February 9, 2022

Recollections on the <u>Postnikov decomposition of simplicial sets</u>

<u>Construction</u> (Postnikov sections):

$p : E \to B$ fibration and E, B Kan complexes

$x, y \in E_n$ $\quad x \overset{m}{\sim} y$ if $x|_{sk_m \Delta^n} = y|_{sk_m \Delta^n}$

&

$p(x) = p(y)$

$$P_m E = E / \overset{m}{\sim}$$

<u>Result</u>: We have

$$\begin{tikzcd}
E \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_m E \arrow[r, two heads] & P_{m-1} E \arrow[r, two heads] & \cdots \arrow[r, two heads] & P_0 E \arrow[r, two heads] \arrow[dr] & P_{-1} E \arrow[d, hook] \\
 & & K(\pi_m F, m) \arrow[u] & & & & B
\end{tikzcd}$$

connected component inclusion

where $F \to E \to B$
$\qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}$
$\qquad\qquad\qquad\qquad *$

$$\pi_* P_m E \xrightarrow{\cong} \pi_* P_{m-1} E \xrightarrow{\cong} \cdots \xrightarrow{\cong} \pi_* B \quad \text{for } * > m+1$$

$$\pi_* E \xrightarrow{\cong} \cdots \xrightarrow{\cong} \pi_* P_{m+1} E \xrightarrow{\cong} \pi_* P_m E \quad \text{for } * \leq m.$$

and
$$\begin{tikzcd}
0 \arrow[r] & \pi_{m+1} P_m E \arrow[r] & \pi_{m+1} P_{m-1} E \arrow[d, equal] & & \\
& & \pi_{m+1} B \arrow[r, "\partial"] & \pi_m F \arrow[r] & \pi_m E
\end{tikzcd}$$

$$\Rightarrow \pi_{m+1} P_m E = \mathrm{im}(\pi_{m+1} E \to \pi_{m+1} B)$$

In particular, we have

$$\pi_* E \xrightarrow{\cong} \pi_* P_m E \xrightarrow{\cong} \pi_* P_{m-1} E \quad \text{for } * \leq m-1$$

and

$$\begin{tikzcd}
\pi_m E \arrow[r, "\cong"] & \pi_m P_m E \arrow[r, two heads] \arrow[dr] & \pi_m P_{m-1} E \arrow[d, hook] \\
& & \pi_m B
\end{tikzcd}$$

Notation: $\mathcal{S} = s\mathcal{S}et$ simplicial sets

Recollections (equivariant cohomology)

- X simplicial set

$\Gamma = \pi X$ fundamental groupoid

We have a canonical map $\phi : X \to B\Gamma$

or $\phi(x) = x(0) \xrightarrow{[x(01)]} x(1) \xrightarrow{[x(12)]} \cdots \xrightarrow{[x(n-1n)]} x(n)$

for $x \in X_n$

- $A : \Gamma \to Ab$ local coefficient system

$$C^n_\Gamma(X, A) = \left\{ \begin{tikzcd} \alpha : X_n \arrow[rr] \arrow[dr] & & \operatorname{hocolim}_\Gamma A \arrow[dl] \\ & B\Gamma_n & \end{tikzcd} \right\}$$

with

$$\operatorname{hocolim}_\Gamma A = \coprod_{v_0 \to v_1 \to \cdots \to v_n} A(v_0)$$

Hence $\alpha \in C^n_\Gamma(X, A)$ is given by:

$$\alpha : x \longmapsto \underbrace{\alpha(x)}_{\in A(x(0))} \times \underbrace{x(0) \to x(1) \to \cdots \to x(n)}_{\in B\Gamma_n}$$

$C^n_\Gamma(-, A)$ extends to $\mathcal{S}/B\Gamma$

- To $\phi : X \to B\Gamma$, we associate $\sigma \mapsto \check{X}_\sigma$ given by:

$$\begin{tikzcd} \check{X}_\sigma \arrow[r, dashed] \arrow[d, dashed] & B(\Gamma/\sigma) \arrow[d] \\ X \arrow[r] & B\Gamma \end{tikzcd}$$

for $\sigma \in \Gamma$ (the covering system of the space X).

$\mathcal{S}_\Gamma$:= category of functors $\underline{X} : \Gamma \to \mathcal{S}$

We have an adjunction

$$(-)^\vee : \mathcal{S}/_{B\Gamma} \rightleftarrows \mathcal{S}_\Gamma : \operatorname{hocolim}_\Gamma$$

and $C^m_\Gamma(X, A) \cong \operatorname{Mor}_{\Gamma\text{-set}}(\check{X}_m, A)$

Fact: We have:

$$H^m_\Gamma(X, A) \cong [\check{X}, K(A, m)]_{\mathcal{S}_\Gamma}$$

$$\cong [X, \operatorname{hocolim}_\Gamma K(A, m)]_{\mathcal{S}/B\Gamma}$$

$$\check{X}_\sigma \xrightarrow{f} Y_\sigma \quad \longmapsto \quad X \longrightarrow \operatorname{hocolim}_\Gamma Y_\sigma$$

$$(x, \phi(x)_0 \to \cdots \to \phi(x)_n) \mapsto f(x) \quad \Big| \quad x \longmapsto (\varepsilon_*^{-1} f(x), \phi(x)_0 \to \cdots \to \phi(x)_n)$$

- Suppose $X \to B\Gamma$ has a section $\sigma : B\Gamma \to X$

then we take

$$\tilde{C}^*_\Gamma(X, A) = \ker\left(C^*_\Gamma(X,A) \xrightarrow{\sigma^*} C^*_\Gamma(B\Gamma, A)\right)$$

so that we have a split exact sequence:

$$0 \to \tilde{H}^*_\Gamma(X,A) \to H^*_\Gamma(X,A) \to H^*_\Gamma(B\Gamma, A) \to 0$$

For a pair $X \hookrightarrow Y$ with $\pi X \cong \pi Y \cong \Gamma$

we take $H^*_\Gamma(Y, X, A) = \tilde{H}^*_\Gamma(B\Gamma \cup_X Y, A)$.

We have a long exact sequence

$$\cdots \to H^n_\Gamma(Y, X, A) \to H^n_\Gamma(Y, A) \to H^n_\Gamma(X, A)$$

$$\to H^{n+1}_\Gamma(Y, X, A) \to \cdots$$

- Setting: $f : Y \to X$ map in $\mathcal{S}/B\Gamma$

with $\pi Y = \Gamma$ & Y connected.

take $Y \underset{\sim}{\overset{j}{\rightarrowtail}} Z \overset{q}{\twoheadrightarrow} X$

and set $F_v = q^{-1}(fv) = \mathrm{hofib}(Y \to X)$ for $v \in Y$

Let $n \geq 2$.

Assume $f_* : \pi_i(Y,v) \xrightarrow{\cong} \pi_i(X, fv)$ for $i<n$ $\forall v \in Y$

&

$f_* : \pi_n(Y,v) \twoheadrightarrow \pi_n(X, fv)$

Lemma: We then have

$f^* : H^i_\rho(X,A) \xrightarrow{\cong} H^i_\rho(Y,A)$ for $i<n$

&

$$\begin{array}{c}
0 \to H^n_\rho(X,A) \xrightarrow{f^*} H^n_\rho(Y,A) \to \mathrm{Hom}_\rho(\pi_{n+1}(f), A) \\
\downarrow d \\
H^{n+1}_\rho(X,A) \\
\downarrow \\
H^{n+1}_\rho(Y,A)
\end{array}$$

where $\pi_*(f) = \pi_{*-1}(F_v)$ for $v \in Y$.

Def: Take $A = \pi_{n+1}(f)$ and set

$k(f) = d_f(\mathrm{id}_{\pi_{n+1}(f)}) \in H^{n+1}_\rho(X, \pi_{n+1}(f))$

• Observation: For $\theta \in H^{n+1}_{\Gamma}(X, A)$ and the space

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, dashed, "p"'] & \operatorname{hocolim}_{\Gamma} L(A, n+1) \arrow[d] \\ X \arrow[r, "\theta"'] & \operatorname{hocolim}_{\Gamma} K(A, n+1) \end{tikzcd}$$

so that $F_{\sigma} \sim K(A_{\sigma}, n)$

$\Rightarrow \pi_{n+1}(p) = \pi_n K(A, n) = A$,

we get $k(p) = \theta \in H^{n+1}_{\Gamma}(X, \underbrace{\pi_{n+1} p}_{= A})$.

• Construction: we apply this construction

to $\theta = k(f)$ with $A = \pi_{n+1}(f)$.

We get:

$$\begin{tikzcd} & Z \arrow[r, dashed] \arrow[d, dashed, "p"'] & \operatorname{hocolim}_{\Gamma} L(A, n+1) \arrow[d] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "\exists ? g"] & X \arrow[r, "k(f)"'] & \operatorname{hocolim}_{\Gamma} K(A, n+1) \end{tikzcd}$$

Result: The map f admits a lifting $g : Y \to Z$

satisfying $g_* : \pi_{n+1}(f) \xrightarrow{\cong} \pi_{n+1}(p)$

- Consequence: For the Postnikov decomposition of a fibration $p: E \twoheadrightarrow B$ with E connected, we get a ho-Cartesian square:

$$\begin{tikzcd}
P_n E \arrow[r] \arrow[d] & \operatorname{hocolim}_{\Gamma} L(\pi_n F, n+1) \arrow[d] \\
P_{n-1} E \arrow[r, "k_n"'] & \operatorname{hocolim}_{\Gamma} K(\pi_n F, n+1)
\end{tikzcd}$$

for all $n \geq 2$

February 11, 2022

Postnikov decomposition of operads

Recollection: P Λ-operad.

$$\mathrm{cosk}^{\Lambda}_{m} P = \lim_{\substack{u \in \Lambda^{+}(r,m) \\ r \leq m}} P(r)$$

- We have

$$\mathrm{Mor}_{\Lambda\mathcal{O}p}(P, \mathrm{cosk}^{\Lambda}_{m} Q) \cong \mathrm{Mor}_{\Lambda\mathcal{O}p^{\leq m}}(P^{\leq m}, Q^{\leq m})$$

where $\Lambda\mathcal{O}p^{\leq m}$ denotes the category of Λ-operads truncated in arity $\leq m$.

- We also have $\mathrm{ar}^{\#}_{\leq m} : \Lambda\mathcal{O}p \to \Lambda\mathcal{O}p$

such that $\mathrm{Mor}_{\Lambda\mathcal{O}p}(\mathrm{ar}^{\#}_{\leq m} P, Q) \cong \mathrm{Mor}_{\Lambda\mathcal{O}p}(P, \mathrm{cosk}^{\Lambda}_{m} Q)$

For a free Λ-operad $P = \mathrm{Free}(M)$, $M \in \Lambda\mathrm{Seq}$, we get $\mathrm{ar}^{\#}_{\leq m} \mathrm{Free}(M) = \mathrm{Free}(\mathrm{ar}_{\leq m} M)$

with $\mathrm{tr}_{\leq m} M(r) = \begin{cases} M(r) & \text{if } r \leq m \\ \emptyset & \text{otherwise} \end{cases}$

together with the Λ-operator $u^*: M(\ell) \to M(k)$ inherited from M for $k \leq \ell \leq m$.

