ITERATED BAR COMPLEXES AND THE POSET OF PRUNED TREES

ADDENDUM TO THE PAPER: "ITERATED BAR COMPLEXES OF E-INFINITY ALGEBRAS AND HOMOLOGY THEORIES"

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ABSTRACT. The purpose of these notes is to explain the relationship between Batanin's categories of pruned trees and iterated bar complexes.

This article is an appendix of the article [4]. Our purpose is to explain the relationship between Batanin's categories of pruned trees (see [1, 2]) and iterated bar complexes and to revisit some constructions [4] in this formalism. The reader can use this appendix as an informal introduction to the constructions of [4].

These notes is the appendix part of a preliminary version of [4], extracted without changes from this article except that we have removed the appendix mark from paragraph numberings. Thus the reader can easily retrieve references to former versions of [4] in this manuscript.

1. *Level trees and sequences of non-decreasing surjections.* Our first aim is to make explicit the expansion of iterated tensor coalgebras $(T^c\Sigma)^n(M)$, for a connected Σ_*-module M. For this purpose, we use the category *Ord* formed by the non-empty ordered sets $\underline{\mathsf{r}} = \{1,\dots,r\}$, $r \in \mathbb{N}^*$, together with non-decreasing surjections as morphisms. The one-point set $\underline{1}$ forms a final object of *Ord* and we also use the notation $*$ to refer to this object.

For any finite set $\underline{\mathsf{e}}$ (not necessarily equipped with an ordering), we consider the set $\Pi^n(\underline{\mathsf{e}})$ defined by sequences of the form

$$\underline{\mathsf{e}} \xrightarrow{\rho_0} \underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n = \underline{1},$$

where ρ_0 is any surjective map and $\{\underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n\}$ is an n-simplex of the category *Ord* such that $\underline{\mathsf{r}}_n = \underline{1}$ is the final object. Equivalently, if we consider the category *Surj* of non-empty finite sets and surjections, then we have $\underline{\mathsf{e}} \in \mathit{Surj}$ and we can identify $\Pi^n(\underline{\mathsf{e}})$ to a subset of the n-dimensional simplices in the nerve of the comma category $\underline{\mathsf{e}} / \mathit{Ord}$, where we consider the obvious forgetful functor from *Ord* to *Surj*. In the sequel, we also consider the subset $\overline{\Pi}^n(\underline{\mathsf{e}}) \subset \Pi^n(\underline{\mathsf{e}})$ formed by the simplices $\underline{\rho} \in \Pi^n(\underline{\mathsf{e}})$ such that the initial arrow $\rho_0 : \underline{\mathsf{e}} \to \underline{\mathsf{r}}_0$ is a bijection.

An n-simplex $\underline{\rho} \in \Pi^n(\underline{\mathsf{e}})$ defines the structure of a planar tree with inputs indexed by $\underline{\mathsf{e}}$ and $n+1$ levels indexed by $0,\dots,n$. The ordered set $\underline{\mathsf{r}}_i$ gives the vertices of

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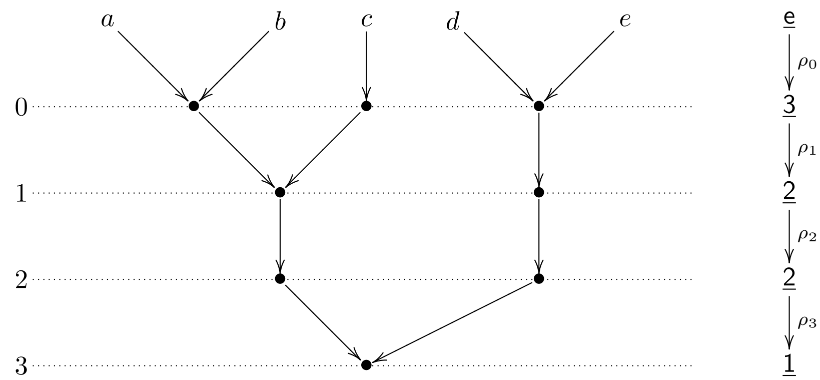


FIGURE 1.

level i of the tree. The surjections $\rho_i : \underline{r}_{i-1} \to \underline{r}_i$, $i = 1, \ldots, n$, determine the internal edges of the tree coming from vertices $u \in \underline{r}_{i-1}$ at level $i-1$ toward vertices $v \in \underline{r}_i$ at level i so that $\rho_i(u) = v$. The surjection $\rho_0 : \underline{e} \to \underline{r}_0$ defines the ingoing edges of the tree coming from inputs $s \in \underline{e}$ toward vertices $v \in \underline{r}_0$ at level 0 so that $\rho_0(s) = v$. An example is displayed in figure 1.

This basic representation is amended as follows: we remove the ingoing edges of the tree; to represent the input information, we prefer to label each vertex at level 0, equivalent to an element $i \in \underline{r}_0$, by the subsets $\underline{e}_i = \rho_0^{-1}(i)$; moreover, we omit superfluous information, like edge orientation and level indices (the marked levels always range from 0 to n, from top to bottom). As an example, the simplified representation of the tree of figure 1 is displayed in figure 2. The simplified representation reflects more properly the role of each element in our constructions.

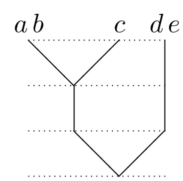


FIGURE 2.

To a simplex $\underline{\rho} = \{\underline{e} \xrightarrow{\rho_0} \underline{r}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{r}_n\} \in \Pi^n(\underline{e})$ we associate a degree $\deg(\underline{\rho}) = r_0 + r_1 + \cdots + r_{n-1}$, which represents the number of internal edges in the tree defined by $\underline{\rho}$.

In [4, §4], we use that the tensor coalgebra $T^c(\Sigma M)$ has an expansion of the form

$$T^c(\Sigma M)(\underline{e}) = \bigoplus_{\substack{\underline{e}_1 \amalg \cdots \amalg \underline{e}_r = \underline{e} \\ r \geq 1}} \Sigma M(\underline{e}_1) \otimes \cdots \otimes \Sigma M(\underline{e}_r).$$

Recall simply that this assertion is immediate from the definition of the tensor product of Σ_*-modules in [4, §0.3].

