quad \Delta_n(h_n, H; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\}).$$

The latter quantity is referred to as a *weighted Lebesgue constant* . Its relation to the classical Lebesgue constant becomes clear by the following considerations: with help of the functions

$$\omega_{Z,n}(z) := \prod_{j=0}^{n} (z - z_{j,n}),$$

each polynomial P of degree at most n may be rewritten as

$$\frac{P(z)}{h_n(z)} = \sum_{j=0}^{n} \frac{\omega_{Z,n}(z)}{(z - z_{j,n}) \cdot \omega'_{Z,n}(z_{j,n})} \cdot \frac{g_n(z_{j,n})}{h_n(z)} \cdot \frac{P(z_{j,n})}{g_n(z_{j,n})}.$$

Consequently, $\Delta_n(g_n, \cdot; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\})$ is equal to a weighted generalization of the classical *Lebesgue function*

$$\Delta_n(h_n, \{z\}; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\}) = \sum_{j=0}^{n} \frac{|\omega_{Z,n}(z)|}{|(z - z_{j,n}) \cdot \omega'_{Z,n}(z_{j,n})|} \cdot \frac{g_n(z_{j,n})}{h_n(z)}. \tag{B.2}$$

Let $L_n(f)$ denote the polynomial of degree at most n which interpolates a given $f \in \mathcal{C}(G)$ at the nodes $z_{0,n}, .., z_{n,n}$. One verifies that the induced operator norm of the interpolation operator $L_n : (\mathcal{C}(G), ||\cdot||_{g_n,G}) \to (\mathcal{C}(G), ||\cdot||_{g_n,G})$ is given by the weighted Lebesgue constant

$$||L_n|| = \Delta_n(g_n, G; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\}).$$

Moreover, for each $f \in \mathcal{C}(G)$ we have

$$||f - L_n(f)||_{g_n,G} \leq (1 + ||L_n||) \cdot \min_{P \in \mathcal{P}_n} ||f - P||_{g_n,G},$$

and therefore the size of the weighted Lebesgue constant will be also important for studying the problem whether $L_n(f)$ tends uniformly to f with respect to a weighted maximum norm for all $f \in \mathcal{C}(G)$.

For a study of nth root asymptotics it is appropriate to employ techniques from Potential theory, where we follow the notation proposed by Nikishin & Sorokin in [NiSo88, Chapter 5]. This chapter is organized as follows: in Section B.1 we recall some basic notation. The exact assumptions for the sets G, H, and for the sequences of density functions are formulated in Section B.2.1. An asymptotic lower bound for the weighted Lebesgue constant in terms of the limit distribution of nodes is established in Section B.2.2. The aim of Section B.3 is to derive an asymptotic lower bound being independent of the limit distribution of nodes, but being attained for a particular limit distribution, for instance for so–called *weighted Fekete nodes.*

B.1 Some basic notation

Denote by $\mathcal{C}(G)'$ the space of continuous linear functionals being dual to $\mathcal{C}(G)$, the set of functions being continuous on a compact set $G \subset \mathbb{C}$, equipped with the maximum norm $||\cdot||_G = ||\cdot||_{1,G}$. By the Riesz theorem, $\mathcal{C}(G)'$, equipped with the operator norm, is isometric to the space $\mathcal{M}_\pm(G)$ of finite charges on G, i.e., real σ–additive functions defined on Borel subsets of G, the isometrie being expressed by the relation

$$\mu(f) = \int f(z)\, d\mu(z), \quad f \in \mathcal{C}(G), \quad \mu \in \mathcal{M}_\pm(G).$$

Furthermore, we denote by $\mathcal{M}(G)$ the set of measures, i.e., charges that take nonnegative values; and by $\mathcal{M}_0(G)$ the subset of probability measures μ for which $\mu(G) = 1$. The support $S = supp(\mu)$ of a measure $\mu \in \mathcal{M}(G)$ is the smallest closed set such that $\mu(S) = \mu(G)$. Note that the operator norm of a linear functional induced by $\mu \in \mathcal{M}(G)$ is given by $\mu(G)$. In particular, $\mathcal{M}_0(G)$ is isometric to a closed subset of the unit ball in $\mathcal{C}(G)'$, which is known to be weakly compact. Consequently, each sequence in $\mathcal{M}_0(G)$ has a weak accumulation point in $\mathcal{M}_0(G)$.

For the following considerations we will require the concept of the Green function, where we follow [NiSo88, Section 5, pp.188]. Recall that a domain D with compact boundary ∂D is called regular if the Dirichlet problem has a solution for each function being continuous on ∂D. We say that a compact set G is regular if each of the connected components of $\mathbb{C} \setminus G$ is regular.

DEFINITION B.1 *Let $G \subset \mathbb{C}$ be compact and regular, and let D be a connected component of $\overline{\mathbb{C}} \setminus G$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the compactification of the complex plane. The* Green function with singularity at a point *$\zeta \in D$, $\zeta \neq \infty$, is a function $g_G(\cdot, \zeta)$ with the following properties*

$$\begin{array}{ll} (a) & g_G(\cdot,\zeta) \text{ is continuous on } \overline{\mathbb{C}} \setminus \{\zeta\}, \\ (b) & g_G(\cdot,\zeta) \text{ vanishes on } \overline{\mathbb{C}} \setminus D, \\ (c) & g_G(\cdot,\zeta) \text{ is harmonic and positive on } D \setminus \{\zeta\} \\ (d) & g(z) := g_G(z,\zeta) - \log(1/|z-\zeta|) \text{ is bounded around } \zeta. \end{array}$$

If $\zeta = \infty$, condition (d) is replaced by the requirement that $g(z) := g_G(z,\infty) - \log|z|$ has to be bounded around infinity. □

This definition of the Green function $g_G(\cdot, \zeta)$ perhaps needs some clarification: Some authors prefer to use instead of G the index D, and in some publications $g_G(\cdot,\zeta)$ is only defined on D or its closure. For instance, $g_G(\cdot, \infty)$ coincides in D with the Green function g_G of Definition 2.8, and is continued to a continuous function on $\mathbb{C}$ by $g_G(z,\infty) = 0$ for all $z \notin D$.

With the notation of Definition B.1, if $\phi(\cdot,\zeta)$ provides a conformal mapping of D on the exterior of the unit disk with pole in ζ, then $g_G(z,\zeta) = \log|\phi(z,\zeta)|$ for all $z \in D$. Let us also recall that the capacity $cap(G)$ of G is obtained by the formula $\lim_{z\to\infty} g_G(z,\infty) - \log|z| = -\log cap(G) =: W(G)$, and $W(G)$ denotes its *Robin constant* (see, e.g., [NiSo88, p.190]).

Following [NiSo88, p.180ff], in the sequel we will restrict ourselves sometimes to sets H having a certain regularity property.

DEFINITION B.2 *Let $H \subset \mathbb{C}$ be closed, $\zeta \in H$, and $H^{(r)} := H \cap \{z \in \mathbb{C} : |z-\zeta| \leq r\}$, $r > 0$. We say that H has the* K–property *at ζ if there are positive numbers K and r_0 such that $W(H^{(r)}) \leq K \cdot \log(1/r)$ for all $0 < r < r_0$.*

We say that H has the K–property if H is compact, and if it has the K–property at any of its points. □

Note that a set with K–property is regular with respect to the Dirichlet problem, and necessarily has positive capacity (see [Tsu59, Corollary 2, p.104]). Moreover, with H_1, H_2, also $H_1 \cup H_2$ has the K–property. Also, a compact set H has the K–property at each of the points lying in its interior. As an example, circular arcs, and segments have the K–property. Moreover, if S is a set with K–property containing $\zeta \in H$ and if $S \cap \{z : |z-\zeta| \leq r_0\} \subset H$ for some r_0, then H has the K–property at ζ.