- Recall that we have $\mathrm{cosk}^{\wedge}_{m-1} P(m) = M(P)(m)$ the Matching object.

Object:

We study the tower

$$\begin{array}{c} \mathrm{Map}_{\Lambda O_p}(P, Q) \to \cdots \to \mathrm{Map}_{\Lambda O_p^{\leq m}}(P^{\leq m}, Q^{\leq m}) \\ \downarrow \\ \mathrm{Map}_{\Lambda O_p^{\leq m-1}}(P^{\leq m-1}, Q^{\leq m-1}) \\ \downarrow \\ \vdots \end{array}$$

and hence, the map

$$\begin{array}{c} \mathrm{Map}_{\Lambda O_p^{\leq m}}(P^{\leq m}, Q^{\leq m}) \\ \downarrow \\ \mathrm{Map}_{\Lambda O_p^{\leq m}}(P^{\leq m}, \mathrm{cosk}_{m-1} Q^{\leq m}) \end{array}$$

Idea: We consider the collection $P_n(\mathrm{cosk}_m^{\wedge} P)$

such that:

$P_n(\mathrm{cosk}_m^{\wedge} P)(r) = P(r)$ for $r < m$

and

$$P(m) \to \cdots \to P_n(\mathrm{cosk}_m^{\wedge} P)(m) \to P_{n-1}(\mathrm{cosk}_m^{\wedge} P)(m) \to$$

$$\begin{array}{ccc} \to P_0(\mathrm{cosk}_m^{\wedge} P)(m) \to & P_{-1}(\mathrm{cosk}_m^{\wedge} P)(m) \\ & \downarrow \wr \\ & M(P)(m) \end{array}$$

is the Postnikov tower of the map

$P(m) \to M(P)(m)$

Claim: $P_n(P^{\leq m}) = P_n(\mathrm{cosk}_m^{\wedge} P)$

inherits the structure of an m-truncated

Λ-operad

so that $P_n(P^{\leq m}) \to P_{n-1}(P^{\leq m})$

is a map of truncated operads

Proof: • For $u: \underline{k} \to \underline{l}$ with $k \leq l < m$,

$$\begin{array}{ccccc} u^*: & P_m(P^{\leq m})(l) & \longrightarrow & P_m(P^{\leq m})(k) \\ & \parallel & & \parallel \\ & P(l) & & P(k) \end{array}$$

So $u^*: P(l) \to P(k)$

• For $u: \underline{k} \to \underline{m}$ with $k < m$,

we take $\underbrace{P(m)/\overset{m}{\sim}}_{P_m(P^{\leq m})(m)} \xrightarrow{u^*} P(k)$

by using that $x \overset{m}{\sim} y \Rightarrow \mu(x) = \mu(y)$

$\Leftrightarrow v^*(x) = v^*(y)$

$\forall v: \underline{l} \to \underline{m}$ $l < m$.

• For $\sigma \in \Sigma_m$, we use $x \overset{m}{\sim} y \Rightarrow \sigma x \overset{m}{\sim} \sigma y$

since $v^*(x) = v^*(y)$ $\forall v: \underline{l} \to \underline{m}$ $l < m$

$$\begin{array}{ccccc} \Rightarrow & v^*(\sigma x) & = & v^*(\sigma y) & \forall v: \underline{l} \to \underline{m} \\ & \parallel & & \parallel & l < m \\ & (\sigma^{-1} v)^*(x) & & (\sigma^{-1} v)^*(y) & \end{array}$$

& $x|_{\Delta^q} = y|_{\Delta^q}$

$$\begin{array}{ccccc} \Rightarrow & \sigma x|_{\Delta^q} & = & \sigma y|_{\Delta^q} & \forall q \leq m. \\ & \parallel & & \parallel & \\ & \sigma(x|_{\Delta^q}) & & \sigma(y|_{\Delta^q}) & \end{array}$$

- For k, ℓ such that $k+\ell-1 < m$,

we take $\circ_i : P_m(P^{\leq m})(k) \times P_m(P^{\leq m})(\ell) \longrightarrow P_m(P^{\leq m})(k+\ell-1)$

given by $\circ_i : P(k) \times P(\ell) \longrightarrow P(k+\ell-1)$

For $k, \ell \leq m$ such that $k+\ell-1 = m$, we consider:

$$P(k) \times P(\ell) \xrightarrow{\circ_i} P(k+\ell-1) \longrightarrow P(k+\ell-1)/\overset{m}{\sim}$$

For $k = m$, $\ell = 1$, we use that

$$x \overset{m}{\sim} y \Rightarrow x \circ_i \theta \overset{m}{\sim} x \circ_i \theta \qquad \forall \theta \in P(1)$$

because:

$$\sigma^* x = \sigma^* y \Rightarrow \sigma^*(x \circ_i \theta) = \sigma^*(y \circ_i \theta)$$

$$\sigma^*(x \circ_i \theta) = \begin{cases} \sigma^* x \circ_j \theta \\ \text{or} \\ \sigma^* x \end{cases} \qquad \sigma^*(y \circ_i \theta) = \begin{cases} \sigma^* y \circ_j \theta & \text{in the case } i = \sigma(j) \\ \text{or} \\ \sigma^* y & \text{in the case } i \notin \sigma(n) \end{cases}$$

&

$$x|_{\Delta^q} = y|_{\Delta^q} \Rightarrow (x \circ_i \theta)|_{\Delta^q} = (y \circ_i \theta)|_{\Delta^q}$$

$$(x \circ_i \theta)|_{\Delta^q} = x|_{\Delta^q} \circ_i \theta|_{\Delta^q} \qquad (y \circ_i \theta)|_{\Delta^q} = y|_{\Delta^q} \circ_i \theta|_{\Delta^q}$$

to get $P(m)/\overset{m}{\sim} \times P(1) \xrightarrow{\circ_i} P(m)/\overset{m}{\sim}$

- For $k=1$ $\ell=n$, we similarly have

$$x \tilde{\sim} y \Rightarrow \theta o_1 x \tilde{\sim} \theta o_1 y$$

and we get $P(1) \times P(n)/\tilde{\sim} \xrightarrow{\circ_i} P(n)/\tilde{\sim}$. //

February 16, 2022

Postnikov invariants of operads

Recollections of the results of Mienné's thesis.

(non-unitary operad case)

- Context: $\mathcal{P}$ an operad such that $\mathcal{P}(0)=\emptyset$ $\mathcal{P}(1)=*$

  $\Gamma = \pi\mathcal{P}$ operad in groupoids such that

  $\Gamma(r)_x = \pi_1(\mathcal{P}(r), x)$ for every $x \in \mathcal{P}(r)$.

- Obsv. The canonical maps $f: \mathcal{P}(r) \to B\Gamma(r)$

  define a morphism of operads in $\mathcal{S} = sSet$.

- An operadic local coefficient system A

  is a collection of functors $A: \Gamma(r) \to Ab$

  tw. symmetric group actions

  $\sigma_*: A(v) \to A(\sigma v)$ for $v \in \Gamma(r)$ $\sigma \in \Sigma_r$

  and operadic composition maps

  $\circ_i: A(u) \oplus A(v) \to A(u \circ_i v)$ for $u \in \Gamma(k)$, $v \in \Gamma(l)$.

  We still assume $A(1) = 0$.

- For an operad $P \in \mathcal{S}Op / B\Gamma$, we then take

$$C^m_\Gamma(P, A) = \left\{ \begin{tikzcd} P_m \arrow[rr, "\alpha"] \arrow[dr] & & \operatorname{hocolim}_\Gamma A \arrow[dl] \\ & B\Gamma_m & \end{tikzcd} \text{ operad map} \right\}$$

thus, $\alpha \in C^m_\Gamma(P, A)$ consist of a collection of maps

$$\alpha : \underset{\in P(n)_m}{x} \longmapsto \underbrace{\underline{\alpha}(x)}_{\in A(x(0))} \times \underbrace{x(0) \to \cdots \to x(m)}_{\in B\Gamma(n)_m}$$

so that $\underline{\alpha}(x \circ_i y) = \underline{\alpha}(x) \circ_i \underline{\alpha}(y)$ in $A(x(0) \circ_i y(0))$

- Obsv.: For $P \in \mathcal{S}Op / B\Gamma$, the collection $\check{P}$ of covering systems

$$\begin{tikzcd} \check{P}_v \arrow[r, dashed] \arrow[d, dashed] & B(P(n)/v) \arrow[d] \\ P(n) \arrow[r] & B\Gamma(n) \end{tikzcd}$$

for $v \in \Gamma(n)$ inherits symmetric group actions

$$\sigma_* : \check{P}_v \longrightarrow \check{P}_{\sigma v} \quad \text{for } \sigma \in \Sigma_n$$

and operadic composition maps:

We denote by $\mathcal{S}Op_\Gamma$ the category formed by the collections of functors $Q : \Gamma(r) \to \mathcal{S}$ equipped with such operad structures.

We still have an adjunction

$$(-)^\vee : \mathcal{S}Op_{/B\Gamma} \rightleftarrows \mathcal{S}Op_\Gamma : \operatorname{hocolim}_\Gamma(-)$$

and
$$C^m_\Gamma(P, A) \simeq \mathrm{Mor}_{SetOp/B\Gamma_m}(P_m, \operatorname{ho.colim}_\Gamma A)$$
$$\simeq \mathrm{Mor}_{SetOp_\Gamma}(\check{P}_m, A).$$

<u>Obsv</u>: $\mathrm{Mor}_{SetOp_\Gamma}(\check{P}_m, A) \simeq \mathrm{Der}_{AbOp_\Gamma}(\mathbb{Z}\check{P}_m, A)$

where we equip $\mathbb{Z}\check{P}(v)$ $v \in \Gamma(r)$ with the augmentation

$$\varepsilon : \mathbb{Z}\check{P}(v) \to \mathbb{Z}_+ = \mathbb{Z}$$
$$\check{x} \mapsto 1$$

and we consider the derivations $\theta : \mathbb{Z}\check{P}_m \to A$, such that

$$\theta(x \circ_i y) = \underbrace{\theta(x) \circ_i 0}_{\theta(x)\circ_i \varepsilon(y)} + \underbrace{0 \circ_i \theta(y)}_{\varepsilon(x)\circ_i \theta(y)} \quad \text{for } x \in \check{P}_m(u),\ u \in \check{P}(v).$$

using that the operations

$$\begin{tikzcd}
A(u) \oplus 0 \arrow[d, hook] \arrow[drr, "\circ_i^r"] & & \\
A(u) \oplus A(v) \arrow[rr, "\circ_i"] & & A(u \circ_i v) \\
0 \oplus A(v) \arrow[u, hook] \arrow[urr, "\circ_i^l"'] & &
\end{tikzcd}$$

provide A with an abelian $\mathcal{C}om$-bimodule structure. //

- Fact:

For a cofibrant operad P over $B\Gamma$, we have:

$$H^n_\Gamma(P, A) \simeq [P, \operatorname{hocolim}_\Gamma K(A,n)]_{\mathcal{S}\mathcal{O}p/B\Gamma}$$

$$\simeq [\check{P}, K(A,n)]_{\mathcal{S}\mathcal{O}_{P\Gamma}} \quad //$$

- Rk: this implies that $H^*_\Gamma(-,A)$ defines a ho-invariant functor on cofibrant operads.

