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Little discs operads, graph complexes and Grothendieck–Teichmüller groups

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Introduction

The operads of little discs (and the equivalent operads of little cubes) were introduced in topology, in the works of Boardman–Vogt [7, 8] and May [47], for the recognition of iterated loop spaces. We refer to the paper [18], in the handbook of algebraic topology, for an account of these applications. We also refer to the literature for the general definition of an operad in a category and for the precise definition of the little discs operads, which we denote by D_n throughout this chapter.

The aim of this chapter is to survey new applications of the little discs operads which were motivated by the works of Kontsevich [42, 43] and Tamarkin [53] on the deformation-quantization of Poisson manifolds and by the Goodwillie–Weiss embedding calculus in topology [34, 58]. For our purpose, we also consider the general class of E_n-operads, which consists of the operads that are weakly-equivalent to the operad of little n-discs (equivalently, to the operad of little n-cubes). Besides, we deal with E_n-operads in the category of differential graded modules, which we similarly define as the class of operads that are weakly-equivalent (quasi-isomorphic) to the operad of singular chains on the little n-discs operad (the chain little n-discs operad) $\mathtt{C}_*(\mathsf{D}_n)$.

In Kontsevich's approach, the proof of the existence of deformation-quantizations of Poisson manifolds reduces to the construction of a comparison map of differential graded Lie algebras between on the one hand, the Hochschild cochain complex, which governs the deformations of an associative algebra structure, and on the other hand, the algebra of polyvector fields, equipped with the Schouten-Nijenhuis bracket of polyvectors, which can be used to govern the deformations of a Poisson structure on a manifold.

Kontsevich used an explicit definition of such a comparison map in his first proof of the existence of deformation-quantizations. The theory of E_2-operads actually occurs in a second generation of proofs of this theorem. The idea is that the differential graded Lie algebra structure of both the Hochschild cochain complex and the algebra of polyvector fields can be integrated into an action of a differential graded E_2-operad and the algebra of polyvector fields is rigid (has a unique realization up to quasi-isomorphism) as an algebra over an E_2-operad.

The action of an E_2-operad on the complex of Hochschild cochains was initially conjectured by Deligne. The proof of the latter statement, now established in a wide context (which includes the topological counterpart of the Hochschild cohomology theory in the stable homotopy theory framework), can be interpreted as a measure of the degree of commutativity of the Hochschild cochain complex, regarded as a derived version of the center of associative algebras.

The algebra of polyvector fields comes actually equipped with the structure of a 2-Poisson algebra, where in general an n-Poisson algebra refers to a form of graded Poisson algebra such that the Poisson bracket is an operation of degree $n-1$ (which is actually the case of the Schouten-Nijenhuis bracket for $n=2$). The operad that governs this category of graded Poisson algebras, the n-Poisson operad Pois_n, represents the homology of the operad of little n-discs $\mathtt{H}_*(\mathsf{D}_n)$. Therefore, the proof that the algebra of polyvector fields inherits an action of an E_2-operad, and actually, the crux of the operadic proof of the existence of deformation-quantizations, is equivalent to an operadic formality claim, which asserts that the chain operads of little 2-discs $\mathtt{C}_*(\mathsf{D}_2)$ is quasi-isomorphic to the 2-Poisson operad Pois_2. In fact, such a statement holds for all $n \geq 2$:

$$\mathtt{C}_*(\mathsf{D}_n) \sim \mathsf{Pois}_n,$$

and one the main objectives of this chapter will be to explain this result in details. For the moment, simply mention that the case $n = 2$ of this formality claim was established by Tamarkin by using the theory of Drinfeld's associators.

This operadic approach gives deep insights on structures carried by the set of solutions of the deformation-quantization problem, when we consider the set of all deformation-quantization functors as a whole. Indeed, from Tamarkin's arguments, we can deduce the more precise result that a formality quasi-isomorphism for the chain operad of little 2-discs (and as a consequence, a deformation-quantization functor for Poisson manifolds) is associated to any Drinfeld associator. This observation hints that the rational version of the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ acts on the moduli space of deformation-quantizations just because the set of Drinfeld's associators $Ass(\mathbb{Q})$ defines a torsor under an action of this group. We explain shortly that this connection reflects a finer identity between the Grothendieck–Teichmüller group and the group of homotopy automorphisms of E_2-operads. Recall simply for the moment that the Grothendieck–Teichmüller group models the relations that can be gained from actions of the absolute Galois group on curves. In deformation-quantization theory, we just consider a pro-algebraic version of this group.

To complete this overview, let us mention that higher dimensional generalizations of the deformation-quantization problem, which involve structures governed by any class of n-Poisson algebras, have been studied by

Calaque–Pantev–Toën–Vaquié–Vezzosi in the realm of derived algebraic geometry (see [14]).

The link between the operads of little discs and the embedding calculus comes from certain descriptions of the Goodwillie–Weiss towers, which are towers of "polynomial" approximations of the embedding spaces $\mathtt{Emb}(M, N)$, where (M, N) is any pair of smooth manifolds (see [34, 57]). We refer to Arone–Ching's paper, in this handbook volume, for a comprehensive introduction to the embedding calculus.

In what follows, we focus on the case of Euclidean spaces $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, and we consider a space of embeddings with compact support $\mathtt{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$, whose elements are the embeddings $f : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ such that there exists a compact domain $K \subset \mathbb{R}^m$ with $f\restriction_{\mathbb{R}^m \setminus K} = i$, where $i : \mathbb{R}^m \to \mathbb{R}^n$ denotes the standard embedding $i : (x_1, \dots, x_m) \to (x_1, \dots, x_m, 0, \dots, 0)$. Then we consider an analogously defined space of immersions with compact support $\mathtt{Imm}_c(\mathbb{R}^m, \mathbb{R}^n)$ and we take the homotopy fiber of the obvious forgetful map $\mathtt{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \to \mathtt{Imm}_c(\mathbb{R}^m, \mathbb{R}^n)$. We use the notation $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ for this space.

In general, one can prove that the Goodwillie–Weiss approximations are weakly-equivalent to mapping spaces of truncated (bi)modules over the little discs operads, where the notion of a truncated operadic (bi)module refers to a a (bi)module which is defined up to some arity only. This result was established by Arone–Turchin in [3], after a pioneering work of Dev Sinha [50] on the particular case of the spaces of long knots $\overline{\mathtt{Emb}}_c(\mathbb{R}, \mathbb{R}^n)$. In the case of the space of embeddings with compact support modulo immersions $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$, one can prove further that the Goodwillie–Weiss approximations are weakly-equivalent to $m+1$-fold loop spaces of mapping spaces of truncated operads with the little m-discs operad as source object and the little m-discs operad as target object. This finer result has been established in full generality by Boavida–Weiss in [10], by an improvement of the methods used in the study of the Goodwillie–Weiss calculus of embedding spaces, while other authors have obtained general results on mapping spaces of (truncated) operadic bimodules which permit one to recover this delooping relation from the results obtained by Sinha and Arone–Turchin (see the articles of Dwyer–Hess [24] and Turchin [56] for the case $m = 1$, and the article of Ducoulombier–Turchin [23] for the case of general $m \geq 1$).

In the case $n - m \geq 3$, we can use convergence statements to deduce an equivalence of total spaces from the operadic interpretation of the Goodwillie–Weiss tower, so that we have a weak homotopy equivalence:

$$\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \sim \Omega^{m+1}\,\mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n),$$

where $\mathtt{Map}^h_{\mathcal{O}p}(-,-)$ denotes a derived mapping space bifunctor on the category of operads in topological spaces.

The formality of the little discs operads over the rationals can be used to determine the rational homotopy type of the operadic derived mapping spaces $\mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n)$ which occur in this description of the embedding spaces $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$. For this purpose, we use the fact that we have a rational homotopy equivalence $\mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n) \sim_{\mathbb{Q}} \mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n^{\mathbb{Q}})$ as soon as $n - m \geq 3$, where $D_n^{\mathbb{Q}}$ denotes a rationalization of the topological operad of little n-discs D_n which is given by an operadic extension of the Sullivan rational homotopy theory of spaces (we explain this construction with more details later on). In fact, we use an improved version of the formality which implies that this rational operad $D_n^{\mathbb{Q}}$ has a model $\langle \mathtt{H}^*(D_n) \rangle$ which is determined by the rational cohomology of the operad of little n-discs $\mathtt{H}^*(D_n) = \mathtt{H}^*(D_n, \mathbb{Q})$ (equivalently, by the dual object of the n-Poisson operad Pois_n). This result gives an effective approach to compute the rational homotopy of the operadic mapping spaces $\mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n)$, and hence, to compute the rational homotopy of the embedding spaces $\overline{Emb}^c(\mathbb{R}^m, \mathbb{R}^n)$ by the Goodwillie–Weiss theory of embedding calculus.

Besides the homotopy of the mapping spaces $\mathtt{Map}^h_{\mathcal{O}p}(D_m, D_n^{\mathbb{Q}})$, we can compute the rational homotopy type of the spaces of homotopy automorphisms $\mathtt{Aut}^h_{\mathcal{O}p}(D_n^{\mathbb{Q}})$ in the category of operads. We will actually see that $\mathtt{Aut}^h_{\mathcal{O}p}(D_2^{\mathbb{Q}})$ is weakly-equivalent to a semi-direct product $GT(\mathbb{Q}) \ltimes \mathtt{SO}(2)^{\mathbb{Q}}$, where $GT(\mathbb{Q})$ is the rational Grothendieck–Teichmüller group, and this statement gives a theoretical explanation for the occurrence of the rational Grothendieck–Teichmüller group in deformation-quantization. We have an analogue of this result in the realm of profinite homotopy theory. We explain both statements in this chapter.

In fact, the main objective of this survey is to explain the result of these computations of mapping spaces and of homotopy automorphism spaces of operads. We mainly address this subject. We organize our account as follows.

We devote the first section of our survey to the particular case $n = 2$ of the homotopy theory of E_n-operads. We will explain that the little 2-discs operad has a model given by an operad shaped on braid groupoids. We use this model to obtain our weak equivalence between the Grothendieck–Teichmüller group and the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{O}p}(D_2^{\mathbb{Q}})$.

We explain the formality of E_n-operads in the second section and we tackle the applications to the computation of the rational homotopy of mapping spaces of E_n-operads in the third section. We use graph complexes in the proof of the formality of E_n-operads. We therefore retrieve graph complexes, the graph complexes alluded to in the title of this paper, in our expression of the rational homotopy of the mapping spaces of E_n-operads. The ultimate goal of this survey is precisely to explain this graph complex description of the rational homotopy type of the mapping spaces of E_n-operads.

In general, in this chapter, we use the term 'differential graded module' and the language of differential graded algebra, rather than the language of chain complexes. In fact, we only use the expression '(co)chain complex' for specific constructions of differential graded modules, like the singular complex of a topological space, the Hochschild cochain complex, ... For short, we also use the prefix 'dg' for any category of structured objects that we may form within a base category of differential graded modules (like dg-algebras, dg-operads, ...).

In what follows, we generally define a differential graded module (thus, a dg-module for short) as the structure, equivalent to a (possibly unbounded) chain complex, which consists of a module M equipped with a $\mathbb{Z}$-graded decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and with a differential $\delta : M \to M$ such that $\delta(M_*) \subset M_{*-1}$. If necessary, we use the phrase 'lower graded dg-modules' to refer to the objects of this category of dg-modules.

In some cases, we also deal with 'upper graded dg-modules', which are modules M equipped with a $\mathbb{Z}$-graded decomposition of the form $M = \bigoplus_{n\in\mathbb{Z}} M^n$ and with a differential $\delta : M \to M$ such that $\delta(M^{*-1}) \subset M^*$. In general, we can use the standard correspondence $M_* = M^{-*}$ to convert an upper graded dg-module structure into a lower graded dg-module structure, but we prefer to keep upper graded dg-module structures when this representation is the usual convention in the literature (for instance, in rational homotopy theory).

We equip the category of dg-modules with its standard tensor product so that this category inherits a symmetric monoidal structure, with a symmetry operator defined by using the usual sign rule of homological algebra,

In our study, we freely use the language and the results of the theory of model categories. In particular, in what follows, we rather use the generic term 'weak equivalence' for the class of quasi-isomorphisms, because the quasi-isomorphisms represent the class of weak equivalences of the usual model categories of dg-objects (dg-modules, dg-algebras, dg-operads, ...).

1.1 Braids and the homotopy theory of E_2-operads

We devote this section to the study of the homotopy of E_2-operads.

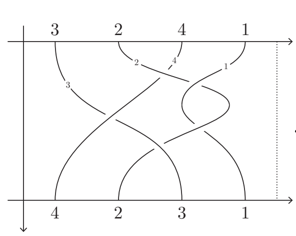
In general, we have a homotopy equivalence of spaces $\mathsf{D}_n(r) \xrightarrow{\sim} \mathsf{F}(\mathring{\mathbb{D}}^n, r)$, for each $r \in \mathbb{N}$, where we consider the underlying spaces of the operad of little n-discs $\mathsf{D}_n(r)$, and $\mathsf{F}(\mathring{\mathbb{D}}^n, r)$ denotes the configuration space of r points in $\mathring{\mathbb{D}}^n$. In the case $n = 2$, this result implies that $\mathsf{D}_2(r)$ forms an Eilenberg–MacLane space $K(P_r, 1)$, where P_r is the pure braid group on r strands in $\mathring{\mathbb{D}}^2$.

The first purpose of this section is to explain that we can elaborate on this result in order to get a model of the class of E_2-operads in the category of operads in groupoids. In short, we check that we have a relation $\mathsf{D}_2 \sim \mathtt{B}(\mathsf{CoB})$, where we consider the classifying spaces of a certain operad in groupoids, the operad of colored braids CoB. We will see that this operad CoB governs the category of strict braided monoidal categories as a category of algebras. We use a variant of this operad, the operad of parenthesized braids, which we associate to the category of general braided monoidal categories, in order to define the Grothendieck–Teichmüller group as a group of automorphisms of an operad in the category of groupoids. We will explain that, when we pass to topological spaces, this identity gives an equivalence between the space of homotopy automorphisms of the little 2-discs operad and a semi-direct product of the Grothendieck–Teichmüller group with the group of rotations. The statement of this result is the second and main objective of this section. To complete this survey, we also explain the definition of the notion of a Drinfeld associator from the viewpoint of the theory of operads.

In our constructions, we deal with versions of the Grothendieck–Teichmüller which are associated to various completions of operads in groupoids, and as a consequence, we actually consider various completions of the little 2-discs operad (namely, the profinite completion and the rationalization) when we examine the relationship between the Grothendieck–Teichmüller group and the homotopy of E_2-operads. In this section, we use a simple definition of these completion operations which we form at the level of the groupoid models of our operads. In the next section, we revisit the definition of the particular case of the rationalization of operads by using the Sullivan rational homotopy theory of spaces.