DEFINITION B.3 *The logarithmic potential of a measure μ with compact support is the function*

$$V[\mu](z) := \int \log \frac{1}{|t-z|} \, d\mu(t).$$

Here the integral is to be thought of in the Lebesgue sense, and $V[\mu](z) \in (-\infty; +\infty]$. □

The potential $V[\mu]$ is harmonic (and therefore continuous) in $\mathbb{C}\setminus supp(\mu)$, and superharmonic in $\mathbb{C}$ (see [NiSo88, Theorem 1.2]). Some further properties are summarized in

THEOREM B.4 *Let the sequence of measures $(\mu_n)_n \subset \mathcal{M}(G)$ converging weakly to $\mu \in \mathcal{M}(G)$, i.e.,*

$$\int f(z)\, d\mu_n(z) \to \int f(z)\, d\mu(z)$$

for all $f \in \mathcal{C}(G)$. Furthermore, suppose that $H \subset \Omega$ is compact.

(a) *If $\mu_n \in \mathcal{M}_0(G)$, then the sequence $(V[\mu_n])_n$ converges to $V[\mu]$ locally uniformly in each connected component of $\overline{\mathbb{C}} \setminus G$.*

(b) *If $(z_n)_n \subset \mathbb{C}$ converges to $z \in \mathbb{C}$, then*

$$V[\mu](z) \le \liminf_{n\to\infty} V[\mu_n](z_n). \tag{B.3}$$

(c) *Let $f \in \mathcal{C}(H)$. If H has the K–property at $\zeta \in H$, and if there exists a set $H_0 \subset H$ with $\mathrm{cap}(H_0) = 0$ and*

$$V[\mu](z) + f(z) \ge 0, \quad z \in H \setminus H_0,$$

then we have $V[\mu](\zeta) + f(\zeta) \ge 0$.

(d) *If $(f_n)_n \subset \mathcal{C}(H)$ converges uniformly to $f \in \mathcal{C}(H)$, and if H has the K–property, then*

$$\lim_{n\to\infty} \min_{z\in H}(V[\mu_n](z) + f_n(z)) = \min_{z\in H}(V[\mu](z) + f(z)).$$

Proof: Parts (b), (c), (d) are cited from [NiSo88, Theorem 1.1, Theorem 4.1, Theorem 4.3], respectively. In order to show (a), let D be a connected component of $\overline{\mathbb{C}} \setminus G$, and define $u_n := V[\mu_n] - V[\mu]$, $n \ge 0$. One easily verifies that the sequence $(u_n)_n$ is uniformly bounded on closed subsets of D, and each u_n is harmonic on D. Thus $(u_n)_n$ is a normal family on D. Moreover, by assumption we have pointwise convergence of $(u_n)_n$ to the zero function on D, and consequently $(u_n)_n$ converges uniformly on each compact subset of D to the zero function. □

B.2 A lower bound for the weighted Lebesgue constant

B.2.1 Formulation of the assumptions

Let us shortly recall the given data as mentioned already in the introduction of Chapter B. The set $\Omega \subset \mathbb{C}$ is open, $G, H \subset \Omega$ are compact. We will always assume implicitly that the set H has the K–property described in Definition B.2 (though many of the subsequent results also hold for more general sets). Additional assumptions for the set G will be mentioned explicitly.

Our second assumption concerns the sequences of density functions: we suppose that

$$\frac{1}{n}\cdot\log g_n \to f_G, \quad \text{and} \quad \frac{1}{n}\cdot\log h_n \to f_H,$$

uniformly in G, and in H, respectively, and therefore $f_G \in \mathcal{C}(G)$, $f_H \in \mathcal{C}(H)$. In order to simplify notation, in the sequel we will not write down all dependencies, but use instead the abbreviations

$$\delta_n := \log \Delta_n(h_n, H; g_n, G)^{1/n}, \quad \delta_{Z,n} := \log \Delta_n(h_n, H; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\})^{1/n}, \tag{B.4}$$

where $Z = (z_{j,n})_{0\leq j\leq n}$ is a triangular array of nodes being elements of G.

Let us also specify families of nodes $Z = (z_{j,n})_{0\leq j\leq n}$ which we wish to consider. First $z_{j,n} \in G$, and $z_{0,n}, .., z_{n,n}$ are distinct for each $n \geq 0$. Introducing the zero counting measures $\tau_n = \tau_{Z,n}$ of $\omega_{Z,n}$ by

$$\tau_n(f) := \frac{1}{n+1}\cdot\sum_{j=0}^{n} f(z_{j,n}), \quad f \in \mathcal{C}(G),$$

we see that $(\tau_n)_n \subset \mathcal{M}_0(G)$, and therefore $(\tau_n)_n$ is weakly compact. We assume that there is at most one weak limit point (which can be always satisfied by eventually considering subsequences), and denote this limit by $\tau = \tau_Z \in \mathcal{M}_0(G)$.

B.2.2 A lower bound in terms of the distribution of nodes

For a given family of nodes $Z = (z_{j,n})_{0\leq j\leq n}$ in G, let

$$G_Z := \{\zeta \in \mathbb{C} : \liminf_{n\to\infty} \min_{0\leq j\leq n} |z_{j,n} - \zeta| = 0\},$$
$$G'_Z := \{\zeta \in \mathbb{C} : \limsup_{n\to\infty} \min_{0\leq j\leq n} |z_{j,n} - \zeta| = 0\}.$$

Note that G_Z, G'_Z are both compact, $G'_Z \subset G_Z \subset G$, and G_Z is the set of accumulation points of $\{z_{j,n} : 0 \leq j \leq n\}$. Moreover, *supp*$(\tau) \subset G'_Z$, where as before $\tau = \tau_Z$ denotes the asymptotic distribution of nodes. Since $(\tau_{Z,n})_n$ converges weakly to τ_Z, we know from Theorem B.4(a) that

$$V[\tau_{Z,n}](z) = -\frac{1}{n+1}\log|\omega_{Z,n}(z)| \to V[\tau_Z](z),$$

locally uniformly in each connected component of $\overline{\mathbb{C}} \setminus G_Z$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the compactification of the complex plane.

We first want to show that the number of accumulation points of $(\delta_{Z,n})_n$ only depends on the family of nodes Z.

THEOREM B.5 *For $n \geq 0$, let*

$$a_{Z,n} = a_n := \max_{0\leq j\leq n} (f_G(z_{j,n}) + V[\tau_{j,n}](z_{j,n})),$$

where $\tau_{j,n}$ denotes the zero counting measure of the polynomial $\omega_{Z,n}(z)/(z-z_{j,n})$, i.e.,

$$\int f(z)\, d\tau_{j,n}(z) = \frac{1}{n}\sum_{k\neq j} f(z_{k,n}), \quad f \in \mathcal{C}(G).$$

Then

$$\lim_{n\to\infty}(\delta_{Z,n} - a_{Z,n}) = -\min_{z\in H}(f_H(z) + V[\tau](z)).$$

