Derived division functors and mapping spaces

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2/1/2003

Introduction

The normalized cochain complex of a simplicial set $N^*(Y)$ is endowed with the structure of an E_∞ algebra. More specifically, we prove in a previous article that $N^*(Y)$ is an algebra over the Barratt-Eccles operad (*cf.* [2]). According to M. Mandell, under reasonable completeness assumptions, this algebra structure determines the homotopy type of Y (*cf.* [15]). In this article, we construct a model of the mapping space $\operatorname{Map}(X, Y)$. For that purpose, we extend the formalism of Lannes' T functor (*cf.* [12]) in the framework of E_∞ algebras. Precisely, in the category of algebras over the Barratt-Eccles operad, we have a division functor $-\oslash N_*(X)$ which is left adjoint to the functor $\operatorname{Hom}_{\mathbf{F}}(N_*(X), -)$. We prove that the associated left derived functor $-\oslash^L N_*(X)$ is endowed with a quasi-isomorphism $N^*(Y)\oslash^L N_*(X) \xrightarrow{\sim} N^* \operatorname{Map}(X, Y)$.

Summary

§1. *Results.* — We give a detailed introduction to our results in this section.

§2. *Cofibrant resolutions and division functors.* — We recall the construction of cofibrant objects in the context of E_∞ algebras. This notion occurs in the definition of a left derived functor.

§3. *The example of loop spaces.* — We consider the case of a division functor $-\oslash K$ where K is the reduced chain complex of the circle $K = \tilde{N}(S^1)$. We make explicit the associated left derived functor $N^*(Y)\oslash^L \tilde{N}_*(S^1)$ which provides a model of loop spaces ΩY.

§4. *The case of Eilenberg-MacLane spaces.* — We prove our main result in section 4. First, we construct a quasi-isomorphism $N^*(Y)\oslash^L N_*(X) \to N^* \operatorname{Map}(X, Y)$ in the case of an Eilenberg-MacLane space $Y = K(\mathbf{Z}/2, n)$. Then, we proceed by induction on the Postnikov tower of Y.

§5. *The Eilenberg-Zilber equivalence.* — We observe that the morphism $N^*(X\times Y) \to N^*(X)\otimes N^*(Y)$ provided by the classical Eilenberg-Zilber equivalence is not compatible with E_∞ algebra structures. We show how to overcome this difficulty in section 5. The results of this section allow to define a morphism $N^*(Y)\oslash^L N_*(X) \to N^* \operatorname{Map}(X, Y)$ functorial in X and Y. The functoriality property is crucial in the proof of our main theorem.

§1. Results

1.1. *Conventions*

We fix a ground field $\mathbf{F}$. We work in the category of differential graded modules over

$\mathbf{F}$ (for short, dg-modules). To be more precise, we consider either upper graded modules $V = V^*$ or lower graded modules $V = V_*$. The relation $V_d = V^{-d}$ makes a lower grading equivalent to an upper grading. In general, the differential of a dg-module is denoted by $\delta : V \to V$. The cohomology of an upper graded dg-module is denoted by $H^*(V)$.

We consider the classical tensor product of dg-modules which equips the category of dg-modules with the structure of a closed symmetric monoidal category. According to the classical rule, we assume that the symmetry isomorphism $V \otimes W \to W \otimes V$ involves a sign, which we denote by $\pm$ without more specification. The dg-module of homogeneous morphisms, denoted by $\mathrm{Hom}_{\mathbf{F}}(V, W)$, is the internal hom in the category of dg-modules. To be explicit, a homogeneous morphism of upper degree d is a map $f : V \to W$ which raises upper degrees by d. The differential of $f \in \mathrm{Hom}_{\mathbf{F}}(V, W)$ is given by the graded commutator $\delta(f) = f\delta - \pm\delta f$. The dual dg-module of a dg-module, denoted by $V^\vee$, is defined by the relation $V^\vee = \mathrm{Hom}_{\mathbf{F}}(V, \mathbf{F})$.

1.2. *The Barratt-Eccles operad*

We consider the differential graded Barratt-Eccles operad, which we denote by the letter $\mathcal{E}$. The classical Barratt-Eccles operad, which we denote by $\mathcal{W}$, is the simplicial operad whose term $\mathcal{W}(r)$ is the homogeneous bar construction of the symmetric group Σ_r (*cf.* [1], [2]). The term $\mathcal{E}(r)$ of the associated differential graded operad is just the normalized chain complex of $\mathcal{W}(r)$. In particular, the dg-module $\mathcal{E}(r)$ is a non-negatively lower graded chain complex.

To be explicit, an n-dimensional simplex in $\mathcal{W}(r)$ is an $n+1$-tuple $(w_0, \ldots, w_n)$, where $w_i \in \Sigma_r$, for $i = 0, \ldots, n$. We have $d_i(w_0, \ldots, w_n) = (w_0, \ldots, \widehat{w_i}, \ldots, w_n)$ and $s_j(w_0, \ldots, w_n) = (w_0, \ldots, w_j, w_j, \ldots, w_n)$. The set $\mathcal{W}(r)$ is equipped with the diagonal action of Σ_r.

Let us consider the case $r = 2$. The chain complex $\mathcal{E}(2)_*$ is the standard free resolution of the trivial representation of Σ_2. The elements of Σ_2 (the identity permutation and the transposition) are denoted by $id \in \Sigma_2$ and $\tau \in \Sigma_2$ respectively. Let $\theta_d \in \mathcal{W}(2)_d$ be the simplex such that $\theta_d = (id, \tau, id, \tau, id, \ldots)$. The elements $\theta_d = (id, \tau, id, \tau, id, \ldots)$ and $\tau \cdot \theta_d = (\tau, id, \tau, id, \tau, \ldots)$ are the non-degenerate simplicies of $\mathcal{W}(2)_d$. Therefore, we have $\mathcal{E}(2)_d = \mathbf{F}_2[\Sigma_2]\,\theta_d$ where $\delta(\theta_d) = \tau \cdot \theta_{d-1} + (-1)^{d-1}\theta_{d-1}$.

The operad structure is specified by a composition product

$$\mathcal{E}(r) \otimes \mathcal{E}(s_1) \otimes \cdots \otimes \mathcal{E}(s_r) \to \mathcal{E}(s_1 + \cdots + s_r)$$

which arises from an explicit substitution process of permutations (for more details, we refer to [2]). The composite of $\rho \in \mathcal{E}(r)$ with $\sigma_1 \in \mathcal{E}(s_1), \ldots, \sigma_r \in \mathcal{E}(s_r)$ is denoted by $\rho(\sigma_1, \ldots, \sigma_r) \in \mathcal{E}(s_1 + \cdots + s_r)$.

The set $\mathcal{W}(1)$ is reduced to a point. The associated basis element is denoted by $1 \in \mathcal{E}(1)$, because this element is a unit for the operad composition product. The set $\mathcal{W}(0)$ is also reduced to a point. The associated basis element is denoted by $\eta \in \mathcal{E}(0)$, because this element determines unit morphisms in the context of algebras over the Barratt-Eccles operad.

1.3. *Algebras*

We consider algebras and coalgebras over the Barratt-Eccles operad. As a reminder, an $\mathcal{E}$-algebra is a dg-module A equipped with an evaluation product $\mathcal{E}(r) \otimes A^{\otimes r} \to A$.

Equivalently, an element $\rho \in \mathcal{E}(r)$ determines a multilinear operation $\rho : A^{\otimes r} \to A$, which maps an r-tuple of elements $a_1, \ldots, a_r \in A$ to an element denoted by $\rho(a_1, \ldots, a_r) \in A$. The operad unit $1 \in \mathcal{E}(1)$ is supposed to give the identity operation $1(a_1) = a_1$. Furthermore, the composite of an operation $\rho : A^{\otimes r} \to A$ with $\sigma_1 : A^{\otimes s_1} \to A, \ldots, \sigma_r : A^{\otimes s_r} \to A$ has to agree with $\rho(\sigma_1, \ldots, \sigma_r) : A^{\otimes s_1 + \cdots + s_r} \to A$, the operation supplied by the composition product of the operad (as indicated by our notation). The evaluation product is a morphism of dg-modules. Therefore, we have the derivation relation $\delta(\rho(a_1, \ldots, a_r)) = (\delta\rho)(a_1, \ldots, a_r) + \sum_{i=1}^{r} \pm \rho(a_1, \ldots, \delta a_i, \ldots, a_r)$. The evaluation product is also assumed to be invariant under the action of the symmetric group. Therefore, we have the relation $(w\rho)(a_1, \ldots, a_r) = \pm \rho(a_{w(1)}, \ldots, a_{w(r)})$.

The basis element $\eta \in \mathcal{E}(0)$ yields a morphism $\eta : \mathbf{F} \to A$. (This morphism is also determined by an element $1 \in A$ denoted as the unit of the algebra A.) In fact, the ground field $\mathbf{F}$ has a unique algebra structure and is an initial object of the category of $\mathcal{E}$-algebras.

Dually, an $\mathcal{E}$-coalgebra is a dg-module K equipped with a coproduct $\mathcal{E}(r) \otimes K \to K^{\otimes r}$. Hence, an element $\rho \in \mathcal{E}(r)$ determines a co-operation $\rho^* : K \to K^{\otimes r}$. Furthermore, the ground field $\mathbf{F}$ is equipped with an $\mathcal{E}$-coalgebra structure. In fact, it represents the final object in the category of $\mathcal{E}$-coalgebras, because the element $\eta \in \mathcal{E}(0)$ determines an augmentation morphism $\eta^* : K \to \mathbf{F}$, for all $\mathcal{E}$-coalgebras K.

As for classical algebras, the linear dual of an $\mathcal{E}$-coalgebra is an $\mathcal{E}$-algebra. The converse assertion holds if we assume that an $\mathcal{E}$-algebra is finite or is equipped with a profinite topology.

1.4. *Cochain algebras*

The normalized chain complex of a simplicial set $N_*(X)$ is equipped with the structure of a coalgebra over the differential graded Barratt-Eccles operad $\mathcal{E}$ (*cf.* [2]).

Precisely, for X a simplicial set, we have a natural coproduct $\mathcal{E}(r) \otimes N_*(X) \to N_*(X)^{\otimes r}$ such that the co-operation ${\theta_0}^* : N_*(X) \to N_*(X)^{\otimes 2}$ determined by the element $\theta_0 \in \mathcal{E}(2)_0$ is the classical Alexander-Whitney diagonal. Thus, the dual operation $\theta_0 : N^*(X) \otimes N^*(X) \to N^*(X)$ agrees with the classical cup-product.

Furthermore, because we have the derivation relation $\delta(\theta_i) = \tau \cdot \theta_{i-1} + (-1)^{i-1}\theta_{i-1}$, the higher operation $\theta_i : N^*(X) \otimes N^*(X) \to N^{*+i}(X)$ associated to the element $\theta_i \in \mathcal{E}(2)$ is a representative of the classical cup-i-product (*cf.* [20]). Consequently, if $x \in N^n(X)$ is a representative of a class $c \in H^n(X)$, then $\theta_{n-i}(x, x) \in N^{n+i}(X)$ is a representative of the ith reduced square $\mathrm{Sq}^i(c) \in H^{n+i}(X)$.

The coalgebra augmentation $\eta^* : N_*(X) \to \mathbf{F}$ is equivalent to the canonical augmentation $N_*(X) \to N_*(pt)$, because the module $N_*(pt) = \mathbf{F}$ is identified with the final coalgebra.

1.5. *The tensor structure*

The chain complex $\mathcal{E}(r)$ is equipped with a diagonal $\Delta : \mathcal{E}(r) \to \mathcal{E}(r) \otimes \mathcal{E}(r)$. We have explicitly $\Delta(w_0, \ldots, w_n) = \sum_{k=0}^{n} (w_0, \ldots, w_k) \otimes (w_k, \ldots, w_n)$. As an example, if we consider the element $\theta_d \in \mathcal{E}(2)_d$ introduced in paragraph 1.2, then we obtain $\Delta(\theta_d) = \sum_{k=0}^{d} \theta_k \otimes \tau^k \cdot \theta_{d-k}$.

The diagonal of $\mathcal{E}$ commutes with the operad composition product of $\mathcal{E}$. Consequently, if A and B are $\mathcal{E}$-algebras, then the tensor product $A \otimes B$ is equipped with the structure of an $\mathcal{E}$-algebra. The elements of $\mathcal{E}(r)$ operate on $A \otimes B$ through the diagonal of $\mathcal{E}(r)$. To be more explicit, the operation $\rho : (A \otimes B)^{\otimes r} \to (A \otimes B)$ associated to an element $\rho \in \mathcal{E}(r)$ statisfies the formula

$$\rho(a_1 \otimes b_1, \ldots, a_r \otimes b_r) = \sum\nolimits_i \pm \rho^i_{(1)}(a_1, \ldots, a_r) \otimes \rho^i_{(2)}(b_1, \ldots, b_r),$$

where $\Delta(\rho) = \sum_i \rho^i_{(1)} \otimes \rho^i_{(2)}$.

Similarly, if A is an $\mathcal{E}$-algebra and K is an $\mathcal{E}$-coalgebra, then the dg-module of homogeneous morphisms $\mathrm{Hom}_{\mathbf{F}}(K, A)$ is equipped with the structure of an $\mathcal{E}$-algebra. In fact, if K is a finite dimensional module, then K is equivalent to the linear dual of an $\mathcal{E}$-algebra $B = K^\vee$. The dg-module $\mathrm{Hom}_{\mathbf{F}}(K, A)$ is equivalent to the tensor product of A with an $\mathcal{E}$-algebra $A \otimes B = A \otimes K^\vee$. Therefore, in the finite dimensional case, the dg-module $\mathrm{Hom}_{\mathbf{F}}(K, A)$ is equipped with the structure of an $\mathcal{E}$-algebra according to the construction of the paragraph above.

Suppose given $u_1, \ldots, u_r \in \mathrm{Hom}_{\mathbf{F}}(K, A)$. In general, for $\rho \in \mathcal{E}(r)$, the homogeneous morphism $\rho(u_1, \ldots, u_r) \in \mathrm{Hom}_{\mathbf{F}}(K, A)$ is the sum

$$\rho(u_1, \ldots, u_r) = \sum\nolimits_i \pm \rho^i_{(1)} \cdot u_1 \otimes \cdots \otimes u_r \cdot {\rho^i_{(2)}}^*$$

of the composites

$$K \xrightarrow{{\rho^i_{(2)}}^*} K \otimes \cdots \otimes K \xrightarrow{u_1 \otimes \cdots \otimes u_r} A \otimes \cdots \otimes A \xrightarrow{\rho^i_{(1)}} A,$$

where $\Delta(\rho) = \sum_i \rho^i_{(1)} \otimes \rho^i_{(2)}$.

1.6. Proposition

Let K be an $\mathcal{E}$-coalgebra. The functor $\mathrm{Hom}_{\mathbf{F}}(K, -)$ *has a left adjoint. More explicitly, for any $\mathcal{E}$-algebra A, there is an $\mathcal{E}$-algebra $A \oslash K$ such that* $\mathrm{Hom}_{\mathcal{E}}(A \oslash K, -) \simeq \mathrm{Hom}_{\mathcal{E}}(A, \mathrm{Hom}_{\mathbf{F}}(K, -))$.

This result follows readily from the special adjoint functor theorem (*cf.* [14, Chapter V]). The functor $\mathrm{Hom}_{\mathbf{F}}(K, -)$ preserves all limits of $\mathcal{E}$-algebras because the forgetful functor from the category of $\mathcal{E}$-algebras to the category of $\mathbf{F}$-modules creates limits.

1.7. *A closed model category structure*

The category of $\mathcal{E}$-algebras is equipped with the structure of a closed model category. By definition, a weak-equivalence is a morphism of $\mathcal{E}$-algebras $f : A \to B$ which induces an isomorphism in cohomology $H^*(f) : H^*(A) \to H^*(B)$ (namely, we assume that $f : A \to B$ is a quasi-isomorphism). A fibration is a morphism of $\mathcal{E}$-algebras which is surjective in all degrees. A cofibration is a morphism of $\mathcal{E}$-algebras which has the left-lifting property with respect to acyclic fibrations. We prove that these definitions provide a full model structure on the category of $\mathcal{E}$-algebras in [2].

