FUNCTOR HOMOLOGY AND OPERADIC HOMOLOGY

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Abstract. The purpose of these notes is to define an equivalence between the natural homology theories associated to operads and the homology of functors over certain categories of operators (PROPs) related to operads.

Introduction

The aim of these notes is to prove that the natural homology theory associated to an operad is equivalent the homology of a category of functors over a certain category of operators associated to our operad.

Recall that an operad P in a symmetric monoidal category $\mathcal{C}$ basically consists of a sequence of objects $\mathsf{P}(n) \in \mathcal{C}$, $n \in \mathbb{N}$, of which elements $p \in \mathsf{P}(n)$ (whenever the notion of an element makes sense) intuitively represent operations on n inputs and with 1 output:

$$A^{\otimes n} = \underbrace{A \otimes \cdots \otimes A}_{n} \xrightarrow{p} A,$$

for any $n \in \mathbb{N}$. In short, an operad is defined axiomatically as such a sequence of objects $\mathsf{P} = \{\mathsf{P}(n), n \in \mathbb{N}\}$ equipped with an action of the symmetric group Σ_n on the term $\mathsf{P}(n)$, for each $n \in \mathbb{N}$, together with composition products which are shaped on composition schemes associate with such operations. The notion of an operad is mostly used to define a category of algebras, which basically consists of objects $A \in \mathcal{C}$ on which the operations of our operad $p \in \mathsf{P}(n)$ act. We use the term of a P-algebra, and the notation ${}_{\mathsf{P}}\mathcal{C}$, to refer to this category of algebras associated to any given operad P. Recall simply that the usual category of associative algebras in a category of modules over a ring $\Bbbk$, the category of (associative and) commutative algebras, and the category of Lie algebras, are associated to operads, which we respectively denote by $\mathsf{P} = \mathsf{As}, \mathsf{Com}, \mathsf{Lie}$. In the Lie algebra case, there is just an issue when 2 is not invertible in the ground ring and we want to encode the vanishing relation $[x, x] = 0$ in a structure defined by an operad (see [2]).

In good cases, we get that the category of operads in a symmetric monoidal category equipped with a model structure $\mathcal{C}$ inherits a model structure itself, and the category of algebras ${}_{\mathsf{P}}\mathcal{C}$ over an operad P inherits a natural model structure as well, at least when the operad P is good enough. For instance, in the case where the base category $\mathcal{C}$ is a category of differential graded modules $\mathcal{C} = dg\,\mathcal{M}od$ over a ground ring $\Bbbk$, we get a model structure on the category of associative algebras without any further assumption on $\Bbbk$, while the commutative algebras and the Lie algebras form a model category only when we have $\mathbb{Q} \subset \Bbbk$. The model structure on

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the category of algebras ${}_{P}\mathcal{C}$ is generally well defined when the operad P is cofibrant (as an operad), and as we soon explain, this general statement will actually be sufficient for our purpose.

The model structures on the category of operads and on the categories of algebras over operads are defined by a general adjunction process. The homotopy category of these model categories represents a localization with respect to a class of weak-equivalences which is essentially created in the base category. In the case of operads, we formally define a weak-equivalence as a morphism of operads $\phi : P \xrightarrow{\sim} Q$ which forms a weak-equivalence in the base category term-wise $\phi : P(n) \xrightarrow{\sim} Q(n)$, $n \in \mathbb{N}$. In the case of algebras over an operad P, we similarly define a weak-equivalence as morphism of P-algebras $\phi : A \xrightarrow{\sim} B$ which forms a weak-equivalence in the base category (when we forget about the action of the operad). We use these model structures to define the natural homology theory associated to an operad P.

Let us briefly recall the definition of this homology in the trivial coefficient case, and when the base category $\mathcal{C}$ is the category of dg-modules $\mathcal{C} = dg\,\mathcal{M}od$, for a fixed ground ring $\Bbbk$. We then assume that our operad P is equipped with an augmentation $\epsilon : P \to I$, where I refers to the unit operad, which has ${}_{I}\mathcal{C} = \mathcal{C}$ as associated category of algebras. We have a natural functor of indecomposables on the category of P-algebras $\epsilon_! : {}_{P}\mathcal{C} \to \mathcal{C}$ which is defined as the left adjoint of the obvious restriction functor $\epsilon^* : {}_{I}\mathcal{C} \to {}_{P}\mathcal{C}$ associated with the augmentation $\epsilon : P \to I$, and we can easily check that the pair $\epsilon_! : {}_{P}\mathcal{C} \rightleftarrows \mathcal{C} : \epsilon^*$ defines a Quillen adjunction whenever we have a well-defined model structure on the category of P-algebras. We define the homology of a P-algebra A by the homology of the image of this P-algebra under the derived functor of the functor of indecomposables:

$$\mathtt{H}^{P}_{*}(A) = \mathtt{H}_{*}(\mathtt{L}\,\epsilon_! A).$$

The homology with coefficients $\mathtt{H}^{P}_{*}(A, M)$ is also defined by a model category construction for any pair (A, M) consisting of a P-algebra A and of a corepresentation M on this P-algebra A. We will review this construction (and the general definition of a corepresentation for algebras over operads) later on. We also refer to [1] for the introduction of the notion of a corepresentation.

The theory of Koszul operads implies that this operadic homology theory, which we define by methods of homotopical algebra reduces to the classical Hochschild homology when $P = As$ is the operad of associative algebras (for any choice of a ground ring), to the Harrison homology when $P = Com$ is the operad of commutative algebras (and the ground ring satisfies $\mathbb{Q} \subset \Bbbk$), to the Chevalley-Eilenberg homology when $P = Lie$ is the operad of Lie algebras (and we still assume $\mathbb{Q} \subset \Bbbk$).

In the case where the category of P-algebras is not equipped with a model structure, we may still define a good homology theory for P-algebras by picking a cofibrant resolution $R \xrightarrow{\sim} P$ of our operad P, and forming our homology in the model category of algebras associated with this operad R. We use the notation $\mathtt{H}\Gamma^{P}_{*}(A, M)$ for this homology theory, defined on pairs (A, M) where A is a P-algebra and M is a corepresentation of A, and such that:

$$\mathtt{H}\Gamma^{P}_{*}(A, M) = \mathtt{H}^{R}_{*}(A, M),$$

where we consider the homology over a cofibrant resolution $R \xrightarrow{\sim} P$ of our operad P. We also refer to this homology theory as the Γ-homology of P-algebras. We just need to observe that the result of this construction does not depend on the choice of the cofibrant resolution $R \xrightarrow{\sim} P$.

The Γ-homology $\mathtt{H\Gamma}^{P}_{*}(A,M)$ reduces to the ordinary homology $\mathtt{H}^{P}_{*}(A,M)$ if the category of P-algebras is equipped with a model structure, but we get a new homology theory otherwise (for instance, when $P = \mathit{Com}, \mathit{Lie}$ and $\mathbb{Q} \not\subset \Bbbk$). In the case of the commutative operad $P = \mathit{Com}$, we actually retrieve the Γ-homology theory of A. Robinson and S. Whitehouse (see [8, 10]), initially introduced to study obstruction problems in stable homotopy. We refer to E. Hoffbeck's work [4] for the definition of the Γ-homology theory in the general case of operads in dg-modules.

The main purpose of this article is to prove that the homology theory associated to an operad $\mathtt{H}^{P}_{*}(-)$, and more generally, the Γ-homology $\mathtt{H\Gamma}^{P}_{*}(-)$ is equivalent to a homology of functors over a category of operators which we associate to the operad P. Let us outline the correspondence in the trivial coefficient case, and yet, when the base category is the category of dg-modules $\mathcal{C} = dg\,\mathcal{M}od$.

The category of operators Γ_P, which we use in the trivial coefficient case, has the ordinals $\underline{n} = \{1 < \cdots < n\}$ as objects and its homomorphisms $f \in \Gamma_P(\underline{m}, \underline{n})$ model maps

$$A^{\otimes m} \xrightarrow{w^*} A^{\otimes m} \xrightarrow{\bigotimes_{i=1}^{n} p_i} A^{\otimes n} ,$$

where w^* is a tensor permutation, and we consider a tensor product of operations $p_i : A^{\otimes m_i} \to A$, $i = 1, \ldots, n$, arising from our operad P, and such that $m_1 + \cdots + m_n = m$. In the literature, this object Γ_P, which is actually an enriched symmetric monoidal category over the base category $\mathcal{C}$, is also referred to as the PROP associated to the operad P. We use that any algebra $A \in {}_P\mathcal{C}$ determines a (covariant) functor $\mathtt{R}_A : \Gamma_P \to \mathcal{C}$ which takes the value $\mathtt{R}_A(\underline{n}) = A^{\otimes n}$, on any object $\underline{n} \in \Gamma_P$. We consider, on the other hand, the (contravariant) functor $\mathtt{1} : \Gamma_P^{op} \to \mathcal{C}$ such that $\mathtt{1}(\underline{1}) = \Bbbk$ and $\mathtt{1}(\underline{n}) = 0$ for $n \neq 1$. Then our result reads as follows:

Theorem A. *Let P be an operad in dg-modules such that each term $P(r)$, $r \in \mathbb{N}$, forms a cofibrant object in the category of dg-modules. We assume that P is equipped with an augmentation $\epsilon : P \to I$. We then have an identity:*

$$\mathtt{H\Gamma}^{P}_{*}(A) = \mathtt{Tor}^{\Gamma_P}_{*}(\mathtt{1}, \mathtt{R}_A),$$

for any P-algebra in dg-modules A which is cofibrant as an object of the category of dg-modules.

