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Koszul duality of $\mathrm{E_n}$-operads

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Abstract. The goal of this paper is to prove a Koszul duality result for E_n-operads in differential graded modules over a ring. The case of an E_1-operad, which is equivalent to the associative operad, is classical. For $n > 1$, the homology of an E_n-operad is identified with the n-Gerstenhaber operad and forms another well known Koszul operad. Our main theorem asserts that an operadic cobar construction on the dual cooperad of an E_n-operad $\mathtt{E}_n$ defines a cofibrant model of $\mathtt{E}_n$. This cofibrant model gives a realization at the chain level of the minimal model of the n-Gerstenhaber operad arising from Koszul duality.

Most models of E_n-operads in differential graded modules come in nested sequences $\mathtt{E}_1 \subset \mathtt{E}_2 \subset \cdots \subset \mathtt{E}_\infty$ homotopically equivalent to the sequence of the chain operads of little cubes. In our main theorem, we also define a model of the operad embeddings $\mathtt{E}_{n-1} \hookrightarrow \mathtt{E}_n$ at the level of cobar constructions.

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Introduction

This work is concerned with E_n-operads in differential graded modules. In that context, an E_n-operad refers to a differential graded operad weakly-equivalent to the chain operad of Boardman-Vogt' little n-cubes.

The topological little cubes operads have been introduced at the origin of the theory of operads in homotopy theory [7, 30]. Since then, many actions of E_n-operads have been discovered in algebra and the idea is now well established that the notion of an E_n-operad gives the right device to understand the degree of commutativity of a multiplicative structure. To give only one reference motivating our study, we cite [24] for a comprehensive account of a program connecting E_n-operads, topological field theories and motivic

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Galois groups. To be more specific, this reference hints for the existence of strong connections between the groups of homotopy automorphisms of E_n-operads and the Grothendieck-Teichmüller group (see also [35] for results in that direction in the characteristic zero setting). The proper definition of homotopy automorphism groups, like many constructions of homotopical algebra, supposes to deal with cofibrant objects. The aim of the present article is to determine the structure of particular cofibrant models of E_n-operads, for any choice of ground ring $\Bbbk$ – in what follows, we fix $\Bbbk = \mathbb{Z}$.

The bar duality of operads [15, 16] shows that any augmented differential graded operad $\mathtt{P}$ has a quasi-free model given by a cobar construction $\mathtt{B}^c(\mathtt{D})$ on a cooperad $\mathtt{D}$ determined by $\mathtt{P}$ up to weak-equivalence. The cofibrant models of E_n-operads which we determine in this article are precisely given by a construction of this form $\mathtt{B}^c(\mathtt{D}_n)$, where $\mathtt{D}_n = \Lambda^{-n}\,\mathtt{E}_n^\vee$ is the n-fold operadic desuspension of the dual cooperad in $\mathbb{Z}$-modules of a certain E_n-operad $\mathtt{E}_n$, the notation Λ referring to the operadic suspension operation (a twisted degree shift) and $(-)^\vee$ to the duality of $\mathbb{Z}$-modules. (The definition of this model is explained with full details in §§0.2-0.4.)

The notion of Koszul operad, introduced in [16], refers to certain good operads $\mathtt{P}$ for which we have a cooperad $\mathtt{K}(\mathtt{P})$, the Koszul dual of $\mathtt{P}$, such that the cobar construction $\mathtt{B}^c(\mathtt{K}(\mathtt{P}))$ defines a minimal quasi-free model of $\mathtt{P}$. Recall simply that the existence of a minimal quasi-free model of this form forces the operad $\mathtt{P}$ to have a quadratic presentation which fully determines the cooperad $\mathtt{K}(\mathtt{P})$. The operad of commutative algebras $\mathtt{C}$ and the operad of Lie algebras $\mathtt{L}$ are classical examples of Koszul operads satisfying $\mathtt{K}(\mathtt{C}) = \Lambda^{-1}\,\mathtt{L}^\vee$ and $\mathtt{K}(\mathtt{L}) = \Lambda^{-1}\,\mathtt{C}^\vee$. The theory of Koszul operads is established in a characteristic zero setting in the original reference [16], but we prove in [12] that the notion of a Koszul operad has a suitable generalization so that the above assertions make sense in any category of modules over a ring.

In the case $n = 1$, we can take the operad of associative algebras $\mathtt{E}_1 = \mathtt{A}$ as an instance of E_1-operad. The associative operad $\mathtt{A}$ is a basic example of Koszul operad (see [16]) for which we have $\mathtt{K}(\mathtt{A}) = \Lambda^{-1}\,\mathtt{A}^\vee$. Thus, in the starting case $n = 1$, we already have a Koszul duality result giving a weak-equivalence $\epsilon_1 : \mathtt{B}^c(\Lambda^{-1}\,\mathtt{A}^\vee) \xrightarrow{\sim} \mathtt{A}$.

For $n > 1$, the homology of the operad of little n-cubes forms an operad in graded modules isomorphic to the operad of n-Gerstenhaber algebras $\mathtt{G}_n$ (see [10]). Thus, we have $H_*(\mathtt{E}_n) = \mathtt{G}_n$ for any E_n-operad $\mathtt{E}_n$. The operad $\mathtt{G}_n$ is Koszul, like the associative operad, and its dual $\mathtt{K}(\mathtt{G}_n) = \Lambda^{-n}\,\mathtt{G}_n^\vee$ is the operadic n-fold desuspension of the cooperad $\mathtt{G}_n^\vee$ dual to $\mathtt{G}_n$ in $\mathbb{Z}$-modules (see [15, 27]). This duality result amounts to the existence of a weak-equivalence in the category of differential graded operads $\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$, where $\mathtt{G}_n$ is viewed as a differential graded operad equipped with a trivial differential.

In the characteristic zero setting, the formality theorem of [24] still asserts that $\mathtt{G}_n$ is weakly-equivalent to the chain operad of little n-cubes and forms itself an instance of E_n-operad. Consequently, in the characteristic zero context, we also have a Koszul duality result for E_n-operads for all $n > 1$ yielded by the result for the n-Gerstenhaber operad $\mathtt{G}_n$. But, in the context of $\mathbb{Z}$-modules, addressed in this paper, we have to introduce new ideas, because the formality theorem does not hold any longer (for instance, just because the differential graded modules $\mathtt{E}_n(r)$ are not formal as representations of the symmetric groups). In a sense, we prove that the weak-equivalence of Koszul duality $\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$ is realized by a morphism at the chain level $\psi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \xrightarrow{\sim} \mathtt{E}_n$. To state the realization condition properly we consider a natural spectral sequence $E^1 = \mathtt{B}^c(H_*(\mathtt{D})) \Rightarrow H_*(\mathtt{B}^c(\mathtt{D}))$ associated to the cobar construction. In short, we assume that ψ_n restricts to the Koszul duality equivalence $\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$ on the edge of this spectral sequence, where we take $\mathtt{D} = \mathtt{D}_n = \Lambda^{-n}\,\mathtt{E}_n^\vee$ (more detailed explanations are given in §§0.2-0.4).

The full statement of our main theorem is given next. In fact, we prove more than just the realization of Koszul duality equivalences at the chain level. Indeed, usual models of E_n-operads come in nested sequences

$$\mathtt{E}_1 \subset \mathtt{E}_2 \subset \cdots \subset \mathtt{E}_n \subset \cdots \subset \operatorname*{colim}_n \mathtt{E}_n = \mathtt{E}, \tag{1}$$

where $\mathtt{E}$ is an E_∞-operad, an operad weakly-equivalent to the operad of commutative algebras $\mathtt{C}$, and our objective is also to give the model of the operad embeddings $\iota : \mathtt{E}_{n-1} \hookrightarrow \mathtt{E}_n$ at the level of the cobar constructions $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$.

In our approach, we use a particular E_∞-operad $\mathtt{E}$ equipped with a filtration of that form: the Barratt-Eccles operad. For this operad, we also have an identity $\mathtt{E}_1 = \mathtt{A}$. In a previous work with C. Berger [5], we made explicit an operad morphism $\sigma : \mathtt{E} \to \Lambda^{-1}\,\mathtt{E}$ on the Barratt-Eccles operad. In the present article, we coin the expression 'suspension morphism' to refer to this morphism and its byproducts.

In §0.1, we observe that σ maps $\mathtt{E}_n$ into $\Lambda^{-1}\,\mathtt{E}_{n-1}$ and yields an operad morphism $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ for every $n > 1$. Consider the morphism $\sigma^* : \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^\vee) \to \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ induced by $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ for each $n > 1$. The full statement of our main theorem reads:

Theorem. *We have a sequence of weak-equivalences*

$$\psi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \xrightarrow{\sim} \mathtt{E}_n, \quad n \geq 1, \tag{2}$$

beginning with the Koszul duality equivalence of the associative operad for $n = 1$*, and such that:*

(a) *the diagram*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\Lambda^{-1}\,\mathtt{E}_1^\vee) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-2}\,\mathtt{E}_2^\vee) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) & \xrightarrow{\sigma^*} & \cdots \\
{\scriptstyle \psi_1 = \epsilon_1}\downarrow{\scriptstyle \sim} & & {\scriptstyle \psi_2}\downarrow{\scriptstyle \sim} & & & & {\scriptstyle \psi_n}\downarrow{\scriptstyle \sim} & & \\
\mathtt{E}_1 & \hookrightarrow & \mathtt{E}_2 & \hookrightarrow & \cdots & \hookrightarrow & \mathtt{E}_n & \hookrightarrow & \cdots
\end{array}$$

commutes;

(b) *the restriction of the homology morphism induced by ψ_n on the edge of the spectral sequence*

$$E^1 = \mathtt{B}^c(H_*(\Lambda^{-n}\,\mathtt{E}_n^\vee)) \Rightarrow H_*(\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee))$$

agrees with the Koszul duality equivalence of the Gerstenhaber operad

$$\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$$

for each $n > 1$.

Thus, the morphisms $\sigma^* : \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^\vee) \to \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ define a model of the operad embeddings $\iota : \mathtt{E}_{n-1} \hookrightarrow \mathtt{E}_n$ at the level of cobar constructions.

The present paper is not the first work aiming to understand the structure of cofibrant models of E_n-operads.

The work of E. Getzler and J. Jones [15], at the beginning of the theory of Koszul operads, already gives an approach to the problem. The existence of a Koszul duality equivalence at the homology level, the Koszul duality of the Gerstenhaber operads $\mathtt{G}_n$, is established in [15]. But Getzler-Jones propose another construction, not directly related to Koszul duality, in view towards the definition of cofibrant E_n-operads. In summary, they define a certain topological E_n-operad, called the Fulton-MacPherson operad by them, which is cofibrant as a topological operad (see [32]). This operad inherits a natural cell decomposition. Their proposal consists in forming a chain complex from this cell structure in order to obtain a cofibrant E_n-operad in differential graded modules. But to carry out this part of the program completely, one faces the difficulty (explained for instance in [2, 28]) that the decomposition of the Fulton-MacPherson operad does not give a proper regular cell structure because the boundary of n-cells does not lie in the skeleton of dimension $< n$.

The work of P. Hu [23] also includes a Koszul duality statement for E_n-operads. But several relations implicitly used along [23, §§3-4], where the differential graded context ("the algebraic story") is considered, appear to be valid in homotopy categories only. This is not enough for the constructions of homotopical algebra used in this reference and this flaw has to be fixed in order to apply the Koszul duality proposal of [23].

Our construction of cofibrant models of E_n-operads does not rely on the ideas and on the operads considered in these earlier approaches (as regards the construction itself since we are greatly influenced by the presentation of [15] for the general theory of Koszul operads). Nevertheless, we have to work out difficulties of the same nature. Indeed, our construction is based on the introduction of suitable abstract cell-like structures, called $\mathcal{K}$-operads, which are not genuine cell complexes in our examples, but rigidify our operads enough to define the weak-equivalences of the theorem. Basically, a $\mathcal{K}$-operad is just a collection of diagrams over an operad in posets, to which the letter $\mathcal{K}$ refers, together with an operadic composition structure shaped on the internal composition structure of $\mathcal{K}$. The crux of our proof lies in the verification that the cobar construction $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ inherits a cofibrant $\mathcal{K}$-operad structure when we take the nth layer of the Barratt-Eccles operad for $\mathtt{E}_n$.

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Contents

In the prologue, we review the definition of the Barratt-Eccles operad, of the operadic cobar construction, and we explain the statement of our main result with more details.

A first step towards the definition of weak-equivalences (2) is the construction of morphisms towards the commutative operad:

$$\phi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \to \mathtt{C}\,. \tag{3}$$

This step is carried out in §1.

In the interlude §2, we revisit the definition of universal cell structures, introduced by C. Berger in [3], and which refines filtration (1). Then we introduce the already alluded to notion of $\mathcal{K}$-operad by starting from an abstract interpretation of these cell structures in terms of diagram operads. The operad in posets $\mathcal{K}$ which shapes these diagram operads is just the poset attached to Berger's universal cell decomposition. In §3, we prove that each operad $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ inherits a $\mathcal{K}$-operad structure.

An E_∞-operad $\mathtt{E}$ is, by definition, equipped with a weak-equivalence of operads $\epsilon : \mathtt{E} \xrightarrow{\sim} \mathtt{C}$ and the morphisms (3) admit a lifting to $\mathtt{E}$. In §4, we use the $\mathcal{K}$-structures of §§2-3 to prove that coherent liftings $\tilde{\phi}_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \to \mathtt{E}$ restrict to a sequence of morphisms $\psi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \to \mathtt{E}_n$ as stated in our theorem. Then we use the Koszul duality of the Gerstenhaber operads to conclude that these morphisms are weak-equivalences.

In the epilogue, we explain applications of our results to the definition of models of the cochain algebras of spheres $\bar{N}^*(S^m)$.

Conventions and background

Throughout this paper, we deal with operads in the category of differential graded modules. The topological operad of little n-cubes, for which we adopt the notation $\mathtt{C}_n$, is the only example of operad considered in the paper which does live in this category.

The ring of integers $\mathbb{Z}$ forms our ground ring. For us a differential graded module (a dg-module for short) refers to a lower $\mathbb{Z}$-graded $\mathbb{Z}$-module C equipped with a differential, usually denoted by $\delta : C \to C$, that lowers degrees by 1. The category of dg-modules is denoted by $\mathcal{C}$. We equip this category with its standard model structure (see for instance [22, §2.3]) for which a morphism is a weak-equivalence if it induces an isomorphism in homology, a fibration if it is degreewise surjective.

We adopt conventions of [13] and we refer to this article for a survey of the homotopy theory of operads in dg-modules and further bibliographical references. To help the reader, we have included a short glossary of notation at the end of the article. Simply mention that we adopt the notation Σ_r for the symmetric group on r letters, and the terminology of Σ_*-object for the structure, underlying an operad, formed by a collection of dg-modules $M = \{M(r)\}_{r\in\mathbb{N}}$ such that $M(r)$ is equipped with an action of the symmetric group Σ_r, for all $r \in \mathbb{N}$.

We only consider operads $\mathtt{P}$ such that $\mathtt{P}(0) = 0$ and $\mathtt{P}(1) = \mathbb{Z}$. We say that an operad $\mathtt{P}$ is connected when it satisfies these conditions (some other references use the terminology 'simply connected' for this notion, but we can drop the adverb 'simply'). We use the notation $\mathcal{O}_1$ to refer to the category of connected operads. Any connected operad $\mathtt{P}$ is equipped with an augmentation $\epsilon : \mathtt{P} \to \mathtt{I}$ just given by the identity of $\mathbb{Z}$ in arity $r = 1$. We use the notation $\bar{\mathtt{P}}$ to refer to the augmentation ideal of $\mathtt{P}$. We obviously have $\bar{\mathtt{P}}(0) = \bar{\mathtt{P}}(1) = 0$ and $\bar{\mathtt{P}}(r) = \mathtt{P}(r)$ for $r \geq 2$.

We apply an extension of the operadic Koszul duality of [16] to connected operads over rings. Our reference for this generalization is [12].

We also use the survey of [13], where we study the generalization of the bar duality of operads in the context of unbounded dg-modules since we deal with operads defined in that category of dg-modules. The arguments of [12] only work for non-negatively graded objects, but this will be the case of the operads and cooperads to which we apply the results of that reference.

Preliminaries: the statement of the main result

The purpose of this section is to review the definition of objects involved in the statement of our main theorem. First, we review the definition of the Barratt-Eccles operad, the filtration of the Barratt-Eccles by E_n-operads, and we explain the construction of our suspension morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$. Then we briefly recall the definition of the operadic cobar construction and we explain the statement of our Koszul duality theorem, the main objective of this work.

0.1. The Barratt-Eccles operad and the suspension morphism

The original Barratt-Eccles operad, defined in [1], is an E_∞-operad in simplicial sets formed by the universal Σ_r-bundles $E\Sigma_r$, where Σ_r denotes the group of permutations of $\{1, \ldots, r\}$. Throughout this paper, we use the dg-operad $\mathtt{E}$ defined by the normalized complexes of this simplicial operad

$$\mathtt{E}(r) = N_*(E\Sigma_r).$$

But, we take the convention, contrary to the standard definition, that $\mathtt{E}$ forms a connected operad so that $\mathtt{E}(0) = 0$.

The purpose of the next paragraphs is only to recall the main features of the operad $\mathtt{E}$ used in the statement of our main theorem. For more details on this operad, we refer to the paper [5] of which we take our conventions.

0.1.1. The Barratt-Eccles operad. Recall that an E_∞-operad in dg-modules is a dg-operad $\mathtt{E}$ together with a weak-equivalence $\epsilon : \mathtt{E} \xrightarrow{\sim} \mathtt{C}$, where $\mathtt{C}$ is the operad of commutative algebras. In most references, an E_∞-operad $\mathtt{E}$ is also assumed to be cofibrant as a Σ_*-object.

For the Barratt-Eccles operad:

- The dg-module $\mathtt{E}(r)$ is spanned in degree d by non-degenerate d-simplices $(w_0, \dots, w_d)$ of permutations $w_i \in \Sigma_r$, where a simplex $(w_0, \dots, w_d)$ is non-degenerate if we have $w_j \neq w_{j+1}$ for every j. The differential of $\mathtt{E}(r)$ is given by the usual formula

$$\delta(w_0, \dots, w_d) = \sum_{i=0}^{d} \pm(w_0, \dots, \widehat{w_i}, \dots, w_d).$$

- The composition structure is yielded by an explicit substitution process on permutations (the definition of this part of the structure is not needed until §3.2 and we put off the corresponding recollections until that section).
- The operad weak-equivalence $\epsilon : \mathtt{E} \xrightarrow{\sim} \mathtt{C}$ towards the commutative operad $\mathtt{C}$ is given by the standard chain augmentations $\epsilon : N_*(E\Sigma_r) \to \mathbb{Z}$, which are defined by:

$$\epsilon(w_0, \dots, w_d) = \begin{cases} 1, & \text{if } d = 0, \\ 0, & \text{otherwise.} \end{cases}$$

0.1.2. The degree 0 part of the Barratt-Eccles operad. The Barratt-Eccles operad vanishes in degree $* < 0$ and consists of finitely generated free $\mathbb{Z}$-modules in degree $* \geq 0$.

The degree 0 parts of the chain complexes $\mathtt{E}(r)$ form a suboperad of the Barratt-Eccles operad such that $\mathtt{E}(r)_0 = \mathbb{Z}[\Sigma_r]$, the free $\mathbb{Z}$-module generated by the permutations of $\{1, \dots, r\}$. This suboperad can be identified with the operad of associative algebras $\mathtt{A}$ for which we also have $\mathtt{A}(r) = \mathbb{Z}[\Sigma_r]$. Hence, the Barratt-Eccles operad sits in a factorization

$$\begin{array}{ccc} \mathtt{A} & \longrightarrow & \mathtt{C} \\ & \searrow \qquad \nearrow_{\sim} & \\ & \mathtt{E} & \end{array}$$

of the morphism $\alpha : \mathtt{A} \to \mathtt{C}$ which represents the usual embedding $\alpha^* : {}_{\mathtt{C}}\mathcal{C} \hookrightarrow {}_{\mathtt{A}}\mathcal{C}$ from the category of commutative algebras ${}_{\mathtt{C}}\mathcal{C}$ to the category of associative algebras ${}_{\mathtt{A}}\mathcal{C}$.

0.1.3. The little cubes filtration of the Barratt-Eccles operad. The Barratt-Eccles chain operad has a filtration by suboperads $\mathtt{E}_n \subset \mathtt{E}$ which form a nested sequence of operads (1) weakly-equivalent to the sequence of the chain operads of little cubes. The definition of this filtration, in the simplicial setting, goes back to [34]. The existence of weak-equivalences with little cubes operads is established in [3]. For our purpose, we briefly recall an explicit definition of the filtration at the chain level.

In our study of the Barratt-Eccles operad, we specify a permutation $w \in \Sigma_r$ by the ordered sequence of its values $(w(1),\dots,w(r))$. For a pair $\{i,j\} \subset \{1,\dots,r\}$, we use the notation $w|_{ij}$ to refer to the permutation of $\{i,j\}$ formed by the occurrences of $\{i,j\}$ in the sequence $(w(1),\dots,w(r))$. For instance, we have $(3,1,2)|_{12} = (1,2)$, $(3,1,2)|_{13} = (3,1)$, $(3,1,2)|_{23} = (3,2)$.

To a simplex $\underline{w} = (w_0,\dots,w_d)$, we associate the number

$$\mu_{ij}(\underline{w}) = \#\{k < d \text{ such that } w_k|_{ij} \neq w_{k+1}|_{ij}\}.$$

In plain words, we just take the number of variations of the sequence $\underline{w}|_{ij} = (w_0|_{ij},\dots,w_d|_{ij})$ to define $\mu_{ij}(\underline{w})$. To give an example, for the simplex

$$\underline{w} = ((1,2,3),(3,1,2),(3,2,1)),$$

we obtain

$$\begin{aligned} \underline{w}|_{12} &= ((1,2),(1,2),(2,1)) \Rightarrow \mu_{12}(\underline{w}) = 1,\\ \underline{w}|_{13} &= ((1,3),(3,1),(3,1)) \Rightarrow \mu_{13}(\underline{w}) = 1,\\ \underline{w}|_{23} &= ((2,3),(3,2),(3,2)) \Rightarrow \mu_{23}(\underline{w}) = 1. \end{aligned}$$

The dg-module $\mathtt{E}_n(r)$ is spanned by the simplices of permutations $\underline{w} = (w_0,\dots,w_d)$ such that $\mu_{ij}(\underline{w}) < n$ for all pairs $\{i,j\} \subset \{1,\dots,r\}$. For instance, we have $\underline{w} = ((1,2,3),(3,1,2),(3,2,1)) \in \mathtt{E}_2(3)$ since we observe that $\mu_{12}(\underline{w}) = \mu_{13}(\underline{w}) = \mu_{23}(\underline{w}) = 1$ for this simplex.

For any $\underline{w} = (w_0,\dots,w_d)$, we clearly have:

$$(\mu_{ij}(\underline{w}) = 0\ (\forall ij)) \Rightarrow (w_0 = w_1 = \dots = w_d).$$

Hence, the subcomplex $\mathtt{E}_1(r) \subset \mathtt{E}(r)$ reduces to the degree 0 part of $\mathtt{E}(r)$ and this observation implies:

0.1.4. Proposition. *The suboperad $\mathtt{E}_1$ of the Barratt-Eccles operad $\mathtt{E}$ is identified with the operad of associative algebras $\mathtt{A}$.* □

To conclude this subsection, we explain the definition of the suspension morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$.

0.1.5. Suspensions. First of all, recall that the operadic suspension of an operad $\mathtt{P}$ is an operad $\Lambda\,\mathtt{P}$ such that:

$$\Lambda\,\mathtt{P}(r) = \Sigma^{1-r}\,\mathtt{P}(r)^{\pm},$$

where Σ refers to the standard suspension of the category of dg-modules and the exponent $\pm$ refers to a twist of the Σ_r-action by the signature of permutations (see [15, §1.3]).

Let $sgn : \mathtt{E}(r)_{r-1} \to \mathbb{Z}$, $r \in \mathbb{N}$, be the cochain of degree $r-1$ which vanishes when the sequence $(w_0(1), \dots, w_{r-1}(1))$, formed by the first term of each permutation w_i, is not a permutation of $(1, \dots, r)$ and takes the value

$$sgn(w_0, \dots, w_{r-1}) = \pm 1$$

assigned by the signature of that permutation otherwise. For instance, we have the result $sgn((\underline{1}, 2, 3), (\underline{3}, 1, 2), (\underline{3}, 2, 1)) = 0$ since $(1, 3, 3)$ is not a permutation. To give another example, we have $sgn((\underline{1}, 2, 3), (\underline{3}, 1, 2), (\underline{2}, 3, 1)) = -1$ since $(1, 3, 2)$ is an odd permutation.

In [5, Proposition 3.2.9], we prove that the cap products with these cochains

$$sgn \cap (w_0, \dots, w_d) = sgn(w_0, \dots, w_{r-1}) \cdot (w_{r-1}, \dots, w_d)$$

define an operad morphism

$$\sigma : \mathtt{E} \to \Lambda^{-1}\,\mathtt{E}\,.$$

This morphism is the suspension morphism (of the Barratt-Eccles chain operad) considered in the introduction.

For our purpose, we use the following crucial observation which has been made to the author by M. Mandell (in Spring 2002):

0.1.6. Observation. *The suspension morphism $\sigma : \mathtt{E} \to \Lambda^{-1}\,\mathtt{E}$ admits factorizations*

$$\begin{array}{ccccccccc}
\mathtt{E}_1 & \hookrightarrow & \mathtt{E}_2 & \hookrightarrow \cdots \hookrightarrow & \mathtt{E}_n & \hookrightarrow \cdots \hookrightarrow & \mathtt{E} & , \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & \\
\Lambda^{-1}\,\mathtt{I} & \hookrightarrow & \Lambda^{-1}\,\mathtt{E}_1 & \hookrightarrow \cdots \hookrightarrow & \Lambda^{-1}\,\mathtt{E}_{n-1} & \hookrightarrow \cdots \hookrightarrow & \Lambda^{-1}\,\mathtt{E} &
\end{array}$$

where $\mathtt{I}$ denotes the composition unit of the category of Σ_*-objects.

This observation follows from a straightforward inspection of the definition of the filtration in §0.1.3.

The morphism $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ gives an analogue, at the level of chain operads, of the map $\beta_n : \mathtt{C}_n(X) \to \Omega\,\mathtt{C}_{n-1}(\Sigma X)$ of [30, Proposition 5.4], where $\mathtt{C}_n(X)$ is the monad on topological spaces associated to the operad of little n-cubes.

In a sense, the actual goal of this paper is to give a new interpretation of the morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ in terms of the homotopy of E_n-operads.

0.2. Operadic cobar constructions and homological Koszul duality

Throughout the paper, we only consider connected cooperads $\mathtt{D}$ which, just like connected operads, have $\mathtt{D}(0) = 0$ and $\mathtt{D}(1) = \mathbb{Z}$. These assumptions prevent most difficulties arising in the duality between operads and cooperads (see [13, §§3.1-3.3]).

A connected cooperad $\mathtt{D}$ is naturally coaugmented over the composition unit of Σ_*-objects and the coaugmentation coideal of $\mathtt{D}$ is identified with the

Σ_*-object such that

$$\bar{\mathtt{D}}(r) = \begin{cases} 0, & \text{if } r = 0, 1, \\ \mathtt{D}(r), & \text{otherwise.} \end{cases}$$

The aim of this subsection is to study the cobar construction of cooperads $\mathtt{D}$ whose homology $H_*(\mathtt{D})$ forms a Koszul cooperad (we say that $\mathtt{D}$ is homologically Koszul). In the next subsection, we apply this analysis to E_n-operads $\mathtt{E}_n$ and their dual cooperads $\mathtt{D} = \Lambda^{-n}\,\mathtt{E}_n^{\vee}$, but we assume that $\mathtt{E}$ is any (homologically Koszul) cooperad for the moment.

To begin with, we briefly review the definition of the free operad $\mathtt{F}(M)$ associated to a Σ_*-object M (at least to recall some conventions), and we review the definition of the cobar construction of cooperads $\mathtt{B}^c(\mathtt{D})$.

The article [12] is our reference for the Koszul duality of operads in the context of modules over a ring. In the general definition of Koszul duality given in that work, the cooperads $\mathtt{D}$ are equipped with an extra weight grading $\mathtt{D} = \bigoplus_{s=0}^{\infty} \mathtt{D}_{(s)}$. In this article, we only deal with the special case of Koszul operads and cooperads for which the natural arity grading

$$\mathtt{D}_{(s)}(r) = \begin{cases} \mathtt{D}(r), & \text{if } s = r - 1, \\ 0, & \text{otherwise,} \end{cases}$$

can be taken as weight grading. Therefore we do not mention such weight gradings in what follows.

The definition of the free operad and of the cobar construction is also examined with great detail in the context of modules over a ring in our reference [12]. For the cobar construction, we also consider the reference [13, §3.6] where an extension of the cobar complex in the context of unbounded dg-modules is studied.

0.2.1. The free operad. Throughout this paper, we only consider free operads $\mathtt{F}(M)$ associated to Σ_*-objects M such that $M(0) = M(1) = 0$. This requirement ensures that the free operad $\mathtt{F}(M)$ is connected as an operad and the mapping $M \mapsto \mathtt{F}(M)$ defines the left adjoint of the augmentation ideal functor $\mathtt{P} \mapsto \bar{\mathtt{P}}$.

In short, the components of the free operad $\mathtt{F}(M)$ are defined by direct sums

$$\mathtt{F}(M)(r) = \bigoplus_{\tau \in \Theta(r)} \tau(M)/\equiv \tag{4}$$

running over a category of r-trees $\Theta(r)$, where $\tau(M)$ is a dg-module, associated to each $\tau \in \Theta(r)$, formed by a tree-wise tensor product of the generating Σ_*-object M. In principle, the sum is divided out by the action of tree isomorphisms but we can replace the large categories of r-trees $\Theta(r)$ by discrete skeletons in order to avoid this quotient process when we assume $M(0) = M(1) = 0$.

The free operad has a natural weight decomposition

$$\mathtt{F}(M) = \bigoplus_{m=0}^{\infty} \mathtt{F}_m(M),$$

where $\mathtt{F}_m(M)$ is a Σ_*-subobject of $\mathtt{F}(M)$ formed by the summands $\tau(M)$ such that τ has m vertices. We have $\mathtt{F}_0(M) = I$, $\mathtt{F}_1(M) = M$ and the universal morphism of the free operad $\eta : M \to \mathtt{F}(M)$ is represented by a identification of M with the summand $\mathtt{F}_1(M) \subset \mathtt{F}(M)$.