The sum ranges over all integers $r \in \mathbb{N}^*$ and all partitions $\underline{\mathsf{e}}_1 \amalg \cdots \amalg \underline{\mathsf{e}}_r = \underline{\mathsf{e}}$. If we assume $M(0) = 0$, then the sum reduces to partitions such that $\underline{\mathsf{e}}_i \neq \emptyset$, for every $i \in \{1, \dots, r\}$. Clearly, a partition of this form is equivalent to a surjection $\rho : \underline{\mathsf{e}} \to \underline{\mathsf{r}}$ such that $\underline{\mathsf{e}}_i = \rho^{-1}(i)$ and the summation set can be replaced by the set of all surjections $\rho : \underline{\mathsf{e}} \to \underline{\mathsf{r}}$.

This expansion of $T^c(\Sigma M)$ has the following straightforward generalization for iterated coalgebras:

2. **Proposition.** *Let M be any connected Σ_*-object. The terms of the n-fold tensor coalgebra $(T^c\Sigma)^n(M)$ have an expansion of the form*

$$(T^c\Sigma)^n(M)(\underline{\mathsf{e}}) = \bigoplus_{\underline{\rho}=\{\underline{\mathsf{e}}\xrightarrow{\rho_0}\underline{\mathsf{r}}_0\xrightarrow{\rho_1}\cdots\xrightarrow{\rho_n}\underline{\mathsf{r}}_n=*\}} \Sigma^{\deg(\underline{\rho})}\Big\{\bigotimes_{i\in\underline{\mathsf{r}}_0} M(\underline{\mathsf{e}}_i)\Big\},$$

for every finite set $\underline{\mathsf{e}}$, where the sum ranges over the simplices of $\Pi^n(\underline{\mathsf{e}})$, and we set $\underline{\mathsf{e}}_i = \rho_0^{-1}(i)$ for each simplex $\underline{\rho} = \{\underline{\mathsf{e}} \xrightarrow{\rho_0} \underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n = *\}$. □

In the particular case of the composition unit $M = I$, for which

$$I(r) = \begin{cases} \mathbb{k}, & \text{if } r = 1,\\ 0, & \text{otherwise,}\end{cases}$$

we obtain

$$T^n(\underline{\mathsf{e}}) = (T^c\Sigma)^n(I)(\underline{\mathsf{e}}) = \bigoplus_{\overline{\Pi}(\underline{\mathsf{e}})} \mathbb{k}\{\underline{\mathsf{e}} \xrightarrow{\simeq} \underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n = *\},$$

the free graded $\mathbb{k}$-module spanned by the simplices of $\Pi(\underline{\mathsf{e}})$ whose initial arrow $\underline{\mathsf{e}} \xrightarrow{\rho_0} \underline{\mathsf{r}}_0$ is bijective.

In the particular case of the commutative operad $M = \mathtt{C}$, for which

$$\mathtt{C}(\underline{\mathsf{e}}) = \mathbb{k}, \quad \forall\, \underline{\mathsf{e}} \neq \emptyset,$$

we obtain

$$T^n \circ \mathtt{C}(\underline{\mathsf{e}}) = (T^c\Sigma)^n(\mathtt{C})(\underline{\mathsf{e}}) = \bigoplus_{\Pi(\underline{\mathsf{e}})} \mathbb{k}\{\underline{\mathsf{e}} \xrightarrow{\rho_0} \underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n = *\},$$

the free graded $\mathbb{k}$-module spanned by $\Pi^n(\underline{\mathsf{e}})$. Throughout this appendix, we apply the convention to specify the generating operation of the dg-module $\mathtt{C}(\underline{\mathsf{e}})$ by the commutative word $e_1 \cdots e_r$ defined by the elements of $\underline{\mathsf{e}} = \{e_1, \dots, e_r\}$.

We can extend the tree representation to an iterated tensor coalgebra $(T^c\Sigma)^n(M)$ on any Σ_*-module M: label the vertices at level 0 by the corresponding factors $x_i \in M(\underline{\mathsf{e}}_i)$, $i \in \underline{\mathsf{r}}_0$, which occur in the expansion of an element $\xi \in (T^c\Sigma)^n(M)(\underline{\mathsf{e}})$.

As an example, in the case $M = \mathtt{C}$, the tree of figure 2 represents an element $x_0 \in (T^c\Sigma)^3(\mathtt{C})(\underline{\mathsf{e}})$ defined inductively by:

$$\begin{aligned}
&x_0 = y_1 \otimes y_2 \in (T^c\Sigma)^2(\mathtt{C})(\{a,b,c\}) \otimes (T^c\Sigma)^2(\mathtt{C})(\{d,e\}),\\
&y_1 = y_1 \in (T^c\Sigma)(\mathtt{C})(\{a,b,c\}),\\
&y_2 = y_2 \in (T^c\Sigma)(\mathtt{C})(\{d,e\}),\\
&z_1 = ab \otimes c \in \mathtt{C}(\{a,b\}) \otimes \mathtt{C}(\{c\}),\\
&z_2 = de \in \mathtt{C}(\{d,e\}).
\end{aligned}$$

Proposition 2 gives as an immediate corollary:

3. **Proposition.** *We have an isomorphism of functors on finite sets and bijections*
$$T^n(\underline{e}) = Bij(\{1,\dots,r\},\underline{e}) \otimes G^n(r),$$
where $G^n(r)$ denotes the free graded $\Bbbk$-module
$$G^n(r) = \bigoplus \Bbbk\{\underline{r} = \underline{r}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{r}_n = *\},$$
and T^n forms a free object in the category of graded Σ_*-modules. □

The object $G^n(r) \subset T^n(r)$ is identified with the generating collection of [4, Proposition 2.8].