1.1.1 The operad of colored braids

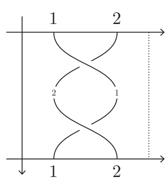
Briefly recall that a braid on r-strands is an isotopy class of paths $\alpha : [0,1] \to \mathsf{F}(\mathring{\mathbb{D}}^2 \times [0,1], r)$ with $\alpha(t) = (\alpha_1(t), \dots, \alpha_r(t)) \in \mathsf{F}(\mathring{\mathbb{D}}^2 \times [0,1], r)$ such that $\alpha_i(t) = (z_i(t), t)$ for each $t \in [0,1]$, and where we assume that $\alpha(0) = (\alpha_1(0), \dots, \alpha_r(0))$ (respectively, $\alpha(1) = (\alpha_1(1), \dots, \alpha_r(1))$) is a permutation of fixed contact points $((z_1^0, 0), \dots, (z_r^0, 0))$ (respectively, $((z_1^0, 1), \dots, (z_r^0, 1))$) on the equatorial line $y = 0$ of the disc $\mathbb{D}^2 \times \{0\}$ (respectively, $\mathbb{D}^2 \times \{1\}$). Thus, we have $z_i^0 = (x_i^0, 0)$ for $i = 1, \dots, r$, and by convention we can also assume that the contact points are ordered so that $x_1^0 < \dots < x_r^0$. In what follows, we use the usual representation of the isotopy class of a braid in terms of a diagram which is given by a projection onto the plane $y = 0$ in the space $\mathring{\mathbb{D}}^2 \times [0,1]$. The assumption $\alpha(t) \in \mathsf{F}(\mathring{\mathbb{D}}^2 \times [0,1], r)$ is equivalent to the requirement that we have $z_i(t) \neq z_j(t)$ for all pairs $i \neq j$. In this definition, we assume that the strands of a braid are indexed by the set $\{1, \dots, r\}$. This assumption is not standard in the definition of a braid, but we use this convention in our definition of colored braids. Intuitively, the indices $i \in \{1, \dots, r\}$ are colors which we assign to the strands of our braids, as in the following picture:

$\alpha =$


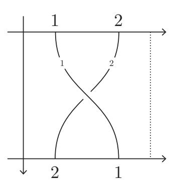
. (1.1)

Formally, the operad of colored braids is an operad in the category of groupoids $\mathsf{CoB} \in \mathcal{G}rd\,\mathcal{O}p$ whose components $\mathsf{CoB}(r)$ are groupoids with the permutations on r letters as objects and the isotopy classes of colored braids as morphisms. The source (respectively, the target) of a morphism is the permutation of the set $\{1,\dots,r\}$ that corresponds to the permutation of the contact points $((z_1^0,\epsilon),\dots,(z_r^0,\epsilon))$ in the sequence $\alpha(0)=(\alpha_1(0),\dots,\alpha_r(0))$ (respectively, $\alpha(1)=(\alpha_1(1),\dots,\alpha_r(1))$). For instance, the above example of colored braid depicts a morphism with the permutation $s=(1\ 3\ 4)$ as source object and the permutation $t=(1\ 4)$ as target object. The composition of braids is given by the standard concatenation operation on paths. Note simply that this operation preserves the indexing when we consider a pair of composable morphisms in our groupoid. Note also that our convention is to orient braids from the top to the bottom and we compose braids accordingly.

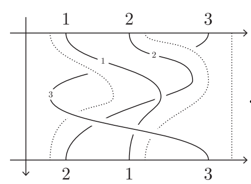
The action of the symmetric group Σ_r on $\mathsf{CoB}(r)$ is given by the obvious re-indexing operation of the strands of our braids. The operadic composition operations $\circ_i : \mathsf{CoB}(k)\times\mathsf{CoB}(l)\to\mathsf{CoB}(k+l-1)$ are functors which are defined on morphisms by a cabling operation on the strands of our braids. In brief, to define a composite $\alpha\circ_i\beta$, where $\alpha\in\mathsf{CoB}(k)$ and $\beta\in\mathsf{CoB}(l)$, we insert the braid β on the ith strand of the braid α, as in the example given in the following picture:



$\circ_1$



$=$



. (1.2)

The operadic unit $1\in\mathsf{CoB}(1)$ is the trivial braid with one strand. By convention, we also assume that the component of arity zero of the colored braid operad is identified with the one-point set $\mathsf{CoB}(0)=*$.

In the introduction of this section, we mentioned that this operad CoB governs the category of strict braided monoidal categories. We give more explanations on this interpretation of the colored braid operad later on, when we explain a similar interpretation of an operad that governs the category of general braided monoidal categories (see Theorem 1.1.5).

The following theorem gives the connection between the operad of little 2-discs and the operad of colored braids:

1.1.2 Theorem (see [26, Theorem I.5.3.4])**.** *We have an equivalence in the category of operads in groupoids $\pi\,\mathsf{D}_2\sim\mathsf{CoB}$, where $\pi\,\mathsf{D}_2$ is the operad in groupoids defined by the fundamental groupoids $\pi\,\mathsf{D}_2(r)$ of the spaces of little 2-discs $\mathsf{D}_2(r)$, $r\in\mathbb{N}$.*

In the category of operads in groupoids $\mathcal{G}rd\,\mathcal{O}p$, we say that a morphism is an equivalence $\phi:\mathsf{P}\xrightarrow{\sim}\mathsf{Q}$ when this morphism defines an equivalence of categories arity-wise $\phi:\mathsf{P}(r)\xrightarrow{\sim}\mathsf{Q}(r)$, for each $r\in\mathbb{N}$. Then we say that operads in groupoids $\mathsf{P},\mathsf{Q}\in\mathcal{G}rd\,\mathcal{O}p$ are equivalent when these operads can be connected by a zigzag of equivalences $P \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdot \dots \cdot \xrightarrow{\sim} Q$ in the category of operads in groupoids.

We just use that the fundamental groupoid functor is strongly symmetric monoidal in order to equip the collection of fundamental groupoids $\pi\, D_2 = \{\pi\, D_2(r), r \in \mathbb{N}\}$ with an operad structure. We refer to the cited reference [26, Theorem I.5.3.4] for the explicit definition of a zigzag of equivalences of operads in groupoids between this object $\pi\, D_2$ and the colored braid operad CoB.

We can apply the classifying space functor $\mathtt{B}(-)$ to go back from the category of groupoids towards the category of spaces (or towards the category of simplicial sets). This functor $\mathtt{B} : \mathcal{G}rd \to \mathcal{T}op$ is also strongly symmetric monoidal, and hence, preserves operad structures. For our purpose, we consider the operad $\mathtt{B}(CoB)$ defined by the collection of the classifying spaces of the colored braid groupoids $\mathtt{B}(CoB(r))$, $r \in \mathbb{N}$. We have the following result:

1.1.3 Theorem (Z. Fiedorowicz [25], see also [26, §I.5.2])**.** *We have a weak equivalence $D_2 \sim \mathtt{B}(CoB)$ in the category of topological operads.*

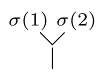
This theorem is established in [25] by arguments of covering theory. In [26, §I.5.3], it is explained that we can also deduce this weak equivalence relation $D_2 \sim \mathtt{B}(CoB)$ from the result of the previous theorem. In brief, the observation that each space $D_2(r)$ is an Eilenberg–MacLane space implies that we have a weak equivalence of spaces $D_2(r) \sim \mathtt{B}(\pi\, D_2(r))$, in each arity $r \in \mathbb{N}$. We can elaborate on the proof of this relation to establish that we actually have a weak equivalence of operads $D_2 \sim \mathtt{B}(\pi\, D_2)$ between the operad of little 2-discs D_2 and the classifying space of the fundamental groupoid operad $\pi\, D_2$. Then we just use that the equivalence of operads in groupoids of Theorem 1.1.2 induces a zigzag of weak equivalences of operads in topological spaces $\mathtt{B}(\pi\, D_2) \sim \mathtt{B}(CoB)$ when we pass to classifying spaces.

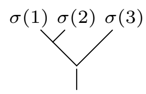
1.1.4 The operad of parenthesized braids

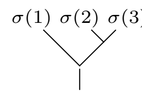
The operads of colored braids are not sufficient for our purpose. To define the Grothendieck–Teichmüller group, we need a variant of this operad, which we call the parenthesized braid operad PaB.

The objects of the colored braid operad form an operad in sets $\mathtt{Ob}\, CoB$, which is identified with the permutation operad , the operad Π defined by the collection of the symmetric groups $\Pi = \{\Sigma_r, r \in \mathbb{N}\}$. The permutation operad also represents a set-theoretic version of the operad of unital associative algebras, or in other words, the operad in sets that governs the structure of a monoid.

To define the operad of parenthesized braids, we just take a pullback of the operad of coloured braids $PaB = \omega^*\, CoB$ along a morphism $\omega : \Omega \to \Pi$, where Ω is the operad in sets that governs the category of non-commutative magmas with a fixed unit element (in another terminology, the category of non-commutative non-associative monoids). This operad Ω, the magma operad, has one free generator in arity two $\mu \in \Omega(2)$, equipped with a free action of the symmetric group Σ_2, and an extra arity-zero element $* \in \Omega(0)$ such that $\mu \circ_1 * = 1 = \mu \circ_2 *$, where $1 \in \Omega(1)$ denotes the operadic unit. In positive arity, the elements of this operad $\pi \in \Omega(r)$ are formal operadic composites of the operations $\mu \in \Omega(2)$ and $(1\ 2) \in \Omega(2)$. The result of these operadic composition operations can also be represented as planar binary rooted trees with r leaves indexed by the values of a permutation on r letters $\sigma = (\sigma(1), \dots, \sigma(r))$, as in the following examples:

$\sigma \cdot \mu =$ 

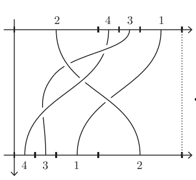
, $\sigma \cdot \mu \circ_1 \mu =$ 

, $\sigma \cdot \mu \circ_2 \mu =$ 

, ...

In arity zero, we just take $\Omega(0) = *$.

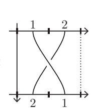
The morphism $\omega : \Omega \to \Pi$, which we consider in the our pullback operation $\mathsf{PaB} = \omega^* \mathsf{CoB}$, carries $\mu \in \Omega(2)$ to the identity permutation on 2 letters $id_2 \in \Sigma_2$. The groupoids $\mathsf{PaB}(r)$ underlying this operad $\mathsf{PaB} = \omega^* \mathsf{CoB}$ are defined by taking $\mathtt{Ob}\, \mathsf{PaB}(r) := \Omega(r)$ and $\mathtt{Mor}_{\mathsf{PaB}(r)}(p, q) := \mathtt{Mor}_{\mathsf{CoB}(r)}(\omega(p), \omega(q))$ for the morphism sets, for all $p, q \in \Omega(r)$. The operadic composition operations are defined by taking an obvious lifting of the composition operations of the operad of colored braids. In [26, §I.6.2], we represent a parenthesized braid by a braid whose contact points form the centers of a dyadic decomposition of the axis $y = 0$ in the disc $\mathring{\mathbb{D}}^2$ (a decomposition obtained by dividing intervals into equal pieces), because one can observe that such decompositions are in bijection with the elements of the magma operad. For instance, the following braid

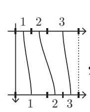
$\beta =$ 

. (1.3)

represents a morphism of the groupoid PaB with the object $p = (1\ 2\ 4) \cdot \mu \circ_2 (\mu \circ_1 \mu)$ as source and the object $q = (1\ 4\ 2\ 3) \cdot \mu \circ_1 (\mu \circ_1 \mu)$ as target.

In the operad PaB, we consider the morphisms

$\tau =$ 

and $\alpha =$ 

, (1.4)

which we call the braiding and the associator respectively.

We aim to give an interpretation of the parenthesized braid operad in classical algebraic language. The object $\mu \in \Omega(2)$ can be regarded as an abstract operation on 2 variables $\mu = \mu(x_1, x_2)$. We use the notation of a tensor product for this operation $\mu(x_1, x_2) = x_1 \otimes x_2$, because we are going to see that μ represents a universal tensor product operation within the operad PaB. The element $(1\,2)\mu = \mu(x_2, x_1)$ represents an operation $\mu(x_2, x_1) = x_2 \otimes x_1$, where the variables (x_1, x_2) are transposed when we use this variable interpretation of our operation. We also get that $\mu \circ_1 \mu = \mu(\mu(x_1, x_2), x_3)$ represents the result of the substitution of the variable x_1 by the operation $\mu = \mu(x_1, x_2)$ in $\mu = \mu(x_1, x_2)$, while $\mu \circ_2 \mu = \mu(x_1, \mu(x_2, x_3))$ represents the result of the substitution of the second variable x_2 by the same operation $\mu = \mu(x_1, x_2)$ with an index shift of the variables. We equivalently have $\mu(\mu(x_1, x_2), x_3) = (x_1 \otimes x_2) \otimes x_3$ and $\mu(x_1, \mu(x_2, x_3)) = x_1 \otimes (x_2 \otimes x_3)$. We accordingly get that the braiding $\tau = \tau(x_1, x_2)$ represents an isomorphism such that

$$\tau(x_1, x_2) : x_1 \otimes x_2 \to x_2 \otimes x_1 \tag{1.5}$$

in the morphism set $\mathtt{Mor}_{\mathsf{PaB}(2)}(\mu, (1\,2)\mu)$ of our operad in groupoids PaB, while the associator $\alpha = \alpha(x_1, x_2, x_3)$ represents an isomorphism such that

$$\alpha(x_1, x_2, x_3) : (x_1 \otimes x_2) \otimes x_3 \to x_1 \otimes (x_2 \otimes x_3) \tag{1.6}$$

in $\mathtt{Mor}_{\mathsf{PaB}(3)}(\mu \circ_1 \mu, \mu \circ_2 \mu)$. The operadic composition formulas $\mu \circ_1 * = 1 = \mu \circ_2 *$ are equivalent to the relations

$$x_1 \otimes * = x_1 = * \otimes x_1, \tag{1.7}$$

so that the arity zero object $* \in \mathit{\Omega}(0) = \mathtt{Ob}\,\mathsf{PaB}(0)$ can be interpreted as a unit object with respect to this tensor product operation $\mu(x_1, x_2) = x_1 \otimes x_2$.

We easily see that the braiding and the associator satisfy the following coherence relations with respect to this unit object:

$$\alpha(*, x_1, x_2) = \alpha(x_1, *, x_2) = \alpha(x_1, x_2, *) = id \quad \text{and} \quad \tau(x_1, *) = id = \tau(*, x_1). \tag{1.8}$$

We easily check, moreover, that the associator satisfies the pentagon relation

$$\begin{aligned} x_1 \otimes \alpha(x_2, x_3, x_4) \cdot \alpha(x_1, x_2 \otimes x_3, x_4) \cdot \alpha(x_1, x_2, x_3) \otimes x_4 \\ = \alpha(x_1, x_2, x_3 \otimes x_4) \cdot \alpha(x_1 \otimes x_2, x_3, x_4) \end{aligned} \tag{1.9}$$

in PaB, as well as the hexagon relations

$$\begin{aligned} x_2 \otimes \tau(x_1, x_3) \cdot \alpha(x_2, x_1, x_3) \cdot \tau(x_1, x_2) \otimes x_3 \\ = \alpha(x_2, x_3, x_1) \cdot \tau(x_1, x_2 \otimes x_3) \cdot \alpha(x_1, x_2, x_3), \end{aligned} \tag{1.10}$$

$$\begin{aligned} \tau(x_1, x_3) \otimes x_2 \cdot \alpha(x_1, x_3, x_2)^{-1} \cdot x_1 \otimes \tau(x_2, x_3) \\ = \alpha(x_3, x_1, x_2)^{-1} \cdot \tau(x_1 \otimes x_2, x_3) \cdot \alpha(x_1, x_2, x_3)^{-1}. \end{aligned} \tag{1.11}$$

Note however that the isomorphism $\tau(x_1, x_2)$ is not involutive in the sense that $\tau(x_2, x_1) \cdot \tau(x_1, x_2) \neq id$, because the braid which represents this isomorphism in the braid group is not involutive either.

From this examination, we conclude that the object $\mu(x_1, x_2) = x_1 \otimes x_2 \in \mathtt{Ob}\,\mathsf{PaB}(2)$ can be interpreted as an abstract tensor product operation that can be used to govern the structure of a braided monoidal category with a strict unit, which is given by the arity zero element of our operad $* \in \Omega(0) = \mathtt{Ob}\, \mathsf{PaB}(0)$, but where we have a general associativity isomorphism $\alpha = \alpha(x_1, x_2, x_3)$, which is depicted in Eqn. 1.4 together with the braiding isomorphism $\tau = \tau(x_1, x_2)$. The operad PaB, equipped with these generating elements, is actually the universal operad that governs such structures, as shown in the following statement:

1.1.5 Theorem (see [26, Theorem I.6.2.4]). *Let $\mathsf{M} \in \mathcal{C}at\,\mathcal{O}p$ be an operad in the category of small categories $\mathcal{C} = \mathcal{C}at$. Fixing an operad morphism $\phi : \mathsf{PaB} \to \mathsf{M}$ amounts to fixing a unit object $e \in \mathtt{Ob}\, \mathsf{M}(0)$, a product object $m \in \mathtt{Ob}\, \mathsf{M}(2)$, an associativity isomorphism $a \in \mathtt{Mor}_{\mathsf{M}(3)}(m \circ_1 m, m \circ_2 m)$, and a braiding isomorphism $c \in \mathtt{Mor}_{\mathsf{M}(2)}(m, (1\ 2)m)$ which satisfy the strict unit relations $m \circ_1 e = 1 = e \circ_1 m$ together with the coherence constraints of the unit, pentagon and hexagon relations of Eqn. 1.7-1.11 in the operad M.*

This theorem is established in the cited reference. The morphism associated to the quadruple (m, e, a, c) given in the theorem is obviously determined by the formulas $\phi(\mu) = m$, $\phi(*) = e$, $\phi(\alpha) = a$ and $\phi(\tau) = c$. The claim is that this assignments determines a well-defined morphism on PaB. The proof of this result follows from a combination of an operadic interpretation of the MacLane coherence theorem and of the classical presentation of the braid group by generators and relations. We have an analogous statement for the operad of colored braids CoB. In this case, we just require $a = id$ in our statement, because we represent the tensor product operation by the identity permutation on 2-letters $id_2 \in \Sigma_2$ in the object-sets of this operad CoB and this operation satisfies a strict associativity relation.