The abstract definition of the free operad implies the existence of a morphism $\lambda_* : \mathtt{F}(\bar{\mathtt{P}}) \to \mathtt{P}$ attached to any connected operad $\mathtt{P}$. In what follows, we use that the restriction of λ_* to a summand $\tau(\bar{\mathtt{P}})$ associated to a tree with two vertices is identified with a partial operadic composition operation $\circ_e : \mathtt{P}(s) \otimes \mathtt{P}(t) \to \mathtt{P}(s+t-1)$. For the moment, simply note that the composition structure of a connected operad $\mathtt{P}$ is fully determined by a morphism $\lambda_2 : \mathtt{F}_2(\bar{\mathtt{P}}) \to \mathtt{P}$ representing the restriction of λ_* to the quadratic part $\mathtt{F}_2(\bar{\mathtt{P}})$ of the free operad $\mathtt{F}(\bar{\mathtt{P}})$.

0.2.2. Cooperads and the cobar construction. The Σ_*-object underlying the free operad $\mathtt{F}(M)$ can be equipped with a cooperad structure, instead of an operad structure, and represents the cofree cooperad associated to M too. For detailed explanations on this observation (and other definitions of this paragraph), we refer to [13, §§3.1-3.5]. Simply note that the composition structure of a cooperad $\mathtt{D}$ can be defined by a morphism $\rho_* : \mathtt{D} \to \mathtt{F}(\bar{\mathtt{D}})$. The projection of this morphism onto the summand $\mathtt{F}_2(\bar{\mathtt{D}})$ gives a morphism $\rho_2 : \bar{\mathtt{D}} \to \mathtt{F}_2(\bar{\mathtt{D}})$, dual to the quadratic composition morphism of an augmented operad, which also suffices to determine the composition structure of $\mathtt{D}$.

The dual dg-modules of an operad $\mathtt{P}^\vee(r) = \mathit{Hom}_{\mathcal{C}}(\mathtt{P}(r), \mathbb{Z})$ inherit such a coproduct $\rho_2 : \mathtt{P}^\vee \to \mathtt{F}_2(\bar{\mathtt{P}}^\vee)$ when each dg-module $\mathtt{P}(r)$ forms a degreewise finitely generated free $\mathbb{Z}$-module, because we have a natural duality isomorphism $\mathtt{F}_2(\bar{\mathtt{P}}^\vee) \simeq \mathtt{F}_2(\bar{\mathtt{P}})^\vee$ (see [12, Proposition 3.1.4 and §3.6.1]). Hence, we obtain that the $\mathbb{Z}$-dual dg-modules of an operad $\mathtt{P}$ inherit a cooperad structure under the usual assumption of duality in module categories.

The cobar construction is a quasi-free operad $\mathtt{B}^c(\mathtt{D}) = (\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}), \partial)$ defined by the addition of an operadic derivation of degree -1

$$\partial : \mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}) \to \mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})$$

to the natural differential of the free operad $\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})$. The operadic derivation relation, which reads $\partial(p \circ_e q) = \partial(p) \circ_e q + \pm p \circ_e \partial(q)$, implies that any derivation of a free operad $\partial : \mathtt{F}(M) \to \mathtt{F}(M)$ is determined by its restriction to the generating Σ_*-object $M \subset \mathtt{F}(M)$. The bar differential is defined on generators

$$\Sigma^{-1}\bar{\mathtt{D}} \xrightarrow{\partial|_{\Sigma^{-1}\bar{\mathtt{D}}}} \mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})$$

by a desuspension of the quadratic coproduct $\rho_2 : \bar{\mathtt{D}} \to \mathtt{F}_2(\bar{\mathtt{D}})$ (see for instance [13, §3.6]).

0.2.3. The filtration of the cobar construction. In [12, §3.1.10], we explain that the twisting derivation ∂ satisfies $\partial(\mathtt{F}_s(\Sigma^{-1}\bar{\mathtt{D}})) \subset \mathtt{F}_{s+1}(\Sigma^{-1}\bar{\mathtt{D}})$ so that the cobar construction forms a cochain complex of dg-modules with the homogeneous components of the free operad $\mathtt{B}^c(\mathtt{D})^s = \mathtt{F}_s(\Sigma^{-1}\bar{\mathtt{D}})$ as components. In what follows, we essentially use the spectral sequence naturally associated to this complex of dg-modules, but we prefer to quickly review the definition of this spectral sequence in different terms, because we need to keep track of operadic composition structures which are broken by the cochain complex representation.

The free operad $\mathtt{F}(M)$ is equipped with the increasing filtration such that $F_{-s}\,\mathtt{F}(M) = \bigoplus_{r\geq s}\mathtt{F}_r(M)$. The condition $M(0) = M(1) = 0$ implies that $F_{-s}\,\mathtt{F}(M)(n) = 0$ when $s > n-1$, for any fixed arity $n \geq 1$. Moreover, we clearly have $F_{-s}\,\mathtt{F}(M) = \mathtt{F}(M)$ for $s \leq 0$.

We equip the cobar construction with this filtration

$$\mathtt{B}^c(\mathtt{D}) = F_0\,\mathtt{B}^c(\mathtt{D}) \supset F_{-1}\,\mathtt{B}^c(\mathtt{D}) \supset \cdots \supset F_{-s}\,\mathtt{B}^c(\mathtt{D}) \supset \cdots,$$

inherited from the free operad $\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})$, and so that $F_{-s}\,\mathtt{B}^c(\mathtt{D})(n) = 0$ for $s > n-1$. We have already recalled that the twisting derivation of the cobar construction satisfies $\partial(\mathtt{F}_s(\Sigma^{-1}\bar{\mathtt{D}})) \subset \mathtt{F}_{s+1}(\Sigma^{-1}\bar{\mathtt{D}})$ and hence, preserves the filtration of the cobar construction. The filtration is also obviously preserved by the composition product of the free operad $\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})$ and we have $E^0_{-s}\,\mathtt{B}^c(\mathtt{D}) = F_{-s}\,\mathtt{B}^c(\mathtt{D})/F_{-s-1}\,\mathtt{B}^c(\mathtt{D}) = \mathtt{F}_s(\Sigma^{-1}\bar{\mathtt{D}})$. From these observations all together, we conclude:

0.2.4. Fact (see [12, Lemma 3.6.2]**).** *We have a strongly convergent spectral sequence of operads*

$$E^r(\mathtt{B}^c(\mathtt{D})) \Rightarrow H_*(\mathtt{B}^c(\mathtt{D})),$$

naturally associated to the cobar construction $\mathtt{B}^c(\mathtt{D})$*, and such that* $E^1 = H_*(\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}))$ *is the homology of the free operad on* $\Sigma^{-1}\bar{\mathtt{D}}$ *with the weight grading inherited from the free operad.*

In addition, we have:

0.2.5. Fact (see [12, §§3.6.1-3.6.2]**).** *If each component* $\mathtt{D}(r)$ *of a cooperad* $\mathtt{D}$ *forms a cofibrant dg-module and each component* $H_*(\mathtt{D}(r))$ *of the homology of* $\mathtt{D}$ *forms a free graded* $\mathbb{Z}$*-module, then we have a Künneth isomorphism*

$$\mathtt{F}(\Sigma^{-1}H_*(\bar{\mathtt{D}})) \xrightarrow{\simeq} H_*(\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}))$$

and the E^1*-term of the spectral sequence can be identified with the cobar construction of the homology of* $\mathtt{D}$*:*

$$(E^1, d^1) = (\mathtt{F}(\Sigma^{-1}H_*(\bar{\mathtt{D}})), \partial) = \mathtt{B}^c(H_*(\mathtt{D})).$$

0.2.6. Homologically Koszul cooperads. The spectral sequence of Fact 0.2.4 can be applied to a graded cooperad $\mathtt{H}$ equipped with a trivial internal differential $\delta = 0$. In this case, the spectral sequence degenerates at E^2 and simply reduces to the definition of natural homogeneous weight components $E^0_{-s}H_*(\mathtt{B}^c(\mathtt{H}))$ in the homology of the bar construction.

By definition, a graded cooperad $\mathtt{H}$ is Koszul if we have the relation

$$E^0_{-s}H_*(\mathtt{B}^c(\mathtt{H}))(n) = \begin{cases} H_*(\mathtt{B}^c(\mathtt{H}))(n), & \text{if } s = n-1,\\ 0, & \text{otherwise,}\end{cases}$$

for each fixed arity $n \in \mathbb{N}$, together with the basic requirement that both $\mathtt{H}$ and $H_*(\mathtt{B}^c(\mathtt{H}))$ consist of projective $\mathbb{Z}$-modules (see [16, §4.1.3] for the original definition of operadic Koszul duality in the characteristic zero context, and [12, §5.2] for the definition in the present setting). Now we say that a dg-cooperad $\mathtt{D}$ is homologically Koszul if its homology cooperad $\mathtt{H} = H_*(\mathtt{D})$ is Koszul. The $\mathbb{Z}$-modules $H_*(\mathtt{D})(n)$ are implicitly assumed to be projective when we say that $H_*(\mathtt{D})$ is a Koszul cooperad. For our purpose, we also assume that the components $\mathtt{D}(n)$ of a homologically Koszul cooperad $\mathtt{D}$ are cofibrant dg-modules. Then we obtain the following result:

0.2.7. Proposition. *If a cooperad $\mathtt{D}$ is homologically Koszul, then the spectral sequence of Fact 0.2.4 degenerates at E^2 and we have a natural weak-equivalence of operads*

$$\eta : \mathtt{B}^c(H_*(\mathtt{D})) \xrightarrow{\sim} H_*(\mathtt{B}^c(\mathtt{D})),$$

where the homology operad $H_*(\mathtt{B}^c(\mathtt{D}))$ is regarded as a dg-operad equipped with a trivial differential. This morphism η is determined on generators of the quasi-free operad $\mathtt{B}^c(H_*(\mathtt{D})) = (\mathtt{F}(\Sigma^{-1}H_*(\bar{\mathtt{D}})), \partial)$ by the homology of the natural embedding $\Sigma^{-1}\bar{\mathtt{D}}(n) \subset \mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}})(n)$ in arity $n = 2$ and by the null morphism in arity $n \neq 2$.

The morphism of the proposition $\eta : \mathtt{B}^c(H_*(\mathtt{D})) \xrightarrow{\sim} H_*(\mathtt{B}^c(\mathtt{D}))$ is identified with an edge morphism

$$\mathtt{B}^c(H_*(\mathtt{D}))(n) \to E^1_{1-n}(\mathtt{B}^c(\mathtt{D}))(n) \to E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n)$$

associated to the spectral sequence of Fact 0.2.4, where we apply a relation

$$E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n) = H_*(\mathtt{B}^c(\mathtt{D}))(n)$$

coming from the Koszul condition (see proof of the proposition). For that reason, we refer to this morphism as the edge morphism attached to the homologically Koszul cooperad $\mathtt{D}$.

Proof. To simplify notation we set $\mathtt{H} = H_*(\mathtt{D})$. The spectral sequence $E^r = E^r(\mathtt{B}^c(\mathtt{D})) \Rightarrow H_*(\mathtt{B}^c(\mathtt{D}))$ satisfies $E^1 = \mathtt{B}^c(\mathtt{H})$ by Fact 0.2.5 and we have $E^2 = E^0H_*(\mathtt{B}^c(\mathtt{H}))$. If the cooperad $\mathtt{D}$ is homology Koszul, then the modules $E^2 = E^0_{-s}H_*(\mathtt{B}^c(\mathtt{H}))(n)$ are concentrated on the column $s = n-1$, for each fixed arity n. Hence, our spectral sequence degenerates at E^2 and gives a grading such that

$$E^0_{-s}H_*(\mathtt{B}^c(\mathtt{D}))(n) = \begin{cases} H_*(\mathtt{B}^c(\mathtt{H}))(n), & \text{if } s = n-1,\\ 0, & \text{otherwise,}\end{cases}$$

on the abutment. This relation implies $H_*(\mathtt{B}^c(\mathtt{D}))(n) = E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n)$.

The vanishing of the filtration $F_{-s}\,\mathtt{B}^c(\mathtt{D})(n)$ for $s > n-1$ (see §0.2.3) implies the existence of an edge morphism

$$\mathtt{B}^c(H_*(\mathtt{D}))(n) \to E^1_{1-n}(\mathtt{B}^c(\mathtt{D}))(n) \to E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n)$$

for each $n \in \mathbb{N}$. The degeneracy of the spectral sequence at E^2 amounts to saying that this morphism is a weak-equivalence, the module $E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n)$ being equipped with a trivial differential.

The relation $H_*(\mathtt{B}^c(\mathtt{D}))(n) = E^0_{1-n}H_*(\mathtt{B}^c(\mathtt{D}))(n)$ and the preservation of operadic structures by the spectral sequence implies that our edge morphisms assemble to an operad morphism

$$\eta : \mathtt{B}^c(H_*(\mathtt{D})) \to H_*(\mathtt{B}^c(\mathtt{D})).$$

Moreover, we immediately see from the definition of η as an edge morphism that its restriction to $\Sigma^{-1}H_*(\bar{\mathtt{D}})$ is as asserted in the proposition and this observation completes our verifications. □

0.2.8. Recollections: Koszul duality of operads. There are several equivalent definitions for the notion of a Koszul operad. In the basic definition, essentially dual to the definition of §0.2.6 for cooperads, a graded operad $\mathtt{P}$ (equipped with a trivial differential) is Koszul if the homology of an associated operadic bar construction $\mathtt{B}(\mathtt{P})$ vanishes in a certain range. By theorem, this definition is equivalent to the existence of a weak-equivalence $\epsilon : \mathtt{B}^c(\mathtt{K}(\mathtt{P})) \xrightarrow{\sim} \mathtt{P}$, where $\mathtt{K}(\mathtt{P})$ is a certain graded cooperad (equipped with a trivial differential) naturally associated to $\mathtt{P}$, the Koszul dual cooperad of $\mathtt{P}$ (see [16, §4] for a first statement of this result in the characteristic 0 context, see [12, §5.2] for the generalization to our setting). In what follows, we refer to this weak-equivalence ϵ as the Koszul duality equivalence of the operad $\mathtt{P}$. In applications of this definition, we tacitly assume that the components of a Koszul operad $\mathtt{P}$ and of the associated cooperad $\mathtt{K}(\mathtt{P})$ are projective over the ground ring. These conditions are always satisfied for the operads considered in this article.

The Koszul dual $\mathtt{K}(\mathtt{P})$ of a Koszul operad $\mathtt{P}$ forms a Koszul cooperad in the sense of §0.2.6 (see [12, Lemma 5.2.10]) and we use this observation in each application of the ideas of §§0.2.6-0.2.7.

0.3. Applications to Gerstenhaber operads

The purpose of this section is to review the application of Koszul duality results for the homology of the little n-cubes operads $H_*(\mathtt{C}_n, \mathbb{Z})$, and to study how the constructions go through the isomorphism $H_*(\mathtt{C}_n, \mathbb{Z}) \simeq H_*(\mathtt{E}_n)$ when we take the Barratt-Eccles operad for $\mathtt{E}_n$. Recall that we use the notation $\mathtt{C}_n$ for the topological operad of little n-cubes. The notation $H_*(\mathtt{C}_n, \mathbb{Z})$ refers to the homology of the topological object $\mathtt{C}_n$ with $\mathbb{Z}$-coefficients which is compared to the homology $H_*(\mathtt{E}_n)$ of the differential graded E_n-operad $\mathtt{E}_n$.

The articles [10, 33] are our references for the computation of $H_*(\mathtt{C}_n, \mathbb{Z})$. For a nice introduction to this topic, we also refer to [25, §I.6]. For $n = 1$, we have an identity $H_*(\mathtt{C}_1, \mathbb{Z}) = \mathtt{A}$, where we consider the operad of associative algebras $\mathtt{A}$. For $n > 1$, the operad $H_*(\mathtt{C}_n, \mathbb{Z})$ is identified with another operad $\mathtt{G}_n$ defined by generators and relations, to which we refer as the n-Gerstenhaber operad. The associative operad and the Gerstenhaber operads are Koszul and we also review the definition of the Koszul duality equivalence $\epsilon : \mathtt{B}^c(\mathtt{K}(\mathtt{P})) \xrightarrow{\sim} \mathtt{P}$ associated to these operads in this subsection.

There is no explicit definition of the weak-equivalences connecting the filtration layers of the Barratt-Eccles operad and the chain operads of little n-cubes. Therefore we devote some time to give a representation of structures associated to the homology operad $\mathtt{G}_n = H_*(\mathtt{C}_n, \mathbb{Z})$ in terms of the Barratt-Eccles operad. Naturally we derive all results from the original computations of [10].

To begin with, we recall the presentation of the Gerstenhaber operads by generators and relations and we explain the definition of the isomorphism $\gamma : \mathtt{G}_n \xrightarrow{\simeq} H_*(\mathtt{E}_n)$ for $n > 1$, where we replace the operad of little n-cubes $\mathtt{C}_n$ by the nth layer of the Barratt-Eccles operad $\mathtt{E}_n$.

0.3.1. The Gerstenhaber operads. The Gerstenhaber operad $\mathtt{G}_n$ is defined by a presentation by generators and relations, with a generating operation $\mu = \mu(x_1, x_2)$ of degree 0, another one $\lambda_{n-1} = \lambda_{n-1}(x_1, x_2)$ of degree $n-1$, and so that a permutation of variables yields the symmetry relations

$$\mu(x_1, x_2) = \mu(x_2, x_1) \quad \text{and} \quad \lambda_{n-1}(x_1, x_2) = (-1)^n \lambda_{n-1}(x_2, x_1). \tag{5}$$

For short, we can set $\lambda = \lambda_{n-1}$. Let $\mathbb{Z}\,\mu \oplus \mathbb{Z}\,\lambda$ represent the Σ_*-object spanned by the elements (μ, λ) in arity 2 together with the action of the symmetric group Σ_2 determined by (5). The n-Gerstenhaber operad $\mathtt{G}_n$ is the quotient of the free operad $\mathtt{F}(\mathbb{Z}\,\mu \oplus \mathbb{Z}\,\lambda)$ by the operadic ideal generated by the associativity relation

$$\mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3)), \tag{6}$$

the Jacobi relation

$$\lambda(\lambda(x_1, x_2), x_3) + \lambda(\lambda(x_2, x_3), x_1) + \lambda(\lambda(x_3, x_1), x_2) \equiv 0, \tag{7}$$

and the distribution relation

$$\lambda(\mu(x_1, x_2), x_3) \equiv \mu(\lambda(x_1, x_3), x_2) + \mu(x_1, \lambda(x_2, x_3)). \tag{8}$$

According to this definition, a morphism $\phi : \mathtt{G}_n \to \mathtt{P}$ towards an operad $\mathtt{P}$ is fully determined by elements $\mu, \lambda \in \mathtt{P}(2)$ that satisfy the symmetry relations (5) and relations (6-8) in $\mathtt{P}$. In the next paragraph, we define representative of such elements in the nth layer of the Barratt-Eccles operad for each $n > 1$.

Just recall before that the associative operad $\mathtt{A}$ has a standard presentation by generators and relations too, consisting of a generating operation $\mu = \mu(x_1, x_2)$ in arity 2 (with no symmetry relation) together with the associativity relation $\mu(\mu(x_1, x_2), x_3) \equiv \mu(x_1, \mu(x_2, x_3))$ as generating relation.

0.3.2. The representatives of generating operations in the Barratt-Eccles operad. In arity 2, the Barratt-Eccles operad $\mathtt{E}$ is spanned by the alternate simplices

$$\mu_d = (\underbrace{\mathrm{id},\tau,\mathrm{id},\tau,\dots}_{\in\Sigma_2^{\times d+1}}) \quad\text{and}\quad \tau\mu_d = (\underbrace{\tau,\mathrm{id},\tau,\mathrm{id},\dots}_{\in\Sigma_2^{\times d+1}}),\quad d\in\mathbb{N},$$

where id is the identity permutation of $(1,2)$ and τ is the transposition $\tau = (2,1)$. Moreover we have $\delta(\mu_d) = \tau\mu_{d-1} + (-1)^d\mu_{d-1}$. Hence the dg-module $\mathtt{E}(2)$ can be identified with the usual free resolution of the trivial Σ_2-module:

$$\mathbb{Z}\,\mu_0\oplus\mathbb{Z}\,\tau\mu_0 \xleftarrow{\tau-1} \mathbb{Z}\,\mu_1\oplus\mathbb{Z}\,\tau\mu_1 \xleftarrow{\tau+1} \mathbb{Z}\,\mu_2\oplus\mathbb{Z}\,\tau\mu_2 \xleftarrow{\tau-1} \cdots .$$

According to the definition of §0.1.3, we have $\mu_d\in\mathtt{E}_n(2)$ if and only if $d<n$. Hence, the dg-module $\mathtt{E}_n(2)$ is identified with a truncation of $\mathtt{E}(2)$ and we have

$$H_d(\mathtt{E}_n(2)) = \begin{cases}\mathbb{Z}, & \text{if } d=0,n-1,\\ 0, & \text{otherwise.}\end{cases}$$

Thus we retrieve the standard description of the homology of the space $\mathtt{C}_n(2)$.

The cycles $\mu=\mu_0$ and $\lambda_{n-1} = \mu_{n-1} + (-1)^{n-1}\tau\mu_{n-1}$ define generating homology classes of $H_*(\mathtt{E}_n(2))$.

The operation $\mu=\mu_0$, which belongs to $\mathtt{E}_1(2)$, satisfies the relation of an associative product in the Barratt-Eccles operad and gives a representative of the generating operation of the associative operad in $\mathtt{E}_1$ when we apply the identity $\mathtt{A}=\mathtt{E}_1$. For $n>1$, the homology class associated to μ_0 in $H_*(\mathtt{E}_n)$ satisfies $\tau\mu_0\equiv\mu_0$ since $\tau\mu_0-\mu_0$ is the boundary of μ_1. Hence we obtain that $\mu=\mu_0$ represents an associative and commutative product in $H_*(\mathtt{E}_n)$, for every $n>1$. The other relations of the Gerstenhaber operad are also satisfied in $H_*(\mathtt{E}_n)$:

0.3.3. Proposition. *The homology classes associated to $\mu\in\mathtt{E}_n(2)$ and $\lambda_{n-1}\in\mathtt{E}_n(2)$ satisfy all identities of the generating operations of the n-Gerstenhaber operad in $H_*(\mathtt{E}_n)$.* □

The same result holds in the homology of little cubes operad (see [10, Theorem 1.2] or [33, §5.4(5) and Theorem 6.6]). The weak-equivalences giving the isomorphism $H_*(\mathtt{E}_n)\simeq H_*(\mathtt{C}_n,\mathbb{Z})$ are not explicit, but the definition of λ_{n-1} in terms of a sequence of products μ_d such that $\delta(\mu_d) = \tau\mu_{d-1}+(-1)^d\mu_{d-1}$ fully determines the homology class of λ_{n-1}. Therefore, our representative of this homology class in $\mathtt{E}_n$ satisfies the same relation as in the homology of the little n-cubes operad.

Because of the definition of $\mathtt{G}_n$ by generators and relations, Fact 0.3.3 implies the existence of an operad morphism $\gamma:\mathtt{G}_n\to H_*(\mathtt{E}_n)$, for all $n>1$. We have moreover:

0.3.4. Theorem (corollary of [10, Theorem 3.1]**).** *The morphism $\gamma:\mathtt{G}_n\to H_*(\mathtt{E}_n)$ which maps the generating operations of $\mathtt{G}_n$ to our homology classes $\mu\in H_0(\mathtt{E}_n(2))$ and $\lambda_{n-1}\in H_{n-1}(\mathtt{E}_n(2))$ is an isomorphism for all $n>1$.* □

This theorem is established for the homology of the topological little n-cubes operad in the cited reference [10] (we also refer to [33] for a new proof of this result). The result for the Barratt-Eccles operad follows since the isomorphism $H_*(\mathtt{E}_n) \simeq H_*(\mathtt{C}_n, \mathbb{Z})$ gives a correspondence between representatives of the generating operations of the Gerstenhaber operad in $H_*(\mathtt{E}_n)$ and $H_*(\mathtt{C}_n, \mathbb{Z})$.

According to this theorem, any morphism $\phi : H_*(\mathtt{E}_n) \to \mathtt{P}$ towards an operad $\mathtt{P}$ is determined by its evaluation on $\mu \in H_0(\mathtt{E}_n(2))$ and $\lambda_{n-1} \in H_{n-1}(\mathtt{E}_n(2))$ since these operations generate $\mathtt{G}_n = H_*(\mathtt{E}_n)$. For the embedding $\iota : \mathtt{E}_{n-1} \to \mathtt{E}_n$ and the suspension morphism $\sigma : \mathtt{E}_n \to \Lambda^{-1}\mathtt{E}_{n-1}$, we obtain by a straightforward inspection:

0.3.5. Proposition.

(a) *The morphism* $\iota_* : H_*(\mathtt{E}_{n-1}) \to H_*(\mathtt{E}_n)$ *induced by the embedding* $\iota : \mathtt{E}_{n-1} \hookrightarrow \mathtt{E}_n$ *satisfies*

$$\iota_*(\mu) = \mu \quad \text{and} \quad \iota_*(\lambda_{n-1}) = 0, \quad \text{for each } n > 1.$$

For $n = 1$, *we also have* $\iota_*(\mu) = \mu$.

(b) *The morphism* $\sigma_* : H_*(\mathtt{E}_n) \to H_*(\Lambda^{-1}\mathtt{E}_{n-1})$ *induced by the suspension morphism* $\sigma : \mathtt{E}_n \to \Lambda^{-1}\mathtt{E}_{n-1}$ *satisfies*

$$\sigma_*(\mu) = 0 \quad \text{and} \quad \sigma_*(\lambda_{n-1}) = \lambda_{n-2}, \quad \text{for each } n > 2.$$

For $n = 2$, *we have the same formulas provided we define the degree* 0 *bracket* $\lambda_0 = \lambda_0(x_1, x_2)$ *by the commutator of the associative product* $\mu = \mu(x_1, x_2)$ *in* $H_*(\mathtt{E}_1) = \mathtt{A}$. □

To simplify notation, we omit to mark the operadic desuspension in the formula of $\sigma_*(\lambda_{n-1})$. Note however that this desuspension reverses the parity of the operation λ_{n-2}.

We aim to study the Koszul dual of the operads $\mathtt{E}_n$. At the homology level, we have the following statement:

0.3.6. Fact (see [15, Theorem 3.1]**,** [16, §2.1.11 and §4.2.7]**).**

(a) *The associative operad* $\mathtt{A}$ *is Koszul with* $\mathtt{K}(\mathtt{A}) = \Lambda^{-1}\mathtt{A}^\vee$. *Let* $\mu^\vee \in \mathtt{A}^\vee(2)$ *be the dual element of* $\mu \in \mathtt{A}(2)$. *The Koszul duality equivalence* $\epsilon : \mathtt{B}^c(\Lambda^{-1}\mathtt{A}^\vee) \xrightarrow{\sim} \mathtt{A}$ *is determined by*

$$\epsilon(\mu^\vee) = \mu.$$

(b) *The Gerstenhaber operad* $\mathtt{G}_n$ *is Koszul with* $\mathtt{K}(\mathtt{G}_n) = \Lambda^{-n}\mathtt{G}_n^\vee$ *as a Koszul dual cooperad, for each* $n > 1$. *Let* $\mu^\vee, \lambda_{n-1}^\vee \in \mathtt{G}_n^\vee(2)$ *be the dual basis of* $\mu, \lambda_{n-1} \in \mathtt{G}_n(2)$. *The Koszul duality equivalence* $\epsilon : \mathtt{B}^c(\Lambda^{-n}\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$ *is determined by*

$$\epsilon(\mu^\vee) = \lambda_{n-1} \quad \text{and} \quad \epsilon(\lambda_{n-1}^\vee) = \mu.$$

In this statement, we also omit to mark operadic suspensions on elements though the suspension operation modifies degrees and symmetric group actions. Recall that a Koszul duality equivalence is supposed to vanish on generators of arity $r > 2$ in general (see §0.2.6). Therefore the formulas of the fact fully determine the Koszul duality equivalences associated to the associative and Gerstenhaber operads.

Assertion (a) about the associative operad is very classical (see [16, §4.2.7]) and holds in the context of modules over a ring (see [12, §5.2]). Assertion (b) about the Gerstenhaber operad is established in [15, Theorem 3.1] and in [27] by another argument. To see that the result holds in the context of modules over a ring, we first have to check (according to [12, §5.2.8]) that the underlying $\mathbb{Z}$-modules of the commutative operad are projective (obvious) as well as the underlying $\mathbb{Z}$-modules of the Lie operad (see [8, 31] or use the argument of [12, §5.2.8]). In this setting, assertion (b) can also be deduced from the commutative and Lie operad cases addressed in [12, Theorems 6.5-6.7] and the argument of [27]. An alternative and direct proof of Fact 0.3.6 follows from [21] (adapt the examples of this reference to check that the associative operad and the Gerstenhaber operads are all Poincaré-Birkhoff-Witt for any choice of ground ring).

The next proposition shows that the Koszul duality equivalences of Gerstenhaber operads fit nicely together:

0.3.7. Proposition. *The Koszul duality gives weak-equivalences*

$$\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$$

so that the diagram

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\Lambda^{-1}\,\mathtt{A}^\vee) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-2}\,\mathtt{G}_2^\vee) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) & \xrightarrow{\sigma^*} & \cdots \\
{\scriptstyle \epsilon_1=\epsilon}\downarrow{\scriptstyle \sim} & & {\scriptstyle \epsilon_2}\downarrow{\scriptstyle \sim} & & & & {\scriptstyle \epsilon_n}\downarrow{\scriptstyle \sim} & & \\
\mathtt{A} & \xrightarrow[\iota_*]{} & \mathtt{G}_2 & \xrightarrow[\iota_*]{} & \cdots & \xrightarrow[\iota_*]{} & \mathtt{G}_n & \xrightarrow[\iota_*]{} & \cdots
\end{array}$$

commutes, where:

- *the lower row morphisms ι_* are induced by the operad embeddings $\iota : \mathtt{E}_{n-1} \to \mathtt{E}_n$,*
- *the upper row morphisms σ^* are defined by the application of the functors $\mathtt{B}^c(\Lambda^{-n} H_*(-)^\vee) = \mathtt{B}^c(H_*(\Lambda^n -)^\vee)$ to the suspension morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$*

Proof. The cobar construction is a quasi-free operad $\mathtt{B}^c(\mathtt{D}) = (\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}), \partial)$ by definition and it is sufficient to check the commutativity of each square on generating elements. Recall that the Koszul duality equivalences are supposed to vanish on generating elements of arity $r > 2$. Thus we only have to check identities

$$\iota_*\epsilon_{n-1}(\gamma^\vee) = \epsilon_n\sigma^*(\gamma^\vee)$$

when $\gamma^\vee$ is a basis element of $\Lambda^{-1}\,\mathtt{A}^\vee(2)$ for $n = 2$, respectively $\Lambda^{1-n}\,\mathtt{G}_{n-1}^\vee(2)$ for $n > 2$. This verification is immediate from the formulas of Proposition 0.3.5 and Fact 0.3.6. □

0.4. Statement of the main theorems

The upper and lower rows of the diagram of Proposition 0.3.7 are defined at the chain level. The main task of this paper is to prove that the vertical morphisms have a realization at the chain level too:

Theorem A. *We have a sequence of morphisms*

$$\psi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \to \mathtt{E}_n, \quad n \geq 1,$$

beginning with the Koszul duality equivalence of the associative operad for $n = 1$, and such that:

(a) *the diagram*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\Lambda^{-1}\,\mathtt{E}_1^\vee) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-2}\,\mathtt{E}_2^\vee) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) & \xrightarrow{\sigma^*} & \cdots, \\
{\scriptstyle \psi_1=\epsilon_1}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_n} & & \\
\mathtt{E}_1 & \xhookrightarrow[\iota]{} & \mathtt{E}_2 & \xhookrightarrow[\iota]{} & \cdots & \xhookrightarrow[\iota]{} & \mathtt{E}_n & \xhookrightarrow[\iota]{} & \cdots
\end{array}$$

commutes;

(b) *the composite of morphism induced by ψ_n in homology with the edge morphism of Proposition 0.2.7 reduces to the Koszul duality equivalence*

$$\epsilon_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) \xrightarrow{\sim} \mathtt{G}_n$$

of propositions 0.3.6-0.3.7.