4. *The representation of the $\mathcal{K}$-structure on $\Pi^n(\underline{e})$.* Recall that $\mathcal{K}(\underline{e})$, the $\underline{e}$-component of the complete graph operad $\mathcal{K}$, denotes a poset formed by pairs (μ,σ), where $\mu = \{\mu_{ef}\}_{ef}$ is a collection of non-negative integers $\mu_{ef} \in \mathbb{N}$ associated to each $\{e,f\} \subset \underline{e}$ and σ is a bijection $\sigma : \{1,\dots,r\} \to \underline{e}$ or equivalently, an ordering $\sigma = \{e_1 < \cdots < e_r\}$ of the set $\underline{e}$. The relation $(\mu,\sigma) \leq (\nu,\tau)$ holds in $\mathcal{K}(\underline{e})$ if, for all $\{e,f\} \subset \underline{e}$, we have $\mu_{ef} < \nu_{ef}$ or $(\mu_{ef},\sigma_{ef}) = (\nu_{ef},\tau_{ef})$ where we consider the restriction of the orderings σ and τ to the pair $\{e,f\}$.

The dg-modules $(T^n)_\kappa \subset T^n(r)$ introduced in [4, §4.2] to provide the Σ_*-object T^n with a $\mathcal{K}$-structure have an intuitive geometric definition. Let $\underline{\rho}$ be a simplex of the form
$$\underline{e} \xrightarrow[\simeq]{\rho_0} \underline{r}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{r}_n = *,$$
where ρ_0 is a bijection, so that $\underline{\rho}$ represents an element of $T^n(\underline{e})$.

The bijection $\rho_0^{-1} : \underline{r}_0 \to \underline{e}$, inverse to ρ_0, determines an ordering $\sigma(\underline{\rho}) = (\rho_0^{-1}(1),\dots,\rho_0^{-1}(r))$ of the elements of $\underline{e}$. For elements $e,f \in \underline{e}$, we define $\mu_{ef}(\underline{\rho})$ as the largest m such that $\rho_m \dots \rho_0(e) \neq \rho_m \dots \rho_0(f)$. In the tree representation of $\underline{\rho}$, this definition amounts to saying that the edge-paths joining e and f to the root of the tree meet at level $\mu_{ef}(\underline{\rho}) + 1$.

As an example, the next tree is associated to the ordering $\sigma(\underline{\rho}) = (i,j,k)$ and the weights $\mu_{ij}(\underline{\rho}) = 0$, $\mu_{jk}(\underline{\rho}) = 2$, $\mu_{ki}(\underline{\rho}) = 2$.

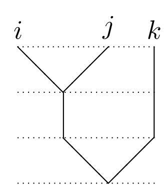


FIGURE 3.

If we check the definition of the dg-module $(T^n)_\kappa$ in [4, §4.2], then we readily obtain:

5. **Proposition.** *The element $\kappa(\underline{\rho}) = (\mu,\sigma) \in \mathcal{K}(\underline{e})$ defined by the collection $\mu = \{\mu_{ef}(\underline{\rho})\}_{ef}$ and the ordering $\sigma = \sigma(\underline{\rho})$ represents the lowest element $\kappa \in \mathcal{K}(\underline{e})$ such that $\underline{\rho} \in (T^n)_\kappa$.* □

In other words, we have $\underline{\rho} \in (T^n)_\kappa$ if and only if $\kappa \geq \kappa(\underline{\rho})$. Thus this proposition gives an alternative definition of the dg-module $(T^n)_\kappa$, for which we immediately have $\kappa \leq \lambda \Rightarrow (T^n)_\kappa \subset (T^n)_\lambda$.

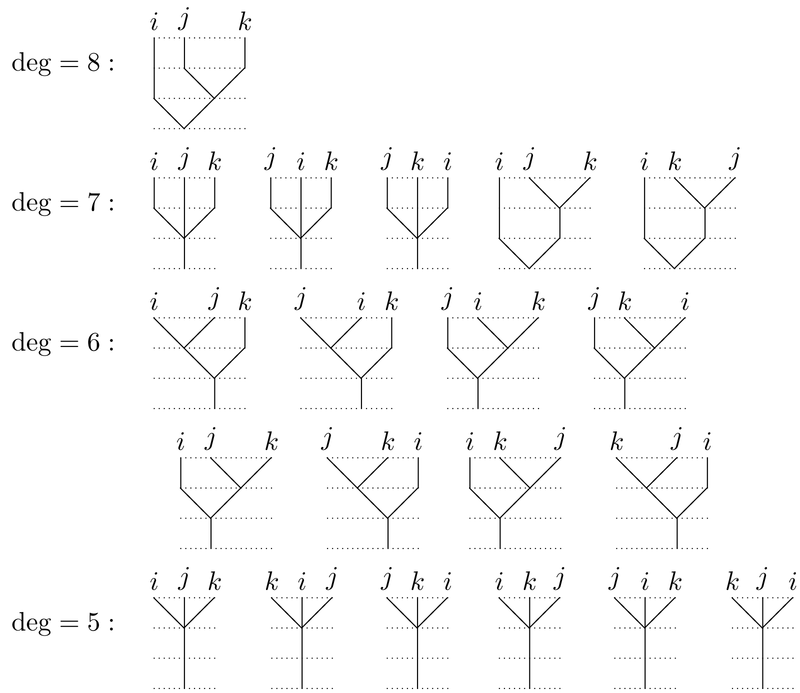


FIGURE 4.

As an illustration, we display in figure 4 the elements of the cell $(T^n)_\kappa \subset T^n(\{i,j,k\})$ such that $\mu_{ij} = \mu_{ik} = 2$, $\mu_{jk} = 1$ and $\sigma = (i,j,k)$.

Recall that the commutative operad $\mathtt{C}$ is equipped with a constant $\mathcal{K}$-structure such that $\mathtt{C}_\kappa = \mathtt{C}(r)$, for all $\kappa \in \mathcal{K}(r)$. In [4, §4.4], we use the composition structure of the posets $\mathcal{K}(r)$ to provide the right $\mathtt{C}$-module $T^n \circ \mathtt{C}$ with a $\mathcal{K}$-structure. If we check the definition of this composition structure, then we immediately obtain that the tree of figure 2 belongs to the cells $(T^3 \circ \mathtt{C})_\kappa$ such that

$$\begin{aligned}
&\mu_{ab} = \mu_{de} = 0\\
&\mu_{ac} = \mu_{bc} = 0\\
&\mu_{cd} = \mu_{ce} = 2\\
&\mu_{da} = \mu_{db} = \mu_{ea} = \mu_{eb} = 2\\
&\text{and}\quad \sigma = (a,b,c,d,e) \quad\text{or}\quad \sigma = (b,a,c,d,e) \quad\text{or}\quad \sigma = (a,b,c,e,d) \quad \dots
\end{aligned}$$

(recall that the order between (a,b) and (d,e) does not matter in $\mathtt{C}$).