We remind the reader that an acyclic fibration (respectively, an acyclic cofibration) denotes a morphism which is both a fibration (respectively, a cofibration) and a weak equivalence. An $\mathcal{E}$-algebra F is cofibrant if the initial morphism $\eta : \mathbf{F} \to F$ is a cofibration. Similarly, an $\mathcal{E}$-algebra A is fibrant if the final morphism $A \to 0$ is a fibration. But, this property holds for any $\mathcal{E}$-algebra, because the final object in the category of $\mathcal{E}$-algebras is the 0 module. A cofibrant resolution of an $\mathcal{E}$-algebra A is a cofibrant algebra F equipped with an acyclic fibration $F \xrightarrow{\sim} A$.

1.8. *The homotopy category of $\mathcal{E}$-algebras*

The category of $\mathcal{E}$-algebras is denoted by $\mathcal{E}\,\mathcal{A}lg$. The associated homotopy category is denoted by $\mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg)$. The set of algebra morphisms from $A \in \mathcal{E}\,\mathcal{A}lg$ to $B \in \mathcal{E}\,\mathcal{A}lg$ is denoted by $\mathrm{Hom}_{\mathcal{E}}(A, B)$. The morphism set in the homotopy category is denoted by $[A, B]_{\mathcal{E}}$. In general, we have $[A, B]_{\mathcal{E}} = \mathrm{Hom}_{\mathcal{E}}(F, B)/\sim$, where F is a cofibrant resolution of A, and where $\sim$ refers to the homotopy relation in the category of $\mathcal{E}$-algebras. In our context, it is possible to make this relation explicit.

Precisely, the construction of paragraph 1.5 makes the definition of a path-object in the category of $\mathcal{E}$-algebras easy. Explicitly, for any $\mathcal{E}$-algebra A, the module $\mathrm{Hom}_{\mathbf{F}}(N_*(\Delta^1), A)$ is a path object of A, because we have a diagram

$$\underbrace{\mathrm{Hom}_{\mathbf{F}}(N_*(pt), A)}_{=A} \xrightarrow[s_0]{\sim} \mathrm{Hom}_{\mathbf{F}}(N_*(\Delta^1), A) \overset{d_0}{\underset{d_1}{\rightrightarrows}} \underbrace{\mathrm{Hom}_{\mathbf{F}}(N_*(pt), A)}_{=A}$$

deduced from the cosimplicial structure of $\Delta^\bullet$. We call the $\mathcal{E}$-algebra $\mathrm{Hom}_{\mathbf{F}}(N_*(\Delta^1), A)$ a cylinder object of A, because we consider tacitly the opposite category of $\mathcal{E}$-algebras.

Suppose given a pair of parallel morphisms $f : F \to B$ and $g : F \to B$, where F is a cofibrant $\mathcal{E}$-algebra. These morphisms are homotopic $f \sim g$ if and only if there is a morphism $h : F \to \mathrm{Hom}_{\mathbf{F}}(N_*(\Delta^1), B)$ such that $d_0 h = f$ and $d_1 h = g$.

1.9. *Derived functors*

Clearly, the functor $\mathrm{Hom}_{\mathbf{F}}(K, -) : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ carries fibrations to fibrations and preserves all weak-equivalences of $\mathcal{E}$-algebras. By adjunction, the functor $- \oslash K : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ carries (acyclic) cofibrations to (acyclic) cofibrations. As a consequence, the functors $- \oslash K : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ and $\mathrm{Hom}_{\mathbf{F}}(K, -) : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ determine a pair of derived adjoint functors

$$- \overset{L}{\oslash} K : \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg) \to \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbf{F}}(K, -) : \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg) \to \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg).$$

In this article, the notation $A \oslash^L K$ refers also to a representative of the image of $A \in \mathcal{E}\,\mathcal{A}lg$ under the left derived functor of $- \oslash K : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$. Such a representative is provided by an $\mathcal{E}$-algebra $F \oslash K \in \mathcal{E}\,\mathcal{A}lg$, where F is a cofibrant resolution of A. According to this convention, the right derived functor of $\mathrm{Hom}_{\mathbf{F}}(K, -) : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ does not differ from itself, because all $\mathcal{E}$-algebras are fibrant objects.

1.10. Theorem

Let X and Y be simplicial sets. We assume that $\pi_n(Y)$ is a finite p-group for $n \geq 0$. We have a quasi-isomorphism $N^*(Y) \oslash^L N_*(X) \to N^*(\mathrm{Map}(X, Y))$.

As in the classical situation of Lannes' T functor, we can introduce profinite structures in order to remove the finiteness assumption (*cf.* [16]).

1.11. *The comparison morphism*

We define a natural morphism $N^*(Y) \oslash^L N_*(X) \to N^*(\mathrm{Map}(X, Y))$.

An $\mathcal{E}$-algebra A has a universal cofibrant resolution $F_A \xrightarrow{\sim} A$ (*cf.* [8]). We recall the definition of this cofibrant algebra in section 2.3. We consider the universal cofibrant resolution $F_X = F_{N^*(X)}$ of the cochain algebra of a simplicial set $A = N^*(X)$.

By functoriality, the evaluation map $X \times \mathrm{Map}(X,Y) \to Y$ gives rise to a morphism of $\mathcal{E}$-algebras $F_Y \to F_{X\times \mathrm{Map}(X,Y)}$. Hence, we have a commutative diagram

$$\begin{array}{ccc} F_Y & \longrightarrow & F_{X\times\mathrm{Map}(X,Y)} \\ \sim\downarrow & & \sim\downarrow \\ N^*(Y) & \longrightarrow & N^*(X\times\mathrm{Map}(X,Y)) \end{array}$$

The Eilenberg-Zilber equivalence provides a morphism of dg-modules (the shuffle morphism):

$$N^*(X \times \mathrm{Map}(X,Y)) \to N^*(X) \otimes N^*(\mathrm{Map}(X,Y)).$$

But, the shuffle morphism is not a morphism of $\mathcal{E}$-algebras. Nevertheless, according to the next theorem, we have a morphism of $\mathcal{E}$-algebras

$$F_{X\times\mathrm{Map}(X,Y)} \to N^*(X) \otimes N^*(\mathrm{Map}(X,Y))$$

which is cohomologically equivalent to the shuffle morphism (we refer to theorem 5.1 for a more precise statement).

We assume that X is a finite simplicial set so that $\mathrm{Hom}_{\mathbf{F}}(N_*(X), -) = N^*(X) \otimes -$. We consider the adjoint of the composite morphism

$$F_Y \to F_{X\times\mathrm{Map}(X,Y)} \to N^*(X) \otimes N^*(\mathrm{Map}(X,Y)).$$

This morphism

$$F_Y \oslash N_*(X) \to N^*(\mathrm{Map}(X,Y))$$

is functorial in X and Y. Hence, we are done in this case.

We observe that both functors $X \mapsto F_Y \oslash N_*(X)$ and $X \mapsto N^*(\mathrm{Map}(X,Y))$ preserves colimits. Therefore, the comparison morphism is also defined for X infinite by a colimit argument. Furthermore, the general case of theorem 1.10 is implied by the case of a finite simplicial set X.

1.12. Theorem

The shuffle morphism $N^*(X\times Y) \to N^*(X)\otimes N^*(Y)$ is not a morphism of $\mathcal{E}$-algebras, but it extends to a morphism of $\mathcal{E}$-algebras $F_{X\times Y} \to N^*(X) \otimes N^*(Y)$ functorially in X and Y.

1.13. *On pointed simplicial sets and augmented algebras*

We have similar statements for pointed spaces. In this context, we consider the reduced Barratt-Eccles operad $\tilde{\mathcal{E}}$ which is defined by the relation

$$\tilde{\mathcal{E}}(r) = \begin{cases} 0, & \text{if } r = 0, \\ \mathcal{E}(r), & \text{otherwise.} \end{cases}$$

An algebra over the reduced Barratt-Eccles operad is equivalent to the augmentation ideal of an augmented $\mathcal{E}$-algebra. Precisely, an augmented $\mathcal{E}$-algebra A is equipped with a morphism of $\mathcal{E}$-algebras $\epsilon : A \to \mathbf{F}$. Since the augmentation $\epsilon : A \to \mathbf{F}$ is supposed to preserve the unit morphisms, we have a splitting $A = \mathbf{F} \oplus \tilde{A}$, where $\tilde{A} = \ker(\epsilon : A \to \mathbf{F})$ is the augmentation ideal of A. The augmentation ideal $\tilde{A}$ is preserved by the operations $\rho : A^{\otimes r} \to A$ where $r \geq 1$. Therefore, the module $\tilde{A}$ is equipped with the structure of an $\tilde{\mathcal{E}}$-algebra. Conversely, the evaluation products $\mathcal{E}(r) \otimes A^{\otimes r} \to A$ are determined by the algebra structure of the augmentation ideal $\tilde{A}$, because the evaluation of an operation at an algebra unit $\rho(a_1, \ldots, \eta(1), \ldots, a_r) \in A$ agrees with the action of the composite operation $\rho(1, \ldots, \eta, \ldots, 1) \in \mathcal{E}(r-1)$ (see [7, Section 1] for more details). We have dual observations for coalgebras.

For a simplicial set X, the choice of a basepoint $pt \to X$ determines a morphism of $\mathcal{E}$-algebras $N^*(X) \to N^*(pt) = \mathbf{F}$. Hence, the cochain complex of a pointed simplicial set is an augmented $\mathcal{E}$-algebra. The reduced cochain complex $\tilde{N}^*(X)$ is the augmentation ideal of $N^*(X)$.

1.14. Theorem

Let X and Y be pointed simplicial sets. We assume that $\pi_n(Y)$ is a finite p-group for $n \geq 0$. We have a quasi-isomorphism $\tilde{N}^*(Y) \oslash^L \tilde{N}_*(X) \to \tilde{N}^*(\mathrm{Map}_*(X, Y))$.

1.15. *The division functor in the context of the reduced Barratt-Eccles operad*

We assume that $\tilde{F}$ is the augmentation ideal of a cofibrant $\mathcal{E}$-algebra $F = \mathbf{F} \oplus \tilde{F}$. We assume that $\tilde{K}$ is the coaugmentation coideal of an $\mathcal{E}$-coalgebra $K = \mathbf{F} \oplus \tilde{K}$. In this context, the construction of paragraph 1.5 provides a functor in the category of algebras over the reduced Barratt-Eccles operad $\mathrm{Hom}_{\mathbf{F}}(\tilde{K}, -) : \tilde{\mathcal{E}}\,\mathcal{A}lg \to \tilde{\mathcal{E}}\,\mathcal{A}lg$. In the theorem, we consider the associated adjoint functor $- \oslash \tilde{K} : \tilde{\mathcal{E}}\,\mathcal{A}lg \to \tilde{\mathcal{E}}\,\mathcal{A}lg$.

Let us introduce the augmented algebra $F \oslash \tilde{K} \in \mathcal{E}\,\mathcal{A}lg$ associated to $\tilde{F} \oslash \tilde{K} \in \tilde{\mathcal{E}}\,\mathcal{A}lg$ and such that $F \oslash \tilde{K} = \mathbf{F} \oplus \tilde{F} \oslash \tilde{K}$. We have a cocartesian square of augmented algebras

$$\begin{array}{ccc} F & \longrightarrow & F \oslash K \\ \downarrow & & \downarrow \\ \mathbf{F} & \longrightarrow & F \oslash \tilde{K} \end{array}$$

as proved by an easy inspection. If we assume that $F = F_Y$ is a cofibrant model of $N^*(Y)$, then we conclude from this observation that theorem 1.14 is a consequence of theorem 1.10. More precisely, the morphism $F_Y \oslash \tilde{N}_*(X) \to N^*(\mathrm{Map}_*(X, Y))$ is a quasi-isomorphism, because the cofibration sequence $F_Y \to F_Y \oslash N_*(X) \to F_Y \oslash \tilde{N}_*(X)$ is a model of the fibration sequence $\mathrm{Map}_*(X, Y) \to \mathrm{Map}(X, Y) \to Y$ (*cf.* [15, Section 5]).

1.16. *Mandell's comparison result*

The map $X \mapsto N^*(X)$ determines a contravariant functor $N^*(-) : \mathcal{S}^{op} \to \mathcal{E}\,\mathcal{A}lg$. This functor has a left adjoint $G : \mathcal{E}\,\mathcal{A}lg \to \mathcal{S}^{op}$. These functors determine a pair of derived adjoint functors

$$LG : \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg) \to \mathrm{Ho}(\mathcal{S})^{op} \qquad \text{and} \qquad N^*(-) : \mathrm{Ho}(\mathcal{S})^{op} \to \mathrm{Ho}(\mathcal{E}\,\mathcal{A}lg).$$

To be explicit, given any $\mathcal{E}$-algebra A, we set $G(A) = \mathrm{Hom}_{\mathcal{E}}(A, N^*(\Delta^\bullet))$. The adjunction relation reads

$$\mathrm{Hom}_{\mathcal{S}}(X, G(A)) = \mathrm{Hom}_{\mathcal{E}}(A, N^*(X)).$$

We have also $LG(A) = \mathrm{Hom}_{\mathcal{E}}(F, N^*(\Delta^\bullet))$, where F is a cofibrant resolution of A.

A simplicial set X is resolvable by $\mathcal{E}$-algebras if the adjonction unit $X \to LG(N^*(X))$ is a weak-equivalence. According to Mandell (*cf.* [15]), this property holds if X is connected, nilpotent, p-complete and has a finite p-type (provided the ground field is the algebraic closure $\mathbf{F} = \bar{\mathbf{F}}_p$).

In our framework, we have the following result:

1.17. Theorem

The mapping spaces $\mathrm{Map}(X, Y)$*, where* X *and* Y *are as in theorem 1.14, are resolvable by* $\mathcal{E}$*-algebras.*

We have a canonical map $\mathrm{Map}(X, Y) \to \mathrm{Hom}_{\mathcal{E}}(F_Y, N^*(X) \otimes N^*(\Delta^\bullet))$. This map is equivalent to a morphism $\mathrm{Map}(X, Y) \to \mathrm{Hom}_{\mathcal{E}}(F_Y \oslash N_*(X), N^*(\Delta^\bullet))$. To be precise, an n-dimensional morphism $u \in \mathrm{Map}(X, Y)_n$ is a map $u : X \times \Delta^n \to Y$ and determines a morphism of $\mathcal{E}$-algebras

$$F_Y \to F_{X \times \Delta^n}$$

which can be composed with the morphism

$$F_{X \times \Delta^n} \to N^*(X) \otimes N^*(\Delta^n)$$

of theorem 1.12.

It is straighforward to extend the arguments of Mandell and to prove that this canonical map $\mathrm{Map}(X, Y) \to \mathrm{Hom}_{\mathcal{E}}(F_Y, N^*(X) \otimes N^*(\Delta^\bullet))$ is a weak equivalence. This assertion is also related to an unpublished work of Dwyer and Hopkins.

§2. Cofibrant resolutions and division functors

2.1. Free algebras and division functors

2.1.1. *Free algebras*

The free $\mathcal{E}$-algebra generated by an $\mathbf{F}$-module V is denoted by $\mathcal{E}(V)$. Let us remind the reader that a realization of $\mathcal{E}(V)$ is provided by a generalization of the symmetric algebra (*cf.* [8], [9]). To be explicit, we have

$$\mathcal{E}(V) = \bigoplus_{r=0}^{\infty} \mathcal{E}_{(r)}(V) \qquad \text{where} \qquad \mathcal{E}_{(r)}(V) = (\mathcal{E}(r) \otimes V^{\otimes r})_{\Sigma_r}.$$

Hence, a tensor $\rho \otimes (v_1 \otimes \cdots \otimes v_r) \in \mathcal{E}(r) \otimes V^{\otimes r}$ gives rise to an element of $\mathcal{E}(V)$ which we denote by $\rho(v_1, \ldots, v_r) \in \mathcal{E}(V)$. By construction, we have the relation $(w\rho)(v_1, \ldots, v_r) = \pm\rho(v_{w(1)}, \ldots, v_{w(r)})$. Observe that $V = \mathcal{E}_{(1)}(V) \subset \mathcal{E}(V)$, because an element $v \in V$ can be identified with the tensor $1(v) \in \mathcal{E}(V)$, where $1 \in \mathcal{E}(1)$ is the unit of the Barratt-Eccles operad.