• Suppose that $P \to B\Gamma$ has a section $\sigma : B\Gamma \to P$ in $\mathcal{SO}p / B\Gamma$.

Then we take:

$$\tilde{C}^n_\Gamma(P,A) = \ker\left(C^n_\Gamma(P,A) \xrightarrow{\sigma^*} C^n_\Gamma(B\Gamma, A)\right)$$

and we again have a split short exact sequence when we pass to the cohomology:

$$0 \to \tilde{H}^*_\Gamma(P,A) \to H^*_\Gamma(P,A) \to H^*_\Gamma(B\Gamma, A) \to 0.$$

Prop: $\tilde{H}^*_\Gamma(-,A)$ carries the we between the operads of $\sigma : B\Gamma \to P$ is a cofibration to isomorphism.

• For a cofibration $f : P \rightarrowtail Q$ of operads / ?,

we set:

$$H^*_\Gamma(Q, P, A) := \tilde{H}^*_\Gamma(B\Gamma \vee_P Q, A)$$

where we take the pushout:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ B\Gamma \arrow[r, tail, "\sigma"] \arrow[rd, bend right, "="'] & B\Gamma \vee_P Q \arrow[d, dashed] \\ & B\Gamma \end{tikzcd}$$

which is endowed with a canonical $B\Gamma \xrightarrow{\sigma} B\Gamma \vee_P A$

• Obs: We have a short exact sequence:

$$0 \to \tilde{C}^{\bullet}_{\Gamma}(B\Gamma \vee_{P} Q, A) \to C^{\bullet}_{\Gamma}(Q, A) \to C^{\bullet}_{\Gamma}(P, A) \to 0$$

which gives a long exact sequence in cohomology:

$$\cdots \to H^n_{\Gamma}(Q, P, A) \to H^n_{\Gamma}(Q, A) \to H^n_{\Gamma}(P, A)$$
$$\to H^{n+1}_{\Gamma}(Q, P, A) \to \cdots //$$

• Reminder: We assume that our operads are non-unitary connected, in the sense that $P(0) = \emptyset$ $P(1) = *$ & $\Gamma(0) = \emptyset$ $\Gamma(1) = *$ for the associated groupoid operad Γ.

In this context, the operad $\mathbb{Z}\check{P}$ is endowed with an augmentation $\varepsilon: \mathbb{Z}\check{P} \to I$ where $I(0) = 0$ $I(1) = \mathbb{Z}$ and $I(n) = 0$ for $n > 1$.

• Def: For $Q \in \mathcal{S}Op_{\Gamma}$, we define

$$\mathrm{Indec}\,\mathbb{Z}Q(t) = \mathrm{coker}\left(\bigoplus_{\substack{u \circ_i v \to t \\ u \in \Gamma(k) \\ v \in \Gamma(l) \\ k, l \geq 2}} \mathbb{Z}Q(u) \otimes \mathbb{Z}Q(v) \xrightarrow{\circ_i} \mathbb{Z}Q(t) \right)$$

The mapping $Q \mapsto \mathrm{Indec}\, Q$

is left adjoint to the functor $(-)_+$

which carries a Γ-symmetric sequence

$M \in \mathrm{Seq}_\Gamma$ to the operad st

$M_+(0) = 0 \quad M_+(1) = \mathbb{Z}$

$M_+(u) = M(u)$ for $u \in \Gamma(n)$ $n \geq 2$

tw the trivial composition operations

$\circ_i = 0 : M_+(u) \otimes M_+(v) \to M_+(u \circ_i v)$

for $u \in \Gamma(k)$ $v \in \Gamma(\ell)$ with $k, \ell \geq 2$.

- For a cofibration $\phi : P \to Q$ in $\mathcal{S}\mathcal{O}p / B\Gamma$,

we consider

$$\mathrm{Indec}(\phi) = \mathrm{Indec}\left(I \vee_{\mathbb{Z}P^\vee} \mathbb{Z}Q^\vee \right)$$

$$= 0 \oplus_{\mathrm{Indec}\, \mathbb{Z}P^\vee} \mathrm{Indec}\, \mathbb{Z}Q^\vee$$

$$= \mathrm{coker}\left(\mathrm{Indec}\, \mathbb{Z}P^\vee \to \mathrm{Indec}\, \mathbb{Z}Q^\vee \right)$$

Lemma:

If $A(n) = 0$ for $n \neq m$, then we have:

$$C^m_\Gamma(Q, P, A) \cong \mathrm{Hom}_{\Sigma_m \rtimes \Gamma(m)}(\mathrm{Indec}\,\phi(m)_m, A(m))$$

$\mathrm{Indec}\,\phi(m)_m$ = the collection $\mathrm{Indec}\,\phi(u)$ with $u \in \Gamma(m)$.

and hence

$$H^*_\Gamma(Q, P, A) \cong H^*\mathrm{Hom}_{\Sigma_m \rtimes \Gamma(m)}(\mathrm{Indec}\,\phi(m), A(m))$$

Proof:

We use that $C^m_\Gamma(Q, P, A) = \tilde{C}^m_\Gamma(B\Gamma \vee_P Q, A)$

consists of the morphisms

$$\theta : E\Gamma_m \vee_{P^\vee_m} Q^\vee_m \longrightarrow A$$

such that $\theta|_{E\Gamma_m} \equiv 0$,

and are equivalent to morphisms $\theta : Q^\vee_m \longrightarrow A$

satisfying $\theta|_{P^\vee_m} \equiv 0$.

When $A(n) = 0$ for $n \neq m$, such a morphism θ

is fully determined by a map $f : \check{Q}_m(m) \to A(m)$

equivalent to a morphism of $\Sigma_m \propto \Gamma(m)$-abelian groups: $f: \mathbb{Z}P^\vee_m(m) \longrightarrow A(m)$

and the condition $\theta(x \circ_i y) = \theta(x) \circ_i \theta(y)$ is equivalent to

$$f(x \circ_i y) = 0 \quad \text{for } x \in \mathbb{Z}P^\vee_m(u) \quad y \in \mathbb{Z}P^\vee_m(v)$$
$$u \in \Gamma(k) \quad v \in \Gamma(l)$$
$$\text{with } k+l-1 = m.$$

The identity of the lemma follows. //

Prop: Let
$$\begin{tikzcd} P \arrow[r, tail, "\phi"] \arrow[dr] & Q \arrow[d] \\ & B\Gamma \end{tikzcd}$$

be so that

0) $\pi P \sim \pi Q \sim \Gamma$

1) $\phi: P(r) \xrightarrow{\sim} Q(r)$ for $r < m$

2) $\phi: P(m) \longrightarrow Q(m)$ is m-connected:

we have $\pi_* P(m) \xrightarrow{\simeq} \pi_* Q(m)$ for $* < m$

&

$\pi_m P(m) \twoheadrightarrow \pi_m Q(m)$

or equivalently $\pi_* \operatorname{hofib}(\phi: P(m) \to Q(m)) = 0$.

Then:

1) $H_* \,\mathrm{Indec}(\phi)(r) = 0$ for $r < m$

2) $H_* \,\mathrm{Indec}(\phi)(m) = 0$ for $* \leq m$
&
$H_{m+1} \mathrm{Indec}(\phi)(\sigma) = \underbrace{\pi_m(\mathrm{hofib}(\phi : \check{P}(\sigma) \to \check{Q}(\sigma)), \sigma)}_{\pi_{m+1}(\phi)}$
for $\sigma \in \Gamma(m)$

<u>Proof</u> (idea): We can assume

$Q_b = P_b \vee \mathrm{Free}(M)$

with $M \in s^{-}\mathrm{Set}\,\mathrm{Seq}\,/\,B\Gamma$

such that $M(r) = \emptyset$ for $r < m$

We then have $Q(r) = P(r)$ for $r < m$,

which yields (1)

and $Q(m)_b = P(m)_b \amalg M(m)$

with $\mathrm{Indec}(\phi)(m) = \mathbb{Z}\check{M}(m)$.

We therefore retrieve the classical relative Hurewicz theorem.

Prop. In the situation of the previous proposition,

for A such that $A(n) = 0$ for $n \neq m$,

we have $H^*_\Gamma(P, A) \cong H^*_\Gamma(Q, A)$ for $* < m$

and an exact sequence:

$$0 \to H^m_\Gamma(Q, A) \xrightarrow{\phi^*} H^m_\Gamma(P, A) \to \mathrm{Hom}_{\Sigma_m \wr \Gamma(m)}(\pi_{m+1}(\phi), A(m))$$

$$\xrightarrow{d_\phi} H^{m+1}_\Gamma(Q, A) \xrightarrow{\phi^*} H^{m+1}_\Gamma(P, A).$$

Proof (idea)

We use the hypercohomology exact sequence

$$E = \mathrm{Ext}^*_{\Sigma_m \wr \Gamma(m)}(H^* \mathrm{Indec}(\phi)(m), A(m))$$

$$\Downarrow$$

$$H^*(\mathrm{Hom}_{\Sigma_m \wr \Gamma(m)}(\mathrm{Indec}(\phi)(m), A(m)))$$

and the result of the previous proposition & lemma. //

February 17, 2022

<u>Def</u>: For an operad morphism $\phi: P \to Q$ as in the previous propositions yet, we take

$$A = \pi_{n+1}(\phi)$$

and we set

$$k(\phi) = d_\phi(id_{\pi_{n+1}(\phi)}) \in H^{n+1}_\Gamma(Q, A)$$

<u>Obsv</u>: For (an m-truncated operad) Q st $\pi Q = \Gamma$ and a cohomology class $\theta \in H^{n+1}_\Gamma(Q, A)$ / where $A(r) = 0$ for $r < m$

we form the pullback

$$\begin{array}{ccc} R & \dashrightarrow & \operatorname{hocolim}_\Gamma L(A, n+1) \\ \psi \downarrow & & \downarrow \\ Q & \xrightarrow{\theta} & \operatorname{hocolim}_\Gamma K(A, n+1) \end{array}$$

We have $fib_v(R \xrightarrow{\psi} P) \simeq K(A(v), n)$ for any $v \in \Gamma(m)$, and hence $\pi_{n+1}(\psi) = A$

~~(We form a factorization $R \xrightarrow{\psi} S \xrightarrow{\sim} Q$)~~

~~(to fix the cofibration assump)~~

We pick factorizations:

$$\begin{tikzcd}
I \arrow[r, tail, dashed] & R' \arrow[r, two heads, dashed, "\sim"] \arrow[d, tail, dashed, "\psi'"'] & R \arrow[r] \arrow[d] & \operatorname{hocolim}_{\Gamma} L(A, n+1) \arrow[d, two heads] \\
& Q' \arrow[r, two heads, dashed, "\sim"] \arrow[rr, bend right, dashed, "\theta'"'] & Q \arrow[r] & \operatorname{hocolim}_{\Gamma} K(A, n+1)
\end{tikzcd}$$

to fit the setting of the previous propositions.

We then have $k(\psi') = \theta'$

Construction:

We consider a cofibration of (n-truncated) operads $\phi : P \rightarrowtail Q$ that fulfill the assumptions of the previous proposition.