6. *Morphisms of non-decreasing surjections.* Our next aim is to give a representation of the differential of the iterated bar module $B^n_{\mathtt{C}}$ in terms of trees. To make this representation more conceptual, we provide the set $\Pi^n(\underline{\mathsf{e}})$ with a category structure.

Define a morphism $f : \underline{\rho} \to \underline{\sigma}$ between simplices

$$\underline{\rho} = \{\underline{\mathsf{e}} \xrightarrow{\rho_0} \underline{\mathsf{r}}_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_n} \underline{\mathsf{r}}_n = *\} \quad\text{and}\quad \underline{\sigma} = \{\underline{\mathsf{e}} \xrightarrow{\sigma_0} \underline{\mathsf{s}}_0 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_n} \underline{\mathsf{s}}_n = *\},$$

as a commutative diagram

$$\begin{array}{ccccccccc}
\underline{e} & \xrightarrow{\rho_0} & \underline{r}_0 & \xrightarrow{\rho_1} & \underline{r}_1 & \xrightarrow{\rho_2} \cdots & \xrightarrow{\rho_n} & * \\
{=}\downarrow & & \vdots\, f_0 & & \vdots\, f_1 & & & \downarrow{=} \\
\underline{e} & \xrightarrow{\sigma_0} & \underline{s}_0 & \xrightarrow{\sigma_1} & \underline{s}_1 & \xrightarrow{\sigma_2} \cdots & \xrightarrow{\sigma_n} & *
\end{array}$$

so that each map f_m, $m = 0, \ldots, n-1$, is surjective and non-decreasing on fibers $\rho_{m+1}^{-1}(i) \subset \underline{r}_m$, for $i \in \underline{r}_{m+1}$.

An example of morphism is displayed in figure 5.

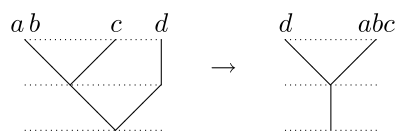


FIGURE 5.

As every map ρ_i, σ_i is supposed to be surjective, an immediate induction shows that each map f_i, $i = 0, \ldots, n$, is completely determined from the previous one f_{i-1} by the commutativity of the diagram

$$\begin{array}{ccc}
\underline{r}_{i-1} & \xrightarrow{\rho_i} & \underline{r}_i \\
f_{i-1}\,\vdots & & \vdots\, f_i \\
\underline{s}_{i-1} & \xrightarrow{\sigma_i} & \underline{s}_i
\end{array},$$

where we adopt the convention $\underline{r}_{-1} = \underline{e}$ and $f_{-1} = \mathrm{id}$ for $i = -1$. Thus we have:

7. **Proposition.** *The category $\Pi^n(\underline{e})$ forms a poset.* □

8. *The covering relation for non-decreasing surjections.* To follow usual conventions, we say that an element $\underline{\rho}$ covers $\underline{\sigma}$, and we write $\underline{\rho} \succ \underline{\sigma}$, if we have $\underline{\rho} > \underline{\sigma}$ in $\Pi^n(\underline{e})$ and there is no element $\underline{\tau} \in \Pi^n(\underline{e})$ such that $\underline{\rho} > \underline{\tau} > \underline{\sigma}$.

Observe that a relation $\underline{\rho} > \underline{\sigma}$ can be refined into a chain $\underline{\rho} = \underline{\rho}_0 > \cdots > \underline{\rho}_d = \underline{\sigma}$ such that $\underline{\rho}_i$ is obtained by merging two vertices of $\underline{\rho}_{i-1}$. As an example, we display all refinements of the morphism of figure 5 in figure 6.

Note that the relation $\underline{\rho}_i > \underline{\rho}_{i-1}$ can obviously not be refined. Hence, we conclude that $\underline{\sigma}$ is covered by $\underline{\rho}$ if and only if $\underline{\sigma}$ is obtained by merging two vertices of $\underline{\rho}$.

Recall that the degree of an element $\underline{\sigma} \in \Pi^n(\underline{e})$ represents the number of internal edges in the tree associated to $\underline{\sigma}$. Since there is a bijection between internal edges and vertices at level $0, \ldots, n-1$, we obtain that $\underline{\sigma}$ is covered by $\underline{\rho}$ if and only if $\deg(\underline{\sigma}) = \deg(\underline{\rho}) - 1$. As an illustration, we display in figure 7 the set of all trees $\underline{\sigma}$ such that $\underline{\rho} \succ \underline{\sigma}$ for a given tree $\underline{\rho}$.

The observations of this paragraph are summarized by the following statement:

9. **Proposition.** *The poset $\Pi^n(\underline{e})$ is graded by the degree $\deg : \Pi^n(\underline{e}) \to \mathbb{N}$ in the sense that a chain $\underline{\rho}_0 > \cdots > \underline{\rho}_d$ is maximal in $\Pi^n(\underline{e})$ if and only if $d = \deg(\underline{\rho}_0) - \deg(\underline{\rho}_d)$.* □

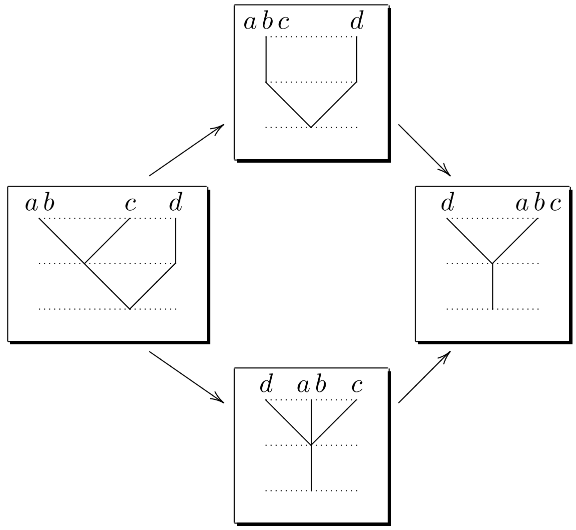


FIGURE 6.

