THE UNIVERSAL HOPF OPERADS OF THE BAR CONSTRUCTION

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ABSTRACT. The goal of this memoir is to prove that the bar complex B(A) of an E-infinity algebra A is equipped with the structure of a Hopf E-infinity algebra, functorially in A. We observe in addition that such a structure is homotopically unique provided that we consider unital operads which come equipped with a distinguished 0-ary operation that represents the natural unit of the bar complex. Our constructions rely on a Reedy model category for unital Hopf operads. For our purpose we define a unital Hopf endomorphism operad which operates functorially on the bar complex and which is universal with this property. Then we deduce our structure results from operadic lifting properties. To conclude this memoir we hint how to make our constructions effective and explicit.

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Introduction

This memoir is a sequel of the article [B1] in which we proved that the classical bar complex $B(A)$ can be equipped with the structure of an E_∞-algebra if A is an E_∞ algebra.

To be precise we consider operads in the category of differential graded modules (dg-modules for short) and an E_∞-operad refers to a dg-operad $\mathcal{E}$ equivalent to the operad of associative and commutative algebras $\mathcal{C}$. An E_∞-algebra refers to an algebra over any given E_∞-operad. Similarly one considers A_∞-operads, defined as dg-operads equivalent to the operad of associative algebras $\mathcal{A}$, and A_∞-algebras, defined as algebras over an A_∞-operad. For our purpose we fix a specific model $\mathcal{K}$ of an A_∞-operad. Namely we consider the classical chain operad of Stasheff's associahedra. The classical notion of an A_∞-algebra, defined by a collection of higher associative products $\mu_r : A^{\otimes r} \to A$, is equivalent to the structure of an algebra over this A_∞ operad. Recall that the bar complex is defined precisely for algebras equipped with such operations. For any E_∞-operad $\mathcal{E}$ there exists a homotopically unique morphism $\mathcal{K} \to \mathcal{E}$. Once we fix such a morphism any $\mathcal{E}$-algebra is provided with the structure of a $\mathcal{K}$-algebra and hence has an associated bar complex $B(A)$.

In [B1] we consider only the chain structure of the bar complex $B(A)$. But, classically, one identifies $B(A)$ with the tensor coalgebra generated by $\Sigma\bar{A}$, the suspension of the augmentation ideal of A. One observes in addition that the bar differential $\partial : B(A) \to B(A)$ is defined by a coalgebra coderivation so that the bar complex $B(A)$ forms a dg-coalgebra. Therefore a natural aim consists in extending the constructions of [B1] in the framework of dg-coalgebras.

For this purpose we consider operads in the ground symmetric monoidal category of dg-coalgebras, usually called Hopf operads, and algebras over operads in this category. The algebras over a Hopf operad $\mathcal{P}$ in the category of dg-coalgebras are usually referred to as Hopf $\mathcal{P}$-algebras. In fact, one can incidentally forget coalgebra structures and consider algebras over Hopf operads in the underlying category of dg-modules and similarly for other structures. Therefore, as a general rule, the objects defined in the category of dg-coalgebras are referred to by the qualifier Hopf. Otherwise we consider tacitely an underlying structure in the category of dg-modules.

For a commutative algebra the classical shuffle product of tensors defines a morphism of differential graded coalgebras $\smile: B(A) \otimes B(A) \to B(A)$. Consequently, the bar complex of a commutative algebra $B(A)$ is equipped with the structure of a commutative Hopf algebra. The goal of this memoir is precisely to extend this structure result to E_∞-algebras.

Sketch of the memoir results. Recall briefly that a Hopf operad $\mathcal{P}$ is defined by a collection of dg-coalgebras $\mathcal{P}(r)$, acted on by the symmetric group Σ_r, equipped with operad composition products $\mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r)$ which are morphisms in the category of coalgebras. Similarly, a Hopf $\mathcal{P}$-algebra consists of a dg-coalgebra Γ equipped with evaluation products $\mathcal{P}(r) \otimes \Gamma^{\otimes r} \to \Gamma$ which are morphisms in the category of coalgebras (we refer to §1.5 for detailed recalls).

Observe that the bar coalgebra $\Gamma = B(A)$ is equipped with a natural unit morphism $\mathbb{F} \to B(A)$. Therefore, for the purposes of this memoir, it is natural to consider unital Hopf operads, equipped with a distinguished unital operation $* \in \mathcal{P}(0)$, and operad actions $\mathcal{P}(r) \otimes B(A)^{\otimes r} \to B(A)$ for which the unital operation $* : \mathbb{F} \to B(A)$ agrees with the natural unit of the bar complex $\mathbb{F} \to B(A)$. This unit requirement gives the analogue of a boundary condition for the construction of operad actions on $B(A)$. Indeed, if we restrict ourself to non-negatively graded Hopf operads, then the unit requirement implies the assumption of the uniqueness theorem of [B1].

As in [B1] we consider also connected operads which, in the unital context, satisfy $\mathcal{P}(0) = \mathbb{F}$ and $\mathcal{P}(1) = \mathbb{F}$.

To recapitulate, we suppose given an E_∞-operad $\mathcal{E}$ in the category of dg-modules, equipped with a fixed operad morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ denotes Stasheff's chain operad, and we consider the bar coalgebra $B(A)$ for A an algebra over $\mathcal{E}$. Then let $\mathcal{Q}$ denote a unital non-negatively graded Hopf E_∞-operad. Our main goal is to prove the following theorem:

Theorem A.

(a) *The bar complex of an $\mathcal{E}$-algebra $B(A)$ can be equipped with the structure of a Hopf $\mathcal{Q}$-algebra functorially in A and so that the unital operation $\mathcal{Q}(0) \to B(A)$ agrees with the natural unit of the bar complex $\mathbb{F} \to B(A)$ provided that $\mathcal{Q}$ is a Reedy cofibrant object in the category of unital Hopf E_∞-operads.*

(b) *Any such $\mathcal{Q}$-algebra structure where $\mathcal{Q}$ is connected and non-negatively graded satisfies the requirement of the uniqueness theorem of [B1]. More explicitly, if the unit condition of claim (a) is satisfied and the operad $\mathcal{Q}$ is connected and non-negatively graded, then, for a commutative algebra A, the $\mathcal{Q}$-algebra structure of $B(A)$ reduces automatically to the classical commutative algebra structure of $B(A)$, the one given by the shuffle product of tensors.*

The proof of the existence claim (a) follows the same lines of argument as in the framework of dg-modules. Namely we introduce first a universal unital Hopf operad, the Hopf endomorphism operad of the bar construction $\mathrm{HopfEnd}_B^{\mathcal{P}}$, which operates functorially on the bar complex of algebras over a given operad $\mathcal{P}$. More precisely, we prove the following result:

Theorem B. *Let $\mathcal{P}$ denote an operad (in dg-modules) equipped with an operad morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ denotes Stasheff's chain operad. There is a universal unital Hopf operad $\mathcal{Q} = \mathrm{HopfEnd}_B^{\mathcal{P}}$ such that the bar complex of a $\mathcal{P}$-algebra $B(A)$ is equipped with the structure of a Hopf algebra over $\mathcal{Q}$, functorially in $A \in \mathcal{P}\,\mathrm{Alg}$.*

More precisely, the Hopf operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ operates on the coalgebra $B(A)$ functorially in $A \in \mathcal{P}\,\mathrm{Alg}$ and so that the unital operation $* : \mathbb{F} \to B(A)$ agrees with the unit of $B(A)$. Furthermore, we have a one-to-one correspondence between such Hopf operad actions and morphisms of unital Hopf operads $\rho : \mathcal{Q} \to \mathrm{HopfEnd}_B^{\mathcal{P}}$.

For the sake of completeness, we should point out that the map $\mathcal{P} \mapsto \mathrm{HopfEnd}_B^{\mathcal{P}}$ defines a functor on the category of operads under $\mathcal{K}$.

As in the context of dg-modules, any coalgebra Γ has an associated endomorphism Hopf operad defined by

$$\mathrm{HopfEnd}_\Gamma(r) = \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$$

where $\mathit{HopfHom}(K, L)$ denotes an appropriate internal hom-object in the category of coalgebras. By definition a Hopf $\mathcal{Q}$-algebra structure is equivalent to a Hopf operad morphism $\mathcal{Q} \to \mathrm{HopfEnd}_\Gamma$. One can adapt this construction for unital coalgebras so that $\mathrm{HopfEnd}_\Gamma$ forms a unital Hopf operad and a morphism $\rho : \mathcal{Q} \to \mathrm{HopfEnd}_\Gamma$ preserves the distinguished unital operations if and only if in the equivalent Hopf $\mathcal{Q}$-algebra structure the unital operation $* : \mathbb{F} \to \Gamma$ agrees with the unit of Γ.

The Hopf endomorphism operad of the bar construction is defined by the coend of the bifunctors

$$\mathrm{HopfEnd}_{B(A)}(r) = \mathit{HopfHom}(B(A)^{\otimes r}, B(A))$$

where A ranges over the category of $\mathcal{P}$-algebras. The assertions of theorem B are immediate from this construction. Notice that this endomorphism operad is $\mathbb{Z}$-graded and non-connected, like any endomorphism object.

The classical commutative algebra structure of the bar complex gives a morphism $\nabla_c : \mathcal{C} \to \mathrm{HopfEnd}_B^{\mathcal{C}}$ and the existence assertion of theorem A is equivalent to the lifting problem:

$$\begin{tikzcd} & & \mathrm{HopfEnd}_B^{\mathcal{E}} \arrow[d] \\ \mathcal{Q} \arrow[r] \arrow[urr, dashed, "\exists ?"] & \mathcal{C} \arrow[r, "\nabla_c"] & \mathrm{HopfEnd}_B^{\mathcal{C}} \end{tikzcd} .$$

As in [B1] we introduce another universal operad $\mathrm{HopfOp}_B^{\mathcal{P}}$, the Hopf operad of natural operations of the bar complex, that forms a suboperad of $\mathrm{HopfEnd}_B^{\mathcal{P}}$ and that agrees with this one only if the ground field is infinite. In general this operad is endowed with better homotopical properties than the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$. More specifically, we prove the following property:

Theorem C. *The functor $\mathcal{P} \mapsto \mathrm{HopfOp}_B^{\mathcal{P}}$ maps a fibration, respectively an acyclic fibration, of (non-unital) operads under $\mathcal{K}$ to a Reedy fibration, respectively an acyclic Reedy fibration, of unital Hopf operads.*

In this statement and in theorem A we refer to a particular model structure on the category of Hopf operads, the Reedy model structure, in which we have cofibrations, fibrations and weak-equivalences endowed with the classical lifting properties. In fact, we introduce a new model structure that differs from the general adjoint model structures considered in [4] and which is more appropriate in the unital context. Usually, for a solvable operadic lifting problem

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "i"'] & \mathcal{P} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "g"] \arrow[ur, dashed, "h"] & \mathcal{Q} \end{tikzcd} ,$$

the lifting $h : \mathcal{B}(r) \to \mathcal{P}(r)$ can be constructed effectively by induction on $r \in \mathbb{N}$. In the unital context one can observe that the operadic composites with a unital operation $* \in \mathcal{P}(0)$ provide the underlying Σ_*-module of a unital operad $\mathcal{P}$ with operations $\partial_i : \mathcal{P}(r) \to \mathcal{P}(r-1)$ that decrease the operadic arity. Consequently, in the inductive construction, one can assume that the composites $\mathcal{B}(r) \xrightarrow{h} \mathcal{P}(r) \xrightarrow{\partial_i} \mathcal{P}(r-1)$ are specified so that the lift-component $h : \mathcal{B}(r) \to \mathcal{P}(r)$ is indeed obtained by lifting a matching morphism

$$(\mu, p) : \mathcal{P}(r) \to \mathbb{M}\mathcal{P}(r) \times_{\mathbb{M}\,\mathcal{Q}(r)} \mathcal{Q}(r)$$

for a natural notion of matching objects associated to unital operads. Accordingly, for our purposes it is natural to let an operad morphism $p : \mathcal{P} \to \mathcal{Q}$ be a fibration if for all $r \in \mathbb{N}$ the matching morphism $(\mu, p) : \mathcal{P}(r) \to \mathbb{M}\mathcal{P}(r) \times_{\mathbb{M}\,\mathcal{Q}(r)} \mathcal{Q}(r)$ is a fibration in the underlying ground category (in the category of dg-coalgebras for Hopf operads, dg-modules otherwise). We call this class of fibrations the Reedy fibrations in order to distinguish them from the classical fibrations. We have also a class of Reedy cofibrations characterized by the left lifting property with respect to acyclic Reedy fibrations as usual, where we consider the usual weak-equivalences. We prove precisely that these class of morphisms define a model structure on unital operads.

To be precise, we prove the axioms of a model structure only for non-negatively graded unital Hopf operads. But we mention that an endomorphism operad $\mathrm{HopfOp}_B^{\mathcal{P}}$

is $\mathbb{Z}$-graded like any endomorphism object. In fact, we extend abusively the notion of a fibration, respectively of an acyclic fibration, to such operads by the left-lifting property requirement. Precisely, we let a morphism of (possibly $\mathbb{Z}$-graded) unital Hopf operads be a fibration, respectively acyclic fibration, if it satisfies the left-lifting property with respect to acyclic cofibrations, respectively cofibrations, of non-negatively graded unital Hopf operads. Equivalently, we have a truncation functor tr_+ which gives a right adjoint to the category embedding of $\mathbb{N}$-graded unital Hopf operads into the category of all $\mathbb{Z}$-graded unital Hopf operads. Abusively, we let a morphism of unital Hopf operads be a fibration, respectively an acyclic fibration, if its truncation defines a fibration, respectively an acyclic fibration, in the category of non-negatively graded unital Hopf operads.

Finally, the existence claim of theorem A is a corollary of theorem C. Explicitly, we observe that the Hopf operad morphism $\nabla_c : \mathcal{C} \to \mathrm{HopfEnd}_B^{\mathcal{C}}$ associated to the classical shuffle algebra structure factorizes through $\mathrm{HopfOp}_B^{\mathcal{C}}$. Then we consider the lifting problem

$$\begin{tikzcd} & & \mathrm{HopfOp}_B^{\mathcal{E}} \arrow[r] \arrow[d, "\sim"] & \mathrm{HopfEnd}_B^{\mathcal{E}} \arrow[d] \\ \mathcal{Q} \arrow[r] \arrow[urr, dashed, "\exists"] & \mathcal{C} \arrow[r, "\nabla_c"] & \mathrm{HopfOp}_B^{\mathcal{C}} \arrow[r] & \mathrm{HopfEnd}_B^{\mathcal{C}} \end{tikzcd}$$

which has automatically a solution if $\mathcal{Q}$ is cofibrant as the augmentation of an E_∞-operad $\mathcal{E} \xrightarrow{\sim} \mathcal{C}$ induces an acyclic fibration of unital Hopf operads

$$\mathrm{HopfOp}_B^{\mathcal{E}} \xrightarrow{\sim} \mathrm{HopfOp}_B^{\mathcal{C}}$$

by theorem C.

Notice that the commutative operad forms a final object in the category of Hopf operads. For the uniqueness claim we observe that any morphism of unital Hopf operads $\nabla : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{C}}$ where $\mathcal{Q}$ is connected and non-negatively graded makes the diagram

$$\begin{tikzcd} \mathcal{Q} \arrow[d] \arrow[dr, "\nabla"] & \\ \mathcal{C} \arrow[r, "\nabla_c"'] & \mathrm{HopfOp}_B^{\mathcal{C}} \end{tikzcd} .$$

commute. As a consequence, for a given E_∞-operad $\mathcal{E}$, any morphism of unital Hopf operads $\nabla : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$ such that $\mathcal{Q}$ is connected and non-negatively graded defines a lifting of $\nabla_c : \mathcal{C} \to \mathrm{HopfOp}_B^{\mathcal{C}}$.

Toward effective constructions. The lifting process can be made effective for certain E_∞ operads. Indeed we observed in [11] that the bar complex of an algebra over the so-called surjection operad $\mathcal{X}$ forms a Hopf algebra over the Barratt-Eccles operad $\mathcal{E}$. This structure result gives an explicit instance of a morphism $\mathcal{E} \to \mathrm{HopfOp}_B^{\mathcal{X}}$. Recall that the surjection operad is an instance an E_∞-operad in the category of dg-modules and the Barratt-Eccles operad is an instance of a Hopf E_∞-operad. We are particularly interested in these operads as, on one hand, we proved in [3] (see also [19]) that the surjection operad, as well as the Barratt-Eccles operad, operates on the cochain complex of simplicial sets $C^*(X)$, and, on the other hand, we observed in [B1] that the bar complex $B(C^*(X))$ is equivalent as an E_∞ algebra to $C^*(\Omega X)$, the cochain complex of the loop space of X.

Nevertheless the Barratt-Eccles operad is not cofibrant and theorem A does not apply to this operad. One could check that the bar complex of an algebra over the Barratt-Eccles operad is not acted on by the Barratt-Eccles operad itself.

Therefore we consider the so-called Boardman-Vogt' construction $\mathcal{Q} = W(\mathcal{E})$ which provides an explicit cofibrant replacement of the Barratt-Eccles operad and for which theorem A applies. Indeed for the Barratt-Eccles operad $\mathcal{E}$ a lifting $\nabla : W(\mathcal{E}) \to \mathrm{HopfOp}_B^{\mathcal{E}}$ can be constructed by an explicit induction process so that we can obtain effectively a Hopf $W(\mathcal{E})$-algebra structure on the bar complex of algebras over the Barratt-Eccles operad. This morphism seems to factorize through a simplicial decomposition $W_\Delta = W_\Delta(\mathcal{E})$ (in fact the chain operad associated to the simplicial Boardman-Vogt construction of the simplicial Barratt-Eccles operad) of the Boardman-Vogt dg-operad $W_\square = W(\mathcal{E})$. To be more precise, we seem to have closed formulas for a morphism $\nabla_\delta : W_\Delta \to \mathrm{HopfOp}_B^{W_\Delta}$ that fits a commutative diagram

$$\begin{tikzcd} W_\Delta \arrow[r, "\nabla_\delta"] & \mathrm{HopfOp}_B^{W_\Delta} \arrow[d] \\ W_\square \arrow[u] \arrow[r, "\nabla"] & \mathrm{HopfOp}_B^{\mathcal{E}} \end{tikzcd} .$$

This result, obtained as an application of explicit formulas given in this memoir, should be confirmed and published in a subsequent article.

Further prospects. The functorial constructions addressed in this memoir and in the previous article have the drawback to yield only global formulas, valid for all algebras in a category. But one would like to control the structure of the bar construction for algebra subclasses or for a particular subclass of operations. Our idea to address this issue is to introduce cellular operads obtained by a limit-colimit decomposition of an E_∞-operads. To motivate this idea, observe that a quotient $\mathcal{Q}$ of an operad $\mathcal{P}$ is associated to a subcategory of the category of $\mathcal{P}$-algebras; a suboperad $\mathcal{R}$ is associated to a larger category of algebras as the $\mathcal{R}$-algebras support less operations. To generalize this process, we shall consider functor operads (in the sense of [20]) on an operad in the category of categories. In the text, we give a few indications on how our constructions might be extended to this context.

The crux is to provide the operad of universal bar operations with appropriate cellular structures. As such, the cell categories that one would like to consider have to be motivated by the internal structure of the operad of universal bar operation itself. A good example includes the G-cellular operads of [1, 2] that give rise to sequences of E_n-operads.

Memoir organization. The first part of this memoir (sections §1-§2) is of a general interest. In this part, we give a comprehensive account of Reedy model categories of unital operads. The general theory is set in §1. The main theorems regarding the Reedy model category of unital operads in dg-modules and dg-coalgebras, the model categories used in the memoir, are also stated in this section. Section §2 is devoted to the Boardman-Vogt construction, a construction which returns explicit cofibrant replacements in operad categories. Though we do not use this construction explicitly in the memoir, we give a detailed account of it for the sake of completeness and for subsequent references.

The goal of the second part of the memoir (sections §1-§2), is to define precisely the universal Hopf operads introduced in the memoir introduction and to study the internal structure of these objects. The first section of this part, section §3, is still of general interests: we define an appropriate notion of cocellular objects in order to obtain classes of effective fibrations in the context of $\mathbb{Z}$-graded dg-coalgebras. The core of the memoir is formed by section §4 in which we prove our main theorem sketched in the introduction. Namely: we define the internal hom object $\mathit{HopfHom}(K, L)$ of coalgebras; we prove the existence of the universal

Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$; we define the suboperad of universal bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$ and we study the structure of these objects; more specifically, we prove that the functor $\mathcal{P} \mapsto \mathrm{HopfOp}_B^{\mathcal{P}}$ preserves fibrations and acyclic fibrations; then we prove our main existence and uniqueness theorem.

In the last part of this memoir (section §5), we give an elementary interpretation of the structure results obtained in §4. More particularly, we make explicit a recursive construction that yields an effective E_∞-structure on the bar complex. For motivations, we suggest the reader to have a glance at the introduction of this section and theorems §5.D-§5.E before to start a thorough reading of the memoir.

We refer to the section introductions for detailed sketches of the content of each part.

The operadic framework

§1. On unital and Hopf operads

§1.1. **Introduction and general conventions.** In general the objects that we consider belong to the category of dg-modules over a ground ring $\mathbb{F}$, always assumed to be a field, and fixed once and for all.

We adopt the conventions of [B1] except that in this memoir we may find more convenient to deal with *augmented unitary algebras* instead of *non-unitary algebras*. To be precise, as we equip the bar construction $B(A)$ with a Hopf algebra structure, we find better to assume $B_0(A) = \mathbb{F}$ so that the functor $A \mapsto B(A)$ targets to a category of augmented unitary objects. On the other hand, the complex $B(A)$ is more naturally defined for a non-unitary algebra A. Therefore we shall assume on the contrary that the source of the functor $A \mapsto B(A)$ lives in a category of non-unitary objects.

Anyway, as stated in the memoir introduction, we shall consider *unital operads* $\mathcal{P}$ which come equipped with a distinguished *unital operation* $* \in \mathcal{P}(0)$ that generates $\mathcal{P}(0)$ so that $\mathcal{P}(0) = \mathbb{F}*$ (for an operad in a module category, the usual case for the operads considered in this memoir). As explained in [B1, §3.2.1], this assumption $\mathcal{P}(0) = \mathbb{F}$ implies that the ground field $\mathbb{F}$ defines the initial object in the category of $\mathcal{P}$-algebras. In this context an augmented $\mathcal{P}$-algebra refers simply to an algebra A equipped with a fixed $\mathcal{P}$-algebra morphism $\epsilon : A \to \mathbb{F}$. The kernel of this augmentation morphism defines the augmentation ideal of A, denoted by $\bar{A}$. Recall that $\bar{A}$ is an algebra over the *reduced operad* $\overline{\mathcal{P}}$ such that $\overline{\mathcal{P}}(r) = 0$ if $r = 0$ and $\overline{\mathcal{P}}(r) = \mathcal{P}(r)$ otherwise. Furthermore, the map $A \mapsto \bar{A}$ defines an equivalence of categories from the category of augmented $\mathcal{P}$-algebras to the category of $\overline{\mathcal{P}}$-algebras.

On the other hand, we consider always *unitary operads* $\mathcal{P}$, equipped with a unit operation $1 \in \mathcal{P}(1)$. In the literature, the unital and unitary terminologies are used interchangeably. The reader should not be confused: in this memoir, we use always these terminologies in the sense specified in this introduction.

As in [B1], we may consider connected operads for which the module $\mathcal{P}(1)$ is spanned by $1 \in \mathcal{P}(1)$. In general the connectedness assumption is not necessary but it can simplify certain constructions.

Naturally, a morphism of unital operads is assumed to preserve the unital operation $* \in \mathcal{P}(0)$. Accordingly, there is an initial object in the category of unital operads $*$ defined explicitly by $*(r) = \mathbb{F}$ for $r = 0, 1$ and $*(r) = 0$ otherwise. The category of augmented algebras over this initial operad is equivalent to the category of dg-modules since any augmented algebra over $*$ has the form $V_+ = \mathbb{F} \oplus V$ for a dg-module V. Observe that the category of unital operads is also endowed with a terminal object given by the commutative operad $\mathcal{C}$ (see §1.2.1).

The aim of this section is to fix our conventions in regard to unital and Hopf operads and to prove the fundamental properties of the categories formed by these objects. More specifically, as explained in the memoir introduction, we introduce a new model structure for unital operads which gives an appropriate framework for the constructions of Hopf operad actions on the bar complex. Recall briefly that our model category structure is deduced from a generalization of the classical Reedy constructions and differs from the model structure defined in [4] like classical Reedy model structures differ from adjoint model structures.

Here is the plan of this section. For our purposes we need more insights into the structure of the underlying Σ_*-module of a unital-operad and we devote the next subsection §1.2 to this topic. Then we perform the construction of the Reedy model structure of unital operads in two steps: first, in §1.3 we define a model structure at the Σ_*-module level; then in §1.4 we transfer this structure to unital operads by the classical adjunction process. In the last subsection §1.5 we generalize these constructions to unital Hopf operads after recalls and precisions on this notion.

§1.2. **On Λ_*-modules and unital operads.** Classically, one observes that the operadic composites with a unital operation $\partial_i(p) = p \circ_i *$, defined for $i = 1, \dots, r$ if $p \in \mathcal{P}(r)$, provide the underlying Σ_*-module of a unital operad $\mathcal{P}$ with morphisms $\partial_i : \mathcal{P}(r) \to \mathcal{P}(r-1)$ such that $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$. The structure of a *preoperad*, introduced in [1], consists precisely of a Σ_*-module M equipped with such morphisms $\partial_i : M(r) \to M(r-1)$. As explained in *loc. cit.* one can observe that the structure of a preoperad is equivalent to a contravariant functor $M : \Lambda_*^{\mathrm{op}} \to \mathrm{Mod}$, where Λ_* denotes the category formed by the finite sets $\underline{r} = \{1, \dots, r\}$ and the injective maps $u : \{1, \dots, r\} \to \{1, \dots, s\}$. In this memoir we prefer to adopt the terminology of a Λ_*-*module* for this structure which is more consistent with our terminology of a Σ_*-*module* for a collection of Σ_r-modules. Similarly, the set of morphisms $u : \{1, \dots, r\} \to \{1, \dots, s\}$ in Λ_* is denoted by Λ_r^s so that $\Lambda_r^r = \Sigma_r$. In this memoir, we consider tacitely only right Λ_*-modules, that are contravariant functors on the category Λ_*, in contrast to left Λ_*-modules that are covariant functors. Accordingly, the Λ_*-action on a Λ_*-module restricts naturally to a right action of the symmetric groups Σ_*. On the other hand, we assume by convention that a Σ_*-module is acted on by permutations on the left, but left and right actions are equivalent for group actions and this convention difference does not create any actual difficulty.

In this subsection we make explicit the relationship between Σ_*-modules and Λ_*-modules and between non-unital operads and unital operads. To be precise, we consider the following categories:

(a) the category $\Sigma_* \mathrm{Mod}_0$ formed by the *non-unital* Σ_*-*modules* – the Σ_*-modules M such that $M(0) = 0$; the category $\Sigma_* \mathrm{Mod}_0^1$ formed by the unitary objects of $\Sigma_* \mathrm{Mod}_0$ – a unitary object in $\Sigma_* \mathrm{Mod}_0$ consists of a Σ_*-module $M \in \Sigma_* \mathrm{Mod}_0$ equipped with a unit element $1 \in M(1)$; and the category $\Sigma_* \mathrm{Mod}_0^1 / \overline{\mathcal{C}}$ formed by unitary objects $M \in \Sigma_* \mathrm{Mod}_0^1$ equipped with an augmentation $\epsilon : M \to \overline{\mathcal{C}}$ over the underlying Σ_*-module of the reduced commutative operad $\overline{\mathcal{C}}$;

(b) the category $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0$ formed by the *non-unital* Λ_*-*modules* – the Λ_*-modules M such that $M(0) = 0$; the category $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0^1$ formed by the unitary objects of $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0$ – a unitary object in $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0$ consists of a Λ_*-module equipped with a unit element $1 \in M(1)$; and the category $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0^1 / \overline{\mathcal{C}}$ formed by unitary objects $M \in \Sigma_* \mathrm{Mod}_0^1$ equipped with an augmentation $\epsilon : M \to \overline{\mathcal{C}}$;

(c) the category Op_0^1 formed by the *non-unital unitary operads* $\mathcal{P}$ – the unitary operads $\mathcal{P}$ such that $\mathcal{P}(0) = 0$; and the category $\mathrm{Op}_0^1 / \overline{\mathcal{C}}$ formed by the operads $\mathcal{P} \in \mathrm{Op}_0^1$ equipped with an augmentation $\epsilon : M \to \overline{\mathcal{C}}$ over the reduced commutative operad $\overline{\mathcal{C}}$;

(d) the category Op_*^1 formed by the *unital unitary operads* $\mathcal{P}$ – the unitary operads $\mathcal{P}$ such that $\mathcal{P}(0) = \mathbb{F} *$ for a distinguished operation $* \in \mathcal{P}(0)$.