The $\mathtt{Tor}$-functor of this theorem is the homology of the derived functor of the coend over the (enriched) category Γ_P:

$$\mathtt{F} \otimes_{\Gamma_P} \mathtt{G} = \int^{\underline{n} \in \Gamma_P} \mathtt{F}(\underline{n}) \otimes \mathtt{G}(\underline{n}),$$

which we may also regard as a generalized tensor product construction (as hinted by our notation).

In the non-trivial coefficient case, we deal with a pointed version of our category of operators Γ_P^+ of which objects are the pointed ordinals $\underline{n}_+ = \{1 < \cdots < n\} \amalg \{0\}$, for $n \in \mathbb{N}$. To any pair (A, M), where A is a P-algebra and M is a corepresentation of A, we associate a (covariant) functor $\mathtt{R}_{A,M} : \Gamma_P^+ \to \mathcal{C}$ such that $\mathtt{R}_{A,M}(\underline{n}_+) = M \otimes A^{\otimes n}$, for each $\underline{n}_+ \in \Gamma_P^+$. We then get a result of the following form:

Theorem B. *Let P be an operad in dg-modules such that each term $P(r)$, $r \in \mathbb{N}$, forms a cofibrant object in the category of dg-modules. We have an identity:*

$$\mathtt{H\Gamma}^{P}_{*}(A,M) = \mathtt{Tor}^{\Gamma_P^+}_{*}(\omega_P, \mathtt{R}_{A,M}),$$

for any P-algebra in dg-modules A and any representation of this P-algebra M such that both A and M are cofibrant as dg-modules.

The object ω_P which occurs in this statement is a contravariant functor such that:

$$\omega_P \otimes_{\Gamma_P^+} \mathtt{R}_{A,M} = M \otimes_{\mathtt{U}_P(A)} \Omega_P(A),$$

where $\mathtt{U}_P(A)$ and $\Omega_P(A)$ are the operadic enveloping algebra and the Kähler form functors considered in the definition of the homology of algebras over an operad.

The results of these theorems generalize several statements of the literature. First, the category of operators Γ_{Com}^+ associated to the operad of commutative algebras $P = Com$ can be identified with the classical (opposite of the) Segal category of finite pointed sets Γ. In this case, we retrieve a theorem of T. Pirashvili and B. Richter [6] (see also [9] for a similar result), which precisely asserts that the E_∞-homology of a commutative algebra is equivalent to a $\mathtt{Tor}$-functor over the Segal category Γ. (The E_∞-homology is the homology theory associated to a cofibrant resolution of the operad of commutative algebras Com.) In the case of the associative operad $P = As$, the category of operators Γ_{As}^+ is identified with Loday's category of finite non-commutative pointed sets and we retrieve a theorem of T. Pirashvili and B. Richter asserting that the Hochschild homology is equivalent to a $\mathtt{Tor}$-functor over that category (see [7]).

0. Background

We give a brief overview of the conventions which we follow all along these notes. We mostly adopt the point of views and the language of the book [3], to which we refer for further details and explanations on the background of our constructions.

0.1. *On the category of dg-modules.* We fix a (commutative) ground ring $\Bbbk$, once and for all. We assume that all our modules are defined over this ring $\Bbbk$. We similarly take the tensor product of modules over $\Bbbk$ as a primitive tensor structure $\otimes = \otimes_\Bbbk$ from which we derive all our symmetric monoidal category constructions.

The category of differential graded modules which we consider in these notes (we also say dg-modules for short) $\mathcal{C} = dg\,\mathcal{M}od$ consist of modules C equipped with a decomposition $C = \bigoplus_{n\in\mathbb{Z}} C_n$ into lower graded homogeneous components C_n, $n \in \mathbb{Z}$, and with a differential, usually denoted by $\delta : C \to C$, which lowers degrees by one and satisfies the standard relation $\delta^2 = 0$.

We equip the category of dg-modules with its standard symmetric monoidal structure, where the tensor product is inherited from the category of modules over the ground ring, and the symmetry isomorphism satisfies the usual sign formula of differential graded algebra. We give a more comprehensive reminder on these concepts in §0.3.

We equip the category of dg-modules with the standard projective model structure, where the weak-equivalences are the morphisms which induce an iso in homology, and the fibrations are the morphisms which are surjective in each degree.

In what follows, we generally need minimal cofibration conditions in order to make our constructions work. We usually have to assume that the structured objects which we consider (operads, algebras) form, at least, cofibrant objects in the category of dg-modules. If the ground ring is a field, then every dg-module is cofibrant, and this condition is void. If this is not the case, then we will tacitely assume that the dg-module cofibration requirement is fulfilled: by convention, all objects which we consider in these notes consist of cofibrant objects in the base category of dg-modules. We only recall this convention in the statement of our main theorems.

0.2. *On operads and modules over operads.* We adopt the notation $\mathcal{O}$ for the category of operads in the base category (of dg-modules) $\mathcal{C}$, and the notation $\mathcal{M}$ for the category, underlying the category of operads, formed by the collections $M = \{M(n), n \in \mathbb{N}\}$, where $M(n) \in \mathcal{C}$ is an object of the base category equipped with an action of the symmetric group Σ_n, for any $n \in \mathbb{N}$. We also use the expression of a symmetric sequence to refer to the objects $M = \{M(n), n \in \mathbb{N}\}$, of this category $\mathcal{M}$.

We generally follow the conventions of the book [3] for our constructions on the category of operads and symmetric sequences (as already specified in the introduction of this section). We use in particular that an operad is identified with a monoid with respect a certain monoidal structure defined by a composition operation $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ on the category of symmetric sequences $\mathcal{M}$. We also adopt the notation I for the unit object of this monoidal structure on $\mathcal{M}$. In this approach, an operad formally consists of a symmetric sequence $P \in \mathcal{M}$ equipped with a unit morphism $\eta : I \to P$ and a product $\mu : P \circ P \to P$ that fulfill the standard unit and associativity relations in the category of symmetric sequences $\mathcal{M}$.

The symmetric sequence I, which we actually already considered in the introduction of these notes, defines the initial object of the category of operads. In our module setting, we basically have $I(1) = \mathbb{k}$ and $I(n) = 0$ for $n \neq 1$.

We also use the composition operation $\circ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ to define the notion of a right module (respectively, left module) over an operad P. We then consider symmetric sequences $M \in \mathcal{M}$ equipped with a morphism $\rho : M \circ P \to M$ (respectively, $\lambda : P \circ M \to M$) which fulfill basic unit and associativity relations with respect to the internal unit and composition structure of the operad P. We also say that this morphism $\rho : M \circ P \to M$ (respectively, $\lambda : P \circ M \to M$) defines a right (respectively, left) action of the operad P on M. We adopt the notation $\mathcal{M}_P$ (respectively, ${}_P\mathcal{M}$) for this category of a right P-modules (respectively, left P-modules). We have a similarly defined category of (P, Q)-bimodules, denoted by ${}_P\mathcal{M}_Q$, for any pair of operads $P, Q \in \mathcal{O}$, which consists of symmetric sequences equipped with a left P-action and a right Q-action that commute to each other.

We equip the category of operads (in dg-modules) $\mathcal{O}$ with the model structure such that the weak-equivalences $\phi : P \xrightarrow{\sim} Q$ are the morphisms which define a weak-equivalence term-wise $\phi : P(n) \xrightarrow{\sim} Q(n)$, $n \in \mathbb{N}$ (we also say arity-wise), in the base category (of dg-modules) $\mathcal{C}$, and similarly as regards the fibrations.

The category of symmetric sequences $\mathcal{M}$ and the category of right modules over an operad $\mathcal{M}_P$ are equipped with similarly defined model structures (we just assume that the operad P consists of cofibrant objects in the base category $P(n) \in \mathcal{C}$, $n \in \mathbb{N}$, in the case of the category of right P-modules $\mathcal{M}_P$). We also have a model structure on the category of left modules ${}_P\mathcal{M}$ and on the category of bimodules ${}_P\mathcal{M}_Q$ at least when P is cofibrant as an operad. These model categories may actually be regarded as generalizations of the model category of P-algebras which we consider in a subsequent paragraph.

0.3. *On symmetric monoidal category structures and algebras over operads.* The tensor product $C \otimes D$ of dg-modules $C, D \in dg\,\mathcal{M}od$ is equipped with the grading such that $(C\otimes D)_n = \bigoplus_{p+q=n} C_p\otimes D_q$, for any $n\in\mathbb{Z}$, and with the differential such that $\delta(x\otimes y) = \delta(x)\otimes y + \pm x\otimes\delta(y)$, for any homogeneous tensor $x\otimes y\in C_p\otimes D_q$, where $\pm$ is a sign $\pm = (-1)^p$ produced by the commutation of the differential δ (which is a homogeneous homomorphism of degree -1) and the element x (which is homogeneous of degree p by assumption).

By convention, we more generally assume that any transposition of homogeneous factors x and y of degree p and q in a tensor product of dg-modules produces a sign $\pm = (-1)^{pq}$ (which we do not make explicit in general), and we equip the tensor product of dg-modules with a symmetry isomorphism $\tau : C\otimes D\to D\otimes C$ determined from this commutation rule. We often work in symmetric monoidal categories $\mathcal{E}$ equipped with a symmetric monoidal functor $\eta : \mathcal{C}\to\mathcal{E}$ over the base category of dg-modules $\mathcal{C} = dg\,\mathcal{M}od$. We say in this situation that the category $\mathcal{E}$ forms a symmetric monoidal category over the category of dg-modules $\mathcal{C} = dg\,\mathcal{M}od$. We generally abusively do not mark the functor η when we consider the image of a dg-module $C\in\mathcal{C}$ in such a category $\mathcal{E}$.