In this statement, the notation $\sigma^* : \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^\vee) \to \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ refers to the image of the suspension morphism $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ under the functor $\mathtt{B}^c(\Lambda^{-n}(-)^\vee) = \mathtt{B}^c((\Lambda^n -)^\vee)$.

To justify the second assertion of the theorem, note that the cooperad $\Lambda^{-n}\,\mathtt{E}_n^\vee$ is homologically Koszul, because, according to Fact 0.3.6, the homology operads $\mathtt{G}_n = H_*(\mathtt{E}_n)$ are Koszul with dual $\Lambda^{-n}\,\mathtt{G}_n^\vee = H_*(\Lambda^{-n}\,\mathtt{E}_n^\vee)$.

Theorem A is only one part of our main theorem, stated in the introduction of the article, but the other assertions of this statement will be automatically fulfilled as soon as we establish Theorem A:

Theorem B. *The morphisms $\psi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \to \mathtt{E}_n$ constructed in Theorem A are automatically weak-equivalences.*

Proof. According to the statement of Theorem A, the morphism induced by ψ_n in homology fits in a commutative diagram

$$\begin{array}{ccc}
\mathtt{B}^c(\Lambda^{-n}\,\mathtt{G}_n^\vee) & \xrightarrow[\sim]{\eta} & H_*(\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)), \\
{\scriptstyle \epsilon_n}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \psi_{n*}} \\
\mathtt{G}_n & \xrightarrow[\simeq]{} & H_*(\mathtt{E}_n)
\end{array}$$

where η denotes the edge morphism of Fact 0.2.7. The conclusion of the theorem is an immediate consequence of the fact that ϵ_n and η are both weak-equivalences. $\square$

In §1.3, we check that each morphism $\sigma^* : \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^\vee) \to \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ is a cofibration of operads too. This observation implies that each operad $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ is cofibrant as an operad (with respect to the standard model structure of [6, 19]). Thus, we deduce from our results that $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ defines a cofibrant replacement of $\mathtt{E}_n$ in the category of operads.

The following corollary of theorems A-B is also worth mentioning:

Corollary. *The result of theorems A-B extends formally to $n = \infty$ when $\Lambda^{-\infty}\,\mathtt{E}_\infty^\vee$ is defined as the colimit of the cooperads $\Lambda^{-n}\,\mathtt{E}_n^\vee$.*

To simplify, we explain how this statement follows from theorems A-B, but this case $n = \infty$ can be obtained more quickly (see brief explanations in §1.3.8).

Proof. By standard observations, the forgetful functor from cooperads to Σ_*-objects creates colimits, the forgetful functor from operads to Σ_*-objects create sequential colimits and the cobar construction, from cooperads to operads, preserves all colimits. Therefore, we have $\operatorname{colim}_n \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee) \simeq \mathtt{B}^c(\Lambda^{-\infty}\,\mathtt{E}_\infty^\vee)$ and the weak-equivalences of Theorem A yield a weak-equivalence

$$\mathtt{B}^c(\Lambda^{-\infty}\,\mathtt{E}_\infty^\vee) \xrightarrow{\sim} \mathtt{E}_\infty = \mathtt{E}$$

by passing to the colimit $n \to \infty$. □

The operad $\mathtt{B}^c(\Lambda^{-\infty}\,\mathtt{E}_\infty^\vee) = \operatorname{colim}_n \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ is also cofibrant (since we prove that each morphism $\sigma^* : \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^\vee) \to \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^\vee)$ is a cofibration) and, hence, forms a cofibrant E_∞-operad according to our result.

Theorems A-B give an answer to the question of §0.1: the suspension morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ correspond to the embeddings $\iota : \mathtt{E}_{n-1} \to \mathtt{E}_n$ under the bar duality of operads. In the epilogue, we develop this remark to give an intrinsic representation of the morphism $\nabla_{S^m} : \mathtt{E} \to End_{\bar{N}^*(S^m)}$ which gives the action of an E_∞-operad on the cochain complex of the m-sphere S^m.

The next sections §§1-4 are entirely devoted to the proof of Theorem A. From now on, we use the short notation $\mathtt{D}_n = \Lambda^{-n}\,\mathtt{E}_n^\vee$ to refer to the dual cooperad of $\mathtt{E}_n$. In §3.2 and §4.2, we also use the notation $\mathtt{M}_n = \Sigma^{-1}\,\mathtt{D}_n$ to refer to the generating Σ_*-object of $\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\Sigma^{-1}\,\mathtt{D}_n), \partial)$.

1. First step: applications of bar duality and obstruction theory

The purpose of this section is to prove the following lemma giving the first step towards the proof of Theorem A:

Lemma A. *The composite $\phi_1 = \alpha\epsilon_1$ of the morphisms $\mathtt{B}^c(\mathtt{D}_1) \xrightarrow{\epsilon_1} \mathtt{A} \xrightarrow{\alpha} \mathtt{C}$, where ϵ_1 is the Koszul duality equivalence of the associative operad $\mathtt{E}_1 = \mathtt{A}$, has extensions*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\mathtt{D}_1) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_2) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow{\sigma^*} & \cdots . \\
{\scriptstyle\phi_1}\downarrow & & \vdots{\scriptstyle\exists\phi_2} & & & & \vdots{\scriptstyle\exists\phi_n} & & \\
\mathtt{C} & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots
\end{array}$$

The proof of this lemma relies on obstruction theory arguments. In §1.1, we explain that the construction of the morphisms ϕ_n amounts to the definition of elements $\omega_n(r) \in \mathtt{E}_n(r)_{\Sigma_r}$ satisfying boundary equations. In §§1.2-1.3, we review Cohen's computation of $H_*(\mathtt{E}_n(r)_{\Sigma_r} \otimes_{\mathbb{Z}} \mathbb{F}) = H_*(\mathtt{C}_n(r)_{\Sigma_r}, \mathbb{F})$ for $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_p$ in order to prove the vanishing of the homological obstructions to the definition of $\omega_n(r)$ and we reach our conclusion.

1.1. The obstruction problem

Since the cobar construction of a cooperad forms a quasi-free operad $\mathtt{B}^c(\mathtt{D}) = (\mathtt{F}(\Sigma^{-1}\bar{\mathtt{D}}), \partial)$, any morphism $\phi : \mathtt{B}^c(\mathtt{D}) \to \mathtt{P}$ towards an operad $\mathtt{P}$ is fully determined by a homomorphism of degree -1

$$\theta : \bar{\mathtt{D}} \to \mathtt{P}$$

that represents the restriction of ϕ to the generating Σ_*-object of $\mathtt{B}^c(\mathtt{D})$ (see [15, §2.3]). We call this homomorphism θ the twisting cochain associated to ϕ.

The commutation of the morphism ϕ with differentials reduces to a sequence of equations

$$\delta(\theta(r)) + \sum_{\substack{s+t-1=r\\ s,t\geq 2}} \theta(s) \smile \theta(t) = 0, \tag{9}$$

where $\theta(r) : \mathtt{D}(r) \to \mathtt{P}(r)$ represents the component of the twisting cochain θ in arity $r \geq 2$ and $\smile$ is a certain operation on homomorphisms $f(r) : \mathtt{D}(r) \to \mathtt{P}(r)$ (see also [15, §2.3]). To define a morphism $\phi : \mathtt{B}^c(\mathtt{D}) \to \mathtt{P}$, a natural idea is to construct the homomorphisms $\theta(r) : \mathtt{D}(r) \to \mathtt{P}(r)$ by induction on r. The obstruction to the existence of $\theta(r)$ is represented by the homology class of $q(\theta)(r) = \sum_{s+t=r-1} \theta(s) \smile \theta(t)$ in $\mathit{Hom}_{\mathcal{C}}(\mathtt{D}, \mathtt{P})$.

For the cooperad $\mathtt{D} = \mathtt{D}_n$ and the operad $\mathtt{P} = \mathtt{C}$ involved in our construction, we have the following reduction of this obstruction problem:

Proposition 1.1.A.

(a) *We have a bijective correspondence between operad morphisms*

$$\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$$

and collections of elements

$$\omega_n(r) \in \Lambda^n\, \mathtt{E}_n(r)_{\Sigma_r}, \quad r \geq 2,$$

of degree $\deg(\omega_n(r)) = -1$ *and so that*

$$\delta(\omega_n(r)) + \sum_{\substack{s+t-1=r\\ s,t\geq 2}} \Big\{\sum_{i=1}^{s} \pm\omega_n(s) \circ_i \omega_n(t)\Big\} = 0$$

holds for every $r \geq 2$.

(b) *The commutativity of the diagrams*

$$\begin{array}{ccc} \mathtt{B}^c(\mathtt{D}_{n-1}) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) \\ & {\scriptstyle\phi_{n-1}}\searrow \quad \swarrow{\scriptstyle\phi_n} & \\ & \mathtt{C} & \end{array}$$

amounts to the verification of equations

$$\omega_{n-1}(r) = \sigma(\omega_n(r)), \quad r \in \mathbb{N},$$

for the elements $\omega_n(r)$ *associated to the morphisms* ϕ_n.

Proof. We have by definition

$$\mathtt{D}_n(r) = \Lambda^{-n}\,\mathtt{E}_n^\vee(r) = (\Lambda^n\,\mathtt{E}_n(r))^\vee = \mathit{Hom}(\Lambda^n\,\mathtt{E}_n(r), \mathbb{Z}).$$

Since $\mathtt{E}_n(r)$ is a degreewise finitely generated free Σ_r-module, the definition of a Σ_r-equivariant homomorphism $\theta(r) : \mathtt{D}_n(r) \to \mathtt{C}(r)$ of degree -1 amounts to the definition of an element $\omega(r) \in \Lambda^n\,\mathtt{E}_n(r)_{\Sigma_r}$, of degree -1, and so that

$$\theta(r)(c) = \sum_{s\in\Sigma_r} \pm c(s\cdot\omega(r)),$$

for all $c \in \mathtt{D}_n(r)$. The sign comes from the symmetry isomorphism $\Lambda^n\,\mathtt{E}_n(r) \otimes \Lambda^{-n}\,\mathtt{E}_n^\vee(r) \simeq \Lambda^{-n}\,\mathtt{E}_n^\vee(r) \otimes \Lambda^n\,\mathtt{E}_n(r)$ involved in this relation.

By a direct application of the formula of [13, §3.7] for the product $\theta(s) \smile \theta(t)$, we see that the equation of twisting cochains (9) amounts to the equation of assertion (a) for the elements $\omega(r) \in \Lambda^n\,\mathtt{E}_n(r)_{\Sigma_r}$, $r \in \mathbb{N}$, associated to $\theta(r)$. Therefore the conclusion of assertion (a) immediately follows from the correspondence between morphisms $\phi : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ and twisting cochains $\theta : \bar{\mathtt{D}}_n \to \mathtt{C}$.

The commutativity of the diagram of assertion (b) holds if and only if the twisting cochains associated to the morphisms $\phi_n\sigma^*$ and ϕ_{n-1} agree. By taking the restriction of these morphisms $\phi_{n-1}\sigma^*$ and ϕ_n to the generating Σ_*-objects of the cobar constructions, we immediately see that the equation $\phi_n\sigma^* = \phi_{n-1}$ amounts to the commutativity of the diagram

$$\begin{array}{ccc} \mathtt{D}_{n-1} & \xrightarrow{\sigma^*} & \mathtt{D}_n \ , \\ & {\scriptstyle\theta_{n-1}}\searrow \quad \swarrow{\scriptstyle\theta_n} & \\ & \mathtt{C} & \end{array}$$

where θ_n refers to the twisting cochain associated to ϕ_n. The conclusion of assertion (b) readily follows since we have

$$\theta_n(\sigma^* c) = \theta_{n-1}(c) \Leftrightarrow \sum_{s\in\Sigma_r} \pm c(s\cdot\sigma(\omega_n(r))) = \sum_{s\in\Sigma_r} \pm c(s\cdot\omega_{n-1}(r))$$

when the twisting cochains θ_n are associated to collections $\omega_n(r)$, $r \geq 2$. □

The next lemmas are used in §1.3 to conclude to the existence of collections $\omega_n(r)$, $r\in\mathbb{N}$, that fulfil the requirements of Proposition 1.1.A.

Lemma 1.1.B. *The mapping $\sigma : \mathtt{E}_n(r) \to \Lambda^{-1}\mathtt{E}_{n-1}(r)$ is surjective for each $r\in\mathbb{N}$ and every $n\geq 2$.*

Proof. Let $\underline{w} = (w_0,\dots,w_d) \in \mathtt{E}(r)$. Suppose $w_0 = (i_1,\dots,i_r)$. Form the sequence of permutations

$$\begin{aligned} t_r = w_0 = (i_1,i_2,i_3,i_4,\dots,i_r),\\ t_{r-1} = (i_2,i_1,i_3,i_4,\dots,i_r),\quad t_{r-2} = (i_3,i_2,i_1,i_4,\dots,i_r),\quad \dots\\ \dots\quad t_1 = (i_r,i_{r-1},i_{r-2},\dots,i_1),\end{aligned}$$

and the simplex $\underline{t} = (t_1,\dots,t_{r-1},w_0,\dots,w_d)$. We have $\mu_{ij}(\underline{t}) = \mu_{ij}(\underline{w})+1$, $\forall ij$, and hence "$\underline{w}\in\mathtt{E}_{n-1}(r) \Rightarrow \underline{t}\in\mathtt{E}_n(r)$". Moreover, we have clearly $\sigma(\underline{t}) = sgn\cap\underline{t} = \pm\underline{w}$ by definition of the cochain $sgn : \mathtt{E}(r)\to\mathbb{Z}$. □

Lemma 1.1.C. *We have $H_d(\ker\{\sigma_* : \Lambda^n\mathtt{E}_n(r)_{\Sigma_r} \to \Lambda^{n-1}\mathtt{E}_{n-1}(r)_{\Sigma_r}\}) = 0$ for $d\geq -2$ and for all $r>2$.*

The proof of this lemma is postponed to the end of the next subsection, because we have to review homology computations of [10] in order to reach our conclusion.

1.2. Applications of Cohen's results

In this subsection (and in this subsection only), we have to deal with dg-modules (and algebras) over extensions of $\mathbb{Z}$. To be explicit, we take a field $\mathbb{F}$ which is either the field of rationals $\mathbb{Q}$, or a finite primary field $\mathbb{F}_p$, and we consider the category of dg-modules over $\mathbb{F}$ for which we adopt the notation $\mathcal{C}_{\mathbb{F}}$. Note that every dg-module of $\mathcal{C}_{\mathbb{F}}$ is cofibrant since $\mathbb{F}$ is a field. The extension of the Barratt-Eccles operad to $\mathbb{F}$

$$(\mathtt{E}_n\otimes_{\mathbb{Z}}\mathbb{F})(r) = \mathtt{E}_n(r)\otimes_{\mathbb{Z}}\mathbb{F}$$

defines an operad in $\mathcal{C}_{\mathbb{F}}$ and we consider the category of algebras over this operad in $\mathcal{C}_{\mathbb{F}}$.

The theorems of [10] gives the equivariant homology of the spaces $\mathtt{C}_n(r)$ when we take $\mathbb{F} = \mathbb{Q},\mathbb{F}_p$ as coefficients. The result is written in terms of natural operations acting on the homology of $\mathtt{C}_n$-spaces. The purpose of this subsection is to review the definition of these homology operations in order to compute the action of our suspension morphism on $H_*((\Lambda^n\mathtt{E}_n\otimes_{\mathbb{Z}}\mathbb{F})(r)_{\Sigma_r})$. Then we apply standard arguments of homological algebra to prove the vanishing property of Lemma 1.1.C. For this aim, we only determine the action of the suspension morphism in degree $* = -2$, but the full action is computable by our argument line.

The computations of [10] rely on the equivariant homology of the topological little n-cubes operads, but we need a description of the obtained result in terms of the Barratt-Eccles operad. Therefore we first revisit the general definition of homology operations acting on the homology of algebras over an operad in order to obtain an abstract characterization of the homology operations associated to E_n-operads.

1.2.1. The monad of homology operations associated to an operad. There are several equivalent definitions of the structure of an algebra over an operad $\mathtt{P}$. For the purpose of this subsection, we use the definition, going back to the introduction of operads [30], in terms of algebras over a monad $S(\mathtt{P}) : E \mapsto S(\mathtt{P}, E)$, naturally associated to $\mathtt{P}$, and given by the generalized symmetric algebra:

$$S(\mathtt{P}, E) = \bigoplus_{r=0}^{\infty} (\mathtt{P}(r) \otimes E^{\otimes r})_{\Sigma_r}.$$

In view of this definition, the structure of a $\mathtt{P}$-algebra is determined by a morphism $\lambda : S(\mathtt{P}, A) \to A$ satisfying natural associativity and unit relations with respect to structure morphisms attached to the monad $S(\mathtt{P})$. Recall that $S(\mathtt{P}, E)$ also represents the free $\mathtt{P}$-algebra associated to E and characterized by the standard universal property of free objects.

Suppose now that $\mathtt{P}$ is a Σ_*-cofibrant operad in the category of dg-modules $\mathcal{C}_{\mathbb{F}}$. Then we have a monad $T(\mathtt{P}) : E \mapsto T(\mathtt{P}, E)$ on the category of graded $\mathbb{F}$-modules together with a natural isomorphism $H_*(S(\mathtt{P}, E)) \simeq T(\mathtt{P}, H_*(E))$, for every $E \in \mathcal{C}_{\mathbb{F}}$. The structure morphism of a $\mathtt{P}$-algebra $\lambda : S(\mathtt{P}, A) \to A$ induces a morphism

$$T(\mathtt{P}, H_*(A)) = H_*(S(\mathtt{P}, A)) \xrightarrow{\lambda_*} H_*(A)$$

which provides the homology module $H_*(A)$ with the structure of an algebra over this monad $T(\mathtt{P})$. Thus, we obtain that $T(\mathtt{P})$ represents a monad of homology operations associated to $\mathtt{P}$.

Naturally, the evaluation of the monad $T(\mathtt{P})$ on a graded $\mathbb{F}$-module E is defined by the formula $T(\mathtt{P}, E) = H_*(S(\mathtt{P}, E))$, where we regard the graded module E as a dg-module equipped with a trivial differential in $S(\mathtt{P}, E)$. The existence of an isomorphism $H_*(S(\mathtt{P}, E)) \simeq T(\mathtt{P}, H_*(E))$ is implied by the degeneracy of the natural spectral sequence $E^2 = T(\mathtt{P}, H_*(E)) \Rightarrow H_*(S(\mathtt{P}, E))$ associated to $S(\mathtt{P}, E)$. To provide the functor $T(\mathtt{P}, E) = H_*(S(\mathtt{P}, E))$ with a monad structure, we essentially use that this isomorphism $H_*(S(\mathtt{P}, E)) \simeq T(\mathtt{P}, H_*(E))$ satisfies suitable coherence properties.

1.2.2. Homology classes and operations. The definition of the monad $T(\mathtt{P})$ is a reminiscence of the approach of [11, 17] for the definition of homotopical operations associated to simplicial algebras over monads. For the purpose of this article, we only need the overall idea that elements of $T(\mathtt{P}, E) = H_*(S(\mathtt{P}, E))$, where E runs over graded $\mathbb{F}$-modules, represent natural operations acting on the homology of $\mathtt{P}$-algebras (we refer to [29] for the origin of this idea).

To be explicit, let $E = \bigoplus_{i=1}^{m} \mathbb{F}\, e_i$ be the graded $\mathbb{F}$-module generated by homogeneous elements e_i of degree $\deg(e_i) = d_i$ together with a trivial differential $\delta(e_i) = 0$. The elements $\pi \in H_d(S(\mathtt{P}, E))$ represent homology operations $q_\pi : H_{d_1}(A) \times \cdots \times H_{d_m}(A) \to H_d(A)$. Formally, any choice of representatives $a_i \in A_{d_i}$ of homology classes $c_i \in H_{d_i}(A)$ determines a dg-module morphism $c : E \to A$ such that $c(e_i) = a_i$. The evaluation of q_π on the classes $(c_1, \ldots, c_m)$ is determined by the image of $\pi \in H_d(S(\mathtt{P}, E))$ under the morphism

$$H_*(S(\mathtt{P}, E)) \xrightarrow{S(\mathtt{P},c)_*} H_*(S(\mathtt{P}, A)) \xrightarrow{\lambda_*} H_*(A)$$

induced by $c : E \to A$ and the structure morphism of the $\mathtt{P}$-algebra A. The homology class $\pi \in H_d(S(\mathtt{P}, E))$ represents itself the evaluation of this operation q_π on the generating elements e_i in the homology of the free $\mathtt{P}$-algebra $H_*(S(\mathtt{P}, E))$ when we identify these elements $e_i \in E$ with fundamental homology classes of $H_*(S(\mathtt{P}, E))$.

1.2.3. Homology operations and weight gradings. For a graded module $E = \mathbb{F}\, e_1 \oplus \cdots \oplus \mathbb{F}\, e_m$, the free $\mathtt{P}$-algebra $S(\mathtt{P}, E)$ inherits a natural splitting

$$S(\mathtt{P}, E) = \bigoplus_{(r_1,\ldots,r_m)} S_{(r_1,\ldots,r_m)}(\mathtt{P}, E) \tag{10}$$

such that $S_{(r_1,\ldots,r_m)}(\mathtt{P}, E) = \mathtt{P}(r) \otimes_{\Sigma_{r_*}} (\mathbb{F}\, e_1)^{\otimes r_1} \otimes \cdots \otimes (\mathbb{F}\, e_m)^{\otimes r_m}$, where we set $r = r_1 + \cdots + r_m$ and $\Sigma_{r_*} = \Sigma_{r_1} \times \cdots \times \Sigma_{r_m}$. Moreover, we have an obvious identity

$$S_{(r_1,\ldots,r_m)}(\mathtt{P}, E) = \Sigma^d\, \mathtt{P}(r)_{\Sigma_{r_*}}^{\pm^{odd}},$$

where $d = r_1 \deg(e_1) + \cdots + r_m \deg(e_m)$ and $\pm^{odd}$ refers to the possible insertion of signs for permutations of homogeneous elements e_i of odd degree. Hence, any homology class $c \in H_*(\mathtt{P}(r)_{\Sigma_{r_*}}^{\pm^{odd}})$ determines a whole collection of classes $\pi_c \in H_*(S(\mathtt{P}, E))$ by suspension and a whole collection of homology operations

$$q_{\pi_c} : H_{d_1}(A) \times \cdots \times H_{d_m}(A) \to H_d(A)$$

such that $d = r_1 d_1 + \cdots + r_m d_m + \deg(c)$. To be more explicit, we can fix an operation $p \in \mathtt{P}(r)$ representing the homology class $c \in H_*(\mathtt{P}(r)_{\Sigma_{r_*}}^{\pm^{odd}})$. The evaluation of q_{π_c} on the homology classes of given cycles $a_i \in A_{d_i}$ is yielded by the element

$$p(\underbrace{a_1, \ldots, a_1}_{r_1}, \ldots, \underbrace{a_m, \ldots, a_m}_{r_m}) \in A_d,$$

where we use the standard notation for the pointwise evaluation of operations in A.

The collection $(r_1, \ldots, r_m) \in \mathbb{N}^m$ is a weight naturally associated to c and to the corresponding operations q_{π_c}. Splitting (10) also yields a splitting of the homology module $T(\mathtt{P}, E) = H_*(S(\mathtt{P}, E))$ into homogeneous weight components. For our computations we essentially need the observation that the evaluation of operations q_{π_c} is in an obvious sense homogeneous with respect to weights. In the case of a single variable operation $c \in H_i(\mathtt{P}(r)^{\pm^{odd}}_{\Sigma_r})$, this homogeneity relation reads

$$a \in H_d(S_{(s_1,\dots,s_m)}(\mathtt{P}, E)) \Rightarrow q_{\pi_c}(a) \in H_{rd+i}(S_{(rs_1,\dots,rs_m)}(\mathtt{P}, E)).$$

To achieve the objective of this section, we need to understand the homology modules $H_*((\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r})$ associated to $\mathbb{F}$-extensions of the suspended operads $\Lambda^n \mathtt{E}_n$. For this purpose, we use that $H_*((\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r})$ is identified with the weight r component of the module of homology operations $T(\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F}\, x) = H_*(S(\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F}\, x))$, where we consider one variable x in degree 0. The suspension of operads satisfies $S(\Lambda\, \mathtt{P}, E) = \Sigma S(\mathtt{P}, \Sigma^{-1} E)$ by [15, §1.3]. Therefore we also have an identity

$$H_*((\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r}) = H_{*-n}((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r) \otimes_{\Sigma_r} (\mathbb{F}\, x_{-n})^{\otimes r}),$$

where x_{-n} is now a variable of degree $-n$. In our study, we rather consider this object $H_*((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r) \otimes_{\Sigma_r} (\mathbb{F}\, x_{-n})^{\otimes r})$ because we derive our statements from the results of [10] written in terms of operations associated to unsuspended E_n-operads.

1.2.4. Rational homology operations associated to E_n-algebras. In §§0.3.2-0.3.3, we have already mentioned the existence of operations giving a product $\mu : H_*(A) \otimes H_*(A) \to H_*(A)$ of degree 0 and a bracket $\lambda_{n-1} : H_*(A) \otimes H_*(A) \to H_{*+n-1}(A)$ of degree $n-1$ on the homology of any $\mathtt{E}_n$-algebra A. These operations are associated to elements $\mu \in H_0(\mathtt{E}_n(2))$ and $\lambda_{n-1} \in H_{n-2}(\mathtt{E}_{n-1}(2))$ defined over $\mathbb{Z}$. In the representation of §1.2.1, we form the free $\mathtt{E}_n$-algebra on two generating elements $S(\mathtt{E}_n, \mathbb{Z}\, x_1 \oplus \mathbb{Z}\, x_2)$ and we use the natural embedding

$$\mathtt{E}_n(2) \otimes (\mathbb{Z}\, x_1) \otimes (\mathbb{Z}\, x_2) \subset \mathtt{E}_n(2) \otimes_{\Sigma_2} (\mathbb{Z}\, x_1 \oplus \mathbb{Z}\, x_2)^{\otimes 2}$$

to define the homology classes

$$\mu(x_1, x_2), \lambda_{n-1}(x_1, x_2) \in H_*(S(\mathtt{E}_n, \mathbb{Z}\, x_1 \oplus \mathbb{Z}\, x_2))$$

associated to these operations.

In the case $\mathbb{F} = \mathbb{Q}$, all natural operations on the homology of an $\mathtt{E}_n$-algebra are composites of these binary operations because we have Künneth isomorphisms

$$H_*(S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{Q}, E)) \simeq S(H_*(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{Q}), H_*(E)) \simeq S(H_*(\mathtt{E}_n) \otimes_{\mathbb{Z}} \mathbb{Q}, H_*(E)),$$

for every $E \in \mathcal{C}_{\mathbb{Q}}$, and we have already mentioned that $\mathtt{G}_n = H_*(\mathtt{E}_n)$ is the operad generated by these operations.

1.2.5. Higher homology operations in the case of finite primary fields. In the case of a finite primary field $\mathbb{F} = \mathbb{F}_p$, the homology of an $\mathtt{E}_n$-algebra $H_*(A)$ inherits higher operations. Notably, we have an operation of one variable $\xi_{n-1} : H_d(A) \to H_{pd+(n-1)(p-1)}(A)$ which gives an analogue, with respect to the bracket λ_{n-1}, of the Frobenius of restricted Lie algebras. To be precise, this operation $\xi_{n-1} : H_d(A) \to H_{pd+(n-1)(p-1)}(A)$ vanishes for $d + n - 1$

odd in the case $p > 2$. In the representation of §§1.2.2-1.2.3, this Frobenius operation is associated to a homology class $c \in H_{(n-1)(p-1)}((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}_p)(p)_{\Sigma_p})$.

The homology operations associated to E_n-algebras are completely determined in [10, §2] in the case of a finite primary field $\mathbb{F} = \mathbb{F}_p$. For our purpose, the important fact is that all natural operations on the homology of an E_n-algebra are composites of the product μ, of the bracket λ_{n-1}, of the Frobenius $\xi_{n-1} : H_d(A) \to H_{pd+(n-1)(p-1)}(A)$ and of other operations of one variable $q : H_d(A) \to H_{pd+i}(A)$ (a subset of the Araki-Kudo-Dyer-Lashof operations) which, in the representation of §1.2.3, are associated to homology classes

$$c \in H_i((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}_p)(p)^{\pm^{odd}}_{\Sigma_p}),$$

of the same weight as the Frobenius, but such that $i < (n-1)(p-1)$ is less than the degree of the class associated to the Frobenius. From the last mentioned observation and the result of [10, Theorem 3.1], we obtain:

1.2.6. Proposition. *In all cases $\mathbb{F} = \mathbb{Q}, \mathbb{F}_p$, the homology module*

$$H_d((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r) \otimes_{\Sigma_r} (\mathbb{F}\, x_{-n})^{\otimes r})$$

(a) *vanishes in degree $d > 1 - r - n$;*
(b) *and is spanned in degree $d = 1 - r - n$ by elements of the form*

$$\xi^l_{n-1}(\gamma_m) = \xi^l_{n-1}(\lambda_{n-1}(\cdots \lambda_{n-1}(\lambda_{n-1}(x_{-n}, x_{-n}), x_{-n}), \ldots, x_{-n}))$$

with $r = p^l m$, where m denotes the number of occurrences of the variable in the Lie monomial (simply forget the power of the Frobenius ξ^l_{n-1} in the case $\mathbb{F} = \mathbb{Q}$ or when $1-m-n$ is odd in the case $\mathbb{F} = \mathbb{F}_p$, $p > 2$). □

Naturally, the Lie monomials $\gamma_m(x_{-n}, \ldots, x_{-n})$ vanish for $m > 2$, because of the Jacobi relation.

Our goal is to compute the action of the suspension morphism $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ on this top homology component. For this aim, we use that σ_* preserves the composition of homology operations (because σ is an operad morphism) and we determine the morphism

$$H_*(S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, E)) \xrightarrow{\sigma_*} H_*(S(\Lambda^{-1}\,\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, E)) = H_{*+1}(S(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, \Sigma E))$$

on representatives of the bracket and Frobenius operations, for any $E \in \mathcal{C}_{\mathbb{F}}$.