The evaluation product of the free $\mathcal{E}$-algebra $\mathcal{E}(r) \otimes \mathcal{E}(V)^{\otimes r} \to \mathcal{E}(V)$ is induced by the composition product of the Barratt-Eccles operad $\mathcal{E}(r) \otimes \mathcal{E}(s_1) \otimes \cdots \otimes \mathcal{E}(s_r) \to \mathcal{E}(s_1 + \cdots + s_r)$. One should observe that the element $\rho(v_1, \ldots, v_r) \in \mathcal{E}(V)$ represents also the image of $v_1,\dots,v_r \in V$ under the operation $\rho : \mathcal{E}(V)^{\otimes r} \to \mathcal{E}(V)$ (this follows from the unit relation in the Barratt-Eccles operad).

The free $\mathcal{E}$-algebra is characterized by the adjunction relation $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(V),A) = \mathrm{Hom}_{\mathbf{F}}(V,A)$. We make the $\mathcal{E}$-algebra morphism $\phi_f : \mathcal{E}(V) \to A$ associated to an $\mathbf{F}$-module morphism $f : V \to A$ explicit in the paragraph 2.1.3 below. Let us recall that a colimit of free algebras is a free algebra. More precisely, because of the adjunction relation, we obtain $\operatorname{colim}_i \mathcal{E}(V_i) = \mathcal{E}(\operatorname{colim}_i V_i)$.

We observe that the adjunction relations $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(V\otimes K),B) = \mathrm{Hom}_{\mathbf{F}}(V\otimes K,B) = \mathrm{Hom}_{\mathbf{F}}(V,\mathrm{Hom}_{\mathbf{F}}(K,B)) = Hom_{\mathcal{E}}(\mathcal{E}(V),\mathrm{Hom}_{\mathbf{F}}(K,B))$ gives immediately:

2.1.2. Lemma

We have $\mathcal{E}(V) \oslash K = \mathcal{E}(V\otimes K)$.

2.1.3. *Free algebra morphisms*

By the universal property of a free algebra, we assume that a morphism of dg-modules $f : V \to A$, where A is an $\mathcal{E}$-algebra, extends to one and only one morphism of $\mathcal{E}$-algebras, which we denote by $\phi_f : \mathcal{E}(V) \to A$. In fact, if $\phi : \mathcal{E}(V) \to A$ is a morphism of $\mathcal{E}$-algebras, then we have the relation $\phi(\rho(v_1,\dots,v_r)) = \rho(\phi(v_1),\dots,\phi(v_r))$. Therefore, a morphism of $\mathcal{E}$-algebras $\phi : \mathcal{E}(V) \to A$ is determined by its restriction to $V \subset \mathcal{E}(V)$ and is equivalent to a morphism of $\mathbf{F}$-modules $f : V \to A$. Furthermore, the algebra morphism associated to $f : V \to A$ can be defined explicitly by the formula

$$\phi_f(\rho(v_1,\dots,v_r)) = \rho(f(v_1),\dots,f(v_r)).$$

(Hence, for $v \in V$, we have $\phi_f(v) = f(v)$.)

We make explicit the morphism $\phi_f \oslash K : \mathcal{E}(V\otimes K) \to \mathcal{E}(W\otimes K)$ associated to a morphism of free algebras $\phi_f : \mathcal{E}(V) \to \mathcal{E}(W)$. We denote also the morphism of dg-modules which is equivalent to $\phi_f \oslash K : \mathcal{E}(V\otimes K) \to \mathcal{E}(W\otimes K)$ by the notation $f \oslash K : V\otimes K \to \mathcal{E}(W\otimes K)$.

2.1.4. Lemma

The image of an element $v\otimes c \in V\otimes K$ under the morphism $\phi_f \oslash K : \mathcal{E}(V\otimes K) \to \mathcal{E}(W\otimes K)$ is determined as follows. We assume

$$f(v) = \sum\nolimits_i \rho^i(w^i_{(1)},\dots,w^i_{(r)}) \in \mathcal{E}(W).$$

We consider the diagonal $\Delta(\rho^i) = \sum_j \rho^j_{(1)} \otimes \rho^j_{(2)}$ of the operations $\rho^i \in \mathcal{E}(r)$ which occur in the expansion above. We take the image of $c \in K$ under the cooperations ${\rho^j_{(1)}}^* : K \to K^{\otimes r}$:

$${\rho^j_{(1)}}^*(c) = \sum\nolimits_k c^k_{(1)} \otimes \dots \otimes c^k_{(r)} \in K^{(r)}.$$

We have finally:

$$(f \oslash K)(v\otimes c) = \sum\nolimits_{ijk} \pm \rho^j_{(2)}(w^i_{(1)}\otimes c^k_{(1)},\dots,w^i_{(r)}\otimes c^k_{(r)}) \in \mathcal{E}(W\otimes K).$$

The verification of this assertion is straightforward from the explicit form of the adjunction relation $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(V\otimes K), A) = \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(V), \mathrm{Hom}_{\mathbf{F}}(K, A))$ and from the definition of the $\mathcal{E}$-algebra structure of $\mathrm{Hom}_{\mathbf{F}}(K, A)$.

2.2. Cell algebras and division functors

2.2.1. *Cell extensions*

We consider the suspension sequence $\Sigma^* V \to C^* V \to V$ of an upper graded differential module V. We have $C^* V = \mathbf{F}\, e \otimes V \oplus \mathbf{F}\, b \otimes V$, where $\deg(e) = 1$, $\deg(b) = 0$ and $\delta(b) = e$. The suspension $\Sigma^* V$ is identified with the module $\mathbf{F}\, e \otimes V \subset C^* V$. Similarly, we have an isomorphism $V \simeq \mathbf{F}\, b \otimes V$ and the module V can be identified with a quotient of $C^* V$. We have a morphism of $\mathcal{E}$-algebras $\mathcal{E}(\Sigma^* V) \to \mathcal{E}(C^* V)$ induced by the canonical morphism $\Sigma^* V \to C^* V$.

In the context of algebras over the Barratt-Eccles operad, a cell extension is an $\mathcal{E}$-algebra morphism $A \to A \vee_f \mathcal{E}(C^* V)$ provided by a cocartesian square

$$\begin{array}{ccc} \mathcal{E}(\Sigma^* V) & \longrightarrow & \mathcal{E}(C^* V) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \vee_f \mathcal{E}(C^* V) \end{array}$$

where the left-hand side arrow is determined by a morphism of dg-modules $f : \Sigma^* V \to A$ (the analogue of an attaching map).

The next assertion is a general property of left adjoint functors:

2.2.2. Lemma

The division functor preserves cell extensions. We have explicitly $\big(A \vee_f \mathcal{E}(C^* V)\big) \oslash K = (A \oslash K) \vee_{f \oslash K} \mathcal{E}(C^* V \otimes K)$.

2.2.3. *Cell algebras*

A cell algebra F is a colimit of a sequence of cell extensions

$$\mathbf{F} = F^{(-1)} \to F^{(0)} \to \cdots \to F^{(n)} \to \cdots$$

Hence, we have explicitly $F^{(n)} = F^{(n-1)} \vee_{f^{(n)}} \mathcal{E}(C^* V^{(n)})$ where $f^{(n)} : \Sigma^* V^{(n)} \to F^{(n-1)}$.

A cell algebra is a cofibrant object in the category of $\mathcal{E}$-algebras. In fact, it is known that the cofibrant objects in the category of $\mathcal{E}$-algebras can be characterized as retracts of cell algebras (*cf.* [15, Section 2]). Furthermore, any $\mathcal{E}$-algebra A has a cofibrant resolution $F \xrightarrow{\sim} A$ where F is a cell algebra.

More generally, a morphism $A \to F$ is a relative cell inclusion if we have a sequence of cell extensions

$$A = F^{(-1)} \to F^{(0)} \to \cdots \to F^{(n)} \to \cdots$$

such that $F = \mathrm{colim}_n F^{(n)}$.

The next assertion is an immediate consequence of lemma 2.2.2:

2.2.4. Lemma

The division functor preserves cell structures. More explicitly, if F is a cell algebra, as in the paragraph above, then we have $F \oslash K = \operatorname{colim}_n F^{(n)} \oslash K$, where $F^{(n)} \oslash K = F^{(n-1)} \oslash K \vee_{f^{(n)} \oslash K} \mathcal{E}(C^* V^{(n)} \otimes K)$.

This lemma allows to determine the image of an $\mathcal{E}$-algebra $A \in \mathcal{E}\,\mathcal{A}lg$ under the derived division functor $- \oslash^L K : \operatorname{Ho}(\mathcal{E}\,\mathcal{A}lg) \to \operatorname{Ho}(\mathcal{E}\,\mathcal{A}lg)$. Just recall that a representative of $A \oslash^L K \in \mathcal{E}\,\mathcal{A}lg$ is given by an $\mathcal{E}$-algebra $F \oslash K \in \mathcal{E}\,\mathcal{A}lg$, where F is a cofibrant resolution of A.

2.3. Almost free algebras and division functors

2.3.1. *Free algebra derivations*

A homogeneous morphism $d : A \to A$, where A is an $\mathcal{E}$-algebra, is a derivation if we have the relation $d(\rho(a_1, \ldots, a_r)) = \sum_{i=1}^r \pm \rho(a_1, \ldots, d(a_i), \ldots, a_r)$. In the case of a free $\mathcal{E}$-algebra $A = \mathcal{E}(V)$, a homogeneous morphism $h : V \to \mathcal{E}(V)$ extends to one and only one derivation $d_h : \mathcal{E}(V) \to \mathcal{E}(V)$. In fact, as for free algebra morphisms, a derivation $d : \mathcal{E}(V) \to \mathcal{E}(V)$ is determined by its restriction to $V \subset \mathcal{E}(V)$, because of the derivation relation. Furthermore, the derivation $d_h : \mathcal{E}(V) \to \mathcal{E}(V)$ can be defined explicitly by the formula

$$d_h(\rho(v_1, \ldots, v_r)) = \sum_{i=1}^r \pm \rho(v_1, \ldots, h(v_i), \ldots, v_r).$$

(Hence, for $v \in V$, we have $d_h(v) = h(v)$.)

2.3.2. *Almost free algebras*

The free $\mathcal{E}$-algebra $F = \mathcal{E}(V)$ is equipped with the differential $\delta : \mathcal{E}(V) \to \mathcal{E}(V)$ induced by the differential of V and by the differential of the Barratt-Eccles operad $\mathcal{E}$. Explicitly, the differential of an element $\rho(v_1, \ldots, v_r) \in \mathcal{E}(V)$ is given by the formula

$$\delta(\rho(v_1, \ldots, v_r)) = (\delta\rho)(v_1, \ldots, v_r) + \sum_{i=1}^r \pm \rho(v_1, \ldots, \delta(v_i), \ldots, v_r).$$

An almost free $\mathcal{E}$-algebra F is a free $\mathcal{E}$-algebra $F = \mathcal{E}(V)$ equipped with a differential which differs from the canonical one by a derivation. To be precise, an almost free $\mathcal{E}$-algebra $F = \mathcal{E}(V)$ is equipped with a differential $\delta_h : \mathcal{E}(V) \to \mathcal{E}(V)$ such that $\delta_h = \delta + d_h$, where $d_h : \mathcal{E}(V) \to \mathcal{E}(V)$ is a derivation associated to a homogeneous morphism of degree 1. The differential $\delta_0 : \mathcal{E}(V) \to \mathcal{E}(V)$ associated to the 0 morphism reduces to the canonical differential.

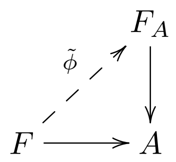
2.3.3. *Morphisms of almost free algebras*

We consider a morphism $\phi : F \to A$ where F is an almost free algebra $F = \mathcal{E}(V)$. In this situation, we assume the relation $\delta\phi = \phi\delta_h$, where $\delta_h : \mathcal{E}(V) \to \mathcal{E}(V)$ denotes the differential of F. We have necessarily $\phi = \phi_f$ where $f : V \to A$ is a homogeneous morphism of degree 0. But, in the case of an almost free algebra, the morphism $f : V \to A$ is not supposed to preserve the differentials of dg-modules, because the relation $\delta\phi_f = \phi\delta_h$ does not depend on this property.

2.3.4. *The strict category of almost free algebras*

We consider the category whose objects are almost free algebras and whose morphisms are strict morphisms of almost free algebras. By definition, a morphism of almost free algebras $\phi_f : \mathcal{E}(V) \to \mathcal{E}(W)$ is strict if we have $\phi_f(V) \subset W$ (equivalently, if the morphism $\phi_f : \mathcal{E}(V) \to \mathcal{E}(W)$ is induced by a morphism of dg-modules $f : V \to W$).

We have an obvious forgetful functor from the strict category of almost free $\mathcal{E}$-algebras to the category of all $\mathcal{E}$-algebras. We claim that this forgetful functor has a right adjoint. To be more explicit, to any $\mathcal{E}$-algebra $A \in \mathcal{E}\,\mathcal{A}lg$, we associate an almost free algebra $F_A = \mathcal{E}(C_A)$ together with a morphism $F_A \to A$. Furthermore, any morphism $F \to A$, where $F = \mathcal{E}(V)$ is almost free, has a unique factorization



such that $\tilde{\phi} : F \to F_A$ is a strict morphism of almost free algebras.

In fact, the augmentation morphism $F_A \to A$ is a quasi-isomorphism, and we explain in section 2.4 that the $\mathcal{E}$-algebra F_A is a canonical cell resolution of A (*cf.* proposition 2.4.4). The purpose of the next paragraphs is to recall the construction of the almost free algebra F_A in connection with the operadic bar construction of Getzler-Jones (*cf.* [8]) and Ginzburg-Kapranov (*cf.* [9]).

2.3.5. *The structure of the bar cooperad*

To be precise, we consider the bar cooperad $\mathcal{BE}$ introduced by E. Getzler and J. Jones (*cf.* [8]). We recall that the structure of a cooperad consists of a sequence $\mathcal{BE}(r)$, $r \in \mathbf{N}$, where $\mathcal{BE}(r)$ is a representation of the symmetric group Σ_r; together with decomposition coproducts

$$\mathcal{BE}(n) \to \mathcal{BE}(r) \otimes \mathcal{BE}(s_1) \otimes \cdots \otimes \mathcal{BE}(s_r),$$

defined for $n \in \mathbf{N}$ and $s_1 + \cdots + s_r = n$; but, the structure is such that for a fixed element of $\mathcal{BE}(n)$ only finitely many of these coproducts are non zero. We assume the dual equivariance and associativity properties of an operad composition product. Therefore, we may also consider the operad $\mathcal{BE}^\vee$ formed by the dual representations $\mathcal{BE}(r)^\vee$, $r \in \mathbf{N}$, of the modules $\mathcal{BE}(r)$, $r \in \mathbf{N}$. (But, these structures are not equivalent.)

Similarly, the structure of a $\mathcal{BE}$-coalgebra consists of a dg-module V equipped with a coproduct $V \to \bigoplus_{r=0}^\infty (\mathcal{BE}(r) \otimes V^{\otimes r})_{\Sigma_r}$. (We observe that the module of coinvariants $(\mathcal{BE}(r) \otimes V^{\otimes r})_{\Sigma_r}$ is isomorphic to a module of invariants $(\mathcal{BE}(r) \otimes V^{\otimes r})^{\Sigma_r}$, because, in the case of the Barratt-Eccles operad, the module of coefficients $\mathcal{BE}(r)$ is a regular representation of Σ_r.) Equivalently, we have a coproduct $V \to \mathcal{BE}(r) \otimes V^{\otimes r}$ for $r \in \mathbf{N}$; but, the structure is such that for a fixed element of V only finitely many of these coproducts are non zero. Finally, a coalgebra over a cooperad $\mathcal{BE}$ is equivalent a coalgebra over a dual operad $\mathcal{BE}^\vee$ together with a certain nilpotence assumption. The construction

$$\mathcal{BE}(V) = \bigoplus_{r=0}^{\infty} \mathcal{BE}_{(r)}(V), \qquad \text{where} \qquad \mathcal{BE}_{(r)}(V) = (\mathcal{BE}(r) \otimes V^{\otimes r})_{\Sigma_r},$$

provides a realization of the cofree $\mathcal{BE}$-coalgebra.