Theorem 1.1.5 implies that the category of algebras governed by the operad PaB in the category of categories is identified with a category of braided monoidal categories with a strict unit but general associativity isomorphisms. The operad of colored braids CoB has a similar interpretation (already mentioned in the introduction), but we then consider the category of braided monoidal categories with strict associativity identities instead of associativity isomorphisms. We refer to [26, §I.6.2] for more detailed explanations on these topics.

1.1.6 The Grothendieck–Teichmüller group

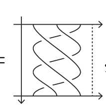
The Grothendieck–Teichmüller group is defined as a group of automorphisms of the parenthesized braid operad. To be more precise, we have to consider completions of this operad in applications. These completions operations are performed at the groupoid level. In what follows, we mainly consider the case of the Malcev completion, which we denote by $G^{\widehat{}}_{\mathbb{Q}}$ for any groupoid $G \in \mathcal{G}rd$, and the profinite completion, which we denote by $G^{\widehat{}}$ (yet another natural example of completion operation is the p-profinite completion, but we do not consider this variant of the profinite completion in this survey). In all cases, the considered completion operation does not change the object sets of our groupoids and is a natural generalization, for groupoids, of the corresponding classical completion operation on groups. Recall simply that the Malcev completion of groups is an extension of the classical rationalization of abelian groups which combines a pro-nilpotent completion with a rationalization operation. In the case of a free group for instance, we can identify the elements of the Malcev completion with infinite products of iterated commutators with rational exponents. (We refer to [26, §I.8] for a detailed survey of this subject.)

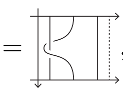
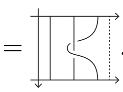
To define the rationalization of our operad $\mathsf{PaB}^{\wedge}_{\mathbb{Q}}$, we just perform the arity-wise completion operation $\mathsf{PaB}^{\wedge}_{\mathbb{Q}}(r) = \mathsf{PaB}(r)^{\wedge}_{\mathbb{Q}}$. Then we define the rational Grothendieck–Teichmüller group $GT(\mathbb{Q})$ as the group of automorphisms of the operad $\mathsf{PaB}^{\wedge}_{\mathbb{Q}}$ which reduce to the identity mapping on the object sets of our operad. In principle, we regard the object $\mathsf{PaB}^{\wedge}_{\mathbb{Q}}$ as an operad in a category of Malcev complete groupoids, where the morphisms satisfy a continuity constraint, and we assume that our automorphisms satisfy such a condition in the definition of the Grothendieck–Teichmüller group. But all morphisms are automatically continuous in the case of the Malcev completion of the operad PaB (see [26, Proposition I.10.1.5]), and therefore, we can neglect this issue in what follows.

We use a similar construction to define the profinite completion of our operad $\mathsf{PaB}^{\wedge}$ and the profinite Grothendieck–Teichmüller group $GT^{\wedge}$. (We just need to take care of the continuity constraints in the definition of morphisms in this case.) We examine the definition of automorphisms on these completions of the parenthesized braid operad to get more insights into the definition of these Grothendieck–Teichmüller groups. We explain our constructions in full details in the case of the rational Grothendieck–Teichmüller group only, because the profinite analogues of these constructions is obvious.

Any morphism $\phi : \mathsf{PaB} \to \mathsf{PaB}^{\wedge}_{\mathbb{Q}}$ admits a unique extension to the completed operad $\phi^{\wedge} : \mathsf{PaB}^{\wedge}_{\mathbb{Q}} \to \mathsf{PaB}^{\wedge}_{\mathbb{Q}}$. By Theorem 1.1.5, such a morphism $\phi : \mathsf{PaB} \to \mathsf{PaB}^{\wedge}_{\mathbb{Q}}$ is fully determined by giving a triple (m, a, c) such that $m = \phi(\mu)$, $a = \phi(\alpha)$ and $c = \phi(\tau)$. Note that we automatically have $\phi(*) = *$ since $\mathsf{PaB}(0) = * \Rightarrow \mathsf{PaB}(0)^{\wedge}_{\mathbb{Q}} = *$. For our purpose, we also set $m = \phi(\mu) = \mu$ since we only consider morphisms that are given by the identity mapping on objects in the definition of the Grothendieck–Teichmüller group.

We necessarily have $\phi(\tau) = \tau \cdot \tau^{2\nu}$ for some parameter $\nu \in \mathbb{Q}$, where we identify $\tau^2 \in \mathtt{Mor}_{\mathsf{PaB}(2)}(\mu, \mu)$ with an element of the pure braid group on 2 strands P_2 and we the expression $\tau^{2\nu}$ with $\nu \in \mathbb{Q}$ to represent an element in the Malcev completion of this group $(P_2)^{\wedge}_{\mathbb{Q}}$. We similarly have $\phi(\alpha) = \alpha \cdot f$ for some morphism $f \in \mathtt{Mor}_{\mathsf{PaB}(3)^{\wedge}_{\mathbb{Q}}}(\mu \circ_1 \mu, \mu \circ_1 \mu)$, which is represented by an element of the Malcev completion of the pure braid group on three strand $(P_3)^{\wedge}_{\mathbb{Q}}$. We have $P_3 = \langle K \rangle \times \langle x_{12}, x_{23} \rangle$, where K denotes a central element in P_3, which is defined by the expression:

$K =$  , (1.12) while x_{12} and x_{23} denote the pure braids such that:

$x_{12} =$  , $\quad x_{23} =$  . (1.13)

The notation $\langle - \rangle$, which we use in this expression $P_3 = \langle K \rangle \times \langle x_{12}, x_{23} \rangle$, refers to the free group generated by a collection of elements.

We can easily deduce from the unit relation $a \circ_2 * = id$ that the associator $\phi(\alpha) = \alpha \cdot f$ has no factor in $\langle K \rangle_{\widehat{\mathbb{Q}}}$. Hence, our morphism ϕ is determined by an assignment of the form:

$$\phi(\mu) := \mu, \quad \phi(\tau) := \tau \cdot \tau^{2\nu} = \tau^{\lambda}, \quad \phi(\alpha) := \alpha \cdot f(x_{12}, x_{23}), \tag{1.14}$$

where we set $\lambda = 1 + 2\nu$ for $\nu \in \mathbb{Q}$ and we assume $f = f(x_{12}, x_{23}) \in \langle x_{12}, x_{23} \rangle_{\widehat{\mathbb{Q}}}$. In what follows, we use the notation of a formal series on two abstract variables $f = f(x, y)$ to represent this element f in the Malcev completion of the free group $F = \langle x_{12}, x_{23} \rangle$.

One can prove that the unit relations $a \circ_1 e = id = a \circ_2 e$ in the coherence constraints of Theorem 1.1.5 are equivalent to the identities:

$$f(x, 1) = x = f(1, x), \tag{1.15}$$

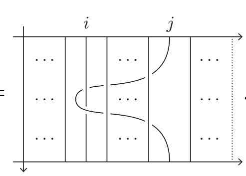
while the hexagon relations are equivalent to the following system of equations

$$f(x, y) \cdot f(y, x) = 1, \tag{1.16}$$
$$f(x, y) \cdot x^{\nu} \cdot f(z, x) \cdot z^{\nu} \cdot f(y, z) \cdot y^{\nu} = 1, \tag{1.17}$$

for a triple of variables (x, y, z) such that $zyx = 1$. The pentagon equation is equivalent to the following relation in the Malcev completion of the pure braid group on 4 strands $(P_4)_{\widehat{\mathbb{Q}}}$ (respectively, in the profinite completion $\widehat{P_4}$):

$$f(x_{23}, x_{34}) f(x_{13} x_{12}, x_{34} x_{24}) f(x_{12}, x_{23}) = f(x_{12}, x_{24} x_{23}) f(x_{23} x_{13}, x_{34}), \tag{1.18}$$

where in general, we use the notation x_{ij} for the pure braid group elements such that:

$x_{ij} =$  . (1.19)

(We refer to [22] and to [26, Proof of Proposition I.11.1.4] for a more detailed analysis of these equations.)

The composition of morphisms corresponds to the following operation on this set of pairs $(\lambda, f(x, y))$:

$$(\lambda, f(x, y)) * (\mu, g(x, y)) = (\lambda\mu, f(x, y) \cdot g(x^{\lambda}, f(x, y)^{-1} \cdot y^{\lambda} \cdot f(x, y))). \tag{1.20}$$

Thus, an element of the Grothendieck–Teichmüller group $\gamma \in GT(\mathbb{Q})$, which corresponds to a morphism $\phi : \mathsf{PaB} \to \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}}$ that induce an isomorphism on the Malcev completion $\phi^{\widehat{}}: \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\simeq} \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}}$, can be uniquely determined by giving a pair $(\lambda, f) \in \mathbb{Q} \times \langle x_{12}, x_{23}\rangle^{\widehat{}}_{\mathbb{Q}}$, which satisfies the constraints of Eqn. 1.15-1.18 and which is invertible with respect to this composition operation. A necessary and sufficient condition for this invertibility condition is given by $\lambda \in \mathbb{Q}^{\times}$ (see [22] and [26, Proposition I.11.1.5]).

The elements of the profinite Grothendieck–Teichmüller group $GT^{\widehat{}}$have a similar representation as pairs $(\lambda, f) \in \mathbb{Z}^{\widehat{}} \times \langle x_{12}, x_{23}\rangle^{\widehat{}}$, where we now consider the profinite completion of the integers $\mathbb{Z}$ for the parameter λ and the profinite completion of the free group $\langle x_{12}, x_{23}\rangle$ for the formal series $f = f(x_{12}, x_{23})$. (We just lack a simple characterization of the invertibility of morphisms in the profinite setting.)

This representation of the elements of the Grothendieck–Teichmüller group in terms of pairs (λ, f) and the equations of Eqn. 1.15-1.18 are actually Drinfeld's original definition of the Grothendieck–Teichmüller group in [22]. The correspondence between this definition and the operadic definition which we summarize in this paragraph is established with full details in the book [26, §I.11.1], but the ideas underlying this operadic interpretation were already implicitly present in Drinfeld's work [22]. We also refer to [5] for another formalization of this interpretation, which uses ideas close to the language of universal algebra. In the introduction of this chapter, we mentioned that the Grothendieck–Teichmüller group was defined by using ideas of the Grothendieck program in Galois theory. In fact, we have an embedding $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow GT^{\widehat{}}$which is defined by using an action of the absolute Galois group on genus zero curves with marked points (see [22]). For the rational Grothendieck–Teichmüller group, a result of F. Brown's (see [13]) implies that we have an analogous embedding $Gal_{MT(\mathbb{Z})} \hookrightarrow GT(\mathbb{Q})$, where $Gal_{MT(\mathbb{Z})}$ now denotes the motivic Galois group of a category of integral mixed Tate motives (see also [21, 55] for the definition of this group and for the definition of this mapping).

We go back to the definition of the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ in terms of operad isomorphisms $\phi^{\widehat{}}_{\gamma} : \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\simeq} \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}}$. We can regard the classifying space operad $\mathsf{E}_2^{\mathbb{Q}} = \mathtt{B}(\mathsf{PaB}^{\widehat{}}_{\mathbb{Q}})$ as a model for the rationalization of the E_2-operad $\mathsf{E}_2 = \mathtt{B}(\mathsf{PaB})$. We deduce from the functoriality of the classifying space construction that any element $\gamma \in GT(\mathbb{Q})$ induces an automorphism $\phi^{\widehat{}}_{\gamma} : \mathsf{E}_2^{\mathbb{Q}} \xrightarrow{\sim} \mathsf{E}_2^{\mathbb{Q}}$ at the topological operad level, and hence, defines an element in the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2^{\mathbb{Q}})$. We claim that this correspondence $\gamma \mapsto \phi^{\widehat{}}_{\gamma}$ induces a bijection when we pass to the group of homotopy classes of homotopy automorphisms. We can deduce this statement from the following more precise statement:

1.1.7 Theorem (B. Fresse [26, Theorem III.5.2.5])**.** *We have* $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2^{\mathbb{Q}}) \sim GT(\mathbb{Q}) \ltimes \mathtt{B}(\mathbb{Q})$.

The factor $\mathtt{B}(\mathbb{Q})$ in the expression of this theorem corresponds to a rationalization of the group of rotations $\mathtt{SO}(2) \sim \mathtt{B}(\mathbb{Z})$ which acts on the little 2-discs model of the class of E_2-operads $\mathsf{E}_2 = \mathsf{D}_2$ by rotating the configurations of little 2-discs in this operad D_2. We equip this factor $\mathtt{B}(\mathbb{Q})$ with the obvious additive group structure.

We have $\mathtt{Mor}_{\mathsf{PaB}(2)^{\wedge}_{\mathbb{Q}}}(\mu, \mu) = (P_2)^{\wedge}_{\mathbb{Q}}$, and as a consequence, any element of the Grothendieck–Teichmüller group $\gamma \in GT(\mathbb{Q})$ determines an automorphism of the Malcev completion of the pure braid group $(P_2)^{\wedge}_{\mathbb{Q}} \simeq \mathbb{Q}$ through its action on the automorphism group of the object $\mu \in \mathtt{Ob}\,\mathsf{PaB}(2)$. We use this observation to determine the action of the Grothendieck–Teichmüller group on the group $\mathtt{B}(\mathbb{Q})$ that we consider in the definition of the semi-product $GT(\mathbb{Q}) \ltimes \mathtt{B}(\mathbb{Q})$. (We refer to [26, §III.5.2] for details.)

Let us insist that we consider derived homotopy automorphism spaces in the statement of this theorem. In the model category approach, these homotopy automorphism spaces are defined by taking the actual spaces of homotopy automorphism spaces associated to a cofibrant-fibrant replacement $\mathsf{R}_2^{\mathbb{Q}}$ of our operad $\mathsf{E}_2^{\mathbb{Q}}$. To associate an element of this derived homotopy automorphism space to an element of the Grothendieck–Teichmüller group $\gamma \in GT(\mathbb{Q})$, we use the fact that an automorphism $\phi_{\gamma}^{\wedge} : \mathsf{E}_2^{\mathbb{Q}} \to \mathsf{E}_2^{\mathbb{Q}}$ automatically admits a lifting to this cofibrant-fibrant replacement $\mathsf{R}_2^{\mathbb{Q}}$. The claim is that all homotopy automorphisms of this cofibrant-fibrant model $\mathsf{R}_2^{\mathbb{Q}}$ are homotopic to such morphisms, and that this correspondence gives all the homotopy of the space $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2^{\mathbb{Q}})$ up to the factor $\mathtt{B}(\mathbb{Q})$.

The book [26, §§III.1-5] gives a proof of this result by using spectral sequence methods and an operadic cohomology theory which provides approximations of our homotopy automorphism spaces. This method is close to the methods which are used in the next sections, when we study the homotopy automorphism spaces of E_n-operads for any value of the dimension parameter $n \geq 2$.