The category of dg-modules $\mathcal{E} = dg\,\mathcal{M}od$ trivially forms a symmetric monoidal category over itself. The category of (symmetric) collections $\mathcal{E} = \mathcal{M}$ and the category of right modules over an operad $\mathcal{E} = \mathcal{M}_{\mathsf{P}}$ are other instances of symmetric monoidal categories over $\mathcal{C} = dg\,\mathcal{M}od$ which we use in our constructions. In both cases $\mathcal{E} = \mathcal{M},\mathcal{M}_{\mathsf{P}}$, we consider the obvious functor $\eta : \mathcal{C}\to\mathcal{E}$ which identifies any object $C\in\mathcal{C}$ with a symmetric sequence, concentrated in arity zero, such that $C(0) = C$ and $C(n) = 0$ for $n\neq 0$.

In turn, we equip the category of symmetric sequences with a tensor product operation $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, which extends the tensor product of the base category of dg-modules $\mathcal{C} = \mathcal{M}$, and maps any pair of objects $\mathsf{M},\mathsf{N}\in\mathcal{M}$ to the symmetric sequence such that $(\mathsf{M}\otimes\mathsf{N})(n) = \bigoplus_{p+q=n}\Sigma_n\otimes_{\Sigma_p\times\Sigma_q}\mathsf{M}(p)\otimes\mathsf{N}(q)$, for each $n\in\mathbb{N}$. We generally define the tensor product of an object $C\in\mathcal{C}$ with a set $S\in\mathcal{S}et$ as a sum of copies of this object C, indexed by the elements $w\in S$, and formally denoted by $w\otimes C$. We identify $\Sigma_p\times\Sigma_q$ with the subgroup of Σ_n, $n = p+q$, formed by the permutations $s\times t\in\Sigma_p\times\Sigma_q$ such that s acts on $\{1,\dots,p\}$ and t acts on $\{p+1,\dots,p+q\}$. When we take a tensor product over the group $\Sigma_p\times\Sigma_q$ in our expression, we just perform an appropriate quotient to identify the action of these permutations $s\times t\in\Sigma_p\times\Sigma_q$ by right translation on Σ_n with their natural action on the tensor product $\mathsf{M}(p)\otimes\mathsf{N}(q)$. We explicitly set $(w\cdot s\times t)\otimes(x\otimes y)\equiv w\otimes(sx\otimes ty)$, for every $w\in\Sigma_n$ and $x\otimes y\in\mathsf{M}(p)\otimes\mathsf{N}(q)$. We go back to this construction in §2. We refer to [3] for the definition of the symmetry isomorphism $c : \mathsf{M}\otimes\mathsf{N}\xrightarrow{\simeq}\mathsf{N}\otimes\mathsf{M}$ associated to this tensor structure.

The tensor product of the category of right modules over an operad, let $\otimes : \mathcal{M}_{\mathsf{P}}\times\mathcal{M}_{\mathsf{P}}\to\mathcal{M}_{\mathsf{P}}$, is defined by observing that the symmetric sequence $\mathsf{M}\otimes\mathsf{N}$ inherits a natural right P-action when we have $\mathsf{M},\mathsf{N}\in\mathcal{M}_{\mathsf{P}}$. This claim basically follows from the observation that the tensor product satisfies the distribution relation $(\mathsf{M}\otimes\mathsf{N})\circ\mathsf{P} = \mathsf{M}\circ\mathsf{P}\otimes\mathsf{N}\circ\mathsf{P}$ with respect to the composition product (see [3]).

The category of P-algebras in the base category $\mathcal{C}$ consists of the left P-modules $A\in{}_{\mathsf{P}}\mathcal{M}$, which are concentrated in arity 0, and are equivalent to objects in the base category $A\in\mathcal{C}$ when we forget about the operad action. We can also define P-algebra structures in any category $\mathcal{E}$ which is symmetric monoidal over $\mathcal{C}$. We just use the structure functor $\eta : \mathcal{C}\to\mathcal{E}$ to transport the operad P into the category $\mathcal{E}$ and to give a sense to this notion. We may actually see that a left module $\mathsf{M}\in{}_{\mathsf{P}}\mathcal{M}$

over an operad P in the base category $\mathcal{C}$ is equivalent to an algebra over P in the category of symmetric sequences $\mathcal{E} = \mathcal{M}$, and a (P, Q)-bimodule $M \in {}_P\mathcal{M}_Q$ is similarly identified with an algebra over P in the category of right Q-modules $\mathcal{E} = \mathcal{M}_Q$.

We already mentioned that the category of left modules over an operad ${}_P\mathcal{M}$, and the category of bimodules ${}_P\mathcal{M}_R$ are equipped with a natural model structure, at least when the operad P is cofibrant (and the operad Q consists of cofibrant dg-modules in the bimodule case). We may regard these model categories as extensions of a model category of P-algebras ${}_P\mathcal{C}$, where the weak-equivalences $\phi : A \xrightarrow{\sim} B$ are the P-algebra morphisms which form a weak-equivalence in the base category (of dg-modules) when we forget about the operad action, and the fibrations $\phi : A \twoheadrightarrow B$ are defined similarly.

We still have a well-defined semi-model structure on the category of P-algebras ${}_P\mathcal{C}$, with the same class of weak-equivalences and fibrations, when P is cofibrant as a symmetric sequence. The expression semi-model refers to the observation that, in this situation, the lifting and factorization properties of model categories can only be guaranteed for morphisms with a cofibrant object as domain. This is actually enough for most constructions of homotopical algebra.

0.4. *Relative composition products and functors.* We have a relative composition product operation $M \circ_P N$, defined by a standard reflexive coequalizer construction:

$$M \circ_P N = \operatorname{coker}\big(M \circ P \circ N \underset{d_1}{\overset{d_0}{\rightrightarrows}} M \circ N \big), \quad s_0 : M \circ N \to M \circ P \circ N \tag{1}$$

for any object $M \in \mathcal{M}_P$ equipped with a right action of an operad P, and any object $N \in {}_P\mathcal{M}$ equipped with a left P-action. In our expression (1), the 0-face d_0 is yielded by the right action of P on M, the 1-face d_1 is yielded by the left action of P on N and the degeneracy s_0 is provided by the unit morphism of the operad $\eta : I \to P$.

If M is just a right P-module and N is a left P-module, then the outcome of this operation is an object of the category of symmetric sequences $M \circ_P N \in \mathcal{M}$. If M is endowed with an additional left R-action and forms an (R, P)-bimodule, then $M \circ_P N$ still inherits a left R-module structure. If N is endowed with an additional right Q-action and forms a (P, Q)-bimodule, then we similarly obtain that $M \circ_P N$ inherits a right R-module structure, and similarly in the case where both M and N are equipped with bimodule structures.

We may still observe that $M \circ_P A$ is concentrated in arity 0 in the case where $N = A$ is left P-module, concentrated in arity 0, equivalent to a P-algebra in the base category $\mathcal{C}$. We accordingly get that the operation $\mathtt{S}_P(M)(A) = M \circ_P A$ defines a functor such that $\mathtt{S}_P(M) : {}_P\mathcal{C} \to \mathcal{C}$, and similarly, when $M \in {}_R\mathcal{M}_P$, we obtain that the operation $\mathtt{S}_P(M)(A) = M \circ_P A$ defines a functor $\mathtt{S}_P(M) : {}_P\mathcal{C} \to {}_R\mathcal{C}$.

We may see that the tensor product and the (relative) composition products on modules over operads actually reflect obvious point-wise tensor product and composition operations on functors on this form. We can more formally check that the obvious relations $(M \otimes N) \circ_P R = (M \circ_P R) \otimes (N \circ_P R)$, $P \circ_P M = M$ and $(M \circ_P N) \circ_Q R = M \circ_P (N \circ_Q R)$, extending the standard associativity relations of relative tensor products, hold for the tensor product and the composition product of modules over operads whenever these relations make sense.

In what follows, we also use the symmetric monoidal structure on the category of right modules over an operad $\mathcal{M}_P$ to define associative algebras and module structures in this category. The mapping $\mathtt{S}_P : M \mapsto \mathtt{S}_P(M)$ carries such objects to associative algebra functors and module functors on the category of P-algebras.

The main instance of an associative algebra structure in the category of right modules over an operad P which we consider in these notes is the right P-module $\mathtt{U}_P$ which represents the enveloping algebra functor $\mathtt{U}_P : A \mapsto \mathtt{U}_P(A)$ on the category of P-algebras. In the definition of the homology of P-algebras, we also consider a right P-module Ω_P, which represents a functor of Kähler forms on the category of P-algebras, and which forms a left module over the algebra $\mathtt{U}_P$ in the category of right P-modules.

0.5. *The left derived functor of the relative composition product.* We establish in [3] that the bifunctor $- \circ_P - : (M, N) \mapsto M \circ_P N$ has a homotopy invariant left derived functor $- \circ^{\mathtt{L}}_P -$, which we may compute by taking a cofibrant resolution $R_M \xrightarrow{\sim} M$ of the object M in the category of right P-modules. We explicitly set:

$$M \circ^{\mathtt{L}}_P N = R_M \circ_P N, \tag{1}$$

for any $M \in \mathcal{M}_P$ and $N \in {}_P\mathcal{M}$. If the operad P and the right P-module M are good enough (cofibrant as symmetric sequences), then we can also compute this left derived functor by taking a cofibrant resolution $R_N \xrightarrow{\sim} N$ of the object N in the category of left P-modules. We explicitly have an identity:

$$M \circ^{\mathtt{L}}_P N = M \circ_P R_N\,. \tag{2}$$

In all cases, we have to assume, at least, that the objects $M(n)$, $P(n)$ and $N(n)$ which form our collections M, P and N are cofibrant as dg-modules.