2.3.6. *The structure of almost free algebras*

The bar cooperad $\mathcal{BE}$ is characterized by the following properties. The structure of a $\mathcal{BE}$-coalgebra $\mathcal{BE}(r) \otimes V \to V^{\otimes r}$ is equivalent to a differential $\delta_h : \mathcal{E}(V) \to \mathcal{E}(V)$. Furthermore, a morphism of $\mathcal{BE}$-coalgebras is equivalent to a strict morphism of almost free algebras $\phi_f : \mathcal{E}(V) \to \mathcal{E}(W)$. In fact, a morphism $f : V \to W$ preserves $\mathcal{BE}$-coalgebra structures if and only if the associated morphism of free algebras $\phi_f : \mathcal{E}(V) \to \mathcal{E}(W)$ preserves the differentials of almost free algebras.

The almost free algebra $F_A = \mathcal{E}(C_A)$ is associated to a $\mathcal{BE}$-coalgebra C_A which has the following structure. The cofree $\mathcal{BE}$-coalgebra $\mathcal{BE}(A)$ is equipped with a differential $\delta : \mathcal{BE}(A) \to \mathcal{BE}(A)$ induced by the internal differential of A and by the differential of $\mathcal{BE}$. As in the context of almost free algebras over an operad, we consider a differential $\delta_A : \mathcal{BE}(A) \to \mathcal{BE}(A)$ which differs from $\delta : \mathcal{BE}(A) \to \mathcal{BE}(A)$ by a specific coderivation of the cofree cooperad $\mathcal{BE}(A)$. (In fact, the coderivation is determined by the algebra structure of A.) The coalgebra C_A consists of the cofree coalgebra $\mathcal{BE}(A)$ equipped with this differential $\delta_A : \mathcal{BE}(A) \to \mathcal{BE}(A)$.

2.3.7. *Almost free algebras and division functors*

We can make explicit the image of an almost free algebra under a division functor (as for cell algebras). To be precise, we observe that the division functor preserves almost free algebras. In fact, if $F = \mathcal{E}(V)$, then we have $F \oslash K = \mathcal{E}(V \otimes K)$. It is also straighforward to determine the derivation $d_h \oslash K : \mathcal{E}(V \otimes K) \to \mathcal{E}(V \otimes K)$ which yields the differential of $F \oslash K$. The morphism of dg-modules which determines this derivation is also denoted by $h \oslash K : V \otimes K \to \mathcal{E}(V \otimes K)$. In fact, we obtain the same result as in lemma 2.1.4:

2.3.8. Lemma

The image of an element $v \otimes c \in V \otimes K$ under the derivation $d_h \oslash K : \mathcal{E}(V \otimes K) \to \mathcal{E}(V \otimes K)$ is determined as follows. We assume

$$h(v) = \sum\nolimits_i \rho^i(v^i_{(1)}, \dots, v^i_{(r)}) \in \mathcal{E}(V).$$

We consider the diagonal $\Delta(\rho^i) = \sum_j \rho^j_{(1)} \otimes \rho^j_{(2)}$ of the operations $\rho^i \in \mathcal{E}(r)$ which occur in the expansion above. We take the image of $c \in K$ under the cooperations ${\rho^j_{(1)}}^* : K \to K^{\otimes r}$:

$${\rho^j_{(1)}}^*(c) = \sum\nolimits_k c^k_{(1)} \otimes \dots \otimes c^k_{(r)} \in K^{(r)}.$$

We have finally:

$$(h \oslash K)(v \otimes c) = \sum\nolimits_{ijk} \pm \rho^j_{(2)}(v^i_{(1)} \otimes c^k_{(1)}, \dots, v^i_{(r)} \otimes c^k_{(r)}) \in \mathcal{E}(V \otimes K).$$

2.4. Cell algebras are almost free

2.4.1. *Cell extensions of almost free algebras*

Let us consider a cell extension $F \to F \vee_f \mathcal{E}(C^* V)$, where $F = \mathcal{E}(U)$ is almost free and is equipped with the differential $\delta_h : \mathcal{E}(U) \to \mathcal{E}(U)$. In this case, the algebra $F \vee_f \mathcal{E}(C^* V)$ is also almost free. Precisely, we obtain $F \vee_f \mathcal{E}(C^* V) = \mathcal{E}(U \oplus V)$, because the free algebra functor preserves colimits. Furthermore, the morphism $h' : U \oplus V \to \mathcal{E}(U \oplus V)$ which determines the differential of $F \vee_f \mathcal{E}(C^* V)$ maps the module $V \subset U \oplus V$ into $\mathcal{E}(U) \subset \mathcal{E}(U \oplus V)$. In fact, this map is determined by the derivation of F on $U \subset U \oplus V$ and by the attaching morphism on $V \subset U \oplus V$. Precisely, the map $h' : U \oplus V \to \mathcal{E}(U \oplus V)$ is the sum of the composites

$$U \xrightarrow{h} \mathcal{E}(U) \hookrightarrow \mathcal{E}(U \oplus V) \qquad \text{and} \qquad V \simeq \Sigma^* V \xrightarrow{f} \mathcal{E}(U) \hookrightarrow \mathcal{E}(U \oplus V).$$

By induction, we deduce from these observations that a cell algebra is always an almost free algebra. Conversely, we have the following result:

2.4.2. Proposition

A cell algebra is equivalent to an almost free algebra $F' = \mathcal{E}(U')$, where the dg-module U' has a filtration

$$0 = U^{(-1)} \subset U^{(0)} \subset \cdots \subset U^{(n)} \subset \cdots \subset U'$$

such that the morphism $h' : U' \to \mathcal{E}(U')$, which determines the differential of $F' = \mathcal{E}(U')$, verifies $h'(U^{(n)}) \subset \mathcal{E}(U^{(n-1)})$.

Proof:

Suppose given an almost free algebra $F' = \mathcal{E}(U')$ as in the proposition above. We consider the algebra $F^{(n)} = \mathcal{E}(U^{(n)})$ together with the differential determined by the restriction of $h' : U' \to \mathcal{E}(U')$ to $U^{(n)} \subset U'$. We fix a splitting $U^{(n)} = U^{(n-1)} \oplus V^{(n)}$. We consider the restriction of $h : U' \to \mathcal{E}(U')$ to $V^{(n)} \subset U'$. According to the observations of paragraph 2.4.1, this gives a morphism of dg-modules $f^{(n)} : \Sigma^* V^{(n)} \to F^{(n-1)}$ such that $F^{(n)} = F^{(n-1)} \vee_{f^{(n)}} \mathcal{E}(C^* V^{(n)})$.

Proposition 2.4.2 can be generalized to a relative context. Namely, for an almost free algebra $F = \mathcal{E}(U)$, a relative cell inclusion $F \to F'$ is equivalent to a strict morphism $\phi_f : \mathcal{E}(U) \to \mathcal{E}(U')$, where the dg-module V has a filtration

$$U = U^{(-1)} \subset U^{(0)} \subset \cdots \subset U^{(n)} \subset \cdots \subset U'$$

such that the map $h' : U' \to \mathcal{E}(U')$, which determines the differential of $F' = \mathcal{E}(U')$, verifies $h'(U^{(n)}) \subset \mathcal{E}(U^{(n-1)})$.

2.4.3. Lemma

Suppose given a strict morphism $\phi_f : F \to F'$ of almost free algebras $F = \mathcal{E}(U)$ and $F' = \mathcal{E}(U')$ induced by an injective morphism of dg-modules $f : U \to U'$. If the almost free algebra $F' = \mathcal{E}(U')$ is equipped with a cell structure, then this morphism $\phi_f : F \to F'$ is equivalent to a relative cell inclusion (and is a cofibration).

Proof:

As in proposition 2.4.2, we let $h' : U' \to \mathcal{E}(U')$ denote the map which determines the differential of $F' = \mathcal{E}(U')$. The cell structure of $F' = \mathcal{E}(U')$ is determined by a filtration such that

$$0 = U^{(-1)} \subset U^{(0)} \subset \cdots \subset U^{(n)} \subset \cdots \subset U'.$$

We can assume that U is a sub-dg-module of U' and $F = \mathcal{E}(U)$ is a sub-dg-algebra of $F' = \mathcal{E}(U')$. We have then $h'(U) \subset \mathcal{E}(U)$ so that the differential of F' verifies $d_{h'}(F) \subset F$. Clearly, the filtration

$$U = U + U^{(-1)} \subset U + U^{(0)} \subset \cdots \subset U + U^{(n)} \subset \cdots \subset U'$$

gives the inclusion morphism $\mathcal{E}(U) \to \mathcal{E}(U')$ the structure of a relative cell inclusion.

We consider the almost free algebra $F_A = \mathcal{E}(\mathcal{B}\mathcal{E}(A))$ introduced in paragraph 2.3.4. We prove the following result:

2.4.4. Proposition

The universal almost free resolution of an $\mathcal{E}$-algebra $F_A = \mathcal{E}(\mathcal{B}\mathcal{E}(A))$ has a cell structure. Consequently, the construction $A \mapsto F_A$ provides a cofibrant resolution for all $\mathcal{E}$-algebras A.

Proof:

We consider the module $\mathcal{B}\mathcal{E}_n(A) \subset \mathcal{B}\mathcal{E}(A)$ generated by the degree n components of the bar construction $\mathcal{B}\mathcal{E}_n(A) = \bigoplus_{r=0}^{\infty} (\mathcal{B}\mathcal{E}_n(r) \otimes A^{\otimes r})_{\Sigma_r}$. We have an exhaustive filtration $0 = U^{(-1)} \subset U^{(0)} \subset \cdots \subset U^{(n)} \subset \cdots \subset \mathcal{B}\mathcal{E}(A)$ where $U^{(n)} = \bigoplus_{m \leq n} \mathcal{B}\mathcal{E}_m(A)$. The internal differential of A maps $\mathcal{B}\mathcal{E}_n(A)$ to $\mathcal{B}\mathcal{E}_n(A)$. The differential of the bar construction maps $\mathcal{B}\mathcal{E}_n(A)$ to $\mathcal{E}(\bigoplus_{m \leq n-1} \mathcal{B}\mathcal{E}_m(A))$ (*cf.* [8]). The proposition follows.

The next statement is an immediate corollary of lemma 2.4.3 and proposition 2.4.4.

2.4.5. Proposition

Let $A \to B$ be an injective morphism of $\mathcal{E}$-algebras. The induced morphism $F_A \to F_B$ has a relative cell structure. Consequently, this morphism is a cofibration in the category of $\mathcal{E}$-algebras.

§3. The example of loop spaces

3.1. *The adjoint functors associated to the circle coalgebra*

As in paragraph 1.13, we work in the category of algebras over the reduced Barratt-Eccles operad $\tilde{\mathcal{E}}$ (equivalent to the category of augmented $\mathcal{E}$-algebras). We replace an augmented $\mathcal{E}$-algebra A by its augmentation ideal $\tilde{A}$, which is equipped with the structure of an $\tilde{\mathcal{E}}$-algebra.

We consider the reduced chain complex of the circle $\tilde{K} = \tilde{N}_*(S^1)$. In this case, the module $\mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(S^1), \tilde{B})$ is denoted by $\Sigma^* \tilde{B} = \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(S^1), \tilde{B})$. and is called the suspension algebra of $\tilde{B}$. Similarly, the division algebra $\tilde{A} \oslash \tilde{N}_*(S^1)$ is denoted by $\Omega^* \tilde{A} = \tilde{A} \oslash \tilde{N}_*(S^1)$ amd is called the loop algebra of A. We have the adjunction relation

$$\mathrm{Hom}_{\tilde{\mathcal{E}}}(\Omega^* \tilde{A}, \tilde{B}) = \mathrm{Hom}_{\tilde{\mathcal{E}}}(\tilde{A}, \Sigma^* \tilde{B}).$$

The next propositions give the homotopy significance of suspension and loop algebra functors.

3.2. Proposition

We assume B is an $\mathcal{E}$-algebra equipped with an augmentation $B \to \mathbf{F}$. We consider the associated augmentation ideal $\tilde{B}$, which is an algebra over the reduced Barratt-Eccles operad $\tilde{\mathcal{E}}$. The suspension algebra $\Sigma^* \tilde{B}$ is a representative of the loop object of $\tilde{B}$ in the homotopy category of $\tilde{\mathcal{E}}$-algebras. (Hence, this is a suspension object in the opposite category.)

Proof:

We consider the cone algebra $C^* \tilde{B} = \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{B})$, which is an acyclic algebra together with a natural fibration $C^* \tilde{B} \to \tilde{B}$. The suspension algebra is identified with the kernel of this fibration. Hence, we have a cartesian square

$$\begin{array}{ccc} \Sigma^* \tilde{B} & \longrightarrow & C^* \tilde{B} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{B} \end{array}$$

and, as a conclusion, the suspension algebra satisfies the classical definition of a loop object in a closed model category (*cf.* [17]).

3.3. Proposition

We assume F is a cofibrant $\mathcal{E}$-algebra equipped with an augmentation $F \to \mathbf{F}$. We consider the associated augmentation ideal $\tilde{F}$, which is a cofibrant algebra over the reduced Barratt-Eccles operad $\tilde{\mathcal{E}}$. The loop algebra $\Omega^* \tilde{F}$ is a representative of the suspension of $\tilde{F}$ in the homotopy category of $\tilde{\mathcal{E}}$-algebras. (Hence, this is a loop object in the opposite category.)

Proof:

We consider the algebra $P^* \tilde{F} = \tilde{F} \oslash \tilde{N}(\Delta^1)$. According to lemma 3.4 below, this algebra is acyclic and, moreover, we have a natural cofibration $\tilde{F} \to P^* \tilde{F}$. Therefore, the algebra $P^* \tilde{F}$ deserves to be called a path algebra of $\tilde{F}$. We observe that the loop algebra $\Omega^* \tilde{F}$ fits in the cocartesian square

$$\begin{array}{ccc} \tilde{F} & \longrightarrow & P^* \tilde{F} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega^* \tilde{F} \end{array}$$

and the conclusion follows. Namely, the loop algebra $\Omega^* \tilde{F}$ satisfies the classical definition of a suspension object in a closed model category (*cf.* [17]).

3.4. Lemma

As in proposition 3.3, we assume that $\tilde{F}$ is a cofibrant $\tilde{\mathcal{E}}$-algebra. The canonical morphism $0 \to \tilde{F} \oslash \tilde{N}_*(\Delta^1)$ is an acyclic cofibration.

Proof:

The morphism $0 \to \tilde{F} \oslash \tilde{N}_*(\Delta^1)$ has the left lifting property with respect to fibrations, because the diagrams

$$\begin{array}{ccc} & & \tilde{A} \\ & \nearrow & \downarrow \\ \tilde{F} \oslash \tilde{N}_*(\Delta^1) & \longrightarrow & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{A}) \\ & \nearrow & \downarrow \\ \tilde{F} & \longrightarrow & \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{B}) \end{array}$$

are equivalent by adjunction. To be more precise, if the morphism $\tilde{A} \to \tilde{B}$ is a fibration, then the induced morphism $\mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{A}) \to \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{B})$ is still surjective. Furthermore, both dg-modules $\mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{A})$ and $\mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{B})$ are clearly acyclic. Hence, the induced morphism $\mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{A}) \to \mathrm{Hom}_{\mathbf{F}}(\tilde{N}_*(\Delta^1), \tilde{B})$ is an acyclic fibration. The lemma follows since the $\mathcal{E}$-algebra $\tilde{F}$ has the left lifting property with respect to such morphisms.

