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Graph complexes and models of chain E_n-operads in odd characteristic

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This talk is based on a work in progress.

There are several usual models of E_n-operads. In the setting of the rational homotopy theory, a model of E_n-operads, used by Kontsevich to prove the formality of E_n-operads in characteristic zero, is given by a differential graded (co)operad of graphs (see [5]). The construction of this cooperad of graphs can be formalized by using a twisting procedure, which reflects a fiberwise integration process of semi-algebraic differential forms on Fulton-MacPherson operads. The main purpose of this talk is to explain the definition of an analogue of the graph cooperad model of E_n-operads in characteristic different from 2.

The construction of this odd characteristic analogue of the cooperad of graphs involves particular E_∞-algebra structures governed by the surjection operad. Recall briefly that the surjection operad, denoted by $\mathbb{E}$ in these notes, is a chain operad spanned in arity r and degree d by the sequences $\underline{u} = (u(1), \dots, u(r+d))$ such that $u(t) \in \{1, \dots, r\}$, for $t = 1, \dots, r+d$. In the chain complex $\mathbb{E}(r)$, we also take $\underline{u} \equiv 0$ when the mapping $t \mapsto u(t)$ does not surject over the set $\{1, \dots, r\}$ or when we have a repetition $u(t) = u(t+1)$ in the sequence $\underline{u}$. The differential of the surjection operad is given by the omission of terms in sequences and the operadic composition operations extend the operadic composition of permutations. The normalized cochain complex $N^*(X)$ of any simplicial set X is equipped with a natural $\mathbb{E}$-algebra structure (see e.g. [1]).

We make the following claim:

Theorem.

(1) *Let A and B be a pair of $\mathbb{E}$-algebras. Let $A \vee B$ denote the coproduct of A and B in the category of $\mathbb{E}$-algebras. We have a natural deformation retract in the category of chain complexes*

$$A \otimes B \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} A \vee B \circlearrowleft H ,$$

where ∇ carries $a \otimes b \in A \otimes B$ to the product of the elements a and b in $A \vee B$, while Δ is induced by an Alexander-Whitney diagonal map on the underlying chain complexes of the surjection operad. The natural transformation Δ is strongly symmetric monoidal (while ∇ is only symmetric up to homotopy).

(2) *In the case of the normalized cochain algebras of simplicial sets $A = N^*(X)$, $B = N^*(Y)$, the map Δ makes the following diagram commute*

$$\begin{array}{ccc} N^*(X) \vee N^*(Y) & \xrightarrow{\mathrm{pr}_X^* + \mathrm{pr}_Y^*} & N^*(X \times Y) , \\ & {\scriptstyle\Delta} \searrow \quad \swarrow {\scriptstyle\nabla^*} & \\ & N^*(X) \otimes N^*(Y) & \end{array}$$

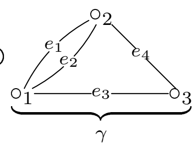
where the right hand side diagonal arrow is yielded by the usual Eilenberg–MacLane morphism, and we consider the morphisms $\mathrm{pr}_X^* : N^*(X) \to N^*(X \times Y)$, $\mathrm{pr}_Y^* : N^*(X) \to N^*(X \times Y)$, *induced by the canonical projections* $\mathrm{pr}_X : X \times Y \to X$, $\mathrm{pr}_Y : X \times Y \to Y$.

We rely on this result to construct a cooperad in chain complexes $\mathbb{E}\mathrm{Gra}_n^c$ such that:

$$\mathbb{E}\mathrm{Gra}_n^c(r) = \bigvee_{ij} \mathbb{E}(\omega_{ij}),$$

for each arity r, where we take, for each pair $\{i,j\} \subset \{1,\dots,r\}$, a free $\mathbb{E}$-algebra $\mathbb{E}(\omega_{ij})$ on one generator ω_{ij} of degree $n-1$. We can represent the elements of this cooperad as tensors $\underline{u} \otimes \gamma$, where γ is a graph based at vertices $\circ_1, \dots, \circ_r$, with edges $e_k = \circ_{i_k}{-}\circ_{j_k}$ corresponding to factors $\omega_{i_k j_k}$, and where $\underline{u} \in \mathbb{E}(m)$ is an element of the surjection operad whose inputs are in bijection with the edges of our graph e_k, $k = 1, \dots, m$. For instance the tensor

$$\underbrace{(1,3,1,2,4)}_{\underline{u}} \otimes$$



$$\in \mathbb{E}\mathrm{Gra}_n^c(3)$$

corresponds to the monomial $\underline{u}(\omega_{12}, \omega_{12}, \omega_{13}, \omega_{23})$ with $\underline{u} = (1,3,1,2,4) \in \mathbb{E}(4)$ in the $\mathbb{E}$-algebra $\mathbb{E}\mathrm{Gra}_n^c(3) = \mathbb{E}(\omega_{12}) \vee \mathbb{E}(\omega_{13}) \vee \mathbb{E}(\omega_{23})$.