We now consider the profinite version of the Grothendieck–Teichmüller group $GT^{\wedge}$. We take the classifying space operad $\mathsf{E}_2^{\wedge} = \mathtt{B}(\mathsf{PaB}^{\wedge})$ as a model for the profinite completion of the E_2-operad $\mathsf{E}_2 = \mathtt{B}(\mathsf{PaB})$ and we use the same construction as in the rational setting to define a mapping from the profinite Grothendieck–Teichmüller group towards the homotopy automorphism space of this object $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{D}_2^{\wedge})$. Then we have the following analogue of the result of Theorem 1.1.8:

1.1.8 Theorem (G. Horel [37]). *We have* $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{D}_2^{\wedge}) \sim GT^{\wedge} \ltimes \mathtt{B}(\mathbb{Z}^{\wedge})$.

The article [37] gives a proof of the result of this theorem by using the correspondence with groupoids. In short, the idea of this paper is to observe that the operad $\mathsf{PaB}^{\wedge}$ represents a cofibrant object with respect to some model structure on the category of operads in groupoids. Then we can use model category arguments (combined with higher category methods) to prove that we can transport the computation of the homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_2^{\wedge})$

to the computation of the homotopy automorphism space associated to this object *PaB* in the category of operads in groupoids.

1.1.9 The Drinfeld–Kohno Lie algebra operad

Besides the colored braid and the parenthesized braid operads, which are defined by using the structures of the braid groups, we consider operads in Lie algebras which are associated to infinitesimal versions of the pure braid groups. To be explicit, in these infinitesimal versions, we consider the Drinfeld–Kohno Lie algebras (also called the Lie algebras of infinitesimal braids), which are defined by a presentation of the form:

$$\mathfrak{p}(r) = \mathbb{L}(t_{ij}, \{i,j\} \subset \{1,\dots,r\}) / < [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] >, \tag{1.21}$$

for $r \in \mathbb{N}$, where $\mathbb{L}(-)$ denotes the free Lie algebra functor, we associate a generator t_{ij} such that $t_{ij} = t_{ji}$ to each pair $\{i \neq j\} \subset \{1,\dots,r\}$, and we take the ideal generated by the commutation relations

$$[t_{ij}, t_{kl}] \equiv 0, \tag{1.22}$$

for all quadruples $\{i,j,k,l\} \subset \{1,\dots,r\}$ such that $\sharp\{i,j,k,l\} = 4$, together with the Yang-Baxter relations

$$[t_{ij}, t_{ik} + t_{jk}] \equiv 0, \tag{1.23}$$

for all triples $\{i,j,k\} \subset \{1,\dots,r\}$ such that $\sharp\{i,j,k\} = 3$. This definition makes sense over any ground ring $\Bbbk$, but from the next paragraph on, we will assume that the ground ring is a field of characteristic zero.

Note that this Lie algebra $\mathfrak{p}(r)$ inherits a weight grading from the free Lie algebra since this ideal is generated by homogeneous relations. If we use the notation $L_m = \mathbb{L}_m(-)$, for the homogeneous component of weight m of the free Lie algebra $L = \mathbb{L}(t_{ij}, \{i,j\} \subset \{1,\dots,r\})$, then we have the decomposition $\mathfrak{p}(r) = \bigoplus_{m\geq 1} \mathfrak{p}(r)_m$, where we set $\mathfrak{p}(r)_m = L_m / L_m \cap < [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] >$, for $m \geq 1$. In fact, we have the identity $\mathfrak{p}(r)_* = gr_*^{\Gamma} P_r$, where on the right-hand side we consider the graded Lie algebra of the sub-quotients of the central series filtration of the pure braid group $gr_*^{\Gamma} P_r$ (see [26, Theorem I.10.0.4] for a detailed proof of this statement).

The collection $\mathfrak{p} = \{\mathfrak{p}(r), r \in \mathbb{N}\}$ inherits the structure of an operad in the category of Lie algebras, where we take the direct sum of Lie algebras to define our symmetric monoidal structure. The action of the symmetric group Σ_r on the Lie algebra $\mathfrak{p}(r)$ is defined, on generators, by the obvious re-indexing operation $\sigma \cdot t_{ij} = t_{\sigma(i)\sigma(j)}$, for all $\sigma \in \Sigma_r$. The composition products are given by Lie algebra morphisms of the form

$$\circ_i : \mathfrak{p}(k) \oplus \mathfrak{p}(l) \to \mathfrak{p}(k+l-1), \tag{1.24}$$

defined for all $k, l \in \mathbb{N}$ and $i \in \{1,\dots,k\}$, and which satisfy the equivariance, unit and associativity relations of operads in the category of Lie algebras. For generators $t_{ab} \in \mathfrak{p}(k)$ and $t_{cd} \in \mathfrak{p}(l)$, we explicitly set:

$$t_{ab} \circ_i 0 = \begin{cases} t_{a+l-1b+l-1}, & \text{if } i < a < b, \\ t_{ab+l-1} + \cdots + t_{a+l-1b+l-1}, & \text{if } i = a < b, \\ t_{ab+l-1}, & \text{if } a < i < b, \\ t_{ab} + \cdots + t_{aj+l-1}, & \text{if } a < i = b, \\ t_{ab}, & \text{if } a < b < i, \end{cases} \tag{1.25}$$

and

$$0 \circ_i t_{cd} = t_{c+i-1d+i-1} \quad \text{for all } i. \tag{1.26}$$

The operadic unit is just given by the zero morphism $0 : 0 \to \mathfrak{p}(1)$ with values the zero object $\mathfrak{p}(1) = 0$.

In fact, these operations reflect the composition structure of the operad of colored braids, in the sense that we can identify the components of homogeneous weight of this operad $\mathfrak{p}(-)_m$, $m \geq 1$, with the fibers of a tower of operads $\mathsf{CoB} / \Gamma_m \mathsf{CoB}$, $m \geq 1$, which we deduce from the central series filtration of the pure braid group. (We refer to [26, §I.10.1] for more explanations on this correspondence.)

We call this operad $\mathfrak{p}$ the Drinfeld–Kohno Lie algebra operad. We consider generalizations of this operad when we study the Sullivan model of E_n-operads. This subsequent study is our main motivation for the recollections of this paragraph, but the Drinfeld–Kohno Lie algebra operad also occurs in the theory of Drinfeld's associators and in the definition of a graded version of the Grothendieck–Teichmüller group. We just give a brief overview of this subject to complete the account of this section.

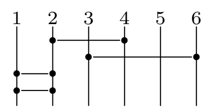
1.1.10 The operad of chord diagrams and associators

To define the set of Drinfeld's associators, we consider an operad in groupoids, the chord diagram operad $\mathsf{CD}^{\widehat{}}_{\Bbbk}$, defined over any characteristic zero field $\Bbbk$, and such that we have the relation $\mathsf{CD}(r)^{\widehat{}}_{\Bbbk} = \exp \hat{\mathfrak{p}}(r)$ for each $r \in \mathbb{N}$, where we consider the exponential group associated to a completion of the Drinfeld Kohno Lie algebra $\hat{\mathfrak{p}}(r)$.

To be more precise, we explained in the previous paragraph that the Drinfeld Kohno Lie algebra admits a weight decomposition $\mathfrak{p}(r) = \bigoplus_{m \geq 1} \mathfrak{p}(r)_m$. To form the completed Lie algebra $\hat{\mathfrak{p}}(r)$, we just replace the direct sum of this decomposition by a product. Thus we have $\hat{\mathfrak{p}}(r) = \prod_{m \geq 1} \mathfrak{p}(r)_m$ so that the elements of this completed Lie algebra $\hat{\mathfrak{p}}(r)$ are represented by infinite series of Lie polynomials (modulo the ideal generated by the commutation and the Yang-Baxter relations). The exponential group $\mathsf{CD}(r)^{\widehat{}}_{\Bbbk} = \exp \hat{\mathfrak{p}}(r)$ consists of formal exponential elements e^{ξ}, $\xi \in \hat{\mathfrak{p}}(r)$, together with the group operation given by the Campbell-Hausdorff formula at the level of the completed Lie algebra

$\hat{\mathfrak{p}}(r)$. We identify this group $\mathit{CD}(r)^{\wedge}_{\Bbbk} = \exp \hat{\mathfrak{p}}(r)$ with a groupoid with a single object when we regard the chord diagram operad $\mathit{CD}^{\wedge}_{\Bbbk} = \{\exp \hat{\mathfrak{p}}(r), r \in \mathbb{N}\}$ as an operad in the category of groupoids. The structure operations of this operad $\mathit{CD}^{\wedge}_{\Bbbk}$ are induced by the structure operations of the Drinfeld–Kohno Lie algebra operad on $\hat{\mathfrak{p}}$.

We also have an identity $\mathit{CD}(r)^{\wedge} = \mathbb{G}(\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r)))$, where we consider the set of group-like elements $\mathbb{G}(-)$ in the complete enveloping algebra of the Drinfeld–Kohno Lie algebra $\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$. Indeed, the group-like elements are identified with actual exponential series of Lie algebra elements $\xi \in \mathfrak{p}(r)$ within the complete enveloping algebras $\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$. The name "chord diagram" comes from the theory of Vassiliev invariants, where a monomial $t_{i_1 j_1} \cdots t_{i_m j_m} \in \hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$ is associated to a diagram with r vertical strands numbered from 1 to r, and l chords corresponding to the factors $t_{i_k j_k}$, as in the following picture:

$t_{12}t_{12}t_{36}t_{24} =$  . (1.27)

The composition products of the chord diagram operad have a simple description in terms of chord diagram insertions too.

In the previous paragraph, we explained that the components of homogeneous weight of the Drinfeld–Kohno Lie algebra operad represent the fibers of a tower decomposition of the parenthesized braid operad (and of the colored braid operad equivalently). In fact, a stronger result holds when we work over a field of characteristic zero. To be more explicit, we consider the Malcev completion of the operad PaB, and a natural extension of this construction for ground fields such that $\mathbb{Q} \subset \Bbbk$. Then we may wonder about the existence of equivalences of operads in groupoids $\phi^{\wedge}_{\alpha} : \mathit{PaB}^{\wedge}_{\Bbbk} \xrightarrow{\sim} \mathit{CD}^{\wedge}_{\Bbbk}$, which would be equivalent to a splitting of this tower decomposition over $\Bbbk$. By Theorem 1.1.5, the morphism of operads in groupoids $\phi_a : \mathit{PaB} \xrightarrow{\sim} \mathit{CD}^{\wedge}_{\Bbbk}$ which would induce such an equivalence on the completion is determined by the choice of a braiding $c \in \exp \hat{\mathfrak{p}}(2)$ and of an associativity isomorphism $a \in \exp \hat{\mathfrak{p}}(3)$. The braiding has the form $c = \exp(\kappa t_{12}/2)$, for some parameter $\kappa \in \Bbbk^{\times}$, since $\hat{\mathfrak{p}}(2) = \Bbbk\, t_{12}$, and one can prove that the associator is necessarily of the form $a = \exp f(t_{12}, t_{23})$, for some Lie power series $f(t_{12}, t_{23}) \in \hat{\mathbb{L}}(t_{12}, t_{23})$. Thus, the existence of an equivalence of operads in groupoids $\phi^{\wedge}_{\alpha} : \mathit{PaB}^{\wedge}_{\Bbbk} \xrightarrow{\sim} \mathit{CD}^{\wedge}_{\Bbbk}$ reduces to the existence of such a Lie power series $f(t_{12}, t_{23}) \in \hat{\mathbb{L}}(t_{12}, t_{23})$ such that $a = \exp f(t_{12}, t_{23})$ satisfies the unit, pentagon and hexagon constraints of Theorem 1.1.5, for some given parameter $\kappa \in \Bbbk^{\times}$.

The set of Drinfeld's associators precisely refers to this particular set of associators $a = \exp f(t_{12}, t_{23})$ which we associate to the chord diagram operad $\mathit{CD}^{\wedge}_{\Bbbk}$. This notion was introduced by Drinfeld in the paper [22], to which we also refer for an explicit expression of the unit, pentagon and hexagon constraints (see also the survey of [26, §I.10.2]). Further reductions occur in the pentagon and hexagon constraints in the definition of Drinfeld's associators. In fact, a result of Furusho (see [30]) implies that the hexagon constraints are satisfied as soon as we have a power series that fulfills the unit and the pentagon constraints.

We have the following main result:

1.1.11 Theorem (V.I. Drinfeld [22])**.** *The set of Drinfeld's associators is not empty, for any choice of field of characteristic zero as ground field* $\Bbbk$ *(including* $\Bbbk = \mathbb{Q}$*), so that we do have an operad morphism* $\phi_\alpha : \mathsf{PaB} \to \mathsf{CD}^{\widehat{}}_{\mathbb{Q}}$ *which induces an equivalence when we pass to the Malcev completion* $\phi^{\widehat{}}_\alpha : \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{\mathbb{Q}}$*.*

In [22], Drinfeld gives an explicit construction of a complex associator by using the monodromy of the Knizhnik–Zamolodchikov connection. This associator, which is usually called the Knizhnik–Zamolodchikov associator in the literature, can also be identified with a generating series of polyzeta values. Descent arguments can be used to establish the existence of a rational associator from the existence of this complex associator (see again [22] and [5] for different proofs of this descent statement), so that the result of this theorem holds over $\Bbbk = \mathbb{Q}$, and not only over $\Bbbk = \mathbb{C}$. Another explicit definition of an associator, defined over the reals, is given by Alekseev–Torossian in [1], by using constructions introduced by Kontsevich in his proof of the formality of the operads of little discs (see [43]).

1.1.12 The operad of parenthesized chord diagrams, the graded Grothendieck–Teichmüller group, and other related objects

The existence of associators can be used to get insights into the structure of the rational Grothendieck–Teichmüller group $GT(\mathbb{Q})$. Indeed, the definition implies that the set of associators inherits a free and transitive action of the rational Grothendieck–Teichmüller group. To go further into the applications of associators, one introduces a parenthesized version of the chord diagram operad $\mathsf{PaCD}^{\widehat{}}_{\mathbb{Q}}$ (by using the same pullback construction as in the case of the parenthesized braid operad PaB) and a group of automorphisms, denoted by $GRT(\mathbb{Q})$, which we associate to this object $\mathsf{PaCD}^{\widehat{}}_{\mathbb{Q}}$. One can easily check that every equivalence of operads in groupoids $\phi^{\widehat{}}_a : \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\sim} \mathsf{CD}^{\widehat{}}_{\mathbb{Q}}$ lifts to an isomorphism $\phi^{\widehat{}}_a : \mathsf{PaB}^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\simeq} \mathsf{PaCD}^{\widehat{}}_{\mathbb{Q}}$ so that the existence of rational associators implies the existence of a group isomorphism $GT(\mathbb{Q}) \simeq GRT(\mathbb{Q})$ by passing to automorphism groups.

This group $GRT(\mathbb{Q})$ is usually called the graded Grothendieck–Teichmüller group in the literature, because this group is identified with the pro-algebraic group associated to a graded Lie algebra such that $\mathfrak{grt} = \bigoplus_{m \geq 0} \mathfrak{grt}_m$. We moreover have $\mathfrak{grt}_m \simeq \mathtt{F}_m\, GT(\mathbb{Q}) / \mathtt{F}_{m+1}\, GT(\mathbb{Q})$, for all $m \geq 1$, for some natural filtration of the Grothendieck–Teichmüller group $GT(\mathbb{Q}) = \mathtt{F}_0\, GT(\mathbb{Q}) \supset \mathtt{F}_1\, GT(\mathbb{Q}) \supset \cdots \supset \mathtt{F}_m\, GT(\mathbb{Q}) \supset \cdots$. We again refer to [22] for a proof of these results (see also the survey of [26, §I.11.4]).

The groups $GT(\mathbb{Q})$, $GRT(\mathbb{Q})$, and the Lie algebra $\mathfrak{grt}$ are also related to other objects of arithmetic geometry and group theory. In §1.1.6, we already recalled that, by a result of F. Brown (see [13]), the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ contains a realization of the motivic Galois group of a category of integral mixed Tate motives $Gal_{MT(\mathbb{Z})}$. In fact, one conjectures that these groups are isomorphic (Deligne-Ihara). Furthermore, one can prove that the Galois group $Gal_{MT(\mathbb{Z})}$ reduces to the semi-direct product of the multiplicative group with the prounipotent completion of a free group on a sequence of generators $s_3, s_5, \dots, s_{2n+1}, \dots$ (see [21]). This result implies that the embedding $Gal_{MT(\mathbb{Z})} \hookrightarrow GT(\mathbb{Q})$ is equivalent to an embedding of the form $\Bbbk \oplus \mathbb{L}(s_3, s_5, \dots, s_{2n+1}, \dots) \hookrightarrow \mathfrak{grt}$ when we pass to the category of Lie algebras. In this context, one can re-express the Deligne-Ihara conjecture as the conjecture that this embedding of Lie algebras is an isomorphism.