Observe that the unit of a Σ_*-module is equivalent to a Σ_*-module morphism $\eta : I \to M$, where $*$ denotes the non-unital Σ_*-module such that $I(1) = \mathbb{F}$ and $I(r) = 0$ for $r \neq 1$. Accordingly, the category of augmented unitary Σ_*-modules $\Sigma_* \mathrm{Mod}_0^* / \overline{\mathcal{C}}$ is defined by a comma category and similarly for the category of augmented unitary Λ_*-modules $\Lambda_*^{\mathrm{op}} \mathrm{Mod}_0^* / \overline{\mathcal{C}}$.

The connections between these categories can be summarized by a diagram of categorical adjunctions

$$\begin{tikzcd}
\Sigma_* \operatorname{Mod}_0^1 / \overline{\mathcal{C}} \arrow[r, shift left, "\mathcal{F}"] \arrow[d, shift right=2] \arrow[d, shift left=2] & \operatorname{Op}_0^1 / \overline{\mathcal{C}} \arrow[l, shift left] \arrow[d, shift right=2] \arrow[d, shift left=2] \\
\Lambda_*^{\mathrm{op}} \operatorname{Mod}_0^1 / \overline{\mathcal{C}} \arrow[u] \arrow[r, shift left, "\mathcal{F}_*"] & \operatorname{Op}_*^1 \arrow[l, shift left] \arrow[u]
\end{tikzcd} .$$

For the upper horizontal adjunction we consider the obvious forgetful functor which maps a non-unital operad $\mathcal{P}$ to its underlying Σ_*-module. The unit element is given by the unit operation $1 \in \mathcal{P}(1)$. This functor has a left-adjoint defined by the classical free operad functor $M \mapsto \mathcal{F}(M)$ – we refer to §1.2.2 for recalls and more precision. In our context this free object is equipped with a natural operad morphism $\epsilon : \mathcal{F}(M) \to \overline{\mathcal{C}}$ induced by the Σ_*-module augmentation $\epsilon : M \to \overline{\mathcal{C}}$ so that $\mathcal{F}$ induces a functor $\mathcal{F} : \Sigma_* \operatorname{Mod}_0^1 / \overline{\mathcal{C}} \to \operatorname{Op}_0^1 / \overline{\mathcal{C}}$. For the bottom horizontal adjunction we consider simply unital versions of these functors – we refer to §1.2.4 for explicit definitions. For vertical adjunctions we consider the canonical forgetful functor from Λ_*-modules to Σ_*-modules and the functor $\mathcal{P} \mapsto \overline{\mathcal{P}}$ which maps a unital operad $\mathcal{P}$ to the associated reduced operad $\overline{\mathcal{P}}$ defined in the introduction of this section. One observes readily that these functors preserve colimits and limits and hence have both a left and a right adjoint. At the module level this adjunction is an instance of a general adjunction relation

$$\begin{tikzcd}
\mathcal{R}_*^{\mathrm{op}} \operatorname{Mod} \arrow[r, shift left=3, "\phi_!"] \arrow[r, shift right=3, "\phi_*"'] & \mathcal{S}_*^{\mathrm{op}} \operatorname{Mod} \arrow[l, "\phi^! = \phi^*"']
\end{tikzcd}$$

associated to a morphism of small categories $\phi : \mathcal{R}_* \to \mathcal{S}_*$. Recall simply that the (right) $\mathcal{S}_*$-module associated to a (right) $\mathcal{R}_*$-module M, also denoted by $\phi_! M = M \otimes_{\mathcal{R}_*} \mathcal{S}_*$, respectively $\phi_* M = \mathit{Hom}_{\mathcal{R}_*}(\mathcal{S}_*, M)$, can be defined by a coend, respectively end, formula. Namely:

$$M \otimes_{\mathcal{R}_*} \mathcal{S}_*(y) = \int_{x \in \mathcal{R}_*} M(x) \otimes \mathcal{S}_y^{\phi(x)},$$

$$\text{respectively} \quad \mathit{Hom}_{\mathcal{R}_*}(\mathcal{S}_*, M)(y) = \int^{x \in \mathcal{R}_*} M(x)^{\mathcal{S}_y^{\phi(x)}},$$

where the notation $K \otimes V$, respectively V^K, refers to the classical tensor product, respectively mapping object, of a dg-module V with a set K. (As explained in [B1], our conventions for ends and coends are converse to the usual one: we use superscripts for ends and subscripts for coends.)

To begin our constructions we study the connection between the category of unital and non-unital operads.

§1.2.1. *Unitary operads versus non-unital operads.* In fact, if we reverse the definition of the reduced operad $\overline{\mathcal{P}}$, then, for a unital operad $\mathcal{P}$, we obtain the relation

$$\mathcal{P}(r) = \begin{cases} \mathbb{F} & \text{if } r = 0, \\ \overline{\mathcal{P}}(r) & \text{otherwise.} \end{cases}$$

Furthermore, the operad composition products $\circ_i : \mathcal{P}(s) \otimes \mathcal{P}(t) \to \mathcal{P}(s + t - 1)$ are determined for $s, t \geq 1$ by the composition products of the reduced operads $\overline{\mathcal{P}}$. As a consequence, the structure of $\mathcal{P}$ is completely determined by the associated reduced operad $\overline{\mathcal{P}}$ and by morphisms $\partial_i : \overline{\mathcal{P}}(r) \to \overline{\mathcal{P}}(r-1)$ that keep track on the composite with the unital operation $* \in \mathcal{P}(0)$ at the level of the reduced operad $\overline{\mathcal{P}}$.

Observe in addition that a unital operad comes equipped with augmentation morphisms $\epsilon : \mathcal{P}(r) \to \mathbb{F}$ which map an operation $p \in \mathcal{P}(r)$ to the composite $p(*, \dots, *) \in \mathcal{P}(0)$ and such that $\epsilon \partial_i = \epsilon$ for all i. As a byproduct, these augmentation morphisms give rise to an operad morphism $\epsilon : \mathcal{P} \to \mathcal{C}$, where $\mathcal{C}$ denotes the operad of commutative algebras (recall that $\mathcal{C}(r) = \mathbb{F}$ for $r \in \mathbb{N}$), so that any unital operad $\mathcal{P}$ is canonically augmented over the commutative operad $\mathcal{C}$. Hence the commutative operad $\mathcal{C}$ defines the terminal object in the category of unital operad as mentioned in the introduction of this section.

Finally, our observations prove that a unital operad $\mathcal{P}$ is equivalent to a non-unital operad $\overline{\mathcal{P}}$ equipped with morphisms $\partial_i : \overline{\mathcal{P}}(r) \to \overline{\mathcal{P}}(r-1)$ for $i = 1, \dots, r$ and $r \geq 2$, such that the associativity and commutativity properties of operad composition products are satisfied for the operations $\partial_i = - \circ_i *$, and with an augmentation morphism $\epsilon : \overline{\mathcal{P}} \to \mathbb{F}$ that preserves all operadic composites including the composites with the unital operation $*$. As a byproduct, one can prove that the functor $\mathcal{P} \mapsto \overline{\mathcal{P}}$ creates small colimits in the category of unital operads. Explicitly, for a diagram of unital operads $\mathcal{P}_\alpha$ one observes readily that the colimit $\operatorname{colim}_\alpha \overline{\mathcal{P}}_\alpha$ is equipped with canonical operations ∂_i and with an augmentation $\epsilon : \operatorname{colim}_\alpha \overline{\mathcal{P}}_\alpha \to \mathbb{F}$ induced by the corresponding operations of the unital operads $\mathcal{P}_\alpha$. As a consequence, this colimit is associated to a unital operad an this operad defines necessarily the colimit of the operads $\mathcal{P}_\alpha$ in the category of unital operads.

§1.2.2. *The adjunction between Σ_*-modules and non-unital operads.* As explained in the introduction of this subsection, we consider the obvious forgetful functor from the category of non-unital unitary operads Op_0^* to the category of non-unital unitary Σ_*-modules $\Sigma_* \operatorname{Mod}_0^*$ which maps an operad $\mathcal{P} \in \mathrm{Op}_0^*$ to its underlying Σ_*-module.

The left adjoint of this functor is given by a variant of the classical free operad $\mathcal{F}(M)$ defined in the literature. Namely, in the unitary context, the universal morphism $\eta : M \to \mathcal{F}(M)$ is supposed to map the unit of M to the unit of the free operad and hence to define a morphism in the category of unitary Σ_*-modules. Furthermore, the adjunction relation supposes that an operad morphism $\phi_f : \mathcal{F}(M) \to \mathcal{P}$ is associated to a Σ_*-module morphism $f : M \to \mathcal{P}$ that preserves unit elements. Equivalently, for the universal property, one assumes that the Σ_*-module morphisms $f : M \to \mathcal{P}$ that preserve unit elements have a unique factorization

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \mathcal{P} \\ & \mathcal{F}(M) \arrow[ur, dashed, "\phi_f"'] & \end{tikzcd}$$

such that $\phi_f : \mathcal{F}(M) \to \mathcal{P}$ is an operad morphism. Accordingly, the free operad associated to a unitary Σ_*-module has to be defined by a quotient of the classical free operad associated to a non-unitary Σ_*-module. Explicitly, for a unitary Σ_*-module M we make the internal unit $1 \in M(1)$ equivalent to the unit of the free operad in $\mathcal{F}(M)$. Up to this quotient process we refer to our article [12], from which we borrow our conventions, for an explicit construction of the free operad that follows closely the original constructions of [13, 15]. The definition of tree structures that occur in this construction is also recalled in §2.2 for the purposes of Boardman-Vogt' W-construction. Recall that $\mathcal{F}(M)$ can be defined simply by the modules spanned by formal expressions $(\dots((x_1 \circ_{i_2} x_2) \circ_{i_3} \dots x_{l-1}) \circ_{i_l} x_l$ which represent composites of generators $x_1 \in M(n_1), \dots, x_l \in M(n_l)$.

§1.2.3. *On unitary and non-unitary Σ_*-modules.* In fact, in our context, we could replace unitary Σ_*-modules by equivalent non-unitary Σ_*-modules. Explicitly, recall that we consider Σ_*-module which are augmented over the reduced commutative operad $\overline{\mathcal{C}}$. For such Σ_*-modules we have a canonical splitting $M(1) = \widetilde{M}(1) \oplus \mathbb{F}\,1$, where $\widetilde{M}(1)$ denotes the cokernel of the morphism $\eta : \mathbb{F} \to M(1)$ defined by the unit element $1 \in M(1)$. Indeed, since $\overline{\mathcal{C}}(1) = \mathbb{F}$, the morphism $\eta : \mathbb{F} \to M(1)$ has a canonical left-inverse given by the augmentation $\epsilon : M(1) \to \overline{\mathcal{C}}(1)$.

Consequently, if we let $\widetilde{M}$ denote the non-unitary Σ_*-module such that $\widetilde{M}(0) = 0$, $\widetilde{M}(1) = \operatorname{coker}(\eta : \mathbb{F} \to M(1))$ and $\widetilde{M}(r) = M(r)$ for $r \geq 1$, then the map $M \mapsto \widetilde{M}$ defines an equivalence from the category $\Sigma_* \operatorname{Mod}_0^* /\overline{\mathcal{C}}$, formed by unitary Σ_*-modules augmented over the reduced commutative operad, to the category $\Sigma_* \operatorname{Mod}_0 /\widetilde{\mathcal{C}}$, formed by non-unitary Σ_*-modules which are augmented over the augmentation ideal of the commutative operad. Furthermore, the free operad $\mathcal{F}(M)$ associated to a unitary Σ_*-module M is clearly isomorphic to the free operad $\mathcal{F}(\widetilde{M})$ associated to the equivalent non-unitary Σ_*-module $\widetilde{M}$.

For our purpose it is more natural to deal with unitary objects and we do not use this relationship. Nevertheless one can conclude immediately from these observations that the free operad functor $M \mapsto \mathcal{F}(M)$ satisfies the same homological properties as the free operad $\mathcal{F}(\widetilde{M})$. For instance, the functor $M \mapsto \mathcal{F}(M)$ maps weak-equivalences of unitary Σ_*-modules to weak-equivalences of operads.

§1.2.4. *The adjunction between Λ_*-modules and unital operads.* The functor $M \mapsto \mathcal{F}_*(M)$ considered in the introduction of this subsection is defined as the left adjoint of the functor $\mathcal{P} \mapsto \overline{\mathcal{P}}$ from the category of unital operads Op_*^1 to the category of augmented non-unital unitary Λ_*-modules $\Lambda_*^{\mathrm{op}} \operatorname{Mod}_0 /\overline{\mathcal{C}}$. The operad $\mathcal{F}_*(M)$ associated to a given non-unital module can also be characterized by the usual universal property. In fact, the reduced operad associated to $\mathcal{F}_*(M)$ can be identified with the usual free operad $\mathcal{F}(M)$ so that

$$\mathcal{F}_*(M)(r) = \begin{cases} \mathbb{F} & \text{if } r = 0, \\ \mathcal{F}(M)(r) & \text{otherwise.} \end{cases}$$

Accordingly, for $s, t \geq 1$ the composition products $\circ_i : \mathcal{F}_*(M)(s) \otimes \mathcal{F}_*(M)(t) \to \mathcal{F}_*(M)(s+t-1)$ are given by the usual formal composition process of the free operad. The composites with the unital operation $- \circ_i * : \mathcal{F}_*(M)(r) \to \mathcal{F}_*(M)(r-1)$ are induced by the morphisms $\partial_i : M(r) \to M(r-1)$ for $r \geq 2$ and by the augmentation morphism $\epsilon : M(r) \to \mathbb{F}$ for $r = 1$. Explicitly, for a generator $x \in M(r)$ we set $x \circ_1 * = \epsilon(x)$ if $r = 1$, $x \circ_i * = \partial_i(x)$ for $r \geq 2$ and we observe that these operations $- \circ_i * : M(r) \to \mathcal{F}_*(M)(r-1)$ have a unique extension to $\mathcal{F}_*(M)$ that satisfies the commutativity properties of operad composition products.

§1.2.5. *On unital Λ_*-modules.* Sometimes we can find convenient to consider *unital unitary Λ_*-modules* – explicitly, unital Λ_*-modules N such that $N(0) = \mathbb{F}*$ for a distinguished element $* \in N(0)$ such that $\partial_1(1) = *$. Equivalently, a unital Λ_*-module consists of a Λ_*-module N equipped with a unit morphism $\eta : * \to N$, where $*$ denotes the underlying Λ_*-module of the initial unital operad, such that $\eta : \mathbb{F} \to N(0)$ is iso. Clearly, the underlying Λ_*-module of a unital operad is unital.

Observe that a unital Λ_*-module N is equipped with a canonical augmentation $\epsilon : N \to \mathcal{C}$ like a unital operad. Explicitly, the map $p \mapsto p(*, \dots, *)$ defined for a unital operad in §1.2.1 can be identified with the Λ_*-module operation $\eta_0^* : N(r) \to N(0)$ associated to the initial map $\eta_0 : \emptyset \to \{1, \dots, r\}$. If we assume $N(0) = \mathbb{F}$, then this operation gives a morphism from N to the constant Λ_*-module $\mathbb{F}$, that represents the underlying Λ_*-module of the commutative operad $\mathcal{C}$.

The definition of the reduced operad can also be extended to unital Λ_*-modules. Explicitly, for a unital Λ_*-module N we consider the Λ_*-module $\overline{N}$ such that $\overline{N}(0) = 0$ and $\overline{N}(r) = N(r)$ for $r \geq 1$. Clearly, this object forms a non-unital unitary Λ_*-module. Since any unital Λ_*-module is augmented over $\mathcal{C}$, the map $N \mapsto \overline{N}$ defines a functor $(-)^- : \Lambda_*^{\mathrm{op}} \operatorname{Mod}_*^1 \to \Lambda_*^{\mathrm{op}} \operatorname{Mod}_0^1 /\overline{\mathcal{C}}$, where $\Lambda_*^{\mathrm{op}} \operatorname{Mod}_*^1$ denotes the category of unital unitary Λ_*-modules.

The functor $N \mapsto \overline{N}$ has clearly a left adjoint $M \mapsto M_+$. Explicitly, the unital Λ_*-module M_+ associated to an augmented non-unital Λ_*-module M is defined by $M_+(0) = \mathbb{F}$ and $M_+(r) = M(r)$ for $r \geq 1$. The operations $\partial_i : M_+(r) \to M_+(r)$ are induced by the corresponding operations of M and by the augmentation $\epsilon : M(1) \to \mathbb{F}$. Clearly, these functors yield inverse adjoint equivalences of categories:

$$(-)_+ : \Lambda_*^{\mathrm{op}} \operatorname{Mod}_0^1 \underset{\simeq}{\overset{\simeq}{\rightleftarrows}} \Lambda_*^{\mathrm{op}} \operatorname{Mod}_*^1 : (-)^- .$$

§1.2.6. *Connected unital operads.* As mentioned in the section introduction, we may consider *connected unital operads* $\mathcal{P}$ such that the module $\mathcal{P}(1)$ is spanned by the operad unit $1 \in \mathcal{P}(1)$. Let Op_*^* denote the category formed by these objects. The category embedding $\mathrm{i}_*^1 : \operatorname{Op}_*^* \to \operatorname{Op}_*^1$ preserves clearly colimits and limits and hence admits both a right and a left adjoint denoted by $\mathrm{s}_*^1 : \operatorname{Op}_*^1 \to \operatorname{Op}_*^*$ and $\mathrm{c}_*^1 : \operatorname{Op}_*^1 \to \operatorname{Op}_*^*$ respectively. We give an explicit construction of these functors in the next paragraph. We check in addition that the adjunction unit $\eta : \mathcal{P} \to \mathrm{s}_*^1 \mathrm{i}_*^1(\mathcal{P})$, respectively the adjunction augmentation $\epsilon : \mathrm{c}_*^1 \mathrm{i}_*^1(\mathcal{Q}) \to \mathcal{Q}$, is an isomorphism for all connected unital operad $\mathcal{P}$, respectively $\mathcal{Q}$. Consequently, we have an adjunction ladder:

$$\begin{tikzcd} \operatorname{Op}_*^* \arrow[r, "\mathrm{i}_*^1"] & \operatorname{Op}_*^1 \arrow[l, bend right, "\mathrm{c}_*^1"'] \arrow[l, bend left, "\mathrm{s}_*^1"] \end{tikzcd}$$

such that $\mathrm{c}_*^1 \mathrm{i}_*^1 = \mathrm{Id} = \mathrm{s}_*^1 \mathrm{i}_*^1$. In forthcoming constructions non-connected operads $\mathcal{Q}$ can be replaced by the associated connected object $\mathrm{s}_*^1(\mathcal{Q})$. Therefore we could restrict ourself to connected unital operads.

To conclude, the map $\mathcal{P} \mapsto \overline{\mathcal{P}}$ and the free operad $M \mapsto \mathcal{F}_*(M)$ restrict to adjoint functors

$$\mathcal{F}_* : \operatorname{dg} \Lambda_*^{\mathrm{op}} \operatorname{Mod}_0^* /\overline{\mathcal{C}} \rightleftarrows \operatorname{dg} \operatorname{Op}_*^* : (-)^-,$$

where $\Lambda_*^{\mathrm{op}} \operatorname{Mod}_0^*$ denotes the category formed by the connected non-unital unitary Λ_*-modules, the non-unital unitary Λ_*-modules M such that $M(0) = 0$ and $M(1) = \mathbb{F}$.

§1.2.7. *The adjunction between connected and non-connected unital objects.* We give an explicit construction of the connected operad $\mathrm{s}_*^1(\mathcal{Q}) \in \operatorname{Op}_*^*$, respectively $\mathrm{c}_*^1(\mathcal{P}) \in \operatorname{Op}_*^*$, associated to a unital operad $\mathcal{Q} \in \operatorname{Op}_*^1$, respectively $\mathcal{P} \in \operatorname{Op}_*^1$. In fact, we define an adjunction ladder for the underlying categories of unital unitary Λ_*-modules:

$$\begin{tikzcd} \Lambda_*^{\mathrm{op}} \operatorname{Mod}_*^* \arrow[r, "\mathrm{i}_*^1"] & \Lambda_*^{\mathrm{op}} \operatorname{Mod}_*^1 \arrow[l, bend right, "\mathrm{c}_*^1"'] \arrow[l, bend left, "\mathrm{s}_*^1"] \end{tikzcd} .$$

One can check readily that the connected unital unitary Λ_*-module $\mathrm{s}_*^1(\mathcal{Q}) \in \operatorname{Mod}_*^*$ associated to a unital operad $\mathcal{Q} \in \operatorname{Op}_*^1$ forms a suboperad of $\mathcal{Q}$ and similarly the Λ_*-module $\mathrm{c}_*^1(\mathcal{P}) \in \operatorname{Mod}_*^*$ forms a quotient operad of $\mathcal{P}$. Hence, for operads, the map $\mathcal{Q} \mapsto \mathrm{s}_*^1(\mathcal{Q})$, respectively $\mathcal{P} \mapsto \mathrm{s}_*^1(\mathcal{P})$, gives rise to a right, respectively left, adjoint of the category embedding $\mathrm{i}_*^1 : \operatorname{Op}_*^* \to \operatorname{Op}_*^1$.

For $r \geq 1$ and $i = 1, \dots, r$, we consider the Λ_*-module operation $\eta_i^* : N(r) \to N(1)$ associated to the map $\eta_i : \{1\} \to \{1, \dots, r\}$ such that $\eta_i(1) = i$. For an operad we have equivalently $\eta_i^*(p) = p(*, \dots, 1, \dots, *)$, where unital operations $*$ are substituted to the entries $k \neq i$ of the operation. For $r \geq 1$, let $\mathrm{s}_*^1(N)(r)$ denotes the submodule of $N(r)$ defined by the pullback diagram

$$\begin{tikzcd} \mathrm{s}_*^1(N)(r) \arrow[r, dashed] \arrow[d, dashed] & N(r) \arrow[d, "(\eta_i^*)_i"] \\ \mathbb{F}^{\times r} \arrow[r, "1^{\times r}"] & N(1)^{\times r} \end{tikzcd}$$

in which we consider the morphism $1 : \mathbb{F} \to N(1)$ defined by the unit element $1 \in N(1)$. One checks readily that these dg-modules $\mathrm{s}_*^1(N)(r)$ define a Λ_*-submodule of N. Furthermore, any morphism of unitary unital Λ_*-module $f : M \to N$ such that M is connected factorizes through $\mathrm{s}_*^1(N)$ since we have a commutative diagram

$$\begin{tikzcd} M(r) \arrow[r, "f"] \arrow[d, "\eta_i^*"] & N(r) \arrow[d, "\eta_i^*"] \\ M(1) = \mathbb{F} \arrow[r, "1"] & N(1) \end{tikzcd}$$

for all $i = 1, \dots, r$. Therefore the functor $N \mapsto \mathrm{s}_*^1(N)$ satisfies the adjunction relation $\mathrm{Hom}_{\Lambda_*^{\mathrm{op}} \mathrm{Mod}_*^1}(\mathrm{i}_*^1(M), N) = \mathrm{Hom}_{\Lambda_*^{\mathrm{op}} \mathrm{Mod}_*}(M, \mathrm{s}_*^1(N))$. According to this construction, we have clearly $M = \mathrm{s}_*^1 \mathrm{i}_*^1(M)$ for any connected unital unitary Λ_*-module M.

The other connected Λ_*-module $\mathrm{c}_*^1(M)$ associated to a unital unitary Λ_*-module M can clearly be defined by $\mathrm{c}_*^1(M)(r) = \mathbb{F}$ for $r = 0, 1$ and $\mathrm{c}_*^1(M)(r) = M(r)$ for $r \geq 2$. For $r = 2$, the operations $\partial_1, \partial_2 : \mathrm{c}_*^1(M)(2) \to \mathbb{F}$ are given by the augmentation morphism $\epsilon : M(2) \to \mathbb{F}$. The relation $\mathrm{c}_*^1 \mathrm{i}_*^1(N) = N$, that holds for a connected Λ_*-module N, is also immediate from this definition.

§1.3. **The Reedy model structure for Λ_*-modules.** The aim of this section is to prove the following theorem:

Theorem §1.A. *The category of ($\mathbb{N}$ or $\mathbb{Z}$-graded) dg-Λ_*-modules* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$ *is equipped with the structure of a cofibrantly generated model category such that a morphism $f : M \to N$ is a weak-equivalence, respectively a cofibration, if f defines a weak-equivalence, respectively a cofibration, in the category of dg-Σ_*-modules* $\mathrm{dg}\,\Sigma_*\,\mathrm{Mod}$.

In this statement we consider the category $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$ formed by all Λ_*-modules which do not satisfy necessarily $M(0) = 0$. On the other hand, for our purposes we need a model structure on the category of non-unital Λ_*-modules $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0$ and on the associated comma category of augmented unitary Λ_*-modules $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0^1 / \overline{\mathcal{C}}$. In fact, we use the following claim in order to equip $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0$ with the structure of a model subcategory of $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$:

§1.3.1. **Proposition.** *The category embedding* $\mathrm{i}_0^\Lambda : \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0 \hookrightarrow \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$ *admits a left and a right adjoint*

$$\begin{tikzcd} \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0 \arrow[r, "\mathrm{i}_0^\Lambda"] & \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod} \arrow[l, bend right, "\mathrm{c}_0^\Lambda"'] \arrow[l, bend left, "\mathrm{s}_0^\Lambda"] \end{tikzcd}$$

such that $\mathrm{c}_0^\Lambda\, \mathrm{i}_0^\Lambda = \mathrm{Id} = \mathrm{s}_0^\Lambda\, \mathrm{i}_0^\Lambda$. *Furthermore, the functor* $\mathrm{i}_0^\Lambda\, \mathrm{c}_0^\Lambda$ *preserves cofibrations and all weak-equivalences in* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$.

Proof. The proof of this proposition is straightforward. The construction of non-unital Λ_*-modules $\mathrm{c}_0^\Lambda(M)$, respectively $\mathrm{s}_0^\Lambda(N)$, is similar to the construction of the connected unital Λ_*-module $\mathrm{c}_*^1(M)$, respectively $\mathrm{s}_*^1(N)$, given in §1.2.7.

Explicitly, for $\mathrm{c}_0^\Lambda(M)$ we set simply $\mathrm{c}_0^\Lambda(M)(r) = 0$ for $r = 0$ and $\mathrm{c}_0^\Lambda(M)(r) = M(r)$ for $r \geq 1$. According to this construction, the functor c_0^Λ preserves clearly cofibrations, acyclic cofibrations and all week-equivalences in $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$.

For $\mathrm{s}_0^\Lambda(N)$ we consider the operation $\eta_0^* : N(r) \to N(0)$ associated to the map $\eta_0 : \emptyset \to \{1, \ldots, r\}$. Then, for $r \geq 0$, we set explicitly $\mathrm{s}_0^\Lambda(N)(r) = \ker(\eta_0^* : N(r) \to N(0))$. More categorically, the module $\mathrm{s}_0^\Lambda(N)(r)$ can be defined by the pullback diagram

$$\begin{tikzcd} \mathrm{s}_0^\Lambda(N)(r) \arrow[r, dashed] \arrow[d, dashed] & N(r) \arrow[d, "\eta_0^*"] \\ 0 \arrow[r, "0"] & N(0) \end{tikzcd} .$$

□

Then, assuming theorem §1.A, we obtain:

§1.3.2. **Proposition.** *The category of non-unital dg-Λ_*-modules* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0$ *forms a model subcategory of* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$ *so that a morphism $f : M \to N$ is a weak-equivalence, respectively a cofibration, a fibration, of non-unital dg-Λ_*-modules if and only if f defines a weak-equivalence, respectively a cofibration, a fibration, in* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$.

Furthermore, this category is cofibrantly generated by the morphisms $\mathrm{c}_0^\Lambda(i) : \mathrm{c}_0^\Lambda(A) \to \mathrm{c}_0^\Lambda(B)$ associated to a generating set of cofibrations, respectively acyclic cofibrations, in $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$.

Proof. This assertion is a straightforward consequence of the adjunction relation and the invariance of c_0^Λ with respect to cofibrations and weak-equivalences. □

The category of augmented unitary Λ_*^{op}-modules $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0^1/\overline{\mathcal{C}}$, which is a comma category associated to $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}_0$, is equipped with a canonical induced model structure as usual.

In fact, theorem §1.A holds for a category of Λ_*-objects in any cofibrantly generated model category and not only in the category of dg-modules. Therefore we make our arguments as general as possible though we consider explicitly only dg-modules (and dg-coalgebras in the next section).

In a first stage we make explicit the definition of a fibration in the category of Λ_*-modules. Then we prove that the lifting properties (M4.i-ii) are satisfied, we specify generating collections of cofibrations and acyclic cofibrations and we deduce the factorization properties (M5.i-ii) from the classical small objects argument. The properties (M1-3) are inherited from the ground model category.