We apply the previous pullback construction to $\theta = k(\phi)$ and $A = \pi_{n+1}(\phi)$

$$\begin{tikzcd}
& R \arrow[r] \arrow[d, dashed, "\psi"] & \operatorname{hocolim}_{\Gamma} L(A, n+1) \arrow[d, two heads] \\
P \arrow[r, tail, "\phi"'] \arrow[ur, dashed, "\exists \tilde{\phi}"] & Q \arrow[r, "k(\phi)"'] & \operatorname{hocolim}_{\Gamma} K(A, n+1)
\end{tikzcd}$$

Result: There is a morphism Φ which makes the diagram commute (up to ho). and which induces an iso:

$$\tilde{\Phi} : \pi_n(\underbrace{\mathrm{hofib}(\phi)}_{=\pi_{n+1}(\phi)}) \longrightarrow \pi_n(\mathrm{fib}\,\psi)$$

Proof: We have:

$$\begin{array}{ccc}
\mathrm{Hom}_{\Sigma_m \ltimes \Gamma(m)}(\pi_{m+1}(\phi), A(m)) & & \ni \mathrm{id} \\
\Big\downarrow d_\phi & & \\
H^{m+1}_{\Gamma}(Q, A) \cong [Q, \mathrm{hocolim}_{\Gamma} K(A, m+1)] & & \ni k(\phi) \\
\Big\downarrow \phi^* & & \Big\downarrow \\
H^{m+1}_{\Gamma}(P, A) \cong [P, \mathrm{hocolim}_{\Gamma} K(A, m+1)] & & \ni k(\phi) \circ \phi
\end{array}$$

Hence $k(\phi) \circ \phi \simeq *$. This implies the existence of $\tau : P \longrightarrow R$ satisfying $\psi \circ \tau = \phi$.

We have:

$$\begin{tikzcd}
\pi_{n+1} Q(n) \arrow[r] \arrow[d, "="] & \pi_n \operatorname{hofib}(\phi) \arrow[r] \arrow[d] & \pi_n P(n) \arrow[r, "\phi_*"] \arrow[d, "\psi_*"] & \pi_n Q(n) \arrow[r] \arrow[d, "="] & 0 \\
\pi_{n+1} Q(n) \arrow[r] & \pi_n \operatorname{fib}(\psi) \arrow[r] & \pi_n R(n) \arrow[r] & \pi_n Q(n) \arrow[r] & 0
\end{tikzcd}$$

~~where we use the assumption that~~

~~ϕ_*~~

We form the factorizations:

$$\begin{tikzcd}
I \arrow[r, tail, dashed] & R' \arrow[dd, dashed, "\psi'"] \arrow[dr, dashed, "\sim"] & & \\
P \arrow[ur, dashed, "\exists \tau'"] \arrow[rr, "\tau"] \arrow[dd, tail, "\varphi"'] & & R \arrow[r] \arrow[dd, two heads, "\psi"] & \operatorname{hocolim} L(A, n+1) \arrow[dd] \\
& Q' \arrow[dr, dashed, "\sim"] & & \\
Q \arrow[ur, dashed, "\exists \varphi'"] \arrow[rr, "="] & & Q \arrow[r, "k(\varphi)"'] & \operatorname{hocolim} K(A, n+1)
\end{tikzcd}$$

We use the functoriality of the short exact sequence that gives the invariant.

We have a commutative diagram:

$$\mathrm{id} \longmapsto k(\varphi)$$

$$\begin{tikzcd}
0 \arrow[r] & H^n_\Gamma(Q,A) \arrow[r] & H^n_\Gamma(P,A) \arrow[r] & \mathrm{Hom}(\pi_{n+1}(\varphi),A) \arrow[r, "d_\varphi"] & H^{n+1}_\Gamma(Q,A) \\
0 \arrow[r] & H^n_\Gamma(Q',A) \arrow[r] \arrow[u, "\simeq"', "\varphi'^*"] & H^n_\Gamma(R',A) \arrow[r] \arrow[u, "\tau'^*"] & \mathrm{Hom}(\pi_{n+1}(\varphi'),A) \arrow[r, "d_{\varphi'}"'] \arrow[u, "\tau'_\#"] & H^{n+1}_\Gamma(\varphi',A) \arrow[u, "\simeq"', "\varphi'^*"]
\end{tikzcd}$$

We deduce from the previous observation

that $k(\varphi') = d_{\varphi'}(\mathrm{id}_{\pi_{n+1}(\varphi')})$

satisfies $(\varphi')^* k(\varphi') = k(\varphi)$

(when we go through $\pi_{n+1}(\varphi') \cong A$)

We therefore have $d_\varphi(\tau'_\#(\mathrm{id}_{\pi_{n+1}(\varphi')}) - \mathrm{id}_{\pi_{n+1}(\varphi)}) = 0$

We pick $\zeta \in H^n_\Gamma(P,A) \cong [P, \operatorname{hocolim}_\Gamma K(A,n+1)]$

such that $\zeta \longmapsto \mathrm{id}_{\pi_{n+1}(\varphi)} - \tau'_\#(\mathrm{id}_{\pi_{n+1}(\varphi')})$.

We modify τ by considering:

$$P \xrightarrow{\tau \times \zeta} R \times_{B\Gamma} \operatorname{hocolim}_\Gamma K(A,n) \xrightarrow{+} R$$

using the principal action of the abelian group in operads $K(A, n)$ on R.

We then have

$$(\tau \times \zeta)'_{\#}\left(id_{\pi_{n+1}(\varphi')} \right) = id_{\pi_{n+1}(\varphi)}$$

for the corresponding morphism

$$(\tau \times \zeta)' : P \to R'$$

We conclude that $(\tau \times \zeta)'$ (and hence $T \times \zeta$) induces an iso:

$$(\tau \times \zeta)'_{*} : \pi_{n+1}(\varphi) \longrightarrow \pi_{n+1}(\varphi')$$

as required. //

<u>Consequence</u>: (of a non-unitary <u>connected</u> operad P)

For the operadic Postnikov decomposition,

we have a ho-Cartesian square

of operads:

$$\begin{array}{ccc} P_n(P^{\leq m}) & \longrightarrow & \operatorname{hocolim}_{\Gamma} L(A, n+1) \\ \downarrow & & \downarrow \\ P_{n-1}(P^{\leq m}) & \xrightarrow[k_{n,m}]{} & \operatorname{hocolim}_{\Gamma} K(A, n+1) \end{array}$$

for each (n, m)

with $A(\sigma) = \pi_n(P(m), \sigma)$ for $\sigma \in \Gamma(m)$.

February 18, 2022

The definition of Postnikov invariants for Λ-operads

Reminder:

We consider operads equipped with operators

$u^*: P(\underline{l}) \to P(\underline{k})$ for $u: \underline{k} \to \underline{l}$ injective.

We can assume $P(\underline{0}) = *$ or consider instead

that $P(\underline{0}) = \emptyset$, assuming that each term $P(\underline{k})$

is endowed with an augmentation map

$\varepsilon: P(\underline{k}) \to *$, which is automatically given by

the terminal map in the case of operads in $\mathcal{Y}$.

We still assume $P(\underline{1}) = *$.

We focus on the setting of m-truncated

operads $\Lambda Op^{\le m}$

$$\begin{tikzcd} \Lambda Op \arrow[r, "t_m"] & \Lambda Op^{\le m} \arrow[l, bend right, dashed, "\exists s^{\#}_m"'] \arrow[l, bend left, dashed, "\exists c_m"] \end{tikzcd}$$

We have $\Lambda Op \xrightarrow{t_m} \Lambda Op^{\le m}$

with $c_m t_m = cosk^{\Lambda}_m$ and $s^{\#}_m t_m = ar^{\#}_m$.

* We also deal with:

$$\Lambda\mathcal{O}_p^{\leq m} \underset{c}{\overset{r_\#}{\leftarrow}} \xrightarrow{t} \Lambda\mathcal{O}_p^{\leq m-1}$$

For $Q \in \Lambda\mathcal{O}_p^{\leq m-1}$ we have

$$cQ(m) = \lim_{\substack{u\in\Lambda^+(n,m)\\ n<m}} Q(n)_u$$

$$cQ(n) = Q(n) \quad \text{for } n<m.$$

* We use $\lim_{\substack{u\in\Lambda^+(n,m)\\ n<m}} Q(n)_u \cong \lim_{\substack{u\in\Lambda(n,m)\\ n<m}} Q(n)_u$

to provide $cQ(m)$ with a Σ_m-action.

Equivalently, for $s\in\Sigma_m$, $u\in\Lambda^+(n,m)$, we have

$su = vt$ with $v\in\Lambda^+(n,m)$ $t\in\Sigma_n$.

We take $s(x_u) = (tx)_v$ for $x\in Q(n)_u$.

We determine $u^*: cQ(m) \to Q(n)$ for $u\in\Lambda^+(n,m)$

by $cQ(m) \to Q(n)_u = Q(n)$

(the canonical projection).

$k, \ell < m$

* For $k+\ell-1 = m$, we determine

$$Q(k) \times Q(\ell) \xrightarrow{\circ_i} cQ(m) \longrightarrow Q(r)_u$$
$$p \quad q \longmapsto (p \circ_i q)_u .$$

by using that $u \in \Lambda^+(r, m)$ has either

a unique decomposition

$u = v \circ_{v(j)} w$ for some $v \in \Lambda^+(p,k)$ $w \in \Lambda^+(q,\ell)$
$p \leq k$ $q \leq \ell$

(with $i = v(j)$)

or a unique decomposition

$u = v \circ_{v(j)} o$ for some $v \in \Lambda^+(p,k)$ $o \in \Lambda^+(0,\ell)$.
$p \leq k$

(with $i = v(j)$ again).

We take

$(p \circ_i q)_u = (v^* p) \circ_j (w^* q)$ in the first case

$(p \circ_i q)_u = v^* p \circ_j *$ in the second case

* We just provide cQ with the structure operations of Q in arity $r < m$.

. Obsv.

0/ We have $P \in \Lambda\mathcal{O}p^{\leq m} \Rightarrow \pi P \in \mathcal{G}rd\,\Lambda\mathcal{O}p^{\leq m}$

1/ We consider the obvious Λ-operad analogue of the notion of an operadic local coefficient system,

and for $P \in \Lambda\mathcal{O}p^{\leq m}/_{B\Gamma}$, we take for $C^*_\Gamma(P,A)$

the set of operad maps

$$\begin{tikzcd}
\alpha : P_n \arrow[rr] \arrow[dr] & & \operatorname{hocolim}_\Gamma \arrow[dl] \\
& B\Gamma &
\end{tikzcd}$$

which also commute with the action of the operators u^*, for $u \in \Lambda(\underline{k},\underline{l})$.

2/ The covering system $\check{P}$ of an operad $\Lambda\mathcal{O}p^{\leq m}/B\Gamma$

forms a Γ-equivariant Λ-operad,

so that

$C^m_\Gamma(P,A) \simeq \mathrm{Mor}_{\mathcal{S}et\Lambda\mathcal{O}p_\Gamma}(\check{P}_m, A)$, and in the identity

$$\mathrm{Mor}_{\mathcal{S}et\Lambda\mathcal{O}p_\Gamma}(\check{P}_m, A) \simeq \mathrm{Der}_{\mathcal{A}b\Lambda\mathcal{O}p_\Gamma}(\mathbb{Z}\check{P}_m, A)$$

We deal with operadic derivations

$\Theta : \mathbb{Z}\check{P}_m \to A$

such that

$\Theta(u^* p) = u^* \Theta(p) \quad \forall u \in \Lambda(\underline{k}, \underline{m})$.