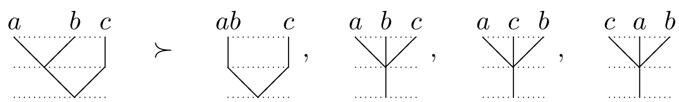


FIGURE 7.

Recall that the iterated bar module $B^n_{\mathtt{C}}$ is defined by the n-fold application of the bar construction of commutative algebras to the commutative operad $\mathtt{C}$, which is viewed as a commutative algebra in the category of right $\mathtt{C}$-modules (see [4, §1]). Hence, we obtain that the module $B^n_{\mathtt{C}}$ is a twisted object of the form $B^n_{\mathtt{C}} = ((T^c\Sigma)^n(\mathtt{C}), \partial_\gamma)$ where ∂_γ is a twisting homomorphism determined by the differentials of the iterated bar construction B^n (see again [4, §1] for details). Observe that a merging of vertices at level $m = 1, \ldots, n-1$ in a tree $\underline{\rho} \in \Pi(\underline{\mathtt{e}})$ corresponds to a shuffle product on the iterated tensor coalgebra $(T^c\Sigma)^m(\mathtt{C})$, and hence to a differential at the $(m+1)$th-stage of iteration of the bar construction. Similarly, a merging of vertices at level $m = 0$ corresponds to a product $\mu : \mathtt{C}(\underline{\mathtt{e}}_i) \otimes \mathtt{C}(\underline{\mathtt{e}}_{i+1}) \to \mathtt{C}(\underline{\mathtt{e}}_i \amalg \underline{\mathtt{e}}_{i+1})$ and hence to a differential at the 1st-stage of iteration of the bar construction. Thus we obtain:

10. **Proposition.** *The differential $\partial_\gamma : T^n \circ \mathtt{C}(\underline{\mathtt{e}}) \to T^n \circ \mathtt{C}(\underline{\mathtt{e}})$ is defined on simplices $\underline{\rho} \in \Pi^n(\underline{\mathtt{e}})$ by the formula:*

$$\partial_\gamma(\underline{\rho}) = \sum_{\underline{\rho} \succ \underline{\sigma}} \pm \underline{\sigma},$$

where the sum ranges over the simplices $\underline{\sigma} \in \Pi^n(\underline{\mathtt{e}})$ which are covered by $\underline{\rho}$. □

To determine the signs $\pm$, the easiest is to associate the suspension symbols in the expansion of $T^n \circ \mathtt{C}(\underline{\mathtt{e}})$ to internal edges of the tree. The internal edges are canonically ordered from bottom to top, and from left to right. The sign is determined by the commutation of the merging operation (which has degree -1)

with the edges and by the permutation of edges in the shuffle operation. To illustrate proposition 10, we have displayed the differential of the tree of figure 7 in figure 8.

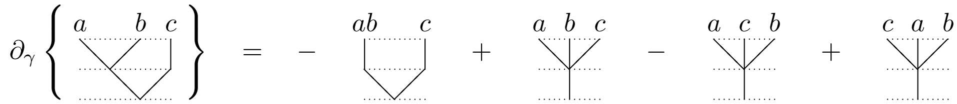


FIGURE 8.

11. *Remark.* In the definition of the poset $\Pi^n(\underline{e})$, we can allow all non-decreasing mappings $\rho : \underline{r} \to \underline{s}$ instead of non-decreasing surjections only. This variant of $\Pi^n(\underline{e})$ is identified with a large category of trees $\underline{\Pi}^n(\underline{e})$, used in [1, 2] to define of a categorical model for E_n-operads. Note that the enlarged category $\underline{\Pi}^n(\underline{e})$ does not form a poset, unlike $\Pi^n(\underline{e})$.

The poset $\Pi^n(\underline{e})$ and the enlarged category $\underline{\Pi}^n(\underline{e})$ occur in several other references of the literature (we refer to the bibliography of [1]).

Let A be a unital commutative algebra. The quotient $A/\Bbbk$ forms a non-unital algebra associated to A. The bar complex $B(A/\Bbbk)$ defined in [4, §1.5] and used throughout [4] can be identified with the normalization $N_*(\underline{B}(A))$ of a simplicial bar complex associated to A. The n-fold simplicial bar complex $\underline{B}^n(A)$ is associated to a right $\mathtt{C}$-module $\underline{B}^n_{\mathtt{C}}$ shaped on the large category $\underline{\Pi}^n(\underline{e})$.

12. *Extension of the bar differential to the Barratt-Eccles operad.* To complete this appendix, we study the lifting of the twisting homomorphism $\partial_\gamma : T^n \circ \mathtt{C} \to T^n \circ \mathtt{C}$ to the iterated tensor coalgebra on the Barratt-Eccles operad.

Recall that the Barratt-Eccles dg-operad (see [3]) is defined by the normalized complexes $\mathtt{E}(r) = N_*(E\Sigma_r)$ spanned in degree d by the d-simplices of permutations $(w_0, \ldots, w_d)$ divided out by the module of degenerate simplices $(w_0, \ldots, w_j, w_j, \ldots, w_d)$. As usual in this article, we specify a permutation $w \in \Sigma_r$, or an ordering $w : \{1, \ldots, r\} \to \underline{e}$, by the sequence of its values $w(1) \ldots w(r)$. In the graphical presentation of this appendix, we write simplices of permutations $(w_0, \ldots, w_d)$ vertically

$$\begin{matrix} w_0(1) \ldots w_0(r) \\ \vdots \qquad \vdots \\ w_d(1) \ldots w_d(r) \end{matrix}$$

and we represent the elements of $(T^c\Sigma)^n(\mathtt{E})$ by n-level planar trees with inputs decorated by simplices of permutations:

The augmentation $\epsilon : \mathtt{E} \to \mathtt{C}$ is the morphism which vanishes in degree $d > 0$ and simply forgets the ordering of permutations in degree $d = 0$. The section $\iota : \mathtt{C} \to \mathtt{E}$, considered in [4, §1.4], maps any commutative word $e_1 \cdots e_r \in \mathtt{C}(\underline{e})$, where $\underline{e} = \{e_1, \ldots, e_r\}$ is any set equipped with an ordering $e_1 < \cdots < e_r$, to the sequence $\iota(e_1 \cdots e_r) = e_1 \cdots e_r$ defined by the ordering of $\underline{e}$. The chain-homotopy $\nu : \mathtt{E}(\underline{e}) \to \mathtt{E}(\underline{e})$ simply inserts the ordered sequence $e_1 \cdots e_r$ at the bottom level of simplices.