The general idea of our construction is provided by a generalization of the classical Reedy model structure in the presence of automorphisms. To be precise, in the usual definition of Reedy structures one considers functors $F : I \to \mathcal{C}$ on a fixed small category I equipped with a grading and a decomposition $I = \overrightarrow{I}\overleftarrow{I}$, where the direct Reedy category $\overrightarrow{I}$, respectively the inverse Reedy category $\overleftarrow{I}$, denote subcategories of I formed by collections of morphisms that increase, respectively decrease, the grading. One can observe that classical constructions, given in Reedy's original article [22] and in the modern monographs [17, 18], still work for a category I that contains automorphisms and such that any morphism $f \in I$ has a unique decomposition $f = \alpha u \beta$ in which $\alpha \in \overrightarrow{I}$, $\beta \in \overleftarrow{I}$ and u is an automorphism. This extension of Reedy structures does not seem to occur in the literature in full generality though an instance is supplied by the strict model category of Γ-spaces defined in [7]. Therefore we give detailed arguments for the category $I = \Lambda_*^{\mathrm{op}}$. By contravariance, we have in this case $\overleftarrow{I} = \overrightarrow{\Lambda}_*^{\mathrm{op}}$ for a Reedy direct subcategory of Λ_* and $\overrightarrow{I}$ is trivial. The latter property simplifies some constructions.

In the next sections we call the model structure supplied by theorem §1.A the *Reedy model structure* in order to distinguish this model category from the classical adjoint model structure (the cofibrantly generated model category of [17, Section 11.6]) from which it differs. To be precise, recall that the classical adjoint model structure is defined by a transfer of model structures from dg-modules to Λ_*-modules through a composite adjunction

$$\mathrm{dg}\,\mathrm{Mod}^{\mathbb{N}} \underset{\phi^!}{\overset{\phi_!}{\rightleftarrows}} \mathrm{dg}\,\Sigma_*\,\mathrm{Mod} \underset{\psi^!}{\overset{\psi_!}{\rightleftarrows}} \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}\,.$$

Explicitly, one let a morphism $f : M \to N$ defines a weak-equivalence, respectively a fibration, in the adjoint model category $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}$ if $\phi^!\psi^!(f)$ defines a weak-equivalence, respectively a fibration, in $\mathrm{dg}\,\mathrm{Mod}^{\mathbb{N}}$. According to this definition, the Reedy model category and the adjoint model category have the same weak-equivalences but different cofibrations and fibrations. In fact, a Reedy fibration defines a fibration in the adjoint model category and hence the Reedy model category has less fibrations but more cofibrations than the adjoint model category. As a byproduct, the identity functor yields a pair of adjoint derived equivalences between the two homotopy categories. Thus the Reedy model structure of Λ_*-modules is different but Quillen equivalent to the adjoint model structure like a classical Reedy category (see [17, Section 15.6]).

§1.3.3. *The category* Λ_*. Recall that a Λ_*-module is equivalent to a contravariant functor on the category Λ_* formed by the finite sets $\underline{r} = \{1, \dots, r\}$ and the injective maps $u : \{1, \dots, r\} \to \{1, \dots, s\}$. This equivalence is also a consequence of the decomposition of Λ_* into a direct subcategory and a subcategory of isomorphisms that we make explicit in this paragraph.

Namely let $\overrightarrow{\Lambda}_*$ denote the subcategory of Λ whose morphisms $\alpha \in \overrightarrow{\Lambda}_r^s$ are the non-decreasing injections $\alpha : \{1, \dots, r\} \to \{1, \dots, s\}$. One observes readily that any morphism $u \in \Lambda_r^s$ has a unique $\overrightarrow{\Lambda}_*\Sigma_*$-decomposition $u = \alpha\sigma$ in which $\sigma \in \Sigma_r$ and $\alpha \in \overrightarrow{\Lambda}_r^s$. Moreover, the category $\overrightarrow{\Lambda}_*$ is the category generated by the injections $d^i : \{1, \dots, r-1\} \to \{1, \dots, r\}$ that avoid $i \in \{1, \dots, r\}$ endowed with the relations $d^j d^i = d^i d^{j-1}$ for $i < j$. With respect to permutations $\sigma \in \Sigma_r$, we have also a relation of the form $\sigma d^i = d^{\sigma(i)}\partial_i(\sigma)$ which gives the $\overrightarrow{\Lambda}_*\Sigma_*$-decomposition of $u = \sigma d^i$. As a consequence, as stated in the introduction §1.2, the structure of a Λ_*-module is indeed equivalent to a Σ_*-module equipped with operations $\partial_i : M(r) \to M(r-1)$ that satisfy the relations above.

Anyway the category $\overrightarrow{\Lambda}_*$ forms clearly a direct Reedy category and we have a generalized Reedy decomposition $\Lambda_* = \overrightarrow{\Lambda}_*\Sigma_*$. As explained previously, we adapt the classical construction of Reedy model structures to this context. In the next paragraphs we let $\Lambda_{*<r}$, respectively $\overrightarrow{\Lambda}_{*<r}$, denote the comma category of morphisms $\alpha : \underline{r}' \to \underline{r}$ in Λ_*, respectively $\overrightarrow{\Lambda}_*$, such that $r' < r$. Observe that the $\overrightarrow{\Lambda}_*\Sigma_*$-decomposition property implies readily that $\overrightarrow{\Lambda}_{*<r}$ is cofinal in $\Lambda_{*<r}$.

§1.3.4. *Matching modules and fibrations of* Λ_**-modules.* In order to obtain an explicit characterization of fibrations we need to introduce a notion of a *matching* *object* in the category of Λ_*-modules. Explicitly, the matching object of a Λ_*-module is defined by the collection of dg-modules $\mathbb{M}M(r)$, $r \in \mathbb{N}$, such that

$$\mathbb{M}M(r) = \lim_{\alpha:\underline{r}'\xrightarrow{\leq}\underline{r}} M(r'),$$

where the limit ranges over the category $\Lambda_{*<r}$ or, equivalently, over the category $\overrightarrow{\Lambda}_{*<r}$ which is cofinal in $\Lambda_{*<r}$ by the observation of the previous paragraph. As usual the dg-module $\mathbb{M}M(r)$ can equivalently be defined by an equalizer

$$\mathbb{M}M(r) = \ker\big(\textstyle\prod_{1\leq i\leq r} M(r-1) \overset{d^0}{\underset{d^1}{\rightrightarrows}} \prod_{1\leq i<j\leq r} M(r-2) \big),$$

where $d^0(x_i)_i = (\partial_i x_j)_{i<j}$ and $d^1(x_i)_i = (\partial_{j-1} x_i)_{i<j}$.

The matching modules are endowed with natural morphisms $\mu : M(r) \to \mathbb{M}M(r)$ induced by the morphisms $\alpha^* : M(r) \to M(r')$ on the component of the limit indexed by $\alpha \in \Lambda^r_{r'}$. Explicitly, if we represent an element of the limit by a collection $(x_\alpha)_\alpha$, where $\alpha \in \Lambda^r_{r'}$, then, for $x \in M(r)$, we have $\mu(x)_\alpha = \alpha^*(x) \in M(r')$. The matching module $\mathbb{M}M$ is also equipped with a canonical Λ_*-module structure such that $\mu : M \to \mathbb{M}M$ defines a morphism of Λ_*-modules. Explicitly, for any morphism $u : \underline{r} \to \underline{s}$, if we let $y = (y_\beta)_\beta$ denote an element of $\mathbb{M}M(s) = \lim_\beta M(s')$, then we have an associated element $u^*(y) = (u^*(y)_\alpha)_\alpha$ in $\mathbb{M}M(r) = \lim_\alpha M(r')$ which can be defined by the collection: $u^*(y)_\alpha = y_{u\alpha}$.

We let a morphism of Λ_*-modules $p : M \to N$ be a fibration if, for all $r \in \mathbb{N}$, the natural morphism

$$(\mu, p) : M(r) \to \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r)$$

defines a fibration in the category of dg-modules and hence in the category of Σ_r-modules for the classical model structure of Σ_r-modules.

As stated previously, the properties (M1-3) of a model category are inherited from the ground category of dg-modules. The first non-trivial verification is supplied by the following claim:

§1.3.5. **Claim.** *The properties (M4.i-ii) hold for the class of weak-equivalences, cofibrations and fibrations specified in theorem §1.A and paragraph §1.3.4. Explicitly, in a commutative diagram of Λ_*-modules*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & M \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & N \end{tikzcd},$$

the lift $h : B \to M$ exists provided that p is an acyclic fibration (M4.i), respectively provided that i is an acyclic cofibration (M4.ii).

Proof. In both cases we construct by induction over $r \in \mathbb{N}$ a morphism $h : B(r) \to M(r)$ that commutes with the action of morphisms $u \in \Lambda^r_s$ such that $s \geq r$. Explicitly, if the morphisms h are defined for $s < r$, then we have a well defined induced morphism $h : \mathbb{M}B(r) \to \mathbb{M}M(r)$ so that we obtain a commutative diagram

$$\begin{tikzcd} A(r) \arrow[r] \arrow[d, tail, "i"'] & M(r) \arrow[d, "(\mu,p)"] \\ B(r) \arrow[r] \arrow[ur, dashed, "h"] & \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r) \end{tikzcd}$$

in the category of Σ_r-modules.

The right-hand side morphism is a fibration of Σ_r-modules by definition and the left-hand side morphism a cofibration. If i is an acyclic cofibration, then, by definition, the left-hand side morphism is obviously an acyclic cofibration as well so that the lift exists in this case since the lifting property (M4.ii) is known to be satisfied in the adjoint model category Σ_r-modules.

If p is an acyclic cofibration, then we claim that the right-hand side morphism is also an acyclic fibration. In fact, if we forget the symmetric group action, then the dg-modules $\mathbb{M}M(r)$ can be identified with the matching modules of the functor underlying M on the direct Reedy category $\overrightarrow{\Lambda}_*$. Therefore we deduce from the classical theory that the morphism $(\mu, p) : M(r) \to \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r)$ defines an acyclic fibration in the category of dg-modules (see for instance [17, Proposition 15.3.14]) and hence in the category of Σ_r-modules. Finally, the lift exists in this case as well again since the lifting property (M4.i) is known to be satisfied in the adjoint model category Σ_r-modules.

By construction, our morphism $h : M(r) \to N(r)$ makes the diagram

$$\begin{tikzcd} M(r) \arrow[r, "h"] \arrow[d] & N(r) \arrow[d] \\ \mathbb{M}M(r) \arrow[r] & \mathbb{M}N(r) \end{tikzcd}$$

commute. This property implies immediately that h commutes with all morphisms $\alpha \in \Lambda^r_s$ such that $s < r$ and hence with all morphisms $\alpha \in \Lambda^r_s$ since h is Σ_r-equivariant as well by construction. $\square$

Now we aim to define generating cofibrations and acyclic cofibrations in the category of Λ_*-modules so that we can deduce (M5.i-ii) from the small object argument. For this purpose we extend the tensor product of dg-modules with simplicial sets $C \otimes K$ to Λ_*-sets and we consider for K the obvious generators of the category of Λ_*-sets supplied by the classical Yoneda lemma and the associated latching Λ_*-sets that give representatives for the matching space modules. We define these objects explicitly in the nexts paragraphs.

§1.3.6. *Generating Λ_*-sets.* As usual, for any object $r \in \mathbb{N}$, we have a canonical Λ_*-set, denoted by Λ^r, defined by the functor corepresented by r and such that $\Lambda^r(s) = \Lambda^r_s$ for $r \in \mathbb{N}$. By definition, the map $r \mapsto \Lambda^r$ defines a covariant functor from Λ_* to the category of Λ_*-sets. Therefore we can form the associated latching object which are defined explicitly by the colimit

$$\mathbb{L}\Lambda^r = \operatorname{colim}_{\alpha:\underline{r}' \xrightarrow{<} \underline{r}} \Lambda^{r'}.$$

As usual for a functor, this colimit can be determined pointwise so that $\mathbb{L}\Lambda^r(s)$ is determined by the equivalent colimit in the category of sets:

$$\mathbb{L}\Lambda^r(s) = \operatorname{colim}_{\alpha:\underline{r}' \xrightarrow{<} \underline{r}} \Lambda^{r'}_s \quad \text{for all } s \in \mathbb{N}.$$

As in the definition of a matching module, we can assume that the colimit ranges over the category $\Lambda_{*<r}$ or, equivalently, over the category $\overrightarrow{\Lambda}_{*<r}$ which is cofinal in $\Lambda_{*<r}$. We have a canonical morphism $\lambda : \mathbb{L}\Lambda^r \to \Lambda^r$ for $r \in \mathbb{N}$. In fact, this morphism is either trivial or an isomorphism. To be precise, we obtain:

§1.3.7. **Observation.**

(a) *For $s < r$, we have $\mathbb{L}\Lambda^r(s) = \Lambda^r(s) = \Lambda^r_s$ and $\lambda : \mathbb{L}\Lambda^r(s) \to \Lambda^r(s)$ is an isomorphism.*

(b) *For $s = r$, we have $\mathbb{L}\Lambda^r(r) = \emptyset$ and $\Lambda^r(r) = \Lambda^r_r = \Sigma_r$ so that $\lambda : \mathbb{L}\Lambda^r(r) \to \Lambda^r(r)$ is an initial morphism in the category of sets.*

(c) *For $s > r$, we have $\mathbb{L}\Lambda^r(s) = \Lambda^r(s) = \emptyset$.*

Proof. The assertions (b-c) are immediate since $\Lambda^r_s = \emptyset$ for $s > r$ and only assertion (a) deserves a verification. In fact, for $s < r$, the map

$$\lambda : \operatorname{colim}_{\alpha:\underline{r}' \xrightarrow{\leq} \underline{r}} \Lambda^{r'}_s \to \Lambda^r_s$$

admits a canonical section that maps an element $u \in \Lambda^r_s$ to the identity morphism $\mathrm{id} \in \Lambda^s_s$ in the component of the colimit indexed by $u : \underline{s} \to \underline{r}$. One checks readily that this section gives also a left-inverse for the map λ, so that λ is an isomorphism. □

§1.3.8. *The tensor product with a Λ_*-set.* In general the tensor product $C \otimes A$ of an object C with a Λ_*-set A is the Λ_*-module defined by the coproducts

$$C \otimes A(r) = \bigoplus_{\alpha \in A(r)} C.$$

In the context of dg-modules the tensor product $C \otimes A(r)$ can be also be represented by the tensor product of C with the free module spanned by the set $A(r)$. Hence an element of $C \otimes A(r)$ can be represented by a tensor $c \otimes \alpha$, where $c \in C$ and $\alpha \in A(r)$.

For a morphism $i : C \to D$ in the category of dg-modules, we form the pushout of Λ_*-modules

$$\begin{tikzcd}
C \otimes \mathbb{L}\Lambda^r \arrow[r] \arrow[d] & D \otimes \mathbb{L}\Lambda^r \arrow[d] \\
C \otimes \Lambda^r \arrow[r] & C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r
\end{tikzcd}$$

and we consider the natural morphism

$$C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r \to D \otimes \Lambda^r$$

induced by $i \otimes \Lambda^r : C \otimes \Lambda^r \to D \otimes \Lambda^r$ and $D \otimes \lambda : D \otimes \mathbb{L}\Lambda^r \to D \otimes \Lambda^r$. We prove precisely that such morphisms give a set of generating arrows in the category of Λ_*-modules:

§1.3.9. **Lemma.** *The morphisms*

$$C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r \to D \otimes \Lambda^r$$

associated to a generating set of cofibrations, respectively acyclic cofibrations, of the category of dg-modules $i : C \to D$ define a generating set of cofibrations, respectively acyclic cofibrations, in the category of Λ_*-modules.

The proof of this lemma is split up into a sequence of claims and can be compared with the classical construction of [17, Section 15.6]. First, we observe that the morphisms considered are indeed cofibrations, respectively acyclic cofibrations, in the category of Λ_*-modules:

§1.3.10. **Claim.** *If $i : C \to D$ is a cofibration, respectively an acyclic cofibration, in the category of dg-modules, then the associated morphism*

$$C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r \to D \otimes \Lambda^r$$

is a cofibration, respectively an acyclic cofibration, in the category of Λ_*-modules.

Proof. This assertion is a consequence of observation §1.3.7. Explicitly, from this statement one deduces that the morphism

$$C \otimes \Lambda^r(s) \bigoplus_{C \otimes \mathbb{L}\Lambda^r(s)} D \otimes \mathbb{L}\Lambda^r(s) \to D \otimes \Lambda^r(s)$$

is an isomorphism for $s < r$, can be identified with $i \otimes \Sigma_r : C \otimes \Sigma_r \to D \otimes \Sigma_r$ for $s = r$ and vanishes for $s > r$. Consequently, this morphism is a cofibration, respectively an acyclic cofibration, of Σ_s-modules in all cases: this assertion is trivial in the cases $s < r$ and $s > r$ and holds by definition (of cofibrations in the category of Σ_r-modules) in the case $s = r$. □

Then we check that the sets of arrows introduced in the lemma detect acyclic fibrations, respectively fibrations. For this purpose we observe that the tensor products $C \otimes \Lambda^r$ and $C \otimes \mathbb{L}\Lambda^r$ are characterized by natural adjunction relations. Namely:

§1.3.11. **Observation.** *For a dg-module C and a Λ_*-module M, we have the adjunction formulas*

$$\mathrm{Hom}_{\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}}(C \otimes \Lambda^r, M) = \mathrm{Hom}_{\mathrm{dg}\,\mathrm{Mod}}(C, M(r))$$
$$\textit{and} \quad \mathrm{Hom}_{\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}}(C \otimes \mathbb{L}\Lambda^r, M) = \mathrm{Hom}_{\mathrm{dg}\,\mathrm{Mod}}(C, \mathbb{M}M(r))$$

and the morphism

$$\mathrm{Hom}_{\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}}(C \otimes \Lambda^r, M) \xrightarrow{C \otimes \lambda^*} \mathrm{Hom}_{\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{Mod}}(C \otimes \mathbb{L}\Lambda^r, M)$$

induced by the latching morphism $\lambda : \mathbb{L}\Lambda^r \to \Lambda^r$ agrees with the morphism induced by the matching morphism $\mu : M(r) \to \mathbb{M}M(r)$.

Proof. These adjunction claims are formal consequences of the definition of the Λ_*-sets Λ^r, $\mathbb{L}\Lambda^r$ and of the Yoneda lemma. □

As a corollary, we obtain:

§1.3.12. **Observation.** *For a morphism of Λ_*-modules $p : M \to N$, the lifting problem*

$$\begin{tikzcd} C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ D \otimes \Lambda^r \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

is equivalent to an adjoint lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d] & M(r) \arrow[d, "(\mu,p)"] \\ D \arrow[r] \arrow[ur, dashed] & \mathbb{M}M(r) \otimes_{\mathbb{M}N(r)} N(r) \end{tikzcd}$$

in the category of dg-modules.

Proof. This observation is a formal consequence of the adjunction relations of observation §1.3.11. □

This observation implies immediately that the arrows introduced in lemma §1.3.9 detect acyclic fibrations and fibrations and achieve the proof of that statement. □

The next claim is a consequence of lemma §1.3.9 and of the classical small object argument:

§1.3.13. **Claim.** *The properties (M5.i-ii) hold for the class of weak-equivalences, cofibrations and fibrations specified in theorem §1.A and paragraph §1.3.4. Explicitly, any morphism of Λ_*-modules $\phi : M \to N$ admits a factorization $\phi = pi$ such that i is a cofibration and p an acyclic fibration (M5.i), respectively such that i is an acyclic cofibration and p a fibration (M5.ii).*

Proof. The arguments are classical. We refer to [17, Section 10.5]. □

This claim achieves the proof of theorem §1.A. □

§1.4. **The Reedy model structure for unital operads.** In this section we check that the Reedy model structure of Λ_*-modules can be transferred to operads through the adjunction

$$\mathcal{F}_* : \operatorname{dg}\Lambda_*^{\mathrm{op}}\operatorname{Mod}_0^1/\overline{C} \rightleftarrows \operatorname{dg}\operatorname{Op}_*^1 : (-)^-$$

defined in §1.2.4. Explicitly, we have the following theorem:

Theorem §1.B. *The category of ($\mathbb{N}$ or $\mathbb{Z}$-graded) unital operads $\operatorname{dg}\operatorname{Op}_*^1$ is equipped with the structure of a cofibrantly generated model category such that a morphism $\phi : \mathcal{P} \to \mathcal{Q}$ is a weak-equivalence, respectively a fibration, if ϕ forms a Reedy weak-equivalence, respectively a Reedy fibration, in the category of Λ_*-modules $\operatorname{dg}\Lambda_*^{\mathrm{op}}\operatorname{Mod}_0^1$.*

The cofibrations are characterized by the left lifting property with respect to acyclic fibrations as usual.

Our model structure differs from the adjoint model structure defined in [5] like the Reedy model structure of Λ_*-modules differs from the adjoint model structure. Nevertheless one can observe again that the identity functor yields a pair of adjoint derived equivalences between the two model categories.

Theorem §1.B follows from the general transfer principle considered in *loc. cit.* since we observe in lemma §1.3.9 that the category of Λ_*-modules is cofibrantly generated and the scheme of the proof of theorem §1.B is classical. As in [4], we have essentially to check that the following property is satisfied:

§1.4.1. **Claim.** *Suppose given a pushout of unital operads*

$$\begin{tikzcd} \mathcal{F}_*(C) \arrow[r, "f"] \arrow[d, tail, "i"', "\sim"] & \mathcal{P} \arrow[d, "j"] \\ \mathcal{F}_*(D) \arrow[r, "g"] & \mathcal{Q} \end{tikzcd}$$

such that $i : \mathcal{F}_*(C) \to \mathcal{F}_*(D)$ is a morphism of free operads induced by an acyclic cofibration of Λ_*-modules $i : C \xrightarrow{\sim} D$. The morphism $j : \mathcal{P} \to \mathcal{Q}$ is a weak-equivalence as well.

Proof. This property is inherited from the analogous property of non-unital operads, for which we can refer to [4, 16], since we observe that the functor $\mathcal{Q} \mapsto \overline{\mathcal{Q}}$ from the category of unital operads to the category of non-unital operads creates pushouts. □

According to the transfer principle (see [4, 9]), this verification achieves the proof of theorem §1.B. □

Theorem §1.B holds for the category of connected unital dg-operads $\operatorname{dg}\operatorname{Op}_*^*$. In this case we consider the category $\operatorname{dg}\Lambda^{\mathrm{op}}\operatorname{Mod}_0^*/\overline{\mathcal{C}}$, introduced in §1.2.6, formed by the augmented non-unital unitary Λ_*-modules M such that $M(1) = \mathbb{F}$. According to observations of this paragraph, we have an adjunction relation between this category and connected unital operads:

$$\mathcal{F}_* : \operatorname{dg}\Lambda_*^{\mathrm{op}}\operatorname{Mod}_0^*/\overline{\mathcal{C}} \rightleftarrows \operatorname{dg}\operatorname{Op}_*^* : (-)^-.$$

Furthermore, the functor $\mathcal{P} \mapsto \overline{\mathcal{P}}$ creates also pushouts in $\operatorname{dg}\operatorname{Op}_*^*$, so that the argument above remains valid for $\operatorname{dg}\operatorname{Op}_*^*$.

On the other hand, one can deduce from the adjunction relations of §1.2.7 that the category of connected unital dg-operads $\operatorname{dg}\operatorname{Op}_*^*$ forms a model subcategory of $\operatorname{dg}\operatorname{Op}_*^1$ so that a morphism of connected unital dg-operads defines a Reedy cofibration, respectively a Reedy fibration, a weak-equivalence, in $\operatorname{dg}\operatorname{Op}_*^*$ if and only if it defines a Reedy cofibration, respectively a Reedy fibration, a weak-equivalence, in $\operatorname{dg}\operatorname{Op}_*^1$. As a byproduct, we have the following statement:

§1.4.2. **Fact.** *The functor $\mathrm{s}_*^1 : \operatorname{dg}\operatorname{Op}_*^1 \to \operatorname{dg}\operatorname{Op}_*^*$, right adjoint to the category embedding $\mathrm{i}_*^1 : \operatorname{dg}\operatorname{Op}_*^1 \to \operatorname{dg}\operatorname{Op}_*^*$, preserves fibrations, acyclic fibrations and all weak-equivalences between fibrant operads.*

This claim could also be deduced from an analogous assertion for Λ_*-modules since operad fibrations, respectively weak-equivalences, are just fibrations, respectively weak-equivalences, in the Λ_*-module category. To be precise, the statement above is implied by the analogous assertion for the category embedding $\mathrm{i}_*^1 : \operatorname{dg}\Lambda_*^{\mathrm{op}}\operatorname{Mod}_*^* \hookrightarrow \operatorname{dg}\Lambda_*^{\mathrm{op}}\operatorname{Mod}_*^1$.

§1.5. **On Hopf Λ_*-modules and unital Hopf operads.** Recall that the notion of an operad makes sense in any symmetric monoidal category. In this memoir, if no category is specified, then an operad refers to an operad in the category of dg-modules $\operatorname{dg}\operatorname{Mod}$ since this category forms our ground monoidal category. On the other hand, as explained in the introduction, we consider also *Hopf operad* structures which are precisely operads in the category of augmented coassociative dg-coalgebras, denoted by CoAlg_+^a. The purpose of this section is to make clear the structure of a Hopf operad and to prove the analogue of theorem §1.B for Hopf operads. Namely we prove that unital Hopf operads form a model category.

§1.5.1. *On (unital) Hopf operads.* Explicitly, a Hopf operad consists of a collection of coalgebras $\mathcal{P}(r) \in \operatorname{CoAlg}_+^a$ such that the operad composition products $\circ_i : \mathcal{P}(s) \otimes \mathcal{P}(t) \to \mathcal{P}(s+t-1)$ define morphisms in CoAlg_+^a as well as the isomorphisms $w : \mathcal{P}(r) \to \mathcal{P}(r)$ defined by the action of permutations $w \in \Sigma_r$. As usual for bialgebra structures, one can equivalently assume that the diagonals $\Delta : \mathcal{P}(r) \to \mathcal{P}(r) \otimes \mathcal{P}(r)$ define an operad morphism, where the tensor product $\mathcal{P}(r) \otimes \mathcal{P}(r)$ is equipped with place-by-place composition products. The distinguished unital operation $* \in \mathcal{P}(0)$ and the operad unit $1 \in \mathcal{P}(1)$ are supposed to define group-like elements in $\mathcal{P}$.

One can observe that the composition products preserve the coalgebra augmentations $\epsilon : \mathcal{P}(r) \to \mathbb{F}$ if and only if these morphisms define a morphism to the operad of commutative algebras $\mathcal{C}$. Accordingly, any Hopf operad $\mathcal{P}$ is automatically augmented over the commutative operad $\mathcal{C}$. Furthermore, for a unital Hopf operad $\mathcal{P}$, we deduce from these observations that the coalgebra augmentations agree with the augmentation morphisms $\epsilon : \mathcal{P}(r) \to \mathbb{F}$ introduced in the introduction of this section and defined by the operadic composites $\epsilon(p) = p(*, \ldots, *)$.

Similarly, a *Hopf algebra over $\mathcal{P}$* refers to an algebra over $\mathcal{P}$ in the category of augmented coassociative dg-coalgebras. Hence a Hopf algebra consists of a coalgebra $\Gamma \in \operatorname{CoAlg}_+^a$ equipped with evaluation products $\mathcal{P}(r) \otimes \Gamma^{\otimes r} \to \Gamma$ that form morphisms in CoAlg_+^a. As stated previously for the composition products of a Hopf operad, one can equivalently assume that the diagonal of Γ defines a morphism of $\mathcal{P}$-algebras $\Delta : \Gamma \to \Gamma \otimes \Gamma$, where the tensor product $\Gamma \otimes \Gamma$ is equipped with a diagonal action of $\mathcal{P}(r)$.

If $\mathcal{P}$ is a unital Hopf operad, then a Hopf $\mathcal{P}$-algebra Γ comes equipped with a unit morphism $\eta : \mathbb{F} \to \Gamma$ yielded by the unital operation $* \in \mathcal{P}(0)$. Consequently, the underlying coalgebra of a Hopf algebra over a unital operad defines an object in the category CoAlg_*^a of *augmented unitary coalgebras*. By convention, if a given augmented unitary coalgebra Γ is equipped with the structure of a Hopf $\mathcal{P}$-algebra, where $\mathcal{P}$ is a unital Hopf operad, then we assume that the unit of Γ agrees with the unital operation.

§1.5.2. *Hopf Λ_*-modules.* Observe that the underlying Λ_*-module of a unital Hopf operad defines a Λ_*-object in the category of augmented coassociative dg-coalgebras. This structure is called a *Hopf Λ_*-module* according to our usual conventions. Clearly, a Hopf Λ_*-module Γ is automatically augmented over the constant Λ_*-module since the coalgebra augmentation gives a morphism $\epsilon : \Gamma \to \mathbb{F}$ as in the case of unital Hopf operads.