In our comments on Theorem 1.1.11, we also explained that the Knizhnik–Zamolodchikov associator represents the generating series of polyzeta values. The polyzeta values satisfy certain equations, called the regularized double shuffle relations, which can be expressed in terms of the Knizhnik–Zamolodchikov associator, and one conjectures that all relations between polyzetas follow from the double shuffle relations and from the fact that the Knizhnik–Zamolodchikov associator defines a group-like power series. By a result of Furusho [31], the pentagon condition for associators implies the regularized double shuffle relations. This result implies that the Grothendieck–Teichmüller group embeds in a group defined by solutions of regularized double shuffle relations with a degeneration condition, and one conjectures again that this embedding is an isomorphism.

The theory of associators is also used by Alekseev–Torossian in the study of the solutions of the Kashiwara–Vergne conjecture, a problem about the Campbell-Hausdorff formula motivated by questions of harmonic analysis. These authors notably proved in [1] that the set of Drinfeld's associators embeds into the set of solutions of the Kashiwara–Vergne conjecture. In particular, one can deduce the existence of such solutions from the existence of associators. In addition, one can prove that the action of the Grothendieck–Teichmüller group on Drinfeld's associators lifts to the set of solutions of the Kashiwara–Vergne conjecture. This action is still free so that we get that the Grothendieck–Teichmüller group embeds into a group of automorphisms associated to this set of solutions. The conjecture is that this group embedding is an isomorphism, yet again.

1.2 The rational homotopy of E_n-operads and formality theorems

The goal of this section is to explain the definition of rational models of E_n-operads and a characterization of the class of E_n-operads up to rational homotopy equivalence. In what follows, we just focus on the case $n \geq 2$, because we can put the case $n = 1$ apart. Indeed, we have $\mathsf{D}_1 \sim \mathit{\Pi}$, where we regard the permutation operad $\mathit{\Pi} = \{\Sigma_r, r \in \mathbb{N}\}$ (see §1.1.4) as a discrete operad in topological spaces. Hence, the class of E_1-operads is also identified with the class of operads that are weakly-equivalent to this discrete operad $\mathit{\Pi}$, and such a class of objects is fixed by the rationalization.

Recall that a map of simply connected topological spaces is a rational homotopy equivalence $f : X \xrightarrow{\sim_{\mathbb{Q}}} Y$ if this map induces a bijection on homotopy groups $f_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. In what follows, we consider a generalization of this notion in the context of spaces which, like the underlying spaces of the little 2-disc operad, are (connected but) not necessarily simply connected. In this case, we assume that a rational homotopy equivalence also induces an isomorphism on the Malcev completion of the fundamental group $f_* : \pi_1(X, x)^{\widehat{}}_{\mathbb{Q}} \xrightarrow{\cong} \pi_1(Y, f(x))^{\widehat{}}_{\mathbb{Q}}$.

In the context of operads, we just consider operad morphisms $\phi : \mathsf{P} \xrightarrow{\sim_{\mathbb{Q}}} \mathsf{Q}$ which define a rational homotopy equivalence of spaces arity-wise $\phi : \mathsf{P}(r) \xrightarrow{\sim_{\mathbb{Q}}} \mathsf{Q}(r)$. and we write $\mathsf{P} \sim_{\mathbb{Q}} \mathsf{Q}$ when our objects P and Q can be connected by a zigzag of such rational homotopy equivalences. We aim to determine the class of operads such that $\mathsf{R} \sim_{\mathbb{Q}} \mathsf{D}_n$.

We develop a rational homotopy theory of operads to address this problem. We rely on the Sullivan rational homotopy of spaces, which we briefly review in the next paragraph. We explain the construction of an operadic extension of the Sullivan model afterwards. We eventually check that the n-Poisson cooperad, the dual structure of the n-Poisson operad, defines a Sullivan model of the little n-discs operad D_n, and as such determines a model for the class of E_n-operads up to rational homotopy. We need a cofibrant resolution of the n-Poisson cooperad to perform computations with this model. We will explain that such a cofibrant resolution is given by the Chevalley–Eilenberg cochain complex of a graded version of the Drinfeld–Kohno Lie algebra operad of the previous section. We actually consider another resolution in our construction, namely a cooperad of graphs, and we also explain the definition of this object. We use the latter model in the next section, when we explain a graph complex description of the rational homotopy type of mapping spaces of E_n-operads.

In order to apply the methods of rational homotopy theory, we take $\Bbbk = \mathbb{Q}$ as a ground ring for our categories of modules from now on, and we also consider the cohomology with coefficients in this field $\mathtt{H}^*(-) = \mathtt{H}^*(-, \mathbb{Q})$. We similarly take $\mathtt{H}_*(-) = \mathtt{H}_*(-, \mathbb{Q})$ for the homology.

1.2.1 Recollections on the Sullivan rational homotopy theory of spaces

Recall that we call 'upper graded dg-module' the structure formed by a module M equipped with a decomposition such that $M = \bigoplus_{n \in \mathbb{Z}} M^n$ and with a differential $\delta : M \to M$ such that $\delta(M^{*-1}) \subset M^*$. We say that such a dg-module M is non-negatively graded when we have $M^n = 0$ for $n < 0$. Let $dg^* \mathcal{C}om$ be the category of commutative algebras in upper non-negatively graded dg-modules (the category of commutative cochain dg-algebras for short). The Sullivan model for the rational homotopy of a space takes values in this category $dg^*\mathcal{C}om$ and is obtained by applying the Sullivan functor of PL differential forms, a version of the de Rham cochain complex which is defined over $\mathbb{Q}$ (instead of $\mathbb{R}$) and on the category of simplicial complexes (or simplicial sets), instead of the category of smooth manifolds (see [52]). For our purpose, we consider the simplicial set variant of this functor:

$$\Omega^* : s\mathcal{S}et^{op} \to dg^*\mathcal{C}om\,. \tag{1.28}$$

In the particular case of a simplex $\Delta^n = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$, we explicitly have:

$$\Omega^*(\Delta^n) = \mathbb{Q}[x_1, \dots, x_n, dx_1, \dots, dx_n], \tag{1.29}$$

where $dx_1, \dots, dx_n$ represents the differential of the variables $x_1, \dots, x_n$ in this commutative cochain dg-algebra.

The Sullivan functor $\Omega^* : s\mathcal{S}et^{op} \to dg^*\mathcal{C}om$ has a left adjoint

$$\mathtt{G}_\bullet : dg^*\mathcal{C}om \to s\mathcal{S}et^{op}, \tag{1.30}$$

which is given by the formula $\mathtt{G}_\bullet(A) = \mathtt{Mor}_{dg^*\mathcal{C}om}(A, \Omega^*(\Delta^\bullet))$, for any commutative cochain dg-algebra $A \in dg^*\mathcal{C}om$, and this pair of adjoint functors $(\mathtt{G}_\bullet, \Omega)$ defines a Quillen adjunction. (We refer to [12] for this application of the formalism of model categories to Sullivan's constructions.) Then we set:

$$\langle A\rangle := \text{derived functor of } \mathtt{G}_\bullet(A) = \mathtt{Mor}_{dg^*\mathcal{C}om}(R_A, \Omega^*(\Delta^\bullet)), \tag{1.31}$$

where $R_A \xrightarrow{\sim} A$ is any cofibrant resolution of A in $dg^*\mathcal{C}om$. If X satisfies reasonable finiteness and nilpotence assumptions, then the space

$$X^{\mathbb{Q}} := \langle \Omega^*(X)\rangle \tag{1.32}$$

defines a rationalization of the space X in the sense that we have the identities

$$\pi_*(X^{\mathbb{Q}}) := \begin{cases} \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}, & \text{for } * \geq 2, \\ \pi_1(X)^{\widehat{}}_{\mathbb{Q}}, & \text{for } * = 1, \end{cases} \tag{1.33}$$

where we again use the notation $(-)^{\widehat{}}_{\mathbb{Q}}$ for the Malcev completion functor on groups. Besides, one can prove that the unit of the derived adjunction relation between the functors $\mathtt{G}_\bullet$ and Ω^* defines a map $\eta : X \to X^{\mathbb{Q}}$ which corresponds to the usual rationalization map at the level of these homotopy groups.

1.2.2 The category of Hopf cochain dg-cooperads

To extend the Sullivan model to operads, the idea is to consider cooperads in the category of commutative cochain dg-algebras, where the cooperad is a structure which is dual to an operad in the sense of the theory of categories.

In general, a cooperad in a symmetric category $\mathcal{C}$ consists of a collection of objects $C = \{C(r), r \in \mathbb{N}\}$, together with an action of the symmetric group Σ_r on $C(r)$, for each $r \in \mathbb{N}$, and composition coproducts

$$\circ_i^* : C(k+l-1) \to C(k) \otimes C(l), \tag{1.34}$$

defined for all $k, l \in \mathbb{N}$, $i \in \{1, \dots, k\}$, and which satisfy equivariance, unit and coassociativity relations dual to the equivariance, unit and coassociativity axioms of operads. To handle difficulties, we consider a subcategory of cooperads such that $C(0) = C(1) = \mathbb{1}$ where $\mathbb{1}$ is the unit object of our base category, and we use the notation $\mathcal{O}p_{*1}^c$ for this category of cooperads. This restriction enables us to simplify some constructions, because the composition coproducts are automatically conilpotent when we put the component of arity zero apart and we assume $C(1) = \mathbb{1}$. In some cases, we consider a category of cooperads $\mathcal{O}p_{*N}^c$ such that we still have $C(0) = \mathbb{1}$, but where $C(1)$ may not reduce to the unit object. More care is necessary in this case, and we actually assume an extra conilpotence condition for the composition coproducts that involve the component of arity one. (We refer to [27] for the precise expression of this conilpotence condition.)

In [26, §II.12] the author considers a category of Λ-cooperads, whose objects have no term in arity zero, but a diagram structure over the category of finite ordinals and injective maps which extends the action of the symmetric groups on ordinary operads. This category of Λ-cooperads is isomorphic to the category of cooperads which we consider in this paragraph $\mathcal{O}p_{*1}^c$, so that the results of this reference [26] can immediately be transposed to our setting. (The structure of a Λ-cooperad is used to overcome technical difficulties that occurs in the construction of the Sullivan model of operads, but we can neglect these issues in this overview.)

We use the name 'Hopf cochain dg-cooperad' for the category of cooperads in the category of commutative cochain dg-algebras $\mathcal{C} = dg^* \mathcal{C}om$ and we also adopt the notation $dg^* \mathcal{H}opf\, \mathcal{O}p_{*1}^c$ for this category of cooperads. We also consider a category of operads in simplicial sets satisfying $P(0) = P(1) = *$ in order to deal with the restrictions imposed by the definition of our category of cooperads in our model. We use the notation $s\mathcal{S}et\, \mathcal{O}p_{*1}$ for this category of operads. We have the following statement:

1.2.3 Theorem (B. Fresse [26, §II.10, §II.12])**.**

– *The letf adjoint of the Sullivan functor $\mathtt{G}_\bullet : dg^* \mathcal{C}om \to s\mathcal{S}et^{op}$ induces a functor $\mathtt{G}_\bullet : dg^* \mathcal{H}opf\, \mathcal{O}p_{*1}^c \to s\mathcal{S}et\, \mathcal{O}p_{*1}^{op}$ from the category of Hopf cochain dg-cooperads $dg^* \mathcal{H}opf\, \mathcal{O}p_{*1}^c$ to the category of operads in simplicial sets $s\mathcal{S}et\, \mathcal{O}p_{*1}^{op}$. For an object $A \in dg^* \mathcal{H}opf\, \mathcal{O}p_{*1}^c$, we set*

$$\mathtt{G}_\bullet(A)(r) = \mathtt{G}_\bullet(A(r))$$

and we use the fact that $\mathtt{G}_\bullet(-)$ is strongly symmetric monoidal to equip the collection of these simplicial sets $\mathtt{G}_\bullet(A) = \{\mathtt{G}_\bullet(A(r)), r \in \mathbb{N}\}$ with the structure of an operad.

– *This functor* $\mathtt{G}_\bullet : dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c \to s\mathcal{S}et\,\mathcal{O}p_{*1}^{op}$ *admits a right adjoint*

$$\Omega^*_\sharp : s\mathcal{S}et\,\mathcal{O}p_{*1}^{op} \to dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c$$

and the pair of functors $(\mathtt{G}_\bullet, \Omega^*_\sharp)$ *defines a Quillen adjunction.*

– *For a cofibrant operad* $P \in s\mathcal{S}et\,\mathcal{O}p_{*1}$ *such that* $\mathtt{H}^*(P(r))$ *forms a finite dimensional* $\mathbb{Q}$*-module in each arity* $r \in \mathbb{N}$ *and in each degree* $* \in \mathbb{N}$*, we have a weak equivalence*

$$\Omega^*_\sharp(P)(r) \xrightarrow{\sim} \Omega^*(P(r))$$

between the component of arity r *of the Hopf cochain dg-cooperad* $\Omega^*_\sharp(P) \in dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c$ *and the image of the space* $P(r)$ *under the Sullivan functor* $\Omega^*(-)$*, for any* $r \in \mathbb{N}$*.*

The first claim of this theorem follows from the observation that the functor $\mathtt{G}_\bullet(-)$ is strongly symmetric monoidal. The functor $\Omega^*(-)$, on the other hand, is only weakly monoidal. To be more precise, in the case of this functor, we have a Künneth morphism $\nabla : \Omega^*(X) \otimes \Omega^*(Y) \to \Omega^*(X \times Y)$ which is a quasi-isomorphism but not an isomorphism. Hence, for an operad in simplicial sets P, we only get that the composition product $\circ_i : P(k) \times P(l) \to P(k+l-1)$ induces a morphism which fits in a zigzag of morphisms of commutative cochain dg-algebras $\Omega^*(P(k+l-1)) \xrightarrow{\circ_i^*} \Omega^*(P(k) \times P(l)) \xleftarrow{\sim} \Omega^*(P(k)) \otimes \Omega^*(P(l))$. The idea is to use the adjoint lifting theorem (see for instance [11, §4.5]) to produce the functor of the second claim of the theorem $\Omega^*_\sharp : s\mathcal{S}et\,\mathcal{O}p_{*1}^{op} \to dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c$ and to fix this problem. Then the crux lies in the verification of the third claim, for which we refer to the cited reference.

For an operad in simplicial sets $P \in s\mathcal{S}et\,\mathcal{O}p_{*1}$, we now set:

$$P^{\mathbb{Q}} := \langle \mathtt{R}\,\Omega^*_\sharp(P) \rangle,$$

where we use the notation $\mathtt{R}\,\Omega^*_\sharp(-)$ for the right derived functor of the functor of the previous theorem $\Omega^*_\sharp : s\mathcal{S}et\,\mathcal{O}p_{*1}^{op} \to dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c$, and we again use the notation $\langle - \rangle$ for the left derived functor of the Sullivan realization on operads $\mathtt{G}_\bullet : dg^*\,\mathcal{H}opf\,\mathcal{O}p_{*1}^c \to s\mathcal{S}et\,\mathcal{O}p_{*1}^{op}$. The equivalence $\Omega^*_\sharp(P)(r) \sim \Omega^*(P(r))$ implies that we have the following result at the level of this realization:

1.2.4 Theorem (B. Fresse [26, Theorem II.10.2.1 and Theorem II.12.2.1])**.** *For any operad* $P \in s\mathcal{S}et\,\mathcal{O}p_{*1}$ *such that* $\mathtt{H}^*(P(r)) = \mathtt{H}^*(P(r), \mathbb{Q})$ *forms a finite dimensional* $\mathbb{Q}$*-module in each arity* $r \in \mathbb{N}$ *and in each degree* $* \in \mathbb{N}$*, we have:*

$$P^{\mathbb{Q}}(r) \sim P(r)^{\mathbb{Q}},$$

where we consider the component of arity r *of the operad* $P^{\mathbb{Q}}$ *on the left-hand side and the Sullivan rationalization of the space* $P(r)$ *on the right-hand side.*

For the operad of little n-discs D_n, we now set $\mathtt{R}\,\Omega^*_\sharp(D_n) = \Omega^*_\sharp(E_n)$, where E_n is any cofibrant model of E_n-operad in simplicial sets such that $E_n(0) = E_n(1) = *$, and we still write $D_n^{\mathbb{Q}} = \langle \mathtt{R}\,\Omega^*_\sharp(D_n)\rangle$. To apply the rational homotopy theory to the class of E_n-operads, we aim to determine the model of these objects $\mathtt{R}\,\Omega^*_\sharp(D_n)$.