In what follows, we also use a $\mathtt{Tor}$-functor notation for the homology of this derived relative composition product:

$$\mathtt{Tor}^P_*(M, N) = \mathrm{H}_*(M \circ^{\mathtt{L}}_P N). \tag{3}$$

0.6. *The explicit definition of an operad structure.* The action of an operad P over an algebra A can be explicitly defined in terms of operations

$$p : A^{\otimes r} \to A, \tag{1}$$

associated to any operad element $p \in P(r)$, satisfying natural equivariance, unit, and associativity constraints. In what follows, we also write $p(a_1, \dots, a_r) \in A$ for the image of a tensor $a_1 \otimes \dots \otimes a_r \in A^{\otimes r}$ under any such operation $p \in P(r)$ arising from an operad.

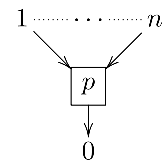
We have a similar representation for the composition structure of an operad and the action of an operad on a module. We generally get that a morphism of symmetric sequences $\mu : M \circ N \to P$, where M, N and P are any object in this category $\mathcal{M}$, is equivalent to a collection of morphisms

$$\mu : M(r) \otimes N(n_1) \otimes \dots \otimes N(n_r) \to P(n_1 + \dots + n_r), \tag{2}$$

defined for all $r \geq 0$, $n_1, \dots, n_r \geq 0$, and which satisfy natural equivariance constraints. We also adopt the notation $x(y_1, \dots, y_r) \in P(n_1 + \dots + n_r)$ for the image of elements $x \in M(r)$, $y_1 \in N(n_1), \dots, y_r \in N(n_r)$, under such a composition operation. In the case of the composition product $\mu : P \circ P \to P$ of an operad P, a composite element $p(q_1, \dots, q_r) \in P(n_1 + \dots + n_r)$ simply reflects a composition of operations of the form (1) acting on a category of algebras. We use this interpretation in several constructions. We will moreover use that the unit morphism of an operad $\eta : I \to P$ is equivalent to an element $1 \in P(1)$ which represents an identity operation.

We also represent the operations $p \in P(n)$ encoded by an operad P by a box picture:

(3)



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We go back to this representation in §2, where we explain the definition of the category of operators associated to an operad.

1. Homology and modules over operads

We assume that P is an operad in the category of dg-modules $dg\,\mathcal{M}od$. We revisit the definition of the homology theory $\mathtt{H}^{P}_{*}(A, M)$ for a pair (A, M) such that A is a P-algebra and M is a corepresentation of this P-algebra A. We also recall the definition of this concept of a corepresentation for an algebra over an operad. We review the definition of a representation of an algebra over an operad first.

1.1. *Representations and enveloping algebras of algebras over an operad.* In short, a representation of a P-algebra A is an object of the base category $M \in \mathcal{C}$ equipped with operations

(1) $$p : A^{\otimes n-1} \otimes M \to M,$$

associated to any $p \in P(n)$, and which satisfy a natural extension of the basic symmetry, unit, and associativity relations of the structure of an algebra over an operad. In what follows, we also write $p(a_1, \dots, a_{n-1}, \xi) \in M$ for the image of a tensor $a_1 \otimes \cdots \otimes a_{n-1} \otimes \xi \in M$ under the product operation (1). In the definition of these operations (1), we only use symmetry relations with respect to the first $n-1$ inputs, marked by the algebra A, of the operation $p \in P(n)$. In practice, we use the full symmetry relations of operads to move the factor $\xi \in M$ to other inputs of our operation $p \in P(n)$.

To any representation M of a P-algebra A, we can associate a P-algebra over A, denoted by $A \ltimes M$, which has the direct sum $A \ltimes M = A \oplus M$ as underlying dg-module, and which is equipped with an P-algebra structure that extends the action of the operad P on A. We simply use the operations (1) to define the action of operations $p \in P(n)$ on a tensor with a single factor in M, and we assume that the operations involving multiple factors in M vanish in $A \ltimes M$. These objects $A \ltimes M = A \oplus M$, which we associate to representations of a P-algebras M, are actually equivalent to abelian group objects in the category of P-algebras over A.

We adopt the notation $\mathcal{R}_P(A)$ for the category of representations of a P-algebra A. We use that this category is isomorphic to a category of left modules over an associative algebra $\mathtt{U}_P(A)$, naturally associated to $A \in {}_P\mathcal{C}$, and usually referred to as the enveloping algebra of the P-algebra A. We basically define $\mathtt{U}_P(A)$ as the dg-module spanned by formal elements $p(a_1, \dots, a_{n-1}, -)$, where $p \in P(n)$,

$a_1, \dots, a_{n-1} \in A$, modulo natural symmetry, unit and associativity relations reflecting the symmetry, unit and associativity relations of the action of a P-algebra on a representation (1).

The mark $-$ in the expression of an element $p(a_1, \dots, a_{n-1}, -) \in \mathtt{U}_{\mathsf{P}}(A)$ denotes an input which we leave free for the action on representations. The multiplication of the enveloping algebra is given by the composition at this free input. We explicitly have:

$$(2)\quad p(a_1, \dots, a_{m-1}, -) \cdot q(b_1, \dots, b_{n-1}, -) = (p \circ_m q)(a_1, \dots, a_{m-1}, b_1, \dots, b_{n-1}, -),$$

for any pair of elements $p(a_1, \dots, a_{m-1}, -), q(b_1, \dots, b_{n-1}, -) \in \mathtt{U}_{\mathsf{P}}(A)$, where $p \circ_n q \in \mathsf{P}(m + n - 1)$ is the notation for the operadic composite operation involved in this multiplication process. We generally set $p \circ_k q = p(1, \dots, q, \dots, 1)$, for the operation defined by composing $p \in \mathsf{P}(m)$ with $q \in \mathsf{P}(n)$ at position k and where we plug operadic units $1 \in \mathsf{P}(1)$ in the remaining positions (see §0.6).

1.2. *The enveloping algebra module associated to an operad.* We already mentioned that the enveloping algebra is identified with a functor associated to an algebra $\mathtt{U}_{\mathsf{P}}$ in the category of right modules over the operad P. We explicitly have:

$$(1)\qquad \mathtt{U}_{\mathsf{P}}(A) = \mathtt{U}_{\mathsf{P}} \circ_{\mathsf{P}} A,$$

where we consider the relative composition product of §0.4. We define this object $\mathtt{U}_{\mathsf{P}}$ as a right P-module by the identity:

$$(2)\qquad \mathtt{U}_{\mathsf{P}}(n) = \mathsf{P}(n + 1),$$

for any arity $n \in \mathbb{N}$, so that the elements $u \in \mathtt{U}_{\mathsf{P}}(n)$ are identified with operations $p \in \mathsf{P}(n + 1)$ with a distinguished input and n remaining inputs which we use for the definition of the right P-action on $\mathtt{U}_{\mathsf{P}}$.

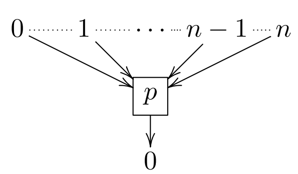
In other contexts, we use the notation $\mathsf{P}[1]$ for this operadic shifting construction $\mathsf{P}[1](n) = \mathsf{P}(n + 1)$. We may also write $p(x_1, \dots, x_n, -) \in \mathsf{P}[1](n)$ for the element of this shifted object $\mathsf{P}[1]$ associated to any $p \in \mathsf{P}(n + 1)$. We then use the mark $-$ to indicate the distinguished input (which arises from the shifting process), and variables $x_1, \dots, x_n$ for the remaining inputs of our operation in $\mathsf{P}[1](n)$. We keep the same ordering as in the previous paragraph for the moment, but we more usually move the distinguished input to the first position when we adopt the point of view of this shifting construction. The multiplication of the enveloping algebra is defined at the level of the shifted object $\mathsf{P}[1]$ by the formula:

$$(3)\qquad p(x_1, \dots, x_m, -) \cdot q(x_1, \dots, x_n, -) = (p \circ_{m+1} q)(x_1, \dots, x_{m+n}, -),$$

for any $p(x_1, \dots, x_m, -) \in \mathsf{P}[1](m)$, $q(x_1, \dots, x_n, -) \in \mathsf{P}[1](n)$.

We also depict an element of the enveloping algebra module by a variant of the box representation of §0.6:

(4)



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We then use the index 0 to mark the distinguish input of our operation $p \in \mathsf{P}[1](n)$. We notably adopt this representation when we define the category of pointed operators associated to an operad (see §**??**).

1.3. *Kähler forms.* We associate a dg-module of Kähler forms $\Omega_{\mathsf{P}}(A)$ to any P-algebra $A \in {}_{\mathsf{P}}\mathcal{C}$. We define this object as the left $\mathtt{U}_{\mathsf{P}}(A)$-module spanned by formal elements da, $a \in A$, to which we assign the degree $\deg(da) = \deg(a) - 1$, modulo relations of the form

$$dp(a_1, \ldots, a_n) \equiv \sum_{i=1}^{n} \pm p(a_1, \ldots, da_i, \ldots, a_n), \tag{1}$$

when we consider the result of an operation $p(a_1, \ldots, a_n) \in A$ in the P-algebra A. We just use the representation operation $p : A^{\otimes n-1} \otimes \Omega_{\mathsf{P}}(A) \to \Omega_{\mathsf{P}}(A)$ equivalent to the action of the enveloping algebra element $p(a_1, \ldots, \xi, \ldots, a_n) \in \mathtt{U}_{\mathsf{P}}(A)$ on $M = \Omega_{\mathsf{P}}(A)$ when we form the expression $p(a_1, \ldots, da_i, \ldots, a_n) \in \Omega_{\mathsf{P}}(A)$.