Proof: In view of (B.2), we have to take the maximum for $z \in H$ of the expression

$$\begin{aligned}
&\frac{1}{n}\cdot\log\sum_{j=0}^{n}\frac{|\omega_{Z,n}(z)|}{|(z-z_{j,n})\cdot\omega'_{Z,n}(z_{j,n})|}\cdot\frac{g_n(z_{j,n})}{h_n(z)}\\
&= \frac{1}{n}\cdot\log\max_{0\le j\le n}\frac{|\omega_{Z,n}(z)|}{|(z-z_{j,n})\cdot\omega'_{Z,n}(z_{j,n})|}\cdot\frac{g_n(z_{j,n})}{h_n(z)} + \frac{\log\eta_n(z)}{n}\\
&= \max_{0\le j\le n}(V[\tau_{j,n}](z_{j,n}) - V[\tau_{j,n}](z) + \log g_n(z_{j,n})^{1/n} - \log h_n(z)^{1/n}) + \frac{\log\eta_n(z)}{n}
\end{aligned}$$

with $\eta_n(z) \in [1; n+1]$ according to well–known inequalities between vector Hölder norms. Hence, by the uniform convergence of the sequences of density functions we get

$$\delta_{Z,n} = \max_{0\le j\le n}\Big(f_G(z_{j,n}) + V[\tau_{j,n}](z_{j,n}) - \min_{z\in H}(f_H(z) + V[\tau_{j,n}](z)) + \epsilon_{j,n}\Big), \tag{B.5}$$

where $\lim_{n\to\infty}\max_{0\le j\le n}\epsilon_{j,n} = 0$. Note that $(\tau_{j(n),n})_n$ also converges weakly to τ for any integers $j(n) \in \{0,...,n\}$. Let the maximum on the right hand side of (B.5) be attained for $j = j(n)$, then with help of Theorem B.4(d) we arrive at

$$\begin{aligned}
&\limsup_{n\to\infty}(\delta_{Z,n} - a_{Z,n}) + \min_{z\in H}(f_H(z) + V[\tau](z))\\
&= \limsup_{n\to\infty}(f_G(z_{j(n),n}) + V[\tau_{j(n),n}](z_{j(n),n}) - a_{Z,n}) \le 0.
\end{aligned}$$

In order to obtain the opposite inequality, suppose that the maximum in $a_{Z,n}$ is attained for $j = j(n)$. Then

$$\delta_{Z,n} - a_{Z,n} \ge -\min_{z\in H}(f_H(z) + V[\tau_{j(n),n}](z)) + \epsilon_{j(n),n},$$

and the assertion follows by applying again Theorem B.4(d). □

Lemma B.6 *We have*

$$\limsup_{n\to\infty} a_{Z,n} \ge \sup_{\zeta\in G_Z}(f_G(\zeta) + V[\tau](\zeta)), \quad \liminf_{n\to\infty} a_{Z,n} \ge \sup_{\zeta\in G'_Z}(f_G(\zeta) + V[\tau](\zeta)).$$

Proof: For each $\zeta \in G_Z$ there exist a subsequence of nodes $(z_{j_k,n_k})_k$ converging to ζ. With help of (B.3) we may conclude that

$$\limsup_{n\to\infty} a_{Z,n} \ge \liminf_{k\to\infty}(f_G(z_{j_k,n_k}) + V[\tau_{j_k,n_k}](z_{j_k,n_k})) \ge f_G(\zeta) + V[\tau](\zeta),$$

leading to the first part of the assertion. The second part follows by using a similar argument: Let

$$\liminf_{n\to\infty} a_{Z,n} = \lim_{k\to\infty} a_{Z,n_k},$$

then for each $\zeta \in G'_Z$ there exists a sequence of integers j_k such that $z_{j_k,n_k} \to \zeta$ for $k \to \infty$. Hence the second assertion again is a consequence of (B.3). □

As a consequence of Theorem B.5 and Lemma B.6 we have

COROLLARY B.7 *If there exist a* $\zeta \in \mathbb{C}$ *with* $V[\tau](\zeta) = \infty$, *then* $\liminf_{n\to\infty} \delta_{Z,n} = +\infty$.

Proof: Note that $V[\tau]$ is finite on $\mathbb{C} \setminus supp(\tau)$, consequently, $\zeta \in supp(\tau) \subset G'_Z$, and the assertion follows from Lemma B.6. □

We finally describe a class of nodes where we may establish the exact nth root behaviour.

THEOREM B.8 *Let the family of nodes* $Z = (z_{j,n})_{0\le j\le n}$ *be defined by* $z_{j,n} = \phi(j/n)$, *where* ϕ *maps conformally a neighborhood of* $[0;1]$ *on some subset of* G. *Then* $G_Z = G'_Z = \mathrm{supp}(\tau) = \phi([0;1])$, $V[\tau](z) = -\int \log|z - \phi(t)|\,dt$, *and*

$$\lim_{n\to\infty} \delta_{Z,n} = -\min_{z\in H}(f_H(z) + V[\tau](z)) + \sup_{z\in G_Z}(f_G(z) + V[\tau](z)).$$

Proof: We first notice that, for all $f \in \mathcal{C}(G)$, the function $f \circ \phi$ is continuous on $[0;1]$. In particular, it is Riemann integrable on $[0;1]$ and thus

$$\int f(z)\,d\tau_n(z) = \frac{1}{n+1}\sum_{j=0}^{n} f(\phi(j/n)) \to \int_0^1 f(\phi(t))\,dt.$$

This yields the explicit formula for $V[\tau]$, and that for G_Z, G'_Z is trivial.

In the second part of the proof we want to show that for each sequence of nodes $(z_{j_k,n_k})_k$ converging to ζ we have $V[\tau_{j_k,n_k}](z_{j_k,n_k}) \to V[\tau](\zeta)$. Notice that, by applying the reasoning of the proof of Lemma B.6, this property together with Theorem B.5 implies the assertion.

For a given $\epsilon > 0$, let $U_\epsilon := \{z \in G : |z - \zeta| < \epsilon\}$. We introduce the restriction of τ, τ_{j_k,n_k} on U_ϵ, denoted by τ^*, τ_k^*, and put $\tau' := \tau - \tau^*$, $\tau'_k := \tau_{j_k,n_k} - \tau_k^*$. Obviously, $(\tau'_k)_k \subset \mathcal{M}(G \setminus U_\epsilon)$ tends weakly to $\tau' \in \mathcal{M}(G \setminus U_\epsilon)$. From Theorem B.4(a) we know that $V[\tau'_k]$ converges to $V[\tau']$ locally uniformly in U_ϵ, in particular

$$\lim_{k\to\infty} V[\tau'_k](z_{j_k,n_k}) = V[\tau'](\zeta).$$

We still have to examine τ_k^*, τ^*. Let $\zeta = \phi(t_0)$, then, by assumption on ϕ, the function $t \to \log|\phi(t_0) - \phi(t)|$ is Riemann integrable on $[0;1]$. In particular, $|V[\tau^*](\zeta)|$ tends to zero for $\epsilon \to 0$. Thus for the assertion of the Lemma it is sufficient to show that

$$|V[\tau_k^*](z_{j_k,n_k})| \le c(\epsilon), \quad k \ge 0, \tag{B.6}$$

with $c(\epsilon)$ tending to zero for $\epsilon \to 0$. First we may assume without loss of generality that $|\phi(t_1) - \phi(t_2)| < 1$ for any two $t_1, t_2 \in [0;1]$. It follows that $\log|z_{j,n} - z_{\ell,n}| < 0$, and

$$V[\tau_k^*](z_{j_k,n_k}) \geq 0, \quad k \geq 0.$$

By assumption on ϕ there exists a constant $c_1 > 0$ such that, for all $0 \leq j, \ell \leq n$,

$$|z_{j,n} - z_{\ell,n}| \geq c_1 \cdot \frac{|\ell - j|}{n}.$$

Consequently, there exists a constant $c_2 > 0$ such that, for all $\epsilon \leq \epsilon_0$ and for all $n \geq 0$, at most $c_2 \cdot \epsilon \cdot n$ nodes out of $\{z_{0,n}, .., z_{n,n}\}$ lie in U_ϵ. We obtain for any $k \geq 0$ and for any sufficiently small ϵ

$$V[\tau_k^*](z_{j_k,n_k}) = -\frac{1}{n_k} \prod_{z_{\ell,n_k} \in U_\epsilon, \ell \neq j} \log|z_{j_k,n_k} - z_{\ell,n_k}|$$

$$\leq -\frac{2}{n_k} \cdot \prod_{m=1}^{[c_2 \cdot \epsilon \cdot n_k - 1]} \log(\frac{c_1 \cdot m}{n_k}) \leq -2 \cdot \int_0^{c_2 \cdot \epsilon} \log(c_1 \cdot t)\, dt =: c(\epsilon),$$

leading to (B.6). □

Let us illustrate the results of this section with help of two examples.