The element $\mu = (\mathrm{id}) \in \mathtt{E}(2)$, where id is the identity permutation, satisfies the relations of an associative product operation and determines a morphism $\eta : \mathtt{A} \to \mathtt{E}$

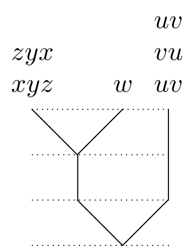
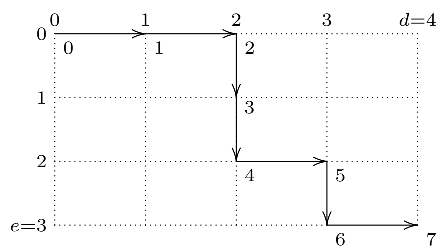


FIGURE 9.

lifting the usual morphism $\alpha : \mathtt{A} \to \mathtt{C}$ from the associative operad $\mathtt{A}$ to the commutative operad $\mathtt{C}$. The composite $\mu((u_0, \dots, u_d), (v_0, \dots, v_e))$, where $(u_0, \dots, u_d) \in \mathtt{E}(r), (v_0, \dots, v_e) \in \mathtt{E}(s)$, is represented by words $(u_{i_0} v_{j_0}, \dots, u_{i_{d+e}} v_{j_{d+e}})$, where $u_{i_*} v_{j_*}$ is just the concatenation of the orderings u_{i_*} and v_{j_*} and

$$(i_0, j_0) \to (i_1, j_1) \to \cdots \to (i_{d+e}, j_{d+e})$$

ranges over paths of the form



in the diagram $(0, \dots, d) \times (0, \dots, e)$. The refer can check [3] for a comprehensive definition of the composition structure of the Barratt-Eccles operad in the dg-setting. In the next examples of computation, we only need the definition of these particular composites $\mu((u_0, \dots, u_d), (v_0, \dots, v_e))$.

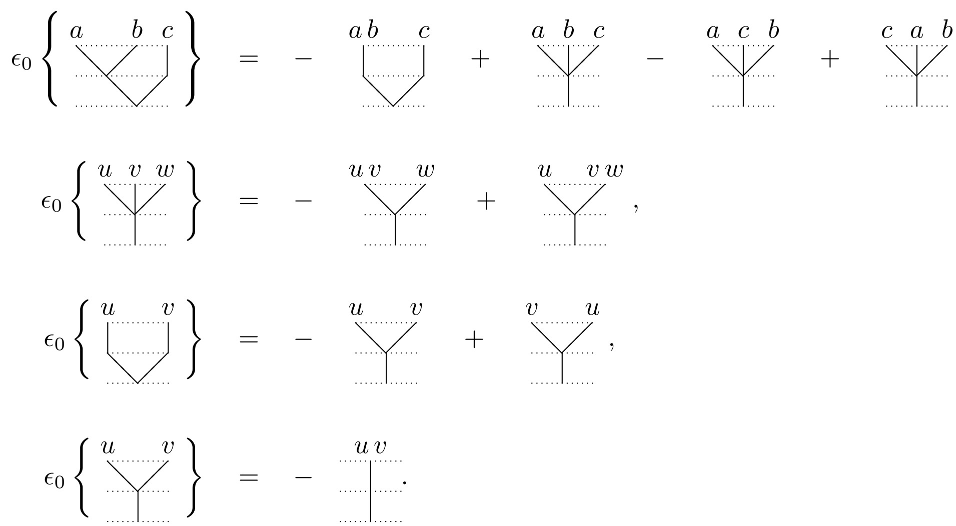
In [4, §2], we explain how to lift the homomorphism $\partial_\gamma : T^n \circ \mathtt{C} \to T^n \circ \mathtt{C}$ to a map $\partial_\epsilon : T^n \circ \mathtt{E} \to T^n \circ \mathtt{E}$ determining the twisting homomorphism of an iterated bar module $B^n_{\mathtt{E}} = (T^n \circ \mathtt{E}, \partial_\epsilon)$. The main purpose of this paragraph is to explain this construction on examples.

In [4, §2.4], we define the homomorphism of right $\mathtt{E}$-modules $\partial_\epsilon : T^n \circ \mathtt{E} \to T^n \circ \mathtt{E}$ as the extension (with respect to right $\mathtt{E}$-actions) of a homomorphism of collections $\epsilon_* : G^n(r) \to (T^c\Sigma)^n(\mathtt{E})(r)$ whose terms $\epsilon_* = \sum_m \epsilon_m$ are determined inductively by the formulas

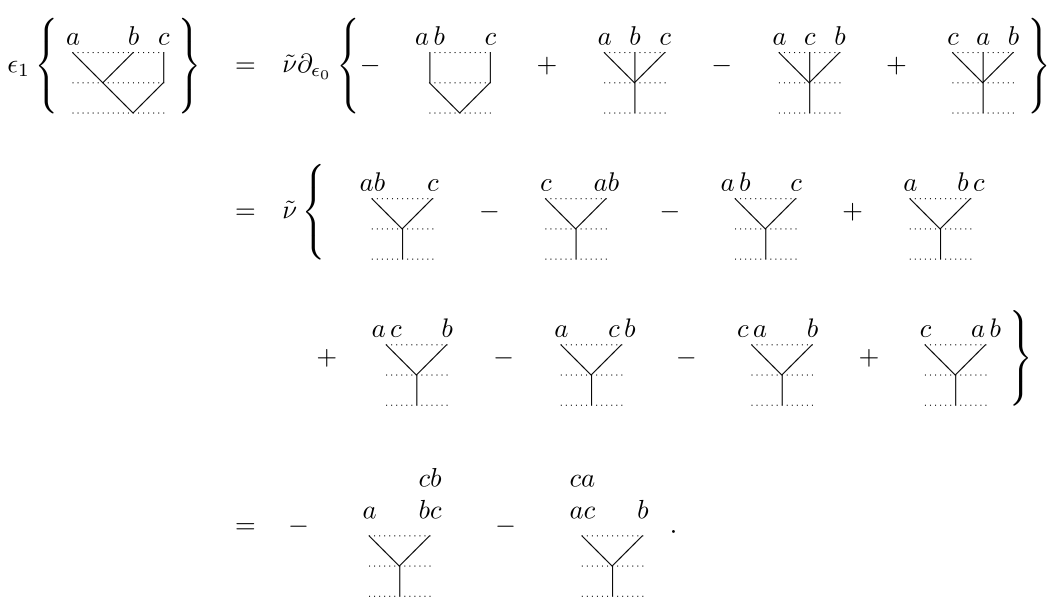
$$\epsilon_0 = \tilde{\iota}\gamma \quad \text{and} \quad \epsilon_m = \sum_{p+q=m-1} \tilde{\nu}(\partial_{\epsilon_p} \cdot \epsilon_q),$$