We consider the left derived functor $L\Omega^* : \mathrm{Ho}(\tilde{\mathcal{E}}\,\mathcal{A}lg) \to \mathrm{Ho}(\tilde{\mathcal{E}}\,\mathcal{A}lg)$ such that $L\Omega^* = - \oslash^L \tilde{N}_*(S^1)$. In the case $X = S^1$, theorem 1.14 asserts that the algebra $L\Omega^* \tilde{N}^*(Y) = \tilde{N}^*(Y) \oslash^L \tilde{N}_*(S^1)$ is quasi-isomorphic to the cochain algebra $\tilde{N}^*(\Omega Y) = \tilde{N}^* \operatorname{Map}_*(S^1, Y)$, provided the homotopy groups $\pi_n(Y)$, $n \in \mathbf{N}$, are finite p-groups. In fact, in the particular case $X = S^1$, it is not difficult to state a more precise result:

3.5. Theorem

Let Y be a pointed simplicial set. We assume that the cohomology modules $H^n(Y)$ are finite dimensional and that the fundamental group $\pi_1(Y)$ is a finite p-group. We have a quasi-isomorphism $L\Omega^* \tilde{N}^*(Y) \xrightarrow{\sim} \tilde{N}^*(\Omega Y)$.

Proof:

We deduce this result from a theorem of Mandell (*cf.* [15, Lemma 5.2]). We recall the main arguments in order to be careful with the assumptions of the theorem. We work with augmented $\mathcal{E}$-algebras.

We fix a cofibrant augmented $\mathcal{E}$-algebra F_Y (equivalent to a cofibrant $\tilde{\mathcal{E}}$-algebras $\tilde{F}_Y$) together with a quasi-isomorphism $F_Y \xrightarrow{\sim} N^*(Y)$. We consider the cocartesian square of augmented $\mathcal{E}$-algebras

$$\begin{array}{ccc} F_Y & \longrightarrow & P^* F_Y \\ \downarrow & (M) & \downarrow \\ \mathbf{F} & \longrightarrow & \Omega^* F_Y \end{array}$$

We compare this diagram to the commutative square of cochain algebras

$$\begin{array}{ccc} N^*(Y) & \longrightarrow & N^*(PY) \\ \downarrow & (C) & \downarrow \\ \mathbf{F} & \longrightarrow & N^*(\Omega Y) \end{array}$$

Precisely, we have a morphism from the cocartesian square (M) to the commutative square (C). In general, we assume that F_Y is the universal resolution of $N^*(Y)$ and we deduce this assertion from theorem 1.12. But, in the current situation, we can deduce the existence of fill-in morphisms from straighforward arguments. Explicitly, the composite morphism $F_Y \to N^*(Y) \to N^*(PY)$ extends to a morphism of augmented $\mathcal{E}$-algebras $P^*F_Y \to N^*(PY)$, because the canonical morphism $N^*(PY) \to \mathbf{F}$ is an acyclic fibration, and the cofibration $F_Y \to P^*F_Y$ has the left lifting property with respect to acyclic fibrations. We have an induced morphism $\Omega^*F_Y \to N^*(\Omega Y)$ because the diagram (M) is cocartesian. We prove that this morphism is a quasi-isomorphism.

The cocartesian diagram (M) gives rise to a strongly convergent spectral sequence $E^r_{(M)} \Rightarrow H^*(\Omega^* F_Y)$, such that $E^2_{(M)} = \mathrm{Tor}_*^{H^*(F_Y)}(\mathbf{F}, H^*(P^*F_Y)) = \mathrm{Tor}_*^{H^*(Y)}(\mathbf{F},\mathbf{F})$ (*cf.* [15, Corollary 3.6]). We compare this spectral sequence with the classical Eilenberg-Moore spectral sequence $E^r_{(C)} \Rightarrow H^*(\Omega Y)$. We have $E^2_{(C)} = \mathrm{Tor}_*^{H^*(Y)}(\mathbf{F}, H^*(PY)) = \mathrm{Tor}_*^{H^*(Y)}(\mathbf{F},\mathbf{F})$. The assumption of the theorem ensures the strong convergence of the classical Eilenberg-Moore spectral sequence (*cf.* [4]). The morphism of commutative squares $(M) \to (C)$ gives rise to a morphism of spectral sequences $E^r_{(M)} \to E^r_{(C)}$. Therefore, we obtain an isomorphism $H^*(\Omega^*F_Y) \xrightarrow{\simeq} H^*(\Omega Y)$, because the E^2 stage of these spectral sequences agree. This achieves the proof of theorem 3.5.

The purpose of the next paragraphs is to make the loop algebra functor explicit. Our result is stated in proposition 3.9. In fact, we recover a construction of Smirnov (*cf.* [21], [22]).

First, we remind the reader of results about the structure of the chain coalgebra of the circle (*cf.* [2]).

3.6. *The circle coalgebra*

We make explicit the structure of the reduced chain coalgebra $\tilde{K} = \tilde{N}_*(S^1)$ of the circle $S^1 = \Delta^1/\partial\Delta^1$. We have $\tilde{N}_*(S^1) = \mathbf{F}\, e_1$, where $\deg(e_1) = 1$.

We consider the cocycle $\epsilon_r : \mathcal{E}(r)_* \to \mathbf{F}$ introduced in article [2]. We have explicitly $\epsilon_r(w_0, \dots, w_d) = \pm 1$, if the sequence $(w_0(1), \dots, w_d(1))$ is a permutation of $(1, \dots, r)$, and $\epsilon_r(w_0, \dots, w_d) = 0$, otherwise. According to this definition, we assume that the cochain $\epsilon_r : \mathcal{E}(r)_* \to \mathbf{F}$ is zero in degree $* \neq r-1$.

For any $\rho \in \mathcal{E}(r)$, the operation $\rho^* : \tilde{N}_*(S^1) \to \tilde{N}_*(S^1)^{\otimes r}$ satisfies the formula $\rho^*(e_1) = \epsilon_r(\rho)\, e_1^{\otimes r}$.

3.7. *The suspension functor*

We make explicit the structure of the suspension algebra $\Sigma^* A$. In fact, we have a suspension functor on dg-modules $V \mapsto \Sigma^* V$ such that $\Sigma^* V$ is deduced from V by a shift of degree.

We set explicitly $\Sigma^* V^* = V^{*-1}$. We let $\Sigma^* v \in \Sigma^* V$ denote the element associated to $v \in V$. Clearly, our definition of the suspension functor in the category of $\tilde{\mathcal{E}}$-algebras agrees with the definition of the suspension functor for dg-modules.

We consider the cap product with the cocycle $\epsilon_r : \mathcal{E}(r)_* \to \mathbf{F}$. We obtain a morphism of dg-modules $\epsilon_r \cap - : \mathcal{E}(r)_* \to \mathcal{E}(r)_{*-r+1}$. We have explicitly $\epsilon_r \cap \rho = \sum_i \epsilon_r(\rho^i_{(1)})\, \rho^i_{(2)}$, where $\Delta(\rho) = \sum_i \rho^i_{(1)} \otimes \rho^i_{(2)}$ is the diagonal of $\rho \in \mathcal{E}(r)$. We deduce from the construction of paragraph 1.5 that the operation $\rho^* : (\Sigma^* A)^{\otimes r} \to \Sigma^* A$ satisfies the relation

$$\rho(\Sigma^* a_1, \ldots, \Sigma^* a_r) = \Sigma^*(\epsilon_r \cap \rho(a_1, \ldots, a_r)).$$

We deduce the structure of a loop algebra $\Omega^* F$ from claim 2.3.8. We introduce the following definition in order to state our result.

3.8. *The suspension of the Barratt-Eccles operad*

As in article [2], we consider the suspension of the Barratt-Eccles operad $\Lambda^* \mathcal{E}$ such that $\Lambda^* \mathcal{E}(r)_* = \operatorname{sgn}(r) \otimes \mathcal{E}(r)_{*-r+1}$, where $\operatorname{sgn}(r)$ denotes the signature representation of the symmetric group Σ_r. An algebra over the operad $\Lambda^* \mathcal{E}$ is equivalent to the suspension $\Sigma^* A$ of an $\mathcal{E}$-algebra. Furthermore, for free algebras, we have the relation $\Lambda^* \mathcal{E}(V) = \Sigma^* \mathcal{E}(V^{*+1})$.

The $\mathcal{E}$-algebra structure of a suspension $\Sigma^* A$ can be deduced from the canonical $\Lambda^* \mathcal{E}$-algebra structure by a restriction process (*cf.* [2]). Precisely, the cap product operations $\epsilon_r \cap - : \mathcal{E}(r)_* \to \mathcal{E}(r)_{*-r+1}$ define an operad morphism $\epsilon_* \cap - : \mathcal{E} \to \Lambda^* \mathcal{E}$. Consequently, if A is an $\mathcal{E}$-algebra, then the canonical evaluation product $\Lambda^* \mathcal{E}(r) \otimes (\Sigma^* A)^{\otimes r} \to \Sigma^* A$ restricts to

$$\mathcal{E}(r) \otimes (\Sigma^* A)^{\otimes r} \to \Lambda^* \mathcal{E}(r) \otimes (\Sigma^* A)^{\otimes r} \to \Sigma^* A.$$

We recover clearly the formula of paragraph 3.7 from this construction.

3.9. Proposition

If F is an almost free algebra over the Barratt-Eccles operad $\mathcal{E}$, then $\Sigma^* \Omega^* F$ is an almost free algebra over the suspension of the Barratt-Eccles operad $\Lambda^* \mathcal{E}$. More precisely, if $F = \mathcal{E}(V)$, then we have $\Sigma^* \Omega^* F = \Lambda^* \mathcal{E}(V)$. Moreover, if the differential of F is determined by a morphism

$$V \xrightarrow{h} \mathcal{E}(V)$$

(as explained in paragraph 2.3.2), then the almost free algebra $\Sigma^* \Omega^* F$ is equipped with the differential associated to the composite

$$V \xrightarrow{h} \mathcal{E}(V) \xrightarrow{\epsilon_* \cap -} \Lambda^* \mathcal{E}(V).$$

Hence, we consider the morphism $\mathcal{E}(V) \xrightarrow{\epsilon_* \cap -} \Lambda^* \mathcal{E}(V)$ induced by the cap product with the cocycles $\epsilon_r : \mathcal{E}(r)_* \to \mathbf{F}$ of paragraph 3.6.

§4. The case of Eilenberg-MacLane spaces

In this section, we determine the image of the cofibrant model of an Eilenberg-MacLane space $K(\mathbf{Z}/p, n)$ under a division functor $- \oslash K : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$. We prove the following result:

4.1. Lemma

The canonical morphism $F_{K(\mathbf{Z}/p,n)} \oslash N_*(X) \longrightarrow N^* \operatorname{Map}(X, K(\mathbf{Z}/p, n))$ is a quasi-isomorphism.

We replace the universal almost free algebra $F = F_{K(\mathbf{Z}/p,n)}$ by an equivalent cell algebra $F = F_n$ introduced by Mandell. We prove that we have a quasi-isomorphism

$$F_n \oslash N_*(X) \xrightarrow{\sim} N^* \operatorname{Map}(X, K(\mathbf{Z}/p, n))$$

by a generalization of Mandell's arguments.

The general idea consists in introducing Steenrod operations and unstable algebra structures. In fact, the mapping space $\operatorname{Map}(X, K(\mathbf{Z}/p, n))$ is a generalized Eilenberg-MacLane space. Consequently, the cohomology algebra $H^*(\operatorname{Map}(X, K(\mathbf{Z}/p, n)), \mathbf{F})$ is identified with a free object in the category of unstable algebras over the classical Steenrod algebra. Our purpose is to identify the cohomology algebra $H^*(F_n \oslash N_*(X))$ with the same free object.

4.1. The big Steenrod algebra

We recall the definition and properties of Steenrod operations in the context of algebras over the Barratt-Eccles operad. For simplicity, we assume that the ground field $\mathbf{F}$ has characteristic $p = 2$. For the general case, we refer to the article of Mandell (*cf.* [15, Sections 11-12]).

4.1.1. *Steenrod operations*

The cohomology $H^*(A)$ of an algebra over the Barratt-Eccles operad A is endowed with Steenrod operations $\operatorname{Sq}^i : H^n(A) \to H^{n+i}(A)$ defined for $i \in \mathbf{Z}$. If a class $c \in H^n(A)$ is represented by an element $a \in A^n$, then the reduced square $\operatorname{Sq}^i(c) \in H^{n+i}(A)$ is represented by the product $\theta_{n-i}(a, a) \in A^{n+i}$. The reduced squares $\operatorname{Sq}^i(c)$ where $i > n$ are 0. Let us observe that the top operation $\operatorname{Sq}^n(c)$ is identified with the Frobenius $\operatorname{Sq}^n(c) = c^2$. In general, the Steenrod operations are quadratic in regard to scalar multiplication. Explicitly, for $\lambda \in \mathbf{F}$, we have $\operatorname{Sq}^i(\lambda\, c) = \lambda^2 \operatorname{Sq}^i(c)$. On the other hand, for $c_1, c_2 \in H^*(A)$, we have $\operatorname{Sq}^i(c_1 + c_2) = \operatorname{Sq}^i(c_1) + \operatorname{Sq}^i(c_2)$. In the case $A = N^*(X)$ and $H^*(A) = H^*(X, \mathbf{F}) = H^*(X, \mathbf{F}_2) \otimes_{\mathbf{F}_2} \mathbf{F}$, we recover the classical Steenrod operations. In this context, we have $\operatorname{Sq}^0(c \otimes \lambda) = c \otimes \lambda^2$ and $\operatorname{Sq}^i(c \otimes \lambda) = 0$ for $i < 0$.

In the general framework of $\mathbf{Z}$-graded Steenrod operations, the Adem relation reads

$$\operatorname{Sq}^i \operatorname{Sq}^j = \sum\nolimits_{k \in \mathbf{Z}} \binom{j-k-1}{i-2k} \operatorname{Sq}^{i+j-k} \operatorname{Sq}^k$$

(where $i < 2j$ are given). In fact, we may assume that the sum ranges over $i - j + 1 \leq k \leq i/2$ because the binomial coefficient vanishes outside this range. Similarly, the Cartan formula reads

$$\operatorname{Sq}^n(a \cdot b) = \sum\nolimits_{k \in \mathbf{Z}} \operatorname{Sq}^k(a) \cdot \operatorname{Sq}^{n-k}(b).$$

This sum is also finite, because we have $\operatorname{Sq}^k(a) = 0$ for $k > \deg(a)$ and $\operatorname{Sq}^{n-k}(b) = 0$ for $n - k > \deg(b)$.

4.1.2. *The big Steenrod algebra*

The big Steenrod algebra B is generated by the operations Sq^i, where $i \in \mathbf{Z}$. We have explicitly $B = \mathbf{F}\langle \operatorname{Sq}^i,\ i \in \mathbf{Z} \rangle / (\text{Adem})$, where (Adem) denotes the two-sided ideal generated by the Adem relations above. The product of Steenrod monomials is extended to $\mathbf{F}$ coefficients according to the relation $\mathrm{Sq}^i \lambda = \lambda^2 \mathrm{Sq}^i$ (for any $\lambda \in \mathbf{F}$). The classical Steenrod algebra A can be identified with the quotient of the big Steenrod algebra B by the two-sided ideal generated by the relations $\mathrm{Sq}^0 = 1$ and $\mathrm{Sq}^i = 0$ for $i < 0$. In fact, according to Mandell, the left ideal generated by $1 + \mathrm{Sq}^0$ is a two-sided ideal in B and contains the elements Sq^i such that $i < 0$. Consequently, we have $A = B/B(1 + \mathrm{Sq}^0)$.

4.1.3. *Unstable modules*

The classical definition of an unstable module (*cf.* [19, Chapter 1]) is generalized in the context of modules over the big Steenrod algebra. To be precise, an unstable B-module (an unstable module over the big Steenrod algebra) is a graded $\mathbf{F}$-module M^* equipped with quadratic operations $\mathrm{Sq}^i : M^n \to M^{n+i}$, (so that M^* is a module over the algebra B) such that $\mathrm{Sq}^i : M^n \to M^{n+i}$ vanishes for $i > n$. Accordingly, an unstable module over the big Steenrod algebra is equipped with the structure of a restricted $\mathbf{F}$-module. By definition, a restricted $\mathbf{F}$-module is a graded $\mathbf{F}$-module M^* equipped with a Frobenius $\phi : M^* \to M^{2*}$. In the case of unstable modules, we consider the top Steenrod operation $\mathrm{Sq}^n : M^n \to M^{2n}$.