We already know that $\sigma_*(\lambda_{n-1}) = \lambda_{n-2}$ in $H_*(\mathtt{E}_{n-1}(2))$. We immediately deduce from this assertion:

1.2.7. Proposition. *Let $\mathbb{F}$ be any ring over $\mathbb{Z}$. For any pair of elements (a, b) in the homology of a free $\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}$-algebra $A = S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, E)$, we have the relation*

$$\sigma_*(\lambda_{n-1}(a, b)) = \lambda_{n-2}(\sigma_*(a), \sigma_*(b))$$

in $H_*(S(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, \Sigma E))$. □

We also have:

1.2.8. Proposition. *Let $\mathbb{F} = \mathbb{F}_p$ be any finite primary field. For any class a in the homology of a free $\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}$-algebra $A = S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, E)$, we have the relation*

$$\sigma_*(\xi_{n-1}(a)) = \xi_{n-2}(\sigma_*(a))$$

in $H_*(S(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, \Sigma E))$.

Proof. To prove this proposition, we use that the homology module

$$H_{(p-1)(n-1)}((\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(p)_{\Sigma_p})$$

is one dimensional for each $n > 1$ and has a generating homology class represented by the operation $\xi_{n-1}(x)$, where x is a variable of degree 0 (see [10, Theorem 3.1]).

Note that the image of $\xi_{n-1}(x)$ under the morphism σ_* is defined by an element of the homology module

$$H_{(p-1)(n-1)}(\Lambda^{-1}(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(p)_{\Sigma_p}) = H_{(p-1)(n-1)-(p-1)}((\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(p)_{\Sigma_p}),$$

which is also one dimensional. Thus, the image of $\xi_{n-1}(x)$ under the suspension morphism

$$\begin{aligned} H_*(S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F}\, x)) \xrightarrow{\sigma_*} H_*(S(\Lambda^{-1}\, &\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F}\, x)) \\ &= H_{*+1}(S(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F}\, y)) \end{aligned}$$

is a multiple of $\xi_{n-2}(y)$, where the variable y represents a one-fold suspension of x. By definition of the operation ξ_{n-1}, we have a relation

$$\sigma_*(\xi_{n-1}(a)) = C^{\mathrm{st}} \cdot \xi_{n-2}(\sigma_*(a))$$

for any class a in the homology of a free $\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}$-algebra $S(\mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F}, E)$, where C^{st} is a multiplicative constant (possibly null).

According to [10, Theorem 1.3], the Frobenius operation ξ_{n-1} satisfies the restriction relation $\lambda_{n-1}(\xi_{n-1}(a), b) = \mathrm{ad}^p_{n-1}(a)(b)$, for every pair of (independent) variables (a, b), where we set $\mathrm{ad}_{n-1}(a) = \lambda_{n-1}(a, -)$. By applying the morphism σ_* to this relation, we obtain an equation:

$$\begin{aligned} & C^{\mathrm{st}} \cdot \lambda_{n-2}(\xi_{n-2}(\sigma_*(a)), \sigma_*(b)) = \mathrm{ad}^p_{n-2}(\sigma_*(a))(\sigma_*(b)) \\ \Rightarrow \quad & C^{\mathrm{st}} \cdot \mathrm{ad}^p_{n-2}(\sigma_*(a))(\sigma_*(b)) = \mathrm{ad}^p_{n-2}(\sigma_*(a))(\sigma_*(b)), \end{aligned}$$

from which we deduce the identity $C^{\mathrm{st}} = 1$. Hence, we conclude that we have an identity $\sigma_*(\xi_{n-1}(a)) = \xi_{n-2}(\sigma_*(a))$, for every a. □

By the identity $H_d(\Lambda^n \mathtt{E}_n(r)_{\Sigma_r} \otimes_{\mathbb{Z}} \mathbb{F}) = H_{d-n}((\mathtt{E}_n(r) \otimes_{\Sigma_r} (\mathbb{F}\, x_{-n})^{\otimes r})$ and Proposition 1.2.6, these propositions imply:

1.2.9. Lemma. *The morphism*

$$\sigma_* : H_d(\Lambda^n \mathtt{E}_n(r)_{\Sigma_r} \otimes_{\mathbb{Z}} \mathbb{F}) \to H_d(\Lambda^{n-1} \mathtt{E}_{n-1}(r)_{\Sigma_r} \otimes_{\mathbb{Z}} \mathbb{F})$$

induced by the suspension $\sigma : \mathtt{E}_n \to \Lambda^{-1} \mathtt{E}_{n-1}$ is an isomorphism in degree $d = 1 - r$, for each primary field $\mathbb{F} = \mathbb{Q}, \mathbb{F}_p$. □

Then we obtain:

1.2.10. Lemma. *We have*

$$H_d(\ker\{\sigma_* : \Lambda^n \mathtt{E}_n(r)_{\Sigma_r} \to \Lambda^{n-1} \mathtt{E}_{n-1}(r)_{\Sigma_r}\} \otimes_{\mathbb{Z}} \mathbb{F}) = 0$$

when $d \geq -2$ and $r > 2$, for each primary field $\mathbb{F} = \mathbb{Q}, \mathbb{F}_p$.

Proof. Since $\sigma_* : \Lambda^n \mathtt{E}_n(r)_{\Sigma_r} \to \Lambda^{n-1} \mathtt{E}_{n-1}(r)_{\Sigma_r}$ forms a surjective morphism of free $\mathbb{Z}$-modules in all degrees by Lemma 1.1.B, we have

$$\begin{aligned} &\ker\{\sigma_* : \Lambda^n \mathtt{E}_n(r)_{\Sigma_r} \to \Lambda^{n-1} \mathtt{E}_{n-1}(r)_{\Sigma_r}\} \otimes_{\mathbb{Z}} \mathbb{F} \\ &\qquad = \ker\{\sigma_* : (\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r} \to \Lambda^{n-1}(\mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r}\} \end{aligned}$$

and we obtain a short exact sequence of dg-modules

$$0 \to \ker\{\sigma_*\} \otimes_{\mathbb{Z}} \mathbb{F} \to (\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r} \xrightarrow{\sigma_*} (\Lambda^{n-1} \mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r} \to 0.$$

Note that the homology modules considered in Lemma 1.2.9 are trivial when $d > 1 - r$ and we have $-2 = 1 - r$ for $r = 3$ and $-2 > 1 - r$ for $r > 3$. In the only non-trivial case $d = -2 = 1 - r$, we obtain a homology exact sequence

$$\begin{aligned} 0 = H_{-1}((\Lambda^{n-1} \mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r}) &\to H_{-2}(\ker\{\sigma_*\} \otimes_{\mathbb{Z}} \mathbb{F}) \\ &\to H_{-2}((\Lambda^n \mathtt{E}_n \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r}) \xrightarrow{\sigma_*} H_{-2}((\Lambda^{n-1} \mathtt{E}_{n-1} \otimes_{\mathbb{Z}} \mathbb{F})(r)_{\Sigma_r}) \end{aligned}$$

and the result of Lemma 1.2.9 implies our conclusion. □

Since the vanishing property of Lemma 1.2.10 holds for every primary field $\mathbb{F} = \mathbb{Q}, \mathbb{F}_p$, we conclude that the homology of the unreduced dg-module

$$\ker\{\sigma_* : \mathtt{E}_n(r)_{\Sigma_r} \to \Lambda^{-1} \mathtt{E}_{n-1}(r)_{\Sigma_r}\}$$

vanishes in degree $d \geq -2$ and this finishes the proof of Lemma 1.1.C. □

1.3. Conclusion of the first step

We can now solve the obstruction problems stated in §1.1:

1.3.1. Lemma. *We have a full set of elements $\omega_n(r) \in \Lambda^n \mathtt{E}_n(r)_{\Sigma_r}$, $r \geq 2$, for $n \geq 1$, satisfying the requirements of Proposition 1.1.A:*

$$(11)\quad \delta(\omega_n(r)) + \sum_{\substack{s+t-1=r\\ s,t\geq 2}} \Bigl\{\sum_{i=1}^{s} \pm\omega_n(s) \circ_i \omega_n(t)\Bigr\} = 0, \quad \textit{for all } r > 2,\ n > 1,$$

$$(12)\quad \sigma(\omega_n(r)) = \omega_{n-1}(r),$$

with $\deg(\omega_n(r)) = n(r-1) - 1$.

Proof. We define the elements $\omega_n(r)$, $r = 2, 3, \dots$, by induction on n and r. For $\omega_n(2) \in \Lambda^n \mathtt{E}_n(2)_{\Sigma_2}$, we take the coinvariance class of the alternate simplex $(\mathrm{id}, \tau, \mathrm{id}, \dots)$ when is n odd, respectively $(\tau, \mathrm{id}, \tau, \dots)$ when n is even. We easily check that $\delta(\mathrm{id}, \tau, \mathrm{id}, \dots) \equiv \delta(\tau, \mathrm{id}, \tau, \dots) \equiv 0$ in $\Lambda^n \mathtt{E}_n(2)_{\Sigma_2}$ and we also obviously have $\sigma(\mathrm{id}, \tau, \mathrm{id}, \dots) = (\tau, \mathrm{id}, \dots)$, respectively $\sigma(\tau, \mathrm{id}, \tau, \dots) = (\mathrm{id}, \tau, \dots)$. (Note: for odd n, we crucially use the change of parity produced by the operadic suspension to obtain the vanishing of the differential.)

For $n = 1$, we have $\omega_1(2) = \mu \in \Lambda\,\mathtt{E}_1(2)$ and we also take $\omega_1(r) = 0$ for $r > 2$. This element corresponds to the usual twisting cochain $\theta_1 : \Lambda^{-1}\bar{\mathtt{A}}^\vee \to \mathtt{A}$. In fact, our choice for $\omega_n(2)$ is forced by the definition of $\omega_1(2)$, since our analysis shows that σ induces an isomorphism between the top components of the dg-modules $\Lambda^n\,\mathtt{E}_n(2)_{\Sigma_2}$ in which the elements $\omega_n(2)$ lie. (Remark: we use a refinement of this observation in §4.2.)

Now, suppose we have defined suitable elements $\omega_m(s)$ for $m < n$ and for $m = n$ and $2 \leq s < r$. Since σ is surjective (by Lemma 1.1.B), we can pick an element $\omega_n^0(r)$ such that $\sigma(\omega_n^0(r)) = \omega_{n-1}(r)$.

For an element of the form $\omega_n(r) = \omega_n^0(r) + \chi_n(r)$, with $\chi_n(r) \in \ker\sigma_*$, Equation (11) amounts to:

$$\delta(\chi_n(r)) = -\sum_{s+t-1=r}\Big\{\sum_{i=1}^{s} \pm\omega_n(s)\circ_i\omega_n(t)\Big\} - \delta(\omega_n^0(r)). \tag{13}$$

The suspension morphism maps the right-hand side of this equation to

$$\begin{aligned}\pm\sum_{s+t-1=r}&\Big\{\sum_{i=1}^{s} \pm\sigma(\omega_n(s))\circ_i\sigma(\omega_n(t))\Big\} + \pm\delta(\sigma(\omega_n^0(r)))\\ &= \pm\sum_{s+t-1=r}\Big\{\sum_{i=1}^{s} \pm\omega_{n-1}(s)\circ_i\omega_{n-1}(t)\Big\} + \pm\delta(\omega_{n-1}(r)) = 0.\end{aligned}$$

By induction, we also have:

$$\delta\Big\{\sum_{s+t-1=r}\Big\{\sum_{i=1}^{s} \pm\omega_n(s)\circ_i\omega_n(t)\Big\}\Big\} = 0.$$

The operations involved in the formation of this relation have an abstract categorical interpretation. Therefore the coherence of signs in our equations is guaranteed by the internal coherence of the symmetric monoidal structure of dg-modules.

Since we prove that the homology of $\ker\sigma_*$ vanishes in degree $n(r-1)-2$, Equation (13) admits a solution $\chi_n(r) \in \ker\sigma_*$ and this completes the definition of the element $\omega_n(r)$. □

And from Proposition 1.1.A we conclude:

1.3.2. Theorem. *We have a sequence of operad morphisms:*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\mathtt{D}_1) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_2) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow{\sigma^*} & \cdots.\\
\downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\exists\phi_2} & & & & \downarrow{\scriptstyle\exists\phi_n} & & \\
\mathtt{C} & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots
\end{array}$$

as asserted in Lemma A. □

Hence, the proof of Lemma A is complete. □

Lemma 1.1.B can be improved to:

1.3.3. Lemma. *The suspension morphism $\sigma : \mathtt{E}_n(r) \to \Lambda^{-1}\,\mathtt{E}_{n-1}(r)$ is surjective and its kernel forms a projective Σ_r-module in all degrees and for every $r \in \mathbb{N}$.* □

(Use that $\mathtt{E}_{n-1}(r)$ forms a free Σ_r-module in each degree to define a Σ_r-equivariant section of σ.)

We also have:

1.3.4. Lemma. *Let $n \in \mathbb{N}$. The underlying chain complex of $\mathtt{E}_n(r)$ is bounded, for each $r \in \mathbb{N}$.*

Proof. Let $\underline{w} = (w_0, \dots, w_d)$ be a simplex of $\mathtt{E}_n(r)$. Since $(w_0, \dots, w_d)$ is supposed to be non-degenerate, we have $w_k|_{ij} \neq w_{k+1}|_{ij}$ for some pair ij (otherwise we would have $w_k = w_{k+1}$). Hence, the weights of $\underline{w}$ satisfy the relation $d \leq \sum_{ij} \mu_{ij}(\underline{w})$, from which we deduce the inequality $d < nr(r-1)/2$ since we have $\mu_{ij}(\underline{w}) < n$ for all pairs ij by definition of $\mathtt{E}_n(r)$. □

These statements imply:

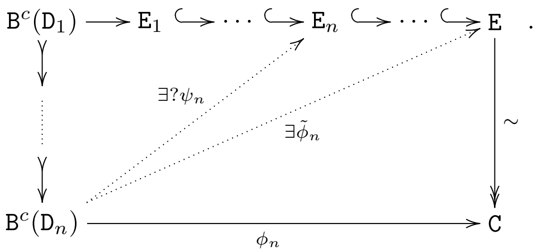
1.3.5. Proposition. *The morphism $\sigma^* : \mathtt{D}_{n-1} \to \mathtt{D}_n$ defines a cofibration of Σ_*-objects.*

Proof. Use the characterization of [22, §§2.3.6-2.3.9] for cofibrations in a category of dg-modules over a ring. □

Hence, according to [13, Proposition 1.4.13], we obtain:

1.3.6. Proposition. *The morphism $\sigma^* : \mathtt{B}^c(\mathtt{D}_{n-1}) \to \mathtt{B}^c(\mathtt{D}_n)$ induced by the suspension morphism defines a cofibration of operads.* □

From this proposition, we deduce the existence of a lifting $\tilde{\phi}_n$ in the diagram:



But the desired morphism is ψ_n.

1.3.7. Why do we not try to prove the existence of ψ_n by the same method as ϕ_n? The morphisms ψ_n are associated to twisting cochains $\eta_n : \bar{\mathtt{D}}_n \to \mathtt{E}_n$ such that $\iota\eta_{n-1} = \eta_n\sigma^*$, or equivalently to Σ_r-coinvariant elements of degree -1

$$\nu_n(r) \in \{\Lambda^n\,\mathtt{E}_n(r) \otimes \mathtt{E}_n(r)\}_{\Sigma_r}$$

such that $\mathrm{id} \otimes \iota(\nu_{n-1}(r)) = \sigma \otimes \mathrm{id}(\nu_n(r))$. The obstructions to the existence of $\nu_n(r)$ are represented by homology classes of degree -2 in

$$H_*(\ker\{(\sigma \otimes \mathrm{id})_* : \{\Lambda^n\,\mathtt{E}_n(r) \otimes \mathtt{E}_n(r)\}_{\Sigma_r} \to \{\Lambda^{n-1}\,\mathtt{E}_{n-1}(r) \otimes \mathtt{E}_n(r)\}_{\Sigma_r}\}).$$

The problem is that this homology does not vanish in degree -2. Hence, obstructions might occur in the construction of our morphism ψ_n.

The idea is to refine the lifting argument in order to go through these obstructions. For this aim, we use cell structures that refine filtration (1) of the Barratt-Eccles operad $\mathtt{E}$. The overall purpose of the notion of $\mathcal{K}$-operad, introduced in the next section, is to give a global abstract interpretation of these cell structures in a form suitable for our applications.

To conclude this section, observe that propositions 1.3.5-1.3.6 give as a corollary:

Proposition 1.3.A. *The cooperad* $\mathtt{D}_\infty = \operatorname{colim}_n \mathtt{D}_n$ *is cofibrant as a* Σ_**-object and the associated cobar construction* $\mathtt{B}^c(\mathtt{D}_\infty)$ *is cofibrant as an operad.* □

1.3.8. Remark. In the prologue, we observe that the main results of the paper, theorems A-B, imply that $\mathtt{B}^c(\mathtt{D}_\infty)$ forms an E_∞-operad, but this assertion can already be gained from the results of this section. Indeed, we can take the colimit $n \to \infty$ of the morphisms ϕ_n of Theorem 1.3.2 (Lemma A) to define a morphism $\phi_\infty : \mathtt{B}^c(\mathtt{D}_\infty) \to \mathtt{C}$. In order to prove that ϕ_∞ is a weak-equivalence, we observe first that $H_*(\mathtt{D}_\infty) = \operatorname{colim}_n H_*(\mathtt{D}_n) = \Lambda^{-1}\mathtt{L}^\vee$ because the verifications of Proposition 0.3.7 imply that the morphism $\sigma_* : H(\mathtt{D}_n) \to H_*(\mathtt{D}_{n-1})$ reduces to a composite

$$\Lambda^{-n}\mathtt{G}_n^\vee \to \Lambda^{-1}\mathtt{L}^\vee \hookrightarrow \Lambda^{1-n}\mathtt{G}_{n-1}^\vee .$$

Then we readily check that the composite of ϕ_∞ with the edge morphism of Proposition 0.2.7 reduces to the Koszul duality equivalence for the commutative operad and the argument of Theorem B can be applied to reach the conclusion.

2. Interlude: operads shaped on complete graph posets

The cell structures we are going to use have been introduced by C. Berger in [3, 4] as a device to compare nested sequences of operads to the sequence of little cubes operads. The existence of a homotopy equivalence between the topological realization of Smith's filtration of the Barratt-Eccles operad and little cubes operads has arisen as a significant success of Berger's article.

The cell structure is defined for operads in topological spaces in the original reference. The input data consists of a topological operad $\mathtt{P}$ whose term $\mathtt{P}(2)$ is a cellular model of the infinite dimensional sphere S^∞. Berger's idea was to use restriction operations, included in the composition structure of $\mathtt{P}$, in order to derive a cell decomposition of each $\mathtt{P}(r)$ from the cell structure of $\mathtt{P}(2)$. The universal poset $\mathcal{K}(r)$ underlying the decomposition

$$\mathtt{P}(r) = \cup_{\kappa\in\mathcal{K}(r)} \mathtt{P}(\kappa) \tag{14}$$

is defined by complete edge-label graphs with r vertices. Such an operad inherits a natural filtration by subcomplexes $\mathtt{P}_n(r) = \cup_{\kappa\in\mathcal{K}_n(r)} \mathtt{P}(\kappa)$, where $\mathcal{K}_n(r)$ is a subposet of $\mathcal{K}$ defined by bounding edge-labels. The existence of a homotopy equivalence between filtration layers $\mathtt{P}_n$ and little cubes operads $\mathtt{C}_n$

is ensured when: the cells $\mathsf{P}(\kappa)$ are contractible; the cell inclusions $\mathsf{P}(\alpha) \subset \mathsf{P}(\beta)$ satisfy standard cofibrancy conditions. In many situations, we obtain spaces $\mathsf{P}(\kappa)$ such that the contractibility condition (or the cofibrancy condition) is not satisfied. The idea of [3] is to discard these bad cells and to check whether decomposition (14) can not reduces to a union of good cells along a poset equivalent to $\mathcal{K}(r)$. In fact, such refinements are necessary to handle the cell structure of the genuine little cubes operads (see [3, 9]).

The applications of this paper require to introduce an opposite paradigm, because we use cell decompositions (14) for the purpose of constructing morphisms and not to determine the homotopy type of operads. For that reason, we completely revisit definitions of [3]. In summary, we use that the complete graph posets form an operad in posets, called the complete graph operad $\mathcal{K}$, and we define a category of diagram operads, called $\mathcal{K}$-operads, whose objects are diagram sequences $\mathsf{P}(r) = \{\mathsf{P}(\kappa)\}_{\kappa\in\mathcal{K}(r)}$, $r \in \mathbb{N}$, equipped with a composition structure shaped on the composition structure of $\mathcal{K}$. The first example of a $\mathcal{K}$-operad, of which we introduce the idea in §2.1.3, is not the Barratt-Eccles operad, but the commutative operad itself viewed as a constant $\mathcal{K}$-diagram: $\mathsf{C}(\kappa) \equiv \mathsf{C}(r)$, for all $\kappa \in \mathcal{K}(r)$. The basic idea of the constructions of §§3-4, where we complete the proof of our main theorem, is to interpret the contractibility condition in terms of a model structure within the category of $\mathcal{K}$-operads.

The notion of a $\mathcal{K}$-operad makes sense in any ambient symmetric monoidal category. This setting comprises the category of topological spaces, used in [3], but also the category of dg-modules. For our purpose, we focus on applications to operads in dg-modules.

In §2.1, we review the definition of the complete graph operad $\mathcal{K}$ and we give the definition of our category of $\mathcal{K}$-operads. In §2.2, we define the model structure on the category of $\mathcal{K}$-operads.

2.1. Operads shaped on complete graph posets

The complete graph operad $\mathcal{K}$ is defined in [3, §1.5 and §2.5(b)]. The name refers to a nice representation of the elements of this operad by complete edge-label graphs. Since the structure of $\mathcal{K}$ is essential for our purpose, we prefer to review the main features of this operad first. We give the definition of a $\mathcal{K}$-operad afterwards.

As we only deal with connected operads, we will assume $\mathcal{K}(0) = \emptyset$ and this convention differs from [3].

2.1.1. The complete graph operad. The rth term of the complete graph operad $\mathcal{K}$ is the set of pairs $\kappa = (\mu, \sigma)$ where $\mu = \{\mu_{ij}\}_{ij}$ is a collection of non-negative integers $\mu_{ij} \in \mathbb{N}$, indexed by pairs $\{i, j\} \subset \{1, \dots, r\}$, and σ is a permutation of $\{1, \dots, r\}$. As in §0.1.3, we use the notation $\sigma|_{ij}$ for the permutation of $\{i, j\}$ defined by the occurrences of $\{i, j\}$ in the sequence $\sigma = (\sigma(1), \dots, \sigma(r))$.

The pair $\kappa = (\mu, \sigma)$ is represented by an edge-label graph, with $\{1, \dots, r\}$ as a vertex set and one edge for each pair $\{i, j\}$, each edge being equipped with a weight, defined by the non-negative integer $\mu_{ij} \in \mathbb{N}$, and an orientation, defined by the permutation $\sigma|_{ij} \in \{(i,j),(j,i)\}$. In the paper, we also refer to the elements $\kappa = (\mu,\sigma)$ as oriented weight systems.

Note that the collection of orientations $\{\sigma|_{ij}\}_{ij}$ is sufficient to determine the permutation σ, but only the collections $\{\theta_{ij} \in (i,j),(j,i)\}_{ij}$ which assemble to a global ordering of the set $\{1,\dots,r\}$ are associated to permutations $\sigma \in \Sigma_r$. The comparison relation $(\mu,\sigma) \leq (\nu,\tau)$ in $\mathcal{K}(r)$ is defined by the requirement that:

$$(\mu_{ij} < \nu_{ij}) \qquad \text{or} \qquad (\mu_{ij},\sigma|_{ij}) = (\nu_{ij},\tau|_{ij}),$$

for all pairs $\{i,j\} \subset \{1,\dots,r\}$.

The symmetric group Σ_r acts on $\mathcal{K}(r)$ by poset morphisms. The element $w\kappa \in \mathcal{K}(r)$ returned by the action of a permutation $w \in \Sigma$ on a pair $\kappa = (\mu,\sigma) \in \mathcal{K}(r)$ is defined by the pair $w\kappa = (w\mu, w\sigma)$, where $w\mu$ is the collection such that $w\mu_{ij} = \mu_{w^{-1}(i)w^{-1}(j)}$ and $w\sigma$ is the composite of σ and w in the symmetric group. This definition amounts to applying the permutation w to the vertices of the edge-label graph κ.

The collection of complete graph posets $\mathcal{K}(r)$ is also equipped with partial composition products

$$\mathcal{K}(s) \times \mathcal{K}(t) \xrightarrow{\circ_e} \mathcal{K}(s+t-1), \quad e = 1,\dots,s,$$

and has the full structure of an operad in posets. We do not recall the explicit construction of this composition structure and we refer to [3, §2.5(b)] for this part of the definition. We only consider axiomatic properties of these partial composition products in our constructions and some applications to the Barratt-Eccles operads (see §2.1.7) for which we can also refer to the existing literature. Simply note that $\mathcal{K}(1)$ is reduced to one point so that $\mathcal{K}$ is connected as an operad in posets.

2.1.2. The category of $\mathcal{K}$-operads. A $\mathcal{K}$-diagram of dg-modules is a collection of functors $M : \mathcal{K}(r) \to \mathcal{C}$ on the posets $\mathcal{K}(r)$, $r \in \mathbb{N}$. We use the notation $M(\kappa)$ for the image of an element $\kappa \in \mathcal{K}(r)$ under $M : \mathcal{K}(r) \to \mathcal{C}$ and the notation $i_* : M(\alpha) \to M(\beta)$ for the morphism of dg-modules associated to an order relation $\alpha \leq \beta$ in $\mathcal{K}(r)$.

A $\mathcal{K}$-operad is a $\mathcal{K}$-diagram $\mathtt{P}$ equipped with

(a) actions of the symmetric groups Σ_r, $r \in \mathbb{N}$, defined by collections of morphisms

$$\mathtt{P}(\kappa) \xrightarrow{w_*} \mathtt{P}(w\kappa),$$

associated to each permutation $w \in \Sigma_r$, and so that the diagram

$$\begin{array}{ccc} \mathtt{P}(\alpha) & \xrightarrow{w_*} & \mathtt{P}(w\alpha) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ \mathtt{P}(\beta) & \xrightarrow[w_*]{} & \mathtt{P}(w\beta) \end{array}$$

commutes whenever we have a relation $\alpha \leq \beta$,

(b) partial composition products

$$\mathtt{P}(\alpha)\otimes\mathtt{P}(\beta)\xrightarrow{\circ_e}\mathtt{P}(\alpha\circ_e\beta),$$

defined for $e=1,\dots,s$ when $\alpha\in\mathcal{K}(s)$, and so that the diagram

$$\begin{array}{ccc} \mathtt{P}(\alpha)\otimes\mathtt{P}(\beta) & \xrightarrow{\circ_e} & \mathtt{P}(\alpha\circ_e\beta)\\ {\scriptstyle i_*\otimes i_*}\big\downarrow & & \big\downarrow{\scriptstyle i_*}\\ \mathtt{P}(\gamma)\otimes\mathtt{P}(\delta) & \xrightarrow[\circ_e]{} & \mathtt{P}(\gamma\circ_e\delta)\end{array}$$

commutes whenever we have relations $\alpha\leq\gamma$ and $\beta\leq\delta$,

(c) together a distinguished unit element $1\in\mathtt{P}(1)$,

satisfying a natural extension of the usual equivariance, unit and associativity axioms of operads:

(a) the diagram

$$\begin{array}{ccccc} \mathtt{P}(\alpha)\otimes\mathtt{P}(\beta) & & \xrightarrow{\qquad\circ_e\qquad} & & \mathtt{P}(\alpha\circ_e\beta)\\ {\scriptstyle u_*\otimes v_*}\big\downarrow & & & & \big\downarrow{\scriptstyle (u\circ_e v)_*}\\ \mathtt{P}(u\alpha)\otimes\mathtt{P}(v\beta) & & & & \mathtt{P}((u\circ_e v)(\alpha\circ_e\beta))\\ & {\scriptstyle\circ_{u(e)}}\searrow & & \nearrow{\scriptstyle =} & \\ & & \mathtt{P}(u\alpha\circ_{u(e)}v\beta) & & \end{array}$$

commutes for every $\alpha\in\mathcal{K}(s)$, $\beta\in\mathcal{K}(t)$, $e=1,\dots,s$, and $u\in\Sigma_s$, $v\in\Sigma_t$;

(b) the diagrams

$$\begin{array}{ccc} \Bbbk\,1\otimes\mathtt{P}(\kappa) & \longrightarrow & \mathtt{P}(1)\otimes\mathtt{P}(\kappa)\\ \simeq\big\downarrow & & \big\downarrow{\scriptstyle\circ_1}\\ \mathtt{P}(\kappa) & \xrightarrow[=]{} & \mathtt{P}(1\circ_1\kappa)\end{array}\qquad\begin{array}{ccc} \mathtt{P}(\kappa)\otimes\Bbbk\,1 & \longrightarrow & \mathtt{P}(\kappa)\otimes\mathtt{P}(1)\\ \simeq\big\downarrow & & \big\downarrow{\scriptstyle\circ_e}\\ \mathtt{P}(\kappa) & \xrightarrow[=]{} & \mathtt{P}(\kappa\circ_e 1)\end{array},$$

commute for every $\kappa\in\mathcal{K}(r)$ and $e=1,\dots,r$;

(c) the diagram

$$\begin{array}{ccccc} \mathtt{P}(\alpha)\otimes\mathtt{P}(\beta)\otimes\mathtt{P}(\gamma) & & \xrightarrow{\qquad\mathrm{id}\otimes\circ_f\qquad} & & \mathtt{P}(\alpha)\otimes\mathtt{P}(\beta\circ_f\gamma)\\ {\scriptstyle \circ_e\otimes\mathrm{id}}\big\downarrow & & & & \big\downarrow{\scriptstyle\circ_e}\\ \mathtt{P}(\alpha\circ_e\beta)\otimes\mathtt{P}(\gamma) & & & & \mathtt{P}(\alpha\circ_e(\beta\circ_f\gamma))\\ & {\scriptstyle\circ_{e+f-1}}\searrow & & \nearrow{\scriptstyle =} & \\ & & \mathtt{P}((\alpha\circ_e\beta)\circ_{e+f-1}\gamma) & & \end{array}$$

commutes for every $\alpha \in \mathcal{K}(r)$, $\beta \in \mathcal{K}(s)$, $\gamma \in \mathcal{K}(t)$, $e = 1, \dots, r$, $f = 1, \dots, s$, as well as the diagram

$$\begin{array}{ccccc} \mathtt{P}(\alpha) \otimes \mathtt{P}(\beta) \otimes \mathtt{P}(\gamma) & \xrightarrow{\simeq} & \mathtt{P}(\alpha) \otimes \mathtt{P}(\gamma) \otimes \mathtt{P}(\beta) & \xrightarrow{\circ_f \otimes \mathrm{id}} & \mathtt{P}(\alpha \circ_f \gamma) \otimes \mathtt{P}(\beta) \\ \downarrow{\scriptstyle \circ_e \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \circ_e} \\ \mathtt{P}(\alpha \circ_e \beta) \otimes \mathtt{P}(\gamma) & & & & \mathtt{P}((\alpha \circ_f \gamma) \circ_e \beta) \\ & \searrow{\scriptstyle \circ_{s+f-1}} & & \nearrow{\scriptstyle =} & \\ & & \mathtt{P}((\alpha \circ_e \beta) \circ_{s+f-1} \gamma)) & & \end{array}$$

for every pair $\{e, f\} \subset \{1, \dots, r\}$ such that $e < f$.