We have natural morphisms of cooperads such that

$$\mathbb{E}\mathrm{Gra}_n^c(r) = \bigvee_{ij} \mathbb{E}(\omega_{ij}) \xrightarrow{(1)} \bigvee_{ij} N^*(S_{ij}^{n-1}) \xrightarrow{(2)} N^*(\bigtimes_{ij} S_{ij}^{n-1}) \xrightarrow{(3)} N^*(\mathcal{E}_n(r)),$$

for each arity r, where we consider the variant of the above cooperad of graphs defined by taking a coproduct of copies of the cochain algebra of the $n-1$-sphere $N^*(S_{ij}^{n-1}) = N^*(S^{n-1})$ (instead of the free $\mathbb{E}$-algebras $\mathbb{E}(\omega_{ij})$), we take the operad in simplicial sets given by the product $\bigtimes_{ij} S_{ij}^{n-1}$ of the copies of the sphere $S_{ij}^{n-1} = S^{n-1}$ and the associated cochain cooperad $N^*(\bigtimes_{ij} S_{ij}^{n-1})$, and $\mathcal{E}_n$ is a model of E_n-operad in simplicial sets with $\mathcal{E}_n(0) = *$ (we can take for instance the Barratt-Eccles operad model of [1] for this operad $\mathcal{E}_n$).

We proceed as follows to construct the morphisms of this sequence (1-3). We use the assumption that 2 is invertible in our ring of coefficients to pick a representative of the fundamental class of the sphere $\omega \in N^{n-1}(S^{n-1})$ whose image under the action of the antipode satisfies $\tau^*(\omega) = (-1)^n \omega$. We then form the coproduct of the morphisms of $\mathbb{E}$-algebras $\phi_{ij} : \mathbb{E}(\omega_{ij}) \to N^*(S_{ij}^{n-1})$, which carry the generator ω_{ij} to this element $\omega \in N^{n-1}(S_{ij}^{n-1})$ in our copy of the cochain algebra of the sphere, in order to define the first morphism (1) of our sequence, and we consider the morphisms $\mathrm{pr}_{ij}^* : N^*(S_{ij}^{n-1}) \to N^*(\bigtimes_{ij} S_{ij}^{n-1})$ induced by the canonical projections $\mathrm{pr}_{ij} : \bigtimes_{ij} S_{ij}^{n-1} \to S_{ij}^{n-1}$ in order to get our second morphism (2). We again rely on the result of the previous theorem to ensure that these morphisms preserve cooperad structures (the invariance assumption on our cochain ω with respect to the action of the antipode ensures that the first morphism preserves the symmetric structures of our cooperads). We eventually use the equivalence $\mathcal{E}_n(2) \sim S^{n-1}$ and the map $\psi_{ij} : \mathcal{E}_n(r) \to \mathcal{E}_n(2)$ such that $\psi_{ij}(\underline{w}) = \underline{w}(*, \dots, \underset{i}{1}, \dots, \underset{j}{1}, \dots, *)$ to get a morphism of operads in simplicial sets such that $\psi : \mathcal{E}_n(r) \to \bigtimes_{ij} S^{n-1}_{ij}$. We take the morphism of cochain cooperads induced by this morphism of simplicial operads to get the third morphism (3) of our sequence.