Recall that we have a homotopy equivalence $D_n(r) \sim F(\mathbb{R}^n, r)$ between the underlying spaces of the operad of little n-discs $D_n(r)$ and the configuration spaces of the Euclidean space $F(\mathbb{R}^n, r)$. (In §1.1.1, we use an equivalent homotopy equivalence $D_n(r) \sim F(\mathring{\mathbb{D}}^n, r)$, where we take the open disc $\mathring{\mathbb{D}}^n \cong \mathbb{R}^n$ rather than the Euclidean spaces $\mathbb{R}^n$.) In a first step, we recall the following result about the cohomology algebras of these spaces:

1.2.5 Theorem (V.I. Arnold [2], F. Cohen [20])**.** *Let $n \geq 2$. For each $r \in \mathbb{N}$, the graded commutative algebra $\mathtt{H}^*(D_n(r)) \simeq \mathtt{H}^*(F(\mathbb{R}^n, r))$ has a presentation of the form:*

$$\mathtt{H}^*(F(\mathbb{R}^n, r)) = \frac{\bigwedge(\omega_{ij}, 1 \leq i < j \leq r)}{(\omega_{ij}^2, \omega_{ij}\omega_{ik} - \omega_{ij}\omega_{jk} + \omega_{ik}\omega_{jk})}$$

where the elements ω_{ij} correspond to cohomology classes of degree $n-1$.

In the expression of this theorem, the notation $\bigwedge(-)$ represents the free graded commutative algebra generated by the variables ω_{ij}. The result established by V.I. Arnold in [2] concerns the case $n = 2$ of this statement. The already cited work of F. Cohen [20] gives the general case $n \geq 2$. We also refer to Sinha's survey [51] for a gentle introduction to the computation of this theorem. The classes $\omega_{ij} \in \mathtt{H}^*(F(\mathbb{R}^n, r))$ represent the pullbacks of the fundamental class of the $n-1$-sphere $\omega \in \mathtt{H}^*(\mathbb{S}^{n-1})$ under the maps $\pi_{ij} : F(\mathbb{R}^n, r) \to \mathbb{S}^{n-1}$ such that $\pi_{ij}(a_1, \dots, a_r) = (a_j - a_i)/||a_j - a_i||$. We can also consider unordered pairs $\{i, j\}$ in this definition. We just have $\omega_{ij} = (-1)^n \omega_{ji}$ in this case, since ω_{ij} corresponds to the image of ω_{ji} under the action of the antipodal map on the sphere. In what follows, we refer to these cohomology classes ω_{ij} as the Arnold classes, we refer to the identity $\omega_{ij}\omega_{ik} - \omega_{ij}\omega_{jk} + \omega_{ik}\omega_{jk} = 0$ as the Arnold relation, and we refer to the presentation of the above theorem as the Arnold presentation.

The homology of the operad D_n now inherits the structure of an operad in graded modules. The cohomology $\mathtt{H}^*(D_n)$ inherits a dual cooperad structure, because the homology of the spaces $D_n(r)$ has a finite dimension as a $\mathbb{Q}$-module in each arity and in each degree, so that we have the arity-wise duality relation $\mathtt{H}^*(D_n(r)) = \mathtt{Hom}_{gr\,\mathcal{M}od}(\mathtt{H}_*(D_n(r)), \mathbb{Q})$ in the category of graded modules. Note that we have $D_n(0) = * \Rightarrow \mathtt{H}^*(D_n(0)) = \mathbb{Q}$ and $D_n(1) \sim * \Rightarrow \mathtt{H}^*(D_n(1)) = \mathbb{Q}$, so that the collection $\mathtt{H}^*(D_n) = \{\mathtt{H}^*(D_n(r), r \in \mathbb{N}\}$ satisfies our connectedness condition in the definition of a cooperad. One can easily check that this cooperad structure is compatible with the graded commutative algebra structure of the cohomology, so that the object $\mathtt{H}^*(D_n)$ actually forms a Hopf cooperad in the category of graded modules.

We aim to determine this cooperad structure. We use the following identity, already mentioned in the introduction of this chapter:

1.2.6 Theorem (F. Cohen [20])**.** *For $n \geq 2$, we have an isomorphism of operads in graded modules $\mathtt{H}_*(D_n) \simeq \mathsf{Pois}_n$, where Pois_n is the operad that governs the category of n-Poisson algebras.*

Recall that the structure of an n-Poisson algebra refers to a graded version of Poisson structure where we have a commutative product $\mu(x_1, x_2) = x_1 x_2$ of degree 0 and a Poisson bracket $\lambda(x_1, x_2) = [x_1, x_2]$ of degree $n-1$. This Poisson bracket satisfies the symmetry relation $\lambda(x_1, x_2) = (-1)^n \lambda(x_2, x_1)$, a graded version of the Jacobi identity and of the Poisson distribution relation. The n-Poisson operad Pois_n is defined by the corresponding presentation by generators and relations in the category of operads. Equivalently, we can represent an element of the graded module $\mathsf{Pois}_n(r)$ as a Poisson polynomial $\pi = \pi(x_1, \dots, x_r)$ of degree one in each variable x_i.

For our purpose, we actually consider a unitary version of the n-Poisson operad, where we have an extra arity zero operation $e \in \mathsf{Pois}(0)$ such that $\mu \circ_1 e = 1 = \mu \circ_2 e$ and $\lambda \circ_1 e = 0 = \lambda \circ_2 e$. This operation corresponds to a unit in the structure of an n-Poisson algebra and reflects the identity $D_n(0) = *$ at the level of the topological operad D_n.

We get the following result when we pass to the cohomology:

1.2.7 Proposition. *The cohomology algebras $\mathtt{H}^*(D_n(r))$, $r \in \mathbb{N}$, form a Hopf cooperad in graded modules such that $\mathtt{H}^*(D_n) \cong \mathsf{Pois}_n^c$, where Pois_n^c denotes the cooperad dual to Pois_n in graded modules.*

The n-Poisson cooperad Pois_n^c is explicitly defined by taking the dual graded modules of the components of the n-Poisson operad $\mathsf{Pois}_n^c(r) = \mathtt{Hom}_{gr\,\mathcal{M}od}(\mathsf{Pois}_n(r), \mathbb{Q})$. We take the adjoint morphisms of the composition products of the n-Poisson operad to provided this collection of graded modules $\mathsf{Pois}_n^c(r)$ with a cooperad structure. Therefore, the relation of this proposition $\mathtt{H}^*(D_n) \cong \mathsf{Pois}_n^c$ follows from the result of the previous theorem $\mathtt{H}_*(D_n) \cong \mathsf{Pois}_n$ and the duality between the homology and the cohomology $\mathtt{H}^*(D_n(r)) = \mathtt{Hom}_{gr\,\mathcal{M}od}(\mathtt{H}_*(D_n(r)), \mathbb{Q})$.

Let $\langle -, - \rangle : \mathsf{Pois}_n(r) \otimes \mathtt{H}^*(D(r)) \to \mathbb{Q}$ denote the duality pairing which we obtain by using this relation $\mathtt{H}_*(D_n) \cong \mathsf{Pois}_n$. For a Poisson monomial $\pi(x_1, \dots, x_r) \in \mathsf{Pois}_n(r)$, we have the formula:

$$\langle \omega_{ij}, \pi(x_1, \dots, x_r) \rangle = \begin{cases} 1, & \text{if } \pi(x_1, \dots, x_r) = x_1 \dots [x_i, x_j] \cdots \widehat{x_j} \cdots x_r, \\ 0, & \text{otherwise,} \end{cases}$$

where we consider the generating classes $\omega_{ij} \in \mathtt{H}^*(F(\mathbb{R}^n, r)) = \mathtt{H}^*(D(r))$ of the Arnold presentation of Theorem 1.2.5. This duality relation is immediate in arity 2, because the Poisson bracket operation $\lambda = \lambda(x_1, x_2)$ corresponds to the fundamental class of the $n-1$-sphere in the homology of $D_n(2)$, where we use the relation $D_n(2) \sim F(\mathbb{R}^n, r) \sim \mathbb{S}^{n-1}$. The general formula follows from the fact that the maps $\pi_{ij} : \mathsf{F}(\mathbb{R}^n, r) \to \mathsf{F}(\mathbb{R}^n, 2)$ in the definition of the classes ω_{ij} correspond to composites with the zero-ary operation $* \in \mathsf{D}_n(0)$ which represents our algebra unit $e \in \mathsf{Pois}_n(0)$ when we pass to the n-Poisson operad Pois_n. We refer to the paper [51] for a more thorough study of this duality relation between the n-Poisson polynomials and the elements of the cohomology algebras $\mathtt{H}^*(\mathsf{D}_n(r)) = \mathtt{H}^*(\mathsf{F}(\mathbb{R}^n, r))$ in the Arnold presentation.

We can now regard the object $\mathsf{Pois}_n^c \simeq \mathtt{H}^*(\mathsf{D}_n)$ as a Hopf cochain dg-cooperad equipped with a trivial differential. We have to make explicit a cofibrant resolution of this object for the applications of our methods of the rational homotopy theory of operads. In the next paragraphs, we explain a first definition of such a resolution by using graded analogues of the Drinfeld–Kohno Lie algebra operad of the previous section.

1.2.8 The graded Drinfeld–Kohno Lie algebra operads and the associated Chevalley–Eilenberg cochain complexes

The graded analogues of the Drinfeld–Kohno Lie algebra operad, which we define for every value of the parameter $n \geq 2$, are denoted by $\mathfrak{p}_n$. The ungraded Drinfeld–Kohno Lie algebra operad of §1.1.9 corresponds to the case $n = 2$. Thus, we have $\mathfrak{p} = \mathfrak{p}_2$ with the notation of §1.1.9.

To define the Lie algebras $\mathfrak{p}_n(r)$, we use the same presentation as in Eqn. 1.21:

$$\mathfrak{p}_n(r) = \mathbb{L}(t_{ij}, \{i,j\} \subset \{1,\dots,r\}) / < [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] >, \tag{1.35}$$

but we now take $\deg(t_{ij}) = n-2$ and we assume the graded symmetry relation $t_{ji} = (-1)^n t_{ij}$, for every pair $\{i,j\} \subset \{1,\dots,r\}$. Then we take the same construction as in §1.1.9 to provide these Lie algebras with an action of the symmetric groups and with additive composition products $\circ_i : \mathfrak{p}_n(k) \oplus \mathfrak{p}_n(l) \to \mathfrak{p}_n(k+l-1)$, so that the collection $\mathfrak{p}_n = \{\mathfrak{p}_n(r), r \in \mathbb{N}\}$ inherits the structure of an operad in the category of graded Lie algebras.

Note that the graded Lie algebras $\mathfrak{p}_n(r)$ still inherit a weight grading from the free Lie algebra, and hence, form weight graded objects in the category of graded modules. Besides, we can form a completed version of the operads $\hat{\mathfrak{p}}_n$, as in the case $n = 2$ in §1.1.9, but for $n \geq 3$, we trivially have $\hat{\mathfrak{p}}_n = \mathfrak{p}_n$ because the components of homogeneous weight $m \geq 1$ of the Lie algebras $\mathfrak{p}_n(r)$ are concentrated in a single degree $* = m(n-2)$ and we have $m(n-2) \to \infty$ when $n \geq 3$.

We consider the Chevalley–Eilenberg cochain complexes $\mathtt{C}^*_{CE}(\hat{\mathfrak{g}})$ associated to the complete Lie algebras $\hat{\mathfrak{g}} = \hat{\mathfrak{p}}_n(r)$. The cofibrant objects of the category of commutative cochain dg-algebras are retracts of dg-algebras of the form $R = (\mathbb{S}(V), \partial)$, where $\mathbb{S}(V)$ is the symmetric algebra on an upper graded dg-module V equipped with a filtration $\mathtt{F}_1 V \subset \mathtt{F}_2 V \subset \cdots \supset \mathtt{F}_m V \subset \cdots \subset V$ and where we have a differential ∂ such that $\partial(\mathtt{F}_m V) \subset \mathbb{S}(\mathtt{F}_{m-1} V)$. The Chevalley–Eilenberg cochain complex is precisely defined by an expression of this form $\mathtt{C}^*_{CE}(\mathfrak{g}) = (\mathbb{S}(\mathbb{Q}[-1] \otimes \hat{\mathfrak{g}}^\vee), \partial)$, where $\mathbb{Q}[-1] = \mathbb{Q}\,\mathtt{e}$ denotes the graded module generated by a single element $\mathbb{e}$ in lower degree -1 (equivalently, in upper degree one) and $\hat{\mathfrak{g}}^\vee$ denotes the (continuous) dual of the completed Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{p}}_n(r)$. The differential ∂ is induced by the dual map of the Lie bracket $[-,-]$ on $\hat{\mathfrak{g}}$.

The commutative cochain dg-algebras $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(r))$ inherit an action of the symmetric groups by functoriality of the Chevalley–Eilenberg cochain complex, as well as composition coproducts $\circ_i^* : \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k+l-1)) \to \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k)) \otimes \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(l))$, which are given by the composites of the morphisms $\circ_i^* : \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k+l-1)) \to \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k) \oplus \hat{\mathfrak{p}}_n(l))$ induced by the composition products of the operad $\mathfrak{p}_n$ with the Künneth isomorphisms $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k) \oplus \hat{\mathfrak{p}}_n(l)) \simeq \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(k)) \otimes \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(l))$. Hence, we get that the collection $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n) = \{\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(r)), r \in \mathbb{N}\}$ inherits the structure of Hopf cochain dg-cooperad. In addition, one can prove that this Hopf cochain dg-cooperad is cofibrant (see [26, Theorem II.14.1.7]).

Then we have the following statement:

1.2.9 Theorem (T. Kohno [40])**.** *We have a quasi-isomorphism of commutative cochain dg-algebras*

$$\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(r)) \xrightarrow{\sim} \mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r))$$

such that $t_{ij}^\vee \mapsto \omega_{ij}$ *for each pair* $\{i,j\} \subset \{1,\dots,r\}$ *and* $p^\vee \mapsto 0$ *when* $p^\vee$ *is the dual basis element of a homogeneous Lie polynomial* $p \in \mathfrak{p}_n(r)_m$ *of weight* $m > 1$.

The cited reference gives the case $n = 2$ of this statement. The general result can be deduced from the observation that the cohomology algebra $\mathtt{H}^*(F(\mathring{\mathbb{D}}^n, r)$ forms a Koszul algebra with the enveloping algebra of the Lie algebra $\mathfrak{p}_n(r)$ as dual associative algebra. We refer to [26, Theorem II.14.1.14] for further explanations on this approach.

Now we can easily check that the quasi-isomorphisms of this theorem preserve cooperad structures. Hence, we get the following statement:

1.2.10 Proposition. *The quasi-isomorphisms of Theorem 1.2.9 define a weak equivalence of Hopf cochain dg-cooperads*

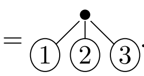
$$\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n) \xrightarrow{\sim} \mathtt{H}^*(D_n) = \mathsf{Pois}_n^c,$$

where we regard the cohomology of the little n*-discs operad* $\mathtt{H}^*(D_n)$ *as a Hopf cochain dg-cooperad equipped with a trivial differential.*

We deduce from this proposition that the object $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n)$ defines a cofibrant resolution of the object $\mathsf{Pois}_n^c = \mathtt{H}^*(D_n)$ in the category of Hopf cochain dg-cooperads. In our constructions, we actually consider a second cofibrant resolution, which is given by a Hopf cochain dg-cooperad of graphs Graphs_n^c, and we explain the definition of this object in the next paragraph.