- We still have a notion of reduced cohomology $\tilde{H}^*_\Gamma(-, A)$ and a notion relative cohomology $H_*(-,-,A)$ with the obvious generalization of the long exact sequence.

For a local coefficient system satisfying

$A(r) = 0$ for $r < m$,

we still have:

$$C^m_\Gamma(Q, P, A) \simeq \mathrm{Hom}_{\Sigma_m \ltimes \Gamma(m)}(\mathrm{Indec}\,\phi(m), A(m))$$

where we use the same construction of

indecomposables as in the Σ-operad context

(we therefore forget about the Λ-structure

when we take $\mathrm{Indec}\,\phi(m)$.)

This extension follows from the observation

that when $A(n)=0$ for $n<m$

every derivation $\Theta: \mathbb{Z}\check{Q}_m \to A$

automatically preserves the Λ-operators

since we have:

$$\begin{array}{ccc} \mathbb{Z}\check{Q}_m(m) & \xrightarrow{\Theta} & A(m) \\ {\scriptstyle u^*}\downarrow & & \downarrow \\ \mathbb{Z}\check{Q}_m(n) & \longrightarrow & 0 \end{array}$$

for $u \in \Lambda(\underline{n}, \underline{m})$ with $n<m$.

- We now consider an operad morphism

$\phi: P \to Q$ in $\mathcal{S}\Lambda\mathcal{O}_p^{\leq m}/B\Gamma$ (over $B\Gamma$)

st 0) $\pi P \sim \pi Q \sim \Gamma$

1) $\phi: P(n) \xrightarrow{\simeq} Q(n)$ for $n<m$

2) ϕ: $\pi_* P(m) \xrightarrow{\simeq} \pi_* Q(m)$ for $*<n$

$\pi_m P(m) \twoheadrightarrow \pi_m Q(m)$

We can still form a cofibrant approximation of this morphism as

$Q_b = P_b \vee \mathcal{F}(M_b)^{\leq m}$ with $M(n) = \emptyset$ for $n < m$

when we forget about faces & Λ-operators.

The Λ-operators on Q_b are determined by maps $d_i : M_b(m) \longrightarrow P_b(m-1)$ on $M_b(m) \subset Q_b(m)$.

We still get for this cofibrant approximation

1) $H_* \operatorname{Indec} \phi(n) = 0$ for $n < m$

2) $H_* \operatorname{Indec} \phi(m) = 0$ for $* \leq m$

&

$H_{m+1} \operatorname{Indec} \phi(v) = \pi_m \operatorname{hofib}(\phi : \check{P}(v) \to \check{Q}(v)), v)$

for $v \in \Gamma(m)$

and we can adapt our construction

of the exact sequence

$$0 \to H^n_\Gamma(Q,A) \xrightarrow{\phi^*} H^n_\Gamma(P,A) \to \mathrm{Hom}_{\Sigma_m \wr P(m)}(\pi_{n+1}(\phi), A(m))$$

$$\xrightarrow{d\phi} H^{n+1}_\Gamma(Q,A) \to H^{n+1}_\Gamma(P,A)$$

and of the Postnikov invariant

$$k(\phi) \in H^{n+1}_\Gamma(P,A) \cong [P, \operatorname{hocolim}_\Gamma K(A,n+1)]_{\mathcal{S}\Lambda Op_P^{\leq m}}$$

for $\quad A(v) = \underbrace{\pi_n(\mathrm{hofib}(\phi : \check{P}(v) \to \check{Q}(v)), v)}_{\pi_{n+1}(\phi)(v)}$.

- For $\theta = k(\phi)$, we then form the pullback

$$\begin{tikzcd} & R \arrow[r] \arrow[d, dashed, "\psi"'] & \operatorname{hocolim}_\Gamma L(A,n+1) \arrow[d] \\ P \arrow[r, "\phi"'] & Q \arrow[r, "\theta = k(\phi)"'] & \operatorname{hocolim}_\Gamma K(A,n+1) \end{tikzcd}$$

in the category $\mathcal{S}\Lambda Op^{\leq m}/_{B\Gamma}$, using trivial operators

$$u^* : K(A(\sigma), n+1) \longrightarrow K(0, n+1)$$

inherited from $A = \pi_{n+1}(\phi)$.

We can apply the same argument as in the Σ-operad setting to get the existence of a map:

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\tau \times \xi"] \arrow[rr, dashed, bend right, "\psi"'] & R \times \underset{B\Gamma}{\mathrm{hocolim}}\, \underset{s}{K(A,n)} \arrow[r] & R
\end{tikzcd}$$

that lifts ϕ and induces an iso:

$$\pi_{n+1}(\phi) \xrightarrow{\simeq} \pi_{n+1}(\psi)$$

- We apply this construction to the tower

m-truncated Λ-operads

$$\begin{tikzcd} P^{\leq m} \arrow[r] & \cdots \arrow[r] & P_n(P^{\leq m}) \arrow[r] & P_{n-1}(P^{\leq m}) \arrow[r] & \cdots \arrow[r] & P_{-1}(P^{\leq m}) \arrow[d, "\sim"] \\ &&&&& cP^{\leq m-1} \end{tikzcd}$$

given by the relative Postnikov decomposition

of the map $P^{\leq m} \to cP^{\leq m-1}$

We get that each map $P_n(P^{\leq m}) \to P_{n-1}(P^{\leq m})$ $n \geq 2$

in this tower fit in a ho-Cartesian square

of m-truncated Λ-operads of the form:

$$\begin{tikzcd} P_n(P^{\leq m}) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\Gamma} L(A, n+1) \arrow[d] \\ P_{n-1}(P^{\leq m}) \arrow[r] & \operatorname{hocolim}_{\Gamma} K(A, n+1) \end{tikzcd}$$

with $A(v) = \pi_n(\mathrm{fib}\,(P(m) \to \underbrace{M(P)(m)}_{= cP^{\leq m-1}(m)}), v)$ //

(assuming that P consists of connected spaces).

February 19, 2022

Applications to mapping spaces of bimodules

Context: We study mapping spaces of the form

$\mathrm{BiMod}_{P}(M, Q)$

where P is a Λ-operad satisfying $P(0) = P(1) = *$

(in applications, we take $P = E_m$)

M is a P-bimodule (also equipped with a Λ-structure)

Q is a Λ-operad, which also satisfies $Q(0) = Q(1) = *$, and is equipped with an operad morphism $\rho: P \to Q$

(in applications, we take $Q = E_n$)

We still have

$\mathrm{BiMod}_{P}(M, Q) = \lim_{m} \mathrm{BiMod}_{P}^{\leq m}(M, Q)$.

and yet, we have

$$\mathrm{BiMod}_P^{\leq m}(M, Q) \to \mathrm{BiMod}_P^{\leq m-1}(M, Q)$$
$$\qquad\qquad\qquad\qquad = \mathrm{BiMod}_P^{\leq m}(M, cQ^{\leq m-1})$$

where we again consider the image of the operad $Q^{\leq m-1} \in \Lambda Op^{\leq m-1}$ under the functor $c : \Lambda Op^{\leq m-1} \to \Lambda Op^{\leq m}$ right adjoint to the truncation $t : \Lambda Op^{\leq m} \to \Lambda Op^{\leq m-1}$ and the morphisms $P^{\leq m} \xrightarrow{\rho} Q^{\leq m} \to cQ^{\leq m-1}$.

Indeed, for a map $f : M^{\leq m} \to cQ^{\leq m-1}$, we necessarily have

$$\begin{tikzcd} M(m) \arrow[r, "f"] \arrow[d, "u^*"] & cQ^{\leq m-1}(m) \arrow[r] \arrow[dr, "u^*"] & Q(r)_u \arrow[d, "="] \\ M(r) \arrow[rr, "f"] & & Q(r) \end{tikzcd}$$

so that f is determined by its components $f : M(r) \to Q(r)$ $r < m$, and from the formulas

$$
\begin{aligned}
f(x \circ_i y)_u &= f(u^*(x \circ_i y)) \\
&= \phi(v^* x) \circ_j f(w^* y) = v^* \phi(x) \circ_j w^* f(y) \\
&\text{or} \\
&= \phi(v^* x) \circ_j * \qquad = v^* \phi(x) \circ_j *
\end{aligned}
$$

$$
\begin{aligned}
f(x \circ_i q)_u &= f(u^*(x \circ_i q)) \\
&= f(v^* x) \circ_j \phi(w^* q) = v^* f(x) \circ_j w^* \phi(q) \\
&\text{or} \\
&= f(v^* x) \circ_j * \qquad = v^* f(x) \circ_j * .
\end{aligned}
$$

we conclude that the extended map

$f : M^{\leq m} \to c\,Q^{\leq m-1}$

automatically preserves the operadic composition products.

Construction:

We can assume $Q^{\leq m} \twoheadrightarrow c\,Q^{\leq m-1}$.

We take the relative Postnikov decomposition of this map of truncated Λ-operads

$$
\begin{tikzcd}
Q^{\leq m} \arrow[r] & \cdots \arrow[r] & P_m(Q^{\leq m}) \arrow[r] & P_{m-1}(Q^{\leq m}) \arrow[r] & \cdots \arrow[r] & P_1(Q^{\leq m}) \arrow[d, "\text{connected}"'] \\
 & & & & & c\,Q^{\leq m-1}
\end{tikzcd}
$$

Remark:

There is no guarantee that $cQ^{\leq m-1}$ is connected, even when Q is connected.