where we already consider the homomorphisms of right $\mathtt{E}$-modules $\partial_{\epsilon_p} : T^n \circ \mathtt{E} \to T^n \circ \mathtt{E}$ extending $\epsilon_p : G^n \to (T^c\Sigma)^n(\mathtt{E})$, for $p < m$. In this expression, the notation $\tilde{\iota}$, respectively $\tilde{\nu}$, refers to the usual extension of the section ι, respectively contracting homotopy ν, to the tensor power $\mathtt{E}^{\otimes r}(\underline{\mathtt{r}}) = \bigoplus \mathtt{E}(\underline{\mathtt{e}}_1) \otimes \cdots \otimes \mathtt{E}(\underline{\mathtt{e}}_r)$, where each component $\underline{\mathtt{e}}_i$ inherits its ordering from the source set $\underline{\mathtt{r}} = \{1 < \cdots < r\}$. Basically, the section ι is applied to all factors of the tensor product in the definition of $\tilde{\iota}$, and in the definition of $\tilde{\nu}$, the contracting homotopy ν is applied to the first factor of degree $d > 0$ while the first factors of degree $d = 0$, which are equivalent to permutations, are simply reordered by an application of the map $\iota\epsilon$. In [4, §2.4], we use the notation $\alpha \triangleright \beta$ to represent a composite of the form $\partial_\alpha \cdot \beta$ occuring in this expression.

For trees occurring in figure 8, we obtain the initial terms:

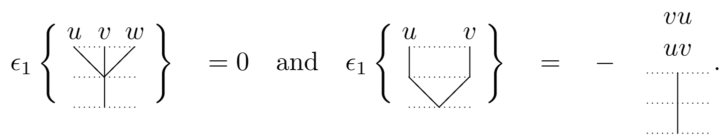


For $\epsilon_1 = \tilde{\nu}(\partial_{\epsilon_0} \cdot \epsilon_0)$, we obtain:

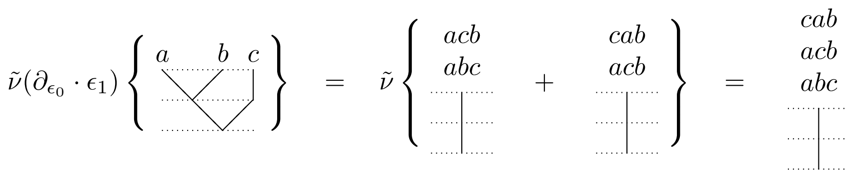


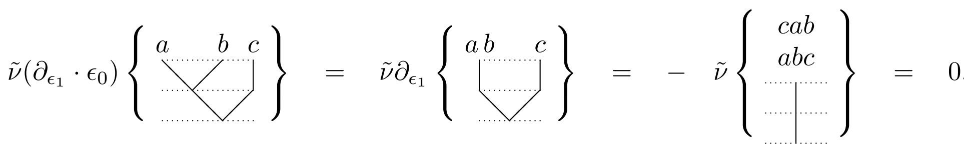
The contracting homotopy $\tilde{\nu}$ is defined relatively to the ordering of variables on the source $\{a < b < c\}$ and we apply operadic substitutions to determine ∂_{ϵ_0} from our previous computations. Terms associated to degenerate simplices, which represent 0 in the Barratt-Eccles operad, are removed from the final result.

We have similarly:

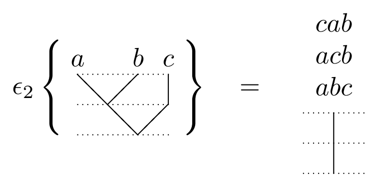


For $\epsilon_2 = \tilde{\nu}(\partial_{\epsilon_1} \cdot \epsilon_0 + \partial_{\epsilon_0} \cdot \epsilon_1)$, we obtain the terms:

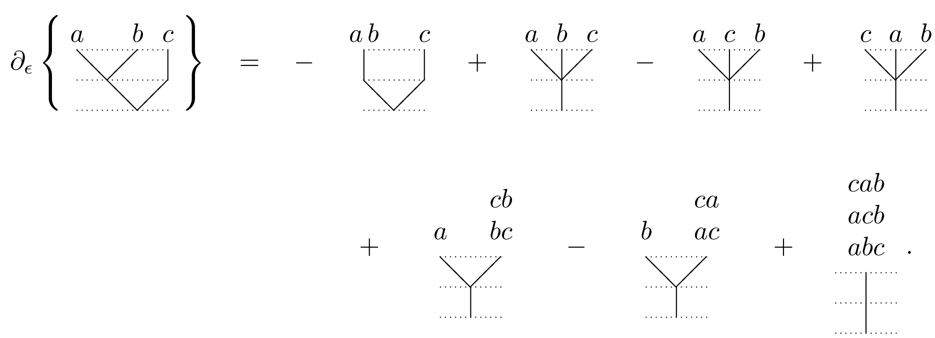




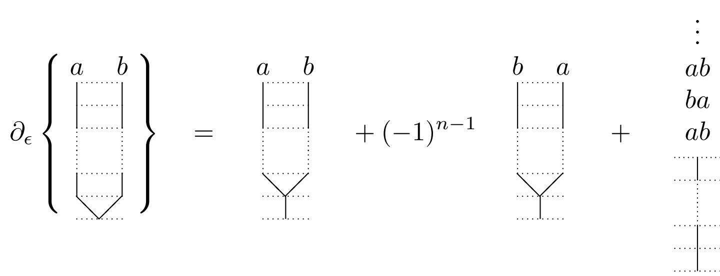
Thus we have:



Finally, the image of the tree of figure 8 under the twisting homomorphism ∂_ϵ is given by:



As another example, we easily obtain:



The simplex $\upsilon_n = (ab, ba, ab, \dots)$ is a representative of the $\cup_n$-product in the Barratt-Eccles operad. Hence we recover the formula used in [4, Lemma 8.11].