To recapitulate, the forgetful functor of unital operads $\mathcal{P} \mapsto \overline{\mathcal{P}}$, considered in §1.2.4, restricts to a functor from the category of unital Hopf operads to the category of non-unital unitary Hopf Λ_*-modules. In the converse direction, if a unitary Λ_*-module Γ is equipped with a coalgebra structure, then the associated free unital operad $\mathcal{F}_*(\Gamma)$ can be equipped with the structure of a unital Hopf operad so that $\mathcal{F}_*(\Gamma)$ satisfies the usual universal property for Hopf Λ_*-modules. In fact, since the diagonal of a Hopf operad is supposed to define an operad morphism, the diagonal of a formal composite $\gamma = (\dots((\gamma_1 \circ_{i_2} \gamma_2) \circ_{i_3} \dots) \circ_{i_l} \gamma_l$ in $\mathcal{F}_*(\Gamma)$ can be written

$$\Delta(\gamma) = \sum_{\gamma_1,\dots,\gamma_r} (\dots((\gamma_1' \circ_{i_2} \gamma_2') \circ_{i_3} \dots) \circ_{i_l} \gamma_l' \otimes (\dots((\gamma_1'' \circ_{i_2} \gamma_2'') \circ_{i_3} \dots) \circ_{i_l} \gamma_l'',$$

where $\Delta(\gamma_i) = \sum_{\gamma_i} \gamma_i' \otimes \gamma_i''$ denotes the diagonal of γ_i in $\Gamma(r)$.

To conclude, the forgetful functor from unital Hopf operads to unital operads creates free objects. As a consequence, for unital Hopf operads, we have an adjunction relation

$$\mathcal{F}_* : \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0^1 \rightleftarrows \mathrm{dg}\,\mathrm{HopfOp}_*^1 : (-)^-,$$

where $\mathcal{F}_*$ denotes the free operad functor defined in §1.2. One can observe in addition that the forgetful functor creates also all small colimits in the category of unital Hopf operads since the forgetful functor from the category of coalgebras to the category of dg-modules has this property.

§1.5.3. *The model category of coalgebras.* According to [14], the category of non-negatively graded dg-coalgebras over a field is equipped with the structure of a model category such that a morphism $\phi : C \to D$ is a weak-equivalence, respectively a cofibration, if ϕ defines a weak-equivalence, respectively a cofibration, in the category of dg-modules. (Recall that a morphism of dg-modules over a field is a cofibration if and only if it is injective.) Furthermore, this category has a set of generating cofibrations, respectively acyclic cofibrations, defined by the collection of injective morphisms $i : C \to D$ such that D is spanned by a finite, respectively countable, set of homogeneous elements.

As specified in §1.3, the Reedy model structure of theorem §1.A can be defined in any cofibrantly generated ground model category. In particular, for the Λ_*-objects in the category of dg-coalgebras we obtain the following theorem:

Theorem §1.C. *The category of $\mathbb{N}$-graded Hopf dg-Λ_*-modules* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$ *is equipped with the structure of a cofibrantly generated model category such that a morphism $\phi : C \to D$ is a weak-equivalence, respectively a cofibration, if ϕ defines a weak-equivalence, respectively a cofibration, in the category of dg-Σ_*-modules* $\mathrm{dg}\,\Sigma_*\,\mathrm{Mod}$.

This model structure for Hopf Λ_*-modules is also cofibrantly generated. To be more explicit, as in the ground category of dg-modules, we have the following lemma:

§1.5.4. **Lemma.** *We have a generating set of cofibrations, respectively acyclic cofibrations, in the category of Hopf Λ_*-modules defined by morphisms*

$$C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r \to D \otimes \Lambda^r$$

associated to a generating set of cofibrations, respectively acyclic cofibrations, of the category of dg-coalgebras $i : C \to D$. □

Recall simply that the forgetful functor creates colimits in the category of coalgebras so that the tensor products $C \otimes K$, where K is a Λ_*-set, as well as the coproducts $C \otimes \Lambda^r \bigoplus_{C \otimes \mathbb{L}\Lambda^r} D \otimes \mathbb{L}\Lambda^r$ have the same realization in the category of dg-coalgebras as in the category of dg-modules.

§1.5.5. *The matching coalgebra of a Hopf Λ_*-module.* The fibrations can be characterized by the right lifting property as usual but, as in the context of dg-modules, one can introduced an appropriate matching object in the category of Hopf Λ_*-modules so that a morphism of Hopf Λ_*-modules $p : C \to D$ is a fibration if and only if the morphisms

$$(\mu, p) : C(r) \to \mathbb{M}C(r) \times_{\mathbb{M}D(r)} D(r)$$

define a fibration in the category of dg-coalgebras for all $r \in \mathbb{N}$. The matching object of a Hopf Λ_*-module $\mathbb{M}C$ is defined by the same limit as in the category of dg-modules

$$\mathbb{M}C(r) = \lim_{\alpha : \underline{r}' \xrightarrow{<} \underline{r}} C(r'),$$

where α ranges over morphisms $\alpha \in \Lambda^r_{r'}$ such that $r' < r$, except that we perform this limit in the category of dg-coalgebras. This matching object $\mathbb{M}C(r)$ can also be defined by an equalizer as in the category of dg-modules, but an equalizer of dg-coalgebras. Namely:

$$\mathbb{M}C(r) = \ker\big(\textstyle\prod_{1 \leq i \leq r} C(r-1) \overset{d^0}{\underset{d^1}{\rightrightarrows}} \prod_{1 \leq i < j \leq r} C(r-2) \big).$$

As in the context of Λ^*-modules, we have an induced model structure on the subcategory of $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$ formed by non-unital Hopf Λ^*-modules Γ. First, we observe that the adjunction relation of proposition §1.3.1 can be extended in the coalgebra context so that we have an adjunction ladder:

$$\begin{tikzcd} \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0 \arrow[r, "\mathrm{i}_0^\Lambda"] & \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod} \arrow[l, bend right, "\mathrm{c}_0^\Lambda"'] \arrow[l, bend left, "\mathrm{s}_0^\Lambda"] \end{tikzcd}.$$

In fact, the non-unital object $\mathrm{c}_0^\Lambda(C)$, respectively $\mathrm{s}_0^\Lambda(D)$, associated to a Hopf Λ_*-module C, respectively D, can be obtained as in the dg-module context except that in the definition of $\mathrm{s}_0^\Lambda(D)$ we perform the pullbacks

$$\begin{tikzcd} \mathrm{s}_0^\Lambda(D)(r) \arrow[r, dashed] \arrow[d, dashed] & D(r) \arrow[d, "\eta_0^*"] \\ 0 \arrow[r, "0"] & D(0) \end{tikzcd}.$$

in the category of coalgebras. As in the context of Λ_*-modules, we observe that the functor $\mathrm{i}_0^\Lambda\,\mathrm{c}_0^\Lambda$ preserves cofibrations and all weak-equivalences of Hopf Λ_*-modules. Then we obtain:

§1.5.6. **Proposition.** *The category of* $\mathbb{N}$*-graded Hopf* Λ_**-modules* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0$ *forms a model subcategory of* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$ *so that a morphism* $f : C \to D$ *is a weak-equivalence, respectively a cofibration, a fibration, in* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0$ *if and only if* f *defines a weak-equivalence, respectively a cofibration, a fibration, in* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$.

Furthermore, this category is cofibrantly generated by the morphisms $\mathrm{c}_0^\Lambda(i) : \mathrm{c}_0^\Lambda(A) \to \mathrm{c}_0^\Lambda(B)$ *associated to a generating set of cofibrations, respectively acyclic cofibrations, in* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$. □

As in the ground category of dg-modules, we obtain that the functor

$$\mathrm{s}_0^\Lambda : \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod} \to \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0$$

preserves fibrations, acyclic fibrations and weak-equivalences between fibrant objects by adjunction.

The category of unitary objects $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0^1$ is also equipped with the canonical model structure of a comma category.

Now we check that the results of §1.4 can be extended to the category of unital Hopf operads. Explicitly, the Reedy model structure of (non-unital) Hopf Λ_*-modules can be transferred to unital Hopf operads through the adjunction

$$\mathcal{F}_* : \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0^1 \rightleftarrows \mathrm{dg}\,\mathrm{HopfOp}_*^1 : (-)^-$$

so that we obtain the following theorem:

Theorem §1.D. *The category of* $\mathbb{N}$*-graded unital Hopf dg-operads* $\mathrm{dg}\,\mathrm{HopfOp}_*^1$ *is equipped with the structure of a cofibrantly generated model category such that a morphism* $\phi : \mathcal{P} \to \mathcal{Q}$ *is a weak-equivalence, respectively a fibration, if* ϕ *forms a Reedy weak-equivalence, respectively a Reedy fibration, in the category of Hopf* Λ_**-modules* $\mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_0^1$.

As in the case of unital operads, we check simply that the assumptions of the transfer principle are satisfied. Essentially, we check the following property:

§1.5.7. **Claim.** *Suppose given a pushout of unital Hopf operads*

$$\begin{tikzcd} \mathcal{F}_*(C) \arrow[r, "f"] \arrow[d, tail, "i"', "\sim"] & \mathcal{P} \arrow[d, "j"] \\ \mathcal{F}_*(D) \arrow[r, "g"] & \mathcal{Q} \end{tikzcd}$$

such that $i : \mathcal{F}_*(C) \to \mathcal{F}_*(D)$ *is a morphism of free operads induced by an acyclic cofibration of Hopf* Λ_**-modules* $i : C \xrightarrow{\sim} D$*. The morphism* $j : \mathcal{P} \to \mathcal{Q}$ *is a weak-equivalence as well.*

Proof. In fact, this property is clearly inherited from the category of unital operads since we observe in §1.5.2 that the forgetful functor creates the free functor $\mathcal{F}_*$ and the small colimits in the category of unital Hopf operads. □

§1.5.8. *Connected unital Hopf operads.* As in the context of operads in dg-modules, we can consider connected unital Hopf operads $\mathcal{P}$ characterized by $\mathcal{P}(1) = \mathbb{F}$ and the full category HopfOp_*^* formed by these objects. The category embedding $\mathrm{i}_*^1 : \mathrm{HopfOp}_*^* \hookrightarrow \mathrm{HopfOp}_*^1$ admits also a left and a right adjoint $\mathrm{c}_*^1, \mathrm{s}_*^1 : \mathrm{HopfOp}_*^1 \to \mathrm{HopfOp}_*^*$ such that $\mathrm{c}_*^1\,\mathrm{i}_*^1 = \mathrm{Id} = \mathrm{s}_*^1\,\mathrm{i}_*^1$.

In fact, we can consider the underlying categories of unital unitary Hopf Λ_*-modules which are associated to these operad categories and for which we have an adjunction ladder:

$$\begin{tikzcd} \Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_*^* \arrow[r, "\mathrm{i}_*^1"] & \Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_*^1 \arrow[l, bend right, "\mathrm{c}_*^1"'] \arrow[l, bend left, "\mathrm{s}_*^1"] \end{tikzcd}.$$

Then one checks readily that the connected unital unitary Hopf Λ_*-module $\mathrm{s}_*^1(\mathcal{Q})$, respectively $\mathrm{c}_*^1(\mathcal{P})$, associated to an operad $\mathcal{Q}$, respectively $\mathcal{P}$, forms a Hopf suboperad of $\mathcal{Q}$, respectively a quotient operad of $\mathcal{P}$, and the maps $\mathcal{Q} \mapsto \mathrm{s}_*^1(\mathcal{Q})$ and $\mathcal{P} \mapsto \mathrm{c}_*^1(\mathcal{P})$ supply required adjoint functors at the operad level.

The connected Hopf Λ_*-module $\mathrm{s}_*^1(N)$ associated to a unital unitary Hopf Λ_*-module $N \in \Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_*^1$ can be defined by the same pullback diagrams as in the category of dg-modules

$$\begin{tikzcd} \mathrm{s}_*^1(N)(r) \arrow[r, dashed] \arrow[d, dashed] & N(r) \arrow[d, "(\eta_i^*)_i"] \\ \mathbb{F}^{\times r} \arrow[r, "1^{\times r}"] & N(1)^{\times r} \end{tikzcd}$$

except that we consider cartesian products in the category of coalgebras and we perform these pullbacks in the same category. The other Hopf Λ_*-module $\mathrm{c}_*^1(M)$ is obtained by the same obvious construction as in the category of dg-modules. Namely we set $\mathrm{c}_*^1(M)(r) = \mathbb{F}$ for $r = 0, 1$ and $\mathrm{c}_*^1(M)(r) = M(r)$ for $r \geq 2$. The operations $\partial_i : \mathrm{c}_*^1(M)(r) \to \mathrm{c}_*^1(M)(r-1)$ are given either by the corresponding operations or by the augmentation of M.

As in the context of operads in dg-modules, we observe that theorem §1.D holds for the category of connected Hopf operads. More precisely, we obtain that the category of connected unital dg-operads $\mathrm{dg}\,\mathrm{HopfOp}_*^*$ forms a model subcategory of $\mathrm{dg}\,\mathrm{HopfOp}_*^1$ such that a morphism of connected unital dg-operads defines a Reedy cofibration, respectively a Reedy fibration, a weak-equivalence, in $\mathrm{dg}\,\mathrm{HopfOp}_*^*$ if and only if it defines a Reedy cofibration, respectively a Reedy fibration, a weak-equivalence, in $\mathrm{dg}\,\mathrm{HopfOp}_*^1$. Furthermore, we have:

§1.5.9. **Fact.** *The functor* $\mathrm{s}_*^1 : \mathrm{dg}\,\mathrm{HopfOp}_*^1 \to \mathrm{dg}\,\mathrm{HopfOp}_*^*$, *right adjoint to the category embedding* $\mathrm{i}_*^1 : \mathrm{dg}\,\mathrm{HopfOp}_*^* \to \mathrm{dg}\,\mathrm{HopfOp}_*^1$, *preserves fibrations, acyclic fibrations and all weak-equivalences between fibrant operads.*

In fact, since fibrations, respectively weak-equivalences, of unital Hopf operads are just fibrations, respectively weak-equivalences, of Hopf Λ_*-modules, we can also deduce this claim from the same assertion at the level of Hopf Λ_*-modules. To be precise, the statement above is implied by the analogous assertion for the category embedding $\mathrm{i}_*^1 : \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_*^* \hookrightarrow \mathrm{dg}\,\Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}_*^1$.

§1.6. **Prospects: cellular operads.** As explained in the memoir introduction, one might be willing to extend the results of this section to operads equipped with a good cellular structure. In regard to our construction, a good notion is supplied by functor operads on an operad $\mathcal{O}$ in Reedy categories. Explicitly, we assume that $\mathcal{O}$ consists of a sequence of categories equipped with a Reedy decomposition $\mathcal{O}(r) = \overrightarrow{\mathcal{O}}(r)\Sigma(r)\overleftarrow{\mathcal{O}}(r)$, where $\overrightarrow{\mathcal{O}}(r)$ is a direct Reedy category, $\overleftarrow{\mathcal{O}}(r)$ is an inverse Reedy category and $\Sigma(r)$ consists of isomorphisms, preserved by the operad structure.

Recall that an operad functor in a category $\mathcal{C}$ consists of a functor $F : \mathcal{O} \to \mathcal{C}$ equipped with a suitable generalization of an operad structure (see [20]). The definitions of this section can be generalized to this context. One has simply to consider the appropriate generalization of the notion of a Λ_*-module (functors

$M : \Lambda\mathcal{O} \to \mathcal{C}$, where $\Lambda\mathcal{O}$ denotes a semi-direct product of categories) and the corresponding matching and latching structures.

§2. On Boardman-Vogt' W-construction

§2.1. **Introduction.** The model category structure implies the existence of Reedy cofibrant replacements in the category of unital Hopf operads. As long as we deal with abstract structure results this purely existence theorem meets our needs. But for effective issues we need to have explicit cofibrant replacements. For the sake of completeness (we do not really use the constructions of this section in the memoir) and for subsequent references, we introduce in this section a differential graded analogue of the Boardman-Vogt W-construction that fulfils this need. Explicitly, we define a functor $\mathcal{P} \mapsto W(\mathcal{P})$, from the category of connected unital dg-operads to itself, endowed with the following features:

Theorem §2.A (Compare with Berger-Moerdijk [5], Boardman-Vogt [6])**.**

(a) *The operad $W(\mathcal{P})$is equipped with a natural morphism $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ which is a weak-equivalence of dg-operads.*

(b) *The operad $W(\mathcal{P})$ is quasi-free as a unital operad. To be more explicit, the operad $W(\mathcal{P})$ is defined by a free operad $\mathcal{F}_*(W'(\mathcal{P}))$ equipped with a differential*
$$\delta + \partial : \mathcal{F}_*(W'(\mathcal{P})) \to \mathcal{F}_*(W'(\mathcal{P}))$$
which is given by the addition of a homogeneous derivation $\partial : \mathcal{F}_*(W'(\mathcal{P})) \to \mathcal{F}_*(W'(\mathcal{P}))$, determined by the operad structure of $\mathcal{P}$, to the canonical differential $\delta : \mathcal{F}_*(W'(\mathcal{P})) \to \mathcal{F}_*(W'(\mathcal{P}))$, induced by the internal differential of $\mathcal{P}$. Furthermore, the morphism $\phi_* : W(\mathcal{P}) \to W(\mathcal{P}')$ associated to an operad morphism $\phi : \mathcal{P} \to \mathcal{P}'$ is the morphism of quasi-free operads induced by the morphism of Λ_*-modules $\phi_* : W'(\mathcal{P}) \to W'(\mathcal{P}')$ associated to ϕ. The morphism $\phi_* : W(\mathcal{P}) \to W(\mathcal{P}')$ defines a cofibration in the Reedy model category of operads provided that $\phi : \mathcal{P} \to \mathcal{P}'$ forms a cofibration in the model category of Λ_*-modules.

(c) *If an operad $\mathcal{P}$ is equipped with the structure of a Hopf operad, then the associated W-construction can still be equipped with a coassociative diagonal functorially in $\mathcal{P}$ so that $\mathcal{P} \mapsto W(\mathcal{P})$ defines a functor from the category of connected unital Hopf operads to itself. Moreover, the augmentation $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ defines a weak-equivalence of unital Hopf operads.*

We refer to §2.3.10 for the general definition of a quasi-free object in the category of unital operads.

In fact, as observed by Berger-Moerdijk in [5], an analogue of the Boardman-Vogt W-construction can be defined in any monoidal model category equipped with a good interval $\mathbb{I}$. Our differential graded W-construction can be identified with an instance of this general W-construction for the category of dg-modules, respectively for the category of dg-coalgebras. On the other hand, the classical topological Boardman-Vogt W-construction, introduced in [6], is defined by a cellular complex. Our differential graded W-construction is precisely the chain complex defined by this cellular object.

In this section, we give a precise account of the definition of $W(\mathcal{P})$, though this construction is not original, in order to give insights into the structure of this operad in the unital context. To be precise, we have to check that $W(\mathcal{P})$ that yield cofibrant resolutions in the Reedy model category as stated in theorem §2.A. An analogous statement for the adjoint model structure is proved in [5] but, for our purpose, we give other arguments. The Boardman-Vogt construction is by definition a cellular object in the ground model category. In [5, 6] the authors deal only with this cellular structure. In this memoir, we prove that the Boardman-Vogt construction $W(\mathcal{P})$ defines a cellular object in the category of operads. Explicitly, we prove the following theorem that implies the cofibrancy claim of theorem §2.A:

Theorem §2.B. *The Boardman-Vogt construction $W(\mathcal{P})$ is the colimit of a sequence of (Hopf) operads*

$$* = W^{-1}(\mathcal{P}) \xrightarrow{j_0} W^0(\mathcal{P}) \xrightarrow{j_1} \dots \xrightarrow{j_d} W^d(\mathcal{P}) \xrightarrow{j_{d+1}} \dots$$
$$\dots \to \operatorname{colim}_d W^d(\mathcal{P}) = W(\mathcal{P})$$

obtained by pushouts

$$\begin{tikzcd} \mathcal{F}_*(C^d(\mathcal{P})) \arrow[r, "f^d"] \arrow[d, "i^d"] & W^{d-1}(\mathcal{P}) \arrow[d, dashed, "j_d"] \\ \mathcal{F}_*(D^d(\mathcal{P})) \arrow[r, dashed, "g^d"] & W^d(\mathcal{P}) \end{tikzcd},$$

where $i^d : \mathcal{F}_*(C^d(\mathcal{P})) \to \mathcal{F}_*(D^d(\mathcal{P}))$ is a morphism of free operads associated to a morphism of unitary (Hopf) Λ_*-modules $i^d : C^d(\mathcal{P}) \to D^d(\mathcal{P})$.

This decomposition is functorial in $\mathcal{P}$ and in addition the canonical morphism of unitary (Hopf) Λ_*-modules

$$(i^d, \phi) : C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P})} D^d(\mathcal{P}) \to D^d(\mathcal{P}')$$

associated to an operad morphism $\phi : \mathcal{P} \to \mathcal{P}'$ is a weak-equivalence, respectively a Reedy cofibration, if ϕ defines a weak-equivalence, respectively a Reedy cofibration, in the category of Λ_*-modules. In particular, the morphism $i^d : C^d(\mathcal{P}) \to D^d(\mathcal{P})$ is a Reedy cofibration if the operad $\mathcal{P}$ forms a Reedy cofibrant object in the category of Λ_*-modules.

The decomposition of theorem §2.B is referred to as the operadic cellular decomposition of the Boardman-Vogt operad $W(\mathcal{P})$.

Theorem §2.B holds in the general framework of [5]. Nevertheless, for simplicity, we make our construction explicit and we give comprehensive proofs only in the context of dg-modules and dg-coalgebras.

We need to recall conventions and results on trees that give the structure of the W-construction. We devote the next subsection §2.2 to this topic. We achieve the definition of the W-construction in the differential graded context and we prove the assertions of theorem §2.A in §2.3. We define the operadic cellular decomposition of $W(\mathcal{P})$ and we prove theorem §2.B in §2.4 of this section. The figures referred to in the text are displayed in the appendix subsection **??**.

As explained, the subsection §2.3 does not provide any original result. Nevertheless our presentation differs from [5, 6] at some points since we aim to introduce another cellular structure on the Boardman-Vogt construction.

Recall again that we do not really use the constructions of this section in the memoir. In regard to the needs of the next parts, this section can be skipped in a first reading. The account of this section is motivated by the sake of references in view of effective constructions of operad actions which are postponed to future articles.

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§2.2. **Cell metric trees.** In the topological context the space $W(\mathcal{P})(r)$ defined by Boardman-Vogt in [6] is a cubical cellular complex formed by trees whose vertices are labeled with operations of $\mathcal{P}$ and whose edges are equipped with a length. The authors of [5] consider in fact a generalization of this cellular construction in other categories than topological spaces.

As stated in the section outline, the aim of this subsection is to recall our conventions for tree structures, borrowed from [12], so that we can make precise the definition of $W(\mathcal{P})$. In addition we prove that the metric trees considered by Boardman-Vogt form a cellular complex endowed with good homotopical properties.

Recall that we consider nothing but the chain complex of Boardman-Vogt' cellular construction. As a byproduct, one can observe simply that $W(\mathcal{P})$ is equipped with a diagonal and forms a Hopf operad if $\mathcal{P}$ is so because $W(\mathcal{P})$ is defined by a cubical cellular complex.

§2.2.1. *The chain interval.* In this memoir we let $\mathbb{I}$ denote the standard cellular complex of the topological interval $[0,1]$. Explicitly, the dg-module $\mathbb{I}$ is generated by elements $0, 1$ in degree 0, by an element 01 in degree 1 and is equipped with the differential such that $\delta(01) = 1 - 0$. One should not be confused by this notation 0 for a basis element because we consider only basis elements in $\mathbb{I}$. Recall that $\mathbb{I}$ defines a cylinder object in the category of dg-modules. To be explicit, we have a morphism $\eta^0 : \mathbb{F} \to \mathbb{I}$, respectively $\eta^1 : \mathbb{F} \to \mathbb{I}$, that maps the ground field to the submodule $\mathbb{F}\,0 \subset \mathbb{I}$, respectively $\mathbb{F}\,1 \subset \mathbb{I}$, and an augmentation $\epsilon : \mathbb{I} \xrightarrow{\sim} \mathbb{F}$ that cancels $01 \in \mathbb{I}$ and such that $\epsilon \cdot \eta^0 = \epsilon \cdot \eta^1 = \mathrm{Id}$.

In addition we have a morphism of dg-modules $\mu : \mathbb{I} \otimes \mathbb{I} \to \mathbb{I}$ induced by the map $(s,t) \mapsto \max(s,t)$ of the topological interval. Explicitly, this morphism satisfies

$$\mu(0,0) = 0, \qquad \mu(01,0) = \mu(0,01) = 01, \qquad \mu(0,1) = \mu(1,0) = \mu(1,1) = 1$$

and vanishes in the other cases.

Observe that $\mathbb{I}$ is equipped with a coassociative diagonal $\Delta : \mathbb{I} \to \mathbb{I} \otimes \mathbb{I}$ defined by the classical formulas

$$\Delta(0) = 0 \otimes 0, \quad \Delta(1) = 1 \otimes 1, \quad \text{and} \quad \Delta(01) = 0 \otimes 01 + 01 \otimes 1.$$

One checks readily that the morphisms η^0, η^1, ϵ and μ are morphisms of dg-coalgebras.

Recall that, according to [4], the background of the Boardman-Vogt construction is provided by a monoidal model category equipped with an interval. The dg-module $\mathbb{I}$ is precisely the standard interval in the category of dg-modules and in the category of dg-coalgebras.

§2.2.2. *Tree structures.* As specified in the introduction, for tree structures, we adopt the conventions of our article [12] and we refer to this reference for more precision.

Recall that an *r-tree* refers to an abstract tree τ defined by a set of vertices $V(\tau)$ and by a set of edges $E(\tau)$, oriented from a source to a target, equipped with one outgoing edge (the *root* of the tree) that targets to an element denoted by 0 and r ingoing edges (the *leaves* of the tree) in one-to-one correspondence with the elements of $\{1,\dots,r\}$ that give the source of these edges (see figure 1).

Formally the source of an edge $e\in\mathcal{E}(\tau)$ is specified by an element $s(e)\in V(\tau)\amalg\{1,\dots,r\}$ and the target by an element $t(e)\in V(\tau)\amalg\{0\}$. For $x\in V(\tau)\amalg\{1,\dots,r\}$, we assume that there is one and only one edge $e\in E(\tau)$ with source $s(e)=x$, so that the functions s,t define a tree properly, and we assume that there is one and only one edge such that $t(e)=0$, so that this edge represents the root of the tree. The leaves of τ are the edges with source $s(e)\in\{1,\dots,r\}$.

Recall that the tree structure is uniquely determined by a partition $V(\tau)\amalg\{1,\dots,r\}=\coprod_{v\in V(\tau)\amalg\{0\}} I_v$, indexed by the set $V(\tau)\amalg\{0\}$, characterized by $x\in I_y$ if and only if x and y are respectively the source and the target of an edge of the tree. Hence the component I_v associated to a vertex v represents the set of entries of the vertex v in the tree τ. In the example represented in figure 5 we have $V(\tau)=\{v_1,v_2,v_3,v_4\}$, and $I_0=\{v_1\}$, $I_{v_1}=\{1,v_2,v_3\}$, $I_{v_2}=\{3,v_4\}$, $I_{v_3}=\{4,5\}$, $I_{v_4}=\{2,6\}$.

The set of *internal edges* of the tree τ, denoted by $E'(\tau)$, consists of the edges e which are neither a leave or the root of τ. Thus an edge e is internal if and only if $s(e),t(e)\in V(\tau)$.

§2.2.3. *Tree morphisms and edge contractions.* In our construction we consider a category of r-trees, denoted by $\Theta(r)$, in which a morphism $f:\tau\to\tau'$ is defined by a contraction of internal edges $e\in E(\tau)$. Formally, a morphism of r-trees $f:\tau\to\tau'$ is defined by a pair of maps $f_V:V(\tau)\amalg\{0,1,\dots,r\}\to V(\tau')\amalg\{0,1,\dots,r\}$ and $f_E:E(\tau)\to E(\tau')\amalg V(\tau')$ endowed with the following properties:

(a) the map f_V is the identity on $\{0,1,\dots,r\}$;
(b) for an edge $e\in E(\tau)$ such that $e\in f_E^{-1}V(\tau')$ we have $f_V(s(e))=f_V(t(e))=f_E(e)$;
(c) for an edge $e\in E(\tau)$ such that $e\in f_E^{-1}E(\tau')$ we have $f_V(s(e))=s(f_E(e))$ and $f_V(t(e))=t(f_E(e))$.

An example of a tree morphism is represented in figure 2. In this example the map f_V is defined by $f_V(v_3)=f_V(v_4)=w_2$ and $f_V(v_1)=f_V(v_2)=w_1$. Observe that the previous assertions imply that the map f_V is onto and f_E induces a one-to-one correspondence between $f_E^{-1}E(\tau')\subset E(\tau)$ and $E(\tau')$. Moreover, the map f_E is clearly uniquely determined by f_V and conversely.