We prolong this sequence of morphisms by using the Koszul duality of E_n-operads in chain complexes (see [3]), which implies the existence of a quasi-isomorphism of cooperads

$$N^*(\mathcal{E}_n) \xrightarrow{\sim} B(\Lambda^n \mathbb{E}_n),$$

where $B(-)$ denotes the operadic bar construction, the notation Λ refers to the operadic suspension, and $\mathbb{E}_n$ is a model of E_n-operads in chain complexes (e.g. we can take the operad of chains on the Barratt-Eccles operad model of E_n-operads $\mathbb{E}_n = N_*(\mathcal{E}_n)$, as in *loc. cit.*) We accordingly have a sequence of morphisms:

$$(*) \qquad \mathbb{E}\mathrm{Gra}^c_n \to N^*(\mathcal{E}_n) \to B(\Lambda^n \mathbb{E}_n) \xrightarrow{\epsilon_*} B(\Lambda^n \mathcal{C}om),$$

where we also adopt the notation $\mathcal{C}om$ for the commutative operad and we consider the morphism induced by the augmentation $\epsilon : \mathbb{E}_n \to \mathcal{C}om$.

For any cooperad $\mathcal{C}$, we have a bijection between the morphisms $\phi_\alpha : \mathcal{C} \to B(\Lambda^n \mathcal{C}om)$ and the set of Maurer-Cartan elements in a differential graded preLie algebra with divided powers $\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om)$ such that:

$$\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om) = \mathrm{Hom}(\mathcal{C}, \Lambda^n \mathcal{C}om),$$

where we take the differential graded hom-object of maps of symmetric sequences $\alpha : \mathcal{C} \to \Lambda^n \mathcal{C}om$ (see [6]). We use a cooperad version of the general twisting procedure of [2] to associate a twisted cooperad $\mathrm{Tw}^c \mathcal{C}$ to any such Maurer-Cartan element $\alpha \in \mathrm{MC}(\mathrm{Dfm}(\mathcal{C}, \Lambda^n \mathcal{C}om))$.

We apply this construction to the case $\mathcal{C} = \mathbb{E}\mathrm{Gra}^c_n$. We then get cooperad morphisms:

$$\mathrm{Tw}^c \mathbb{E}\mathrm{Gra}^c_n \to \mathrm{Tw}^c N^*(\mathcal{E}_n) \to \mathrm{Tw}^c B(\Lambda^n \mathbb{E}_n),$$

by functoriality of the twisting construction. We also have a cooperad morphism $\pi_* : \mathrm{Tw}^c B(\Lambda^n \mathbb{E}_n) \to B(\Lambda^n \mathbb{E}_n)$ induced by composites with arity zero operations $* \in \mathbb{E}_n(0)$ at the level of the operad $\mathbb{E}_n$.

We can identify the elements of the cooperad $\mathrm{Tw}^c \mathbb{E}\mathrm{Gra}^c_n$ with tensors $\underline{u} \otimes \gamma$, where γ is a graph in which we split the set of vertices into a subset of internal vertices, usually denoted in black $\bullet_1, \dots, \bullet_k$, and a subset of external vertices $\circ_1, \dots, \circ_r$, which correspond to operadic inputs in $\mathrm{Tw}^c \mathbb{E}\mathrm{Gra}^c_n$.

We have $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}^c_n, \Lambda^n \mathcal{C}om) = \prod_r (\mathbb{E}\mathrm{Gra}^c_n(r)_{\Sigma_r})^\vee$, where we take the dual $(-)^\vee$ of the modules of graphs $\mathbb{E}\mathrm{Gra}^c_n(r)$ (moded out by the action of the symmetric group). We let $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}^c_n, \Lambda^n \mathcal{C}om)^{conn}$ denote the submodule spanned by the connected graphs inside $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}^c_n, \Lambda^n \mathcal{C}om)$. We can check that this submodule is preserved by the preLie algebra structure with divided powers on $\mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)$. We then make the following crucial observation:

Claim (Theorem in the case $n = 2$. Conjecture in the case $n > 2$)**.**

(1) *The Maurer-Cartan element* $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)$*, which corresponds to the morphism* $\phi_\alpha : \mathbb{E}\mathrm{Gra}_n^c \to B(\Lambda^n\mathcal{C}om)$ *yielded by our sequence (*), satisfies* $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n\mathcal{C}om)^{conn}$*. This relation implies that we can form a well-defined cooperad in chain complexes by taking the quotients*

$$\mathbb{E}\mathrm{Graphs}_n^c(r) = \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c(r)/\equiv,$$

where we mod out the complex $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c(r)$ *by relations such that* $\underline{u}\otimes\gamma \equiv 0$ *when the graph* γ *contains a connected components of internal vertices.*