The mapping $\Omega_{\mathsf{P}} : A \mapsto \Omega_{\mathsf{P}}(A)$ clearly defines a functor on the category of P-algebras, and we actually have an identity:

$$\Omega_{\mathsf{P}}(A) = \Omega_{\mathsf{P}} \circ_{\mathsf{P}} A, \tag{2}$$

for a module Ω_{P} over the algebra $\mathtt{U}_{\mathsf{P}}$ in the symmetric monoidal category of right P-modules $\mathcal{M}_{\mathsf{P}}$. This object Ω_{P} is basically defined, as a right P-module, by the dg-module spanned by formal elements $p(x_1, \ldots, dx_i, \ldots, x_n) \in \Omega_{\mathsf{P}}(n)$, where $p \in \mathsf{P}(n)$ is an operation and $(x_1, \ldots, x_n)$ are variables, modulo the obvious symmetry relations, and where we take the right P-action that reflects the derivation relation (1) on the input dx_i distinguished by the derivation symbol d.

1.4. *corepresentations and homology.* The corepresentations M which we consider in the definition of the homology of a P-algebra A are just right modules over the enveloping algebra $\mathtt{U}_{\mathsf{P}}(A)$. We also use the notation $\mathcal{R}^c_{\mathsf{P}}(A) = \mathcal{M}od_{\mathtt{U}_{\mathsf{P}}(A)}$ for the category of corepresentations associated to any $A \in {}_{\mathsf{P}}\mathcal{C}$.

We now assume that P is cofibrant as a symmetric sequence so that we have at least a well-defined semi-model structure on the category of P-algebras, which is enough to do homotopical algebra. We then define the homology of a P-algebra A with coefficients in a corepresentation $M \in \mathcal{R}^c_{\mathsf{P}}(A)$ as the derived functor construction such that:

$$\mathtt{H}^{\mathsf{P}}_*(A, M) = \mathtt{H}_*(M \otimes_{\mathtt{U}_{\mathsf{P}}(R_A)} \Omega_{\mathsf{P}}(R_A)), \tag{1}$$

where we consider a cofibrant resolution $R_A \xrightarrow{\sim} A$ of the P-algebra A, and the module of Kähler forms $\Omega_{\mathsf{P}}(R_A)$ associated to this object $R_A \in {}_{\mathsf{P}}\mathcal{C}$. We use the morphism $\mathtt{U}_{\mathsf{P}}(R_A) \to \mathtt{U}(A)$ induced by the resolution augmentation $R_A \xrightarrow{\sim} A$ to provide M with a right $\mathtt{U}_{\mathsf{P}}(R_A)$-action.

When P is not cofibrant as a symmetric sequence, we just pick a cofibrant resolution $\mathsf{R} \xrightarrow{\sim} \mathsf{P}$ and we perform this homology construction in the category of R-algebras. We then set:

$$\mathtt{H}\Gamma^{\mathsf{P}}_*(A, M) = \mathtt{H}_*(M \otimes_{\mathtt{U}_{\mathsf{R}}(R_A)} \Omega_{\mathsf{R}}(R_A)), \tag{2}$$

to define the homology of any pair (A, M) such that A is a P-algebra and $M \in \mathcal{R}^c_{\mathsf{P}}(A)$. We then assume that R_A is a cofibrant resolution of A in the category of

R-algebras. We have an identity:

$$M \otimes_{\mathtt{U}_{\mathsf{R}}(R_A)} \Omega_{\mathsf{R}}(R_A) = M \otimes_{\mathtt{U}_{\mathsf{P}}(A)} (\mathtt{U}_{\mathsf{P}}(A) \otimes_{\mathtt{U}_{\mathsf{R}}(R_A)} \Omega_{\mathsf{R}}(R_A)), \tag{3}$$

and we may accordingly consider the object:

$$\mathtt{H}\Gamma^{\mathsf{P}}_*(A, \mathtt{U}_{\mathsf{P}}(A)) = \mathtt{H}_*(\mathtt{U}_{\mathsf{P}}(A) \otimes_{\mathtt{U}_{\mathsf{R}}(R_A)} \Omega_{\mathsf{R}}(R_A)), \tag{4}$$

as a universal coefficient homology associated to A.

We have the following proposition:

1.5. **Theorem.** *We assume that $\mathtt{U}_{\mathsf{P}}$ is cofibrant as a right P-module (which, according to [3] is automatically true when P is cofibrant as an operad). We then have an identity:*

$$\mathtt{H}\Gamma^{\mathsf{P}}_*(A, \mathtt{U}_{\mathsf{P}}(A)) = \mathtt{Tor}^{\mathsf{P}}_*(\Omega_{\mathsf{P}}, A),$$

where we identify the P-algebra A with a left module over the operad P, and we consider the operadic $\mathtt{Tor}$-functor with coefficient in this left P-module.

Proof. □

We assume from now on that P is cofibrant as an operad, so that $\mathtt{H}\Gamma^{\mathsf{P}}_* = \mathtt{H}^{\mathsf{P}}_*$. When this is not the case, we just replace P by the associated cofibrant resolution R, so that $\mathtt{H}\Gamma^{\mathsf{P}}_* = \mathtt{H}\Gamma^{\mathsf{R}}_* = \mathtt{H}^{\mathsf{R}}_*$.

2. The equivalence with functor homology

We now define the equivalence between the operadic homology theory defined in §1.3 and the homology of functors. We first review the the definition of the category of unpointed operators Γ_{P} associated to any operad P, and we explain our definition of the category of pointed operators Γ^+_{P}.

2.1. *The category of operators associated to an operad.* The classical category of unpointed operators Γ_{P}, also called PROP, which we associate to an operad P has the finite ordinals $\underline{n} = \{1 < \cdots < n\}$ as objects, and is equipped with hom-objects such that $\Gamma_{\mathsf{P}}(\underline{m}, \underline{n}) = \mathsf{P}^{\otimes n}(m)$, for any $m, n \in \mathbb{N}$, where we consider the nth power of the symmetric sequence P with respect to the symmetric monoidal structure of §0.3.

By definition of this tensor structure on symmetric sequences, we also have:

$$\Gamma_{\mathsf{P}}(\underline{m}, \underline{n}) = \bigoplus_{m_1 + \cdots + m_n = m} \Sigma_m \otimes_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}} (\mathsf{P}(m_1) \otimes \cdots \otimes \mathsf{P}(m_n)), \tag{1}$$

where we consider the obvious n-fold extension of the tensor product expression of §0.3. We can more explicitly identify $\mathsf{P}^{\otimes n}(m)$ with the dg-module spanned by tensors $w \otimes (p_1 \otimes \cdots \otimes p_n)$ such that $w \in \Sigma_m$, $p_1 \in \mathsf{P}(m_1), \ldots, p_n \in \mathsf{P}(m_n)$, modulo relations such that:

$$(w \cdot s_1 \times \cdots \times s_n) \otimes (p_1 \otimes \cdots \otimes p_n) \equiv w \otimes (s_1 p_1 \otimes \cdots \otimes s_n p_n), \tag{2}$$

for any $s_1 \times \cdots \times s_n \in \Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$, and where we consider the obvious cartesian subgroup embedding $\Sigma_{m_1} \times \cdots \times \Sigma_{m_n} \subset \Sigma_m$ (see §0.3).

Intuitively, such a tensor represents a composite operation:

$$A^{\otimes m} \xrightarrow{w^*} A^{\otimes m} \xrightarrow{\bigotimes_{i=1}^n p_i} A^{\otimes n}, \tag{3}$$

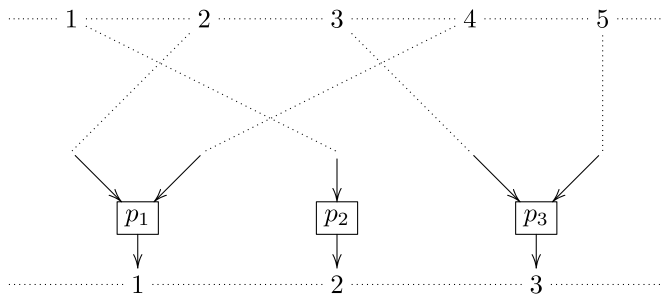

FIGURE 1.

acting on any P-algebra A, where w^* is the tensor permutation associated to $w \in \Sigma_n$, and we consider the tensor product of the operations $p_i : A^{\otimes m_i} \to A$ associated to the elements $p_i \in P(m_i)$, $i = 1, \dots, n$. The hom-objects $\Gamma_P(\underline{m}, \underline{n})$ are equipped with composition products $\circ : \Gamma_P(\underline{k}, \underline{n}) \otimes \Gamma_P(\underline{m}, \underline{k}) \to \Gamma_P(\underline{m}, \underline{n})$ which reflect the composition of such operations. From this interpretation, we immediately get that the collection $\mathtt{R}_A(\underline{n}) = A^{\otimes n}$ defines a functor $\mathtt{R}_A : \Gamma_P \to \mathcal{C}$ when A is a P-algebra.

2.2. *The graphical description of the category of operators associated to an operad.* In what follows, we also represent the homomorphisms of the category Γ_P by a box diagram, such as in the picture of Figure 1, and where:

- the indices of the upper row correspond to the elements of the domain set $\underline{m} = \{1 < \cdots < m\}$, and materialize the inputs of our tensor product operation,
- the indices of the lower row correspond to the elements of the codomain set $\underline{n} = \{1 < \cdots < n\}$, and materialize the outputs of our tensor product operation,
- the graph on the upper part of our figure materializes the permutation w^*,
- and the boxes represent the operations $p_j \in P(m_j)$, $j = 1, \dots, n$.