EXAMPLE B.9 *Let $z_{j,n} = z_j$, $0 \leq j \leq n$, with $(z_j)_j$ having only a finite number N of accumulation points, for instance the geometric nodes $z_j = 2^{-j}$ or the harmonic nodes $z_j = 1/(j+1)$. Then for the weak limit τ of the sequence of zero counting measures τ_n we obtain*

$$\tau(f) = \sum_{j=1}^{N} c_j \cdot f(\zeta_j), \quad f \in \mathcal{C}(G),$$

with suitable c_j, ζ_j. In particular, its potential

$$V[\tau](z) = \sum_{j=1}^{N} c_j \cdot \log \frac{1}{|z - \zeta_j|}$$

has poles, and by Corollary B.7 we may conclude that $\liminf_{n\to\infty} \delta_{Z,n} = \infty$. □

EXAMPLE B.10 *Consider equidistant nodes on $[-1;1]$, i.e., $z_{j,n} = -1 + 2j/n$, $0 \leq j \leq n$. Then by Theorem B.8 we get $G_Z = G'_Z = \mathrm{supp}(\tau) = [-1;1]$, and*

$$V[\tau](z) = -\int_0^1 \log|z - (-1 + 2 \cdot t)|\, dt = -\frac{1}{2} \int_{-1}^{1} \log|z - t|\, dt.$$

Hence

$$V[\tau](z) = 1 - \frac{1-z}{2} \cdot \log|1 - z| - \frac{1+z}{2} \cdot \log|1 + z|, \quad z \in \mathbb{R},$$

and for $z = e^{i\alpha} \in \partial\mathbb{D} \setminus \mathbb{R}$

$$V[\tau](z) = 1 - \frac{\pi}{4} \cdot |\sin(\alpha)| - \frac{1-\cos(\alpha)}{2} \cdot \log|1-z| - \frac{1+\cos(\alpha)}{2} \cdot \log|1+z|.$$

In particular one verifies that

$$\min_{z\in[-1;1]} V[\tau](z) = 1 - \log(2), \quad \max_{z\in[-1;1]} V[\tau](z) = 1, \quad \min_{z\in\partial\mathbb{D}} V[\tau](z) = 1 - \log\sqrt{2} - \frac{\pi}{4}.$$

Using Theorem B.8 and (B.4) we get for the (non–weighted) Lebesgue constant of equidistant nodes on $[-1;1]$

$$\lim_{n\to\infty} \Delta_n(1,[-1;1];1,\{z_{0,n},z_{1,n},..,z_{n,n}\})^{1/n} = 2,$$
$$\lim_{n\to\infty} \Delta_n(1,\mathbb{D};1,\{z_{0,n},z_{1,n},..,z_{n,n}\})^{1/n} = \sqrt{2}\cdot\exp(\pi/4),$$

in accordance with results mentioned in Example 4.2 and Example 4.7. □

B.3 Optimal nodes

The aim of this section is to study families of nodes Z where the quantity $\delta_{Z,n}$ is minimized. A connection to the quantity δ_n is given, and we establish for families of weights required in Section 4.3 an explicit formula for the limit of $(\delta_n)_n$.

In Section B.3.1 we recall the (electrostatical) problem of how to distribute a positive unity charge on a given compact set G such that, in the presence of an external field represented by $f \in \mathcal{C}(G)$, the energy is minimized. Most of the results of this section are cited from [MhSa92], however, according to the restriction to sets having the K–property several results simplify (see [NiSo88]); proofs for these simplifications will be given. In Section B.3.2 we study the family of weighted Fekete nodes. Properties for the quantities $\delta_{n,Z}$ and δ_n are deduced in Section B.3.3.

B.3.1 The equilibrium measure with external field

The energy of $\mu \in \mathcal{M}(G)$ with respect to the external field $f \in \mathcal{C}(G)$ is defined by

$$F(\mu) := \int (V[\mu](z) + 2 \cdot f(z))\, d\mu(z).$$

Consider the extremal problem of finding

$$W(f,G) = \inf\{F(\mu) : \mu \in \mathcal{M}_0(G)\}. \tag{B.7}$$

This problem and related extremal problems have been considered by Gonchar & Rakhmanov [GoRa86] and Mhaskar & Saff [MhSa85] for sets $G \subset \mathbb{R}$ being not necessarily bounded. A discussion of the complex case may be found in [NiSo88, Saf90, MhSa92], where again also the case of unbounded G and not necessarily continues functions f is studied.

The solution of problem (B.7) is characterized in the following Theorem being a summary of [NiSo88, Theorem 4.4] and [MhSa92, Theorem 3.1]

THEOREM B.11 *Let* $\mathrm{cap}(G) > 0$. *Then the following properties hold*

(a) *The quantity* $W(f,G)$ *is finite, moreover, there exists exactly one measure* $\tau = \tau(f,G)$ *with* $F(\tau) = W(f,G)$, *called the equilibrium measure. In particular,* $\mathrm{cap}(\mathrm{supp}(\tau)) > 0$.

(b) *Let* $w(f,G) := W(f,G) - \int f(t)\,d\tau(t)$. *There exists a set* $K(f,G)$ *of capacity zero such that the following equilibrium conditions hold*

$$V[\tau](z) + f(z) \begin{cases} \geq w(f,G) & \text{if } z \in G \setminus K(f,G), \\ \leq w(f,G) & \text{if } z \in \mathrm{supp}(\tau). \end{cases}$$

(c) *If in addition* G *has the K–property, then the exceptional set* $K(f,G)$ *of part (b) is empty, i.e., we have*

$$V[\tau](z) + f(z) \begin{cases} \geq w(f,G) & \text{if } z \in G, \\ = w(f,G) & \text{if } z \in \mathrm{supp}(\tau). \end{cases} \tag{B.8}$$

□

Note that part (c) of Theorem B.11 follows from part (b) together with Theorem B.4(c).

In the case of an unbounded closed set G one may show that as well that *supp*$(\tau(f,G))$ is compact; however, additional restrictions on the weight function at infinity are required ($f(z)/\log|z|$ is asymptotically less than 1 in [GoRa86], and equal to zero in [MhSa85, MhSa92]). Note also that in order to solve (B.7) explicitly, we have to solve the difficult problem of finding *supp*(τ). Afterwards, τ may be determined by a balayage while solving the corresponding Dirichlet problem (see, e.g., [Tsu59, Chapter III.10]). In our applications we will be mostly interested in the simpler case $f(z) = c - V[\mu](z)$ with c a constant (see Example B.15 below).

For the sequel we will need a technical Lemma where we summarize the 'first' maximum principle [NiSo88, Theorem 1.3] and a weak form of the 'second' maximum principle [NiSo88, Theorem 2.5] for potentials (the latter is also referred to as the 'principle of domination').