The free unstable B-module (respectively, the free unstable A-module) generated by a graded $\mathbf{F}$-module V is denoted either by $B \cdot V$ (respectively, $A \cdot V$) or more simply by BV (respectively, AV). The free unstable module generated by a single element e^n of degree n is also denoted by Be^n (respectively, Ae^n). According to Mandell, we have a short exact sequence of left B-modules

$$0 \longrightarrow Be^n \xrightarrow{1+\mathrm{Sq}^0} Be^n \longrightarrow Ae^n \longrightarrow 0$$

and, furthermore, this short exact sequence is split in the category of restricted modules. We observe that this property holds for $n \in \mathbf{Z}$. If $n < 0$, then we have $Ae^n = 0$. Hence, in this case, we just observe that the map $e^n \mapsto (1+\mathrm{Sq}^0)e^n$ gives an isomorphism of unstable B-modules.

4.1.4. *Unstable algebras*

The cohomology of an algebra over the Barratt-Eccles operad has the structure of an unstable B-algebra. To be precise, an unstable B-algebra is an associative and commutative $\mathbf{F}$-algebra R^* equipped with quadratic operations $\mathrm{Sq}^i : R^n \to R^{n+i}$ (so that R^* is a module over the algebra B) such that $\mathrm{Sq}^i : R^n \to R^{n+i}$ vanishes for $i > n$ and such that $\mathrm{Sq}^n : R^n \to R^{2n}$ is identified with the Frobenius of R. We assume in addition that the operations Sq^n verify the Cartan formula with respect to the product of R.

An unstable B-module M has an associated free unstable B-algebra $U(M)$. This algebra has the same construction as the classical free unstable A-algebra (*cf.* [19, Section 3.8]). Explicitly, the algebra $U(M)$ is the quotient of the symmetric algebra generated by M by the ideal generated by the relations $\mathrm{Sq}^n(x) = x^2$, where $x \in M^n$. (Hence, the map $M \mapsto U(M)$ is given by a functor on the category of restricted $\mathbf{F}$-modules.)

The cohomology of a free $\mathcal{E}$-algebra is a free unstable B-algebra. To be more explicit, for any differential graded $\mathbf{F}$-module V, the canonical morphism $H^*(V) \to H^*(\mathcal{E}(V))$ induces an isomorphism

$$U(BH^*(V)) \xrightarrow{\simeq} H^*(\mathcal{E}(V)).$$

Let us remind the reader that the cohomology of an Eilenberg-MacLane space is a free unstable A-algebra. Explicitly, we have a fundamental class $e^n \in H^n(K(\mathbf{Z}/2,n),\mathbf{F})$ together with an isomorphism

$$U(Ae^n) \xrightarrow{\simeq} H^*(K(\mathbf{Z}/2,n),\mathbf{F})$$

(*cf.* [19, Section 9.8]).

4.2. Models of Eilenberg-MacLane spaces and division functors

We recall the construction of the cell algebra F_n. As above, we assume that the ground field $\mathbf{F}$ has characteristic $p = 2$.

4.2.1. *Construction of the cofibrant model of an Eilenberg-MacLane space*

The cell algebra F_n introduced by Mandell is given by a cell extension of the form

$$\begin{array}{ccc} \mathcal{E}(\mathbf{F}\,e^n) & \longrightarrow & \mathcal{E}(\mathbf{F}\,e^n \oplus \mathbf{F}\,b^{n-1}) \\ {\scriptstyle \phi_f}\downarrow & & \downarrow \\ \mathcal{E}(\mathbf{F}\,e^n) & \longrightarrow & F_n \end{array}$$

The morphism $\phi_f : \mathcal{E}(\mathbf{F}\,e^n) \to \mathcal{E}(\mathbf{F}\,e^n)$ maps the element $e^n \in \mathcal{E}(\mathbf{F}\,e^n)$ to a representative of the class $e^n + \mathrm{Sq}^0(e^n) \in H^*(\mathcal{E}(\mathbf{F}e^n))$. For instance, we have $\phi_f(e^n) = e^n + \theta_n(e^n, e^n)$.

This cell algebra F_n is equivalent to an almost free algebra such that $F_n = \mathcal{E}(\mathbf{F}\,e^n \oplus \mathbf{F}\,b^{n-1})$. Furthermore, we have a morphism $F_n \to N^*(K(\mathbf{Z}/2,n))$, which maps the element $e^n \in F_n$ to the fundamental class of $K(\mathbf{Z}/2,n)$.

4.2.2. *The comparison arguments*

According to Mandell, the morphism $F_n \to N^*(K(\mathbf{Z}/2,n))$ is a quasi-isomorphism (*cf.* [15, Theorem 6.2]). We outline Mandell's arguments. We should observe that the result is valid for $n \in \mathbf{Z}$. If $n < 0$, then we have $K(\mathbf{Z}/2,n) = pt$ and we obtain simply $F_n \sim \mathbf{F}$.

We have $H^*(\mathcal{E}(\mathbf{F}\,e^n)) = U(Be^n)$. We consider the morphism of free unstable algebras $\psi_f : U(Be^n) \to U(Be^n)$ determined by $\phi_f : \mathcal{E}(\mathbf{F}\,e^n) \to \mathcal{E}(\mathbf{F}\,e^n)$. One observes precisely that $\psi_f : U(Be^n) \to U(Be^n)$ is induced by the morphism of unstable B-modules $f : Be^n \to Be^n$ such that $f(e^n) = e^n + \mathrm{Sq}^0(e^n)$. We mention in paragraph 4.1.3 that this morphism is split-injective in the category of restricted $\mathbf{F}$-modules. More precisely, if $f^*(Be^n)$ denotes the module obtained by restriction of structure, then we have $f^*(Be^n) = Be^n \oplus Ae^n$. Consequently, the morphism $\psi_f : U(Be^n) \to U(Be^n)$ makes $U(Be^n)$ a free module over $U(Be^n)$. We have explicitly $\psi_f^*(U(Be^n)) = U(Be^n) \otimes U(Ae^n)$.

The cocartesian square of paragraph 4.2.1 gives rise to a spectral sequence $E^r \Rightarrow H^*(F_n)$ such that $E^2 = \mathrm{Tor}_*^{U(Be^n)}(\psi_f^*(U(Be^n)),\mathbf{F})$. We deduce from the result above that this spectral sequence degenerates, because we obtain

$$\mathrm{Tor}_*^{U(Be^n)}(\psi_f^*(U(Be^n)),\mathbf{F}) = \begin{cases} U(Ae^n), & \text{for } * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the cohomology of F_n is a free unstable A-algebra and is generated by the class of the element $e^n \in F_n$. One concludes readily that the cohomology morphism $H^*(F_n) \to H^*(K(\mathbf{Z}/2,n))$ is an isomorphism.

We consider the image of the cell algebra F_n under a division functor $-\oslash K : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$, where K is a given coalgebra. By lemma 2.1.4, we have:

4.2.3. Lemma

The algebra $F_n \oslash K$ fits in a cell extension

$$\begin{array}{ccc} \mathcal{E}(\mathbf{F}\,e^n \otimes K) & \longrightarrow & \mathcal{E}(\mathbf{F}\,e^n \otimes K \oplus \mathbf{F}\,b^{n-1} \otimes K) \\ {\scriptstyle \phi_f \oslash K}\downarrow & & \downarrow \\ \mathcal{E}(\mathbf{F}\,e^n \otimes K) & \longrightarrow & F_n \oslash K \end{array}$$

Furthermore, the morphism $f \oslash K : \mathbf{F}\,e^n \otimes K \to \mathcal{E}(\mathbf{F}\,e^n \otimes K)$ satisfies the relation

$$(f \oslash K)(e^n \otimes c) = e^n \otimes c + \sum_{k=0}^{n} (\tau^k \cdot \theta_{n-k})(e^n \otimes c^i_{(1)}, e^n \otimes c^i_{(2)}),$$

for all $c \in K$, where ${\theta_k}^*(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)} \in K^{\otimes 2}$.

4.2.4. *Construction of the comparison morphism*

We assume that K is the chain complex of a simplicial set $K = N_*(X)$. Furthermore, we restrict ourself to the case of a finite simplicial set X. We extend our constructions to infinite simplicial sets by limit arguments (*cf.* paragraph 1.11).

We construct a morphism $F_n \oslash N_*(X) \to N_* \operatorname{Map}(X, K(\mathbf{Z}/2, n))$ natural in X. We consider the universal resolution of $N^*(K(\mathbf{Z}/2, n))$ and the morphism $F_{K(\mathbf{Z}/2,n)} \oslash N_*(X) \to N^* \operatorname{Map}(X, K(\mathbf{Z}/2, n))$ introduced in paragraph 1.11. We have a quasi-isomorphism $F_n \to F_{K(\mathbf{Z}/2,n)}$, because F_n and $F_{K(\mathbf{Z}/2,n)}$ are both cofibrant resolutions of $N^*(K(\mathbf{Z}/2, n))$. Therefore, we have a natural comparison morphism $F_n \oslash N_*(X) \to N_* \operatorname{Map}(X, K(\mathbf{Z}/2, n))$, which is given by the composite

$$F_n \oslash N_*(X) \to F_{K(\mathbf{Z}/2,n)} \oslash N_*(X) \to N^* \operatorname{Map}(X, K(\mathbf{Z}/2, n)).$$

We generalize the arguments outlined in paragraph 4.2.2 in order to prove that this morphism is a quasi-isomorphism. We still assume that the ground field has characteristic $p = 2$. In fact, we have fixed representatives of the Steenrod squares Sq^n, $n \in \mathbf{Z}$, but we just need to assume the Cartan relation $\mathrm{Sq}^0(a \cdot b) = \sum_{n \in \mathbf{Z}} \mathrm{Sq}^n(a) \cdot \mathrm{Sq}^{-n}(b)$. Therefore, the generalization of our arguments to odd primes is straighforward.

To begin with, we obtain the following result:

4.2.5. Lemma

We have $H^*\big(\mathcal{E}(\mathbf{F}\,e^n \otimes N_*(X))\big) = U\big(B(\mathbf{F}\,e^n \otimes H_*(X))\big)$. The morphism of free unstable algebras

$$\psi_g : U\big(B(\mathbf{F}\,e^n \otimes H_*(X))\big) \to U\big(B(\mathbf{F}\,e^n \otimes H_*(X))\big)$$

determined by

$$\phi_f \oslash N_*(X) : \mathcal{E}(\mathbf{F}\,e^n \otimes N_*(X)) \to \mathcal{E}(\mathbf{F}\,e^n \otimes N_*(X))$$

is induced by the morphism of unstable B-modules

$$g : B(\mathbf{F}\, e^n \otimes H_*(X)) \to B(\mathbf{F}\, e^n \otimes H_*(X))$$

such that

$$g(e^n \otimes c) = e^n \otimes c + \sum\nolimits_{l\in\mathbf{Z}} \mathrm{Sq}^l(e^n \otimes (c\,\mathrm{Sq}^{-l})),$$

for all $c \in H_*(X)$.

Proof:

We would like to determine the cohomology of the morphism

$$f' : \mathbf{F}\, e^n \otimes N_*(X) \to \big(\mathcal{E}(2) \otimes (\mathbf{F}\, e^n \otimes N_* X)^{\otimes 2}\big)_{\Sigma_2}$$

such that $f'(e^n \otimes x) = \sum_{k=0}^n (\tau^k \cdot \theta_{n-k})(e^n \otimes x^i_{(1)}, e^n \otimes x^i_{(2)})$, where $\theta^*_k(x) = \sum_i x^i_{(1)} \otimes x^i_{(2)}$ (*cf.* lemma 4.2.3). We consider homological Steenrod operations $\mathrm{Sq}^k : H_m(X) \to H_{m-k}(X)$. These operations are deduced from the cohomological ones by a duality process. Therefore, we introduce the dual

$$f'^\vee : \big(\mathcal{E}(2)^\vee \otimes (\mathbf{F}\, e_n \otimes N^* X)^{\otimes 2}\big)^{\Sigma_2} \to \mathbf{F}\, e_n \otimes N^*(X)$$

of the morphism f'. We let $(c_i)_i$ denote a basis of $H_*(X, \mathbf{F}_2)$. We fix a representative $x_i \in N_*(X)$ of each class $c_i \in H_*(X)$. We consider dual basis elements $c^i \in H^*(X)$ and dual cocycles $x^i \in N^*(X)$. We have also $\mathcal{E}(2)^\vee = \mathbf{F}[\Sigma_2]\, \theta^k$, where $\theta^k \in \mathcal{E}(2)^\vee$ is dual to $\theta_k \in \mathcal{E}(2)$ and satisfies the relation $\delta(\theta^k) = \theta^{k+1} + \tau\theta^{k+1}$.

The cohomology module

$$H^*\big(\mathcal{E}(2) \otimes (e^n \otimes N_* X)^{\otimes 2}\big)_{\Sigma_2} \subset H^*\big(\mathcal{E}(\mathbf{F}\, e^n \otimes N_*(X))\big) = U\big(B(\mathbf{F}\, e^n \otimes H^*(X))\big)$$

is generated by the cycles $\theta_0(e^n \otimes x_i, e^n \otimes x_j)$ and $\theta_{n-k}(e^n \otimes x_i, e^n \otimes x_i)$, where $k \leq n$. The former represent the products $e^n \otimes c_i \cdot e^n \otimes c_j \in U\big(B(\mathbf{F}\, e^n \otimes H^*(X))\big)$ and the latter represent the Steenrod squares $\mathrm{Sq}^l(e^n \otimes c_i) \in U\big(B(\mathbf{F}\, e^n \otimes H^*(X))\big)$, where $l = k - \deg(c_i) \leq n - \deg(c_i)$. Dually, the homology module

$$H_*\big(\mathcal{E}(2)^\vee \otimes (\mathbf{F}\, e_n \otimes N^* X)^{\otimes 2}\big)^{\Sigma_2} \subset H_*\big(\mathcal{E}(\mathbf{F}\, e^n \otimes N_*(X))^\vee\big)$$

is generated by the cycles $(1+\tau)\theta^0(e_n \otimes x^i, e_n \otimes x^j) + (1+\tau)\theta^0(e_n \otimes x^j, e_n \otimes x^i)$ and $(1+\tau)\theta^{n-k}(e_n \otimes x^i, e_n \otimes x^i)$, where $k \leq n$.

We obtain

$$f'^\vee\big((1+\tau)\theta^0(e_n \otimes x^i, e_n \otimes x^j) + (1+\tau)\theta^0(e_n \otimes x^j, e_n \otimes x^i)\big) \\ = e_n \otimes \theta_n(x^i, x^j) + e_n \otimes \theta_n(x^j, x^i).$$

The element $\theta_n(x^i, x^j) + \theta_n(x^j, x^i) \in N^*(X)$ vanishes in $H^*(X)$, because it corresponds to the differential of $\theta_{n+1}(x^i, x^j) \in N^*(X)$. We obtain also

$$f'^\vee\big((1+\tau)\theta^{n-k}(e_n \otimes x^i, e_n \otimes x^i)\big) = \begin{cases} e_n \otimes \theta_k(x^i, x^i), & \text{for } 0 \leq k \leq n, \\ 0, & \text{for } k < 0, \end{cases}$$

The element $\theta_k(x^i, x^i) \in N^*(X)$ is a representative of the Steenrod square $\mathrm{Sq}^{-l}(c^i) \in H^*(X)$, where $l = k - \deg(c^i)$, as in the paragraph above. We have $\mathrm{Sq}^{-l}(c^i) = \sum_j \lambda_{ij}^{-l} c^j$, where $\deg(c^j) = 2\deg(c^i) - k = -2l + k$.

By duality, the homology morphism $H_*(f')$ maps the element $e^n \otimes c_j \in \mathbf{F}\, e^n \otimes H_*(X)$ to the sum $\sum_l \lambda_{ij}^{-l} \mathrm{Sq}^l(e^n \otimes c_i) \in H^*\big(\mathcal{E}(\mathbf{F}\, e^n \otimes N_*(X))\big)$. By definition of homological Steenrod operations, we have $c_j \mathrm{Sq}^{-l} = \sum_i \lambda_{ij}^{-l} c_i$. Accordingly, the sum above is an expansion of the expression $\sum_l \mathrm{Sq}^l(e^n \otimes c_j \mathrm{Sq}^{-l})$. The summation index l is such that $\deg(c_j) = -2l + k$. Since $0 \leq k \leq n$, we obtain $(-\deg(c_j))/2 \leq l \leq (n - \deg(c_j))/2$. But, the unstability relations implies that the formula $\mathrm{Sq}^l(e^n \otimes (c_j \mathrm{Sq}^{-l}))$ vanishes outsider this range. Therefore, we can assume that the sum ranges over $\mathbf{Z}$ and we obtain the result stated in the lemma.