Intuitively, the edges $e\in E(\tau)$ such that $e\in f_E^{-1}V(\tau')$ are contracted to the vertex $v'=f_E(e)$ by the morphism $f:\tau\to\tau'$ and the other edges are preserved. Our assumptions imply that the leaves and the root are fixed by a morphism of r-trees and, as a consequence, only internal edges are allowed to be contracted. Furthermore, we can observe that the subsets $f_V^{-1}(v')\subset V(\tau)$ and $f_E^{-1}(v')\subset E(\tau)$ associated to a vertex $v'\in V(\tau')$ determine a subtree $\sigma_{v'}$ of τ. In addition the entry set of v' can be identified with the image under the map f_V of the entry set of this subtree $\sigma_{v'}$ (we refer to [12] for the formal definition of these notions). In fact, the tree τ' can be determined by a generalization of the quotient process of [12]. Namely, if we perform a sequence of quotients by the subtrees $\sigma_{v'}$ for $v'\in V(\tau')$, then we obtain the tree τ' up to isomorphism.

Clearly, a morphism of r-trees is an isomorphism if and only if the map f_V is one-to-one and f_E defines a one-to-one correspondence between edge sets.

One can also observe that a morphism of r-trees $f : \tau \to \tau'$ that contracts a single edge $e_0 \in E(\tau)$ is equivalent to the contraction process $\tau \mapsto \tau/e_0$ defined in [12]. To be precise, for an internal edge $e_0 \in E(\tau)$ one considers the tree τ/e_0 obtained by identifying the source $s_0 = s(e_0)$ and the target $t_0 = t(e_0)$ of e_0 in τ to a single vertex $s_0 \equiv t_0$ whose entries are defined by the union $I_{t_0} \setminus \{s_0\} \amalg I_{s_0}$ of the entries of s_0 and t_0 in τ. Hence we have $V(\tau/e_0) = V(\tau)/\{s_0 \equiv t_0\}$ and $E(\tau/e_0) = E(\tau) \setminus \{e_0\}$. Furthermore, the quotient map $(\gamma_{e_0})_V : V(\tau) \to V(\tau/e_0)$ and the map $(\gamma_{e_0})_E : E(\tau) \to E(\tau/e_0) \amalg V(\tau/e_0)$ such that $(\gamma_{e_0})_E(e_0) = \{s_0 \equiv t_0\}$ and $(\gamma_{e_0})_E(e) = e$ for $e \neq e_0$ defines a morphism of r-trees $\gamma_{e_0} : \tau \to \tau/e_0$ which is called an *edge contraction*. As an example, the contraction of the edge $v_3 \to v_1$ in the tree of figure 1 gives the tree represented in figure 3.

Clearly, a morphism of r-trees is a composite of tree isomorphisms and edge contractions.

§2.2.4. *Tree categories.* We call a tree *n-reduced* if all vertices $v \in V(\tau)$ have more than n entries. For instance, a tree τ is called 0-reduced if it has no *terminal vertex* (a vertex v is terminal if $I_v = \emptyset$).

Observe that a 0-reduced r-tree τ has no automorphism. Accordingly, if we fix a representative for each isomorphism class of 0-reduced tree, then we obtain a category, denoted by $\Theta'(r)$, equivalent to the full subcategory of $\Theta(r)$ generated by 0-reduced trees in which all isomorphisms are identities. Hence any morphism in $\Theta'(r)$ is a composite of edge contractions. Moreover, one can observe that $\Theta'(r)$ defines a poset equipped with a terminal element represented by the unique r-tree τ_r with one vertex and no internal edge (see figure 4).

The category $\Theta'(r)$ is also equipped with a natural grading $\mathrm{gr} : \Theta'(r) \to \mathbb{N}$ given by the number of internal edges and we let $\Theta'_d(r)$ denote the subcategory of $\Theta'(r)$ generated by trees of grading $\leq d$. Observe that $\Theta'_0(r)$ contains only of the terminal r-tree τ_r. Clearly, any non-identity morphism of $\Theta'(r)$ decreases the grading so that $\Theta'(r)$ defines an inverse Reedy category.

In fact, in the constructions of this section we consider only the subcategory of $\Theta'(r)$ formed by 1-reduced r-trees. Therefore we introduce the notation $\Theta''(r)$ for this category and we let also $\Theta''_d(r)$ denote the subcategory of $\Theta'(r)$ formed by 1-reduced r-trees with no more than d internal edges.

§2.2.5. *Cell metric trees and length tensors.* In the definition of the Boardman-Vogt complex we consider *cell metric trees τ* equipped with a *length tensor* defined by a tensor product of interval elements indexed by the internal edges of τ:

$$\lambda = \bigotimes_{e \in E'(\tau)} \lambda_e \in \bigotimes_{e \in E'(\tau)} \mathbb{I}\,.$$

In fact, we identify abusively a length tensor with a basis element of the tensor product $\bigotimes_{e \in E'(\tau)} \mathbb{I}$, so that a length tensor is defined by a map $e \mapsto \lambda_e$ which associates to any internal edge $e \in E'(\tau)$ an element $\lambda_e \in \mathbb{I}$. In the representation of a tree we decorate the internal edges e by the corresponding length λ_e as in figure 5.

The module of length tensors is denoted by

$$\mathbb{D}^\tau = \bigotimes_{e \in E'(\tau)} \mathbb{I}\,.$$

In fact, this module is nothing but the cellular complex of the cube build on the internal edges of the tree τ. One observes that the map $\tau \mapsto \mathbb{D}^\tau$ can be extended to a contravariant functor from the category of trees to the category of dg-modules. Explicitly, a tree morphism $f:\tau\to\tau'$ induces a dg-module morphism $f^*:\mathbb{D}^{\tau'}\to\mathbb{D}^{\tau}$ that assigns the length $\lambda_e=0$ to the edges $e\in E'(\tau)$ which are contracted to a vertex by $f:\tau\to\tau'$ and preserves the length of the other edges $e\in E'(\tau)$ which are mapped to an edge of τ'. Formally, a morphism of trees $f:\tau\to\tau'$ yields a partition $E'(\tau)=f_E^{-1}V(\tau')\amalg f_E^{-1}E'(\tau')$ and f induces a bijection from $f_E^{-1}E'(\tau')$ to $E'(\tau')$. The dg-module morphism $f^*:\mathbb{D}^{\tau'}\to\mathbb{D}^{\tau}$ is given by the tensor product of the canonical isomorphism

$$\bigotimes_{e'\in E'(\tau')}\mathbb{I}\simeq\bigotimes_{e\in f_E^{-1}E'(\tau')}\mathbb{I}$$

with the unit map

$$\bigotimes_{e\in f_E^{-1}V(\tau')}\eta^0:\bigotimes_{e\in f_E^{-1}V(\tau')}\mathbb{F}\,0\to\bigotimes_{e\in f_E^{-1}V(\tau')}\mathbb{I}\,.$$

Hence this morphism identifies the module

$$\mathbb{D}^{\tau'}=\bigotimes_{e'\in E'(\tau')}\mathbb{I}$$

with the module of length tensors

$$\Bigl[\bigotimes_{e\in f_E^{-1}E'(\tau')}\mathbb{I}\Bigr]\otimes\Bigl[\bigotimes_{e\in f_E^{-1}V(\tau')}0\Bigr]\subset\bigotimes_{e\in E'(\tau)}\mathbb{I}=\mathbb{D}^{\tau}$$

in which the edges $e\in f_E^{-1}V(\tau')$, that are contracted to a vertex in τ', have length $\lambda_e=0$. As an example, an edge contraction $\gamma_{e_0}:\tau\to\tau/e_0$ induces a dg-module morphism $(\gamma_{e_0})^*:\mathbb{D}^{\tau/e_0}\to\mathbb{D}^{\tau}$ that identifies the module $\mathbb{D}^{\tau/e_0}$ to the module of length tensors $\lambda\in\mathbb{D}^{\tau}$ such that $\lambda_{e_0}=0$.

Finally, as our chain interval $\mathbb{I}$, equipped with a coassociative diagonal, defines an interval in the category of dg-coalgebras and not only to the category of dg-modules, we observe that the construction of this paragraph gives a functor $\tau\mapsto\mathbb{D}^{\tau}$ from the category of trees to the category of dg-coalgebras as well. Explicitly, the diagonal of $\mathbb{D}^{\tau}$ is given by the composite of tensor products of the diagonal of $\mathbb{I}$ with the obvious tensor permutation:

$$\bigotimes_{e\in E'(\tau)}\mathbb{I}\to\bigotimes_{e\in E'(\tau)}(\mathbb{I}\otimes\mathbb{I})\xrightarrow{\simeq}\Bigl[\bigotimes_{e\in E'(\tau)}\mathbb{I}\Bigr]\otimes\Bigl[\bigotimes_{e\in E'(\tau)}\mathbb{I}\Bigr].$$

Equivalently, in the definition of $\mathbb{D}^{\tau}=\bigotimes_{e\in E'(\tau)}\mathbb{I}$ we consider a tensor product in the monoidal category of dg-coalgebras and not only in the category of dg-modules. The morphism $f^*:\mathbb{D}^{\tau'}\to\mathbb{D}^{\tau}$ induced by a tree morphism defines clearly a morphism of dg-coalgebras as the definition of f^* can be deduced from the axioms of symmetric monoidal categories.

§2.2.6. *Coends over the category of trees.* In this section we consider coends

$$\int_{\tau\in\Theta''(r)}\Phi_\tau\otimes\mathbb{D}^{\tau}$$

associated to covariant functors $\tau\mapsto\Phi_\tau$ from the category of 1-reduced trees $\Theta''(r)$ (see §2.2.4) to the category of dg-modules, respectively dg-coalgebras. Recall that the coends in the category of dg-coalgebras are created in the category of dg-modules like all colimits.

We aim to determine the homotopy type of these coends. We prove precisely that $\int_{\tau\in\Theta''(r)}\Phi_\tau\otimes\mathbb{D}^{\tau}$ is homotopy equivalent to the object Φ_{τ_r} associated to the terminal r-tree τ_r. We prove in addition that the coend morphism

$$\phi_* : \int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \to \int_{\tau\in\Theta''(r)} \Psi_\tau \otimes \mathbb{D}^\tau$$

induced by a functor morphism $\phi_\tau : \Phi_\tau \to \Psi_\tau$ forms a cofibration, respectively an acyclic cofibration, if $\phi_\tau : \Phi_\tau \to \Psi_\tau$ is a pointwise cofibration, respectively a pointwise acyclic cofibration.

For these purposes we assume that the category of contravariant functors from the category of trees $\Theta''(r)$ to the category of dg-modules, respectively to the category of dg-coalgebras, is equipped with a Reedy model structure. Explicitly, for contravariant functors $\tau \mapsto C^\tau$ we have a latching object, defined by

$$\mathbb{L}C^\tau = \operatorname{colim}_{\tau \xrightarrow{\neq} \tau'} C^{\tau'},$$

and a functor morphism $\phi_\tau : C^\tau \to D^\tau$ is a Reedy cofibration, respectively an acyclic Reedy cofibration, if and only if, for each tree $\tau \in \Theta''(r)$, the morphism

$$(\phi, \lambda) : C^\tau \bigoplus_{\mathbb{L}C^\tau} \mathbb{L}D^\tau \to D^\tau$$

defines a cofibration, respectively an acyclic cofibration, in the ground category. Fibrations and weak-equivalences are defined pointwise. The definitions are dual for covariant functors $\tau \mapsto \Phi_\tau$ but we do not consider fibrations and matching objects explicitly in this case.

Recall that a morphism of dg-coalgebras is a cofibration, respectively an acyclic cofibration, if and only if it defines a cofibration, respectively an acyclic cofibration, in the category of dg-modules. The same statement holds for functor morphisms since colimits and latching objects in the category of dg-coalgebras are created in the category of dg-modules. As a byproduct, the dg-coalgebra case of the next statements is always an immediate consequence of the dg-module case. Therefore we give proofs in the context of dg-modules and omit to give more precision in the context of dg-coalgebras. On the other hand, one can observe that our assertions and our arguments can be extended to monoidal model categories including both the category of dg-modules and the category of dg-coalgebras.

We deduce our results from forthcoming observations about the functor $\tau \mapsto \mathbb{D}^\tau$ defined by the modules of length tensors.

§2.2.7. *Homotopy properties of the modules of length tensors.* We observe that the functor $\tau \mapsto \mathbb{D}^\tau$ is connected to the constant functor $\mathbb{F}$ by equivalences of functors on $\Theta''(r)$. To be more explicit, for each tree τ we have a morphism $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ induced by $\eta^0 : \mathbb{F} \to \mathbb{I}$. One checks readily that η^0_* commutes with the morphisms $f^* : \mathbb{D}^{\tau'} \to \mathbb{D}^\tau$ induced by a tree morphism $f : \tau \to \tau'$ so that η^0_* defines a morphism of functors on the category of trees. Similarly, one can check that the augmentation $\epsilon : \mathbb{I} \to \mathbb{F}$ induces a morphism of functors $\epsilon_* : \mathbb{D}^\tau \to \mathbb{F}$ such that $\epsilon_* \cdot \eta^0_* = \mathrm{Id}$. Finally, one can observe that $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ and $\epsilon_* : \mathbb{D}^\tau \to \mathbb{F}$ are coalgebra morphisms because so are $\eta^0 : \mathbb{F} \to \mathbb{I}$ and $\epsilon : \mathbb{I} \to \mathbb{F}$. We have the following assertion:

§2.2.8. **Observation.** *The morphism $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ is an acyclic Reedy cofibration in the category of contravariant functors from $\Theta''(r)$ to the category of dg-modules, respectively to the category of dg-coalgebras. As a corollary, the morphism ϵ_* : $\mathbb{D}^\tau \to \mathbb{F}$, which is right inverse to η^0_*, defines a weak equivalence of functors as well.*

Proof. Recall that η^0_* defines an acyclic Reedy cofibration if and only if the morphism

$$(\eta^0_*, \lambda) : \mathbb{F} \bigoplus_{\mathbb{L}\mathbb{F}^\tau} \mathbb{L}\mathbb{D}^\tau \to \mathbb{D}^\tau$$

defines an acyclic cofibration in the category of dg-modules for all $\tau \in \Theta''(r)$.

Clearly, for the constant functor we have $\mathbb{L}\,\mathbb{F}^\tau = 0$ if $\tau \neq \tau_r$, the terminal r-tree, and $\mathbb{L}\,\mathbb{F}^{\tau_r} = \mathbb{F}$. Hence we obtain

$$\mathbb{F} \bigoplus_{\mathbb{L}\,\mathbb{F}^\tau} \mathbb{L}\mathbb{D}^\tau = \mathbb{L}\mathbb{D}^\tau \quad \text{if } \tau \neq \tau_r \qquad \text{and} \qquad \mathbb{F} \bigoplus_{\mathbb{L}\,\mathbb{F}^{\tau_r}} \mathbb{L}\mathbb{D}^{\tau_r} = \mathbb{F} = \mathbb{D}^{\tau_r}.$$

(Recall that $\mathbb{D}^{\tau_r} = \mathbb{F}$ since τ_r has no internal edge.) Consequently, the morphism $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ is an acyclic Reedy cofibration if an only if the canonical map $\mathbb{L}\mathbb{D}^\tau \to \mathbb{D}^\tau$ is an acyclic cofibration of dg-modules (or dg-coalgebras) for $\tau \neq \tau_r$.

This assertion can be deduced from the axioms of monoidal model categories and from the general properties of an interval. On the other hand, in the framework of dg-modules one can observe simply that the dg-module $\mathbb{L}\mathbb{D}^\tau$ is identified with the submodule of $\mathbb{D}^\tau$ spanned by length tensors $\lambda = \bigotimes_e \lambda_e$ in which some edges have length $\lambda_e = 0$. Consequently, the quotient $\mathbb{D}^\tau/\mathbb{L}\mathbb{D}^\tau$ can be identified with a tensor product of dg-modules $\mathbb{F}\,01 \xrightarrow{\partial_1} \mathbb{F}\,1$ which are clearly acyclic. □

By the way, we observe in this proof that the natural morphism

$$\mathbb{L}\mathbb{D}^\tau\tau = \operatorname{colim}_{f:\tau' \xrightarrow{\neq} \tau} \mathbb{D}^{\tau'} \to \mathbb{D}^\tau$$

defines an embedding from the latching object $\mathbb{L}\mathbb{D}^\tau$ to $\mathbb{D}^\tau$. Accordingly, we have the following result:

§2.2.9. **Observation.** *The functor $\tau \mapsto \mathbb{D}^\tau$ defines a Reedy cofibrant object in the category of contravariant functors from the category of trees $\Theta''(r)$ to the category of dg-modules, respectively to the category of dg-coalgebras.* □

We can now prove the results announced in §2.2.6. First, we have the following assertion that arises as a consequence of observation §2.2.8:

§2.2.10. **Claim.** *The coend morphism*

$$\eta^0_* : \int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{F} \to \int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau$$

induced by $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ defines an acyclic cofibration of dg-modules, respectively dg-coalgebras. As a consequence, the morphism $\epsilon_* : \mathbb{D}^\tau \to \mathbb{F}$, which is right inverse to η^0_*, defines an inverse weak equivalence of dg-modules, respectively dg-coalgebras,

$$\epsilon_* : \int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \xrightarrow{\sim} \int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{F}$$

such that $\epsilon_* \eta^0_* = \mathrm{Id}$.

Proof. To be explicit, recall that $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ defines by observation §2.2.8 an acyclic Reedy cofibration of functors on $\Theta''(r)$. Accordingly, the claim is an instance of a general statement that can be deduced from the categorical definition of a coend and from the axioms of monoidal model categories. Explicitly, for a fibration of dg-modules $p : C \twoheadrightarrow D$ the lifting problem

$$\begin{tikzcd}
\int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{F} \arrow[r] \arrow[d] & C \arrow[d, "p"] \\
\int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

is equivalent to an adjoint lifting problem

$$\begin{tikzcd} \mathbb{F} \arrow[r] \arrow[d, "\eta^0_*"'] & Hom(\Phi_\tau, C) \arrow[d, "p_*"] \\ \mathbb{D}^\tau \arrow[r] \arrow[ur, dashed] & Hom(\Phi_\tau, D) \end{tikzcd}$$

in the category of contravariant functors in $\tau \in \Theta''(r)$. This lifting problem admits a solution since a fibration of dg-modules $p : C \to D$ induces a Reedy fibration on internal hom-objects $p_* : Hom(\Phi_\tau, C) \twoheadrightarrow Hom(\Phi_\tau, D)$ and since $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$ is an acyclic Reedy cofibration. Accordingly, our morphism has the left lifting property with respect to fibrations of dg-modules and the conclusion follows. □

As the category of trees admits a terminal object τ_r, we obtain in addition:

§2.2.11. **Observation.** *We have an isomorphism* $\int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{F} \simeq \Phi_{\tau_r}$. □

Accordingly, we obtain finally:

§2.2.12. **Lemma.** *Let $\tau \mapsto \Phi_\tau$ denote a covariant functor from the category of trees to the category of dg-modules, respectively dg-coalgebras.*

The morphism $\epsilon_* : \mathbb{D}^\tau \to \mathbb{F}$ induces a weak-equivalence of dg-modules, respectively dg-coalgebras,

$$\int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \xrightarrow[\epsilon_*]{\sim} \int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{F} \simeq \Phi_{\tau_r},$$

which has a natural inverse equivalence, such that $\epsilon_*\eta^0_* = \mathrm{Id}$, induced by $\eta^0_* : \mathbb{F} \to \mathbb{D}^\tau$. □

As stated in §2.2.6, we have also the following assertion which arises as a consequence of observation §2.2.9:

§2.2.13. **Lemma.** *Let $\phi_\tau : \Phi_\tau \to \Psi_\tau$ denote a pointwise cofibration of covariant functors from the category of trees to the category of dg-modules, respectively to the category of dg-coalgebras. The associated coend morphism*

$$\phi_* : \int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \to \int_{\tau\in\Theta''(r)} \Psi_\tau \otimes \mathbb{D}^\tau$$

defines a cofibration in the category of dg-modules, respectively in the category of dg-coalgebras.

Proof. As in the proof of claim §2.2.10, we observe that a lifting problem in the category of dg-modules

$$\begin{tikzcd} \int_{\tau\in\Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \arrow[r] \arrow[d, "\phi_*"'] & C \arrow[d, two heads, "\sim"', "p"] \\ \int_{\tau\in\Theta''(r)} \Psi_\tau \otimes \mathbb{D}^\tau \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} ,$$

where $p : C \xrightarrow{\sim} D$ is an acyclic fibration, is equivalent to an adjoint lifting problem

$$\begin{tikzcd} \Phi_\tau \arrow[r] \arrow[d, "\phi"'] & Hom(\mathbb{D}^\tau, C) \arrow[d, "p_*"] \\ \Psi_\tau \arrow[r] \arrow[ur, dashed] & Hom(\mathbb{D}^\tau, D) \end{tikzcd}$$

in the category of functors in $\tau \in \Theta''(r)$. Recall that the modules of length tensors $\mathbb{D}^\tau$ define a Reedy cofibrant object in the category of contravariant functors on $\Theta''(r)$. As a consequence, the natural transformation $p_* : Hom(\mathbb{D}^\tau, C) \to Hom(\mathbb{D}^\tau, D)$, induced by an acyclic fibration of dg-modules $p : C \xrightarrow{\sim} D$, defines a Reedy fibration in the category of covariant functors on $\Theta''(r)$. Hence we conclude from our observation that the morphism $\phi_* : \int_{\tau \in \Theta''(r)} \Phi_\tau \otimes \mathbb{D}^\tau \to \int_{\tau \in \Theta''(r)} \Psi_\tau \otimes \mathbb{D}^\tau$ satisfies the left lifting property with respect to acyclic fibrations and defines a cofibration of dg-modules. □

§2.3. **The Boardman-Vogt construction.** As announced in the section introduction, the purpose of this subsection is to give a precise account of the Boardman-Vogt construction in the differential graded context and to prove the properties of this construction asserted in theorem §2.A up to the cofibrancy claim in assertion (b).

In our construction, we consider only connected unital operads $\mathcal{P}$ such that $\mathcal{P}(0) = \mathbb{F}$ and $\mathcal{P}(1) = \mathbb{F}$. In this context we set $W(\mathcal{P})(0) = \mathbb{F}$ and $W(\mathcal{P})(1) = \mathbb{F}$ so that $W(\mathcal{P})$ is still a connected unital operad. In what follows we give first the definition of the dg-modules $W(\mathcal{P})(r)$ for $r \geq 2$. Then we define the partial composites $\circ_i : W(\mathcal{P})(s) \otimes W(\mathcal{P})(t) \to W(\mathcal{P})(s+t-1)$ for operations such that $s, t \geq 2$ and we make a particular case for composites with unital operations. In view of the construction of [5], the connectedness assumption $\mathcal{P}(1) = \mathbb{F}$ is not necessary but simplifies our construction.

The assertion (c) of theorem §2.A, the Hopf operad structure, arises from the definition of $W(\mathcal{P})$. Only assertions (a) and (b) require an actual proof supplied by claim §2.3.9 and claim §2.3.11 respectively.

§2.3.1. *Treewise tensor products.* For a finite set $I = \{i_1, \ldots, i_r\}$, we form the module $\mathcal{P}(I)$ whose elements represent operations in r variables indexed by $\{i_1, \ldots, i_r\}$ instead of the integers $\{1, \ldots, r\}$. For finite sets I, J and an element $i \in I$, we have a partial composite operation $\circ_i : \mathcal{P}(I) \otimes \mathcal{P}(J) \to \mathcal{P}(I \setminus \{i\} \amalg J)$.

As in [12], given a tree τ, we let $\tau(\mathcal{P})$ denote the module of tensors

$$\underline{p} = \bigotimes_{v \in V(\tau)} p_v \in \bigotimes_{v \in V(\tau)} \mathcal{P}(I_v)$$

that represent labelings of vertices of τ by operations $p_v \in \mathcal{P}(I_v)$ whose variables are in bijection with the entry set of the associated vertex I_v. In fact, we shall consider only 1-reduced trees whose vertices have all at least to 2 entries. Thus, for the factors $p_v \in \mathcal{P}(I_v)$ of a labeling, the set I_v contains at least two elements.

One observes that the map $\tau \mapsto \tau(\mathcal{P})$ can be extended to a covariant functor from the category of trees to the category of dg-modules. In particular, for each internal edge $e_0 \in E'(\tau)$, we have a natural morphism

$$(\gamma_{e_0})_* : \tau(\mathcal{P}) \to \tau/e_0(\mathcal{P})$$

from the labellings of the tree τ to the labellings of the tree τ/e_0 obtained by the contraction of the edge e_0. In general a morphism $f_* : \tau(\mathcal{P}) \to \tau'(\mathcal{P})$ induced by a tree morphism $f : \tau \to \tau'$ is a composite of edge contractions $(\gamma_{e_0})_*$ and isomorphisms. Therefore, for our purposes, it is sufficient to make the morphisms $(\gamma_{e_0})_*$ explicit. Let $s_0 = s(e_0)$ and $t_0 = t(e_0)$ denote respectively the source and the target of e_0. By definition, the morphism γ_{e_0} preserves the label of vertices $v \neq s_0, t_0$ which are untouched by the contraction process and labels the collapsed vertex $s_0 \equiv t_0$ in τ/e_0 by the partial composite $p_{t_0} \circ_{s_0} p_{s_0} \in \mathcal{P}(I_{t_0} \setminus \{s_0\} \amalg I_{s_0})$ of the labels $p_{s_0} \in \mathcal{P}(I_{s_0}), p_{t_0} \in \mathcal{P}(I_{t_0})$ of the vertices s_0 and t_0 in τ (see figure 7).

Clearly, for the terminal r-tree τ_r we have a canonical isomorphism $\tau_r(\mathcal{P}) \simeq \mathcal{P}(r)$. Accordingly, for each r-tree τ the terminal morphism $\gamma : \tau \to \tau_r$ induces a morphism $\gamma_* : \tau(\mathcal{P}) \to \mathcal{P}(r)$. Intuitively, a tree labeling represents a formal operadic composite of operations and the morphism $\gamma_* : \tau(\mathcal{P}) \to \mathcal{P}(r)$ is defined by the evaluation of this composite operation in the operad $\mathcal{P}$.

For a Hopf operad $\mathcal{P}$, the construction of this paragraph gives a functor $\tau \mapsto \tau(\mathcal{P})$ from the category of trees to the category of dg-coalgebras and not only to the category of dg-modules. Formally, in the definition of $\tau(\mathcal{P}) = \bigotimes_{v\in V(\tau)} \mathcal{P}(I_v)$ we consider a tensor product in the monoidal category of dg-coalgebras and not only in the category of dg-modules (as in the definition of the module of length tensors). Hence a morphism of r-trees $f : \tau \to \tau'$ induces a coalgebra morphism $f_* : \tau(\mathcal{P}) \to \tau'(\mathcal{P})$ since the definition of f_* can be deduced from the axioms of symmetric monoidal categories and since a Hopf operad refers precisely to an operad in the category of coalgebras. Explicitly, for a fixed r-tree τ, the diagonal of $\tau(\mathcal{P}) = \bigotimes_{v\in V(\tau)} \mathcal{P}(I_v)$ is given by the composite of a tensor product of the internal diagonal of $\mathcal{P}$ with the obvious tensor permutation. Thus, for an element $\underline{p} = \bigotimes_{v\in V(\tau)} p_v \in \tau(\mathcal{P})$, we have

$$\Delta(\underline{p}) = \sum \Big\{ \bigotimes_{v\in V(\tau)} p'_v \Big\} \otimes \Big\{ \bigotimes_{v\in V(\tau)} p''_v \Big\} \in \tau(\mathcal{P}) \otimes \tau(\mathcal{P}),$$

where we consider the expansion $\Delta(p_v) = \sum p'_v \otimes p''_v$ of the diagonal of each factor $p_v \in \mathcal{P}(I_v)$. Accordingly, the diagonal of a labeling is represented by the graphical formula of figure 6.

§2.3.2. *The W-construction.* In this paragraph we define the chain structure of the Boardman-Vogt operad $W(\mathcal{P})$. As specified in the subsection introduction, we set $W(\mathcal{P})(0) = \mathbb{F}$ and $W(\mathcal{P})(1) = \mathbb{F}$ so that $W(\mathcal{P})$ is still a connected unital operad. Consequently, in this paragraph and in the next ones, we consider only the components $W(\mathcal{P})(r)$ such that $r \geq 2$ and we define the structure of the reduced operad associated to $W(\mathcal{P})$. The definition of the partial composition products with unital operations $\circ_i : W(\mathcal{P})(r) \otimes W(\mathcal{P})(0) \to W(\mathcal{P})(r-1)$ is postponed to §2.3.5.