We also say that a $\mathcal{K}$-operad is connected if we have $\mathtt{P}(0) = 0$ and $\mathtt{P}(1) = \mathbb{Z}$. We adopt the notation $\mathcal{O}_1^{\mathcal{K}}$ for the category of connected $\mathcal{K}$-operads. Naturally, a morphism of $\mathcal{K}$-operads $\phi : \mathtt{P} \to \mathtt{Q}$ is a morphism of $\mathcal{K}$-diagrams which preserves the symmetric group action and the composition structure of operads.

2.1.3. Constant $\mathcal{K}$-operads and colimits. Let $\mathtt{P} \in \mathcal{O}_1$ be an ordinary operad. The constant $\mathcal{K}(r)$-diagrams

$$\mathtt{P}(\kappa) = \mathtt{P}(r), \quad \kappa \in \mathcal{K}(r),$$

inherit an obvious $\mathcal{K}$-operad structure. Hence we have a constant diagram functor $cst : \mathcal{O}_1 \to \mathcal{O}_1^{\mathcal{K}}$ from the usual category of operads $\mathcal{O}_1$ to the category of $\mathcal{K}$-operads $\mathcal{O}_1^{\mathcal{K}}$.

In the other direction, any $\mathcal{K}$-operad $\mathtt{P}$ has an associated ordinary operad $\operatorname{colim}_{\mathcal{K}} \mathtt{P}$ defined by the colimits of its underlying $\mathcal{K}(r)$-diagrams:

$$(\operatorname*{colim}_{\mathcal{K}} \mathtt{P})(r) = \operatorname*{colim}_{\kappa \in \mathcal{K}(r)} \mathtt{P}(\kappa).$$

The action of symmetric groups and the composition structure of $\operatorname{colim}_{\mathcal{K}} \mathtt{P}$ is defined by patching the action of symmetric groups and the composition structure of $\mathtt{P}$ on colimits. Hence, we also have a functor $\operatorname{colim}_{\mathcal{K}} : \mathcal{O}_1^{\mathcal{K}} \to \mathcal{O}_1$ from the category of $\mathcal{K}$-operads $\mathcal{O}_1^{\mathcal{K}}$ to the usual category of operads $\mathcal{O}_1$.

In what follows, we say that an operad in dg-modules $\mathtt{P}$ has a $\mathcal{K}$-structure if there is a given $\mathcal{K}$-operad $\mathtt{P}(\kappa)$ such that $\mathtt{P}(r) = \operatorname{colim}_{\kappa \in \mathcal{K}(r)} \mathtt{P}(\kappa)$. Similarly, we say that an operad morphism $f : \mathtt{P} \to \mathtt{Q}$ is realized by a morphism of $\mathcal{K}$-operads if the operads $\mathtt{P}$ and $\mathtt{Q}$ have a $\mathcal{K}$-structure and f is the colimit of a given morphism of $\mathcal{K}$-operads.

Throughout the paper, we take the same notation for the underlying $\mathcal{K}$-operad $\mathtt{P}(\kappa)$ of an operad $\mathtt{P}(r) = \operatorname{colim}_{\kappa \in \mathcal{K}(r)} \mathtt{P}(\kappa)$ equipped with a $\mathcal{K}$-structure. The letter $\mathtt{P}$ can denote either the one or the other object. Usually, the structure to which we refer is clearly determined by the context. Otherwise, we simply use letters $r, s, t, \dots \in \mathbb{N}$ and $\alpha, \beta, \kappa, \dots \in \mathcal{K}(r)$ as dummy variables to mark the distinction.

The usual adjunction between colimits and constant diagrams gives:

2.1.4. Proposition. *The colimit functor* $\operatorname{colim}_{\mathcal{K}} : \mathcal{O}_1^{\mathcal{K}} \to \mathcal{O}_1$ *is left adjoint to the constant functor* $cst : \mathcal{O}_1 \to \mathcal{O}_1^{\mathcal{K}}$.

Proof. Straightforward: check that the augmentation $\epsilon : \operatorname{colim}_{\mathcal{K}(r)}(\mathtt{P}(r)) \to \mathtt{P}(r)$ and the unit $\eta : \mathtt{P}(\kappa) \to \operatorname{colim}_{\kappa\in\mathcal{K}(r)} \mathtt{P}(\kappa)$ yielded by the usual adjunction of colimits preserve operad structures. ☐

2.1.5. The nested sequence associated to a $\mathcal{K}$-operad. The complete graph operad has a nested sequence of suboperads

$$\mathcal{K}_1 \subset \cdots \subset \mathcal{K}_n \subset \cdots \subset \operatorname*{colim}_n \mathtt{K}_n = \mathcal{K}$$

whose terms $\mathcal{K}_n(r)$ consist of oriented weight-systems $\kappa = (\mu,\sigma) \in \mathcal{K}(r)$ satisfying $\max_{ij}(\mu_{ij}) < n$. By convention, we also set $\mathcal{K}_\infty = \mathcal{K}$.

The construction of §2.1.3 can be applied to produce, from the structure of a $\mathcal{K}$-operad $\mathtt{P}$, a sequence of operads

$$\operatorname*{colim}_{\mathcal{K}_1} \mathtt{P} \to \cdots \to \operatorname*{colim}_{\mathcal{K}_n} \mathtt{P} \to \cdots \to \operatorname*{colim}_n \{\operatorname*{colim}_{\mathcal{K}_n} \mathtt{P}\} = \operatorname*{colim}_{\mathcal{K}} \mathtt{P}$$

so that $(\operatorname{colim}_{\mathcal{K}_n} \mathtt{P})(r) = \operatorname{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{P}(\kappa)$. The morphisms $\operatorname{colim}_{\mathcal{K}_{n-1}} \mathtt{P} \to \operatorname{colim}_{\mathcal{K}_n} \mathtt{P}$ are yielded by the poset embeddings $\mathcal{K}_{n-1}(r) \subset \mathcal{K}_n(r)$.

In the next paragraph, we revisit the definition of the filtration layers of the Barratt-Eccles operad in order to prove that they arise from a $\mathcal{K}$-operad structure. In this case, the identity $\mathtt{E}_n(r) = \operatorname{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{E}(\kappa)$ is an abstract reformulation of an observation of [3].

2.1.6. The example of the Barratt-Eccles operad. Recall (see §0.1.3) that we associate to each simplex of permutations $\underline{w} = (w_0,\dots,w_d) \in \Sigma_r^{\times d+1}$ a collection of weights $\mu(\underline{w}) = \{\mu_{ij}(\underline{w})\}_{ij}$ defined by the variation numbers of the sequences $\underline{w}|_{ij} = (w_0|_{ij},\dots,w_d|_{ij})$. Let $\kappa(\underline{w}) = (\mu(\underline{w}),\sigma(\underline{w}))$ be the element of $\mathcal{K}(r)$ defined by this weight collection $\mu(\underline{w})$ and by the last permutation $\sigma(\underline{w}) = w_d$ of the simplex $\underline{w} = (w_0,\dots,w_d)$.

To each $\kappa \in \mathcal{K}(r)$ we associate the module $\mathtt{E}(\kappa) \subset \mathtt{E}(r)$ spanned by the simplices $\underline{w}$ such that $\kappa(\underline{w}) \leq \kappa$.

In [3], an analogous collection of subobjects $\mathtt{W}(\kappa)$ is defined at the level of the simplicial Barratt-Eccles operad $\mathtt{W}$. In this original article, the subobjects $\mathtt{W}(\kappa)$ are defined from the skeletal filtration of $\mathtt{W}(2)$ by an intersection

$$\mathtt{W}(\kappa) = \bigcap_{ij} r_{ij}^{-1}(\operatorname{sk}_{\mu_{ij}} \mathtt{W}(2)),$$

where $r_{ij} : \mathtt{W}(\kappa) \to \mathtt{W}(2)$ is a restriction operation deduced from the operad structure of $\mathtt{W}$. The module $\mathtt{E}(\kappa)$ is just the submodule of $\mathtt{E}(r) = N_*(\mathtt{W}(r))$ spanned by the non-degenerate simplices of $\mathtt{W}(\kappa) \subset \mathtt{W}(r)$ (we use this observation next, in §3.1.3).

The next statement follows from an easy inspection of definitions in the cited references [3, 5]:

2.1.7. Observation (see [3, §1.14 and §2.7(a)] **and** [5, §1.6]**).**

(a) *For any simplex* $\underline{w} = (w_0, \ldots, w_d)$ *and* $k = 0, \ldots, d$*, we have the relation*

$$\kappa(w_0, \ldots, \widehat{w_k}, \ldots, w_d) \leq \kappa(w_0, \ldots, w_d).$$

(b) *For any simplex* $\underline{w} \in \mathtt{E}(r)$ *and any permutation* $s \in \Sigma_r$*, we have the relations* $\mu_{ij}(s \cdot \underline{w}) = \mu_{s^{-1}(i)s^{-1}(j)}(\underline{w})$*,* $\forall ij$*, and* $\sigma(s \cdot \underline{w}) = s \cdot \sigma(\underline{w})$*, from which we deduce*

$$\kappa(s \cdot \underline{w}) = s \cdot \kappa(\underline{w}),$$

where $s \cdot \kappa(\underline{w})$ *refers to the action of* s *on the oriented weight-system associated to* $\underline{w}$*.*

(c) *For any pair of simplices* $\underline{u} \in \mathtt{E}(s)$ *and* $\underline{v} \in \mathtt{E}(t)$ *and any* $e = 1, \ldots, s$*, we have the identity*

$$\kappa(\underline{u} \circ_e \underline{v}) = \kappa(\underline{u}) \circ_e \kappa(\underline{v}).$$

A first application of these relations gives:

2.1.8. Lemma (compare with [3] **and** [5]**).**

(a) *The modules* $\mathtt{E}(\kappa) \subset \mathtt{E}(r)$ *are preserved by the differential of the Barratt-Eccles operad* $\delta : \mathtt{E}(r) \to \mathtt{E}(r)$ *and form a collection of dg-submodules of* $\mathtt{E}(r)$*.*

(b) *The morphism* $w : \mathtt{E}(r) \to \mathtt{E}(r)$ *defined by the action of a permutation* $w \in \Sigma_r$ *on* $\mathtt{E}(r)$ *maps the submodule* $\mathtt{E}(\kappa) \subset \mathtt{E}(r)$ *into* $\mathtt{E}(w\kappa) \subset \mathtt{E}(r)$*, for all* $\kappa \in \mathcal{K}(r)$*.*

(c) *The partial composition products* $\circ_e : \mathtt{E}(s) \otimes \mathtt{E}(t) \to \mathtt{E}(s+t-1)$ *maps the submodule* $\mathtt{E}(\alpha) \otimes \mathtt{E}(\beta) \subset \mathtt{E}(s) \otimes \mathtt{E}(t)$ *into* $\mathtt{E}(\alpha \circ_e \beta) \subset \mathtt{E}(s+t-1)$*, for all* $\alpha \in \mathcal{K}(s)$*,* $\beta \in \mathcal{K}(t)$*.* □

From which we deduce:

2.1.9. Proposition. *The collection of dg-modules* $\{\mathtt{E}(\kappa)\}_{\kappa \in \mathcal{K}(r)}$*,* $r \in \mathbb{N}$*, inherits a* $\mathcal{K}$*-operad structure.* □

We have moreover:

2.1.10. Proposition. *We have a natural isomorphism of operads*

$$\operatorname*{colim}_{\kappa \in \mathcal{K}_n(r)} \mathtt{E}(\kappa) \xrightarrow{\simeq} \mathtt{E}_n(r),$$

for all n*, including* $n = \infty$*.*

Proof. We have clearly $\mathtt{E}(\kappa) \subset \mathtt{E}_n(r)$, for all $\kappa \in \mathcal{K}_n(r)$, and we have by definition $\underline{w} \in \mathtt{E}(\kappa)$ if and only if $\kappa(\underline{w}) \leq \kappa$, for all $\underline{w} \in \mathtt{E}(r)$. We have a map $\mathtt{E}_n(r) \to \operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathtt{E}(\kappa)$ sending a basis element $\underline{w} \in \mathtt{E}_n(r)$ to the same element in the summand $\mathtt{E}_n(\kappa(\underline{w}))$ of the colimit $\operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathtt{E}(\kappa)$. It is easy to check that this mapping gives an inverse bijection of the natural morphism $\operatorname{colim}_{\kappa \in \mathcal{K}_n(r)} \mathtt{E}(\kappa) \to \mathtt{E}_n(r)$ yielded by the embeddings $\mathtt{E}(\kappa) \subset \mathtt{E}_n(r)$. □

In the case $n = \infty$, the proposition asserts that $\operatorname{colim}_{\kappa\in\mathcal{K}(r)} \mathtt{E}(\kappa)$ is isomorphic to $\mathtt{E}(r) = \mathtt{E}_\infty(r)$. Hence, we obtain that the Barratt-Eccles operad $\mathtt{E}$ comes equipped with a $\mathcal{K}$-structure such that the operads of §2.1.5

$$\operatorname*{colim}_{\mathcal{K}_1} \mathtt{E} \to \cdots \to \operatorname*{colim}_{\mathcal{K}_n} \mathtt{E} \to \cdots \to \operatorname*{colim}_{n}\{\operatorname*{colim}_{\mathcal{K}_n} \mathtt{E}\} = \operatorname*{colim}_{\mathcal{K}} \mathtt{E}$$

are identified with the layers $\mathtt{E}_n \subset \mathtt{E}$ of the filtration of §0.1.3.

The simplicial analogue of the identity $\mathtt{E}_n(r) = \operatorname{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{E}(\kappa)$ is used in the definition of the homotopy equivalence between the filtration layers of the Barratt-Eccles operad and the operads of little n-cubes. The proof of the existence of these homotopy equivalences in [3] involves a homotopical study of the $\mathcal{K}$-diagram underlying the Barratt-Eccles operad.

For our purpose, we just record:

2.1.11. Proposition (see [3, §1.14]**).** *Each dg-module* $\mathtt{E}(\kappa) \subset \mathtt{E}(r)$, $\kappa \in \mathcal{K}(r)$, *is contractible so that the augmentation* $\epsilon : \mathtt{E}(r) \xrightarrow{\sim} \mathbb{Z}$ *restricts to a weak-equivalence on* $\mathtt{E}(\kappa)$.

Proof. For any $\kappa = (\mu, \sigma) \in \mathcal{K}(r)$, we consider the chain homotopy $h_\sigma : \mathtt{E}(r) \to \mathtt{E}(r)$ such that $h_\sigma(w_0, \dots, w_d) = (w_0, \dots, w_d, \sigma)$. We easily see that h_σ maps the submodule $\mathtt{E}(\kappa) \subset \mathtt{E}(r)$ into itself (see [3, §1.14]) and hence restricts to a contracting homotopy on $\mathtt{E}(\kappa)$. ☐

2.2. Model structures

We apply a standard process to provide the category of $\mathcal{K}$-operads with a model structure. We study first a category of symmetric $\mathcal{K}$-diagrams $\mathcal{C}_1^{\mathcal{K}\,\Sigma_*}$ which only retain the action of symmetric groups and the $\mathcal{K}$-diagram structure of a connected $\mathcal{K}$-operad. We have an obvious forgetful functor $U : \mathcal{O}_1^{\mathcal{K}} \to \mathcal{C}_1^{\mathcal{K}\,\Sigma_*}$. We adapt the definition of the free operad to prove that this functor has a left adjoint $F : \mathcal{C}_1^{\mathcal{K}\,\Sigma_*} \to \mathcal{O}_1^{\mathcal{K}}$. We check that symmetric $\mathcal{K}$-diagrams form a cofibrantly generated model category and we use the adjunction $F : \mathcal{C}_1^{\mathcal{K}\,\Sigma_*} \rightleftarrows \mathcal{O}_1^{\mathcal{K}} : U$ to transport this model structure to the category of $\mathcal{K}$-operads $\mathcal{O}_1^{\mathcal{K}}$.

2.2.1. The category of symmetric $\mathcal{K}$-diagrams. A symmetric $\mathcal{K}$-diagram is just a sequence of $\mathcal{K}(r)$-diagrams $\{M(\kappa)\}_{\kappa\in\mathcal{K}(r)}$, $r \in \mathbb{N}$, together with symmetric group actions defined by collections of morphisms

$$M(\kappa) \xrightarrow{w_*} M(w\kappa)$$

associated to each permutation $w \in \Sigma_r$, so that the diagram of §2.1.2(a) commutes. A morphism of symmetric $\mathcal{K}$-diagrams $f : M \to N$ is a collection of $\mathcal{K}(r)$-diagram morphisms $f : M(\kappa) \to N(\kappa)$ preserving symmetric group actions.

Let $\mathcal{K}\,\Sigma_*$ be the category formed by oriented weight-systems $\kappa \in \mathcal{K}(r)$ as objects and the composites of order relations $\alpha \xrightarrow{\leq} \beta$ and permutations

$\kappa \xrightarrow{w_*} w\kappa$ as morphisms, with the convention that the diagram

$$\begin{array}{ccc} \alpha & \xrightarrow{w_*} & w\alpha \\ {\scriptstyle\leq}\downarrow & & \downarrow{\scriptstyle\leq} \\ \beta & \xrightarrow[w_*]{} & w\beta \end{array}$$

commutes in $\mathcal{K}\Sigma_*$. This category $\mathcal{K}\Sigma_*$ can be identified with the Grothendieck construction (see for instance [18, §IX.3.4]) of the collection of posets $\mathcal{K}(r)$ viewed as a functor on the small category formed by the disjoint union of symmetric groups together with the non-negative integers as objects. The next assertion is obvious from the definition:

2.2.2. Proposition. *The category of symmetric $\mathcal{K}$-diagrams is isomorphic to the category of functors $M : \mathcal{K}\Sigma_* \to \mathcal{C}$.* □

We adopt the notation $\mathcal{C}_1^{\mathcal{K}\Sigma_*}$ for the category of symmetric $\mathcal{K}$-diagrams such that $M(0) = M(1) = 0$. We only consider symmetric $\mathcal{K}$-diagrams satisfying these conditions in what follows.

By [20, Theorem 11.6.1], any category of functors $F : \mathcal{I} \to \mathcal{C}$ from a small category $\mathcal{I}$ towards a cofibrantly generated model category $\mathcal{C}$ inherits a cofibrantly generated model structure. In the case of symmetric $\mathcal{K}$-diagrams, this statement returns:

2.2.3. Proposition. *The category of symmetric $\mathcal{K}$-diagrams in dg-modules inherits a model structure so that:*

- *the weak-equivalences, respectively fibrations, are the morphisms of symmetric $\mathcal{K}$-diagrams $f : M \to N$ such that $f(\kappa) : M(\kappa) \to N(\kappa)$ forms a weak-equivalence, respectively a fibration, of dg-modules, for each $\kappa \in \mathcal{K}$;*
- *the cofibrations are the morphisms of symmetric $\mathcal{K}$-diagrams which have the left lifting property with respect to acyclic fibrations.*

This model category also inherits a set of generating (acyclic) cofibrations. □

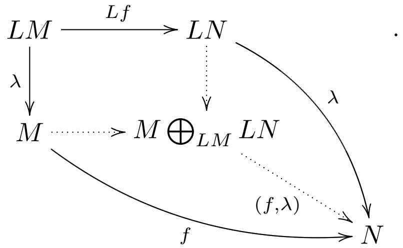
2.2.4. Latching objects. The cofibrations of symmetric $\mathcal{K}$-diagrams can be characterized effectively as retracts of relative cell complexes, as in any cofibrantly generated category. But, in this paper, we use another characterization of cofibrations of symmetric $\mathcal{K}$-diagrams which arises from a generalization of the notion of a Reedy cofibration to categories with isomorphisms.

Any symmetric $\mathcal{K}$-diagram M has a latching object LM defined by the collections of dg-modules

$$LM(\kappa) = \operatorname*{colim}_{\alpha \lneqq \kappa} M(\alpha), \text{ for } \kappa \in \mathcal{K}(r),\ r \in \mathbb{N},$$

where the colimit runs over the subposet $\{\alpha \in \mathcal{K}(r) | \alpha \lneqq \kappa\} \subset \mathcal{K}(r)$. The morphisms $M(\alpha) \to M(\kappa)$ assemble to a natural latching morphism $\lambda : LM(\kappa) \to M(\kappa)$, for each κ. Observe that LM inherits symmetric group actions and structure morphisms $i_* : LM(\alpha) \to LM(\beta)$, for every relation $\alpha \leq \beta$, so that LM forms itself a symmetric $\mathcal{K}$-diagram and the latching morphisms $\lambda : LM(\kappa) \to M(\kappa)$ define a morphism of symmetric $\mathcal{K}$-diagrams $\lambda : LM \to M$.

For a morphism of symmetric $\mathcal{K}$-diagrams $f : M \to N$, we form the relative latching morphisms



We have:

2.2.5. Proposition. *Let $i : M \to N$ be a morphism of symmetric $\mathcal{K}$-diagrams. If the relative latching morphisms*

$$(i, \lambda) : M(\kappa) \bigoplus_{LM(\kappa)} LN(\kappa) \to N(\kappa)$$

are cofibrations of dg-modules for all $\kappa \in \mathcal{K}(r)$ and all $r \in \mathbb{N}$, then i is a cofibration of symmetric $\mathcal{K}$-diagrams.

Proof. The proposition follows from a straightforward extension of the arguments of [20, §15.3.16] (the left lifting property for Reedy cofibrations) in the context of diagrams over categories with isomorphisms. The posets $\mathcal{K}(r)$ form themselves a direct Reedy category with respect to the degree map $\deg : \mathcal{K}(r) \to \mathbb{N}$ such that $\deg(\kappa) = \sum_{ij} \mu_{ij}$, for $\kappa = (\mu, \sigma) \in \mathcal{K}(r)$. The category $\mathcal{K}\,\Sigma_*$ has the disjoint sum of posets $\mathcal{K}(r)$ as direct subcategory and a trivial inverse subcategory. To adapt the arguments of [20, §15.3.16] in our setting, we essentially use the additional equivariance relation $\deg(w\kappa) = \deg(\kappa)$, valid for every permutation w, and the observation that Σ_r acts freely on $\mathcal{K}(r)$ for each $r \in \mathbb{N}$ (this latter assertion explains that no equivariance condition occurs in the statement of our proposition). □

2.2.6. The construction of free $\mathcal{K}$-operads. We adapt the construction of the ordinary free operad, briefly reviewed in §0.2.1, in order to define a free object functor $F : \mathcal{C}_1^{\mathcal{K}\,\Sigma_*} \to \mathcal{O}_1^{\mathcal{K}}$ left adjoint of the forgetful functor $U : \mathcal{O}_1^{\mathcal{K}} \to \mathcal{C}_1^{\mathcal{K}\,\Sigma_*}$. This extension of the construction of free operads is natural, but the existence of possible difficulties in applications (see §2.2.8) motivates us to give a detailed and comprehensive account of the construction. Recall that $\Theta(r)$ denotes the category formed by r-trees and isomorphisms between them and that $\tau(M)$ is an object of treewise tensors, associated to each r-tree $\tau \in \Theta(r)$, and defined for any Σ_*-object M within a symmetric monoidal category $\mathcal{E}$. If necessary, then we refer the reader to [12, §§3.3-3.4] for a detailed account of these constructions in our conventions.

The construction of the free operad can be applied to the complete graph operad $\mathcal{K}$ and returns an operad in posets $\mathtt{F}(\mathcal{K})$ together with an operad morphism $\lambda_* : \mathtt{F}(\mathcal{K}) \to \mathcal{K}$. In this context, the summand $\tau(\mathcal{K})$ of the expansion of the free operad

$$\mathtt{F}(\mathcal{K})(r) = \coprod_{\tau \in \Theta(r)} \tau(\mathcal{K})/\equiv,$$

is the set of collections $\alpha_* = \{\alpha_v\}_{v \in V(\tau)}$, where each $\alpha_v \in \mathcal{K}(r_v)$ is an oriented weight-system associated to a vertex $v \in V(\tau)$ together with a bijection between $\{1, \ldots, r_v\}$ and the ingoing edges of v. Each summand $\tau(\mathcal{K})$, defined by a cartesian product of posets $\mathcal{K}(r_v)$, has an internal poset structure and elements of $\mathtt{F}(\mathcal{K})$ are comparable only when they belong to the same summand $\tau(\mathcal{K})$ (up to tree isomorphisms).

Let M be a symmetric $\mathcal{K}$-diagram. To a tree $\tau \in \Theta(r)$ and a collection $\alpha_* = \{\alpha_v\}_{v \in V(\tau)} \in \tau(\mathcal{K})$, we associate the dg-module

$$\tau(M, \alpha_*) = \bigotimes_{v \in V(\tau)} M(\alpha_v).$$

For comparable elements $\alpha_* \leq \beta_*$ we have a morphism $i_* : \tau(M, \alpha_*) \to \tau(M, \beta_*)$ defined by the tensor product of the morphisms $i_* : M(\alpha_v) \to M(\beta_v)$ associated to the relations $\alpha_v \leq \beta_v$. Hence, we obtain that the collection $\{\tau(M, \alpha_*)\}_{\alpha_* \in \tau(\mathcal{K})}$ defines a functor on the poset $\tau(\mathcal{K})$.

To each oriented weight-system $\kappa \in \mathcal{K}(r)$, we associate the dg-module

$$\tau(M)(\kappa) = \operatorname*{colim}_{\lambda_*(\alpha_*) \leq \kappa} \tau(M, \alpha_*),$$

where the colimit ranges over the subposet of collections $\alpha_* \in \tau(\mathcal{K})$ such that $\lambda_*(\alpha_*) \leq \kappa$. This construction is clearly functorial in $\tau \in \Theta(r)$. The free $\mathcal{K}$-operad $\mathtt{F}(M)$ is defined at $\kappa \in \mathcal{K}(r)$ by the sum

$$\mathtt{F}(M)(\kappa) = \bigoplus_{\tau \in \Theta(r)} \tau(M)(\kappa)/\equiv$$

divided out by the action of automorphisms. Again (see §0.2.1), the assumption $M(0) = M(1) = 0$ implies that we can restrict the expansion of $\mathtt{F}(M)(r)$ to the subcategory of reduced trees in order to obtain a reduced expression, in which no quotient occurs, for the components of the free $\mathcal{K}$-operad $\mathtt{F}(M)$.

For the tree with one vertex of §0.2.1, we have an isomorphism $M(\kappa) \simeq \psi(M)(\kappa)$, for each $\kappa \in \mathcal{K}(r)$, and the identification of $M(\kappa)$ with this summand yields a natural morphism

$$\eta : M(\kappa) \to \mathtt{F}(M)(\kappa).$$

Our purpose is to check:

2.2.7. Proposition. *The collection of dg-modules $\{\mathtt{F}(M)(\kappa)\}_{\kappa \in \mathcal{K}(r)}$, $r \in \mathbb{N}$, inherits the structure of a $\mathcal{K}$-operad and represents the free $\mathcal{K}$-operad associated to M together with the universal morphism $\eta : M \to \mathtt{F}(M)$ defined in §2.2.6.*

Proof. The operad structure of $\mathtt{F}(M)$ is obtained by a natural extension of the standard construction of the free ordinary operad.

First, for comparable elements $\alpha \leq \beta$, the relation $\lambda_*(\gamma_*) \leq \alpha \leq \beta$ implies that each summand of $\tau(M)(\alpha)$ defines a summand of $\tau(M)(\beta)$. Hence, we have morphisms $i_* : \tau(M)(\alpha) \to \tau(M)(\beta)$, for each $\tau \in \Theta(r)$, and an induced morphism $i_* : \mathtt{F}(M)(\alpha) \to \mathtt{F}(M)(\beta)$ at the level of $\mathtt{F}(M)$, so that $\mathtt{F}(M)$ inherits the structure of a $\mathcal{K}$-diagram.

Recall that the action of a permutation on a tree reduces to a reindexing of the ingoing edges. For any permutation $w \in \Sigma_r$, we have a natural isomorphism $w_* : \tau(M, \alpha_*) \xrightarrow{\simeq} (w\tau)(M, w_*(\alpha_*))$, where $w_*(\alpha_*) \in (w\tau)(\mathcal{K})$ is the same as $\{\alpha_v\}_{v \in V(\tau)}$ in $V(w\tau) = V(\tau)$. Since $\lambda_*(\alpha_*) \leq \kappa$ implies $\lambda_*(w_*(\alpha_*)) \leq w\kappa$, these isomorphisms yield an isomorphism $w_* : \tau(M)(\kappa) \xrightarrow{\simeq} (w\tau)(M)(w\kappa)$ on each $\tau(M)$, and an isomorphism $w_* : \mathtt{F}(M)(\kappa) \to \mathtt{F}(M)(w\kappa)$ at the level of the $\mathcal{K}$-diagram $\mathtt{F}(M)$. Hence, we obtain that $\mathtt{F}(M)$ inherits the structure of a symmetric $\mathcal{K}$-diagram.

For trees $\sigma \in \Theta(s), \tau \in \Theta(t)$ and collections $\alpha_* \in \sigma(\mathcal{K}), \beta_* \in \tau(\mathcal{K})$, we have a natural isomorphism $\sigma(M, \alpha_*) \otimes \tau(M, \beta_*) \simeq (\sigma \circ_e \tau)(M, \alpha_* \circ_e \beta_*)$, where $\alpha_* \circ_e \beta_* \in \sigma \circ_e \tau(\mathcal{K})$ represents the composite of α_* and β_* in the free operad $\mathtt{F}(\mathcal{K})$ and $\sigma \circ_e \tau$ represents the standard operadic composite of trees. Since $\lambda_*(\alpha_*) \leq \gamma$ and $\lambda_*(\beta_*) \leq \delta$ implies $\lambda_*(\alpha_* \circ_e \beta_*) \leq \gamma \circ_e \delta$, these isomorphisms assemble to an isomorphism $\sigma(M)(\gamma) \otimes \tau(M)(\delta) \simeq \sigma \circ_e \tau(M)(\gamma \circ_e \delta)$, for each $\gamma \in \mathcal{K}(s)$ and $\delta \in \mathcal{K}(t)$, where $\gamma \circ_e \delta \in \mathcal{K}(s+t-1)$ represents the composite of γ and δ in $\mathcal{K}$, from which we deduce the existence of morphisms $\circ_e$: $\mathtt{F}(M)(\gamma) \otimes \mathtt{F}(M)(\delta) \to \mathtt{F}(M)(\gamma \circ_e \delta)$ which provide $\mathtt{F}(M)$ with the composition structure of a $\mathcal{K}$-operad.