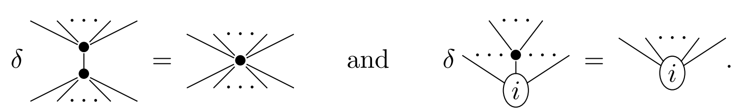
1.2.11 The graph cooperad

The Hopf cochain dg-cooperad Graphs^c_n precisely consists of graphs $\gamma \in \mathsf{Graphs}^c_n(r)$ with unnumbered internal vertices $\bullet$ and external vertices indexed by $1, \ldots, r$, as in the following picture:

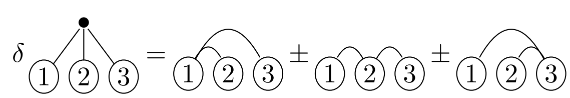
$\gamma =$  (1.36)

The degree of such a graph is determined by assuming that each internal vertex $\bullet$ contributes to the degree by $\deg(\bullet) = n$ and that each edge contributes to the degree by $\deg(-) = 1 - n$ (in the lower grading convention). Thus, we have $\deg(\gamma) = (1 - n)v + ne$ (in the lower grading convention again), where v denotes the number of internal vertices and e denotes the number of edges in the graph $\gamma \in \mathsf{Graphs}^c_n(r)$. In fact, we can regard our graphs as tensor products of symbolic elements given by the internal vertices and by the edges of our objects. In particular, we assume that graphs equipped with odd symmetries vanish in $\mathsf{Graphs}^c_n(r)$. We also assume that each edge is oriented and that a reversal of orientation is equivalent to the multiplication by a sign $(-1)^n$ in $\mathsf{Graphs}^c_n(r)$. For our purpose, we allow graphs with double edges, but not loops (edges with the same origin and endpoint) and we assume that each internal vertex is at least trivalent though the latter conditions are not essential. Besides, we assume that each connected component of our graph contains at least one external vertex.

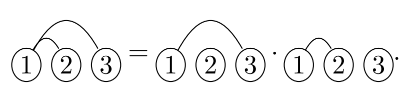
The differential of graphs is defined by contracting edges in order to merge internal vertices together or in order to merge internal vertices with external vertices, as shown schematically in the following picture:

 (1.37)

For instance, we have the formula:

 (1.38)

in $\mathsf{Graphs}^c_n(3)$. The product is given by the amalgamated sum of graphs along external vertices. For instance, we have the formula:

 (1.39)

The cooperad coproduct $\circ_i^* : \mathsf{Graphs}^c_n(k+l-1) \to \mathsf{Graphs}^c_n(k) \otimes \mathsf{Graphs}^c_n(l)$, where we fix $k, l \in \mathbb{N}$, $i \in \{1, \ldots, k\}$, has the form $\circ_i^*(\gamma) = \sum_{\alpha \subset \gamma} \gamma/\alpha \otimes \alpha$, where the sum runs over all the subgraphs $\alpha \subset \gamma$ that contain the external vertices indexed by $i, \ldots, i + l - 1$, and γ/α denotes the graph obtained by collapsing this subgraph to a single external vertex (which we index by i in the result of the operation, while we shift the index of the vertices such that $j > i$ by $j \mapsto j - l + 1$). Note that we have $\mathsf{Graphs}_n^c(1) \neq \mathbb{Q}$ in general, so that our object Graphs_n^c belongs to the extended category of Hopf cochain dg-cooperads $dg^* \mathcal{H}opf\, \mathcal{O}p_{*N}^c$ but not to the category of connected Hopf cochain dg-cooperads $dg^* \mathcal{H}opf\, \mathcal{O}p_{*1}^c$.

We easily see that the commutative cochain dg-algebras of graphs defined in this paragraph $\mathsf{Graphs}_n^c(r)$ have a structure of the form $\mathsf{Graphs}_n^c(r) = (\mathbb{S}(\mathbb{Q}[-1] \otimes \mathsf{ICGraphs}_n^c(r)), \partial)$ (like the Chevalley–Eilenberg cochain dg-algebras of the previous paragraph), where $\mathsf{ICGraphs}_n^c(r)$ is a complex of graphs which are connected when we remove the external vertices inside $\mathsf{Graphs}_n^c(r)$. (In what follows, we refer to such graphs as internally connected graphs.) We just perform an extra degree shift in the definition of this complex of internally connected graphs in order to get a $\mathbb{Q}[-1]$ factor on the generating dg-module of our symmetric algebra (as in the definition of the Chevalley–Eilenberg cochain complex of a Lie dg-algebra). We can actually use this expression to identify the object $\mathsf{ICGraphs}_n^c(r)$ with the dual of an L_∞-algebra (a strongly homotopy Lie algebra). We can use this symmetric algebra structure $\mathsf{Graphs}_n^c(r) = (\mathbb{S}(\mathbb{Q}[-1] \otimes \mathsf{ICGraphs}_n^c(r)), \partial)$ to prove that Graphs_n^c forms a cofibrant object in the category $dg^* \mathcal{H}opf\, \mathcal{O}p_{*N}^c$, and we also have the following proposition:

1.2.12 Proposition (M. Kontsevich [43])**.** *We have a quasi-isomorphism of Hopf dg-cooperads*
$$\mathsf{Graphs}_n^c \xrightarrow{\sim} \mathtt{H}^*(\mathsf{D}_n) = \mathsf{Pois}_n^c$$
which carries the graph $\gamma_{ij} \in \mathsf{Graphs}_n^c(r)$ with a single edge $i \frown j$ to the Arnold class ω_{ij} and which cancel the internally connected graphs with a non-empty set of internal vertices.

The assignment of this proposition determines the map $\mathsf{Graphs}_n^c(r) \xrightarrow{\sim} \mathtt{H}^*(\mathsf{D}_n(r))$ as a morphism of graded commutative algebras since the internally connected graphs generate the object $\mathsf{Graphs}_n^c(r)$ as a graded commutative algebra. (Note that the graphs γ_{ij} of the proposition represent the internally connected graphs with an empty set of internal vertices.) We just check that this map preserves differentials (and hence, gives a well-defined morphism of commutative cochain dg-algebras in each arity $r \in \mathbb{N}$), as well as the cooperad structures, so that our collection of maps define a morphism of Hopf dg-cooperads. We refer to the cited reference [43] and to [46] for a proof that this morphism defines a quasi-isomorphism. Observe simply that the differential identity of Eqn. 1.38 is carried to the Arnold relation in $\mathtt{H}^*(\mathsf{D}_n(3))$.

Recall that we set $\mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_n) = \Omega_\sharp^*(\mathsf{E}_n)$ for the topological operad of little n-discs D_n, where E_n is any cofibrant model of E_n-operad in simplicial sets such that $\mathsf{E}_n(0) = \mathsf{E}_n(1) = *$. We have the following result:

1.2.13 Theorem (B. Fresse and T. Willwacher [29, Theorem A'])**.** *We have the relation*
$$\mathsf{Pois}_n^c \sim \mathtt{R}\,\Omega_\sharp^*(\mathsf{D}_n),$$

in the category of Hopf cochain dg-cooperads.

This theorem asserts that the operad of little n-discs is formal in the sense of our operadic counterpart of the Sullivan rational homotopy theory of spaces. The cited reference [29, Theorem A'] proves an intrinsic formality theorem which implies this operadic formality result in the case $n \geq 3$. (In the next statement, we will explain that the case $n = 2$ of this theorem follows from the existence of Drinfeld's associators.)

The result of this theorem can also be deduced from Kontsevich's proof of the formality of E_n-operads when we pass to real coefficients (see [43]). Indeed, the construction of Kontsevich can be used to define a collection of quasi-isomorphisms $\mathsf{Graphs}^c_n(r) \xrightarrow{\sim} \Omega^*_{sa}(\mathsf{FM}_n(r))$, where FM_n is a model of E_n-operad given a real oriented analogue of the Fulton-MacPherson compactification of the configuration spaces (see [33]) and $\Omega^*_{sa}(-)$ denotes a cochain dg-algebra functor of semi-algebraic forms (see [36, 46]). One can observe that these morphisms can be associated to a strict morphism of Hopf cochain dg-cooperads $\mathsf{Graphs}^c_n \xrightarrow{\sim} \Omega^*_\sharp(\mathsf{FM}_n)$ (by using a general coherence statement of [26, Proposition II.12.1.3]).

The approach of the cited reference [29] does not use this constructions and gives a formality quasi-isomorphism which is defined over the rationals by using obstruction theory methods. The claim of this reference is that the E_n-operads are intrinsically rationally formal for $n \geq 3$ in the sense that every Hopf cochain dg-cooperad A_n which satisfies $\mathtt{H}_*(\mathsf{A}_n) \simeq \mathsf{Pois}^c_n$ and is equipped with an extra-involution operad $J : \mathsf{A}_n \to \mathsf{A}_n$ such that $J(\lambda) = -\lambda$ in the case $4|n$ satisfies $\mathsf{A}_n \sim \mathsf{Pois}^c_n$. We apply this claim to the Hopf cochain dg-cooperad $\mathsf{A}_n = \mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_n)$ to get the statement of the theorem. We have an extension of this formality result for the morphisms $\mathsf{D}_m \to \mathsf{D}_n$ which link the operads of little discs when $n - m \geq 2$ (see [29, Theorem C]).

Recall that we set

$$\mathsf{D}^{\mathbb{Q}}_n := \langle \mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_n)\rangle$$

to define a model for the rationalization of the little n-discs operad in topological spaces. The result of the previous theorem has the following corollary:

1.2.14 Corollary. *We have $\mathsf{D}^{\mathbb{Q}}_n = \langle \mathsf{Pois}^c_n\rangle$, for any $n \geq 2$, where we consider the image of the dual cooperad of the n-Poisson operad Pois^c_n under the operadic upgrading of the Sullivan realization functor $\langle - \rangle$.*

We just use the implication $\mathsf{Pois}^c_n \sim \mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_n) \Rightarrow \langle \mathsf{Pois}^c_n\rangle \sim \langle \mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_n)\rangle$ to get the result of this corollary. This result, together with the observations of Proposition 1.2.10 and Proposition 1.2.12, implies that we can take either $\langle \mathsf{Pois}^c_n\rangle = \mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n))$ or $\langle \mathsf{Pois}^c_n\rangle = \mathtt{G}_\bullet(\mathsf{Graphs}^c_n)$ to get a model of the rationalization $\mathsf{D}^{\mathbb{Q}}_n$.

We have an identity $\mathtt{G}_\bullet(\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(r))) = \mathtt{MC}_\bullet(\hat{\mathfrak{p}}_n(r))$, for each $r \in \mathbb{N}$, where we consider a Maurer–Cartan space associated to the complete Lie algebra $\hat{\mathfrak{p}}_n(r)$ (we review the definition of this construction in the next sections).

Hence, the results of this section gives a simple algebraic model of the rational homotopy type of E_n-operads. In the case $n = 2$, we have an identity $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}) \sim \mathtt{B}(\widehat{\mathsf{CD}_{\mathbb{Q}}})$, where we consider the chord diagram operad of §1.1.10 (recall also that we use the notation $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}_2$ for the ungraded Drinfeld–Kohno Lie algebra operad which occurs in this case $n = 2$). Thus, since we have on the other hand $\mathsf{D}_2^{\mathbb{Q}} = \mathtt{B}(\widehat{\mathsf{PaB}_{\mathbb{Q}}})$ (see §1.1), we can deduce the existence of a weak equivalence $\mathsf{D}_2^{\mathbb{Q}} \sim \langle \mathsf{Pois}^c_n \rangle$ from the operadic interpretation of Drinfeld's associators given in §1.1.10 (see [26, §II.14.2]).

We now examine a counterpart of the formality result of Theorem 1.2.13 in the category of dg-modules $dg^*\mathcal{M}od$. We use the notation $\mathtt{C}_*(-)$ for both the singular complex functor from the category of topological spaces to the category of dg-modules and for the standard normalized chain complex functor on simplicial sets. These functors are lax symmetric monoidal and therefore carry operads in topological spaces (respectively, in simplicial sets) to operads in dg-modules. Furthermore, in the case of a cofibrant operad in simplicial sets R, we have the duality relation $\Omega^*_\sharp(\mathsf{R})^\vee \sim \mathtt{C}_*(\mathsf{R})$ in the category of dg-operads $dg\,\mathcal{O}p_*$ when we consider the dual in dg-modules of the Hopf cochain dg-cooperad $\Omega^*_\sharp(\mathsf{R})$ of Theorem 1.2.13. Therefore, the result of Theorem 1.2.13 implies the following statement, which was also obtained by the authors cited in this statement by other method:

1.2.15 Theorem (D. Tamarkin [54], M. Kontsevich [43]). *We have the relation*

$$\mathsf{Pois}_n \sim \mathtt{C}_*(\mathsf{D}_n),$$

in the category of dg-operads.

This result is exactly the formality theorem mentioned in the introduction of this chapter for the class of E_n-operads in dg-modules. Tamarkin's proof of this theorem, which works in the case $n = 2$, relies on the correspondence between formality equivalences and associators, whereas Kontsevich's proof, which works for every $n \geq 2$ but requires to pass to real coefficients, relies on the definition of semi-algebraic forms associated to graphs (as we explain in our survey of Theorem 1.2.13). In [9], Boavida and Horel have given a new proof of the formality result of this theorem by using a generalization of classical formality criterion of mixed Hodge theory in the context of operads (see also [48] for an application of this approach in the case $n = 2$).

1.3 The rational homotopy of mapping spaces on the operads of little discs

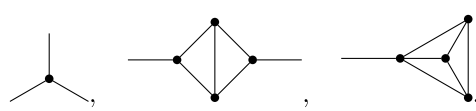
We now tackle the main objective of this chapter, namely the computation of the homotopy of the mapping spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}})$ and of the homotopy automorphism spaces $\mathtt{Aut}^h_{\mathcal{O}p}(D^{\mathbb{Q}}_n)$, for all $n \geq 2$. Thus, we aim to generalize the computation carried out in §1.1 in the case of the automorphism space $\mathtt{Aut}^h_{\mathcal{O}p}(D^{\mathbb{Q}}_2)$. In the case of the mapping spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D^{\mathbb{Q}}_n)$, we are also going to check that we have the relation $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D^{\mathbb{Q}}_n) \sim \mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D_n)^{\mathbb{Q}}$ when $n - m \geq 3$, so that the results explained in this section gives a full computation of the rational homotopy of the mapping spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D_n)$ that occur in the operadic description of the introduction for the embedding spaces $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$.

To carry out these computations, we use the graph complex model Graphs^c_n of the rational homotopy of the operad D_n. Hence, we naturally obtain, as a main outcome, a graph complex description of the homotopy type of the spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D^{\mathbb{Q}}_n)$ and $\mathtt{Aut}^h_{\mathcal{O}p}(D^{\mathbb{Q}}_n)$. In the case of the mapping spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(D_m, D^{\mathbb{Q}}_n)$, we express the result as the Maurer–Cartan space $\mathtt{MC}_\bullet(HGC_{mn})$ associated to a Lie dg-algebra of hairy graphs HGC_{mn}. We explain the definition of this object first and we explain our computation afterwards.

In the case of the automorphism spaces $\mathtt{Aut}^h_{\mathcal{O}p}(D^{\mathbb{Q}}_n)$, we get that our object is homotopy equivalent to a cartesian product of Eilenberg–MacLane spaces (like any H-group in rational homotopy theory). Thus, we can focus on the computation of the homotopy groups in this case. We give a description of these groups in terms of the homology of a non-hairy graph complex GC_n of which we also explain the definition beforehand. This graph complex GC_n is a graded version of a complex introduced by Kontsevich in [41], and therefore, this complex is usually called the Kontsevich graph complex in the literature.