LEMMA B.12 *Let* $E \subset \mathbb{C}$ *be compact,* $\mu \in \mathcal{M}_0(E)$, $\nu \in \mathcal{M}(E)$ *with* $\nu(E) \leq 1$, *and denote by* $\mathcal{D}_\infty(E)$ *the unbounded connected component of* $\mathbb{C} \setminus E$. *Suppose that there exist constants* C_1, C_2 *with*

$$V[\mu](z) \leq \max\{V[\nu](z) + C_1, C_2\}, \quad z \in \mathrm{supp}(\mu).$$

Then

$$V[\mu](z) \leq \max\{V[\nu](z) + C_1, C_2\}, \quad z \in \mathbb{C}.$$

Moreover, in the case $\nu(E) = 1$ we have

$$\textit{either} \quad C_1 \leq 0 \quad \textit{or} \quad \exists z_0 \in \mathcal{D}_\infty(E): \; V[\mu](z_0) \geq V[\nu](z_0) + C_1$$

if and only if

$$C_1 = 0 \quad \textit{and} \quad \forall z \in \mathcal{D}_\infty(E): \; V[\mu](z) = V[\nu](z).$$

Proof: The fact $V[\mu](z) \leq C_2$ for $z \in \mathbb{C}$ is shown in [NiSo88, Theorem 1.3], here we only discuss the rest of the assertion. Let $u(z) := V[\mu](z) - C_1 - V[\nu](z)$. First notice that u is bounded on any compact subset of $\mathbb{C}$, moreover, by definition of the potential of a unit measure with compact support, u is continuous at infinity, with $u(\infty) = -C_1$ in the case $\nu(E) = 1$, and $u(\infty) = -\infty$ otherwise.

Let $D := \mathbb{C} \setminus supp(\mu)$, D_0 any connected component of D. By assumption, u is subharmonic and bounded on D_0, and less or equal to zero on the boundary of D_0. From the maximum principle of subharmonic functions it follows that $u(z) \leq 0$ for all $z \in D_0$, implying the first part of the assertion. If now in particular $\nu(E) = 1$, then u is harmonic in $\mathcal{D}_\infty(E)$, a subset of the unbounded connected component of D, and less or equal to zero on the boundary of $\mathcal{D}_\infty(E)$. From the maximum principle for harmonic functions we may conclude that either $u(\infty) \geq 0$ or $u(z_0) \geq 0$ for a $z_0 \in \mathcal{D}_\infty(E)$ implies that u vanishes on the whole component $\mathcal{D}_\infty(E)$ including infinity. □

With help of Lemma B.12 we may further characterize the weighted equilibrium measure of Theorem B.11

Corollary B.13 *Let G have the K-property.*

(a) *The potential of the weighted equilibrium measure is continuous and bounded above in $\mathbb{C}$.*

(b) *Condition (B.8) characterizes the weighted equilibrium measure in a unique way: if (B.8) holds for any $\tau \in \mathcal{M}_0(G)$ and for any constant $w(f, G)$, then $\tau = \tau(f, G)$.*

(c) *If f is subharmonic in $\mathbb{C}$, and $f(z) - \log(|z|)$ is bounded above in $\mathbb{C}$, then (B.8) takes the form*

$$V[\tau](z) + f(z) = w(f, G), \quad z \in G.$$

Proof: In order to show (a), notice that (B.8) implies the continuity of $V[\tau]$ restricted on the support of $\tau = \tau(f, G)$, and, by the 'principle of continuity' [NiSo88, Theorem 1.4], the potential of the (weighted) equilibrium measure is continuous in $\mathbb{C}$. Moreover, since $supp(\tau)$ is compact, we may find an upper bound for $V[\tau]$ on $supp(\tau)$, which by the first maximum principle of Lemma B.12 also is an upper bound in $\mathbb{C}$.

For part (b), suppose that there exist two measures $\mu_1, \mu_2 \in \mathcal{M}_0(G)$ and constants c_1, c_2 such that for $j = 1, 2$

$$V[\mu_j](z) + f(z) \left\{ \begin{array}{ll} \geq c_j & \text{if } z \in G, \\ = c_j & \text{if } z \in supp(\mu_j). \end{array} \right.$$

Then for $z \in supp(\mu_1)$ we have

$$V[\mu_1](z) - V[\mu_2](z) - c_1 + c_2 \leq (V[\mu_1](z) + f(z) - c_1) - (V[\mu_2](z) + f(z) - c_2) \leq 0,$$

which by Lemma B.12 is valid for all $z \in \mathbb{C}$. Interchanging the role of μ_1, μ_2 leads to the identity $V[\mu_1](z) - V[\mu_2](z) = c_1 - c_2$ for all $z \in \mathbb{C}$, and hence $\mu_1 = \mu_2$ by [Tsu59, Theorem II.25].

It remains to discuss part (c). By assumption, $u := V[\tau] + f$ is subharmonic in $\mathbb{C} \setminus supp(\tau)$ and bounded above, moreover, on the boundary of each connected component of $\mathbb{C} \setminus supp(\tau)$ we have $u = w(f, G)$. From the maximum principle of subharmonic functions we may conclude that $u(z) \leq 0$ for all $z \in \mathbb{C}$, which together with (B.8) gives the assertion. □

EXAMPLE B.14 *Let us first discuss the case of a trivial external field $f = 0$. Here we have*

$$W(0, G) = w(0, G) = W(G) = \log(1/\text{cap}(G)),$$

the classical Robin's constant. In fact, by (B.7) for $f = 0$ one usually defines the capacity of a compact set [Tsu59, p.55]. In the case $f = 0$, the extremal measure $\mu_G := \tau(0, G)$ of Theorem B.11 also is called the equilibrium measure *of G. Denote by $D = \mathcal{D}_\infty(G)$ the unbounded connected component of $\mathbb{C} \setminus G$. From [Tsu59, Theorem II.31] we know that the support of μ_G is a subset of the boundary of D, and hence $\mu_G = \mu_{\partial G} = \mu_{\partial D}$ as well as* $\text{cap}(G) = \text{cap}(\partial G) = \text{cap}(\partial D)$.

Suppose as before that G has the K–property, and therefore D is regular with respect to the Dirichlet problem. Then [Tsu59, Theorem II.37]

$$V[\mu_G](z) = w(0, G) - g_G(z; \infty), \quad z \in \mathbb{C}, \tag{B.9}$$

with $g_G(\cdot; \infty)$ denoting the Green function of G with singularity at infinity (see Definition B.1), in accordance with the characterizations of Corollary B.13. □

EXAMPLE B.15 *Let $H \subset \mathbb{C}$ be compact. We consider the subharmonic weight function $f(z) = c - V[\mu](z)$ with $c \in \mathbb{C}$ a constant and $\mu \in \mathcal{M}_0(H)$, e.g., $f(z) = g_H(z; \infty)$, the Green function with singularity at infinity of a set H having the K–property. From Corollary B.13(b),(c) we know that*

$$f(z) + V[\tau](z) = c + V[\tau - \mu](z)$$

is constant on G if and only if τ equals the weighted equilibrium measure $\tau(f, G)$. We shall construct a measure τ satisfying

$$f(z) + V[\tau](z) = c + \int_{\mathbb{C} \setminus G} g_G(t; \infty)\, d\mu(t) - \int_{\mathbb{C} \setminus G} g_G(z; t)\, d\mu(t), \quad z \in \mathbb{C}. \tag{B.10}$$