From this representation, we see that our hom-object $\Gamma_P(\underline{m}, \underline{n})$ can also be defined by an expression of the form:

$$\Gamma_P(\underline{m}, \underline{n}) = \bigoplus_{f : \underline{m} \to \underline{n}} P(f^{-1}(1)) \otimes \cdots \otimes P(f^{-1}(n)), \tag{1}$$

where f runs over the set of all set-theoretic maps $f : \{1 < \cdots < m\} \to \{1 < \cdots < n\}$, and we set $P(\underline{\mathsf{n}}) = \mathcal{B}ij(\underline{n}, \underline{\mathsf{n}}) \otimes_{\Sigma_n} P(n)$ to define the value of the symmetric sequence underlying P on any finite set $\underline{\mathsf{n}}$ of cardinal n. In our picture, the map f is materialized by the graph that forms the composition scheme of our operation. For instance, in the example depicted in Figure 1, we get a map $f : \{1 < \cdots < 5\} \to \{1 < \cdots < 3\}$ such that $f(1) = 2$, $f(2) = 1$, $f(3) = 3$, $f(4) = 1$ and $f(5) = 3$. The elements $p \in P(f^{-1}(j))$ are equivalent to operations $p = p(x_i)_{i \in f^{-1}(j)}$ on variables x_i indexed by the elements i of the sets $f^{-1}(j)$. For instance, in our example, we deal with operations of the form $p_1 = p_1(x_2, x_4)$, $p_2 = p_2(x_1)$, $p_3 = p_3(x_3, x_5)$.

In this formalism, the image of a tensor $\bigotimes_{i=1}^m a_i \in A^{\otimes m}$ under a homomorphism ϕ of the category Γ_P in §2.1(3) is given by an expression of the form:

$$\phi_*(\textstyle\bigotimes_{i=1}^n a_i) = \bigotimes_{j=1}^n p_j(a_i)_{i \in f^{-1}(j)}, \tag{2}$$

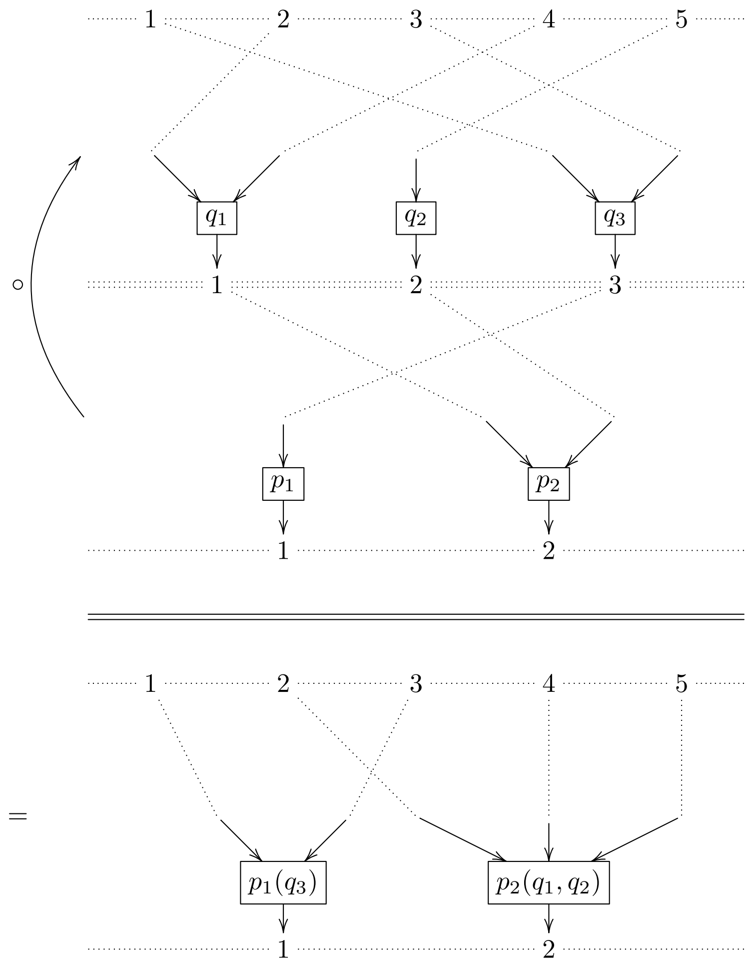

FIGURE 2.

where, for each $j = 1, \ldots, n$, we evaluate the operation $p_j \in P(f^{-1}(j))$ on the factors a_i such that $i \in f^{-1}(j)$. For instance, in the case of the homomorphism $\phi \in \Gamma_P(\underline{5}, \underline{3})$ depicted in Figure 1, we obtain the formula:

$$\phi_*(a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5) = p_1(a_2, a_4) \otimes p_2(a_1) \otimes p_3(a_3, a_5), \tag{3}$$

for any $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \in A^{\otimes 5}$. We may also retrieve this expression by labeling the inputs of our morphism by the elements $a_1, \ldots, a_5 \in A$ in the diagram of Figure 1. We then use the composition scheme depicted in this diagram in order to determine the evaluation process associated with our homomorphism.

We can use the same representation in order to determine the composition scheme of homomorphisms in our category of operators. We give an example of this process in the picture of Figure 2.

2.3. *The category of pointed operators associated to an operad.* The category of pointed operators Γ_P^+ which we associate to an operad P has the pointed finite

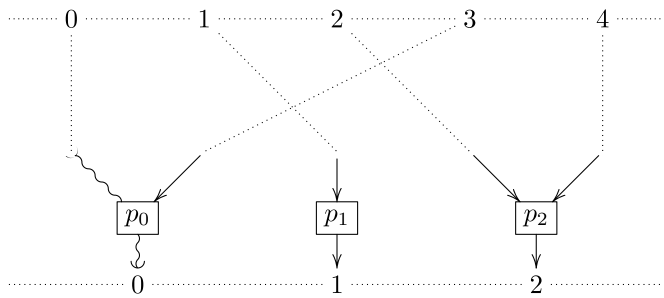

FIGURE 3.

ordinals $\underline{n}_+ = \{1 < \cdots < n\} \amalg \{0\}$ as objects, and is equipped with hom-objects such that $\Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+) = (\mathsf{P}[1]^{op} \otimes \mathsf{P}^{\otimes n})(m)$, for any $m, n \in \mathbb{N}$, where we consider the tensor product of the shifted symmetric sequence $\mathsf{P}[1]$ of §1.2 with the tensor power $\mathsf{P}^{\otimes n}$ in the category of symmetric sequences $\mathcal{M}$. The super-script *op* refers to the consideration of the opposite of the internal multiplication of this algebra $\mathsf{P}[1]$ when we perform compositions of this factor $\mathsf{P}[1]$ in the homomorphisms of the category Γ^+_{P}.

We adapt our description of the unpointed category of operators in §2.2, and we represent the homomorphisms of the category Γ^+_{P} by diagrams of the form of Figure 3, where we now use wavy lines (with particular endings) to mark the composition path of the distinguished input, which is indexed by 0 in our domain and codomain sets.

From this picture, we see that our hom-object $\Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+)$ now admits an expansion of the form:

$$(1) \qquad \Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+) = \bigoplus_{f:\underline{m}_+ \to \underline{n}_+} \mathsf{P}(f^{-1}(0)) \otimes \mathsf{P}(f^{-1}(1)) \otimes \cdots \otimes \mathsf{P}(f^{-1}(n)),$$

where the sum now runs over the set of pointed maps $f : \{1 < \cdots < m\}_+ \to \{1 < \cdots < n\}_+$. In this expression, the factor $\mathsf{P}(f^{-1}(0))$, corresponds to the shifted symmetric sequence $\mathsf{P}[1]$ of our initial definition, and the base point $0 \in f^{-1}(0)$ gives the place of the distinguished input, marked by the symbol $-$ in our algebraic notation. We therefore represent an element $p_0 \in \mathsf{P}(f^{-1}(0))$ by an expression of the form $p_0 = p_0(-, x_i)_{i \in f^{-1}(0) - \{0\}}$, while we keep the same expression as in §2.2 for the other factors $p_j = p_j(x_i)_{i \in f^{-1}(j)}$, $j = 1, \dots, n$, of a homomorphism $\phi \in \Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+)$. In the example depicted in Figure 1, we get the pointed map $f : \{1 < 2 < 3 < 4\}_+ \to \{1 < 2\}_+$ such that $f(1) = 1$, $f(2) = 2$, $f(3) = 0$ and $f(4) = 2$, and we deal with operations $p_0 = p_0(-, x_3) \in \mathsf{P}(f^{-1}(0))$, $p_1 = p_1(x_1) \in \mathsf{P}(f^{-1}(1))$, $p_2 = p_2(x_2, x_4) \in \mathsf{P}(f^{-1}(2))$.

We make explicit the action of such homomorphisms $\phi \in \Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+)$ on the functor $\mathtt{R}_{A,M}$ associated to a pair (A, M) where A is a P-algebra and $M \in \mathcal{R}^c_{\mathsf{P}}(A)$ is a corepresentation, before explaining the definition of the composition operation of this category Γ^+_{P}. This functor $\mathtt{R}_{A,M}$ is defined on objects by the tensor product $\mathtt{R}_{A,M}(\underline{n}_+) = M \otimes A^{\otimes n}$, for any $n \in \mathbb{N}$, where we put the corepresentation M at the position corresponding to the base point $0 \in \underline{n}_+$ We then define the image of a tensor $\xi \otimes (\bigotimes_{i=1}^m a_i) \in M \otimes A^{\otimes m}$ under a homomorphism $\phi \in \Gamma^+_{\mathsf{P}}(\underline{m}_+, \underline{n}_+)$ by the formula:

$$\phi_*(\xi \otimes (\bigotimes_{i=1}^n a_i)) = (\xi \cdot p_0(-,a_i)_{i\in f^{-1}(0)-\{0\}}) \otimes (\bigotimes_{j=1}^n p_j(a_i)_{i\in f^{-1}(j)}), \tag{2}$$

which we explain as follows:

- The evaluation of the distinguished operation $p_0 \in \mathsf{P}(f^{-1}(0))$ on the factors $a_i \in A$, $i \in f^{-1}(0)-\{0\}$, returns an element of the enveloping algebra of our P-algebra $p_0(-,a_i)_{i\in f^{-1}(0)-\{0\}} \in \mathtt{U}_{\mathsf{P}}(A)$, which, in turns, acts on $\xi \in M$ on the right to produce the distinguished factor $\xi \cdot p_0(-,a_i)_{i\in f^{-1}(0)-\{0\}} \in M$ occurring in our formula.
- The remaining factors $p_j(a_i)_{i\in f^{-1}(j)} \in A$, $j = 1,\dots,n$, are determined by the same procedure as in §2.2.