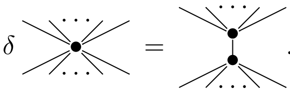
1.3.1 The hairy graph complex

The hairy graph complex HGC_{mn} explicitly consists of formal series of connected graphs with internal vertices $\bullet$, internal edges $\bullet\!-\!\bullet$, which link internal vertices together, and external edges $\bullet\!-$ (the hairs), which are open at one extremity, as in the following examples:

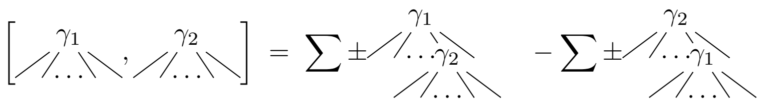


(1.40)

This complex HGC_{mn} is equipped with a lower grading. The degree of a graph $\gamma \in HGC_{mn}$ is determined by assuming that each vertex contributes by $\deg(\bullet) = n$, that each internal edge contributes by $\deg(\bullet\!-\!\bullet) = 1 - n$, that each hair contributes by $\deg(\bullet\!-) = m - n + 1$, and by adding a global degree shift by $-m$. Thus, we have $\deg(\gamma) = nv + (1 - n)e + (m - n + 1)h - m$, where v denotes the number of internal vertices, the letter e denotes the number of internal edges and h denotes the number of hairs of the graph $\gamma \in HGC_{mn}$. The differential of the hairy graph complex is defined by the blow-up of internal vertices:

δ  (1.41)

We equip the hairy graph complex with the Lie bracket such that:

 (1.42)

where the first sum runs over the re-connections of a hair of the graph γ_1 to a vertex of the graph γ_2, and similarly in the second sum, with the role of the graphs γ_1 and γ_2 exchanged. In the case $m = 1$, we have to consider a deformation of this Lie dg-algebra structure which we call the Shoikhet L_∞-structure (a strongly homotopy Lie algebra). We just refer to [59] for the explicit definition of this structure.

In the next theorem, we consider the Maurer–Cartan space $\mathtt{MC}_\bullet(L)$ associated to the Lie dg-algebra $L = HGC_{m,n}$. This simplicial set $\mathtt{MC}_\bullet(L)$ is defined by the sets of flat L-valued PL connections on the simplices Δ^n, $n \in \mathbb{N}$. To be more precise, in the definition of this object $\mathtt{MC}_\bullet(L)$, we generally assume that L forms a complete Lie dg-algebra with respect to a filtration $L = \mathtt{F}_1 L \supset \mathtt{F}_2 L \supset \cdots \supset \mathtt{F}_k L \supset \cdots$ such that $[\mathtt{F}_k L, \mathtt{F}_l L] \subset \mathtt{F}_{k+l} L$. In the case $L = HGC_{mn}$, we assume that $\mathtt{F}_k L = \mathtt{F}_k HGC_{mn}$ consists of power series of graphs $\gamma \in \mathtt{H}_{mn}$ such that $e - v \geq k$, where e denotes the number of edges and v denotes the number of internal vertices in γ. Then we explicitly set:

$$\mathtt{MC}_n(L) = \big\{\omega \in (L \hat{\otimes} \Omega^*(\Delta^n))^1 \,|\, \delta(\omega) + \frac{1}{2}[\omega, \omega] = 0\big\}, \tag{1.43}$$

for every simplicial dimension $n \in \mathbb{N}$, where $(L \hat{\otimes} \Omega^*(\Delta^n))^1$ denotes the component of upper degree 1 in the completed tensor product of the Lie dg-algebra L with the Sullivan cochain dg-algebra of PL forms $\Omega^*(\Delta^n)$. The face and degeneracy operators of this simplicial set are inherited from the simplices. This construction has a natural extension for L_∞-algebras (see for instance [32]). We now have the following main result:

1.3.2 Theorem (B. Fresse, V. Turchin, and T. Willwacher [28, Theorem 1])**.** *For any $n \geq m \geq 2$, we have the relation:*

$$\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) \sim \mathtt{MC}_\bullet(HGC_{mn}),$$

where HGC_{mn} is the hairy graph complex. This relation extends to the case $n > m = 1$ when we equip HGC_{1n} equipped with the Shoikhet L_∞-structure.

The results of the previous section imply that we have the following weak-equivalences:

$$\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}}) \sim \mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(\mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_n), \mathtt{R}\,\Omega^*_\sharp(\mathsf{D}_m)) \tag{1.44}$$

$$\sim \mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(\mathsf{Pois}^c_n, \mathsf{Pois}^c_m), \tag{1.45}$$

where $\mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(-,-)$ denote a derived mapping space bifunctor in the category of Hopf cochain dg-cooperads, we use the Quillen adjunction between the functors $\mathtt{G}_\bullet(-)$ and $\Omega^*_\sharp(-)$ in the first equivalence (1.44) and the formality result of Theorem 1.2.13 in the second equivalence (1.45). These equivalences reduce the proof of Theorem 1.3.2 to a problem of algebra.

To compute the derived mapping space of Hopf cochain dg-cooperads $\mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(\mathsf{Pois}^c_n, \mathsf{Pois}^c_m)$, we need to pick a cofibrant resolution of the object Pois^c_n on the source and a fibrant resolution of the object Pois^c_m on the target. For this purpose, we take the cofibrant Hopf cochain dg-cooperad $\mathsf{R}_n = \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n)$ (see §1.2.8) and we adapt the classical Boardman–Vogt W-construction of operads to define a natural fibrant resolution functor $\mathtt{W}^c(-)$ on the category of Hopf cochain dg-cooperad (see [28, §5] for a detailed definition of this functor). By analyzing the definition of maps on these Hopf cochain dg-cooperads, one sees that the mapping space $\mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(\mathsf{Pois}^c_n, \mathsf{Pois}^c_m)$ is weakly equivalent to the Maurer–Cartan space associated to an L_∞-algebra of biderivations $\mathtt{BiDer}(\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n), \mathtt{W}^c(\mathsf{Pois}^c_m))$ (see [28, §6]).

The object $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n)$ in this complex of biderivations can be replaced by the graph cooperad model of the n-Poisson cooperad $\mathsf{Graphs}^c_n \sim \mathsf{Pois}^c_n$. The connection of the derived mapping space $\mathtt{Map}^h_{dg^*\,\mathcal{H}opf\,\mathcal{O}p^c_{*1}}(\mathsf{Pois}^c_n, \mathsf{Pois}^c_m)$ with the hairy graph complex of the theorem comes from an ultimate reduction of this L_∞-algebra of biderivations, which yields a relation of the form

$$HGC_{mn} \sim \mathtt{BiDer}(\mathsf{Graphs}^c_n, \mathtt{W}^c(\mathsf{Pois}^c_m)) \tag{1.46}$$

in the category of L_∞-algebras (see [28, §8]).

The result of this theorem has the following corollary:

1.3.3 Corollary. *For any $n \geq m \geq 2$ (and for $n > m = 1$), we have the identity:*

$$\pi_*(\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}^{\mathbb{Q}}_n), \omega) = H_{*-1}(HGC^{\omega}_{mn}),$$

for any $\omega \in \mathtt{MC}_0(HGC_{mn})$, where HGC^ω_{mn} is the complex HGC_{mn} equipped with the twisted differential $\delta_\omega = \delta + [\omega, -] + (\text{extra terms in the } L_\infty\text{-case})$.

The identity of this statement follows from the result of Theorem 1.3.2 and from a general result about the homotopy groups of Maurer–Cartan spaces $\mathtt{MC}_\bullet(L)$ for which we refer to [6].

A computation of the rational homotopy groups of the embedding spaces $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$, analogous to the result established in this corollary, is given in [4] (see also [44] for the case $m = 1$ of these computations). These previous computations are based on the interpretation in terms of mapping spaces of operadic bimodules of the Goodwillie–Weiss tower of the embedding spaces $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ (or of the equivalent interpretation of the Goodwillie–Weiss tower in terms of Sinha's cosimplicial model in the case $m = 1$). In [45], the formality of E_n-operads in chain complexes is also used to get a description of the homology of the embedding spaces $\overline{\mathtt{Emb}}_c(\mathbb{R}^1, \mathbb{R}^n)$ in terms of a Hochschild cohomology theory for operads (we apply this Hochschild cohomology theory to the n-Poisson operad). The graph operad model of the n-Poisson can also be used to deduce a graph complex model of the homology of the embedding space $\overline{\mathtt{Emb}}_c(\mathbb{R}^1, \mathbb{R}^n)$ from this algebraic approach.

In fact, we can use the result of the above corollary and the equivalence between the embedding space $\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ and the $m+1$-fold iterated loop space of the operadic mapping space $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)$ given in the introduction to get applications of the result of Theorem 1.3.2 in the theory embedding spaces. For this purpose, we also use the following theorem:

1.3.4 Theorem (B. Fresse, V. Turchin, and T. Willwacher [28, Theorem 15])**.** *In the case $n - m \geq 3$, the space $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)$ is $n - m - 1$ connected, and we moreover have the relation:*

$$\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n)^{\mathbb{Q}} \sim \mathtt{Map}_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_m, \mathsf{D}_n^{\mathbb{Q}})$$

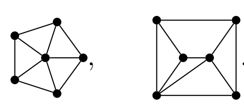
in the homotopy category of spaces.

We refer to the cited reference [28, §10] for the detailed proof of this statement, which relies on an analogous result for spaces, established by Haefliger in [35].

We examine the rational homotopy of the spaces of homotopy automorphisms to complete the result of this section. We first explain the definition of the Kontsevich graph complexes GC_n which occur in this computation.

1.3.5 The Kontsevich graph complex

The definition of the complex GC_n is the same as the definition of the hairy graph complex HGC_{mn}, except that we now consider graphs without hairs, as in the following examples:



(1.47)

We determine the degree of a graph in GC_n by assuming that each vertex contributes by $\deg(\bullet) = n$ and each edge contributes by $\deg(\bullet\!\!-\!\!\bullet) = 1 - n$ as in the case of hairy graphs. We still assume that every vertex of a graph in GC_n is at least trivalent and we do not allow loops (edges with the same origin and endpoint). The differential is defined by the blow-up of vertices again.

The space of homotopy automorphisms $\mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_n^{\mathbb{Q}})$ is the sum of the connected components of the mapping spaces $\mathtt{Map}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_n, \mathsf{D}_n^{\mathbb{Q}})$ associated to the morphisms ϕ which are invertible in the homotopy category of operads. Let $h : \mathtt{Aut}^h_{\mathcal{T}op\,\mathcal{O}p}(\mathsf{D}_n^{\mathbb{Q}}) \to \mathtt{Aut}_{\mathcal{H}opf\,\mathcal{O}p}(\mathtt{H}_*(\mathsf{D}_n, \mathbb{Q}))$ be the natural map which carries any such morphism to the associated homology morphism. For $n \geq 2$, we have a bijection $\mathtt{Aut}_{\mathcal{H}opf\,\mathcal{O}p}(\mathtt{H}_*(\mathsf{D}_n, \mathbb{Q})) = \mathbb{Q}^{\times}$ which is determined by taking the action of an automorphism $\phi \in \mathtt{Aut}_{\mathcal{H}opf\,\mathcal{O}p}(\mathtt{H}_*(\mathsf{D}_n, \mathbb{Q}))$ on the representative of the Poisson bracket operation $\lambda \in \mathsf{Pois}_n$ in the operad $\mathsf{Pois}_n = \mathtt{H}_*(\mathsf{D}_n)$. We get the following result:

1.3.6 Theorem (B. Fresse, V. Turchin, and T. Willwacher [28, Corollary 5]). *For each $\lambda \in \mathbb{Q}^{\times}$, we have the identity:*

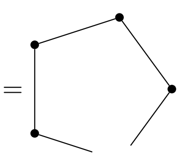
$$\pi_*(h^{-1}(\lambda)) = \mathtt{H}_*(GC_n) \oplus \begin{cases} \mathbb{Q}, & \textit{if } * \equiv -n-1(4), \\ 0, & \textit{otherwise}, \end{cases}$$

where GC_n denotes the Kontsevich graph complex.

We deduce this statement from the result of Theorem 1.3.2, by using that the identity morphism is represented by the Maurer–Cartan element such that $\omega = |$ in the hairy graph complex HGC_{nn}. We just consider versions of the graph complexes GC_n^2 and HGC_{mn}^2 where bivalent vertices are allowed. We have $HGC_{mn}^2 \sim HGC_{mn}$ for any $n \geq m \geq 2$, whereas for the graph complex GC_n^2, we have:

$$\mathtt{H}_*(GC_n^2) = \mathtt{H}_*(GC_n) \oplus \begin{cases} \mathbb{Q}\, L_*, & \text{if } * \equiv -n-1(4), \\ 0, & \text{otherwise}, \end{cases} \tag{1.48}$$

where L_* denotes the homology classes of graphs of the form:

$L_* =$  (1.49)

We easily see that the operation $[\omega, -]$ in the differential $\delta_\omega = \delta + [\omega, -]$ of the twisted complex $(HGC_{nn}^2)^\omega$ associated to the Maurer–Cartan element $\omega = |$ is given by the addition of a hair $|$ to any graph $\gamma \in HGC_{nn}^2$. We can then use a spectral sequence to check that we have a quasi-isomorphism $\mathbb{Q}\,| \oplus \mathbb{Q}[-1] \otimes GC_n^2 \xrightarrow{\sim} HGC_{nn}^2$ where we consider the mapping $\gamma \mapsto \gamma-$ which associates a graph with one hair $\gamma- \in HGC_{nn}^2$ to any graph $\gamma \in GC_n$. We refer to [29, Proposition 2.2.9] for the detailed line of arguments.

In the case $n = 2$, we have $\mathtt{H}_0(GC_n) = \mathfrak{grt}$ by a result of T. Willwacher, where $\mathfrak{grt}$ is the graded Grothendieck–Teichmüller Lie algebra (see [60]). Therefore, in this case, the result of Theorem 1.3.6 reflects the relation that we obtained in Theorem 1.1.7.

1.4 Outlook

Throughout this survey, we have focused on the study of the homotopy of E_n-operads themselves, but one can use variants of the definition of an E_n-operad to associate operadic right module structures to any n-manifold M.

For this purpose, we can use again the Fulton–MacPherson operad FM_n, the model of E_n-operad, given by a real oriented version of the Fulton–MacPherson compactification of the configuration spaces, which was considered by Kontsevich in his proof of the formality of E_n-operads (see §1.2). These Fulton–MacPherson compactifications have a natural generalization for the configuration spaces of manifolds $\mathsf{F}(M,r)$, and when M is a framed manifold, this construction returns a collection of spaces $\mathsf{FM}_M = \{\mathsf{FM}_M(r), r \in \mathbb{N}\}$ which inherits the structure of a right module (in the operadic sense) over the Fulton–MacPherson operad FM_n.

This object is equivalent to constructions used by Ayala–Francis in the definition of the factorization homology of manifolds (see Ayala–Francis's chapter, in this handbook volume, for a survey of this subject). In particular, one can use a relative composition product of the object FM_M over the operad FM_n to compute the factorization homology of any framed manifold M. The methods used by Kontsevich to prove the formality of E_n-operads have been used by several authors to define models of the rational homotopy type of this right module FM_M, and hence, to tackle the rational homotopy computations in factorization homology theory. To cite a few works on this subject, let us mention that a graph complex model of the object FM_M, which extends the graph cooperad of §1.2.11 when M is a simply connected compact manifold without boundary, is defined by Campos–Willwacher in [17], while an extension of Arnold's presentation is used by Idrissi in [38] to get a small model of the object FM_M. Idrissi's result provides a generalization of Knudsen's description of the factorization homology for higher enveloping algebras of Lie algebras [39]. The paper [16] provides an extension of the constructions of [17, 38] for manifolds with boundary, while the paper [15] addresses an extension of the definition of these operadic module structures by using a framed version of the operads of little discs.

In §§1.2-1.3, we entirely focus on the rational homotopy theory framework, but we may wonder which information we may still retrieve by our methods in positive characteristic. For instance, partial formality results have been obtained by Cirici–Horel in [19] when we take an arity-wise truncation of operads below the characteristic of the coefficients (see also [9] for an improvement of these partial formality results). In fact, the E_n-operads are not formal as symmetric operads in chain complexes in positive characteristic, because their components are not formal as representations of the symmetric groups. Nevertheless, we may wonder whether E_n-operads are formal as non-symmetric operads, which is enough for the study of mapping spaces over an E_1-operad. The case $n > 2$ of this question is still open, but Salvatore has proved in [49] that E_2-operads are not formal as non-symmetric operads over $\mathbb{F}_2$.

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