Denote by D_0 the unbounded and by D_k, $k = 1, 2, ..,$ the bounded connected components of $\mathbb{C}\setminus G$, each being regular with respect to the Dirichlet problem. For each D_k, there exist the harmonic measure *(or* mass of balayage*)* $\mu_k(\cdot; z_0) \in \mathcal{M}_0(\partial D_k)$ *such that (see [Tsu59, Theorem III.41] or [StTo92, Section A.VII])*

$$g_G(z;\zeta) = \log \frac{1}{|z-\zeta|} - \int_{\partial D_k} \log \frac{1}{|z-t|}\, d\mu_k(t;\zeta), \quad z, \zeta \in D_k, \quad k \geq 1, \tag{B.11}$$

$$g_G(z;\zeta) = \log \frac{1}{|z-\zeta|} + g_G(\zeta;\infty) - \int_{\partial D_0} \log \frac{1}{|z-t|}\, d\mu_0(t;\zeta), \quad z, \zeta \in D_0, \quad \zeta \neq \infty \tag{B.12}$$

Moreover, for fixed $\zeta \in D_k$, both hand sides of (B.11) and (B.12) (as a function of z) are vanishing on ∂D_k, and harmonic on each component of $\overline{\mathbb{C}}\setminus(D_k \cup \partial D_k)$. Hence, by the maximum principle for harmonic functions, equality (B.11) remains valid for $z \in \mathbb{C}$ and $\zeta \in D_k$. Let

$$\tau := \mu|_G + \sum_k \int_{D_k} \mu_k(\cdot;\zeta)\, d\mu(\zeta),$$

then

$$\tau(\mathbb{C}) = \mu(G) + \sum_k \int_{D_k} \mu_k(\mathbb{C};\zeta)\, d\mu(\zeta) = \mu(G) + \sum_k \mu(D_k) = \mu(\mathbb{C}) = 1$$

and $\operatorname{supp}(\tau) \subset G$. *By using the Fubini Theorem and (B.11), (B.12), we get for all $z \in \mathbb{C}$*

$$\begin{aligned}
& f(z) + V[\tau](z) = c + V[\tau - \mu](z) \\
&= c + \sum_k \left(-\int_{D_k} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) + \int_{\partial D_k} \log \frac{1}{|z-t|} \int_{D_k} d\mu_k(t;\zeta)\, d\mu(\zeta) \right) \\
&= c + \int_{D_0} g_G(t;\infty)\, d\mu(t) - \sum_k \int_{D_k} g_G(z;\zeta)\, d\mu(\zeta) \\
&= c' - \int_{\mathbb{C}\setminus G} g_G(z;\zeta)\, d\mu(\zeta), \quad c' := c + \int_{\mathbb{C}\setminus G} g_G(t;\infty)\, d\mu(t).
\end{aligned}$$

As a consequence, we have found a $\tau \in \mathcal{M}_0(G)$ satisfying (B.10). In particular, $f + V[\tau]$ equals the constant c' on G, and hence $\tau = \tau(f, G)$ with $w(f, G) = c'$. □

To the end of this section, let us show a weighted analogue of the Bernstein–Walsh lemma, where we adapt [MhSa92, Theorem 4.1] to our setting.

Lemma B.16 *Let G have the K–property, and $f_n(z) := \exp(n \cdot f(z))$, $n \geq 0$. Then for each polynomial P of degree at most n and for each $z \in \mathbb{C}$*

$$\log(|P(z)|/||P||_{f_n,\operatorname{supp}(\tau)}) \leq n \cdot w(f, G) - n \cdot V[\tau](z),$$

with the equilibrium measure $\tau = \tau(f, G)$. In particular, $||P||_{f_n,G} = ||P||_{f_n,\operatorname{supp}(\tau)}$. Finally, if $\deg\, P = n$, *then the leading coefficient of P has to be less or equal to $||P||_{f_n,G} \cdot \exp(n \cdot w(f, G))$.*

Proof: Let $P(z) = a \cdot (z - a_1) \cdot .. \cdot (z - a_k)$, $k \leq n$,

$$C_1 := -\frac{1}{n} \cdot \log(a/||P||_{f_n, supp(\tau)}) + w(f, G),$$

and define the measure ν by

$$\int h(z)\, d\nu(z) := \frac{1}{n} \sum_{j=1}^{k} h(a_j).$$

Then $\nu(\mathbb{C}) \leq 1$, and $\nu(\mathbb{C}) = 1$ if and only if $deg\, P = n$. With help of (B.8), we obtain for $z \in supp(\tau)$

$$\begin{aligned} V[\tau](z) - V[\nu](z) - C_1 &= \frac{1}{n} \cdot \log(|P(z)|/||P||_{f_n, supp(\tau)}) + V[\tau](z) - w(f, G) \\ &\leq f(z) + V[\tau](z) - w(f, G) = 0, \end{aligned}$$

and the assertion follows from Lemma B.12. □

A discussion of the case $f = 0$ of the classical Bernstein–Walsh Lemma may be found in Theorem B.22.

B.3.2 Weighted Fekete nodes

Let $G \subset \Omega$ be compact, and $g \in \mathcal{C}(G)$ be a density function. We consider for an integer $n \geq 1$ and $z_0, .., z_n \in G$ the expression

$$F_n[g](z_0, .., z_n) = \frac{\prod_{0 \leq j < k \leq n} |z_j - z_k|}{\prod_{0 \leq j \leq n} g(z_j)}. \tag{B.13}$$

Obviously, $F_n[g] : G^{n+1} \to [0; +\infty)$ is continuous, and therefore attains its maximum on G^{n+1}.

DEFINITION B.17 *A family of nodes $(z_{j,n})_{0 \leq j \leq n}$ is called a family of* Fekete nodes *(with respect to G and to the sequence of densities $(g_n)_n$) if, for all $n \geq 1$, the maximum of $F_n[g_n]$ on G^{n+1} is attained at $(z_{0,n}, .., z_{n,n}) \in G$.*

$(z_{j,n})_{0 \leq j \leq n}$ are called (g, G)–Fekete nodes if $(z_{j,n})_{0 \leq j \leq n}$ is a family of Fekete nodes with respect to G and to the sequence of densities $g_n(z) = \exp(n \cdot g(z))$, $n \geq 0$. In this case, let

$$W_n(g, G) := -\frac{2}{n \cdot (n+1)} \cdot \log F_n[g_n](z_{0,n}, .., z_{n,n}).$$

□

Fekete nodes are not necessarily unique. Let us first mention the following simple observation

Corollary B.18 *If $Z = (z_{j,n})_{0\le j\le n}$ is a family of Fekete nodes with respect to G and to the sequence of densities $(g_n)_n$, then for all $0 \le j \le n$ and for all $z \in G$*

$$\left|\frac{\omega_{Z,n}(z)}{(z - z_{j,n}) \cdot \omega'_{Z,n}(z_{j,n})}\right| \cdot \frac{g_n(z_{j,n})}{g_n(z)} \le 1. \tag{B.14}$$

In particular we have for the corresponding weighted Lebesgue function on G the estimate

$$1 \le \Delta_n(g_n, G; g_n, \{z_{0,n}, ..., z_{n,n}\}) \le n + 1.$$

Proof: We have for all z

$$\frac{F_n[g_n](z_{0,n}, .., z_{j-1,n}, z, z_{j+1,n}, .., z_{n,n})}{F_n[g_n](z_{0,n}, .., z_{n,n})} = \left|\frac{\omega_{Z,n}(z)}{(z - z_{j,n}) \cdot \omega'_{Z,n}(z_{j,n})}\right| \cdot \frac{g_n(z_{j,n})}{g_n(z)}.$$

Notice that $z_{j,n}$ has been chosen such that the ratio is maximized over all $z \in G$, consequently, the ratio is less or equal to 1 for all $z \in G$. The estimate of the Lebesgue function now becomes immediate. □