For $r \geq 2$, the module $W(\mathcal{P})(r)$ is defined by the dg-module coend

$$W(\mathcal{P})(r) = \int_{\tau\in\Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$$

in which the variable τ ranges over the category of 1-reduced r-trees $\Theta''(r)$ introduced in §2.2.4. The map $\mathcal{P} \mapsto W(\mathcal{P})$ defines clearly a functor on the category of dg-operads.

Equivalently, the module $W(\mathcal{P})(r)$ can be defined by the direct sum

$$W(\mathcal{P})(r) = \bigoplus_{\tau\in\Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}^\tau / \equiv$$

together with the relation

$$\underline{p} \otimes f^*\lambda \equiv f_*\underline{p} \otimes \lambda$$

for any r-tree morphism $f : \tau \to \tau'$. Clearly, the relation $\equiv$ is generated by the relations $\underline{p} \otimes (\gamma_{e_0})^*\lambda \equiv (\gamma_{e_0})_*\underline{p} \otimes \lambda$ associated to edge contractions $\gamma_{e_0} : \tau \to \tau/e_0$ since any morphism of $\Theta''(r)$ is a composite of edge contractions.

Recall that the morphism $(\gamma_{e_0})^* : \mathbb{D}^{\tau/e_0} \to \mathbb{D}^\tau$ identifies $\mathbb{D}^{\tau/e_0}$ with the submodule of length tensors $\lambda = \bigotimes_{e\in E'(\tau)} \lambda_e$ such that $\lambda_{e_0} = 0$. Thus, intuitively, the relation $\equiv$ identifies an element $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ equipped with an edge of length $\lambda_{e_0} = 0$ to an element $\gamma_{e_0}(\underline{p}) \otimes \lambda/e_0 \in \tau/e_0(\mathcal{P}) \otimes \mathbb{D}^{\tau/e_0}$ obtained by the contraction of e_0 as represented in figure 7. Accordingly, in the context of dg-modules, any element of $W(\mathcal{P})(r)$ has a normal form $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ for a uniquely determined tree τ such that $\lambda_e = 1$ or $\lambda_e = 01$ for all internal edges $e \in E'(\tau)$.

Finally, we observe that $W(\mathcal{P})(r)$ is a dg-coalgebra if $\mathcal{P}$ is a Hopf operad. Indeed we observe in this section that the functor $\tau \mapsto \mathbb{D}^\tau$ targets to the category of coalgebras and not only to the category of dg-modules as well as the functor $\tau \mapsto \tau(\mathcal{P})$ if $\mathcal{P}$ is a Hopf operad. Our claim follows since the forgetful functors from the category of coalgebras to the category of dg-modules creates coends. These observations prove also that, for a Hopf operad, the general definition of $W(\mathcal{P})(r)$ deduced from monoidal model category axioms agrees with our elementary construction of $W(\mathcal{P})$ within the category of dg-modules. Explicitly, in order to define the diagonal of an element $\underline{p}\otimes\lambda$ in $W(\mathcal{P})$, we can simply put together the diagonal of $\underline{p}$, represented in figure 6, with the diagonal of λ in the cubical complex $\mathbb{D}^\tau$, as in the representation of figure 8.

§2.3.3. *The cellular differential in the W-construction.* We make explicit the differential of an element of $W(\mathcal{P})(r)$. By definition, the module $W(\mathcal{P})(r)$ is equipped with a natural differential $\delta : W(\mathcal{P})(r) \to W(\mathcal{P})(r)$ induced by the internal differential of the operad $\mathcal{P}$ and with a cellular differential $\partial : W(\mathcal{P})(r) \to W(\mathcal{P})(r)$ induced by the differential of the chain interval $\mathbb{I}$. Explicitly, this cellular differential replaces an edge length $\lambda_e = 01$ by a difference $\partial(\lambda_e) = 1 - 0$. The cellular differential of an explicit element of $W(\mathcal{P})(r)$ is represented in figure 9. We have unspecified signs determined by an orientation of the cell associated to the tree τ, equivalent to an ordering of the length tensors $\bigotimes_{e\in E'(\tau)} \lambda_e$, since the differential ∂ is supposed to operate on the module $\mathbb{D}^\tau = \bigotimes_{e\in E'(\tau)} \mathbb{I}$ according to the rules of differential graded algebra. Notice that, according to these definitions, the dg-module $(W(\mathcal{P})(r), \partial)$ forms a cubical complex whose cells are indexed by 1-reduced r-trees.

§2.3.4. *The operad structure of the W-construction.* Observe that $W(\mathcal{P})(r)$ comes equipped with a natural action of the symmetric group Σ_r given by the leaf reindexing of r-trees as usual. In this paragraph we define the operadic composition products
$$\circ_i : W(\mathcal{P})(s) \otimes W(\mathcal{P})(t) \to W(\mathcal{P})(s+t-1)$$
for $s,t \geq 2$ and this definition gives the structure of the reduced operad $\overline{W(\mathcal{P})}$.

The partial composition product $\circ_i$ is deduced from the grafting process of rooted trees extended to cell metric trees. To begin with we recall the definition of the tree $\sigma \circ_i \tau$ obtained by grafting the root of a t-tree τ to the ith leaf of an s-tree σ. Formally, the vertex set of $\sigma\circ_i\tau$ is defined by the union $V(\sigma\circ_i\tau) = V(\sigma) \amalg V(\tau)$ of the vertices of σ and τ and the edge set by the union $E(\sigma\circ_i\tau) = E(\sigma)\amalg E(\tau)/\equiv$ of the edges of σ and τ in which the root e_0 of τ and the ith leaf e_i of σ are identified. For this new edge $\{e_0 \equiv e_i\}$, we set $s(\{e_0 \equiv e_i\}) = s(e_0)$ and $t(\{e_0 \equiv e_i\}) = t(e_i)$. The other edges $e \in E(\sigma) \setminus \{e_i\}$ and $f \in E(\tau) \setminus \{e_0\}$ keep the same source and target in $\sigma\circ_i\tau$. The leaves of $\sigma\circ_i\tau$ are reindexed as usual so that our construction produces an $s+t-1$-tree $\sigma\circ_i\tau$ from an s-tree σ and a t-tree τ.

Observe that the internal edges of $\sigma\circ_i\tau$ are either internal edges of σ and τ or produced by the grafting $\{e_0 \equiv e_i\}$ of the root e_0 of τ with the ith leaf e_i of σ. Consequently, we have a natural morphism
$$\circ_i : \mathbb{D}^\sigma \otimes \mathbb{D}^\tau \to \mathbb{D}^{\sigma\circ_i\tau}$$
that assigns the length $\lambda_{\{e_0\equiv e_i\}} = 1$ to the new internal edge and preserves the length of the other edges of $\sigma\circ_i\tau$. Formally, we have a partition $E'(\sigma\circ_i\tau) = E'(\sigma)\amalg E'(\tau)\amalg\{e_0 \equiv e_i\}$ and the morphism $\circ_i$ is defined by
$$\Bigl[\bigotimes_{e\in E'(\sigma)} \mathbb{I}\Bigr] \otimes \Bigl[\bigotimes_{e\in E'(\tau)} \mathbb{I}\Bigr] \xrightarrow{\mathrm{Id}\otimes\eta^1} \Bigl[\bigotimes_{e\in E'(\sigma)} \mathbb{I}\Bigr] \otimes \Bigl[\bigotimes_{e\in E'(\tau)} \mathbb{I}\Bigr] \otimes \mathbb{F}\,1 \hookrightarrow \bigotimes_{e\in E'(\sigma\circ_i\tau)} \mathbb{I}.$$
One checks readily that this definition gives a morphism of functors in $\tau \in \Theta''(r)$.

On the other hand, we have also a natural isomorphism

$$\circ_i : \sigma(\mathcal{P}) \otimes \tau(\mathcal{P}) \xrightarrow{\simeq} \sigma \circ_i \tau(\mathcal{P})$$

since $V(\sigma \circ_i \tau) = V(\sigma) \amalg V(\tau)$. The pairing of these morphisms gives the required composition product on $W(P)(r) = \int_\tau \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$. The representation of this grafting process is given by figure 10.

One checks readily that the maps $\circ_i$ satisfy the associativity and commutativity properties of partial composition products of an operad. Clearly, the only example of a reduced 1-tree is provided by the unit tree $\downarrow$ equipped with an empty set of vertices and only one edge. This object defines a unit for the partial composition products defined in this paragraph. Accordingly, for $r = 1$ we can extend the definition of §2.3.2 in order to obtain $W(\mathcal{P})(1) = \mathbb{F}$, coherently with our conventions.

Observe that the definition of $\circ_i : \sigma(\mathcal{P}) \otimes \tau(\mathcal{P}) \xrightarrow{\simeq} \sigma \circ_i \tau(\mathcal{P})$ can be deduced from the axioms of symmetric monoidal categories so that this morphism defines a morphism of coalgebras if $\mathcal{P}$ is a Hopf operad. Similarly, the morphism $\circ_i : \mathbb{D}^\sigma \otimes \mathbb{D}^\tau \to \mathbb{D}^{\sigma \circ_i \tau}$ defines a morphism of coalgebras and not only of dg-modules because $\mathbb{I}$ defines an interval in the category of coalgebras and not only in the category of dg-modules. From these observations we conclude that the operadic composition products of the W-construction are morphism of coalgebras so that $W(\mathcal{P})$ forms a Hopf operad. To conclude, for a Hopf operad $\mathcal{P}$, our construction returns an operad $W(\mathcal{P})$ in the category of dg-coalgebras. In fact, our construction agrees with the general construction of [5], defined within the formalism of monoidal model categories, for the category of dg-coalgebras.

§2.3.5. *Unitary operations in the Boardman-Vogt construction.* We define now the partial composites with unital operations

$$\circ_i : W(\mathcal{P})(r) \otimes W(\mathcal{P})(0) \to W(\mathcal{P})(r-1).$$

As specified in the introduction, we set $W(\mathcal{P})(0) = \mathcal{P}(0) = \mathbb{F}$ so that $W(\mathcal{P})$ is still a connected unital operad. For our purpose we consider an expansion

$$W(\mathcal{P})(r) = \bigoplus_\tau \tau(\mathcal{P}) \otimes \mathbb{D}^\tau / \equiv$$

that ranges over all r-trees, unlike the expansion of §2.3.2, but which contains more relations so that this expansion returns the same result for $r \geq 1$ and $W(\mathcal{P})(0) = \mathbb{F}$ for $r = 0$. Roughly, we put relations so that any labeling reduces to the labeling of a reduced tree for $r \geq 1$ or to an element of $\mathcal{P}(0)$ for $r = 0$.

Explicitly, let τ denote a tree equipped with a vertex v_0 such that I_{v_0} is empty and let e_0 denote the unique edge such that $s(e_0) = v_0$. Thus, for a labeling $\underline{p} = \bigotimes_{v \in V(\tau)} p_v \in \tau(\mathcal{P})$, we have $p_{v_0} \in \mathcal{P}(0)$. An element $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ is set to be equivalent to an element of the summand $\tau/e_0(\mathcal{P}) \otimes \mathbb{D}^{\tau/e_0}$ for the tree τ/e_0 obtained by contraction of the edge e_0 according to the process of §2.2.3. By definition, the terminal vertex v_0 disappears in τ/e_0. Explicitly, we consider the morphism $(\gamma_{e_0})_* : \tau(\mathcal{P}) \to \tau/e_0(\mathcal{P})$ defined in §2.3.2 and we let $(\gamma_{e_0})_* : \mathbb{D}^\tau \to \mathbb{D}^{\tau/e_0}$ denote the dg-module morphism induced by the augmentation $\epsilon : \mathbb{I} \to \mathbb{F}$ on the factor of the tensor product $\mathbb{D}^\tau = \bigotimes_{e \in E'(\tau)} \mathbb{I}$ indexed by the edge $e_0 \in E'(\tau)$. Recall that $E'(\tau/e_0) = E'(\tau) \setminus \{e_0\}$ so that we have a morphism

$$\Big[\bigotimes_{e \in E'(\tau) \setminus \{e_0\}} \mathbb{I} \Big] \otimes \mathbb{I} \xrightarrow{\mathrm{Id} \otimes \epsilon} \Big[\bigotimes_{e \in E'(\tau) \setminus \{e_0\}} \mathbb{I} \Big] \xrightarrow{=} \bigotimes_{e \in E'(\tau/e_0)} \mathbb{I}\,.$$

For a tensor $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$, we set precisely

$$\underline{p} \otimes \lambda \equiv (\gamma_{e_0})_*(\underline{p}) \otimes (\gamma_{e_0})_*(\lambda).$$

Let τ denote a tree equipped with a vertex v_0 such that I_{v_0} has only one element and let e_0, respectively e_1, denote the unique edge such that $s(e_0) = v_0$, respectively $t(e_1) = v_0$. For a labeling $\underline{p} = \bigotimes_{v\in V(\tau)} p_v$, the element $p_{v_0} \in \mathcal{P}(I_{v_0})$ denotes necessarily the operad unit $1 \in \mathcal{P}(1)$. We consider the tree $\tau\backslash v_0$ in which the vertex v_0 is removed. Formally, this tree is defined by the vertex set $V(\tau\backslash v_0) = V(\tau)\backslash\{v_0\}$ and by the quotient edge set $E(\tau\backslash v_0) = E(\tau)/\{e_0 \equiv e_1\}$ in which the edges e_0 and e_1 are identified. An element $\underline{p}\otimes\lambda \in \tau(\mathcal{P})\otimes\mathbb{D}^\tau$ is set to be equivalent to an element of the summand $\tau\backslash v_0(\mathcal{P}) \otimes \mathbb{D}^{\tau\backslash v_0}$. Namely we set

$$\underline{p} \otimes \lambda \equiv (\delta_{v_0})_*(\underline{p}) \otimes (\delta_{v_0})_*(\lambda)$$

for morphisms $(\delta_{v_0})_* : \tau(\mathcal{P}) \to \tau\backslash v_0(\mathcal{P})$ and $(\delta_{v_0})_* : \mathbb{D}^\tau \to \mathbb{D}^{\tau\backslash v_0}$ obtained as follows. Since $\mathcal{P}(1) = \mathbb{F}$, we have an isomorphism

$$\bigotimes_{v\in V(\tau)} \mathcal{P}(I_v) \xrightarrow{\simeq} \bigotimes_{v\in V(\tau)\backslash\{v_0\}} \mathcal{P}(I_v)$$

which yields the required morphism $(\delta_{v_0})_* : \tau(\mathcal{P}) \to \tau\backslash v_0(\mathcal{P})$. Equivalently, the tensor $(\delta_{v_0})_*(\underline{p})$ is obtained simply by removing the unit element $p_{v_0} = 1$ from $\underline{p} = \bigotimes_{v\in V(\tau)} p_v$. The length tensor $(\delta_{v_0})_*(\lambda)$ is defined by $(\delta_{v_0})_*(\lambda)_e = \mu(\lambda_{e_0}, \lambda_{e_1})$ for $e = \{e_0 \equiv e_1\}$ and $(\delta_{v_0})_*(\lambda)_e = \lambda_e$ for the other edges of $\tau\backslash v_0$. To be precise, the edges $e = e_0, e_1$ are not necessarily internal. In this case we set by convention $\lambda_e = 1$.

§2.3.6. *The homotopy type of the W-construction – assertion (a) of theorem §2.A.* In this paragraph we define the operad equivalence $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ specified in the claim (a) of theorem §2.A. Recall that the module of length tensors $\mathbb{D}^\tau$ is equipped with natural equivalences

$$\mathbb{F} \xrightarrow{\eta^0_*} \mathbb{D}^\tau \xrightarrow{\epsilon_*} \mathbb{F}$$

such that $\epsilon_*\eta^0_* = \mathrm{Id}$ and η^0_* is an acyclic Reedy cofibration. Furthermore, by lemma §2.2.12, these morphisms yield inverse equivalences of dg-modules (respectively dg-coalgebras in the Hopf operad context):

$$\tau_r(\mathcal{P}) \simeq \int_{\tau\in\Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{F} \underset{\eta^0_*}{\overset{\epsilon_*}{\rightleftarrows}} \int_{\tau\in\Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}^\tau.$$

Recall that $\tau_r(\mathcal{P}) \simeq \mathcal{P}(r)$. Consequently, we obtain morphisms of dg-modules, respectively dg-coalgebras, $\eta : \mathcal{P} \to W(\mathcal{P})$ and $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ such that $\epsilon\eta = \mathrm{Id}$ and which are endowed with the following properties:

§2.3.7. **Claim.** *The morphism $\eta : \mathcal{P} \to W(\mathcal{P})$ defines an acyclic cofibration of dg-modules, respectively dg-coalgebras. As a consequence, the morphism $\epsilon : W(\mathcal{P}) \to \mathcal{P}$, which is right inverse to η, defines a weak equivalence of dg-modules, respectively dg-coalgebras, as well.* □

We check that $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ defines indeed an operad equivalence. In fact, one can obtain a more elementary definition for $\eta : \mathcal{P} \to W(\mathcal{P})$ and $\epsilon : W(\mathcal{P}) \to \mathcal{P}$. Namely the morphism $\eta : \mathcal{P} \to W(\mathcal{P})$ identifies an operation $p \in \mathcal{P}(r)$ with an element of $\tau_r(\mathcal{P})$, the summand of $W(\mathcal{P})(r) = \bigoplus_{\tau\in\Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}^\tau / \equiv$ associated to the terminal r-tree $\tau = \tau_r$. In the converse direction, for a tensor

$$\underline{p} \otimes \lambda = \Big[\bigotimes_{v\in V(\tau)} p_v\Big] \otimes \Big[\bigotimes_{e\in E'(\tau)} \lambda_e\Big] \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau,$$

we have $\epsilon(\underline{p}\otimes\lambda) = 0$ if $\lambda_e = 01$ for an edge $e \in E'(\tau)$. Otherwise, if $\lambda_e = 0$ or 1 for all edges $e \in E'(\tau)$, then the morphism ϵ performs the composite of the operations p_v in $\mathcal{P}$. According to this elementary definition, we obtain:

§2.3.8. **Observation.** *The morphism $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ defines a morphism of unital (Hopf) operads.* $\square$

and finally:

§2.3.9. **Claim** (Assertion (a) of theorem §2.A)**.** *The morphism $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ defines a weak-equivalence of unital (Hopf) operads.* $\square$

Notice that $\eta : \mathcal{P} \to W(\mathcal{P})$ does not form an operad morphism and give only a left inverse of $\epsilon : W(\mathcal{P}) \to \mathcal{P}$ in the category of dg-modules, respectively dg-coalgebras if $\mathcal{P}$ is a Hopf operad.

§2.3.10. *The quasi-free operad property – assertion (b) of theorem §2.A.* In the differential context, we have a natural notion of a quasi-free object in the category of unital operads. Explicitly, a quasi-free unital operad is specified by a pair $\mathcal{Q} = (\mathcal{F}_*(M), \partial)$ and represents a free operad $\mathcal{F}_*(M)$ equipped with a non-canonical differential obtained by the addition of a homogeneous derivation of degree -1

$$\partial : \mathcal{F}_*(M) \to \mathcal{F}_*(M)$$

to the natural differential of the free operad $\delta : \mathcal{F}_*(M) \to \mathcal{F}_*(M)$ induced by the internal differential of M.

In this paragraph we define a graded Λ_*-module $W'(\mathcal{P})$ such that $W(\mathcal{P}) = (\mathcal{F}_*(W'(\mathcal{P})), \partial)$ as claimed by assertion (b) of theorem §2.A. In fact, this Λ_*-module $W'(\mathcal{P})$ is defined by a section of the indecomposable quotient of $W(\mathcal{P})$. The construction of this paragraph makes sense only in the differential graded context and not in the general framework of [5] in which a Boardman-Vogt construction can be defined.

By definition, an element $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ is decomposable for the operadic composition product of $W(\mathcal{P})$ if and only if the length tensor λ contains edges of length $\lambda_e = 1$. Therefore we consider the graded module $\mathbb{D}'^\tau \subset \mathbb{D}^\tau$ spanned by length tensors $\lambda = \bigotimes_e \lambda_e$ such that $\lambda_e = 0$ or $\lambda_e = 01$ for all $e \in E'(\tau)$. This module is not preserved by the differential of $\mathbb{D}^\tau$ but defines a section of the quotient dg-module of $\mathbb{D}^\tau$ in which length tensors λ that contain edges of length $\lambda_e = 1$ are canceled. By an abuse of notation, we identify the section $\mathbb{D}'^\tau \subset \mathbb{D}^\tau$ with this quotient dg-module so that we have a natural morphism of dg-modules $\mathbb{D}^\tau \to \mathbb{D}'^\tau$.

Then let

$$W'(\mathcal{P})(r) = \int_{\tau \in \Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}'^\tau.$$

Clearly, the quotient morphism $\mathbb{D}^\tau \to \mathbb{D}'^\tau$ induces a morphism of dg-modules $W(\mathcal{P}) \to W'(\mathcal{P})$ that identifies the module $W'(\mathcal{P})$ with the indecomposable quotient of $W(\mathcal{P})$. On the other hand, the embedding $\mathbb{D}'^\tau \hookrightarrow \mathbb{D}^\tau$ induces an embedding of graded modules $W'(\mathcal{P}) \hookrightarrow W(\mathcal{P})$ so that $W'(\mathcal{P})$ represents a section of the indecomposable quotient of the operad $W(\mathcal{P})$. One checks easily that $W'(\mathcal{P})$ is preserved by the operadic composites $\partial_i = - \circ_i *$ supplied by the construction of §2.3.5. Consequently, the module $W'(\mathcal{P})$ forms a graded Λ_*-submodule of $W(\mathcal{P})$.

§2.3.11. **Claim** (Assertion (b) of theorem §2.A)**.** *The embedding of graded Λ_*-modules $W'(\mathcal{P}) \hookrightarrow W(\mathcal{P})$ induces an isomorphism of graded operads $\mathcal{F}_*(W'(\mathcal{P})) \simeq W(\mathcal{P})$. Consequently, the operad $W(\mathcal{P})$ is quasi-free and we have*

$$W(\mathcal{P}) = (\mathcal{F}_*(W'(\mathcal{P})), \partial)$$

for an operad derivation $\partial : \mathcal{F}_*(W'(\mathcal{P})) \to \mathcal{F}_*(W'(\mathcal{P}))$.

Proof. In fact, for an element $\underline{p}\otimes\lambda \in \tau(\mathcal{P})\otimes\mathbb{D}^{\tau}$, the edges of length $\lambda_e = 1$ form the edges of a tree σ whose vertices are the subtrees of τ cut by these edges precisely. This process identifies the tensor $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^{\tau}$ with an operadic composite of elements of $W'(\mathcal{P})$ arranged on a tree σ and hence with an element of the free operad generated by $W'(\mathcal{P})$ according to the classical construction of the free operad. Therefore we have an isomorphism $\mathcal{F}_*(W'(\mathcal{P})) \xrightarrow{\simeq} W(\mathcal{P})$ as stated. □

To be more precise, notice that the interval differential ∂ admits a splitting $\partial = \partial_0 + \partial_1$ such that $\partial_0(01) = 0$ and $\partial_1(01) = 1$. Clearly, the component ∂_0 of the interval differential preserves the module $\mathbb{D}'^{\tau} \subset \mathbb{D}^{\tau}$ and defines the differential of the associated quotient module. As a consequence, the cellular differential of the W-construction is endowed with a similar splitting $\partial = \partial_0 + \partial_1$, the component ∂_0 preserves $W'(\mathcal{P})$ and ∂_1 is decomposable. Hence the indecomposable quotient of $W(\mathcal{P})$ can be identified with the graded module $W'(\mathcal{P})$ equipped with a cellular differential defined by ∂_0. In fact, the suspended dg-module $(\Sigma W'(\mathcal{P}), \partial_0)$ can be identified with $\overline{B(\mathcal{P})}$, the operadic bar construction introduced in [14] (see also [12, 15]), and $\overline{W(\mathcal{P})} = (\mathcal{F}(W'(\mathcal{P})), \partial)$ can be identified with $B^c(B(\mathcal{P}))$, the cobar-bar construction of $\mathcal{P}$. In the topological context the suspension of the Σ_*-space $W'(\mathcal{P})$ defined by the operadic indecomposable quotient of $W(\mathcal{P})$ is homeomorphic to the bar construction defined in [8] but the relation with $B^c(B(\mathcal{P}))$ fails in this context.

Clearly, the module $W'(\mathcal{P})$ is preserved by the morphism $\phi_* : W(\mathcal{P}) \to W(\mathcal{P}')$ induced by an operad morphism $\phi : \mathcal{P} \to \mathcal{P}'$. Hence the map $\mathcal{P} \mapsto W(\mathcal{P})$ has the functoriality property claimed by assertion (b) of theorem §2.A. As explained in the introduction, we deduce the cofibrancy claim from theorem §2.B.

§2.4. **The cellular decomposition of the W-construction.** In this subsection we define the operadic cellular decomposition of the operad $W(\mathcal{P})$ announced by theorem §2.B. Namely we specify suboperads $W^d(\mathcal{P}) \subset W(\mathcal{P})$ such that $W(\mathcal{P}) = \operatorname{colim}_d W^d(\mathcal{P})$ together with Λ_*-submodules $C^d(\mathcal{P}) \subset W^{d-1}(\mathcal{P})$ and $D^d(\mathcal{P}) \subset W^d(\mathcal{P})$ so that we have an operad pushout

$$\begin{tikzcd} \mathcal{F}_*(C^d(\mathcal{P})) \arrow[r, "f^d"] \arrow[d, "i^d"] & W^{d-1}(\mathcal{P}) \arrow[d, "j_d"] \\ \mathcal{F}_*(D^d(\mathcal{P})) \arrow[r, "g^d"] & W^d(\mathcal{P}) \end{tikzcd} .$$

For this purpose we consider the quasi-free representation of $W(\mathcal{P})$ defined in §2.3.10. Recall also that the category of trees is equipped with a grading defined by the number of internal edges and we let $\Theta''_d(r)$ denote the full subcategory of $\Theta''(r)$ formed by trees $\tau \in \Theta''(r)$ such that $\operatorname{gr}(\tau) \leq d$. We consider the graded module

$$W'^d(\mathcal{P})(r) = \int_{\tau\in\Theta''_d(r)} \tau(\mathcal{P}) \otimes \mathbb{D}'^{\tau}$$

that forms clearly a Λ_*-submodule of $W'(\mathcal{P})$. The simplest definition of $W^d(\mathcal{P})$ arises from the following straightforward observation:

§2.4.1. **Observation.** *The free unital operad $\mathcal{F}_*(W'^d(\mathcal{P})) \subset \mathcal{F}_*(W'(\mathcal{P}))$ is preserved by the differential of $W(\mathcal{P}) = (\mathcal{F}_*(W'(\mathcal{P})), \partial)$. Consequently, the pairs $W^d(\mathcal{P}) = (\mathcal{F}_*(W'^d(\mathcal{P})), \partial)$ define a nested sequence of quasi-free unital suboperads of $W(\mathcal{P})$ such that $W(\mathcal{P}) = \operatorname{colim}_d W^d(\mathcal{P})$.*

If $\mathcal{P}$ is a Hopf operad, then $\mathcal{F}_*(W'^d(\mathcal{P})) \subset \mathcal{F}_*(W'(\mathcal{P}))$ is preserved by the diagonal of $W(\mathcal{P})$ as well so that $W^d(\mathcal{P}) = (\mathcal{F}_*(W'^d(\mathcal{P})), \partial)$ defines a Hopf suboperad of $W(\mathcal{P})$. □

§2.4.2. *Skeletal filtration and decomposition of length tensors.* For our purposes we give another equivalent definition of $W^d(\mathcal{P})$. This new definition can also be generalized to the framework of [5] unlike the previous construction of $W^d(\mathcal{P})$.

First, we need to introduce a skeletal filtration of $\mathbb{D}^\tau$ which is defined by the submodules $\operatorname{sk}_d \mathbb{D}^\tau \subset \mathbb{D}^\tau$ spanned by the length tensors of degree $\deg(\lambda) \leq d$. In a general framework the module $\operatorname{sk}_d \mathbb{D}^\tau$ is defined by a colimit of tensor products of the form

$$\Big\{\bigotimes_e \mathbb{F}\,0\Big\} \otimes \Big\{\bigotimes_e \mathbb{F}\,1\Big\} \otimes \Big\{\bigotimes_e \mathbb{I}\Big\} \subset \bigotimes_{e \in E'(\tau)} \mathbb{I}$$

with no more than d factors $\mathbb{I}$.

Then, for an element $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$, we observe that the relation $\underline{p} \otimes \lambda \in \mathcal{F}_*(W'^d(\mathcal{P}))(r)$ is equivalent to a condition on the length tensor only $\lambda \in \mathbb{D}^\tau$, like the characterization of a decomposable element given in §2.3.10. Explicitly, as explained in the proof of claim §2.3.11 the edges of length $\lambda_e = 1$ decomposes the length tensor $\lambda \in \mathbb{D}^\tau$ into a composite of elements $\lambda_u \in \mathbb{D}^{\tau_u}$ for subtrees $\tau_u \subset \tau$ arranged on a tree σ. By definition, we have $\underline{p} \otimes \lambda \in \mathcal{F}_*(W'^d(\mathcal{P}))(r)$ if and only if these length tensors satisfy $\deg(\lambda_u) \leq d$ for all $u \in V(\sigma)$.