For instance, in the case of the homomorphism $\phi \in \Gamma^+_{\mathsf{P}}(\underline{4}_+, \underline{2}_+)$ depicted in Figure 3, we obtain the formula:

$$\phi_*(\xi \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4) = (\xi \cdot p_0(-,a_3)) \otimes p_1(a_1) \otimes p_2(a_2,a_4), \tag{3}$$

for any $\xi \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4 \in M \otimes A^{\otimes 4}$.

We equip the category Γ^+_{P} with the composition structure that reflects the composition of operations of this form. We just have to replace the tensor $\xi \otimes (\bigotimes_{i=1}^m a_i)$ in our formula (2) by operations $\xi = q_0 \in \mathsf{P}(g^{-1}(0))$ and $a_i = q_i \in \mathsf{P}(g^{-1}(i))$, $i = 1,\dots,m$, defining the factors of a homomorphism ψ to get the expression of a composite $\phi\psi$ in Γ^+_{P}. We use the right P-module structure of the object $\mathsf{P}[1]$ to form the composite $p_0(-,q_i)_{i\in f^{-1}(0)-\{0\}} \in \mathsf{P}[1]$, which replaces our enveloping algebra element of our formula (2), and we take the multiplication of the algebra in right P-modules $\mathsf{P}[1]$ when we form the product $q_0 \cdot p_0(-,q_i)_{i\in f^{-1}(0)-\{0\}} \in \mathsf{P}[1]$. We just plug $p_0(-,q_i)_{i\in f^{-1}(0)-\{0\}} \in \mathsf{P}[1]$ in the distinguished input of the operation $q_0 \in \mathsf{P}[1]$ according to the definition of this multiplication in §1.2. We note that this operation swaps the positions of the factors p_0 and q_0 and this twist corresponds to the mark *op* in our first expression of the hom-object Γ^+_{P}. We just use the operadic composition operation, in the standard order, to get the remaining factors $p_j(q_i)_{i\in f^{-1}(j)}$ of our composite homomorphism. We give an explicit example of application of this process in the picture of Figure 4.

The definition of the functor $\mathtt{R}_{A,M}$ makes sense in any symmetric monoidal category over the category of dg-modules $\mathcal{E}$. We apply this construction to the operad $A = \mathsf{P}$, which we regard as an algebra over itself in the category of right modules over itself $\mathcal{E} = \mathcal{M}_{\mathsf{P}}$, and to the enveloping algebra module $M = \mathtt{U}_{\mathsf{P}}$, which we regard as a corepresentation of the object $A = \mathsf{P}$ in $\mathcal{E} = \mathcal{M}_{\mathsf{P}}$. We are going to use the identity $\mathtt{U}_{\mathsf{P}} = \mathsf{P}[1]$. We get a functor $\mathtt{R}_{\mathsf{P},\mathsf{P}[1]} : \Gamma^+_{\mathsf{P}} \to \mathcal{M}_{\mathsf{P}}$ towards the category of right P-modules $\mathcal{M}_{\mathsf{P}}$. We have the following observations:

2.4. **Proposition.**

(a) *We have an identity of functors in* $\underline{n}_+ \in \Gamma^+_{\mathsf{P}}$

$$\mathtt{R}_{A,\mathtt{U}_{\mathsf{P}}(A)}(\underline{n}_+) = \mathtt{R}_{\mathsf{P},\mathsf{P}[1]}(\underline{n}_+) \circ_{\mathsf{P}} A$$

for every $A \in {}_{\mathsf{P}}\mathcal{C}$*, where we consider the relative composition product with the object* $\mathtt{R}_{\mathsf{P},\mathsf{P}[1]}(\underline{n}_+)$ *assigned to* $\underline{n}_+ \in \Gamma^+_{\mathsf{P}}$ *in the category of right* P*-modules.*

(b) *The functor* $\mathtt{R}_{\mathsf{P},\mathsf{P}[1]}(-)(m) : \underline{n}_+ \mapsto \mathtt{R}_{\mathsf{P},\mathsf{P}[1]}(\underline{n}_+)(m)$*, defined by considering the terms of a fixed arity* $m \in \mathbb{N}$ *in the right* P*-modules* $\mathtt{R}_{\mathsf{P},\mathsf{P}[1]}(\underline{n}_+) \in \mathcal{M}$*, is*

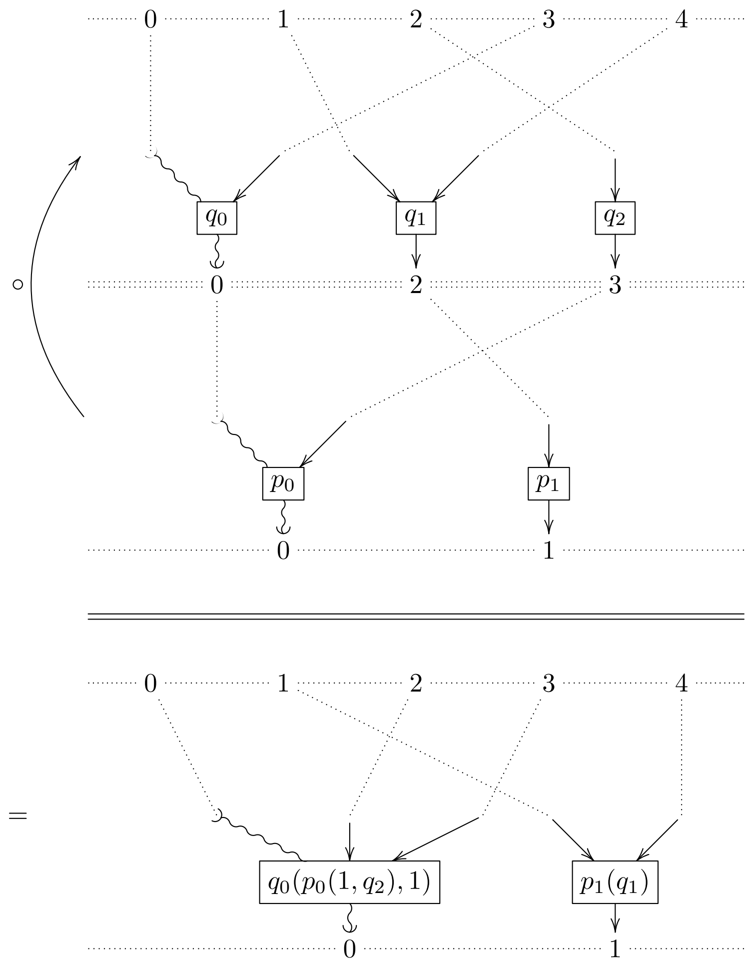

FIGURE 4.

identified with a representable functor:

$$\mathtt{R}_{P,P[1]}(-)(m) = \Gamma^+_P(\underline{m}_+, -)$$

which we associate to the object $\underline{m}_+$ in the category Γ^+_P.

Proof. □

We review the definition of the categorical bar construction before defining the functor homology counterpart of the operadic homology of §1.

2.5. *The categorical bar construction.* We assume that $\mathtt{F} : \Gamma^{op} \to dg\,\mathcal{M}od$ (respectively, $\mathtt{G} : \Gamma \to dg\,\mathcal{M}od$) is any contravariant (respectively, covariant) functor over a category enriched in dg-modules Γ. We will go back to the case of the category $\Gamma = \Gamma^+_P$ later on. We just assume that Γ has the pointed finite ordinals $\underline{n}_+$, $n \in \mathbb{N}$, as object for simplicity, and we denote the hom-objects of this category by $\Gamma(\underline{m}_+, \underline{n}_+)$, for every $m, n \in \mathbb{N}$.

The categorical bar construction $\mathtt{B}(\mathtt{F},\Gamma,\mathtt{G})$ is the simplicial object $\mathtt{B}_\bullet(\mathtt{F},\Gamma,\mathtt{G})$ such that:

$$(1)\quad \mathtt{B}_t(\mathtt{F},\Gamma,\mathtt{G}) = \bigoplus_{\underline{n}_{0+},\dots,\underline{n}_{t+}} \mathtt{F}(\underline{n}_{0+})\otimes\Gamma(\underline{n}_{1+},\underline{n}_{0+})\otimes\cdots\otimes\Gamma(\underline{n}_{t+},\underline{n}_{t-1+})\otimes\mathtt{G}(\underline{n}_{t+}),$$

for every dimension $t\in\mathbb{N}$. The face operators of this simplicial object d_i are given:

- by the right action of the hom-object $\Gamma(\underline{n}_{1+},\underline{n}_{0+})$ on the functor $\mathtt{F}$ for $i=0$
- by the composition operation of the category Γ on the $(i,i+1)$ homomorphism factors, for $i=1,\dots,t-1$,
- by the left action of the hom-object $\Gamma(\underline{n}_{1+},\underline{n}_{0+})$ on the functor $\mathtt{G}$ for $i=t$.