The proof that $\mathtt{F}(M)$ represents the free $\mathcal{K}$-operad associated to M follows from a straightforward generalization of the case of ordinary operads for which we refer to [15, 16] (see also [13, §§1.2.4-1.2.10]). □

2.2.8. Remark. Since the components of the free $\mathcal{K}$-operad $\mathtt{F}(M)$ are defined by colimits, the existence of embeddings $M(\kappa) \subset M(r)$ for all $\kappa \in \mathcal{K}(r)$, $r \in \mathbb{N}$, does not imply the existence of embeddings $\mathtt{F}(M)(\kappa) \subset \mathtt{F}(M)(r)$ at the level of free $\mathcal{K}$-operads (at least in general). Similarly, we do not necessarily have embeddings $\tau(M)(\kappa) \subset \tau(M)(r)$ at the level of treewise tensor products. The embedding relations turn out to be satisfied for the particular operads considered in the next section (see §3.2.17) but we are not going to use this observation. In any case, some care is necessary in §3.2 when we study the realization of structures at the level of free $\mathcal{K}$-operads.

The forgetful functor $U : \mathcal{O}_1^{\mathcal{K}} \to \mathcal{C}_1^{\mathcal{K}\,\Sigma_*}$ creates limits in the category of $\mathcal{K}$-operads. The forgetful functor creates coequalizers and filtered colimits as well, but not all colimits. Nevertheless, any colimit can be identified with a reflexive coequalizer of free $\mathcal{K}$-operads. Hence, we obtain that the category of $\mathcal{K}$-operads has all colimits.

We have moreover:

2.2.9. Theorem. *The category of connected $\mathcal{K}$-operads in dg-modules inherits a model structure so that:*

- *a morphism $f : \mathtt{P} \to \mathtt{Q}$ is a weak-equivalence, respectively a fibration, of $\mathcal{K}$-operads if f defines a weak-equivalence, respectively a fibration, of symmetric $\mathcal{K}$-diagrams (we say that the forgetful functor creates weak-equivalences and fibrations);*
- *the cofibrations of $\mathcal{K}$-operads are the morphisms which have the left lifting property with respect to acyclic fibrations.*

This model category also inherits a set of generating (acyclic) cofibrations defined by the morphisms of free $\mathcal{K}$-operads $\mathtt{F}(i) : \mathtt{F}(M) \to \mathtt{F}(N)$ such that $i : M \to N$ runs over the generating (acyclic) cofibrations of the category of symmetric $\mathcal{K}$-diagrams. □

Proof. Straightforward generalization of the analysis of [19]. The elegant argument of [6, Theorem 3.1] can also be extended to the context of $\mathcal{K}$-operads. □

The assertion of Fact 2.1.11 has the following interpretation:

2.2.10. Proposition. *The augmentation morphism of the Barratt-Eccles operad $\epsilon : \mathtt{E} \to \mathtt{C}$ defines an acyclic fibration of $\mathcal{K}$-operads when we equip $\mathtt{E}$ with the $\mathcal{K}$-structure of §§2.1.6-2.1.10 and the commutative operad $\mathtt{C}$ is viewed as a constant $\mathcal{K}$-operad.* □

One proves further:

2.2.11. Proposition. *The Barratt-Eccles operad, though not cofibrant in the category of $\mathcal{K}$-operads, is cofibrant as a symmetric $\mathcal{K}$-diagram.* □

This latter proposition is only given as a remark and is not used further in the article. To give hints on its proof, simply mention that we prove a more involved but similar cofibrancy statement later on, in §4.1. Therefore we suggest the reader interested in the proof of Proposition 2.2.11 to check the argument lines of §4.1.

The arguments of [3, Theorem 2.8] can be adapted to prove that the nested sequence of §2.1.5 associated to any $\mathcal{K}$-operad satisfying propositions 2.2.10-2.2.11 is equivalent to the sequence of the chain little cubes operads as a sequence of operads in dg-modules. But we do not use $\mathcal{K}$-structures that way. Instead, we are going to use structures, like constant $\mathcal{K}$-operads, for which Proposition 2.2.11 fails.

2.2.12. Quasi-free objects. The last purpose of this subsection is to give an effective construction of cofibrations in the category of $\mathcal{K}$-operads. For this aim, we use a natural generalization of the notion of a quasi-free object in the context of $\mathcal{K}$-operads. The structure of a quasi-free $\mathcal{K}$-operad is defined explicitly by a pair $\mathtt{P} = (\mathtt{F}(M), \partial)$, where $\mathtt{F}(M)$ is a free $\mathcal{K}$-operad and ∂ is a derivation of the free $\mathcal{K}$-operad $\mathtt{F}(M)$ which is added to the natural differential of $\mathtt{F}(M)$ to produce the differential of $\mathtt{P}$. This derivation ∂ consists of a collection of homomorphisms of degree -1

$$\partial : \mathtt{F}(M)(\kappa) \to \mathtt{F}(M)(\kappa), \quad \kappa \in \mathcal{K}(r),$$

commuting with the structure morphisms $i_* : \mathtt{F}(M)(\alpha) \to \mathtt{F}(M)(\beta)$, for every pair $\alpha \leq \beta$, with the action of permutations $w_* : \mathtt{F}(M)(\kappa) \to \mathtt{F}(M)(w\kappa)$ and so that we have the derivation relation $\partial(p \circ_e q) = \partial(p) \circ_e q + \pm p \circ_e \partial(q)$ for every composite in the free operad $\mathtt{F}(M)$. The sum $\delta + \partial$, where δ refers to the natural differential of the free operad $\mathtt{F}(M)$, gives a well defined new differential on $\mathtt{F}(M)$ if and only if the derivation ∂ satisfies the relation $\delta(\partial) + \partial^2 = 0$ in the dg-module $Hom_{\mathcal{C}}(\mathtt{F}(M)(\kappa), \mathtt{F}(M)(\kappa))$, where $\delta(-)$ refers to the differential of this dg-hom object.

The derivation relation implies that ∂ is uniquely determined by its restriction to the generating symmetric $\mathcal{K}$-diagram $M \subset \mathtt{F}(M)$.

The free $\mathcal{K}$-operad associated to a symmetric $\mathcal{K}$-diagram inherits a splitting $\mathtt{F}(M) = \bigoplus_{m=0}^{\infty} \mathtt{F}_m(M)$ like the usual free operad of Σ_*-objects. Moreover, we have $\mathtt{F}_0(M) = \mathtt{I}$, $\mathtt{F}_1(M) = M$ and $\bar{\mathtt{F}}(M) = \bigoplus_{m=1}^{\infty} \mathtt{F}_m(M)$ represents the augmentation ideal of $\mathtt{F}(M)$. In general (see explanations in [13, §1.4.9]), we assume that the restriction $\partial|_M : M \to \mathtt{F}(M)$ satisfies the relation $\partial(M) \subset \bigoplus_{m \geq 2} \mathtt{F}_m(M)$.

2.2.13. Morphism on quasi-free operads. In the case of a quasi-free $\mathcal{K}$-operad $\mathtt{P} = (\mathtt{F}(M), \partial)$, a morphism of $\mathcal{K}$-operads $\phi : (\mathtt{F}(M), \partial) \to \mathtt{Q}$ is uniquely determined by its restriction to $M \subset \mathtt{F}(M)$, just like the derivation ∂. In fact, any homomorphism of degree 0

$$f : M \to \mathtt{Q}$$

gives rise to a unique homomorphism $\phi_f : \mathtt{F}(M) \to \mathtt{P}$ commuting with composition structures. This homomorphism defines a morphism of $\mathcal{K}$-operads $\phi_f : (\mathtt{F}(M), \partial) \to \mathtt{Q}$ if and only if we have the commutation relation $\delta\phi_f = \phi_f\delta + \phi_f\partial$ with respect to the differential of $\mathtt{P} = (\mathtt{F}(M), \partial)$ and the internal differential of $\mathtt{Q}$.

In the next section, we consider morphisms of quasi-free $\mathcal{K}$-operads

$$\phi_f : (\mathtt{F}(M), \partial) \to (\mathtt{F}(N), \partial)$$

induced by morphisms of symmetric $\mathcal{K}$-diagrams

$$M \xrightarrow{f} N \subset \mathtt{F}(N).$$

In that case, we have an identity $\phi_f = \mathtt{F}(f)$, where $\mathtt{F}(f)$ is the morphism of free $\mathcal{K}$-operads associated to f, and the commutation of ϕ_f with differentials amounts to the relation $\partial\phi_f = \phi_f\partial$, because $\phi_f = \mathtt{F}(f)$ automatically commutes with the internal differential of free $\mathcal{K}$-operads when f is a genuine morphism of symmetric $\mathcal{K}$-diagrams.

2.2.14. Quasi-free operad filtrations. By [13, Lemma 1.4.11], the quasi-free operad $\mathtt{P} = (\mathtt{F}(M), \partial)$ associated to a connected Σ_*-object M inherits a canonical filtration by quasi-free operads $\mathtt{P}_{\leq r} = (\mathtt{F}(M_{\leq r}), \partial)$ such that $M_{\leq r} \subset M$ is the Σ_*-object formed by the components $M(n)$ of arity $n \leq r$ of M.

This observation has a straightforward generalization in the context of $\mathcal{K}$-operads: to a symmetric $\mathcal{K}$-diagram M we associate the subobject $M_{\leq r} \subset M$ such that

$$M_{\leq r}(\kappa) = \begin{cases} M(\kappa), & \text{for all } \kappa \in \mathcal{K}(n) \text{ when } n \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Under the assumption $M(0) = 0$, the derivation of $\mathtt{P} = (\mathtt{F}(M), \partial)$ satisfies automatically $\partial(M_{\leq r}) \subset \mathtt{F}(M_{\leq r})$ (compare with [13, Lemma 1.4.11]) so that we have a quasi-free $\mathcal{K}$-operad $\mathtt{P}_{\leq r} = (\mathtt{F}(M_{\leq r}), \partial)$ such that $\mathtt{P}_{\leq r} \subset \mathtt{P}$. Under the assumption $M(0) = M(1) = 0$ and $\partial(M) \subset \bigoplus_{m \geq 2} \mathtt{F}_m(M)$, we have better, namely: $\partial(M_{\leq r}) \subset \mathtt{F}(M_{\leq r-1})$. This assumption is not necessary for the definition of the quasi-free operad $\mathtt{P}_{\leq r}$ but is needed for the proof of the next proposition.

The filtration of a quasi-free $\mathcal{K}$-operad is canonical in the sense that a morphism

$$\phi_f : \underbrace{(\mathtt{F}(M), \partial)}_{\mathtt{P}} \to \underbrace{(\mathtt{F}(N), \partial)}_{\mathtt{Q}}$$

induced by a morphism of symmetric $\mathcal{K}$-diagrams $f : M \to N$ satisfies $\phi_f(\mathtt{P}_{\leq r}) \subset \mathtt{Q}_{\leq r}$. Thus we have a commutative diagram of $\mathcal{K}$-operad morphisms

$$\begin{array}{ccc} \mathtt{P}_{\leq r} & \overset{\phi_f}{\dashrightarrow} & \mathtt{Q}_{\leq r} \\ {\scriptstyle i_r}\downarrow & & \downarrow{\scriptstyle j_r} \\ \mathtt{P} & \underset{\phi_f}{\longrightarrow} & \mathtt{Q} \end{array}$$

for each $r \in \mathbb{N}$. For our purpose, we record the following result:

2.2.15. Proposition. *Suppose that the symmetric $\mathcal{K}$-diagrams M and N are reduced ($M(0) = M(1) = 0$ and $N(0) = N(1) = 0$) and the derivations of the quasi-free operads $\mathtt{P}$ and $\mathtt{Q}$ satisfy $\partial(M) \subset \bigoplus_{m \geq 2} \mathtt{F}_m(M)$ and $\partial(N) \subset \bigoplus_{m \geq 2} \mathtt{F}_m(N)$. If f is a cofibration of symmetric $\mathcal{K}$-diagrams, then every pushout morphism*

$$\mathtt{P} \bigvee_{\mathtt{P}_{\leq r}} \mathtt{Q}_{\leq r} \xrightarrow{(\phi_f, j_r)} \mathtt{Q}, \ r \in \mathbb{N},$$

forms a cofibration of $\mathcal{K}$-operads.

Proof. Easy extension of the arguments of [13, Proposition 1.4.13] to $\mathcal{K}$-operads. □

3. Second step: applications of $\mathcal{K}$-structures

The goal of this section is to prove:

Lemma B.

(a) *Each quasi-free operad* $\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\Sigma^{-1}\mathtt{D}_n), \partial)$ *is equipped with a* $\mathcal{K}$*-structure and comes from a quasi-free* $\mathcal{K}$*-operad for which we have:*

$$\operatorname*{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa) \xrightarrow{\simeq} \operatorname*{colim}_{\kappa\in\mathcal{K}_{n+1}(r)} \mathtt{B}^c(\mathtt{D}_{n+1})(\kappa) \xrightarrow{\simeq} \cdots$$
$$\cdots \xrightarrow{\simeq} \operatorname*{colim}_{\kappa\in\mathcal{K}_\infty(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa) \xrightarrow{\simeq} \mathtt{B}^c(\mathtt{D}_n)(r).$$

(b) *The morphism* $\sigma^* : \mathtt{B}^c(\mathtt{D}_{n-1}) \to \mathtt{B}^c(\mathtt{D}_n)$ *induced by the suspension morphism of the Barratt-Eccles operad is realized by a morphism of quasi-free* $\mathcal{K}$*-operads*

$$(\mathtt{F}(\Sigma^{-1}\mathtt{D}_{n-1})(\kappa), \partial) \xrightarrow{\phi_{\sigma^*}} (\mathtt{F}(\Sigma^{-1}\mathtt{D}_n)(\kappa), \partial)$$

associated to a morphism of symmetric $\mathcal{K}$*-diagrams* $\sigma^* : \mathtt{D}_{n-1}(\kappa) \to \mathtt{D}_n(\kappa)$*.*

Then we have immediately:

Corollary. *The diagram of Lemma A*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\mathtt{D}_1) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_2) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow{\sigma^*} & \cdots . \\
\downarrow{\scriptstyle \phi_1} & & \vdots{\scriptstyle \exists\phi_2} & & & & \vdots{\scriptstyle \exists\phi_n} & & \\
\mathtt{C} & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots & \xrightarrow[=]{} & \mathtt{C} & \xrightarrow[=]{} & \cdots
\end{array}$$

is equivalent to an adjoint diagram in the category of $\mathcal{K}$*-operads, where the commutative operad* $\mathtt{C}$ *is equipped with the structure of a constant* $\mathcal{K}$*-operad.*

The first step towards the proof of Lemma B is to define $\mathcal{K}$-structures on the generating Σ_*-objects $\mathtt{D}_n$ of the quasi-free operads $\mathtt{B}^c(\mathtt{D}_n)$. This step is carried out in §3.1. Besides, we check in that section that the morphism of Σ_*-objects $\sigma^* : \mathtt{D}_{n-1} \to \mathtt{D}_n$ arises from a morphism of symmetric $\mathcal{K}$-diagrams.

From the result of §3.1, we obtain that the underlying free operad of $\mathtt{B}^c(\mathtt{D}_n)$ arises from a free $\mathcal{K}$-operad. We check in §3.2 that the twisting derivation of the cobar construction arises from a derivation of this free $\mathcal{K}$-operad. We conclude from this observation that the quasi-free operad $\mathtt{B}^c(\mathtt{D}_n)$ has a $\mathcal{K}$-structure as claimed by the first assertion of Lemma B. We also check that the morphism of symmetric $\mathcal{K}$-diagrams $\sigma^* : \mathtt{D}_{n-1} \to \mathtt{D}_n$ defined in §3.1, induce morphisms of quasi-free $\mathcal{K}$-operads on cobar constructions. We deduce the second assertion of Lemma B from this verification.

3.1. Definition of $\mathcal{K}$-structures on generating Σ_*-objects

Our first purpose is to define the $\mathcal{K}$-structure of $\mathtt{D}_n$. The idea behind our construction is to use a natural complement $\kappa^\vee$ associated to each oriented weight system κ of $\mathcal{K}_n$ and to associate to κ the submodule of $\mathtt{D}_n$ orthogonal to the complement of $\mathtt{E}_n(\kappa^\vee)$ in $\mathtt{E}_n(r)$. The precise definition of this submodule $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$ is given next. Then we check that the obtained collection $\mathtt{D}_n(\kappa)$ has the required structure to form a symmetric $\mathcal{K}$-diagram and is preserved by the dual of the suspension morphism of the Barratt-Eccles operad.

3.1.1. The complement of oriented weight systems. Let $\kappa = (\mu, \sigma)$ be any oriented weight system in $\mathcal{K}(r)$. The complement of κ in $\mathcal{K}_n$ is the oriented weight system $\kappa^\vee = (n-1-\mu, \sigma)$, where $n-1-\mu$ refers to the collection $\{n-1-\mu_{ij}\}_{ij}$. If $\kappa \in \mathcal{K}_n(r)$, then we have by definition $0 \leq \mu_{ij} < n$, $\forall ij$, from which we deduce $0 \leq n-1-\mu_{ij} < n$, $\forall ij$. Hence, we still have $\kappa^\vee \in \mathcal{K}_n(r)$. Otherwise we can assume that $\kappa^\vee = (n-1-\mu, \sigma)$ has negative weights. Observe simply that the order relation of $\mathcal{K}(r)$ has a natural extension to oriented weight-systems with negative weights. The next easy observation is used repeatedly:

3.1.2. Observation. *For every pair of oriented weight-systems $\alpha, \beta \in \mathcal{K}(r)$, we have $\alpha \leq \beta \Leftrightarrow \beta^\vee \leq \alpha^\vee$.*

3.1.3. The $\mathcal{K}$-structure of generating Σ_*-objects. In our study, we use that $\mathtt{E}_n$ is defined as a free dg-module over the non-degenerate simplices of the simplicial Barratt-Eccles operad and that the dg-modules $\mathtt{D}_n(r)$, dual to $\mathtt{E}_n(r)$, inherit a basis. The simplicial Barratt-Eccles operad $\mathtt{W}$ is defined by the simplicial sets $\mathtt{W}(r)$, $r \in \mathbb{N}$, such that $\mathtt{W}(r)_d = \Sigma^{\times d+1}$, for each dimension $d \in \mathbb{N}$. The subset of nondegenerate simplices of $\mathtt{W}(r)$, denoted by $\mathtt{N}\,\mathtt{W}(r) \subset \mathtt{W}(r)$, consists of simplices $(w_0, \ldots, w_d) \in \mathtt{W}(r)$ such that $w_j \neq w_{j+1}$ for $j = 0, \ldots, d-1$. The Barratt-Eccles chain operad $\mathtt{E}(r)$ is defined by the free $\mathbb{Z}$-module $\mathtt{E}(r) = \mathbb{Z}[\mathtt{N}\,\mathtt{W}(r)]$.

Recall that we have an oriented weight-system $\kappa(\underline{w}) = (\mu(\underline{w}), \sigma(\underline{w}))$ associated to each simplex $\underline{w} = (w_0, \ldots, w_d) \in \mathtt{W}(r)$, such that $\sigma(\underline{w}) = w_d$, the last permutation of $\underline{w}$, and the term $\mu_{ij}(\underline{w})$ of the collection $\mu(\underline{w}) = \{\mu_{ij}(\underline{w})\}_{ij}$ is defined as the number of variations of the sequence $\underline{w}|_{ij} = (w_0|_{ij}, \ldots, w_d|_{ij})$.

Smith's filtration of the simplicial Barratt-Eccles operad is defined by the simplicial sets

$$\mathtt{W}_n(r) = \left\{\underline{w} \in \mathtt{W}(r) | \mu_{ij}(\underline{w}) < n \text{ for all pairs } ij\right\}$$

(see [34]). We have clearly: $\mathtt{E}_n(r) = \mathbb{Z}[\mathtt{N}\,\mathtt{W}_n(r)]$ and $\mathtt{D}_n(r) = \left\{c : \mathtt{N}\,\mathtt{W}_n(r) \to \mathbb{Z}\right\}$, where $\mathtt{N}\,\mathtt{W}_n(r) = \mathtt{W}_n(r) \cap \mathtt{N}\,\mathtt{W}(r)$ denotes the subset of non-degenerate simplices of $\mathtt{W}_n(r)$. For any $\kappa \in \mathcal{K}(r)$, we set

$$\mathtt{D}_n(\kappa) = \left\{c : \mathtt{N}\,\mathtt{W}_n(r) \to \mathbb{Z} \text{ such that } c(\underline{w}) \neq 0 \Rightarrow \kappa(\underline{w}) \geq \kappa^\vee\right\},$$

where $\kappa^\vee$ refers to the complement, in the sense of §3.1.1, of the oriented weight-system κ in $\mathcal{K}_n$.

We have the easy observations:

3.1.4. Lemma.

(a) *If $\alpha \leq \beta$, then we have $\mathtt{D}_n(\alpha) \subset \mathtt{D}_n(\beta)$.*

(b) *The differential of* $\mathtt{D}_n(r)$ *preserves each subobject* $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$ *so that* $\mathtt{D}_n(\kappa)$ *forms a dg-submodule of* $\mathtt{D}_n(r)$.

(c) *For each permutation* $w \in \Sigma_r$*, the action of* w *on* $\mathtt{D}_n(r)$ *maps the submodule* $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$ *into* $\mathtt{D}_n(w\kappa) \subset \mathtt{D}_n(r)$.

Proof. Assertion (a) is an immediate consequence of observation 3.1.2. Assertions (b-c) are consequences of observation 2.1.7. □

From which we deduce:

3.1.5. Proposition. *The dg-modules* $\{\mathtt{D}_n(\kappa)\}_{\kappa \in \mathcal{K}(r)}$*,* $r \in \mathbb{N}$*, form a symmetric* $\mathcal{K}$*-diagram.* □

3.1.6. Dual basis and associated weights. The definition given in §3.1.3 for the dg-module $\mathtt{D}_n(\kappa)$ is more natural for most verifications of the present section. But in the next section, we heavily use the dual basis in $\mathtt{D}_n(r)$ of the basis of non-degenerate simplices $\underline{w} \in \mathtt{N}\,\mathtt{W}_n(r)$ in $\mathtt{E}_n(r)$. The basis element $\underline{s}^\vee$ dual to $\underline{s} \in \mathtt{N}\,\mathtt{W}_n(r)$ is characterized as a map $\underline{s}^\vee : \mathtt{N}\,\mathtt{W}_n(r) \to \mathbb{Z}$ by the relation:

$$\underline{s}^\vee(\underline{w}) = \begin{cases} 1, & \text{if } \underline{w} = \underline{s}, \\ 0, & \text{otherwise,} \end{cases}$$

for each $\underline{w} \in \mathtt{N}\,\mathtt{W}_n(r)$.

To a dual basis element $\underline{s}^\vee$, we associate the complement in $\mathcal{K}_n$ of the oriented weight-system associated to $\underline{s}$. In this particular case $\kappa = \kappa(\underline{s})$, we adopt the notation $\kappa^\vee(\underline{s}) = \kappa^\vee$ for the complementary oriented weight-system of $\kappa(\underline{s})$. Note again that the relations $0 \leq \mu_{ij}(\underline{s}) < n$, which characterize the simplices of $\mathtt{E}_n(r)$, imply $\kappa^\vee(\underline{s}) \in \mathcal{K}_n(r)$.

We easily obtain:

3.1.7. Observation. *For a dual basis element* $\underline{s}^\vee \in \mathtt{D}_n(r)$*, we have* $\underline{s}^\vee \in \mathtt{D}_n(\kappa)$ *if and only if* $\kappa^\vee(\underline{s}) \leq \kappa$.

We already need the dual basis for the next proposition:

3.1.8. Proposition. *The embeddings* $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$ *induce isomorphisms of* Σ_**-objects:*

$$\operatorname*{colim}_{\kappa \in \mathcal{K}_n(r)} \mathtt{D}_n(\kappa) \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \operatorname*{colim}_{\kappa \in \mathcal{K}(r)} \mathtt{D}_n(\kappa) \xrightarrow{\simeq} \mathtt{D}_n(r).$$

Proof. Let $\Psi : \mathtt{D}_n(r) \to \operatorname{colim}_{\kappa \in \mathcal{K}_m(r)} \mathtt{D}_n(\kappa)$ be the morphism of $\mathbb{Z}$-modules mapping a dual basis element $\underline{s}^\vee$ to the same element $\underline{s}^\vee$ in the summand $\mathtt{D}_n(\kappa^\vee(\underline{s}))$ of the colimit. This mapping is well defined for all $m \geq n$ since, for every basis element $\underline{s}^\vee \in \mathtt{D}_n(r)$, we have $\kappa^\vee(\underline{s}) \in \mathcal{K}_n(r)$ (see §3.1.6). This map Ψ is clearly right inverse to the natural morphism $\Phi : \operatorname{colim}_{\kappa \in \mathcal{K}_m(r)}\{\mathtt{D}_n(\kappa)\} \to \mathtt{D}_n(r)$. In the other direction, observe that a dual basis element $\underline{s}^\vee \in \mathtt{D}_n(\kappa)$ in the summand associated to any weight-system κ is identified in the colimit $\operatorname{colim}_{\kappa \in \mathcal{K}_m(r)}\{\mathtt{D}_n(\kappa)\}$ with the same element $\underline{s}^\vee$ coming from the summand $\mathtt{D}_n(\kappa^\vee(\underline{s}))$. Hence, we also have $\Psi\Phi = \mathrm{Id}$, from which we conclude that Ψ is both right and left inverse to Φ so that Φ is necessarily a bijection. □

To complete the results of this subsection, we study the mapping induced by the suspension morphism on $\mathcal{K}$-structures.

3.1.9. Proposition. *The morphism* $\sigma^* : \mathtt{D}_{n-1}(r) \to \mathtt{D}_n(r)$ *is the colimit of a morphism of symmetric* $\mathcal{K}$*-diagrams* $\sigma^* : \mathtt{D}_{n-1}(\kappa) \to \mathtt{D}_n(\kappa)$.

Proof. To obtain this result, we essentially have to check that the morphism $\sigma^* : \mathtt{D}_{n-1}(r) \to \mathtt{D}_n(r)$ maps the submodule $\mathtt{D}_{n-1}(\kappa) \subset \mathtt{D}_{n-1}(r)$ associated to any weight-system $\kappa = (\mu,\sigma) \in \mathcal{K}(r)$ into $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$. Indeed, the components of our morphism $\sigma^* : \mathtt{D}_{n-1}(\kappa) \to \mathtt{D}_n(\kappa)$ are necessarily given by restrictions of $\sigma^* : \mathtt{D}_{n-1}(r) \to \mathtt{D}_n(r)$ and whenever these restrictions exist, they automatically form a morphism of symmetric $\mathcal{K}$-diagrams corresponding to $\sigma^* : \mathtt{D}_{n-1}(r) \to \mathtt{D}_n(r)$ on the colimit.

Let $c \in \mathtt{D}_{n-1}(\kappa)$. We have by definition

$$\begin{aligned}\sigma^*(c)(w_0,\dots,w_d) &= c(sgn \cap (w_0,\dots,w_d))\\ &= sgn(w_0,\dots,w_{r-1}) \cdot c(w_{r-1},\dots,w_d),\end{aligned}$$

for any simplex $\underline{w} = (w_0,\dots,w_d) \in \mathtt{N}\,\mathtt{W}_n(r)$, where $sgn : \mathtt{E}(r) \to \mathbb{Z}$ is the cochain defined in §0.1.5.

We have clearly $(w_{r-1},\dots,w_d) \in \mathtt{N}\,\mathtt{W}(r)$, and we easily see from the definition of $sgn : \mathtt{E}(r) \to \mathbb{Z}$ that $sgn(w_0,\dots,w_{r-1}) \neq 0$ implies

$$\mu_{ij}(w_0,\dots,w_d) \geq \mu_{ij}(w_{r-1},\dots,w_d) + 1 \quad (\forall ij).$$

When this condition is satisfied, we have the implications

$$\begin{aligned}
&c(w_{r-1},\dots,w_d) \neq 0\\
&\qquad \Rightarrow \kappa(w_{r-1},\dots,w_d) \geq (n-2-\mu,\sigma)\\
&\qquad \Rightarrow \mu_{ij}(w_{r-1},\dots,w_d) > n-2-\mu_{ij}\\
&\qquad\qquad\qquad \text{or } (\mu_{ij}(w_{r-1},\dots,w_d), w_d|_{ij}) = (n-2-\mu_{ij}, \sigma|_{ij}) \quad (\forall ij)\\
&\qquad \Rightarrow \mu_{ij}(w_0,\dots,w_d) > n-1-\mu_{ij}\\
&\qquad\qquad\qquad \text{or } (\mu_{ij}(w_{r-1},\dots,w_d), w_d|_{ij}) = (n-1-\mu_{ij}, \sigma|_{ij}) \quad (\forall ij)\\
&\qquad \Rightarrow \kappa(w_0,\dots,w_d) \geq (n-1-\mu,\sigma),
\end{aligned}$$

from which we conclude: $c \in \mathtt{D}_{n-1}(\kappa) \Rightarrow \sigma^*(c) \in \mathtt{D}_n(\kappa)$. □

3.2. Extension of $\mathcal{K}$-structures to quasi-free operads

In order to define the $\mathcal{K}$-operad underlying the cobar construction $\mathtt{B}^c(\mathtt{D}_n)$, we check how the $\mathcal{K}$-structure of $\mathtt{D}_n$ extends to the quasi-free operad

$$\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\Sigma^{-1}\,\mathtt{D}_n), \partial).$$

Throughout this subsection, we adopt the short notation $\mathtt{M}_n = \Sigma^{-1}\,\mathtt{D}_n$ for the generating Σ_*-object of this quasi-free operad and its underlying symmetric $\mathcal{K}$-diagram.

By adjunction, we have clearly:

3.2.1. Observation. *For any symmetric $\mathcal{K}$-diagram M, we have a natural isomorphism*

$$\operatorname{colim}_{\mathcal{K}} \mathtt{F}(M) \simeq \mathtt{F}(\operatorname{colim}_{\mathcal{K}} M),$$

between:

- *the colimit* $\operatorname{colim}_{\mathcal{K}} \mathtt{F}(M)$ *of the free $\mathcal{K}$-operad $F(M)$ associated to the symmetric $\mathcal{K}$-diagram $M(\kappa)$ on the left-hand-side,*
- *the ordinary free operad on the colimit Σ_*-object* $(\operatorname{colim}_{\mathcal{K}} M)(r)$ *associated to the symmetric $\mathcal{K}$-diagram $M(\kappa)$ on the right-hand side.*

Hence, in the case $M = \mathtt{M}_n$, we obtain that the underlying free operad of the cobar construction $\mathtt{B}^c(\mathtt{D}_n)$ has a $\mathcal{K}$-structure. The main task of this subsection is to prove that the twisting differential of the cobar construction is realized at the $\mathcal{K}$-operad level. For this aim, we need at first to analyze the differential of dual basis elements $\underline{s}^\vee$ in the usual cobar construction of the cooperad $\mathtt{D}_n$.