(2) *The morphism* $\mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c \to B(\Lambda^n\mathbb{E}_n)$*, which we deduce from the map* $\pi_* : \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n) \to B(\Lambda^n\mathbb{E}_n)$ *at the level of the bar construction, cancels these elements such that* $\underline{u}\otimes\gamma \equiv 0$*, and therefore induces a morphism*

$$\begin{array}{ccccc} \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c & \longrightarrow & \mathrm{Tw}^c N^*(\mathcal{E}_n) & \longrightarrow & \mathrm{Tw}^c B(\Lambda^n\mathbb{E}_n) \\ \downarrow & & & & \downarrow{\scriptstyle \pi_*} \\ \mathbb{E}\mathrm{Graphs}_n^c & & \cdots\cdots\cdots\cdots\cdots\cdots & \dashrightarrow & B(\Lambda^n\mathbb{E}_n) \end{array}$$

on our quotient cooperad $\mathbb{E}\mathrm{Graphs}_n^c = \mathrm{Tw}^c\mathbb{E}\mathrm{Gra}_n^c/\equiv$.

We then obtain:

Theorem (depending on the validity of the conjectured property of the claim in the case $n > 2$)**.** *The morphism of cooperads yielded by the above construction defines a quasi-isomorphism:*

$$\mathbb{E}\mathrm{Graphs}_n^c \xrightarrow{\sim} B(\Lambda^n\mathbb{E}_n),$$

and hence the cooperad of graphs $\mathbb{E}\mathrm{Graphs}_n^c$ *defines a model of* E_n*-cooperad in chain complexes through the Koszul duality of* E_n*-operads.*

The cooperad $\mathbb{E}\mathrm{Graphs}_n^c$ is a generalization of the (co)operad of graphs defined in [5]. We also refer to [7] for a thorough study of this characteristic zero version of the (co)operad of graphs, which we usually denote by Graphs_n^c. We have, according to this reference, an action of a Lie algebra of graphs GC_n^2 on the cooperad Graphs_n^c, and one can prove that this differential graded Lie algebra GC_n^2 is quasi-isomorphic to the deformation complex of the object Graphs_n^c in the category of Hopf cooperads. We can use the latter result to compute the homotopy of the space of homotopy automorphisms of the rationalization of E_n-operads (see [4]). When we forget about Hopf structures, we still have a result asserting that the deformation complex of the graph cooperad Graphs_n^c is quasi-isomorphic, as a chain complex, to the symmetric algebra generated by the $n+1$-fold suspension of the differential graded module GC_n^2 (up to an extra, global, degree shift).

In our setting, the role of the differential graded Lie algebra of graphs GC_n^2 is yielded by the preLie algebra with divided powers:

$$\mathbb{E}\mathrm{GC}_n^2 = \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n \mathcal{C}om)^{conn}$$

which we equip with a twisted differential determined by our Maurer-Cartan element $\alpha \in \mathrm{Dfm}(\mathbb{E}\mathrm{Gra}_n^c, \Lambda^n \mathcal{C}om)^{conn}$. We can equip this extended graph complex $\mathbb{E}\mathrm{GC}_n^2$ with the structure of an L_∞-algebra with divided powers and we also have an action of this L_∞-algebra with divided powers on the chain cooperad $\mathbb{E}\mathrm{Graphs}_n^c = \mathrm{Tw}^c \mathbb{E}\mathrm{Gra}_n^c / \equiv$. We conjecture that the deformation complex of the object $\mathbb{E}\mathrm{Graphs}_n^c$ (in the category of chain cooperads) is equipped with a natural E_{n+2}-algebra structure and that the action of the L_∞-algebra $\mathbb{E}\mathrm{GC}_n^2$ on $\mathbb{E}\mathrm{Graphs}_n^c$ gives rise to a quasi-isomorphism $U_{E_{n+2}}(\mathbb{E}\mathrm{GC}_n^2) \xrightarrow{\sim} \mathrm{Dfm}(\mathbb{E}\mathrm{Graphs}_n^c, \mathbb{E}\mathrm{Graphs}_n^c)$, where $U_{E_{n+2}}(-)$ denotes the enveloping E_{n+2}-algebra functor from the category of L_∞-algebras (with divided powers) to the category of E_{n+2}-algebras. This conjecture is consistent with the computation of the deformation complex of the graph cooperad Graphs_n^c as a chain complex in characteristic zero.

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