In the sequel of this section we will restrict ourselves to (f, G)–Fekete nodes, $f \in \mathcal{C}(G)$, where it is supposed that G has the K–property. Note that

$$\begin{aligned} W_n(f, G) &= \frac{2}{n \cdot (n + 1)} \sum_{0\le j<k\le n} \log \frac{1}{|z_{j,n} - z_{k,n}|} + \frac{2}{n + 1} \cdot \sum_{j=0}^{n} f(z_{j,n}) \\ &= \frac{2}{n \cdot (n + 1)} \sum_{0\le j<k\le n} \Big(\log \frac{1}{|z_{j,n} - z_{k,n}|} + f(z_{j,n}) + f(z_{k,n})\Big), \end{aligned}$$

corresponding with the quantity $\log(1/\delta_{n+1}(1/\exp\circ f, G))$ of [MhSa92, Eqn.(2.5)]. Some results from [MhSa92] concerning modified transfinite diameter and modified capacity of G with respect to f, adapted to our framework, are summarized in the following

Theorem B.19 *Let G have the K–property.*

(a) *The sequence $(W_n(f, G))_n$ is increasing, with limit $W(f, G)$.*

(b) *Let E be compact satisfying* $\mathrm{supp}(\tau) \subset E \subset G$, *with the weighted equilibrium measure $\tau = \tau(f, G)$. Then (f, E)–Fekete nodes are also (f, G)–Fekete nodes, moreover, (f, G)–Fekete nodes necessarily lie in the compact set*

$$G' := \{z \in G : V[\tau](z) + f(z) = w(f, G)\}.$$

(c) *Let τ_n denote the zero counting measure associated to (f, G)–Fekete nodes $z_{0,n}, .., z_{n,n}$. Then $(\tau_n)_n$ converges weakly to the equilibrium measure $\tau = \tau(f, G)$.*

Proof: Part (a) is cited from [MhSa92, Theorem 5.1, Lemma 5.2]. In order to prove (b), notice that, according to the continuity of $F_n[\exp(n\cdot f(z))]$, the (f,G) Fekete node $z_{j,n}$ is uniquely defined by the requirement

$$|g(z_{j,n})| = \max_{z\in G} |g(z)|, \quad g(z) := \exp(-n\cdot f(z)) \cdot \prod_{k\neq j} (z - z_{k,n}).$$

From Lemma B.16 we know that

$$\max_{z\in supp(\tau)} |g(z)| = \max_{z\in E} |g(z)| = \max_{z\in G} |g(z)|,$$

such that in fact $z_{j,n}$ may be chosen as an element of E. Therefore, (f,E)–Fekete nodes are also (f,G)–Fekete nodes. Also, G' is compact since both f, $V[\tau]$ are continuous on G. If now $z \in G\setminus G'$, then by Lemma B.16 and (B.8)

$$|g(z)| \leq \max_{\zeta\in S} |g(z)| \cdot \exp\Big(n\cdot(-V[\tau](z) - f(z) + w(f,G))\Big) < |g(z_{j,n})|.$$

Let us finally prove part (c) which for the case $G \subset \mathbb{R}$ may be found in [MhSa85, Remark p.90]. Here we will use (B.7) and the uniqueness of the equilibrium measure, however, unfortunately $F(\tau_n) = +\infty$. Following [MhSa92, p.122ff], we consider for $0 \leq j \leq n$

$$G_n := \{z \in \mathbb{C} : dist(z,G) \leq (\pi(n+1))^{-1/2}\}, \quad \Delta_{j,n} := \{z \in \mathbb{C} : |z - z_{j,n}| \leq (\pi(n+1))^{-1/2}\}.$$

As in [MhSa92, Eqn.(5.10)] we may extend $f \in \mathcal{C}(G)$ to a continuous function on the set G_0. Notice that $G_n \subset \Omega$ for sufficiently large n. With $\chi_{j,n}$ denoting the characteristic function of $\Delta_{j,n}$, we define the smoothed measure $\sigma_n \in \mathcal{M}_0(G_n)$ by

$$\sigma_n(B) = \int_B \sum_{j=0}^{n} \chi_{j,n}(z)\, dm(z),$$

where dm denotes the two–dimensional Lebesgue measure, and B is any Borel set.

For each $g \in \mathcal{C}(G_0)$ we have $\lim_{n\to\infty} \int g\, d\tau_n - \int g\, d\sigma_n = 0$, hence it is sufficient to prove that $(\sigma_n)_n$ has the only weak accumulation point $\tau(f,G)$. Mhaskar and Saff showed [MhSa92, Eqn.(5.21)] that

$$W(f,G_n) \leq F(\sigma_n) \leq W_n(f,G) + \epsilon_n,$$

where $\epsilon_n \to 0$. Now the right hand side tends to $W(f,G)$ according to Theorem B.19(a), and the left hand side according to [MhSa92, Theorem 3.3(d)]. Consequently, each accumulation point σ of $(\sigma_n)_n$ satisfies $F(\sigma) = W(f,G)$, proving that necessarily $\sigma = \tau(f,G)$. □

B.3.3 The optimal limit distribution

By definition given in (B.4), we have $\delta_{Z,n} \geq \delta_n$ for all $n \geq 0$. The aim of this section is to show that, for all sets H and G having the K–property, these quantity coincide in the limit if one chooses as Z a family of weighted Fekete nodes. Moreover, we will establish an explicit expression for this limit in terms of the weighted equilibrium measure. In view of the result of Theorems B.5 and B.8, let us first study the following extremal problem

LEMMA B.20 *Let G have the K–property, $f \in \mathcal{C}(G)$. We have for all $z \in \mathbb{C}$ and for all $\mu \in \mathcal{M}_0(G)$*

$$-V[\mu](z) + \sup_{\zeta \in \mathrm{supp}(\mu)} (V[\mu](\zeta) + f(\zeta)) \geq -V[\tau](z) + w(f, G)$$

with the weighted equilibrium measure $\tau = \tau(f, G)$.

Proof: In order to apply Lemma B.12, we take $\nu = \tau = \tau(f, G)$, $E = G$, $\mu \in \mathcal{M}_0(G)$ with potential being bounded on the support of μ (otherwise the assertion is trivially true). Furthermore, let

$$C_1 := \sup_{\zeta \in supp(\mu)} (V[\mu](\zeta) + f(\zeta)) - w(f, G).$$

We get for $z \in supp(\mu) \subset G$ by using (B.8)

$$\begin{aligned} &V[\mu](z) - V[\tau](z) - C_1 \\ &= V[\mu](z) + f(z) - \sup_{\zeta \in supp(\mu)} (V[\mu](\zeta) + f(\zeta)) - V[\tau](z) - f(z) + w(f, G) \leq 0, \end{aligned}$$

and Lemma B.12 yields the assertion. □

THEOREM B.21 *Let G have the K–property, and denote by $\tau = \tau(f_G, G)$ the weighted equilibrium measure. Then we have for each family of nodes Z in G*

$$\liminf_{n \to \infty} \delta_{Z,n} \geq -\min_{z \in H} (V[\tau](z) + f_H(z)) + w(f_G, G).$$

Here, equality is attained for the family Z of (f_G, G)–Fekete nodes

$$\lim_{n \to \infty} \delta_{Z,n} = \lim_{n \to \infty} \delta_n = -\min_{z \in H} (V[\tau](z) + f_H(z)) + w(f_G, G).$$

Moreover, in the particular case $f_H = 0$ and $f_G(z) = g_H(z, \infty)$, $z \in G$, the right hand quantity takes the form

$$-\min_{z \in H} (V[\tau](z) + f_H(z)) + w(f_G, G) = \max_{z \in H} \int_{\mathbb{C} \setminus G} g_G(z, t) \, d\mu_H(t),$$

with $\mu_H \in \mathcal{M}_0(H)$ denoting the (non–weighted) equilibrium measure of H.