In our case $Q = E_d$, we have

$$P_{N_{d,m}}(E_d^{\leq m}) \xrightarrow{\sim} P_{N_{d,m-1}}(E_d^{\leq m}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} P_{-1}(E_d^{\leq m}) \sim *$$

for $N_{d,m} = (d-2)(m-1)$

because we observed that the map

$$E_d(m) \longrightarrow cE_d^{\leq m-1}(m) = M(E_d)(m)$$

is $(d-2)(m-1)$-connected (see §10 of the rational homotopy of mapping spaces of E_d-operads)

Furthermore, we have

$$\pi_* \operatorname{hofib}(E_d(m) \to M(E_d)(m))$$

$$= \bigcap_{i=1}^{m} \ker\left(\pi_* E_d(m) \xrightarrow{-\circ_i *} \pi_* E_d(m-1)\right)$$

by a general result of Bousfield-Kan

(see again §10 of the rational homotopy of mapping spaces

Result:

We have a homotopy cartesian square

$$\begin{tikzcd} P_n(Q^{\leq m}) \arrow[r] \arrow[d] & L(A, n+1) \arrow[d] \\ P_{n-1}(Q^{\leq m}) \arrow[r] & K(A, n+1) \end{tikzcd}$$

for any $n > N_{d,m}$

and

$$\begin{tikzcd} P_n(Q^{\leq m}_{\mathbb{Q}}) \arrow[r] \arrow[d] & L(A_{\mathbb{Q}}, n+1) \arrow[d] \\ P_{n-1}(Q^{\leq m}_{\mathbb{Q}}) \arrow[r] & K(A_{\mathbb{Q}}, n+1) \end{tikzcd}$$

when we pass to the rationalization,

where

$A = \pi_n \operatorname{hofib}(Q(m) \to M(Q)(m))$, $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$,

and we the simplifying result $\pi Q \simeq *$

in our case $Q = E_d$

February 20, 2022

For a bimodule M, we consider:

$$\begin{tikzcd}
\mathrm{BiMod}_{P}^{\leq m}(M, P_m(Q^{\leq m})) \arrow[r] \arrow[d] & \mathrm{BiMod}_{P}^{\leq m}(M, L(A, m+1)) \arrow[d] \\
\mathrm{BiMod}_{P}^{\leq m}(M, P_{m-1}(Q^{\leq m})) \arrow[r] & \mathrm{BiMod}_{P}^{\leq m}(M, K(A, m+1))
\end{tikzcd}$$

Observation:

For coefficients A such that $A(r) = 0$ for $r < m$, and a cofibrant object M in the category of P-bimodules, we still have:

$$\mathrm{BiMod}_{P}^{\leq m}(M, K(A, m+1)) \simeq \mathrm{Map}_{\Sigma_m\text{-Mod}}(\mathrm{Indec}_P \mathbb{Z}M(m), A(m)[m+1])$$

and hence

$$\pi_* \mathrm{BiMod}_{P}^{\leq m}(M, K(A, m+1)) \simeq H^{m+1-*}_{\Sigma_m}(\mathrm{Indec}_P \mathbb{Z}M(m), A(m))$$

where:

- we consider the mapping space $\mathrm{Map}_{\Sigma_m\text{-Mod}}(-,-)$ in the category of Σ_m-modules, so that $N_* \mathrm{Map}_{\Sigma_m\text{-Mod}}(-,-) = \tau_{*\geq 0} \mathrm{Hom}_{\Sigma_m\text{-Mod}}(-,-)$

for the truncation $\tau_{\geq 0}$ in non-negative degrees of the dg-Hom $\mathrm{Hom}_{\Sigma_m\text{-Mod}}(-,-)$

- $A(m)[n+1]$ denotes the graded Σ_m-module defined by putting $A(m)$ in degree $n+1$

- and we set:

$$\mathrm{Indec}_P \mathbb{Z}M(m) = \mathrm{coker}\left(\left(\bigoplus_{\substack{k+l-1=m \\ 2\leq k\leq m}} \mathbb{Z}P(k) \otimes \mathbb{Z}M(l) \right) \oplus \left(\bigoplus_{\substack{k+l-1=m \\ 2\leq l\leq m}} \mathbb{Z}M(k) \otimes \mathbb{Z}P(l) \right) \xrightarrow{\circ_i} \mathbb{Z}M(m) \right)$$

Proof (idea):

We have $\mathrm{BiMod}_P^{\leq m}(M, K(A,n+1))$.

$$= \mathrm{Mor}_{P\text{-BiMod}}(M^{\leq m}, K(A,n+1)^{\Delta^\bullet})$$

We still immediately get that a P bimodule morphism $f : M^{\leq m} \to K(A,n+1)^{\Delta}$

is uniquely determined by a Σ_m-equivariant $\mathbb{Z}$-module map

$$f : \mathbb{Z}M(m) \to K(A(m), m+1)^{\Delta^\bullet},$$

which automatically preserves the Λ-operators. and the preservation of the composition products are equivalent to the relation

$$f(x \circ_i q) = 0 \quad \text{for} \quad x \in M(k) \quad q \in Q(l)$$
$$l \geq 2 \ \& \ k+l-1 = m \Rightarrow k < m$$

$$f(p \circ_i y) = 0 \quad \text{for} \quad p \in P(k) \quad y \in M(l)$$
$$k \geq 2 \ \& \ k+l-1 = m \Rightarrow l < m.$$

We therefore get

$$\mathrm{BiMod}_P^{\leq m}(M, K(A, m+1))_\bullet \cong \underbrace{\mathrm{Mor}_{\Sigma_m\text{-Mod}}(\mathrm{Indec}_P \mathbb{Z}M(m), K(A, m+1)^{\Delta^\bullet})}_{= \mathrm{Map}_{\Sigma_m\text{-Mod}}(\mathrm{Indec}_P \mathbb{Z}M(m), K(A, m+1))}$$

We then use the identity

$$N_* \mathrm{Map}_{\Sigma_m\text{-Mod}}(-,-) \cong \mathrm{Hom}_{dg\, \Sigma_m\text{-Mod}}(N_*(-), N_*(-))$$

and $N_*(K(A, n+1)) = A[n+1]$

to obtain

$$\pi_* \operatorname{BiMod}_p^{\leq m}(M, K(A, n+1)) \simeq H_*(\operatorname{Hom}_{dg\Sigma_m\text{-}Mod}(\operatorname{Indec} \mathbb{Z}M, A[n+1]))$$

$$\overset{\text{degree in } M}{H^{n+1-*}}(\operatorname{Hom}_{\Sigma_m}(\operatorname{Indec} \mathbb{Z}M_*, A))$$

and to get the result of the observation. //

Thm

We assume that for each $m \geq 1$, we have a dimension bound $d(m)$, such that

$H^*_{\Sigma_m}(\operatorname{Indec} \mathbb{Z}M(m), A(m)) = 0$ for $* > d(m)$,

for any choice of coefficient system $A(m) \in \Sigma_m\text{-}Mod$.

We get in this case that the space

$\operatorname{BiMod}_p^{\leq m}(M, Q)$ is nilpotent (componentwise)

We moreover have:

$$\operatorname{BiMod}_p^{\leq m}(M, Q)^{\mathbb{Q}}_{\phi} \simeq \operatorname{BiMod}_p^{\leq m}(M, Q_{\mathbb{Q}})_{\hat{\phi}}$$

for any choise of base point $M^{\leq m} \xrightarrow{\phi} Q^{\leq m} \longrightarrow Q^{\leq m}_{\mathbb{Q}}$

February 21, 2022

<u>Proof:</u>

We argue by induction on the Postnikov decomposition of the operad Q.

Recall that $cQ^{\leq m-1}(m)$ is not necessarily connected. Nevertheless, for $\phi : M^{\leq m} \to Q^{\leq m}$, we have $M^{\leq m} \xrightarrow{\phi} Q^{\leq m} \to P_{-1}(Q^{\leq m}) \hookrightarrow cQ^{\leq m-1}$ and any element in $\mathrm{BiMod}_P^{\leq m}(M, cQ^{\leq m-1})_\phi$ still belong to $\mathrm{BiMod}_P^{\leq m}(M, P_{-1}(Q^{\leq m}))$

$$\mathrm{BiMod}_P^{\leq m}(M, cQ^{\leq m-1}).$$

We set $R = P_n(Q^{\leq m})$, $S = P_{n-1}(Q^{\leq m})$, so that we have a homotopy pullback:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi"'] & L(A, n+1) \arrow[d] \\ S \arrow[r, "\theta"] & K(A, n+1) \end{tikzcd}$$

where $A(m) = \pi_n(\mathrm{hofib}\ Q^{\leq m}(m) \to cQ^{\leq m-1}(m))$

We then have:

$$\begin{tikzcd} R_{\mathbb{Q}} \arrow[r] \arrow[d, "\eta_{\mathbb{Q}}"'] & L(A_{\mathbb{Q}}, \mathrm{mol}) \arrow[d] \\ S_{\mathbb{Q}} \arrow[r] & K(A_{\mathbb{Q}}, \mathrm{mol}) \end{tikzcd}$$

when we pass to the rationalization

We assume by induction that $\mathrm{BiMod}_{\rho}(M, S)_{\varphi}$

is nilpotent, for each $\varphi: M \to S$,

and that we have:

$$\mathrm{BiMod}_{\rho}^{\leq m}(M, S)_{\varphi}^{\mathbb{Q}} \xrightarrow{\sim} \mathrm{BiMod}_{\rho}^{\leq m}(M, S_{\mathbb{Q}})_{\hat{\varphi}}.$$

where $\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[rr, bend right, "\hat{\varphi}"'] & S \arrow[r] & S_{\mathbb{Q}} \end{tikzcd}$

We have a ho-Cartesian square:

$$\begin{tikzcd}
\mathrm{BiMod}_p^{\leq m}(M, R) \arrow[r] \arrow[d, "\eta_*"'] & \mathrm{BiMod}_p^{\leq m}(M, L(A, m+1)) \arrow[d] \\
\mathrm{BiMod}_p^{\leq m}(M, S) \arrow[r, "\theta_*"] & \mathrm{BiMod}_p^{\leq m}(M, K(A, m+1))
\end{tikzcd}$$

and an abelian group action

$$\mathrm{BiMod}_p^{\leq m}(M, K(A,m)) \circlearrowright \mathrm{BiMod}_p^{\leq m}(M, L(A, m+1))$$

so that the map

$$\mathrm{BiMod}_p^{\leq m}(M, L(A, m+1)) \to \mathrm{BiMod}_p^{\leq m}(M, K(A, m+1))$$

actually forms a principal $\mathrm{BiMod}_p^{\leq m}(M, K(A,m))$ fibration (in fact a morphism of simplicial abel. group)

We have a similar observation for the mapping spaces with values in the rationalization

. We fix $\psi \in \mathrm{BiMod}_P^{\leq m}(M, S)$

and $\phi \in \mathrm{BiMod}_P^{\leq m}(M, R)$ such that $p \circ \phi = \psi$.

We set:

$B = \mathrm{BiMod}_P^{\leq m}(M, S)_\psi$ with $\psi \in B$ as a base point

$$E = \mathrm{BiMod}_P^{\leq m}(M, R) \times_{\mathrm{BiMod}_P^{\leq m}(M, S)} \mathrm{BiMod}_P^{\leq m}(M, S)_\psi$$

$\Longleftrightarrow \quad E = p_*^{-1}\left(\mathrm{BiMod}_P^{\leq m}(M, S)_\psi\right) \subset \mathrm{BiMod}_P^{\leq m}(M, R)$

We take $\phi \in E$ as a base point for E

We also set:

$K = \mathrm{BiMod}_P^{\leq m}(M, K(A, n+1))_0$ for the connected

component of the zero map $0 : M \to K(A, n+1)$

and

$$L = \mathrm{BiMod}_P^{\leq m}(M, L(A, n+1)) \times_{\mathrm{BiMod}_P^{\leq m}(M, K(A, n+1))} \mathrm{BiMod}_P^{\leq m}(M, K(A, n+1))_0$$

We accordingly have:

$$\begin{tikzcd} L \arrow[r] \arrow[d, two heads] & \mathrm{BiMod}_p^{\leq m}(M, L(A,m+1)) \arrow[d, two heads] \\ \mathrm{BiMod}_p^{\leq m}(M, K(A,m+1))_0 \arrow[r, hook] & \mathrm{BiMod}_p^{\leq m}(M, K(A,m+1)) \end{tikzcd}$$

We actually have

$$L(A,m+1) \sim * \Rightarrow \mathrm{BiMod}_p^{\leq m}(M, L(A,m+1)) \sim *$$
$$\Rightarrow L = \mathrm{BiMod}_p^{\leq m}(M, L(A,m+1)) \sim *.$$

Observation:

Let $G = \mathrm{BiMod}_p^{\leq m}(M, K(A,m))$

We consider the action $G \circlearrowright L$ yielded by the translation action of the abelian group $K(A,m)$ on $L(A,m+1)$. The map $L \to K = \mathrm{BiMod}_p^{\leq m}(M, K(A,m+1))_0$ forms a G-minimal fibration and we have :

$$\begin{tikzcd}
 & & G \arrow[loop above] \\
\mathrm{BiMod}_P^{\le m}(M,R)_\varphi \arrow[r, hook] \arrow[d] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & L \sim * \arrow[d] \\
\mathrm{BiMod}_P^{\le m}(M,S)_\psi \arrow[r, "="] & B \arrow[r, "k"'] & K
\end{tikzcd}$$

$K \subseteq \mathrm{BiMod}_P^{\le m}(M, K(A, m+1))$ //

Cor: We have a homotopy exact sequence

$$\cdots \to \pi_1(B,\psi) \xrightarrow{k_*} \pi_1(K) \xrightarrow{\partial} \pi_0(E,\varphi) \to \pi_0(B,\psi) \to \cdots \to *$$

$$\pi_1(K) \cong \pi_0 G, \qquad \pi_0(B,\psi) \cong *$$

Hence, the group $\pi_0 G$ acts transitively on $\pi_0(E,\varphi)$ and we have $\partial[g] = [\varphi]$ for $[g] \in \pi_0 G$

$$\iff [g] = \mathrm{im}(\pi_1(B,\psi) \to \pi_0 G)$$

We also get $\pi_0(E,\varphi) \cong \pi_0 G / k_* \pi_1(B,\psi)$.