To make our construction more general, we introduce a submodule $\operatorname{dec}_d \mathbb{D}^\tau \subset \mathbb{D}^\tau$ spanned by decomposable tensors such that $\deg(\lambda_u) \leq d$. Formally, this dg-module $\operatorname{dec}_d \mathbb{D}^\tau$ can be defined by a colimit process. For each decomposition of a tree τ into subtrees τ_u as above, we consider the tensor product

$$\bigotimes_{u \in V(\sigma)} \operatorname{sk}_d \mathbb{D}^{\tau_u}.$$

By definition, we have $E'(\tau) = E'(\sigma) \amalg \Big\{\coprod_{u \in V(\sigma)} E'(\tau_u)\Big\}$. Consequently, we have an embedding

$$\eta^1_* : \bigotimes_{u \in V(\sigma)} \operatorname{sk}_d \mathbb{D}^{\tau_u} \to \bigotimes_{e \in E'(\tau)} \mathbb{I} = \mathbb{D}^\tau$$

given by $\eta^1 : \mathbb{F}\,1 \to \mathbb{I}$ on the factors associated to an edge $e \in E'(\sigma)$. Furthermore, for each decomposition refinement, we have a similar morphism

$$\bigotimes_{u' \in V(\sigma')} \operatorname{sk}_d \mathbb{D}^{\tau'_{u'}} \to \bigotimes_{u \in V(\sigma)} \operatorname{sk}_d \mathbb{D}^{\tau'_u},$$

that commutes with these embeddings, given by $\eta^1 : \mathbb{F}\,1 \to \mathbb{I}$ on the factors associated to an edge

$$e \in \coprod_u E'(\tau_u) \setminus \coprod_{u'} E'(\tau'_{u'}).$$

The module $\operatorname{dec}_d \mathbb{D}^\tau$ is defined by the colimit of these morphisms for all decompositions into subtrees τ_u. This module is endowed with a natural embedding $\operatorname{dec}_d \mathbb{D}^\tau \hookrightarrow \mathbb{D}^\tau$ induced by the embeddings above. The assumption $\lambda_u \in \operatorname{sk}_d \mathbb{D}^{\tau_u}$ ensures that a length tensor that arises in this colimit satisfies $\deg(\lambda_u) \leq d$. The colimit process permits to identify an element $\lambda_u \in \mathbb{D}^{\tau_u}$ that contains an edge of length $(\lambda_u)_e = 1$ to a further decomposition. Hence any element of $\mathcal{F}_*(W'^d(\mathcal{P}))(r)$ is equivalent to a representative $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ such that $\lambda \in \operatorname{dec}_d \mathbb{D}^\tau$.

To conclude, we obtain:

§2.4.3. **Observation.** *We have*

$$W^d(\mathcal{P})(r) = \int_{\tau \in \Theta''(r)} \tau(\mathcal{P}) \otimes \operatorname{dec}_d \mathbb{D}^\tau. \quad \square$$

Furthermore, observe that we have an isomorphism $\mathrm{dec}_d(\sigma \circ_i \tau) \simeq \mathrm{dec}_d(\sigma) \otimes \mathrm{dec}_d(\tau)$ for any pair of trees $\sigma \in \Theta''(s)$ and $\tau \in \Theta''(t)$. Consequently, the composition product of $W^d(\mathcal{P})$ can be obtained by a generalization of the construction of §2.3.4 for the operad $W(\mathcal{P})$.

For our next construction, we record the following assertion which is a tautological consequence of the definition of the modules $\mathrm{dec}_d \mathbb{D}^\tau$:

§2.4.4. **Fact.** *For a tree $\tau \in \Theta''(r)$, we have natural embeddings*

$$\begin{tikzcd}
\vdots \arrow[d, hook] & \vdots \arrow[d, hook] & \\
\mathrm{sk}_{d-1} \mathbb{D}^\tau \arrow[r, hook] \arrow[d, hook] & \mathrm{dec}_{d-1} \mathbb{D}^\tau \arrow[d, hook] \arrow[dr, hook] & \\
\mathrm{sk}_d \mathbb{D}^\tau \arrow[r, hook] \arrow[d, hook] & \mathrm{dec}_d \mathbb{D}^\tau \arrow[r, hook] \arrow[d, hook] & \mathbb{D}^\tau \\
\vdots & \vdots &
\end{tikzcd}.$$

§2.4.5. *The cells of the W-construction.* In order to define the Λ_*-modules $C^d(\mathcal{P})$ and $D^d(\mathcal{P})$, we consider the skeletal filtration of the module of length tensors $\mathbb{D}^\tau$ introduced in the previous paragraph and the subcategory of $\Theta''(r)$ formed by trees with no more than d internal edges. We set precisely

$$C^d(\mathcal{P}) = \int_{\tau \in \Theta''_d(r)} \tau(\mathcal{P}) \otimes \mathrm{sk}_{d-1} \mathbb{D}^\tau \quad \text{and} \quad D^d(\mathcal{P}) = \int_{\tau \in \Theta''_d(r)} \tau(\mathcal{P}) \otimes \mathrm{sk}_d \mathbb{D}^\tau.$$

We have canonical embeddings

$$\begin{tikzcd}
C^d(\mathcal{P}) \arrow[r, "f^d"] \arrow[d, "i^d"'] & W^{d-1}(\mathcal{P}) \arrow[d, hook] \arrow[dr, hook] & \\
D^d(\mathcal{P}) \arrow[r, "g^d"'] & W^d(\mathcal{P}) \arrow[r, hook] & W(\mathcal{P})
\end{tikzcd}.$$

induced by the embeddings of fact §2.4.4 and by the category inclusion $\Theta''_d(r) \subset \Theta''(r)$. One checks readily that $C^d(\mathcal{P})$, respectively $D^d(\mathcal{P})$, is preserved by the operadic composites with unital operations $\partial_i = - \circ_i *$ so that the diagram above defines a commutative diagram of Λ_*-modules. Furthermore, if $\mathcal{P}$ is a Hopf operad, then the dg-module $C^d(\mathcal{P})$, respectively $D^d(\mathcal{P})$, is equipped with a natural coalgebra structure and we obtain a commutative diagram of Hopf Λ_*-modules. The modules $C^d(\mathcal{P})$ and $D^d(\mathcal{P})$ are also unitary and connected since for all $d \geq 0$ the category $\Theta''_d(1)$ is reduced to the unit tree.

We consider the commutative square of (Hopf) operads

$$\begin{tikzcd}
\mathcal{F}_*(C^d(\mathcal{P})) \arrow[r, "f^d"] \arrow[d, "i^d"'] & W^{d-1}(\mathcal{P}) \arrow[d, hook] \\
\mathcal{F}_*(D^d(\mathcal{P})) \arrow[r, "g^d"'] & W^d(\mathcal{P})
\end{tikzcd}$$

in which f^d and g^d are induced by our morphisms of (Hopf) Λ_*-modules f^d and g^d.

§2.4.6. **Claim** (First part of theorem §2.B)**.** *This commutative square defines a pushout in the category of unital (Hopf) operads.*

Proof. We use the relation $W^d(\mathcal{P}) = (\mathcal{F}_*(W'^d(\mathcal{P})), \partial)$ in our proof. This argument is valid only in the differential graded context. Nevertheless one can observe that the claim holds in the framework of [5] though the proof becomes more technical. Therefore we give only a few hints below for a general proof of the claim.

Recall that the functor $\mathcal{P} \mapsto \overline{\mathcal{P}}$, from the category of unital operads to the category of non-unital operads, creates pushouts. Therefore we consider the reduced operads associated to $W^d(\mathcal{P})$ and the free non-unital operads $\mathcal{F}(C^d(\mathcal{P}))$ and $\mathcal{F}(D^d(\mathcal{P}))$. Recall that the reduced operad of a quasi-free unital operad is still a quasi-free object in the category of non-unital operad. For the operad $W^d(\mathcal{P})$ we obtain precisely $\overline{W^d(\mathcal{P})} = (\mathcal{F}(W'^d(\mathcal{P})), \partial)$.

Observe that $W'^d(\mathcal{P})(r) = W'^{d-1}(\mathcal{P})(r) \oplus E^d(\mathcal{P})(r)$, where $E^d(\mathcal{P})(r)$ denotes the submodule of $W'^d(\mathcal{P})(r) = \int_{\tau \in \Theta''(r)} \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ spanned by tensors $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ such that $\mathrm{gr}(\tau) = d$ and $\lambda_e = 01$ for all edges $e \in E'(\tau)$. Equivalently, the module $E^d(\mathcal{P})(r)$ consists of tensors $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ such that the length tensor λ has a top degree $\deg(\lambda) = d$. Clearly, the modules $E^d(\mathcal{P})(r)$ form a graded Σ_*-modules and the splitting $W'^d(\mathcal{P}) = W'^{d-1}(\mathcal{P}) \oplus E^d(\mathcal{P})(r)$ holds in the category of graded Σ_*-modules. Observe that we have a similar splitting $D^d(\mathcal{P})(r) = C^d(\mathcal{P})(r) \oplus E^d(\mathcal{P})(r)$ for the same Σ_*-module $E^d(\mathcal{P})$. If we forget about the differential of the quasi-free operads $W^d(\mathcal{P})$, then these observations imply that the diagram of free graded operads

$$\begin{tikzcd}
\mathcal{F}(C^d(\mathcal{P})) \arrow[r, "f^d"] \arrow[d, "i^d"'] & \mathcal{F}(W'^{d-1}(\mathcal{P})) \arrow[d] \\
\mathcal{F}(D^d(\mathcal{P})) \arrow[r, "g^d"'] & \mathcal{F}(W'^d(\mathcal{P}))
\end{tikzcd}$$

form a pushout. Our claim follows immediately from this assertion. □

For a proof of claim §2.4.6 in the general framework of [5], we use the coend representation of $W^d(\mathcal{P})$ supplied by observation §2.4.3. One can observe further that the free operad $\mathcal{F}_*(D^d(\mathcal{P}))$ is given by a coend of tensor products

$$\bigotimes_u \tau_u(\mathcal{P}) \otimes \mathrm{sk}_d \, \mathbb{D}^{\tau_u},$$

where τ_u ranges over collection of trees in $\Theta''_d(r)$ arranged on a tree σ. This assertion follows from a straightforward interchange of colimits. In addition the map $g^d : \mathcal{F}_*(D^d(\mathcal{P})) \to W^d(\mathcal{P})$ can be identified with a coend morphism induced by canonical morphisms

$$\bigotimes_u \tau_u(\mathcal{P}) \otimes \mathrm{sk}_d \, \mathbb{D}^{\tau_u} \to \rho(\mathcal{P}) \otimes \mathrm{dec}_d \, \mathbb{D}^\rho,$$

where ρ is a tree formed by a composite of the trees τ_u along σ. The free operad $\mathcal{F}_*(C^d(\mathcal{P}))$ and the attaching map $f^d : \mathcal{F}_*(C^d(\mathcal{P})) \to W^{d-1}(\mathcal{P})$ have a similar representation. Then one can use a treewise representation of operadic composites in order to check by hand that $W^d(\mathcal{E})$ satisfies the universal property of an operad pushout.

§2.4.7. **Claim** (Second part of theorem §2.B)**.** *The morphism of unitary (Hopf) Λ_*-modules*

$$(i^d, \phi) : C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P})} D^d(\mathcal{P}) \to D^d(\mathcal{P}')$$

associated to an operad morphism $\phi : \mathcal{P} \to \mathcal{P}'$ is a weak-equivalence, respectively a Reedy cofibration, if ϕ defines a weak-equivalence, respectively a Reedy cofibration, in the category of (Hopf) Λ_*-modules.

Proof. This statement is valid in the framework of [5] like claim §2.4.6 but for simplicity we give arguments that make sense only in the differential graded context.

Consider the module $E^d(\mathcal{P})(r) \subset D^d(\mathcal{P})(r)$, introduced in the proof of claim §2.4.6, spanned by tensors $\underline{p} \otimes \lambda \in \tau(\mathcal{P}) \otimes \mathbb{D}^\tau$ such that $\deg(\tau) = d$ and $\lambda_e = 01$ for all edges $e \in E'(\tau)$. Recall that these modules $E^d(\mathcal{P})(r)$ form a Σ_*-module and we have a natural splitting $D^d(\mathcal{P}) = C^d(\mathcal{P}) \oplus E^d(\mathcal{P})$. In fact, the Σ_*-module $E^d(\mathcal{P})$, equipped with a differential δ induced by the internal differential of $\mathcal{P}$, can be identified with a quotient object of $D^d(\mathcal{P})$ so that we have a short exact sequence

$$C^d(\mathcal{P}) \rightarrowtail D^d(\mathcal{P}) \twoheadrightarrow E^d(\mathcal{P}) \ .$$

Observe that the coproduct $C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P}')} D^d(\mathcal{P})$ admits a similar splitting

$$C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P}')} D^d(\mathcal{P}) = C^d(\mathcal{P}') \oplus E^d(\mathcal{P})$$

so that the morphism of the claim (i^d, ϕ) fits a diagram of short exact sequences of dg-modules

$$\begin{tikzcd}
C^d(\mathcal{P}') \arrow[r, tail] \arrow[d, "="] & C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P}')} D^d(\mathcal{P}) \arrow[r, two heads] \arrow[d, "(i^d,\phi)"] & E^d(\mathcal{P}) \arrow[d, "\phi"] \\
C^d(\mathcal{P}') \arrow[r, tail] & D^d(\mathcal{P}') \arrow[r, two heads] & E^d(\mathcal{P}')
\end{tikzcd} ,$$

where $\phi : E^d(\mathcal{P}) \to E^d(\mathcal{P}')$ denotes the natural morphism induced by $\phi : \mathcal{P} \to \mathcal{P}'$.

By definition, we have

$$E^d(\mathcal{P})(r) = \bigoplus_{\mathrm{gr}(\tau)=d} \tau(\mathcal{P}) \otimes \Big[\bigotimes_{e \in E'(\tau)} 01 \Big] \simeq \bigoplus_{\mathrm{gr}(\tau)=d} \Sigma^d \tau(\mathcal{P}),$$

where the sum ranges over isomorphism classes of reduced r-trees with d internal edges. Recall that a reduced r-tree has no automorphism (see §2.2.4). As a consequence, no relation occurs in the expansion above. Hence if $\phi : \mathcal{P} \to \mathcal{P}'$ is a weak-equivalence of operads, then the induced morphism $\phi : E^d(\mathcal{P}) \to E^d(\mathcal{P}')$ is a weak-equivalence of dg-modules and we deduce from the short exact sequence that the morphism (ϕ, i^d) is a weak-equivalence as well, as stated by the claim.

Recall that a morphism $\phi : \mathcal{P} \to \mathcal{P}'$ is a Reedy cofibration in the category of (Hopf) Λ_*-modules if $\phi : \mathcal{P} \to \mathcal{P}'$ defines a cofibration in the category of Σ_*-modules. Furthermore, in this case the morphism $\phi : \mathcal{P} \to \mathcal{P}'$ can be decomposed into a sequence of Σ_*-modules embeddings

$$\mathcal{P} = M_{-1} \hookrightarrow M_0 \hookrightarrow \ldots \hookrightarrow M_n \hookrightarrow \ldots \hookrightarrow \operatorname{colim}_n M_n = \mathcal{P}'$$

such that $\delta(M_n) \subset M_{n-1}$ and where $M_{n-1} \hookrightarrow M_n$ is a split injective morphism of Σ_*-modules with a projective cokernel. The treewise tensor products of §2.3.1 can be defined for elements of a Σ_*-modules (see [12]) so that we have a sequence of Σ_*-modules $E^d(M_n)$ defined by

$$E^d(M_n)(r) = \bigoplus_{\mathrm{gr}(\tau)=d} \Sigma^d \tau(M_n).$$

One can check that the morphisms $E^d(M_{n-1}) \hookrightarrow E^d(M_n)$ are split injective and have a projective cokernel as well because the symmetric group operates freely on the treewise tensor products of a free Σ_*-module. Furthermore, the cellular differential of $D^d(\mathcal{P}')$ satisfies $\partial(E^d(M_n)) \subset C^d(\mathcal{P}')$ and we have $\delta(E^d(M_n)) \subset E^d(M_{n-1})$ for the differential induced by the internal differential of $\mathcal{P}'$. Therefore the sequence of Σ_*-module inclusions

$$\begin{aligned} C^d(\mathcal{P}') \oplus E^d(\mathcal{P}) = C^d(\mathcal{P}') \oplus E^d(M_{-1}) \hookrightarrow \dots \\ \dots \hookrightarrow C^d(\mathcal{P}') \oplus E^d(M_n) \hookrightarrow \dots \\ \dots \hookrightarrow \operatorname{colim}_n C^d(\mathcal{P}') \oplus E^d(M_n) = C^d(\mathcal{P}') \oplus E^d(\mathcal{P}') \end{aligned}$$

fulfils our requirements for a cofibration decomposition and show that the morphism

$$(i^d, \phi) : C^d(\mathcal{P}') \bigoplus_{C^d(\mathcal{P})} D^d(\mathcal{P}) \to D^d(\mathcal{P}')$$

defines a cofibration in the category of Σ_*-modules. This conclusion achieves the proof of claim §2.4.7. ☐

This claim achieves the proof theorem §2.B and, as a byproduct, of the cofibrancy claim in assertion (b) of theorem §2.A. ☐

§2.5. **Appendix: figures.**

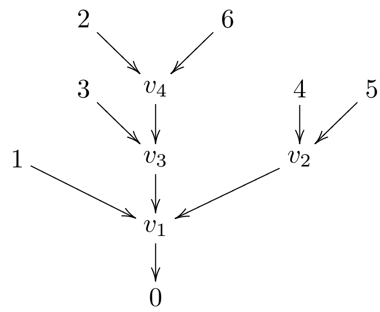

FIGURE 1. The structure of a tree

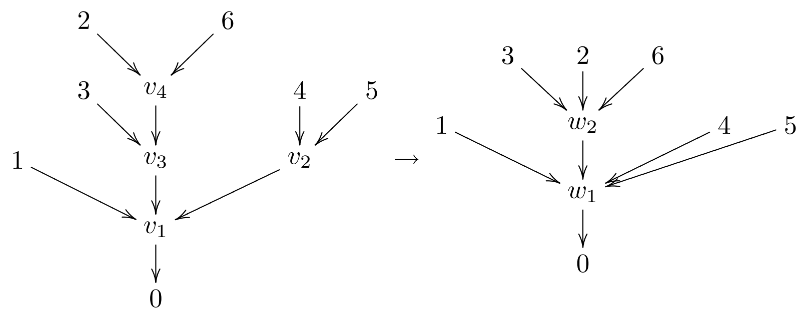

FIGURE 2. An example of a tree morphism

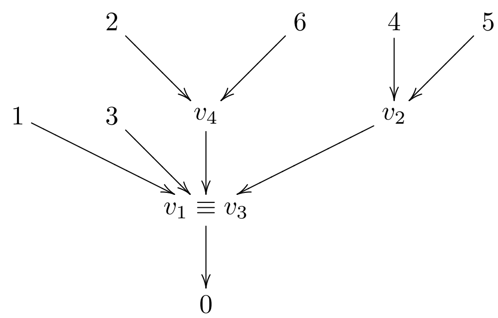

FIGURE 3. An edge contraction

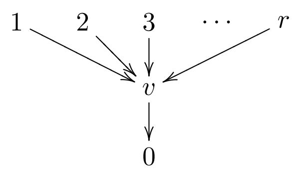

FIGURE 4. The terminal r-tree

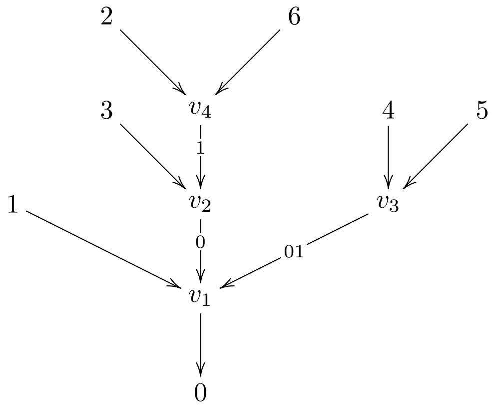

FIGURE 5. A cell metric tree

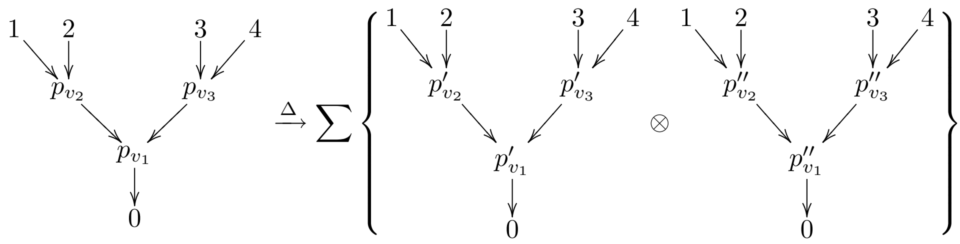

FIGURE 6. The diagonal of a labeled tree

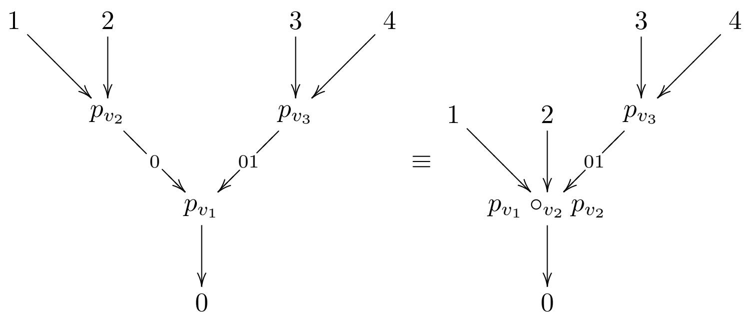

FIGURE 7. The equivalence relation of the W-construction

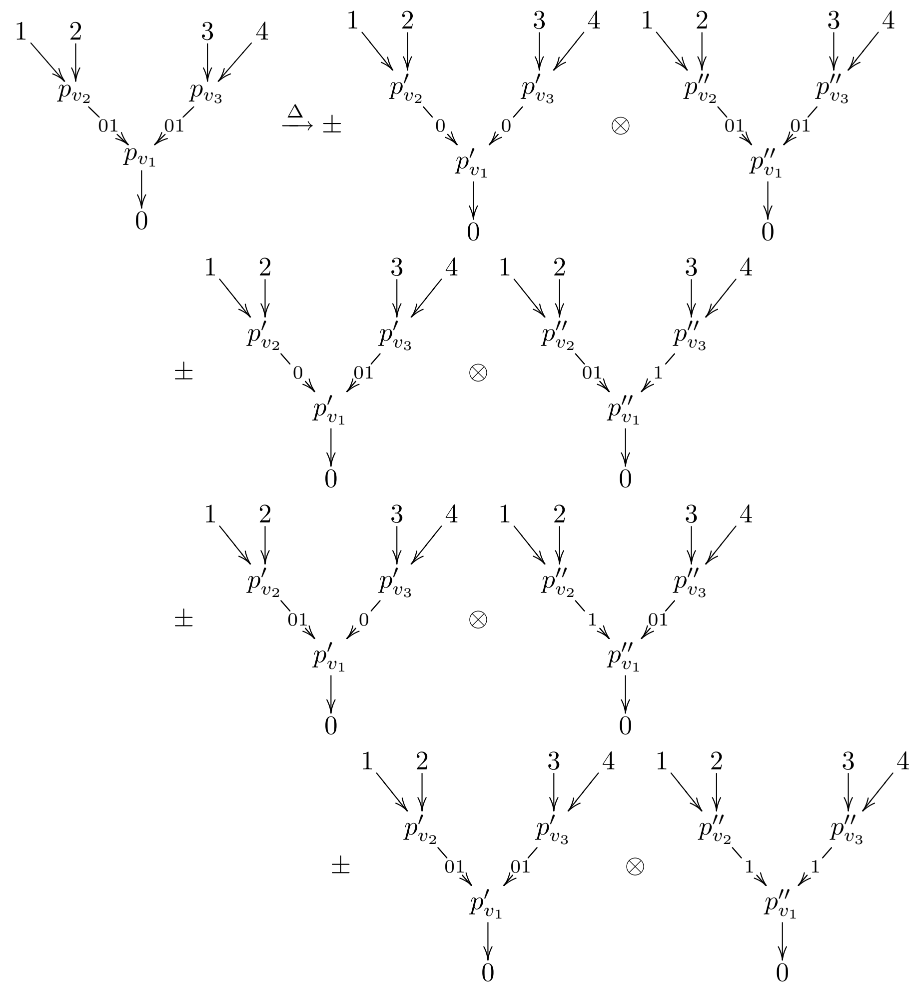

Figure 8. The diagonal of the W-construction

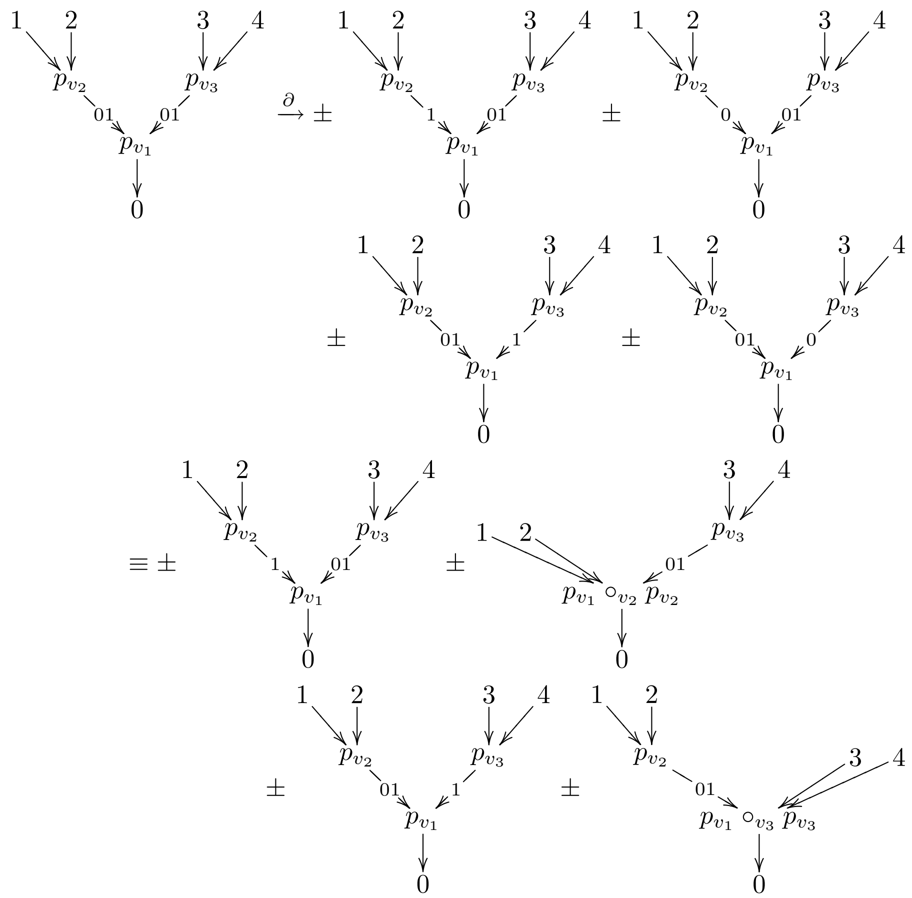

FIGURE 9. The cellular differential of the W-construction

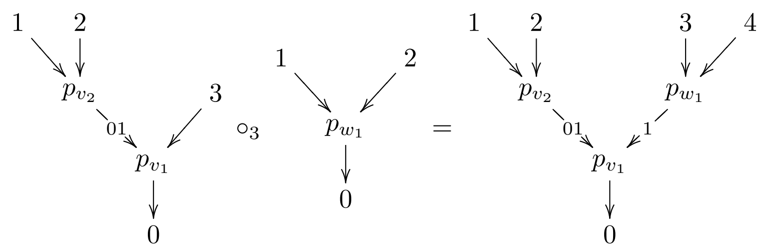

FIGURE 10. The operadic composition product of the W-construction

Construction of Hopf operad actions

§3. Cofree coalgebras and quasi-cofree Hopf Λ_*-modules

§3.1. **Introduction.** The goal of this section is to introduce a suitable notion of *cocellular complex* in order to obtain an effective class of fibrations in the category of Hopf operads. Our cocellular objects do not clearly generate the class of fibrations in the category of Hopf algebras but the *Hopf operad of bar operations* $\mathrm{HopfOp}_B^{\mathcal{P}}$ defined in the next section (our main device for the construction of operad actions) has such a cocellular structure. Hence the framework introduced in this section enables us to prove that the functor $\mathcal{P} \mapsto \mathrm{HopfOp}_B^{\mathcal{P}}$ maps fibrations, respectively acyclic fibrations, of dg-operads to fibrations, respectively acyclic fibrations, of unital Hopf operads. This assertion allows us to deduce the existence of operad morphisms $\mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{P}}$ from model category arguments.