3.2.2. The differential of dual basis elements in the usual cobar construction. According to the description of §§0.2.1-0.2.2, the cobar differential maps a cooperad element $c \in \mathtt{D}(r)$ to a sum of terms $\partial_\tau(c) \in \tau(\Sigma^{-1}\mathtt{D})(r)$, where τ ranges over trees with two vertices. For the $\mathbb{Z}$-dual cooperad of a dualizable operad $\mathtt{D} = \mathtt{P}^\vee$, we have a duality pairing

$$\langle -,-\rangle : \tau(\Sigma^{-1}\mathtt{D}) \otimes \tau(\Sigma\,\mathtt{P}) \to \mathbb{Z}\,.$$

The elements of the dg-module $\tau(\Sigma\,\mathtt{P})(r)$ represent formal operadic composites $p(i_1,\dots,i_e,\dots,i_s) \circ_{i_e} q(j_1,\dots,j_t)$, $p \in \mathtt{P}(s)$, $q \in \mathtt{P}(t)$ together with an input sharing $\{1,\dots,r\} = \{i_1,\dots,\widehat{i_e},\dots,i_s\} \amalg \{j_1,\dots,j_t\}$ determined by the structure of the tree τ. The homomorphism $\partial_\tau : \Sigma^{-1}\mathtt{D} \to \tau(\Sigma^{-1}\mathtt{D})$ is dual to the natural morphism $\lambda_\tau : \tau(\Sigma\,\mathtt{P}) \to \Sigma\,\mathtt{P}$ which forgets suspensions and performs the partial composite $p(i_1,\dots,i_e,\dots,i_s) \circ_{i_e} q(j_1,\dots,j_t)$ in $\mathtt{P}$. To forget suspension, we have to move suspensions on the left first and this produces a sign, but this process does not create any actual problem.

In this proof, we use the literal expression

$$p(i_1,\dots,i_e,\dots,i_s) \circ_{i_e} q(j_1,\dots,j_t)$$

to represent an element of $\tau(\Sigma\,\mathtt{P})$ (or $\tau(\Sigma^{-1}\mathtt{D})$) rather than a graphical representation in terms of treewise tensors. To shorten notation, we also set $p(i_*) = p(i_1,\dots,i_e,\dots,i_s)$ and $q(j_*) = q(j_1,\dots,j_t)$. Recall that the index sharing $\{1,\dots,r\} = \{i_1,\dots,\widehat{i_e},\dots,i_s\} \amalg \{j_1,\dots,j_t\}$ is fixed by the structure of the tree. Note that the index i_e in the expression $p(i_*) \circ_{i_e} q(j_*)$ is a dummy variable.

The element $w(i_1,\dots,i_r)$ associated to a permutation w is equivalent to an ordering $(i_{w(1)},\dots,i_{w(r)})$ of the set $\{i_1,\dots,i_r\}$. The element $\underline{w}(i_1,\dots,i_r)$ associated to a simplex of permutations $\underline{w} = (w_0,\dots,w_d)$ is just the $d+1$-tuple of orderings $\underline{w}(i_*) = (w_0(i_*),\dots,w_d(i_*))$ associated to the permutations of $\underline{w}$.

For permutations $u \in \Sigma_s$, $v \in \Sigma_t$, the composite $u(i_1, \dots, i_e, \dots, i_s) \circ_{i_e} v(j_1, \dots, j_t)$ is defined by the substitution of the value i_e in the ordering $u(i_*) = (i_{u(1)}, \dots, i_e, \dots, i_{u(s)})$ by the sequence $v(j_*) = (j_{v(1)}, \dots, j_{v(t)})$. For instance, we have

$$(1, i_e, 4) \circ_{i_e} (5, 2, 3) = (1, 5, 2, 3, 4).$$

For the Barratt-Eccles chain operad $\mathtt{E}(r) = N_*(\mathtt{W}(r))$ and its suboperad $\mathtt{E}_n \subset \mathtt{E}$, a partial composite of simplices

$$\underline{u}(i_*) = (u_0(i_*), \dots, u_m(i_*)) \quad \text{and} \quad \underline{v}(j_*) = (v_0(j_*), \dots, v_n(j_*))$$

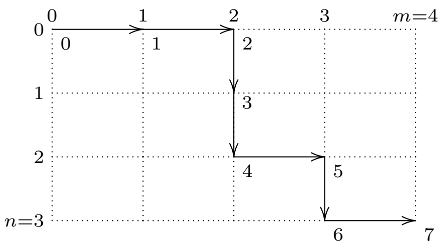
is a signed sum of simplices of the form

$$\underline{w} = (u_{k_0}(i_*) \circ_{i_e} v_{l_0}(j_*), \dots, u_{k_{m+n}}(i_*) \circ_e v_{l_{m+n}}(j_*)),$$

where the index sequence

$$(k_0, l_0) \to (k_1, l_1) \to \cdots \to (k_{m+n}, l_{m+n})$$

ranges over paths



and each $u_{k_*}(i_*) \circ_{i_e} v_{l_*}(j_*)$ is given by the substitution of permutations. We use the notation $\underline{w} \in \underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)$ to mean that a simplex $\underline{w}$ occurs in the expansion of $\underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)$.

The definition of the substitution process for permutations immediately implies that:

3.2.3. Observation. *The mapping* $(u(i_*), v(j_*)) \mapsto u(i_*) \circ_{i_e} v(i_*)$ *is injective.*

Indeed, the ordering $v(j_*)$ is identified with the connected subsequence of $w = u(i_*) \circ_{i_e} v(i_*)$ formed by the occurrences of $\{j_1, \dots, j_t\}$ and we recover $u(j_*)$ by replacing this subsequence by the variable i_e. By an easy generalization of this argument, we obtain:

3.2.4. Observation. *For any fixed sharing* $\{1, \dots, r\} = \{i_1, \dots, \widehat{i_e}, \dots, i_s\} \amalg \{j_1, \dots, j_t\}$ *a non-degenerate simplex* $\underline{w} \in \mathtt{W}(r)$ *occurs in at most one partial composite of non-degenerate simplices* $\underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)$ *and only once if so.*

In light of this analysis, we obtain from the definition of ∂_τ a result of the form:

3.2.5. Fact. *Let $\tau \in \Theta(r)$ be a tree with two vertices. Let $\{1,\dots,r\} = \{i_1,\dots,\widehat{i_e},\dots,i_s\} \amalg \{j_1,\dots,j_t\}$ be the index sharing associated to τ. For a basis element $\underline{s}^\vee$, we have:*

$$\partial_\tau(\underline{s}^\vee) = \begin{cases} \pm \underline{u}^\vee(i_*) \circ_{i_e} \underline{v}^\vee(j_*), & \text{whenever } \underline{s} \in \underline{u}(i_*) \circ_{i_e} \underline{v}(j_*) \\ & \qquad\qquad \text{for some } \underline{u} \in \mathtt{W}(s),\ \underline{v} \in \mathtt{W}(t), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover:

3.2.6. Fact. *If $\underline{s}^\vee$ has a non-trivial cobar differential in $\tau(\mathtt{M}_n)$, then we have the relation $\kappa^\vee(\underline{s}) = \kappa^\vee(\underline{u})(i_*) \circ_{i_e} \kappa^\vee(\underline{v})(j_*)$.*

This assertion is an immediate consequence of the identity $\kappa(\underline{u} \circ_e \underline{v}) = \kappa(\underline{u}) \circ_e \kappa(\underline{v})$ of Proposition 2.1.7.

3.2.7. The differential of dual basis elements in the $\mathcal{K}$-diagram version of the cobar construction. For a fixed $\kappa \in \mathcal{K}(r)$, the summand $\tau(M)(\kappa)$ of a free operad $\mathtt{F}(M)$ is represented by a colimit

$$\tau(M)(\kappa) = \operatorname*{colim}_{\alpha(i_*) \circ_{i_e} \beta(j_*) \leq \kappa} \tau(M, \alpha(i_*) \circ_{i_e} \beta(j_*)),$$

where each $\tau(M, \alpha(i_*) \circ_{i_e} \beta(j_*))$ consists of formal composites $p(i_*) \circ_{i_e} q(j_*)$ such that $p \in M(\alpha)$ and $q \in M(\beta)$. Define the image of a basis element $\underline{s}^\vee \in \mathtt{D}_n(\kappa)$ in the summand $\tau(\mathtt{M}_n)(\kappa) \subset \mathtt{F}(\mathtt{M}_n)(\kappa)$ as the element of the colimit represented by the composite

$$\underline{u}(i_*) \circ_{i_e} \underline{v}(j_*) \in \tau(\mathtt{M}_n, \alpha(i_*) \circ_{i_e} \beta(j_*)),$$

where $\alpha = \kappa^\vee(\underline{u})$, $\beta = \kappa^\vee(\underline{v})$, whenever we have $\underline{s} \in \underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)$. This image is set to be zero otherwise. This definition is coherent with the definition of $\mathcal{K}$-structures since

$$\kappa^\vee(\underline{s}) \leq \kappa \Rightarrow \kappa^\vee(\underline{u})(i_*) \circ_{i_e} \kappa^\vee(\underline{v})(j_*) = \kappa^\vee(\underline{s}) \leq \kappa.$$

Moreover:

3.2.8. Lemma. *The homomorphisms*

$$\mathtt{M}_n(\kappa) \xrightarrow{\partial} \mathtt{F}(\mathtt{M}_n)(\kappa)$$

defined by the process of §3.2.7 preserve $\mathcal{K}$-diagram structures, commute with symmetric group actions, and fit in commutative diagrams

$$\begin{array}{ccc} \mathtt{M}_n(\kappa) & \xrightarrow{\partial} & \mathtt{F}(\mathtt{M}_n)(\kappa) \\ \downarrow & & \downarrow \\ \mathtt{M}_n(r) & \xrightarrow[\partial]{} & \mathtt{F}(\mathtt{M}_n)(r) \end{array},$$

where $\partial : \mathtt{M}_n(r) \to \mathtt{F}(\mathtt{M}_n)(r)$ refers to the twisting homomorphism of the usual cobar construction.

Proof. Immediate from the definition of §3.2.7. □

Thus, our definition gives a well-defined homomorphism of symmetric $\mathcal{K}$-diagrams

$$\mathtt{M}_n \xrightarrow{\partial} \mathtt{F}(\mathtt{M}_n)$$

and we obtain naturally a derivation of the free $\mathcal{K}$-operad underlying the differential of the usual cobar construction when we form the derivation associated to this homomorphism. But we still have to check that this $\mathcal{K}$-operad derivation satisfies the equation $\delta(\partial) = \partial^2 = 0$ in order to achieve the construction of the quasi-free $\mathcal{K}$-operad underlying the cobar construction $\mathtt{B}^c(\mathtt{D}_n)$.

To ease verifications, we use the following observation:

3.2.9. Observation. *Let M be any symmetric $\mathcal{K}$-diagram such that $M(0) = M(1) = 0$. Let τ be any tree.*

(a) *For an oriented weight system of the form $\kappa = \lambda_*(\beta_*)$, where $\beta_* \in \tau(\mathcal{K})$, we have a natural isomorphism*

$$\tau(M, \beta_*) \to \operatorname*{colim}_{\lambda_*(\alpha_*) \leq \kappa} \tau(M, \alpha_*) = \tau(M)(\kappa),$$

because $\beta_* \in \tau(\mathcal{K})$ is the largest element of $\tau(\mathcal{K})$ satisfying $\lambda_*(\beta_*) \leq \kappa$.

(b) *If the symmetric $\mathcal{K}$-diagram M consists of subobjects of a Σ_*-object $M(\kappa) \subset M(r)$ (like $\mathtt{M}_n$), then the natural morphism*

$$\tau(M, \beta_*) \to \tau(M)(r)$$

is an embedding, for every $\beta_* \in \tau(\mathcal{K})$.

We check at first:

3.2.10. Lemma. *The homomorphism of Lemma 3.2.8 satisfies the commutation relation $(\delta\partial + \partial\delta)(\underline{s}^\vee) = 0$ with respect to the internal differential of $\mathtt{D}_n$, for every basis element $\underline{s}^\vee$.*

Proof. We prove that the composites of the diagram

(15)
$$\begin{array}{ccc} \mathtt{M}_n(\kappa) & \xrightarrow{\partial_\tau} & \tau(\mathtt{M}_n)(\kappa) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathtt{M}_n(\kappa) & \xrightarrow[\partial_\tau]{} & \tau(\mathtt{M}_n)(\kappa) \end{array}$$

agree on $\underline{s}^\vee$ (up to the minus sign), for each fixed tree with two vertices τ. We can assume $\kappa = \kappa^\vee(\underline{s})$ since each mapping ∂_τ preserves $\mathcal{K}$-diagram structures.

If $\kappa = \kappa^\vee(\underline{s})$ has the form $\kappa = \alpha(i_*) \circ_{i_e} \beta(j_*)$, where $\alpha(i_*) \circ_{i_e} \beta(j_*)$ is the partial composite representation of an element of $\tau(\mathcal{K})$, then we are done, because the dg-module $\tau(\mathtt{M}_n)(\kappa)$ embeds into $\tau(\mathtt{M}_n)(r)$ by observation 3.2.9 and the commutation with internal differentials is satisfied in the usual cobar construction when $\underline{s}^\vee$ is viewed as an element of $\mathtt{M}_n(r)$.

Otherwise, we have necessarily $\partial_\tau(\underline{s}^\vee) = 0$ by Fact 3.2.6. The internal differential of $\mathtt{M}_n(r)$ is defined on dual basis elements by a formula of the form

$$\delta(\underline{s}^\vee) = \sum_{\underline{s} \in \delta(\underline{w})} \pm \underline{w}^\vee,$$

where we use the notation $\underline{s} \in \delta(\underline{w})$ to assert that the term $\underline{s}$ occurs in the expansion of $\delta(\underline{w})$ for any simplex $\underline{w}$. Note again that $\underline{s}$ occurs only at most once in $\delta(\underline{w})$, because $\underline{w}$ is assumed to be non-degenerate.

From the derivation relation

$$\delta(\underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)) = \delta(\underline{u}(i_*)) \circ_{i_e} \underline{v}(j_*) + \pm \underline{u}(i_*) \circ_{i_e} \delta(\underline{v}(j_*)),$$

we obtain the implication:

$$\begin{aligned} \underline{s} \in \delta(\underline{w}) \quad \text{and} \quad & \underline{w} \in \underline{u}(i_*) \circ_{i_e} \underline{v}(j_*) \\ & \Rightarrow \quad \underline{s} \in \underline{t}(i_*) \circ_{i_e} \underline{v}(j_*) \text{ for some } \underline{t} \in \delta(\underline{u}), \\ & \qquad \text{or } \underline{s} \in \underline{u}(i_*) \circ_{i_e} \underline{t}(j_*) \text{ for some } \underline{t} \in \delta(\underline{v}), \end{aligned}$$

whose conclusion requires that $\kappa^\vee(\underline{s})$ has the form $\kappa^\vee(\underline{s}) = \alpha(i_*) \circ_{i_e} \beta(j_*)$ for some α, β. Hence, if this is not the case, then we also have $\partial_\tau(\underline{w}^\vee) = 0$ for each $\underline{w}$ such that $\underline{s} \in \delta(\underline{w})$. From this observation, we conclude that diagram (15) still commutes when we have $\kappa \neq \alpha(i_*) \circ_{i_e} \beta(j_*)$, $\forall \alpha(i_*) \circ_{i_e} \beta(j_*) \in \tau(\mathcal{K})$. □

3.2.11. Lemma. *The $\mathcal{K}$-operad derivation $\partial : \mathtt{F}(\mathtt{M}_n) \to \mathtt{F}(\mathtt{M}_n)$ induced by the homomorphism of Lemma 3.2.8 satisfies $\partial\partial(\underline{s}^\vee) = 0$, for every generating element $\underline{s}^\vee \in \mathtt{D}_n$.*

Proof. By a straightforward extension of the description of $\partial\partial$ in the usual cobar construction (see for instance [12, §§3.3-3.5] or [13, §1.4.3]), the restriction of the map $\partial\partial : \mathtt{F}(\mathtt{M}_n)(\kappa) \to \mathtt{F}(\mathtt{M}_n)(\kappa)$ to generators $\mathtt{M}_n(\kappa)$ has a component for each summand $\tau(\mathtt{M}_n)$ associated to a tree with 3 vertices and this component is defined by the sum of all composites

$$\mathtt{M}_n(\kappa) \xrightarrow{\partial_\rho} \rho(\mathtt{M}_n)(\kappa) \xrightarrow{\partial_\sigma} \tau(\mathtt{M}_n)(\kappa),$$

where ∂_σ is applied to a vertex $v \in V(\rho)$ such that the blow-up $v \mapsto \sigma$ in the tree ρ gives the tree τ.

By an easy inspection of definitions, we check that any such composite cancels $\underline{s}^\vee$ when $\kappa^\vee(\underline{s})$ is not of the form $\kappa^\vee(\underline{s}) = \lambda_*(\alpha_*)$ for some $\alpha_* \in \tau(\mathcal{K})$. Hence, in this case, the equation $\partial\partial(\underline{s}^\vee) = 0$ holds trivially. Otherwise, the summand $\tau(\mathtt{M}_n)(\kappa)$ embeds into $\tau(\mathtt{M}_n)(r)$ by observation 3.2.9 and since the identity $\partial\partial(\underline{s}^\vee) = 0$ holds in the usual cobar construction, we still have $\partial\partial(\underline{s}^\vee) = 0$ on $\tau(\mathtt{M}_n)(\kappa)$. Thus, we obtain that $\partial\partial(\underline{s}^\vee)$ vanishes in all cases. □

3.2.12. Lemma. *The $\mathcal{K}$-operad derivation $\partial : \mathtt{F}(\mathtt{M}_n) \to \mathtt{F}(\mathtt{M}_n)$ satisfies the assertions of Lemma 3.2.10 and Lemma 3.2.11 for all elements of $\mathtt{F}(\mathtt{M}_n)$ and not only generating elements.*

Proof. Immediate from the derivation relation (straightforward generalization of the usual verification for the twisting derivation of a quasi-free operad). □

From observation 3.2.1 and lemmas 3.2.8-3.2.12, we conclude:

3.2.13. Theorem. *We have a quasi-free $\mathcal{K}$-operad $\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\mathtt{M}_n), \partial)$ whose colimit $\operatorname{colim}_{\kappa\in\mathcal{K}(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa)$ is isomorphic to the usual cobar construction of the (ordinary) cooperad $\mathtt{D}_n$.* □

This theorem gives a first part of the assertions of Lemma B.

For our needs, we prove a strengthened form of observation 3.2.1:

3.2.14. Proposition. *Let M be any symmetric $\mathcal{K}$-diagram. The natural morphism $\eta : M \to \mathtt{F}(M)$ induces a natural isomorphism of operads*

$$\phi_\eta : \mathtt{F}(\operatorname*{colim}_{\mathcal{K}_m} M) \xrightarrow{\simeq} \operatorname*{colim}_{\mathcal{K}_m} \mathtt{F}(M),$$

for every m, including $m = \infty$ in which case we recover the identity of observation 3.2.1.

Proof. From the definition of the free $\mathcal{K}$-operad $\mathtt{F}(M)$, we obtain by interchange and composition of colimits:

$$\begin{aligned}\operatorname*{colim}_{\kappa\in\mathcal{K}_m(r)} \bigl\{\mathtt{F}(M)(\kappa)\bigr\} &= \operatorname*{colim}_{\kappa\in\mathcal{K}_m(r)} \Bigl\{ \bigoplus_{\tau\in\Theta(r)} \tau(M)(\kappa)/\equiv\Bigr\}\\ &= \bigoplus_{\tau\in\Theta(r)} \operatorname*{colim}_{\kappa\in\mathcal{K}_m(r)} \Bigl\{ \operatorname*{colim}_{\lambda_*(\alpha_*)\leq\kappa} \tau(M,\alpha_*)\Bigr\}/\equiv\\ &= \bigoplus_{\tau\in\Theta(r)} \Bigl\{ \operatorname*{colim}_{|\alpha_*|<n} \tau(M,\alpha_*)\Bigr\}/\equiv,\end{aligned}$$

where we use the notation $|(\mu,\sigma)| = \max_{ij}\{\mu_{ij}\}$ for an oriented weight-system (μ,σ) and $|\alpha_*| = \max_{v\in V(\tau)} |\alpha_v|$ for a composite $\alpha_* \in \tau(\mathcal{K})$.

We have clearly

$$|\alpha_*| < n \Leftrightarrow \alpha_* \in \tau(\mathcal{K}_m).$$

Hence, the standard relation

$$\operatorname*{colim}_{I\times J} X(i)\otimes Y(j) \simeq \operatorname*{colim}_{I} X(i) \otimes \operatorname*{colim}_{J} Y(j)$$

for tensor products of colimits gives the identity

$$\operatorname*{colim}_{|\alpha_*|<n} \tau(M,\alpha_*) = \tau(\operatorname*{colim}_{\mathcal{K}_n} M)$$

and the conclusion follows. □

In the case $M = \mathtt{M}_n$, we deduce from this lemma and Proposition 3.1.8:

3.2.15. Proposition. *For the quasi-free $\mathcal{K}$-operad $\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\mathtt{M}_n), \partial)$, we have natural isomorphisms*

$$\operatorname*{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa) \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} \operatorname*{colim}_{\kappa\in\mathcal{K}(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa) \xrightarrow{\simeq} \mathtt{B}^c(\mathtt{D}_n)(r). \quad \square$$

The verification of this proposition finishes the proof of the first assertion of Lemma B. To complete our result, we still have to check:

3.2.16. Proposition. *The morphism of symmetric $\mathcal{K}$-diagrams $\sigma^* : \mathtt{D}_{n-1}(\kappa) \to \mathtt{D}_n(\kappa)$ of Proposition 3.1.9 induces a morphism of quasi-free $\mathcal{K}$-operads*

$$(\mathtt{F}(\mathtt{M}_{n-1}), \partial) \xrightarrow{\phi_{\sigma^*}} (\mathtt{F}(\mathtt{M}_n), \partial)$$

which gives a realization at the $\mathcal{K}$-operad level of the morphism

$$\sigma^* : \mathtt{B}^c(\mathtt{D}_{n-1}) \to \mathtt{B}^c(\mathtt{D}_n)$$

yielded by the suspension morphism of the Barratt-Eccles operad.

Proof. To obtain this result, we essentially have to check that the morphism of free operads $\phi_{\sigma^*} : \mathtt{F}(\mathtt{M}_{n-1}) \to \mathtt{F}(\mathtt{M}_n)$ commutes with twisting derivations. Indeed, we explain in §2.2.12 that this condition is sufficient to form the morphism of quasi-free $\mathcal{K}$-operads considered in the proposition. Then the obtained morphisms automatically fit in a commutative diagram

$$\begin{array}{ccc} (\mathtt{F}(\mathtt{M}_{n-1})(r), \partial) & \xrightarrow{\sigma^*} & (\mathtt{F}(\mathtt{M}_n)(r), \partial) \\ \uparrow & & \uparrow \\ (\mathtt{F}(\mathtt{M}_{n-1})(\kappa), \partial) & \underset{\sigma^*}{\dashrightarrow} & (\mathtt{F}(\mathtt{M}_n)(\kappa), \partial) \end{array} ,$$

and hence, give a realization of $\sigma^* : \mathtt{B}^c(\mathtt{D}_{n-1}) \to \mathtt{B}^c(\mathtt{D}_n)$ at the $\mathcal{K}$-operad level, because this is so on generators by definition of the morphism $\sigma^* : \mathtt{D}_{n-1}(\kappa) \to \mathtt{D}_n(\kappa)$.

First, we check that the identity

$$\partial(\sigma^* \underline{s}^\vee) = \sigma^* \partial(\underline{s}^\vee) \tag{16}$$

holds for any basis elements $\underline{s}^\vee$. For this aim, we can argue exactly as in the proof of Lemma 3.2.10. In our verifications, we just replace the differential δ by the morphism σ, the derivation relation of differentials by the structure relation of morphisms

$$\sigma(\underline{u}(i_*) \circ_{i_e} \underline{v}(j_*)) = \sigma(\underline{u}(i_*)) \circ_{i_e} \sigma(\underline{v}(j_*))$$

and we proceed by exactly the same argument line.

Then we readily deduce from the derivation relation that equation (16) holds for every element of $\mathtt{F}(\mathtt{M}_{n-1})$, not only for basis elements of the generating $\mathcal{K}$-diagram $\mathtt{M}_{n-1}$, and this observation finishes the proof of the proposition. □

The verification of this proposition gives the proof of the second assertion of Lemma B and completes the proof of Lemma B itself. □

3.2.17. Remark. One can extend the argument line of Proposition 3.1.8 and prove that the operads $\mathtt{B}^c(\mathtt{D}_n)$ are cofibrant as $\mathcal{K}$-diagrams. As a byproduct, we have $\mathtt{B}^c(\mathtt{D}_n)(\kappa) \subset \mathtt{B}^c(\mathtt{D}_n)(r)$ for all $\kappa \in \mathcal{K}(r)$, but this inclusion relation is not a simple consequence of the inclusion relation $\mathtt{D}_n(\kappa) \subset \mathtt{D}_n(r)$ at the level of generating objects (see §2.2.8).

On the other hand, one can see that the dg-modules $\mathtt{B}^c(\mathtt{D}_n)(\kappa)$ are not contractible in general. Indeed, we analyze the structure of $\mathtt{D}_n(\kappa)$ for $\kappa \in$

$\mathcal{K}(2)$ next (in §4.2). For $\kappa \in \mathcal{K}(2)$, we have $\mathtt{B}^c(\mathtt{D}_n)(\kappa) = \mathtt{D}_n(\kappa)$, and one can immediately see from the description to come that these dg-modules are not acyclic when $\kappa = (\mu, \sigma)$ satisfies $\mu_{12} \geq n$.

4. Final step: applications of model structures and lifting arguments

The goal of this section is to prove:

Lemma C. *We have morphisms*

$$\begin{array}{ccccccccc}
\mathtt{B}^c(\mathtt{D}_1) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_2) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow{\sigma^*} & \cdots \\
\downarrow{\scriptstyle\kappa} & & \vdots{\scriptstyle\exists\kappa} & & & & \vdots{\scriptstyle\exists\kappa} & & \\
\mathtt{E}_1 & \xrightarrow[\iota]{} & \mathtt{E}_2 & \xrightarrow[\iota]{} & \cdots & \xrightarrow[\iota]{} & \mathtt{E}_n & \xrightarrow[\iota]{} & \cdots
\end{array}$$

lifting the morphisms $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ of Lemma A and satisfying the requirement of assertion (b) in Theorem A.

In §3, we proved that the morphisms $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ of Lemma A are realized by morphisms of $\mathcal{K}$-operads. The rough idea is to apply the lifting argument of §1.3 in order to get morphisms of $\mathcal{K}$-operads $\tilde{\phi}_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}$ lifting $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$. Then we simply take the $\mathcal{K}_n$-colimits of these morphisms and use the identity $\mathtt{E}_n = \operatorname{colim}_{\mathcal{K}_n} \mathtt{E}$ to produce the morphisms $\kappa : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}_n$ of Lemma B.

First of all, we check in §4.1 that the morphism of symmetric $\mathcal{K}$-diagrams $\sigma^* : \mathtt{D}_{n-1} \to \mathtt{D}_n$ which represents the dual of the suspension morphism of the Barratt-Eccles operad is a cofibration (of symmetric $\mathcal{K}$-diagrams). In this situation, we obtain from Proposition 2.2.15 that a natural cobase extension of the morphism of quasi-free operads $\sigma^* : \mathtt{B}^c(\mathtt{D}_{n-1}) \to \mathtt{B}^c(\mathtt{D}_n)$ is a cofibration of $\mathcal{K}$-operads, for each $n > 1$.

Then we apply the model category structure of $\mathcal{K}$-operads in order to define the desired liftings $\tilde{\phi}_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}$ in §4.2 and we complete the proof of Lemma C in §4.3.

4.1. The cofibration statement

The purpose of this subsection is to prove the following proposition:

Proposition 4.1.A. *The morphism of symmetric $\mathcal{K}$-diagrams $\sigma^* : \mathtt{D}_{n-1} \to \mathtt{D}_n$ is a cofibration, for all $n > 1$.*

According to Proposition 2.2.5, the verification of Proposition 4.1.A reduces to the following result:

Lemma 4.1.B. *The relative latching morphism*

$$\mathtt{D}_{n-1}(\kappa) \bigoplus_{L\,\mathtt{D}_{n-1}(\kappa)} L\,\mathtt{D}_n(\kappa) \xrightarrow{(\sigma^*, \lambda)} \mathtt{D}_n(\kappa)$$

is a cofibration of dg-modules for every $\kappa \in \mathcal{K}(r)$, $r \in \mathbb{N}$.

The verification of this lemma is deferred to a series of sublemmas.

4.1.1. Lemma. *The latching morphism $\lambda : L\,\mathtt{D}_n(\kappa) \to \mathtt{D}_n(\kappa)$ induces an isomorphism between the latching object*

$$L\,\mathtt{D}_n(\kappa) = \operatorname*{colim}_{\alpha \lneqq \kappa} \mathtt{D}_n(\alpha)$$

and the submodule

$$\mathrm{Span}\{\underline{s}^\vee | \kappa^\vee(\underline{s}) \lneqq \kappa\} \subset \mathtt{D}_n(\kappa).$$

Proof. We adapt the argument of Proposition 3.1.8. Set

$$S = \mathrm{Span}\{\underline{s}^\vee | \kappa^\vee(\underline{s}) \lneqq \kappa\}.$$

We have clearly $\lambda(L\,\mathtt{D}_n(\kappa)) \subset S$. We have a map

$$\psi : S \to L\,\mathtt{D}_n(\kappa)$$

sending a basis element $\underline{s}^\vee$, $\kappa^\vee(\underline{s}) \lneqq \kappa$, to the same element $\underline{s}^\vee$ in the summand $\mathtt{D}_n(\kappa^\vee(\underline{s}))$ of the latching object. We have clearly $\lambda\psi = \mathrm{id}$. Conversely, a basis element $\underline{s}^\vee \in \mathtt{D}_n(\alpha)$ is identified in the latching object with the same element $\underline{s}^\vee$ coming from $\mathtt{D}_n(\kappa^\vee(\underline{s}))$ since $\underline{s}^\vee \in \mathtt{D}_n(\alpha) \Rightarrow \kappa^\vee(\underline{s}) \leq \alpha$. Hence, we also have $\psi\lambda = \mathrm{id}$. □

4.1.2. Lemma. *The morphism $\sigma^* : \mathtt{D}_{n-1}(r) \to \mathtt{D}_n(r)$ has a retraction τ^* mapping the modules*

$$L\,\mathtt{D}_m(\kappa) \subset \mathtt{D}_m(\kappa) \subset \mathtt{D}_m(r)$$

for $m = n$ into the same modules for $m = n-1$ and preserving the splitting

$$\mathtt{D}_m(\kappa) = \underbrace{\mathrm{Span}\{\underline{s}^\vee | \kappa^\vee(\underline{s}) \lneqq \kappa\}}_{= L\,\mathtt{D}_m(\kappa)} \oplus \mathrm{Span}\{\underline{s}^\vee | \kappa^\vee(\underline{s}) = \kappa\}.$$

Proof. By duality, the mapping $\tau(w_0, \dots, w_d) = (t_0, \dots, t_{r-1}, w_0, \dots, w_d)$ defined in the proof of Lemma 1.1.B gives a morphism

$$\tau^* : \mathtt{D}_n(r) \to \mathtt{D}_{n-1}(r)$$

such that

$$\tau^*((s_0, \dots, s_d)^\vee) = \begin{cases} \pm(s_r, \dots, s_d)^\vee, & \text{if } (s_0, \dots, s_{r-1}) = (t_0, \dots, t_{r-1}), \\ 0, & \text{otherwise.} \end{cases}$$

The permutations $(t_0, \dots, t_{r-1})$ are defined in the proof of Lemma 1.1.B. For our present purpose, we just need the relation

$$\kappa(s_r, \dots, s_d) = (\mu, \sigma) \Rightarrow \kappa(t_0, \dots, t_{r-1}, s_r, \dots, s_d) = (\mu + 1, \sigma)$$

which is also a consequence of identities established in the proof of this Lemma. By inversion, we obtain

$$\begin{aligned} \kappa^\vee(t_1, \dots, t_{r-1}, s_r, \dots, s_d) &= ((n-1) - (\mu+1), \sigma) \\ &= (n - 2 - \mu, \sigma) = \kappa^\vee(s_r, \dots, s_d). \end{aligned}$$

From this identity, we immediately conclude that the mapping τ^* satisfies the requirements of the present lemma. □

Note that the mapping τ^* in Lemma 4.1.2 is not supposed to preserve differentials.