The degeneracy s_j are given by the insertion of an identity homomorphism id at the jth position of the tensor product $\mathtt{B}_{t+1}(\mathtt{F},\Gamma,\mathtt{G})$ for any $j=0,\dots,t$. The sum in our expression (1) runs over all $t+1$-tuples of objects $\underline{n}_{0+},\dots,\underline{n}_{t+}$.

The simplicial object $\mathtt{B}_\bullet(\mathtt{F},\Gamma,\mathtt{G})$ is equipped with an augmentation over the coend:

$$(2)\qquad \mathtt{F}\otimes_\Gamma\mathtt{G} = \int^{\underline{n}_+\in\Gamma}\mathtt{F}(\underline{n}_+)\otimes\mathtt{G}(\underline{n}_+).$$

We can also take the homology (of the normalized complex) of this simplicial object to compute $\mathtt{Tor}$-functors over the category Γ:

$$(3)\qquad \mathtt{Tor}^\Gamma_*(\mathtt{F},\mathtt{G}) = \mathtt{H}_*(\mathtt{B}_\bullet(\mathtt{F},\Gamma,\mathtt{G})).$$

We apply this definition to the category $\Gamma=\Gamma^+_P$ associated to an operad P, and to the functor $\mathtt{G}=\mathtt{R}_{A,M}$ associated to a P-algebra $A\in{}_P\mathcal{C}$ and a corepresentation $M\in\mathcal{R}^c_P(A)$. We can represent the tensors defining the bar complex $\mathtt{B}_\bullet(\mathtt{F},\Gamma^+_P,\mathtt{R}_{A,M})$ associated to these objects by arranging the diagrams of our homomorphisms $\phi_i\in\Gamma^+_P(\underline{n}_{i+},\underline{n}_{i-1+})$ on a series of levels. We label the root of this composite diagram by the factor $\alpha\in\mathtt{F}(\underline{n}_{0+})$ of the left coefficient of our complex, and the leaves by the tensor $\xi\otimes a_1\otimes\cdots\otimes a_n\in M\otimes A^{\otimes n}$ corresponding to the right coefficient $\mathtt{R}_{A,M}(\underline{n}_+)$. We give an example of this representation in the picture of Figure 5.

We now define the contravariant functor ω_P which we use to retrieve the operadic homology of §1 from this categorical bar complex.

2.6. *The functor of Kähler forms.* We take:

$$(1)\qquad \omega_P(\underline{n}_+) = \Omega_P(n),$$

for any object $\underline{n}_+$. We therefore identify the elements $\alpha\in\omega_P(\underline{n}_+)$ with Kähler forms $\alpha=\pi(x_1,\dots,dx_k,\dots,x_n)$, such that $\pi\in P(n)$ and $x_1,\dots,x_n$ represent abstract variables.

We use the right module structure of the object ω_P over the operad P and the left module structure over the algebra $P[1]$ to determine the contravariant action of the homomorphisms $\phi\in\Gamma^+_P(\underline{m}_+,\underline{n}_+)$ on $\omega_P(\underline{n}_+)$. We assume, to be explicit, that ϕ is given by the tensor product of a marked operation $p_0(-,x_i)_{i\in f^{-1}(0)-\{0\}}\in P(f^{-1}(0))$, which represents an element of the algebra $P[1]$, together with formal operations $p_j=p_j(x_i)_{i\in f^{-1}(j)}$, $j=1,\dots,n$. We get the action of this homomorphism ϕ on our Kähler form $\alpha=\pi(x_1,\dots,dx_k,\dots,x_n)$ by plugging the operations p_j, $j=1,\dots,n$, in the input variables $x_1,\dots,x_n$, and composing α on the left with the distinguished operation p_0. We therefore obtain an expression of the following

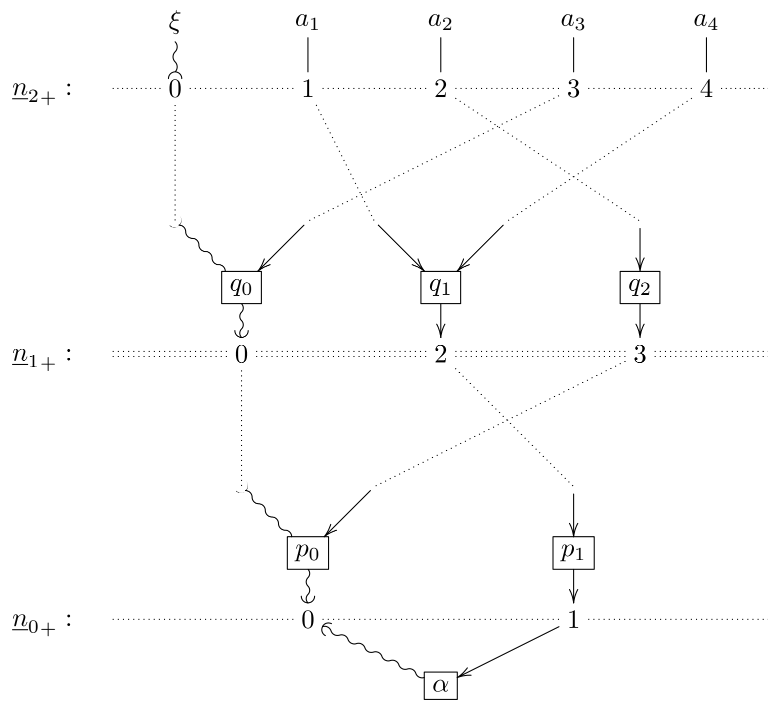

FIGURE 5.

form:

$$\text{(2)}\quad \phi^*(\pi(x_1,\dots,dx_i,\dots,x_n)) \\ = \pm p_0(\pi(p_1(x_i)_{f^{-1}(1)},\dots,dp_k(x_i)_{f^{-1}(k)},\dots,p_n(x_i)_{f^{-1}(n)}),x_i)_{f^{-1}(0)-\{0\}},$$

which we may still reduce by using the derivation relation in the right P-module Ω_P.

For instance, if we assume $\alpha = \pi(dx_1, x_2)$, and we consider the homomorphism $\phi \in \Gamma_P^+(\underline{4}_+, \underline{2}_+)$ depicted in Figure 3, then we get the following formula:

$$\text{(3)}\quad \begin{aligned} \phi^*(\pi(dx_1,x_2)) &= \pm p_0(\pi(dp_1(x_1),p_2(x_2,x_4)),x_3) \\ &= \pm p_0(\pi(p_1(dx_1),p_2(x_2,x_4)),x_3) \\ &= \pm p_0(\pi(p_1,p_2),1)(dx_1,x_2,x_4,x_3). \end{aligned}$$

We have the following proposition:

2.7. **Proposition.** *We have the identity:*

$$\omega_P \otimes_{\Gamma_P^+} \mathtt{R}_{A,M} = M \otimes_{\mathtt{U}_P(A)} \Omega_P(A),$$

for every P-algebra $A \in {}_P\mathcal{C}$, and every corepresentation $M \in \mathcal{R}^c_P(A)$.

Proof. □

We can easily extend the result of Proposition 2.4(a) to the categorical bar complex. We basically get an identity:

$$\mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{A,\mathtt{U}_P(A)}) = \mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{P,P[1]}) \circ_P A$$

for every P-algebra $A \in {}_P\mathcal{C}$, where we consider the functor towards the category right P-module $\mathtt{R}_{P,P[1]}$ associated to the operad P and the shifted object $P[1]$. We check that this categorical bar complex in right P-modules has the following properties:

2.8. **Proposition.** *The object $\mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{P,P[1]})$ is cofibrant as a right P-module, and is equipped with a weak-equivalence:*

$$\mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{P,P[1]}) \xrightarrow{\sim} \Omega_P$$

in the category of left $\mathtt{U}_P$-modules in right P-modules.

Proof. The first assertion readily follows from the characterization of cofibrant objects in the category of right P-modules in [4].

In the case $A = P$ and $M = P[1]$, the result of Proposition 2.7 gives $\omega_P \otimes_{\Gamma_P^+} \mathtt{R}_{P,P[1]} = P[1] \otimes_{P[1]} \Omega_P = \Omega_P$ and the augmentation of the categorical bar complex defines a morphism:

$$\epsilon : \mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{P,P[1]}) \to \Omega_P\, .$$

We aim to check that this morphism is a weak-equivalence.

We readily see that the result of Proposition 2.4(b) has an extension to the categorical bar complex, and that we have

$$\mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{P,P[1]})(m) = \mathtt{B}(\omega_P, \Gamma_P^+, \Gamma_P^+(\underline{m}_+, -))$$

when we consider a term of a fixed arity $m \in \mathbb{N}$, in this right P-module. We conclude that our augmentation ϵ defines a weak-equivalence arity-wise, by using that the categorical bar complex is acyclic on representable functors. □

We use the statement of Theorem 1.5 to get the universal coefficient case $M = \mathtt{U}_P(A)$ of Theorem B from the result of this proposition. We get the general case Theorem B by using the following observation:

2.9. **Proposition.** *We have an identity:*

$$\mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{A,M}) = M \otimes_{\mathtt{U}_P(A)} \mathtt{B}(\omega_P, \Gamma_P^+, \mathtt{R}_{A,\mathtt{U}_P(A)})$$

for every P-algebra A and every corepresentation $M \in \mathcal{R}^c_{\mathcal{P}}(A)$.

Proof. □

3. The trivial coefficient case

We can significantly simplifies our construction in the case where the operad P is equipped with an augmentation $\epsilon : P \to I$ and the coefficients are given by the trivial representation $M = \Bbbk$. We then get the result of Theorem A.

Outlook: the case of operads and algebras in spectra

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