13. *Remark: Liftings of the bar differential to E_∞-operads.* In the expansions of §§2-3, the homomorphism ϵ_* has a component

$$\Bbbk\{\underline{\mathsf{r}} = \underline{\mathsf{r}}_0 \to \cdots \to \underline{\mathsf{r}}_n\} \xrightarrow{\epsilon_*(\underline{\rho},\underline{\sigma})} \Bbbk\{\underline{\mathsf{r}} \to \underline{\mathsf{s}}_0 \to \cdots \to \underline{\mathsf{s}}_n\} \otimes \Big\{\bigotimes_{j\in\underline{\mathsf{s}}_0} \mathtt{E}(\underline{\mathsf{f}}_j)\Big\},$$

for every simplex of the form $\underline{\rho} = \{\underline{\mathsf{r}} = \underline{\mathsf{r}}_0 \to \underline{\mathsf{r}}_1 \to \dots \underline{\mathsf{r}}_n = *\}$ and every simplex $\underline{\sigma} = \{\underline{\mathsf{r}} \xrightarrow{\sigma} \underline{\mathsf{s}}_0 \to \cdots \to \underline{\mathsf{s}}_n\} \in \Pi^n(\underline{\mathsf{r}})$, where $\underline{\mathsf{f}}_j = \sigma^{-1}(j)$, $j \in \underline{\mathsf{s}}_0$. The associated twisting homomorphism has components

$$\Bbbk\{\underline{\mathsf{e}} \to \underline{\mathsf{r}}_0 \to \cdots \to \underline{\mathsf{r}}_n\} \otimes \Big\{\bigotimes_{i\in\underline{\mathsf{r}}_0} \mathtt{E}(\underline{\mathsf{e}}_i)\Big\} \xrightarrow{\partial_*(\underline{\rho},\underline{\sigma})} \Bbbk\{\underline{\mathsf{e}} \to \underline{\mathsf{s}}_0 \to \cdots \to \underline{\mathsf{s}}_n\} \otimes \Big\{\bigotimes_{j\in\underline{\mathsf{s}}_0} \mathtt{E}(\underline{\mathsf{f}}_j)\Big\}$$

formed by composing the operations $p_i \in \mathtt{E}(\underline{\mathsf{e}}_i)$ to the result of the homomorphisms

$$\Bbbk\{\underline{\mathsf{r}}_0 = \underline{\mathsf{r}}_0 \to \cdots \to \underline{\mathsf{r}}_n\} \xrightarrow{\epsilon_*(\underline{\rho},\underline{\sigma})} \Bbbk\{\underline{\mathsf{r}}_0 \to \underline{\mathsf{s}}_0 \to \cdots \to \underline{\mathsf{s}}_n\} \otimes \Big\{\bigotimes_{j\in\underline{\mathsf{s}}_0} \mathtt{E}(\underline{\mathsf{g}}_j)\Big\},$$

for every pair of simplices $(\underline{\rho}, \underline{\sigma})$ such that the diagram

$$\begin{array}{ccc} \underline{\mathsf{e}} & \longrightarrow & \underline{\mathsf{s}}_0 \\ & \searrow \quad \nearrow & \\ & \underline{\mathsf{r}}_0 & \end{array}$$

commutes.

The inductive construction of [4, §2.4] implies readily that $\epsilon_*(\underline{\rho}\ \underline{\sigma})$ vanishes unless $\underline{\rho} \geq \underline{\sigma}$. Indeed, this vanishing property holds for the homomorphism γ which determines the twisting homomorphism of $B^n_{\mathtt{C}}$. Therefore we obtain that the lifting $\epsilon_0(\underline{\rho}, \underline{\sigma}) = \tilde{\iota}\gamma(\underline{\rho}, \underline{\sigma})$ has only components such that $\underline{\rho} \geq \underline{\sigma}$ and so do the next terms $\epsilon_m(\underline{\rho}, \underline{\sigma})$, because $\epsilon_m(\underline{\rho}, \underline{\sigma})$ solves an equation of the form:

$$\delta(\epsilon_m(\underline{\rho}, \underline{\sigma})) = \sum_{\underline{\rho} > \underline{\tau} > \underline{\sigma}} \epsilon_p(\underline{\sigma}, \underline{\tau}) \triangleright \epsilon_q(\underline{\rho}, \underline{\sigma}),$$

where δ refers to the internal differential of $\mathtt{E}$. In the construction of [4, §2.4], we just apply a chain-homotopy on the right hand side to solve this equation. Recall that $\alpha \triangleright \beta$ is a notation for composites of the form $\partial_\alpha \cdot \beta$. The degrees satisfy the relations $m = \deg(\underline{\rho}) - \deg(\underline{\sigma}) - 1$, $p = \deg(\underline{\rho}) - \deg(\underline{\tau}) - 1$ and $q = \deg(\underline{\tau}) - \deg(\underline{\sigma}) - 1$.

In light of the obtained equation, the homomorphism ϵ_* represents a kind of twisting cochain with coefficients in $\mathtt{E}$ on the poset $\Pi^n(\underline{\mathsf{r}})$. Formally, the observations of this paragraph imply that the n-fold bar complex $B^n(A)$ of an $\mathtt{E}$-algebra A forms a diagram over a certain cofibrant replacement of Batanin's category of pruned trees - we refer to the follow up [5] for an explanation of this assertion.

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