[Reminder from Bousfield-Kan, §IX.4:

If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration sequence of based spaces, then we have a homotopy exact sequence:

$$\cdots \longrightarrow \pi_1 E \xrightarrow{p_*} \pi_1 B \xrightarrow{\partial} \pi_0 F \xrightarrow{i_*} \pi_0 E \xrightarrow{p_*} \pi_0 B$$

$$\circlearrowleft$$

$$\pi_1 B$$

such that:

0) $\partial([\alpha]\cdot[\beta]) = [\alpha]\cdot\partial([\beta])$

1) $\underset{\pi_0 E}{p_* [x]} = * \iff [x] = i_* \underset{\pi_0 F}{[y]}$

2) $i_* \underset{\pi_0 F}{[x]} = i_* \underset{\pi_0 F}{[y]} \iff \exists [\alpha] \in \pi_1 B$ st $\underset{\pi_1 B}{[\alpha]} \cdot \underset{\pi_0 F}{[x]} = \underset{\pi_0 F}{[y]}$

In particular $i_*[x] = *$

$\iff \exists [\alpha] \in \pi_1 B$ st $\partial[\alpha] = [x]$.

3) $\partial \underset{\pi_1 B}{[\alpha]} = * \iff \exists \underset{\pi_1 E}{[\beta]} \in \pi_1 E$ st $p_*[\beta] = [\alpha]$.]

Obsv. We moreover have

$$E \simeq \pi_0 E \times \mathrm{BiMod}_{\rho}^{\subseteq m}(M, R)_{\varphi}$$

(as E is the total space of a principal fibration over a connected base space) and $\mathrm{BiMod}_{\mathcal{P}}^{\mathcal{E}_m}(M,R)_\varphi \to \mathrm{BiMod}_{\mathcal{P}}^{\mathcal{E}_m}(M,S)_\varphi$

forms a principal H-fibration, for the simplicial group st:

$$H = \{ g \in G \ / \ \underset{\in \pi_0 G}{[g]} \cdot \underset{\in \pi_0 E}{[\varphi]} = \overset{\text{in } \pi_0 E}{[\varphi]} \}$$

February 26 2022

We let $\hat{B}, \hat{E}, \hat{K}, \hat{L}, \hat{G}, \hat{H}$ denote the counterpart of the spaces B, E, K, L, G, H that we obtain by applying our constructions to the Cartesian square of the rationalized operads:

$$\begin{tikzcd}
R_{\mathbb{Q}} \arrow[r] \arrow[d] & L(A_{\mathbb{Q}}, n+1) \arrow[d] \\
S_{\mathbb{Q}} \arrow[r] & K(A_{\mathbb{Q}}, n+1)
\end{tikzcd}$$

We therefore have:

$$\hat{B} = \mathrm{BiMod}_{P}^{\leq m}(M, S_{\mathbb{Q}})_{\hat{\varphi}}$$

$$\hat{E} = \mathrm{BiMod}_{P}^{\leq m}(M, R_{\mathbb{Q}}) \times_{\mathrm{BiMod}_{P}^{\leq m}(M, S_{\mathbb{Q}})} \mathrm{BiMod}_{P}^{\leq m}(M, S_{\mathbb{Q}})_{\hat{\varphi}}$$

$$\hat{K} = \mathrm{BiMod}_{P}^{\leq m}(M, K(A_{\mathbb{Q}}, n+1))_{0}$$

$$\hat{L} = \mathrm{BiMod}_{P}^{\leq m}(M, L(A_{\mathbb{Q}}, n+1)) \times_{\mathrm{BiMod}_{P}^{\leq m}(M, K(A_{\mathbb{Q}}, n+1))} \mathrm{BiMod}_{P}^{\leq m}(M, K(A_{\mathbb{Q}}, n+1))_{0}$$

$$\hat{G}' = \mathrm{BiMod}_p^{\leq m}(M, K|A_{\mathbb{Q}}, m))$$

$$\hat{H} = \{ g \in G' \,/\, [g].[\hat{\varphi}] = [\hat{\varphi}] \}$$

where we still use the notation $\hat{\varphi}$ for the composite

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[rr, bend right, "\hat{\varphi}"'] & S \arrow[r] & S_{\mathbb{Q}} \end{tikzcd}$$

Proposition:

We have $G \sim \prod_i K(H^{m-i}_{\Sigma_m}(\mathrm{Indec}_p \mathbb{Z}M(m), A(m)), i)$

by the Dold-Thom theorem and our computation of $\pi_* \mathrm{BiMod}_p^{\leq m}(M, K|A, m)$.

We similarly have

$\hat{G} \sim \prod_i K(H^{m-i}_{\Sigma_m}(\mathrm{Indec}_p \mathbb{Z}M(m), A_{\mathbb{Q}}(m)), i)$ //

Corollary:

We have $G_{\mathbb{Q}} \xrightarrow{\sim} \hat{G}$ as we have

$$H^*_{\Sigma_m}(\mathrm{Indec}_p \mathbb{Z}M(m), A(m))_{\mathbb{Q}} = H^*_{\Sigma_m}(\mathrm{Indec}_p \mathbb{Z}M(m), A(m)_{\mathbb{Q}})$$

by flatness of $\mathbb{Q}/\mathbb{Z}$. //

Recall that we assume by induction:

$$\underbrace{\mathrm{BiMod}_{\rho}^{\leq m}(M,S)_{\mathbb{Q}}}_{B_{\mathbb{Q}}} \xrightarrow{\sim} \underbrace{\mathrm{BiMod}_{\rho}^{\leq m}(M,S_{\mathbb{Q}})}_{\hat{B}}$$

We have by functoriality of the homotopy exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1 B \arrow[r] \arrow[d] & \pi_1 K \simeq \pi_0 G \arrow[r] \arrow[d] & \pi_0 E \arrow[r] \arrow[d] & \pi_0 B \simeq * \arrow[d] \\
\cdots \arrow[r] & \pi_1 \hat{B} \arrow[r] & \pi_1 \hat{K} \simeq \pi_0 \hat{G} \arrow[r] & \pi_0 \hat{E} \arrow[r] & \pi_0 \hat{B} \simeq *
\end{tikzcd}$$

We use the identity $H \overset{def}{=} G \times_{\pi_0 G} \mathrm{Ker}(\pi_0 G \to \pi_0 E)$

$$\simeq G \times_{\pi_0 G} \pi_0 G / \pi_1 B$$

and the parallel relation $\hat{H} \simeq \hat{G} \times_{\pi_0 \hat{G}} \pi_0 \hat{G} / \pi_1 \hat{B}$

to obtain $H_{\mathbb{Q}} \xrightarrow{\sim} \hat{H}$.

We now have a diagram of principal fibrations:

$$\begin{tikzcd}
\mathrm{BiMod}_{\rho}^{\leq m}(M,R)_{\phi} \arrow[loop above, "H"] \arrow[r] \arrow[d] & \mathrm{BiMod}_{\rho}^{\leq m}(M,R_{\mathbb{Q}})_{\hat{\phi}} \arrow[loop above, "\hat{H}"] \arrow[d] \\
\mathrm{BiMod}_{\rho}^{\leq m}(M,S)_{\psi} \arrow[r] & \mathrm{BiMod}_{\rho}^{\leq m}(M,S_{\mathbb{Q}})_{\hat{\psi}}
\end{tikzcd}$$

We use that $B = \mathrm{BiMod}_{\rho}^{\leq m}(M,S)_{\psi}$ is nilpotent and $B_{\mathbb{Q}} \xrightarrow{\sim} \hat{B} = \mathrm{BiMod}_{\rho}^{\leq m}(M,S_{\mathbb{Q}})_{\hat{\psi}}$.

The idea is to rely on the decomposition $H \sim \pi_0 H \times H_0$, and:

$$\begin{tikzcd}
\mathrm{BiMod}_{\rho}^{\leq m}(M,R)_{\phi} \arrow[d] \arrow[dd, bend right] & & \\
\mathrm{BiMod}_{\rho}^{\leq m}(M,R)_{\phi}^{\circ} \arrow[rr] \arrow[d] & & L(\pi_0 H, 1) \arrow[d] \\
\mathrm{BiMod}_{\rho}^{\leq m}(M,S)_{\psi} \arrow[r] & \overline{W} H \arrow[r] & K(\pi_0 H, 1)
\end{tikzcd}$$

so that $\mathrm{BiMod}_{\rho}^{\leq m}(M,R)_{\phi} \to \mathrm{BiMod}_{\rho}^{\leq m}(M,R)_{\phi}^{\circ}$ forms an H_0-principal fibration, whereas

$\mathrm{BiMod}_{\rho}^{\leq m}(M,R|_{\phi}^{\circ} \longrightarrow \mathrm{BiMod}_{\rho}^{\leq m}(M,S)_{\psi}$ forms

a $\pi_0 H$-principal fibration

- We rely on the following

general result:

We assume that $p: X \to Y$ is a Γ-principal fibration

where Γ is a discrete abelian group of finite type.