Proof: We may assume without loss of generality that the zero counting measures of the given family of nodes has a weak limit μ (otherwise we consider a subsequence). Since $supp(\mu) \subset G'_Z$, we obtain from Theorem B.5 and Lemma B.6

$$\begin{aligned} &\liminf_{n \to \infty} \delta_{Z,n} \\ &\geq -\min_{z \in H} \Big(V[\mu](z) + f_H(z) \Big) + \sup_{z \in G'_Z} \Big(V[\mu](z) + f_G(z) \Big) \\ &\geq -\min_{z \in H} \Big(V[\mu](z) + f_H(z) \Big) + \sup_{z \in supp(\mu)} \Big(V[\mu](z) + f_G(z) \Big). \end{aligned}$$

Since the minimum is attained in H, the first part of the assertion follows from Lemma B.20.

Consider now the particular case of Z being the family of (f_G, G)–Fejer nodes. We apply again Theorem B.5 together with Theorem B.19(c), where it remains to determine the asymptotics of the sequence $(a_{Z,n})_n$. By definition of weighted Fekete nodes we obtain

$$a_{Z,n} = \max_{0\le j\le n} (f_G(z_{j,n}) + V[\tau_{j,n}](z_{j,n})) = \max_{0\le j\le n} \min_{z\in G}(f_G(z) + V[\tau_{j,n}](z)),$$

which by Theorem B.4(d), Theorem B.19(c) and (B.8) is convergent, with limit $w(f_G, G)$. This yields the desired representation for the limit of $(\delta_{Z,n})_n$. The connection with the limit of $(\delta_n)_n$ is obtained by observing that, for all $n \ge 0$,

$$\begin{aligned}&\Delta_n(h_n, H; g_n, G) \le \Delta_n(h_n, H; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\})\\ &\le \Delta_n(h_n, H; g_n, G) \cdot \Delta_n(g_n, G; g_n, \{z_{0,n}, z_{1,n}, .., z_{n,n}\}),\end{aligned}$$

and thus $\delta_{Z,n} \ge \delta_n \ge \delta_{Z,n} - \log(n+1)^{1/n}$ according to Corollary B.18. Finally, the explicit formula for the particular weight functions f_H, f_G follows from (B.10) established in Example B.15 by using (B.9). □

As the final assertion mentioned in this Appendix, we want to specify some more properties of the sequence $(\delta_n)_n$. In our work, these properties will be only required in the context of trivial density functions, hence the following Theorem is formulated in terms of $\Delta_n(\cdot, G)$ and $\Delta(\cdot, G)$, see Definition 2.5 and Definition 2.8.

THEOREM B.22 *Let G be compact,* $\text{cap}(G) > 0$. *Then for $z \in \mathcal{D}_\infty(G)$ and for $P \in \mathcal{P}_n$ there holds*

$$|P(z)| \le ||P||_G \cdot e^{n \cdot g_G(z)}. \tag{B.15}$$

In particular, for each compact set H we have

$$\Delta_n(H, G)^{1/n} \le \Delta(H, G), \tag{B.16}$$

and

$$\lim_{n\to\infty} \Delta_n(H, G)^{1/n} = \Delta(H, G). \tag{B.17}$$

Finally,

$$\lim_{n\to\infty} \Delta_n(z, G)^{1/n} = \Delta(z, G) \tag{B.18}$$

pointwise for $z \in \mathbb{C}$, and uniformly on compact subsets of $\mathcal{D}_\infty(G)$. If in addition $\mathcal{D}_\infty(G)$ is regular, then (B.18) holds uniformly on compact subsets of $\mathbb{C}$.

Proof: Inequality (B.15) is referred to as the (classical) Bernstein–Walsh Lemma, for a proof see, e.g., [NiSo88, Lemma 5.1]. If G in addition has the K–property, and therefore $\mathcal{D}_\infty(G)$ is regular, then

$$g_G(z) = g_G(z; \infty) = w(0, G) - V[\tau(0, G)](z) \tag{B.19}$$

(see Example B.14), and (B.15) follows by taking $f = 0$ in Lemma B.16.

Using the maximum modulus principle for analytic functions, we obtain $|P(z)| \leq ||P||_G$ for all $z \in \mathbb{C} \setminus \mathcal{D}_\infty(G)$ and for each polynomial P. Thus (B.16) follows from (B.15) by taking the maximum with respect to $z \in H$ and $P \in \mathcal{P}_n$.

By definition of $\Delta(H,G)$, for each $\epsilon > 0$ there exists a $\zeta_\epsilon \in H$ with $\Delta(\zeta_\epsilon, G) \geq \Delta(H,G) - \epsilon$. With help of the pointwise convergence of (B.18) we get

$$\liminf_{n\to\infty} \Delta_n(H,G)^{1/n} \geq \liminf_{n\to\infty} \Delta_n(\zeta, G)^{1/n} \geq \Delta(H,G) - \epsilon.$$

Since $\epsilon > 0$ may be chosen arbitrarily close to zero, assertion (B.17) becomes a consequence of (B.16) (notice also that (B.17) was already proved in Theorem B.21 for the case of G, H having the K–property).

It remains to show (B.18). First notice that (B.18) trivially holds for $z \in \mathbb{C} \setminus \mathcal{D}_\infty(G)$ since then all quantities involved are equal to 1. In order to verify locally uniform convergence in $\mathcal{D}_\infty(G)$ (implying of course also pointwise convergence in $\mathcal{D}_\infty(G)$), according to (B.15) it is sufficient to construct a sequence of 'extremal' polynomials $(p_n)_n$, p_n of degree n, such that

$$\lim_{n\to\infty} \Big(\frac{|p_n(z)|}{||p_n||_G}\Big) = \Delta(z,G) = e^{g_G(z)},$$

locally uniformly in $\mathcal{D}_\infty(G)$. We may construct such a sequence if G has the K–property (for compact sets G with $cap(G) > 0$, such a sequence has been given in [NiSo88, pp.193-195]). Let $Z = (z_{j,n})_{0\leq j\leq n}$ be the family of $(1,G)$–Fekete nodes, and $p_{n+1} = \omega_{Z,n}$, $n \geq 0$. It was shown in Theorem B.19 that the corresponding sequence of zero counting measures $(\tau_n)_n$ converges weakly to the equilibrium measure $\tau = \tau(0,G)$. From Theorem B.4(a) we may conclude that $\log p_n^{-1/n} = V[\tau_{n-1}]$ converges locally uniformly in $\mathcal{D}_\infty(G)$ to $V[\tau]$, whereas by Theorem B.4(d) and (B.8) we get

$$\log ||p_n||_G^{-1/n} = \min_{z\in G} V[\tau_{n-1}](z) \to \min_{z\in G} V[\tau](z) = w(0,G).$$

Thus the desired convergence follows from (B.19).

Finally suppose that $\mathcal{D}_\infty(G)$ is regular, and let $E \subset \mathbb{C}$ be compact. Then, for each $\epsilon > 0$, the set $E_\epsilon := \{z \in E \cap \mathcal{D}_\infty(G) : g_G(z) \geq \epsilon\}$ is a compact subset of $\mathcal{D}_\infty(G)$. We have already established uniform convergence in E_ϵ, and get in addition for $z \in E \setminus E_\epsilon$

$$|\log \Delta_n(z,G)^{1/n} - \log \Delta(z,G)| \leq 2 \cdot \log \Delta(z,G) \leq 2\epsilon.$$

This yields uniform convergence in E, or locally uniform convergence in $\mathbb{C}$. □

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