4.2.6. Lemma

The morphism $g : B(\mathbf{F}\, e^n \otimes H_*(X)) \to B(\mathbf{F}\, e^n \otimes H_*(X))$ is split-injective in the category of restricted $\mathbf{F}$-modules and fits in a short exact sequence of unstable B-modules $0 \longrightarrow B(\mathbf{F}\, e^n \otimes H_*(X)) \longrightarrow B(\mathbf{F}\, e^n \otimes H_*(X)) \longrightarrow A(\mathbf{F}\, e^n \otimes H_*(X)) \longrightarrow 0.$

Proof:

The sequence

$$0 \longrightarrow B(e^n \otimes c) \xrightarrow{1+\mathrm{Sq}^0} B(e^n \otimes c) \longrightarrow A(e^n \otimes c) \longrightarrow 0,$$

is exact, for all $c \in H_*(X)$ (*cf.* paragraph 4.1.3). Furthermore, we have a morphism of restricted modules $r_0 : B(e^n \otimes c) \to B(e^n \otimes c)$ such that $r_0(b(1 + \mathrm{Sq}^0)(e^n \otimes c)) = b(e^n \otimes c)$, for all $b(e^n \otimes c) \in B(\mathbf{F}\, e^n \otimes H_*(X))$. To prove our property, we observe that the morphism $g : B(\mathbf{F}\, e^n \otimes H_*(X)) \to B(\mathbf{F}\, e^n \otimes H_*(X))$ is represented by a triangular matrix which has the element $(1 + \mathrm{Sq}^0)$ on all diagonal entries.

To be more explicit, we construct a morphism of restricted $\mathbf{F}$-modules $r : B(\mathbf{F}\, e^n \otimes H_*(X)) \to B(\mathbf{F}\, e^n \otimes H_*(X))$ such that $r\, g(b(e^n \otimes c)) = b(e^n \otimes c)$ by induction on the degree of $c \in H_*(X)$. Since $c \,\mathrm{Sq}^{-l} = 0$ for $l > 0$ and $c\, \mathrm{Sq}^0 = c$, we obtain

$$g(b(e^n \otimes c)) = b(1 + \mathrm{Sq}^0)(e^n \otimes c) + g'(b(e^n \otimes c)),$$

$$\text{where} \qquad g'(b(e^n \otimes c)) = \sum_{l<0} b\, \mathrm{Sq}^l(e^n \otimes c\, \mathrm{Sq}^{-l}).$$

In this equation, we have $\deg(c\, \mathrm{Sq}^{-l}) = \deg(c) + l < \deg(c)$. Therefore, by induction, we can set $r(b(e^n \otimes c)) = r_0(b(e^n \otimes c)) - r\, g'(r_0(b(e^n \otimes c))$. We obtain readily the identity $r\, g(b(e^n \otimes c)) = b(e^n \otimes c)$.

Similarly, a straightforward induction proves that the cokernel of the morphism $g : B(\mathbf{F}\, e^n \otimes H_*(X)) \to B(\mathbf{F}\, e^n \otimes H_*(X))$ is the module $A(\mathbf{F}\, e^n \otimes H_*(X))$.

As in Mandell's proof (*cf.* paragraph 4.2.2), lemma 4.2.6 has the following formal consequence:

4.2.7. Lemma

Let $U_B^X = U\big(B(\mathbf{F}\, e^n \otimes H_*(X))\big)$ and $U_A^X = U\big(A(\mathbf{F}\, e^n \otimes H_*(X))\big)$. The morphism $\psi_g : U_B^X \to U_B^X$ makes the algebra U_B^X a free module over U_B^X. We have $\psi_g^* U_B^X = U_B^X \otimes U_A^X$.

4.2.8. *The spectral sequence*

The cell extension of lemma 4.2.3 gives rise to a spectral sequence $E^r \Rightarrow H^*(F_n \oslash N_*(X))$ such that $E^2 = \mathrm{Tor}_*^{U_B^X}(\psi_g^* U_B^X, \mathbf{F})$. The lemma above implies that this spectral sequence degenerates at E^2, because we obtain

$$\mathrm{Tor}_*^{U_B^X}(\psi_f^* U_B^X, \mathbf{F}) = \begin{cases} U_A^X, & \text{for } * = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently:

4.2.9. Lemma

We have an isomorphism $U\big(A(\mathbf{F}\, e^n \otimes H_*(X))\big) \xrightarrow{\simeq} H^*(F_n \oslash N_*(X))$.

We can achieve the proof of the main result of this section:

4.2.10. Lemma

The comparison morphism $F_n \oslash N_*(X) \to N^* \mathrm{Map}(X, K(\mathbf{Z}/2, n))$ *(defined in paragraph 4.2.4) is a quasi-isomorphism.*

Proof:

We have also an isomorphism $U\big(A(\mathbf{F}\, e^n \otimes H_*(X))\big) \xrightarrow{\simeq} H^*\big(\mathrm{Map}(X, K(\mathbf{Z}/2, n))\big)$, because $\mathrm{Map}(X, K(\mathbf{Z}/2, n))$ is a generalized Eilenberg-MacLane space. We observe that the comparison morphism preserves the cohomology generators. For that purpose, we consider the adjoint morphism $F_n \to N^*(X) \otimes N^* \mathrm{Map}(X, K(\mathbf{Z}/2, n))$. According to the definition, this morphism occurs as a composite in a diagram

$$\begin{array}{ccccc} F_n & \longrightarrow & F_Y & \longrightarrow & F_{X \times \mathrm{Map}(X,Y)} & \\ & \searrow & \downarrow \sim & & \downarrow \sim & \searrow \\ & & N^*(Y) & \longrightarrow & N^*(X \times \mathrm{Map}(X,Y)) & \longrightarrow & N^*(X) \otimes N^*(\mathrm{Map}(X,Y)) \end{array}$$

where the right triangle commutes in cohomology (*cf.* theorem 5.1). Consequently, the element $e^n \in F_n$ is mapped to a representative of the image of the fundamental class of $K(\mathbf{Z}/2, n)$ under the classical morphism

$$H^*(K(\mathbf{Z}/2, n)) \to H^*(X) \otimes H^* \mathrm{Map}(X, K(\mathbf{Z}/2, n))$$

induced by the evaluation product. By adjunction, we deduce that the comparison morphism $F_n \oslash N_*(X) \to N^* \mathrm{Map}(X, K(\mathbf{Z}/2, n))$ identifies a cycle $e^n \otimes x \in e^n \otimes N_*(X)$ with a generator of $H^* \mathrm{Map}(X, K(\mathbf{Z}/2, n))$. The conclusion follows.

4.3. Fibrations and division functors

The next lemma allows to achieve the proof of theorem 1.10 by a classical induction process (*cf.* [3], [19, Section 9.9]).

4.3.1. Lemma

We are given a fibration sequence $F \to E \to B$ where B, E, F are p-π_*-finite (cf. [19, Section 9.7]). If the comparison morphism $F_Y \oslash N_*(X) \to N^*(\mathrm{Map}(X,Y))$ is a quasi-isomorphism for $Y = B$ and $Y = E$, then the comparison morphism is also a quasi-isomorphism for $Y = F$.

Proof:

We consider the natural morphism $F_B \to F_E$, which is a cofibration by proposition 2.4.5. We form the cocartesian square

$$\begin{array}{ccc} F_B & \longrightarrow & F_E \\ \downarrow & & \downarrow \\ \mathbf{F} & \longrightarrow & \mathbf{F} \vee_{F_B} F_E \end{array}$$

We have also a morphism $F_E \to F_F$ by functoriality of universal cofibrant resolutions. We claim that the induced morphism $\mathbf{F} \vee_{F_B} F_E \to F_F$ is a quasi-isomorphism. This property follows from a result of Mandell (*cf.* [15, Lemma 5.2]). Explicitly, the algebra $\mathbf{F} \vee_{F_B} F_E$ is quasi-isomorphic to $N^*(F)$. Hence, the pushout process above provides a cofibrant resolution of $N^*(F)$, which is necessarily equivalent to the universal one F_F. The division functor $- \oslash N_*(X) : \mathcal{E}\,\mathcal{A}lg \to \mathcal{E}\,\mathcal{A}lg$ preserves cocartesian squares and quasi-isomorphisms of cofibrant algebras, such as $\mathbf{F} \vee_{F_B} F_E \to F_F$. Therefore, we obtain a homotopy cocartesian square

$$\begin{array}{ccc} F_B \oslash N_*(X) & \longrightarrow & F_E \oslash N_*(X) \\ \downarrow & (M) & \downarrow \\ \mathbf{F} & \longrightarrow & F_F \oslash N_*(X) \end{array}$$

This homotopy cocartesian square gives rise to a strongly convergent spectral sequence $E^r_{(M)} \to H^*(F_F \oslash N_*(X))$ such that $E^2_{(M)} = \mathrm{Tor}^{H^*(F_B \oslash N_*(X))}(H^*(F_E \oslash N_*(X)), \mathbf{F})$ (*cf.* [15, Corollary 3.6]).

We compare this spectral sequence to the Eilenberg-Moore spectral sequence $E^r_{(C)} \to H^*(\mathrm{Map}(X,F))$ of the fibration $\mathrm{Map}(X,F) \to \mathrm{Map}(X,E) \to \mathrm{Map}(X,B)$. We have $E^2_{(C)} = \mathrm{Tor}^{H^*(\mathrm{Map}(X,B))}(H^*(\mathrm{Map}(X,E)), \mathbf{F})$ and the finiteness assumption of the lemma ensures the strong convergence of the spectral sequence. By functoriality, the comparison morphism gives a morphism from the commutative square of cofibrant models (M) to the commutative square of cochain algebras (C)

$$\begin{array}{ccc} N^*(\mathrm{Map}(X,B)) & \longrightarrow & N^*(\mathrm{Map}(X,E)) \\ \downarrow & (C) & \downarrow \\ \mathbf{F} & \longrightarrow & N^*(\mathrm{Map}(X,F)) \end{array}$$

As a consequence, we have a morphism of spectral sequence $E^r_{(M)} \to E^r_{(C)}$. If we have isomorphisms

$$H^*(F_B \oslash N_*(X)) \to H^*(\mathrm{Map}(X,B)) \qquad \text{and} \qquad H^*(F_E \oslash N_*(X)) \to H^*(\mathrm{Map}(X,E)),$$

then the E^2-stages of the spectral sequences agree. We conclude that the natural morphism $H^*(F_F \oslash N_*(X)) \to H^*(\mathrm{Map}(X,F))$ is an isomorphism. This achieves the proof of lemma 4.3.1.

§5. The Eilenberg-Zilber equivalence

We consider the classical shuffle morphism $N^*(X \times Y) \to N^*(X)\hat{\otimes} N^*(Y)$ (*cf.* [13, Section VIII.8]), where $N^*(X)\hat{\otimes} N^*(Y)$ is the profinite completion of the tensor product $N^*(X) \otimes N^*(Y)$. We observe that this morphism of dg-modules is not a morphism of $\mathcal{E}$-algebras. Therefore, the purpose of this section is to construct a homotopy Eilenberg-Zilber morphism from the universal almost free algebra $F_{X\times Y}$ to the complete tensor product $N^*(X)\hat{\otimes} N^*(Y)$. We obtain the following result:

5.1. Theorem

Let X and Y be simplicial sets. We have a morphism of $\mathcal{E}$-algebras $F_{X\times Y} \to N^*(X)\hat{\otimes} N^*(Y)$, functorial in X and Y, whose cohomology corresponds to the classical Künneth isomorphism $H^*(X \times Y) \xrightarrow{\simeq} H^*(X)\hat{\otimes} H^*(Y)$. Precisely, the diagram

$$\begin{array}{ccc} F_{X\times Y} & & \\ \downarrow{\scriptstyle\sim} & \searrow & \\ N^*(X \times Y) & \xrightarrow{\sim} & N^*(X)\hat{\otimes} N^*(Y) \end{array}$$

where $N^*(X \times Y) \xrightarrow{\sim} N^*(X)\hat{\otimes} N^*(Y)$ is the classical shuffle morphism, gives rise to a commutative diagram in cohomology.

We have $N^*(X)\hat{\otimes} N^*(Y) = \lim_{K,L} N^*(K)\hat{\otimes} N^*(L)$, where K (respectively, L) ranges over finite simplicial subsets of X (respectively, Y). Therefore, we reduce to the case of finite simplicial sets and omit profinite completions. We construct a homotopy Eilenberg-Zilber morphism as stated in lemma 5.2 below. We prove that the assertion of theorem 5.1 about the Künneth isomorphism follows from the construction (*cf.* lemma 5.5).

5.2. Lemma

Let X and Y be finite simplicial sets. There is a morphism of $\mathcal{E}$-algebras

$$F_{X\times Y} \to N^*(X) \otimes N^*(Y)$$

which is functorial in X and Y and which reduces to the augmentation morphism $F_{pt} \to N^*(pt)$ for $X = Y = pt$.

5.3. *Reminder: the structure of the universal almost free resolution*

We recall that the universal almost free resolution of an algebra A is given by the construction $F_A = \mathcal{E}(C_A)$, where $C_A = \mathcal{B}\mathcal{E}(A)$ is an almost cofree coalgebra over the bar cooperad $\mathcal{B}\mathcal{E}$. This coalgebra is equipped with a differential $\delta_A : \mathcal{B}\mathcal{E}(A) \to \mathcal{B}\mathcal{E}(A)$ which differs from the canonical differential of the cofree coalgebra $\delta : \mathcal{B}\mathcal{E}(A) \to \mathcal{B}\mathcal{E}(A)$ by a coderivation $d_A : \mathcal{B}\mathcal{E}(A) \to \mathcal{B}\mathcal{E}(A)$ determined by the algebra structure of A. We have $\mathcal{B}\mathcal{E}(A) = \bigoplus_r \mathcal{B}\mathcal{E}_{(r)}(A)$, where $\mathcal{B}\mathcal{E}_{(r)}(A) = (\mathcal{B}\mathcal{E}(r) \otimes A^{\otimes r})_{\Sigma_r}$. We have a natural isomorphism $A \simeq \mathcal{B}\mathcal{E}_{(1)}(A)$ and the canonical surjection $\mathcal{B}\mathcal{E}(A) \to \mathcal{B}\mathcal{E}_{(1)}(A) = A$ is identified with the universal morphism $\mathcal{B}\mathcal{E}(A) \to A$. If we consider the canonical section $A = \mathcal{B}\mathcal{E}_{(1)}(A) \to \mathcal{B}\mathcal{E}(A)$, then we obtain a morphism of dg-modules $A \to C_A$ which identifies the algebra A with the indecomposable part of the almost cofree coalgebra C_A.

By definition of the universal almost free resolution, a morphism of $\mathcal{E}$-algebras $F_A \to B$ is equivalent to a morphism of $\mathcal{B}\mathcal{E}$-coalgebras $C_A \to C_B$. Furthermore, the canonical morphism $F_A \to A$ corresponds to the identity of C_A. In particular, the morphism of lemma 5.2 is equivalent to a morphism of $\mathcal{B}\mathcal{E}$-coalgebras $C_{N^*(X\times Y)} \to C_{N^*(X)\otimes N^*(Y)}$, which reduces to the identity morphism for $X = Y = pt$.

We would like to mention that the morphism $C_{N^*(X\times Y)} \to C_{N^*(X)\otimes N^*(Y)}$ is an extension of the classical Eilenberg-Zilber equivalence. Precisely, we obtain the following result:

5.4. Proposition

The morphism of $\mathcal{B}\mathcal{E}$-coalgebras $C_{N^*(X\times Y)} \to C_{N^*(X)\otimes N^*(Y)}$, deduced from lemma 5.2, fits in a commutative diagram of dg-modules

$$\begin{array}{ccc} N^*(X\times Y) & \longrightarrow & N^*(X)\otimes N^*(Y) \\ \downarrow & & \downarrow \\ C_{N^*(X\times Y)} & \longrightarrow & C_{N^*(X)\otimes N^*(Y)} \end{array}$$

where $N^*(X\times Y) \to N^*(X)\otimes N^*(Y)$ is the classical shuffle morphism.