We then have $\pi_i X \xrightarrow{\simeq} \pi_i Y \quad \forall i \geq 2$

and $0 \to \pi_1 X \to \pi_1 Y \to \Gamma \to 0 \Rightarrow \pi_1 X = \ker(\pi_1 Y \to \Gamma)$

We therefore have $\pi_1 Y$ nilpotent $\Rightarrow \pi_1 X$ nilpotent

and $\pi_1 Y \circlearrowright \pi_i Y$ nilpotent for $i \geq 2$

$$\begin{array}{ccc} \pi_1 Y & \circlearrowright & \pi_i Y \\ \cup & & \text{IS} \\ \pi_1 X & \circlearrowright & \pi_i X \end{array}$$

$\Rightarrow \pi_1 X \circlearrowright \pi_i X$ nilpotent

Hence Y nilpotent $\Rightarrow X$ nilpotent

We consider the tower:

$$\begin{tikzcd}
\pi_1 X \arrow[r, hook] \arrow[d, two heads] & \pi_1 Y / \Gamma_{\ell+1} \pi_1 Y \simeq \pi_1 Y \arrow[d, two heads] \\
\pi_1 X_{\langle \ell-1 \rangle} \arrow[r, hook] \arrow[d, two heads] & \pi_1 Y / \Gamma_\ell \pi_1 Y \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
\pi_1 X_{\langle 2 \rangle} \arrow[r, hook] \arrow[d, two heads] & \pi_1 Y / \Gamma_3 \pi_1 Y \arrow[d, two heads] \\
\pi_1 X_{\langle 1 \rangle} \arrow[r, hook] & \pi_1 Y / \Gamma_2 \pi_1 Y \arrow[r, dashed, two heads] & \Gamma
\end{tikzcd}$$

since Γ is abelian

where we set:

$\pi_1 X_{\langle i \rangle} = \text{image}\,(\pi_1 X \to \pi_1 Y / \Gamma_{i+1} \pi_1 Y)$

We have $\pi_1 X_{\langle i \rangle} = \ker\,(\pi_1 Y / \Gamma_{i+1} \pi_1 Y \to \Gamma)$

We set $F_{i+1} \pi_1 X = \ker(\pi_1 X \to \pi_1 X_{\langle i \rangle})$

so that $\pi_1 X_{\langle i \rangle} = \pi_1 X / F_{i+1} \pi_1 X$

We have $F_{i+1} \pi_1 X = \pi_1 X \cap \Gamma_{i+1} \pi_1 Y$, so that

$[F_i \pi_1 X, F_j \pi_1 X] \subset F_{i+j} \pi_1 X \quad \forall i, j \geq 1.$

Let:

$Y_{<i>} = K(\pi_1 Y / \Gamma_{i+1} \pi_1 Y, 1)$

$X_{<i>} = K(\pi_1 X / \Gamma_{i+1} \pi_1 X, 1)$.

We have $\left.\begin{array}{l} Y_{<e>} = P_1 Y \\ X_{<e>} = P_1 X \end{array}\right\}$ — Postnikov sections.

We have $Y_{<i>\,\mathbb{Q}} = K(\pi_1(Y_{<i>})_{\mathbb{Q}}, 1)$

$X_{<i>\,\mathbb{Q}} = K(\underbrace{\pi_1(X_{<i>})_{\mathbb{Q}}}_{\text{Malcev completion.}}, 1)$

and $1 \to \pi_1(X_{<i>})_{\mathbb{Q}} \to \pi_1(Y_{<i>})_{\mathbb{Q}} \to \Gamma_{\mathbb{Q}} \to 1$

by general results about the Malcev completion of nilpotent groups.

Hence, we have:

$$\begin{array}{ccc} X_{<i>\,\mathbb{Q}} & \longrightarrow & L(\Gamma_{\mathbb{Q}}, 1) \\ \downarrow & & \downarrow \\ Y_{<i>\,\mathbb{Q}} & \longrightarrow & K(\Gamma_{\mathbb{Q}}, 1) \end{array} \quad \text{and} \quad \begin{array}{ccc} X_{\mathbb{Q}} & \longrightarrow & L(\Gamma_{\mathbb{Q}}, 1) \\ \downarrow & & \downarrow \\ Y_{\mathbb{Q}} & \longrightarrow & K(\Gamma_{\mathbb{Q}}, 1) \end{array}$$

$\forall i \geq 1$

We apply these results to the $\pi_0 H$-principal fibration $\mathrm{BiMod}_\rho^{\leq m}(M,R)^\circ_\phi \to \mathrm{BiMod}_\rho^{\leq m}(M,S)_\varphi$.

We get that $\mathrm{BiMod}_\rho^{\leq m}(M,R)^\circ_\phi$ is nilpotent, and we have:

$$\begin{tikzcd}
\mathrm{BiMod}_\rho^{\leq m}(M,R)^{\circ\mathbb{Q}}_{\phi} \arrow[r] \arrow[d] & L(\pi_0 H_{\mathbb{Q}},1) \arrow[d] \\
\mathrm{BiMod}_\rho^{\leq m}(M,S)^{\mathbb{Q}}_{\varphi} \arrow[r] & K(\pi_0 H_{\mathbb{Q}},1)
\end{tikzcd}$$

//

By the general results of Bousfield-Kan about principal fibrations with a connected fiber, we also get that $\mathrm{BiMod}_\rho^{\leq m}(M,R)_\phi$ is still nilpotent, and we have:

$$\begin{tikzcd}
\mathrm{BiMod}_\rho^{\leq m}(M,R)^{\mathbb{Q}}_{\phi} \arrow[r] \arrow[d] & W(H_{0\mathbb{Q}}) \arrow[d] \\
\mathrm{BiMod}_\rho^{\leq m}(M,S)^{\circ\mathbb{Q}}_{\phi} \arrow[r] & \overline{W}(H_{0\mathbb{Q}})
\end{tikzcd}$$

//

Corollary:

We have

$$\begin{array}{ccc} \mathrm{BiMod}_P^{\leq m}(M,R)^Q_\phi & \longrightarrow & W(H_Q) \\ \downarrow & & \downarrow \\ \mathrm{BiMod}_P^{\leq m}(M,S)^Q_\psi & \longrightarrow & \bar{W}(H_Q) \end{array}$$

and through $\left| \begin{array}{l} H_Q \xrightarrow{\sim} \widehat{H} \\ \mathrm{BiMod}_P^{\leq m}(M,S)^Q_\psi \xrightarrow{\sim} \mathrm{BiMod}_P^{\leq m}(M,S_Q)_{\hat{\psi}} \end{array} \right.$

we conclude that

$$\mathrm{BiMod}_P^{\leq m}(M,R)^Q_\phi \xrightarrow{\sim} \mathrm{BiMod}_P^{\leq m}(M,R_Q)_{\hat{\phi}} \quad //$$

This result finishes the induction step of the procedure.

We use the relation $H^*_{\Sigma_m}(\mathrm{Indec}\, ZM(m), A(m)) = 0$ for $* > d(m)$

to pass to the limit in the tower.

$$\mathrm{BiMod}_P^{\leq m}(M, Q) \to \cdots \to \mathrm{BiMod}_P^{\leq m}(M, P_m Q)$$
$$\to \mathrm{BiMod}_P^{\leq m}(M, P_{m-1} Q)$$
$$\cdots$$
$$\to \mathrm{BiMod}_P^{\leq m}(M, P_{-1} Q)$$
$$= \mathrm{BiMod}_P^{\leq m-1}(M, Q)$$

in order to conclude that $\mathrm{BiMod}_P^{\leq m}(M, Q)_\varphi$ is nilpotent if $\mathrm{BiMod}_P^{\leq m-1}(M, Q)_\varphi$ is so and that we also have

$$\mathrm{BiMod}_P^{\leq m}(M, Q)_\varphi^{\mathbb{Q}} \xrightarrow{\sim} \mathrm{BiMod}_P^{\leq m}(M, Q_{\mathbb{Q}})_{\hat{\varphi}}$$

if we have the same result at arity level $m-1$.

. If $c(m) = \mathrm{conn}(Q(m) \to c\,Q(m))$ satisfies $c(m) - d(m) \to \infty$, then we can also pass to the limit in the tower

$$\mathrm{BiMod}_P(M, Q) \to \cdots \to \mathrm{BiMod}_P^{\leq m}(M, Q)$$
$$\to \mathrm{BiMod}_P^{\leq m-1}(M, Q)$$
$$\to \cdots \to \mathrm{BiMod}_P^{\leq 1}(M, Q) = *.$$

We get that $\mathrm{BiMod}_P(M,Q)_\varphi$ is nillpotent,

for each $\varphi: M \to Q$, and we have:

$$\mathrm{BiMod}_P(M,Q)^{\mathbb{Q}}_\varphi \xrightarrow{\sim} \mathrm{BiMod}_P(M, Q_{\mathbb{Q}})_\varphi. //$$

February 27, 2022

Reminder: <u>the rationalization map is finite-to-one</u>
(notes on December 11, 2018)

THM: We assume that X is a space of finite type
(the homotopy groups $\pi = \pi_n(X, *)$ are abelian groups of f.t.)
and A is a finite complex.
Then the map $[A, X] \to [A, X_{\mathbb{Q}}]$
is finite to one.

Proof: We use an induction on the Postnikov decomposition of the space X.
We therefore consider a Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & L(\pi, n+1) \sim * \\ \downarrow & & \downarrow \\ Y & \longrightarrow & K(\pi, n+1) \end{array}$$

and we assume that the statement holds for the space Y

We have the homotopy exact sequence:

$$\begin{tikzcd} \to \pi_0 \mathrm{Map}(A, K(\pi, n)) \arrow[r] & \pi_0 \mathrm{Map}(A, X) \arrow[r] & \pi_0 \mathrm{Map}(A, Y) \\ & \circlearrowright \\ & \pi_0 \mathrm{Map}(A, K(\pi, n)) \end{tikzcd}$$

We are given $\hat{f} : A \to X_{\mathbb{Q}}$

We count the number of maps $f : A \to X$ that lift $\hat{f}$ and have the same image in $\pi_0 \mathrm{Map}(A, Y)$.

We have
$$\begin{array}{ccc} \pi_0 \mathrm{Map}(A, X) & \longrightarrow & \pi_0 \mathrm{Map}(A, Y) \\ \cup & & \cup \\ [f_0], [f_1] & \longmapsto & [g_0] = [g_1] \end{array}$$

$\Leftrightarrow$ $\exists [\sigma] \in \pi_0 \mathrm{Map}(A, K(\pi, n)) = H^n(A, \pi)$

such that $[f_1] = [\sigma] \cdot [f_0]$

We have $[\hat{f}_1] = [\hat{f}_0] = [\hat{f}]$ in $\pi_0 \mathrm{Map}(A, X_{\mathbb{Q}})$

$\Leftrightarrow$ $[\hat{\sigma}] = 0$ in $\pi_0 \mathrm{Map}(A, K(\pi_{\mathbb{Q}}, n)) = H^n(A, \pi_{\mathbb{Q}})$
$= H^n(A, \pi) \otimes_{\mathbb{Z}} \mathbb{Q}$

$\Leftrightarrow$ $[\sigma] \in \mathrm{Tors}\, H^n(A, \pi)$.

We just use that $\# \mathrm{Tors}\, H^n(A, \pi) < \infty$ by our finiteness

Extension for bimodules:

For bimodules maps, we similarly get that $\pi_0 \mathrm{BiMod}_P(M, Q) \to \pi_0 \mathrm{BiMod}_P(M, Q_{\mathbb{Q}})$ is finite to one when

1) M is so that $\mathrm{Indec}_P \mathcal{Z}M$ is a finite complex, with a dimension bound $d(m)$ st.
$H^*_{\Sigma_m}(\mathrm{Indec}\,\mathcal{Z}M(m), A(m)) = 0$ for $* > d(m)$.

2) Q consists of spaces of finite type, and $c(m) = \mathrm{conn}(Q(m) \to c\,Q(m))$ is so that $c(m) - d(m) \to \infty$.