Lemmas 4.1.1-4.1.2 imply readily:

4.1.3. Lemma. *The relative latching morphism of Lemma 4.1.B is split injective (as long as we forget differentials).* □

By Lemma 1.3.4, we also have:

4.1.4. Fact. *The dg-modules* $L\,\mathtt{D}_m(\kappa) \subset \mathtt{D}_m(\kappa) \subset \mathtt{D}_m(r)$ *are bounded, for all* $m < \infty$.

This fact and Lemma 4.1.3 imply the conclusion of Lemma 4.1.B and completes the proof of Proposition 4.1.A. □

4.2. The lifting argument

The goal of this subsection is to define the liftings $\tilde{\phi}_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}$ of the morphisms of $\mathcal{K}$-operads $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ yielded by the results of Lemma A and Lemma B. To fulfil the requirements of Theorem A, we have to fix $\tilde{\phi}_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}$ on the 2-ary part of the generating $\mathcal{K}$-diagram $\mathtt{D}_n$ and we analyze this issue first.

Throughout this subsection, we use the short notation $\mathtt{M}_n = \Sigma^{-1}\,\mathtt{D}_n$ for the desuspension of $\mathtt{D}_n$.

4.2.1. The 2-ary components. Recall that $\mathtt{E}(2)$ is identified with the standard Σ_2-free resolution of the trivial representation of Σ_2:

$$\mathbb{Z}\,\mu_0 \oplus \mathbb{Z}\,\tau\mu_0 \xleftarrow{\tau-1} \mathbb{Z}\,\mu_1 \oplus \mathbb{Z}\,\tau\mu_1 \xleftarrow{\tau+1} \mathbb{Z}\,\mu_2 \oplus \mathbb{Z}\,\tau\mu_2 \xleftarrow{\tau-1} \cdots,$$

where μ_d denotes the alternate d-simplex $(\mathrm{id},\tau,\mathrm{id},\dots)$, viewed as an element of $\mathtt{E}(2)$, and τ refers to the transposition of $(1,2)$. For the submodules $\mathtt{E}(\kappa)$ associated to oriented weight systems $\kappa = (\mu,\mathrm{id}),(\mu,\tau) \in \mathcal{K}(2)$, we have clearly:

$$\mathtt{E}(\kappa)_d = \mathtt{E}(2)_d = \mathbb{Z}\,\mu_d \oplus \mathbb{Z}\,\tau\mu_d \quad \text{in degree } d < \mu_{12},$$
$$\mathtt{E}(\mu,\mathrm{id})_d = \mathbb{Z}\,\tau^d\mu_d \quad \text{and} \quad \mathtt{E}(\mu,\tau)_d = \mathbb{Z}\,\tau^{d+1}\mu_d \quad \text{in degree } d = \mu_{12},$$
$$\mathtt{E}(\kappa)_d = 0 \quad \text{in degree } d > \mu_{12}.$$

The submodule $\mathtt{E}_n(2)$ is identified with the truncation of $\mathtt{E}(2)$ in degree $d < n$. The dual dg-module $\mathtt{M}_n(2) = \Sigma^{-1}(\mathtt{D}_n)(2)$ is identified with the chain complex:

$$\mathbb{Z}\,\mu_{n-1}^\vee \oplus \mathbb{Z}\,\tau\mu_{n-1}^\vee \xleftarrow{\tau-1} \mathbb{Z}\,\mu_{n-2}^\vee \oplus \mathbb{Z}\,\tau\mu_{n-2}^\vee \xleftarrow{\tau+1} \cdots \xleftarrow{\tau\pm1} \mathbb{Z}\,\mu_0^\vee \oplus \mathbb{Z}\,\tau\mu_0^\vee$$

where $\mu_d^\vee$ and $\tau\mu_d^\vee$ are dual basis elements, put in degree $n-1-d$, of the alternate simplices $\mu_d = (\mathrm{id},\tau,\mathrm{id},\dots)$ and $\tau\mu_d = (\tau,\mathrm{id},\tau,\dots)$. For oriented weight systems $\kappa = (\mu,\mathrm{id}),(\mu,\tau) \in \mathcal{K}(2)$ such that $\mu_{12} \leq n-1$, we also obtain:

$$\begin{aligned}
&\mathtt{M}_n(\kappa)_d = \mathbb{Z}\,\mu_{n-1-d}^{\vee} \oplus \mathbb{Z}\,\tau\mu_{n-1-d}^{\vee} \quad \text{in degree } d < \mu_{12},\\
&\mathtt{M}_n(\mu,\mathrm{id})_d = \mathbb{Z}\,\tau^{n-1-d}\mu_{n-1-d}^{\vee}\\
&\qquad\qquad \text{and} \quad \mathtt{M}_n(\mu,\tau)_d = \mathbb{Z}\,\tau^{n-1-d+1}\mu_{n-1-d}^{\vee} \quad \text{in degree } d = \mu_{12},\\
&\mathtt{M}_n(\kappa)_d = 0 \quad \text{in degree } d > \mu_{12}.
\end{aligned}$$

In the case $\mu_{12} > n-1$, we have $\mathtt{M}_n(\kappa)_d = \mathtt{M}_n(2)_d$ for every d.

The dual of the suspension morphism of the Barratt-Eccles operad satisfies:

$$\sigma^*(\mu_{n-2-d}^{\vee}) = \tau\mu_{n-1-d}^{\vee} \quad \text{and} \quad \sigma^*(\tau\mu_{n-2-d}^{\vee}) = \mu_{n-1-d}^{\vee},$$

for every $0 \leq d \leq n-2$ and hence identifies $\mathtt{M}_{n-1}(2)$ with a truncation of $\mathtt{M}_n(2)$.

For each $\kappa \in \mathcal{K}_n(2)$, we have a natural morphism

$$\tilde{\phi}_n : \mathtt{M}_n(\kappa) \to \mathtt{E}(\kappa)$$

defined by $\tilde{\phi}_n(\mu_{n-1-d}^{\vee}) = \tau^{n-1}\mu_d$ on basis elements. From our observations, we conclude:

4.2.2. Fact. *In arity $r = 2$, the just defined morphisms form, for each fixed $n \geq 1$, a morphism of $\mathcal{K}(2)$-diagrams*

$$\tilde{\phi}_n : \mathtt{M}_n(\kappa) \to \mathtt{E}(\kappa),$$

commuting with the action of permutations $w \in \Sigma_2$, with augmentations

$$\begin{array}{ccc}
\mathtt{M}_n(\kappa) & \dashrightarrow & \mathtt{E}(\kappa)\ , \\
& \searrow \quad \swarrow & \\
& \mathbb{Z} &
\end{array}$$

and so that the diagram

$$\begin{array}{ccccccc}
\mathtt{M}_1(\kappa) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \mathtt{M}_n(\kappa) & \xrightarrow{\sigma^*} & \cdots \\
\downarrow{\scriptstyle \tilde{\phi}_1} & & & & \downarrow{\scriptstyle \tilde{\phi}_n} & & \\
\mathtt{E}(\kappa) & \xrightarrow[=]{} & \cdots & \xrightarrow[=]{} & \mathtt{E}(\kappa) & \xrightarrow[=]{} & \cdots
\end{array}$$

commutes, for each $\kappa \in \mathcal{K}(2)$.

4.2.3. The bottom filtration layer of a quasi-free operad. In the remainder of this section, we use the notation $M^{(2)}$ to represent the Σ_*-object $M^{(2)} \subset M$ such that

$$M^{(2)}(r) = \begin{cases} M(2), & \text{if } r = 2,\\ 0, & \text{otherwise.}\end{cases}$$

If M has a $\mathcal{K}$-structure, then so does the Σ_*-object $M^{(2)} \subset M$ and we may also use the notation $M^{(2)}$ to refer to the symmetric $\mathcal{K}$-diagram $M^{(2)}(\kappa)$ underlying $M^{(2)}(r)$. If necessary, then we use dummy variables to mark the distinction between the Σ_*-object $M^{(2)}$ and its underlying $\mathcal{K}$-diagram (as explained in §2.1.3).

For a quasi-free operad $\mathtt{P} = (\mathtt{F}(M), \partial)$ such that $M(0) = M(1) = 0$, we have $M_{\leq 1} = 0$ and the submodule $M_{\leq 2}$ of §2.2.14 is reduced to $M_{\leq 2} = M^{(2)}$. Moreover, if we assume $\partial(M) \subset \bigoplus_{m \geq 2} \mathtt{F}_m(M)$, then the observation of §2.2.14

$$\partial(M_{\leq r}) \subset \mathtt{F}(M_{\leq r-1})$$

implies that ∂ vanishes on $M_{\leq 2}$ and the 2nd layer of the quasi-free operad filtration

$$\mathtt{P}_{\leq r} = (\mathtt{F}(M_{\leq r}), \partial) \subset \mathtt{P}$$

is identified with the free operad $\mathtt{P}_{\leq 2} = \mathtt{F}(M^{(2)})$.

The assertion of Fact 4.2.2 implies the existence of a morphism of symmetric $\mathcal{K}$-diagrams $\mathtt{M}_n^{(2)} \xrightarrow{\tilde{\phi}_n} \mathtt{E}^{(2)} \subset \mathtt{E}$, for every $n \geq 1$, which gives rise to a morphism of $\mathcal{K}$-operads:

$$\tilde{\phi}_n : \mathtt{F}(\mathtt{M}_n^{(2)}) \to \mathtt{E}\,.$$

The observations of 4.2.3 imply that $\mathtt{F}(\mathtt{M}_n^{(2)})$ is identified with the 2nd layer of the quasi-free $\mathcal{K}$-operads $\mathtt{B}^c(\mathtt{D}_n) = (\mathtt{F}(\mathtt{M}_n), \partial)$ on which we want to define our morphism.

Note that:

4.2.4. Lemma. *The just defined morphisms of $\mathcal{K}$-operads $\tilde{\phi}_n : \mathtt{F}(\mathtt{M}_n^{(2)}) \to \mathtt{E}$*

(a) *make the diagram*

$$\begin{array}{ccc} \mathtt{F}(\mathtt{M}_n^{(2)}) & \xrightarrow{\tilde{\phi}_n} & \mathtt{E} \\ \downarrow & & \downarrow \\ \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow[\phi_n]{} & \mathtt{C} \end{array}$$

commute, for every $n \geq 1$,

(b) *and are preserved by the morphisms of free $\mathcal{K}$-operads $\sigma^* : \mathtt{F}(\mathtt{M}_{n-1}^{(2)}) \to \mathtt{F}_n(\mathtt{M}_n^{(2)})$ induced by the suspension morphism of the Barratt-Eccles operad.*

Proof. To obtain assertion (a), we are reduced to check, by adjunction, that the restriction of $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ to the generators of arity 2 of the cobar construction agrees with the obvious augmentation $\epsilon : \mathtt{M}_n(2) \to \mathbb{Z}$. For this aim, one can inspect the construction of $\phi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{C}$ in §1.3. On the other hand, we can observe that this identity is forced by the requirement of Lemma A, because the morphisms ϕ_n necessarily vanish in degree $d > 0$ and are determined by ϕ_1 when we take their restriction to $\mathtt{M}_n(2)_0 = \Sigma^{-1}\,\mathtt{D}_n(2)_0$ (see also the explanations given in the proof of Lemma 1.3.1 for this observation).

The second assertion of the lemma is an immediate consequence of the observations of Fact 4.2.2 and the functoriality of free operads. □

The lifting argument, which motivates the constructions of §§2-3, is given in the following proposition:

Proposition 4.2.A. *The morphisms of $\mathcal{K}$-operads ϕ_n deduced from the results of Lemma A and Lemma B admit liftings*

$$\begin{array}{ccc} & & \mathtt{E} \\ & \overset{\exists\tilde{\phi}_n}{\nearrow} & \downarrow \\ \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow[\phi_n]{} & \mathtt{C} \end{array}$$

so that the diagrams

$$\begin{array}{ccccc} \mathtt{B}^c(\mathtt{D}_{n-1}) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\mathtt{D}_n) & \longleftarrow & \mathtt{F}(\mathtt{M}_n^{(2)}) \\ \downarrow{\scriptstyle \tilde{\phi}_{n-1}} & & \downarrow{\scriptstyle \tilde{\phi}_n} & & \downarrow{\scriptstyle \tilde{\phi}_n} \\ \mathtt{E} & \xrightarrow[=]{} & \mathtt{E} & \xleftarrow[=]{} & \mathtt{E} \end{array}$$

commute, for all $n > 1$.

Proof. The lifting $\tilde{\phi}_n$ is defined by induction on n, starting with the Koszul duality augmentation of the associative operad $\mathtt{B}^c(\mathtt{A}) \xrightarrow{\epsilon} \mathtt{A} \subset \mathtt{E}$ in the case $\mathtt{E}_1 = \mathtt{A}$. At the nth stage of the induction process, the already defined morphisms fit in the solid frame of a commutative diagram of $\mathcal{K}$-operads

$$\begin{array}{ccc} \mathtt{B}^c(\mathtt{D}_{n-1}) \bigvee_{\mathtt{F}(\mathtt{M}_{n-1}^{(2)})} \mathtt{F}(\mathtt{M}_n^{(2)}) & \xrightarrow{(\tilde{\phi}_{n-1},\tilde{\phi}_n)} & \mathtt{E} \\ \downarrow & \nearrow & \downarrow{\scriptstyle \sim} \\ \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow[\phi_n]{} & \mathtt{C} \end{array} .$$

Proposition 4.1.A implies, according to Proposition 2.2.15, that the left-hand side vertical morphism is a cofibration of $\mathcal{K}$-operads, where we use the observation of §4.2.3 (again)

$$\mathtt{B}^c(\mathtt{D}_n)_{\leq 2} = \mathtt{F}(\mathtt{M}_n^{(2)})$$

in order to apply Proposition 2.2.15. The lifting axiom of the model category of $\mathcal{K}$-operads implies the existence of a fill-in morphism in this diagram and this gives the desired morphism $\tilde{\phi}_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}$. ☐

4.3. Conclusion of the lifting argument

The morphisms of Proposition 4.2.A give by colimit a sequence of morphisms:

$$(17) \qquad \begin{array}{ccccccc} \operatorname{colim}_{\mathcal{K}_1} \mathtt{B}^c(\mathtt{D}_1) & \xrightarrow{\sigma^*} & \cdots & \xrightarrow{\sigma^*} & \operatorname{colim}_{\mathcal{K}_n} \mathtt{B}^c(\mathtt{D}_n) & \xrightarrow{\sigma^*} & \cdots . \\ \downarrow{\scriptstyle \tilde{\phi}_1} & & & & \downarrow{\scriptstyle \exists\tilde{\phi}_n} & & \\ \operatorname{colim}_{\mathcal{K}_1} \mathtt{E} & \longrightarrow & \cdots & \longrightarrow & \operatorname{colim}_{\mathcal{K}_n} \mathtt{E} & \longrightarrow & \cdots \end{array}$$

The upper row of (17) is isomorphic to $\mathtt{B}^c(\mathtt{D}_1) \xrightarrow{\sigma^*} \mathtt{B}^c(\mathtt{D}_2) \xrightarrow{\sigma^*} \cdots \xrightarrow{\sigma^*} \mathtt{B}^c(\mathtt{D}_n) \xrightarrow{\sigma^*} \cdots$ by Theorem 3.2.13 and the lower row to $\mathtt{E}_1 \hookrightarrow \mathtt{E}_2 \hookrightarrow \cdots \hookrightarrow \mathtt{E}_n \hookrightarrow \cdots$ by Proposition 2.1.10. Hence, our construction returns morphisms $\psi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}_n$.

In arity $r = 2$, the relation $\operatorname{colim}_{\kappa \in \mathcal{K}_n(2)} \mathtt{M}_n(\kappa) \simeq \mathtt{M}_n(2)$ gives, by Proposition 3.2.14, an isomorphism between the colimit over $\mathcal{K}_n$ of the free $\mathcal{K}$-operad $\mathtt{F}(\mathtt{M}_n^{(2)})(\kappa)$, where we consider the underlying symmetric $\mathcal{K}$-diagram of $\mathtt{M}_n^{(2)}$, and the ordinary free operad $\mathtt{F}(\mathtt{M}_n^{(2)})(r)$ on the Σ_*-object associated to this symmetric $\mathcal{K}$-diagram. By definition of the lifting $\tilde{\phi}_n$ in Proposition 4.2.A, the composites

$$\begin{array}{ccccccc}
\mathtt{M}_n^{(2)}(r) & \longrightarrow & \operatorname*{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{F}(\mathtt{M}_n^{(2)})(\kappa) & \longrightarrow & \operatorname*{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{B}^c(\mathtt{D}_n)(\kappa) & \xrightarrow{\tilde{\phi}_n} & \operatorname*{colim}_{\kappa\in\mathcal{K}_n(r)} \mathtt{E}(\kappa) \\
 & & & & \simeq\downarrow & & \downarrow\simeq \\
 & & & & \mathtt{B}^c(\mathtt{D}_n)(r) & \overset{\psi_n}{\dashrightarrow} & \mathtt{E}_n(r) \subset \mathtt{E}(r)
\end{array}$$

agree in arity $r = 2$ with the natural embedding $\mathtt{M}_n(2) \xrightarrow{\simeq} \mathtt{E}_n(2) \subset \mathtt{E}(2)$. From this observation, we obtain:

4.3.1. Fact. *The restriction of our morphism $\psi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}_n$ to the generating Σ_*-object of $\mathtt{B}^c(\mathtt{D}_n)$ in arity $r = 2$ agrees in homology with the morphism*

$$H_*(\mathtt{M}_n)(2) \to H_*(\mathtt{E}_n)(2)$$

mapping $\lambda_{n-1}^\vee$ to the representative of the product μ and $\mu^\vee$ to the representative of the bracket λ_{n-1}, where we use the identity

$$H_*(\mathtt{M}_n) = \Sigma^{-1} H_*(\mathtt{D}_n) = \Sigma^{-1}(\Lambda^{-n}\, \mathtt{G}_n^\vee)$$

and the notation of Proposition 0.3.6 for the dual basis of $\mu, \lambda_{n-1} \in \mathtt{G}_n(2)$ in $\mathtt{G}_n^\vee(2)$.

Recall that the edge morphism $\eta : \mathtt{B}^c(H_*(\mathtt{D}_n)) \to H_*(\mathtt{B}^c(\mathtt{D}_n))$ in the homological requirement of assertion (b) of Theorem A is defined on the generating Σ_*-object $\Sigma^{-1} H_*(\bar{\mathtt{D}}_n)$ by the homology of the natural embedding $\Sigma^{-1}\bar{\mathtt{D}}_n = \mathtt{M}_n \hookrightarrow \mathtt{B}^c(\mathtt{D}_n)$ in arity $r = 2$ and by the null morphism in arity $r \neq 2$, like the Koszul duality equivalence of the n-Gerstenhaber operad $\mathtt{G}_n$. Therefore Fact 4.3.1 implies immediately:

4.3.2. Fact. *The composite of our morphisms $\psi_n : \mathtt{B}^c(\mathtt{D}_n) \to \mathtt{E}_n$ with the edge morphism $\eta : \mathtt{B}^c(H_*(\mathtt{D}_n)) \to H_*(\mathtt{B}^c(\mathtt{D}_n))$ agrees with the Koszul duality equivalence of the n-Gerstenhaber operad, as required by assertion (b) of Theorem A.*

This observation finishes the verification of Lemma C, completes the proof of Theorem A, and hence, of the main result of this article. □

Epilogue

In the prologue, we observe that our main theorem gives an interpretation of the suspension morphisms $\sigma : \mathtt{E}_n \to \Lambda^{-1}\,\mathtt{E}_{n-1}$ in terms of the homotopy of E_n-operads. Namely, the cooperad morphism $\sigma^* : \Lambda^{1-n}\,\mathtt{E}_{n-1}^{\vee} \to \Lambda^{-n}\,\mathtt{E}_n^{\vee}$ associated to σ and the embedding $\iota : \mathtt{E}_{n-1} \to \mathtt{E}_n$ are dual to each other with respect to the cobar construction. This assertion implies that:

- on one side, we have a commutative square

(18)
$$\begin{array}{ccc} \mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_{n-1}^{\vee}) & \xrightarrow{\sigma^*} & \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee}) \\ {\sim}\downarrow{\psi_{n-1}} & & {\sim}\downarrow{\psi_n} \\ \mathtt{E}_{n-1} & \xrightarrow[\iota]{} & \mathtt{E}_n \end{array}$$

  in which vertical arrows are weak-equivalences;
- on the other side, we have a commutative square

(19)
$$\begin{array}{ccccc} \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee}) & \xrightarrow{\iota^*} & \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee}) & \xrightarrow{\simeq} & \Lambda^{-1}\,\mathtt{B}^c(\Lambda^{1-n}\,\mathtt{E}_n^{\vee}) \ , \\ {\sim}\downarrow{\psi'_n} & & & & {\sim}\downarrow{\psi'_{n-1}} \\ \mathtt{E}_n & & \xrightarrow[\sigma]{} & & \Lambda^{-1}\,\mathtt{E}_{n-1} \end{array}$$

  where ι^* represents the image of the operad embedding ι under the functor $\mathtt{B}^c(\Lambda^{-n}(-)^{\vee})$.

The existence of (18) is nothing but the assertion of theorems A-B. Diagram (19) is obtained from (18) by an application of the adjunction relation of the bar duality of operads (see [15, Theorem 2.17]) using that the cobar construction commutes with operadic suspensions. The vertical morphisms in (19) are not the same as the vertical morphisms of (18), but the twisting cochains associated to these morphisms correspond to each other by the duality of $\mathbb{Z}$-modules.

In [5], we prove that the Barratt-Eccles operad acts on the cochain complex of any simplicial set. Our motivation was to give a combinatorial realization of Mandell's cochain model for the p-complete homotopy type of spaces [26]. The purpose of this concluding section is to give an intrinsic interpretation, in terms of the homotopy of E_n-operads, of the action of the Barratt-Eccles operad on the reduced normalized cochain complex of spheres $\bar{N}^*(S^m)$, and hence of the cochain model for the p-complete homotopy of spheres S^m.

The action of $\mathtt{E}$ on $\bar{N}^*(S^m)$ is defined by a morphism $\nabla_{S^m} : \mathtt{E} \to End_{\bar{N}^*(S^m)}$, where $End_{\bar{N}^*(S^m)}$ is the endomorphism operad of $\bar{N}^*(S^m)$. Recall that this operad is given by the dg-hom

$$End_{\bar{N}^*(S^m)}(r) = Hom_{\mathcal{C}}(\bar{N}^*(S^m)^{\otimes r}, \bar{N}^*(S^m))$$

for each arity $r \geq 1$. Let $\mathbb{Z}[d]$ denote the free $\mathbb{Z}$-module of rank 1 put in lower degree d. Since $\bar{N}^*(S^m) = \mathbb{Z}[-m]$, we obtain the identity $End_{\bar{N}^*(S^m)}(r) =$

$Hom_{\mathcal{C}}(\mathbb{Z}[-m]^{\otimes r}, \mathbb{Z}[-m]) = \mathbb{Z}[rm - m]$, from which we deduce

$$End_{\bar{N}^*(S^m)} = \Lambda^{-m}\,\mathtt{C},$$

where $\Lambda^{-m}\,\mathtt{C}$ is the operadic m-fold desuspension of the commutative operad $\mathtt{C}$. Hence, the action of $\mathtt{E}$ on $\bar{N}^*(S^m)$ is represented by a morphism $\sigma_m : \mathtt{E} \rightarrow \Lambda^{-m}\,\mathtt{C}$.

The identity $End_{\bar{N}^*(S^m)}(r) = \mathbb{Z}[rm-m]$ implies that ∇_{S^m} is defined by a cochain of degree $rm - m$ on each $\mathtt{E}(r)$, $r \in \mathbb{N}$. According to [5, Theorem 3.2.4-Proposition 3.2.5], these cochains are nothing but the m-fold cup products $sgn^{\cup m} : \mathtt{E}(r) \rightarrow \mathbb{Z}$ of the cochains sgn defined in §0.1.5. The associativity relation between cup products and cap products implies:

Fact. *The morphism $\sigma_m : \mathtt{E} \rightarrow \Lambda^{-m}\,\mathtt{C}$ giving the action of the Barratt-Eccles operad on $\bar{N}^*(S^m)$ is identified with the composite of the m-fold suspension morphism $\mathtt{E} \xrightarrow{\sigma} \Lambda^{-1}\,\mathtt{E} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \Lambda^{-m}\,\mathtt{E}$ with the morphism $\Lambda^{-m}\,\mathtt{E} \xrightarrow{\sim} \Lambda^{-m}\,\mathtt{C} = End_{\bar{N}^*(S^m)}$ induced by the augmentation of the Barratt-Eccles operad.*

Note that the action of $\mathtt{E}_n \subset \mathtt{E}$ on $\bar{N}^*(S^m)$ vanishes when $n \leq m$. In the case $n > m$, we deduce from (19):

Theorem C. *For every $n > m$, we have a commutative diagram*

$$\begin{array}{ccccccc}
\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee}) & & \xrightarrow{\quad\sim\quad} & & \mathtt{E}_n & \hookrightarrow & \mathtt{E} \\
\downarrow{\scriptstyle \iota^*} & & & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
\vdots & & & & \vdots & & \vdots \\
\downarrow{\scriptstyle \iota^*} & & & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_{n-m}^{\vee}) & \xrightarrow{\simeq} & \Lambda^{-m}\,\mathtt{B}^c(\Lambda^{m-n}\,\mathtt{E}_{n-m}^{\vee}) & \xrightarrow{\sim} & \Lambda^{-m}\,\mathtt{E}_{n-m} & \hookrightarrow & \Lambda^{-m}\,\mathtt{E} \\
 & & & \searrow{\scriptstyle \Lambda^{-m}\phi'_{n-m}} & & & \downarrow{\scriptstyle \sim} \\
 & & & & & & \Lambda^{-m}\,\mathtt{C} = End_{\bar{N}^*(S^m)}
\end{array}$$

giving the action of the E_n-operad $\mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee})$ on $\bar{N}^*(S^m)$.

The morphism ϕ'_{n-m} which occurs in this diagram is given by the composite of ψ'_{n-m} with the augmentation $\epsilon : \mathtt{E} \rightarrow \mathtt{C}$ and does not necessarily agree with our initial morphism ϕ_{n-m}, used as an input for the construction of ψ_{n-m}. Nevertheless, we can easily check, by inspection of our constructions, that ϕ'_{n-m} has the same restriction as ϕ_{n-m} on the edge of the spectral sequence $E^1 = \mathtt{B}^c(\Lambda^{m-n}\,\mathtt{G}_{n-m}^{\vee}) \Rightarrow H_*(\mathtt{B}^c(\Lambda^{m-n}\,\mathtt{E}_{n-m}^{\vee}))$. One can also observe that ϕ'_{n-m} agrees with ϕ_{n-m} up to the indeterminacy of the construction of §1.

On the other hand, we prove in a follow up [14] that each morphism $\phi_n : \mathtt{B}^c(\Lambda^{-n}\,\mathtt{E}_n^{\vee}) \rightarrow \mathtt{C}$ in Lemma A is uniquely determined up to homotopy. The result of this reference also implies that ϕ'_{n-m} is homotopic to ϕ_{n-m}. Hence, we conclude from Theorem C and the uniqueness result of [14] that the action of an E_n-operad on the cochain complex of a sphere $\bar{N}^*(S^m)$ has an intrinsic characterization in terms of the embeddings $\iota : \mathtt{E}_{n-1} \hookrightarrow \mathtt{E}_n$ underlying the definition of an E_n-operad.

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Glossary of notation

$\mathtt{A}$: the associative operad
$\mathtt{B}^c(\mathtt{D})$: the operadic cobar construction of a cooperad (§0.2.2)
$\mathcal{C}$: the base category of dg-modules
$\mathtt{C}$: the commutative operad
$\mathtt{C}_n$: the topological operad of little n-cubes
$\mathtt{D}$: any cooperad
$\mathtt{D}_n$: a short notation for $\mathtt{D}_n = \Lambda^{-n}\,\mathtt{E}_n^\vee$
$\mathtt{E}$: the Barratt-Eccles chain operad (§0.1)
$\mathtt{E}_n$: the suboperad of $\mathtt{E}$ equivalent to the chain operad of little n-cubes (§0.1.3)

$\mathtt{F}(M)$: the free operad on a Σ_*-object (§0.2.1), on a symmetric $\mathcal{K}$-diagram (§§2.2.6-2.2.7)
$\mathtt{G}_n$: the n-Gerstenhaber operad (§0.3)
$\mathcal{K}$: the complete graph operad (§2.1)
$\mathcal{K}_n$: the filtration layers of the complete graph operad (§2.1)
κ: any oriented weight-system (§2.1.1)
$\mathtt{K}(\mathtt{P})$: the Koszul dual of an operad (§0.2.8)
$\mathtt{L}$: the Lie operad
Λ: the operadic suspension (§0.1.5)
$\mathtt{M}_n$: a short notation for $\mathtt{M}_n = \Sigma^{-1}\,\mathtt{D}_n$
$\mathtt{P}$: any operad
Σ: the suspension of dg-modules
Σ_r: the symmetric group on r letters
Θ: the category of trees
$(-)^\vee$: the complement of oriented weight-systems (§3.1.1) or the duality of $\mathbb{Z}$-modules
$\mathtt{W}$: the simplicial Barratt-Eccles operad (§3.1.3)

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