Proof:

Recall that $N^*(X\times Y)$ (respectively, $N^*(X)\otimes N^*(Y)$) is identified with the indecomposable part of $C_{N^*(X\times Y)}$ (respectively, $C_{N^*(X)\otimes N^*(Y)}$). Because of this property, the morphism of coalgebras $C_{N^*(X\times Y)} \to C_{N^*(X)\otimes N^*(Y)}$ induces a morphism of dg-modules $N^*(X\times Y) \to N^*(X)\otimes N^*(Y)$ as in the diagram above. The induced morphism coincides with the classical shuffle morphism because it verifies the same characteristic properties. Namely, we obtain a morphism $N^*(X\times Y) \to N^*(X)\otimes N^*(Y)$ which is functorial in X and Y and which reduces to the identity morphism for $X = Y = pt$ (*cf.* [5]).

We prove the assertion of theorem 5.1 about the Künneth isomorphism by similar arguments:

5.5. Lemma

The morphism provided by lemma 5.2 makes the following diagram commutative in *cohomology*

$$\begin{array}{ccc} F_{N^*(X\times Y)} & & \\ \downarrow{\scriptstyle \sim} & \searrow & \\ N^*(X\times Y) & \xrightarrow{\sim} & N^*(X)\otimes N^*(Y) \end{array}$$

Proof:

The composite of the canonical injections $N^*(X\times Y) \to \mathcal{BE}(N^*(X\times Y)) = C_{N^*(X\times Y)}$ and $C_{N^*(X\times Y)} \to \mathcal{E}(C_{N^*(X\times Y)}) = F_{N^*(X\times Y)}$ defines a morphism of dg-modules $N^*(X \times Y) \to F_{N^*(X\times Y)}$. This morphism is a section of the canonical augmentation $F_{N^*(X\times Y)} \to N^*(X\times Y)$. Hence, the induced morphism $H^*(N^*(X\times Y)) \to H^*(F_{N^*(X\times Y)})$ is an inverse of the cohomology isomorphism $H^*(F_{N^*(X\times Y)}) \xrightarrow{\simeq} H^*(N^*(X \times Y))$. The lemma follows from the observation that the composite

$$N^*(X \times Y) \to F_{N^*(X\times Y)} \to N^*(X)\otimes N^*(Y)$$

coincides with the shuffle morphism. As in the proof of lemma 5.2, this assertion is an immediate consequence of the characteristic properties of the Eilenberg-Zilber equivalence.

We define the morphism $F_{N^*(X\times Y)} \to N^*(X)\otimes N^*(Y)$ and prove lemma 5.2 in the next paragraphs. We adapt the classical methods of acyclic models to our framework. First, we restrict to the case $X = \Delta^m$ and $Y = \Delta^n$. To be precise, the full subcategory of $\mathcal{S}^2$ formed by the pairs (Δ^m, Δ^n), $(m, n) \in \mathbf{N}^2$ is isomorphic to Δ^2. Hence, we consider the covariant functor $\Delta^2 \to \mathcal{S}^2$ provided by the map $(m, n) \mapsto (\Delta^m, \Delta^n)$. We construct a functorial morphism of algebras $F_{N^*(\Delta^m\times\Delta^n)} \to N^*(\Delta^m)\otimes N^*(\Delta^n)$ by induction on dimension. For that purpose, we introduce Reedy structures in the framework of dg-modules and $\mathcal{E}$-algebras (*cf.* [18]). We refer to the monographs [6], [10] and [11] for more background about Reedy's theory.

5.6. *Bisimplicial modules and Reedy structures*

Let V be a bisimplicial (differential graded) module. The matching module $M_{m,n}V$ is formed by sequences of elements $v_i \in V_{m-1,n}$, $0 \leq i \leq m$, and $w_j \in V_{m,n-1}$, $0 \leq j \leq n$, such that $d_k \times 1(v^i) = d_{i-1} \times 1(v_k)$, for $k < i$, and $1 \times d_l(w^j) = 1 \times d_{j-1}(w^l)$, for $l < j$. We have a canonical morphism $V_{m,n} \to M_{m,n}V$ which maps an element $x \in V_{m,n}$ to $(d_0 \times 1(x), \dots, d_m \times 1(x), 1 \times d_0(x), \dots, d_n(x)) \in M_{m,n}V$. We have equivalently $M_{m,n}V = \lim_{(u,v)} V_{m',n'}$, where the limit ranges over proper pairs of injective morphisms $u : [m'] \to [m]$ and $v : [n'] \to [n]$ in the simplicial category.

Dually, we have a latching module $L_{m,n}V$ together with a morphism $L_{m,n}V \to V_{m,n}$. The latching module verifies $L_{m,n}V = \operatorname{colim}_{(u,v)} V_{m',n'}$, where the colimit ranges over proper pairs of surjective morphisms $u : [m] \to [m']$ and $v : [n] \to [n']$.

We have the following property:

5.7. Fact

All bisimplicial dg-modules are Reedy fibrant and Reedy cofibrant. More explicitly, the matching morphism $V_{m,n} \to M_{m,n}V$ is always surjective and the latching morphism $L_{m,n}V \to V_{m,n}$ is always injective.

Proof:

The proof of these assertions is classical. Suppose given $(v_0, \dots, v_m, w_0, \dots, w_n) \in M_{m,n}V$. We construct an element $x \in V_{m,n}$ such that $(d_0 \times 1(x), \dots, d_m \times 1(x), 1 \times d_0(x), \dots, 1 \times d_n(x)) = (v_0, \dots, v_m, w_0, \dots, w_n)$. We assume by induction $v_0 = \dots = v_{i-1} = 0$. We have an element $x' \in V_{m,n}$ such that $d_0 \times 1(x') = \dots = d_{i-1} \times 1(x') = 0$ and $d_i \times 1(x') = v_i$. For instance, we can prove by straighforward verifications that the element $x' = (-1)^i s_0 \times 1(v_i) + (-1)^{i-1} s_1 \times 1(v_i) + \dots + s_i \times 1(v_i)$ satisfies these equations. We set $(v'_0, \dots, v'_m, w'_0, \dots, w'_n) = (v_0, \dots, v_m, w_0, \dots, w_n) - (d_0 \times 1(x'), \dots, d_m \times 1(x'), 1 \times d_0(x'), \dots, 1 \times d_n(x'))$ and we resume the process. Eventually, we prove that the element $(v_0, \dots, v_m, w_0, \dots, w_n) \in M_{m,n}V$ lies in the image of the matching morphism.

Hence, we conclude that the matching morphism $V_{m,n} \to M_{m,n}V$ is surjective. By a similar construction, we prove that the matching morphism of a bicosimplicial module $V^{m,n} \to M^{m,n}V$ is surjective. By duality, this proves also that the latching morphism of a bisimplicial module $L_{m,n}V \to V_{m,n}$ is injective.

5.8. Lemma

The bisimplicial algebra $(m,n) \mapsto N^*(\Delta^m) \otimes N^*(\Delta^n)$ is Reedy fibrant. Explicitly, given $(m,n) \in \mathbf{N}^2$, the limit morphism

$$N^*(\Delta^m) \otimes N(\Delta^n) \to \lim_{(u,v)} N^*(\Delta^{m'}) \otimes N^*(\Delta^{n'}),$$

where $u : [m'] \to [m]$ and $v : [n'] \to [n]$ range over proper pairs of injections, is a fibration of $\mathcal{E}$-algebras.

The bisimplicial algebra $(m,n) \mapsto F_{N^*(\Delta^m \times \Delta^n)}$ is acyclic and Reedy cofibrant. Explicitly, given $(m,n) \in \mathbf{N}^2$, the colimit morphism

$$\operatorname*{colim}_{(u,v)} F_{N^*(\Delta^{m'} \times \Delta^{n'})} \to F_{N^*(\Delta^m \times \Delta^n)},$$

where $u : [m] \to [m']$ and $v : [n] \to [n']$ range over proper pairs of surjections, is an acyclic cofibration of $\mathcal{E}$-algebras.

Proof:

We deduce immediately from the case of dg-modules (*cf.* fact 5.7) that the matching morphism

$$N^*(\Delta^m) \otimes N(\Delta^n) \to \lim_{(u,v)} N^*(\Delta^{m'}) \otimes Nj(\Delta^{n'}),$$

is a fibration, because, in the category of $\mathcal{E}$-algebras, the fibrations are the surjective morphisms.

Consider the case of the bisimplicial algebra $(m,n) \mapsto F_{N^*(\Delta^m \times \Delta^n)}$. The morphisms $\mathbf{F} \to F_{N^*(\Delta^m \times \Delta^n)}$ are clearly weak-equivalences. Hence, we have just to prove that these morphisms form a Reedy cofibration. We have

$$\operatorname*{colim}_{(u,v)} F_{N^*(\Delta^{m'} \times \Delta^{n'})} = \mathcal{E}(\operatorname*{colim}_{(u,v)} C_{N^*(\Delta^{m'} \times \Delta^{n'})}),$$

by construction of colimits in the context of almost free algebras. Furthermore, the latching morphism $\operatorname{colim}_{(u,v)} F_{N^*(\Delta^{m'} \times \Delta^{n'})} \to F_{N^*(\Delta^m \times \Delta^n)}$ is induced by the morphism of dg-modules $\operatorname{colim}_{(u,v)} C_{N^*(\Delta^{m'}\times\Delta^{n'})} \to C_{N^*(\Delta^m\times\Delta^n)}$. We deduce from the case of dg-modules (*cf.* fact 5.7) that this morphism is injective. Finally, our claim follows from the results of section 2.4: since $F_{N^*(\Delta^m\times\Delta^n)} = \mathcal{E}(C_{N^*(\Delta^m\times\Delta^n)})$ is a cell algebra (*cf.* proposition 2.4.4), lemma 2.4.3 implies that the latching morphism $\operatorname{colim}_{(u,v)} F_{N^*(\Delta^{m'}\times\Delta^{n'})} \to F_{N^*(\Delta^m\times\Delta^n)}$ is a relative cell inclusion and is a cofibration of $\mathcal{E}$-algebras.

5.9. Lemma

Suppose given compatible morphisms of algebras

$$F_{N^*(\Delta^{m'}\times\Delta^{n'})} \to N^*(\Delta^{m'})\otimes N^*(\Delta^{n'})$$

for all bisimplices $\Delta^{m'}\times\Delta^{n'}$, such that $m'+n'<N$. We consider a cartesian product $\Delta^m\times\Delta^n$, where $m+n=N$. There is a morphism

$$F_{N^*(\Delta^m\times\Delta^n)} \to N^*(\Delta^m)\otimes N^*(\Delta^n)$$

which makes all functoriality diagrams commute for pairs of morphisms $(u:[m']\to[m], v:[n']\to[n])$ and $(u:[m]\to[m'], v:[n]\to[n'])$ such that $m'+n'<N$.

Proof:

The given morphisms $F_{N^*(\Delta^{m'}\times\Delta^{n'})} \to N^*(\Delta^{m'})\otimes N^*(\Delta^{n'})$ determine a morphism from the latching algebra $F_{N^*(\Delta^{m'}\times\Delta^{n'})}$ to $N^*(\Delta^m)\otimes N^*(\Delta^n)$. Precisely, we consider the composite

$$\operatorname*{colim}_{(u,v)} F_{N^*(\Delta^{m'}\times\Delta^{n'})} \to \operatorname*{colim}_{(u,v)} N^*(\Delta^{m'})\otimes N^*(\Delta^{n'}) \to N^*(\Delta^m)\otimes N^*(\Delta^n).$$

Similarly, we have a morphism from the cofibrant algebra $F_{N^*(\Delta^m\times\Delta^n)}$ to the matching algebra $\lim_{(u,v)} N^*(\Delta^{m'})\otimes N^*(\Delta^{n'})$. In this case, we consider the composite

$$F_{N^*(\Delta^m\times\Delta^n)} \to \lim_{(u,v)} F_{N^*(\Delta^{m'}\times\Delta^{n'})} \to \lim_{(u,v)} N^*(\Delta^{m'})\otimes N^*(\Delta^{n'}).$$

Finally, we obtain a commutative diagram

$$\begin{array}{ccc}
\operatorname{colim}_{(u,v)} F_{N^*(\Delta^{m'}\times\Delta^{n'})} & \longrightarrow & N^*(\Delta^m)\otimes N^*(\Delta^n) \\
\downarrow & \nearrow & \downarrow \\
F_{N^*(\Delta^m\times\Delta^n)} & \longrightarrow & \lim_{(u,v)} N^*(\Delta^{m'})\otimes N^*(\Delta^{n'})
\end{array}$$

We deduce the existence of a fill-in morphism from the results of lemma 5.8. We claim that this morphism satisfies our requirements.

Precisely, by construction, the functoriality diagrams

$$\begin{array}{ccc}
F_{N^*(\Delta^{m'}\times\Delta^{n'})} & \longrightarrow & N^*(\Delta^{m'})\otimes N^*(\Delta^{n'}) \\
\downarrow & & \downarrow \\
F_{N^*(\Delta^m\times\Delta^n)} & \longrightarrow & N^*(\Delta^m)\otimes N^*(\Delta^n)
\end{array}$$

commute for pairs of surjective morphisms $u : [m] \to [m']$ and $v : [n] \to [n']$. The general case follows from the induction assumption, because a morphism in the simplicial category has a factorization $u = u'u''$, where u' is injective and u'' is surjective. Similarly, we prove that our morphism $F_{N^*(\Delta^m \times \Delta^n)} \to N^*(\Delta^m) \otimes N^*(\Delta^n)$ makes all functoriality diagrams commute for pairs of morphisms $u : [m'] \to [m]$ and $v : [n'] \to [n]$, such that $m' + n' < N$.

This achieves the proof of lemma 5.9.

5.10. *Construction of the homotopy Eilenberg-Zilber equivalence*

We consider the category formed by pairs of morphisms $\sigma : \Delta^m \to X$ and $\tau : \Delta^n \to Y$. We have an induced morphism $(\sigma \times \tau)^* : F_{X \times Y} \to F_{\Delta^m \times \Delta^n}$ for each pair $\sigma \times \tau : \Delta^m \times \Delta^n \to X \times Y$. We consider the composites of these morphisms with the natural transformation $F_{\Delta^m \times \Delta^n} \to N^*(\Delta^m) \otimes N^*(\Delta^n)$ supplied by lemma 5.9. Since our constructions are all functorial, we obtain a natural morphism

$$F_{X \times Y} \longrightarrow \lim_{(\sigma,\tau)} N^*(\Delta^m) \otimes N^*(\Delta^n).$$

We recall that $X = \operatorname{colim}_{\sigma:\Delta^m \to X} \Delta^m$ and $Y = \operatorname{colim}_{\tau:\Delta^n \to Y} \Delta^n$ (*cf.* [10]). Consequently, we have

$$N^*(X) \widehat{\otimes} N^*(Y) = \lim_{(\sigma,\tau)} N^*(\Delta^m) \otimes N^*(\Delta^n)$$

and the construction above provides a morphism $F_{X \times Y} \to N^*(X) \widehat{\otimes} N^*(Y)$, which is functorial in X and Y. For $X = \Delta^m$ and $Y = \Delta^n$, this morphism agrees clearly with the map $F_{\Delta^m \times \Delta^n} \to N^*(\Delta^m) \otimes N^*(\Delta^n)$ defined by lemma 5.9. In particular, for $X = Y = \Delta^0$, we can assume that our map $F_{\Delta^0 \times \Delta^0} \to N^*(\Delta^0) \otimes N^*(\Delta^0)$ is given by the augmentation morphism $F_{\mathbf{F}} \to \mathbf{F}$.

The proof of lemma 5.2 is complete.

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Acknowledgements: I have learned about Mandell's work from a course given by Paul Goerss and I am grateful to Ezra Getzler and Paul Goerss for their invitation at Northwestern University in spring 1999.

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