For simplicity, we perform our constructions in the category of coalgebras first and we extend our definitions to Hopf Λ_*-modules next.

In our constructions, we consider $\mathbb{Z}$-graded objects which are outside the coalgebra model category considered in §1. Thus, for our purposes, we have to extend the notion of a fibration and of an acyclic fibration to this context. In fact, we deal with lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

such that $i : A \to B$ is a morphism of non-negatively graded objects. Therefore we distinguish morphisms of (possibly $\mathbb{Z}$-graded) dg-coalgebras $q : C \to D$ such that this lifting problem has a solution for any acyclic cofibration, respectively cofibration, of $\mathbb{N}$-graded dg-coalgebras $i : A \to B$. Clearly, we can also characterize such morphisms of $\mathbb{Z}$-graded coalgebras $q : C \to D$ by the right lifting property with respect to a generating set of acyclic cofibrations, respectively cofibrations, of $\mathbb{N}$-graded dg-coalgebras. By convention we refer abusively to these class of morphisms $q : C \to D$ as fibrations, respectively acyclic fibrations, of dg-coalgebras.

Equivalently, let $\mathrm{dg}_{\mathbb{N}}\,\mathrm{CoAlg}$, respectively $\mathrm{dg}_{\mathbb{Z}}\,\mathrm{CoAlg}$, denote the category of $\mathbb{N}$-graded, respectively $\mathbb{Z}$-graded dg-coalgebras. One can observe that the category embedding $i_+^{\mathrm{dg}} : \mathrm{dg}_{\mathbb{N}}\,\mathrm{CoAlg} \to \mathrm{dg}_{\mathbb{Z}}\,\mathrm{CoAlg}$ has a right adjoint $\mathrm{s}_+^{\mathrm{dg}} : \mathrm{dg}_{\mathbb{Z}}\,\mathrm{CoAlg} \to \mathrm{dg}_{\mathbb{N}}\,\mathrm{CoAlg}$. Clearly, a morphism of coalgebras $q : C \to D$ forms a fibration, respectively an acyclic fibration, according to the convention above if the associated morphism $\mathrm{s}_+^{\mathrm{dg}}(q) : \mathrm{s}_+^{\mathrm{dg}}(C) \to \mathrm{s}_+^{\mathrm{dg}}(D)$ defines a fibration, respectively an acyclic fibration in the model category of $\mathbb{N}$-graded dg-coalgebras.

One observes that the category of augmented coalgebras is equipped with cofree objects characterized by the usual universal property. Namely, for any dg-module V, we have a coalgebra, denoted by $\Gamma(V)$, equipped with a dg-module morphism $\pi : \Gamma(V) \to V$ such that any dg-module morphism $f : \Gamma \to \Gamma(V)$ where Γ is a dg-coalgebra admits one and only one factorization $f = \pi \cdot \nabla_f$ where $\nabla_f : \Gamma \to \Gamma(V)$ is a coalgebra morphism. For our needs we give an explicit construction of this object $\Gamma(V)$ in §3.2. The cofree coalgebra is equipped with a natural differential $\delta : \Gamma(V) \to \Gamma(V)$ induced by the internal differential of V. For our purposes we consider *quasi-cofree coalgebras* which are specified as usual by a cofree coalgebra $\Gamma = \Gamma(V)$ equipped with a coderivation $\partial : \Gamma(V) \to \Gamma(V)$ such that $\delta + \partial$ defines the differential of Γ.

Our goal is to give sufficient conditions for a morphism of quasi-cofree coalgebras

$$\nabla_f : (\Gamma(V), \partial_\alpha) \to (\Gamma(W), \partial_\beta)$$

to be a fibration, respectively an acyclic fibration. Explicitly, we observe that the coderivation of a quasi-cofree coalgebra $(\Gamma(V), \partial_\alpha)$ is equivalent to a morphism of dg-coalgebras $\nabla_\alpha : (\Gamma(V), \partial_\alpha) \to \Gamma(\Delta^1 \wedge V)$, where $\Delta^1 \wedge V$ is the cone of V in the category of dg-modules. For our purpose we determine the structure of morphisms ∇_f that fit a pullback diagram of the form

$$\begin{tikzcd}
(\Gamma(V), \partial_\alpha) \arrow[r] \arrow[d] & \Gamma(\Delta^1 \wedge V) \arrow[d] \\
(\Gamma(W), \partial_\beta) \arrow[r] & \Gamma(\Delta^1 \wedge W) \times_{\Gamma(S^1 \wedge W)} \Gamma(S^1 \wedge V)
\end{tikzcd} ,$$

where we consider natural morphisms of cofree coalgebras

$$\begin{tikzcd}
\Gamma(\Delta^1 \wedge V) \arrow[r] \arrow[d] & \Gamma(S^1 \wedge V) \arrow[d] \\
\Gamma(\Delta^1 \wedge W) \arrow[r] & \Gamma(S^1 \wedge W)
\end{tikzcd}$$

induced by a morphism of dg-modules $f : V \to W$ and a canonical morphism $\sigma \wedge - : \Delta^1 \wedge - \to S^1 \wedge -$. By immediate categorical arguments, we obtain that ∇_f defines a fibration, respectively an acyclic fibration, of dg-coalgebras if $f : V \to W$ is a fibration, respectively an acyclic fibration, of dg-modules. This program is carried out in §3.3. This construction admits a natural and straightforward extension to the category of Hopf Λ_*-modules. For the sake of completeness, we state explicitly the results that we obtain in this context in §3.4.

For our needs, we consider morphisms of quasi-cofree coalgebras $\nabla_f : (\Gamma(V), \partial_\alpha) \to (\Gamma(W), \partial_\beta)$ obtained as the limit of a tower of morphisms

$$\begin{aligned}
(\Gamma(V), \partial_\alpha) = \lim_m (\Gamma(\mathrm{ck}_m V), \partial_\alpha) \to \dots \\
\dots \to (\Gamma(\mathrm{ck}_m V), \partial_\alpha) \xrightarrow{\mathrm{p}_m} (\Gamma(\mathrm{ck}_{m-1} V), \partial_\alpha) \to \dots \\
\dots \to (\Gamma(\mathrm{ck}_{-1} V), \partial_\alpha) = \mathbb{F}
\end{aligned}$$

such that the maps $\mathrm{p}_m : (\Gamma(\mathrm{ck}_m V), \partial_\alpha) \to (\Gamma(\mathrm{ck}_{m-1} V), \partial_\alpha)$ fit coextension diagrams as above.

One can observe that any morphism of quasi-cofree coalgebras where V and W are non-negatively graded has a such a decomposition that arises from the canonical degreewise filtration of dg-modules. Moreover this argument can also be applied to the truncation $\mathrm{s}_+^{\mathrm{dg}} \nabla_f$ of a morphisms of quasi-cofree coalgebras ∇_f because the truncation functor preserves quasi-cofree objects. In the memoir, we do not give this general construction. In fact, in the next section, we define only a specific decomposition

$$\begin{aligned}
\mathrm{HopfOp}_B^{\mathcal{P}} = \lim_m \mathrm{ck}_m \mathrm{HopfOp}_B^{\mathcal{P}} \to \dots \\
\dots \to \mathrm{ck}_m \mathrm{HopfOp}_B^{\mathcal{P}} \to \mathrm{ck}_{m-1} \mathrm{HopfOp}_B^{\mathcal{P}} \to \dots \\
\dots \to \mathrm{ck}_0 \mathrm{HopfOp}_B^{\mathcal{P}} = \mathcal{C},
\end{aligned}$$

obtained similarly as the general construction, but more natural in regard to the Hopf operad of universal bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$. Then we use this construction to prove that the functor $\mathcal{P} \mapsto \mathrm{HopfOp}_B^{\mathcal{P}}$ maps an operad fibration, respectively acyclic fibration, to a fibration, respectively an acyclic fibration, of unital Hopf operads.

Here is the plan of this section. In §3.2 we recall categorical properties of coalgebras: existence of small limits, of cofree objects and adjoint functors. This subsection does not contain any original result but the explicit constructions recalled there allows us to introduce a notion of a quasi-cofree object in the category of coalgebras. In the next subsection §3.3 we study the structure of quasi-cofree coalgebras with the aim to define coalgebra fibrations. In the final subsection §3.4 we extend these results to Hopf Λ_*-modules.

§3.2. **Cofree coalgebras.** As stated, the aim of this subsection is to recall categorical properties of augmented coassociative coalgebras. For our needs the main issue is to give an effective construction of cofree objects. For the sake of precision, we state the definition of these objects in a proposition:

§3.2.1. **Proposition.** *Any dg-module V has an associated cofree augmented coassociative coalgebra $\Gamma(V)$ equipped with a morphism of dg-modules $\pi : \Gamma(V) \to V$ characterized by the classical universal property. Namely any morphism of dg-modules $f : \Gamma \to V$ where Γ is a coalgebra has a unique factorization $f = \pi \cdot \nabla_f$ such that $\nabla_f : \Gamma \to \Gamma(V)$ is a coalgebra morphism.*

We refer to [25] for a proof of this result in the case of non-graded coalgebras, to [13] for the case of non-negatively graded coalgebras and to [24] for a generalization in the context of coalgebras over an operad. In this subsection we give an explicit realization of $\Gamma(V)$ which is a special instance of the construction of the latter reference in the case of coassociative coalgebras. As such this subsection does not contain any original idea and our account is only motivated by the applications of the next subsection.

As explained in the introduction, we have to deal with $\mathbb{Z}$-graded coalgebras. Therefore we prove proposition §3.2.1 in this framework. On the other hand, our assertion holds in both the category of $\mathbb{Z}$-graded coalgebras and the category of $\mathbb{N}$-graded coalgebras. In fact, if V is an $\mathbb{N}$-graded dg-module, then the associated cofree object in the category $\mathbb{Z}$-graded coalgebras turns out to be $\mathbb{N}$-graded and gives also a realization of the cofree object cogenerated by V in the category of $\mathbb{N}$-graded coalgebras.

§3.2.2. *An inductive construction.* We define a nested sequence of dg-modules

$$\cdots \hookrightarrow \Gamma_{r+1}(V) \hookrightarrow \Gamma_r(V) \hookrightarrow \cdots \hookrightarrow \Gamma_1(V) = \prod_{n=0}^{\infty} V^{\otimes n}$$

and we prove that the module $\Gamma_\infty(V) = \bigcap_{r=1}^{\infty} \Gamma_r(V)$ is equipped with a coalgebra structure and represents the cofree coalgebra cogenerated by V.

The modules $\Gamma_r(V)$ are defined by induction. We consider the natural map

$$\{\prod_m V^{\otimes m}\} \otimes \{\prod_n V^{\otimes n}\} \xrightarrow{\nabla_\Pi} \prod_{m,n}\{V^{\otimes m} \otimes V^{\otimes n}\}$$

and the composite

$$\begin{tikzcd} \prod_N V^{\otimes N} \arrow[r, "\{\Delta_N\}"] \arrow[rr, bend left, "\Delta_\Pi"] & \prod_N\{\prod_{m+n=N} V^{\otimes m} \otimes V^{\otimes n}\} \arrow[r, "\simeq"] & \prod_{m,n}\{V^{\otimes m} \otimes V^{\otimes n}\} \end{tikzcd},$$

where $\Delta_N : V^{\otimes N} \to \prod_{m+n=N} V^{\otimes m} \otimes V^{\otimes n}$ denotes the deconcatenation of tensors. In general, the natural map $A \otimes \prod_j B_j \to \prod_j A \otimes B_j$ is an embedding provided that the module A is free over the ground ring. Accordingly, in our context, the map ∇_Π is always injective since our ground ring $\mathbb{F}$ is supposed to be a field. Notice that the map Δ_Π is injective as well.

For $r = 1$, we set $\Gamma_1(V) = \prod_{n=0}^{\infty} V^{\otimes n}$. By induction, we have a module $\Gamma_r(V)$ equipped with an embedding $\iota_r : \Gamma_r(V) \hookrightarrow \prod_{n=0}^{\infty} V^{\otimes n}$. The next module $\Gamma_{r+1}(V)$ is defined by the fiber product

$$\begin{tikzcd} \Gamma_{r+1}(V) \arrow[r, dashed] \arrow[d, dashed] & \Gamma_r(V) \otimes \Gamma_r(V) \arrow[d, "\nabla_\Pi \cdot \iota_r \otimes \iota_r"] \\ \Gamma_r(V) \arrow[r, "\Delta_\Pi \cdot \iota_r"] & \prod_{m,n}\{V^{\otimes m} \otimes V^{\otimes n}\} \end{tikzcd} .$$

The composite map $\nabla_\Pi \cdot \iota_r \otimes \iota_r$ is injective according to the observations above. Therefore the map $\Gamma_{r+1}(V) \to \Gamma_r(V)$, which is defined by a base extension of $\nabla_\Pi \cdot \iota_r \otimes \iota_r$, is injective as well.

As announced, we consider the dg-module $\Gamma_\infty(V) = \bigcap_{r=0}^{\infty} \Gamma_r(V)$ equipped with the embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_{n=0}^{\infty} V^{\otimes n}$.

§3.2.3. **Claim.** *The module $\Gamma_\infty(V)$ is equipped with a diagonal Δ_∞ such that we have a commutative diagram*

$$\begin{tikzcd} \Gamma_\infty(V) \arrow[r, dashed, "\Delta_\infty"] \arrow[d, hook, "\iota_\infty"] & \Gamma_\infty(V) \otimes \Gamma_\infty(V) \arrow[d, hook, "\nabla_\Pi \cdot \iota_\infty \otimes \iota_\infty"] \\ \prod_N V^{\otimes N} \arrow[r, "\Delta_\Pi"] & \prod_{m,n}\{V^{\otimes m} \otimes V^{\otimes n}\} \end{tikzcd} .$$

Proof. By definition of $\Gamma_{r+1}(V)$, for $r \in \mathbb{N}$, we have maps $\Delta_r : \Gamma_{r+1}(V) \to \Gamma_r(V) \otimes \Gamma_r(V)$ which fit a commutative diagram as in the claim statement. Accordingly, these maps restrict to a map from the intersection $\Gamma_\infty(V) = \bigcap_{r=1}^{\infty} \Gamma_r(V)$ to the module $\bigcap_{r=1}^{\infty}\{\Gamma_r(V) \otimes \Gamma_r(V)\}$ and the claim is a consequence of the next assertion. □

§3.2.4. **Claim.** *The natural map*

$$\Gamma_\infty(V) \otimes \Gamma_\infty(V) = \{\bigcap_{r=1}^{\infty} \Gamma_r(V)\} \otimes \{\bigcap_{s=1}^{\infty} \Gamma_s(V)\} \to \bigcap_{r=1}^{\infty}\{\Gamma_r(V) \otimes \Gamma_r(V)\}$$

is an isomorphism.

Proof. This map connects submodules of $\prod_{m,n} V^{\otimes m} \otimes V^{\otimes n}$ and hence represents a submodule inclusion. We prove that any element $\omega \in \bigcap_{r=1}^{\infty}\{\Gamma_r(V) \otimes \Gamma_r(V)\}$ belongs to $\bigcap_{r=1}^{\infty}\{\Gamma_r(V)\} \otimes \bigcap_{s=1}^{\infty}\{\Gamma_r(V)\}$.

There is a finitely generated module $\Omega_1 \subset \Gamma_1(V)$ such that $\omega \in \Omega_1 \otimes \Omega_1$ inside $\Gamma_1(V) \otimes \Gamma_1(V)$. If we let $\Omega_r = \Omega_1 \cap \Gamma_r(V)$, then we have $\{\Omega_1 \otimes \Omega_1\} \cap \{\Gamma_r(V) \otimes \Gamma_r(V)\} = \Omega_r \otimes \Omega_r$ and hence $\omega \in \bigcap_{r=1}^{\infty}\{\Omega_r \otimes \Omega_r\}$. On the other hand, since Ω_1 is finitely generated, the sequence Ω_r is necessary stationary: we have $\Omega_{r_0} = \Omega_\infty \subset \Gamma_\infty(V)$ for some $r_0 < \infty$. Hence the relation $\omega \in \Omega_{r_0} \otimes \Omega_{r_0}$ implies $\omega \in \Gamma_\infty(V) \otimes \Gamma_\infty(V)$. □

§3.2.5. **Lemma.** *The module $\Gamma(V) = \Gamma_\infty(V)$ defines a realization of the cofree augmented coassociative coalgebra cogenerated by V.*

To be precise, the coproduct of $\Gamma(V)$ is given by the map $\Delta_\infty : \Gamma_\infty(V) \to \Gamma_\infty(V)$ of claim §3.2.3, the augmentation $\epsilon : \Gamma(V) \to \mathbb{F}$ is defined by the composite of the embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_n V^{\otimes n}$ with the projection onto the component $n = 0$ of the product $\prod_n V^{\otimes n}$ and the universal morphism $\pi : \Gamma(V) \to V$ is defined by the composite of ι_∞ with the projection onto the component $n = 1$.

Proof. Observe that the diagonal Δ_∞ specified in claim §3.2.3 is coassociative simply because the deconcatenation of tensors is componentwise coassociative. One checks similarly that the composite of the embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_n V^{\otimes n}$ with the projection onto the component $n = 0$ of the product $\prod_n V^{\otimes n}$ defines an augmentation for the diagonal Δ_∞.

We check the universal property of a cofree coalgebra. Let Γ be a coalgebra equipped with a morphism of dg-modules $f : \Gamma \to V$. Consider the map $\widehat{\nabla}_f : \Gamma \to \prod_n V^{\otimes n}$ which maps an element $\gamma \in \Gamma$ to the collection of tensors $\{f^{\otimes n} \cdot \Delta^n(\gamma)\}$, where $\Delta^n(\gamma) \in \Gamma^{\otimes n}$ denotes the n-fold diagonal of $\gamma \in \Gamma$.

We claim that this map $\widehat{\nabla}_f$ admits a sequence of factorizations

$$\begin{tikzcd}
\Gamma \arrow[d, "\nabla_\infty"] \arrow[rr, bend left, "\nabla_{r+1}"] \arrow[rrr, bend left, "\nabla_r"] \arrow[rrrrr, bend left, "\nabla_1 = \widehat{\nabla}_f"] & & & & & \\
\Gamma_\infty(V) \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_{r+1}(V) \arrow[r, hook] & \Gamma_r(V) \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_1(V) = \prod_n V^{\otimes n}
\end{tikzcd}.$$

Indeed, by induction, we are given a map $\nabla_r : \Gamma \to \Gamma_r(V)$ such that $\iota_r \cdot \nabla_r = \widehat{\nabla}_f$. Observe that the map $\widehat{\nabla}_f$ makes the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\widehat{\nabla}_f"] & \Gamma \otimes \Gamma \arrow[d, "\nabla_\Pi \cdot \widehat{\nabla}_f \otimes \widehat{\nabla}_f"] \\
\prod_N V^{\otimes N} \arrow[r, "\Delta_\Pi"] & \prod_{m,n} \{V^{\otimes m} \otimes V^{\otimes n}\}
\end{tikzcd}$$

commute and, as a consequence, our map $\nabla_r : \Gamma \to \Gamma_r(V)$ fits a commutative diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[dr, dashed] \arrow[ddr, bend right, "\nabla_r"'] & \Gamma \otimes \Gamma \arrow[dr, bend left, "\nabla_r \otimes \nabla_r"] & \\
 & \Gamma_{r+1}(V) \arrow[r, dashed] \arrow[d, dashed] & \Gamma_r(V) \otimes \Gamma_r(V) \arrow[d, "\nabla_\Pi \cdot \iota_r \otimes \iota_r"] \\
 & \Gamma_r(V) \arrow[r, "\Delta_\Pi \cdot \iota_r"] & \prod_{m,n} \{V^{\otimes m} \otimes V^{\otimes n}\}
\end{tikzcd}.$$

Accordingly, the existence of $\nabla_{r+1} : \Gamma \to \Gamma_{r+1}(V)$ follows from the fiber product definition of $\Gamma_{r+1}(V)$.

According to this construction, the resulting map

$$\Gamma \xrightarrow{\nabla_\infty} \bigcap_{r=1}^{\infty} \Gamma_r(V) = \Gamma_\infty(V)$$

commutes with the diagonal of $\Gamma_\infty(V)$. Hence we conclude that the map $f : \Gamma \to V$ lifts to a coalgebra morphism $\nabla_\infty : \Gamma \to \Gamma_\infty(V)$. The next assertion implies any coalgebra morphism $\nabla : \Gamma \to \Gamma_\infty(V)$ that lifts the map $f : \Gamma \to V$ defines a factorization of the map $\widehat{\nabla}_f : \Gamma \to \prod_n V^{\otimes n}$ through $\Gamma_\infty(V)$. Therefore the uniqueness property of the coalgebra lifting follows from the injectivity of the canonical embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_n V^{\otimes n}$. This observation achieves the proof of lemma §3.2.5. $\square$

§3.2.6. **Observation.** *The embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_n V^{\otimes n}$ can be identified with the composite*

$$\Gamma_\infty(V) \xrightarrow{\{\Delta^n_\infty\}} \prod_n \Gamma_\infty(V)^{\otimes n} \xrightarrow{\{\pi^{\otimes n}\}} \prod_n V^{\otimes n},$$

where Δ^n_∞ denotes the n-fold diagonal of the cofree coalgebra.

Proof. This assertion is an immediate consequence of the definition of the diagonal given in claim §3.2.3. □

Recall that the category of augmented coassociative coalgebras is denoted by CoAlg_+^a. Beside the construction of cofree coalgebras, we recall that CoAlg_+^a has all small limits. Our constructions are standard for a category of coalgebras over a comonad. First, we have the following classical result:

§3.2.7. **Lemma.** *The forgetful functor from* CoAlg_+^a *to the category of dg-modules creates the equalizers which are reflexive in the category of dg-modules.*

Proof. Explicitly, we consider a pair of coalgebra morphisms

$$d^0, d^1 : \Gamma^0 \to \Gamma^1$$

together with a map $s^0 : \Gamma^1 \to \Gamma^0$ such that $s^0 d^0 = s^0 d^1 = \mathrm{Id}$.

One checks readily that $\ker(d^0 - d^1) \otimes \ker(d^0 - d^1)$ is the equalizer of the morphisms

$$(d^0 \otimes d^0, d^1 \otimes d^1 : \Gamma^0 \otimes \Gamma^0 \to \Gamma^1 \otimes \Gamma^1$$

in the category of dg-modules. Indeed, if the ground ring is a field, then we have $\ker(d^0 - d^1) \otimes \ker(d^0 - d^1) = \ker((d^0 - d^1) \otimes \mathrm{Id}) \cap \ker(\mathrm{Id} \otimes (d^0 - d^1))$. On the other hand, the relation $(d^0 \otimes d^0 - d^1 \otimes d^1)(\gamma) = 0$ implies $(\mathrm{Id} \otimes s^0)(d^0 \otimes d^0 - d^1 \otimes d^1)(\gamma) = (d^0 \otimes \mathrm{Id} - d^1 \otimes \mathrm{Id})(\gamma) = 0$ and symmetrically $(\mathrm{Id} \otimes d^0 - \mathrm{Id} \otimes d^1)(\gamma) = 0$. Hence, for a reflexive pair, we have

$$\ker(d^0 \otimes d^0 - d^1 \otimes d^1) = \ker(d^0 - d^1) \otimes \ker(d^0 - d^1).$$

One deduces from this observation that $\ker(d^0 - d^1)$ forms a subcoalgebra of Γ^0 and the lemma follows. □

Then we obtain:

§3.2.8. **Lemma.** *The category of augmented coalgebras* CoAlg_+^a *has small products.*

Proof. This assertion is classical for coalgebras over a comonad. In fact, one can observe that a product $\prod_\alpha X_\alpha$ can be defined by a reflexive coequalizer of cofree objects. Namely:

$$\prod_\alpha X_\alpha = \ker\Big(\begin{tikzcd} \Gamma(\prod_\alpha X_\alpha) \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Gamma(\prod_\alpha \Gamma(X_\alpha)) \arrow[l, bend right, "s^0"'] \end{tikzcd} \Big),$$

where d^0 is induced by the coalgebra structure morphisms $\rho_\alpha : X_\alpha \to \Gamma(X_\alpha)$ and d^1 is the composite of the comonad coproduct $\nu : \Gamma(X) \to \Gamma(\Gamma(X))$ of the cofree coalgebra functor with the morphism $(p_\alpha)_* : \Gamma(\Gamma(\prod_\alpha X_\alpha)) \to \Gamma(\prod_\alpha \Gamma(X_\alpha))$ induced by the canonical projections. □

Observe also that CoAlg_+^a comes equipped with a final object which is defined by the ground field $* = \mathbb{F}$ since any coalgebra $\Gamma \in \mathrm{CoAlg}_+^a$ is supposed to be augmented over $\mathbb{F}$.

By standard categorical constructions, the existence of all small limits can be deduced from these particular cases (reflexive equalizers, small products and the final object). Hence we obtain the expected proposition:

§3.2.9. **Proposition.** *The category of augmented coalgebras* CoAlg_+^a *has all small limits.* □

As recalled in the introduction of this section, the model structure used in this memoir has been defined for $\mathbb{N}$-graded coalgebras only; we have to consider $\mathbb{Z}$-graded coalgebras but we shall deal only with lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

such that $i : A \to B$ is a morphism of non-negatively graded objects. In fact, one can use the following proposition in order to put such problems into an $\mathbb{N}$-graded framework:

§3.2.10. **Proposition.** *The category embedding $i_+^{\mathrm{dg}} : \mathrm{dg}_{\mathbb{N}} \mathrm{CoAlg}_+^a \hookrightarrow \mathrm{dg}_{\mathbb{Z}} \mathrm{CoAlg}_+^a$ has a right adjoint $\mathrm{s}_+^{\mathrm{dg}} : \mathrm{dg}_{\mathbb{Z}} \mathrm{CoAlg}_+^a \to \mathrm{dg}_{\mathbb{N}} \mathrm{CoAlg}_+^a$.*

In fact, the construction of this functor $\mathrm{s}_+^{\mathrm{dg}} : \mathrm{dg}_{\mathbb{Z}} \mathrm{CoAlg}_+^a \to \mathrm{dg}_{\mathbb{N}} \mathrm{CoAlg}_+^a$ is not essential for our purposes and we do not use this functor explicitly. Therefore we can skip the detailed verification of this proposition. On the other hand, the proposition is a straightforward consequence of the special adjoint functor theorem. Recall simply that colimits in a coalgebra category are created in the ground category of dg-modules. As a consequence, the functor i_+^{dg} preserves colimits. The category of dg-coalgebras has also a set of generators. One can observe more precisely that any $\mathbb{Z}$-graded coalgebra is a colimit of finite dimensional coalgebras as in the non-graded framework, for which we refer to the classical book [25], or as in the $\mathbb{N}$-graded framework, for which we refer to the article [13]. In fact, the proof given in these references can be extended to the $\mathbb{Z}$-graded context.

One can also adapt the classical arguments used for another adjoint functor in §4.2 in order to obtain an explicit realization of the functor $\mathrm{s}_+^{\mathrm{dg}}$. Namely recall first that we have a standard truncation functor on dg-modules. Then, for a cofree coalgebra $\Gamma(V)$, we are clearly forced to set $\mathrm{s}_+^{\mathrm{dg}} \Gamma(V) = \Gamma(\mathrm{s}_+^{\mathrm{dg}} V)$. One checks that the map $V \mapsto \Gamma(\mathrm{s}_+^{\mathrm{dg}} V)$ extends appropriately to a functor on the full subcategory of CoAlg_+^a formed by cofree coalgebras. Finally, in the general case, the truncation $\mathrm{s}_+^{\mathrm{dg}} K$ of a coalgebra K is obtained by an equalizer of cofree coalgebra truncations since any coalgebra K is the equalizer of a natural pair of cofree coalgebra morphisms associated to K.

As mentioned in the introduction, one can also check that the truncation functor preserves quasi-cofree coalgebras, the generalization of cofree coalgebras that we define in the next subsection.

§3.3. **Quasi-cofree coalgebras and coextension diagrams.** In the differential graded context, the cofree coalgebra $\Gamma(V)$ is equipped with a natural differential $\delta : \Gamma(V) \to \Gamma(V)$ induced by the internal differential of V. As explained in the introduction of this section, we consider *quasi-cofree coalgebra* structures defined by a cofree coalgebra $\Gamma = \Gamma(V)$ equipped with a coderivation $\partial : \Gamma(V) \to \Gamma(V)$ such that $\delta + \partial$ defines the differential of Γ.

The following useful assertion generalizes a classical result for the tensor coalgebra:

§3.3.1. **Lemma.** *For any homogeneous morphism $\alpha : \Gamma(V) \to V$, there is a unique coderivation $\partial_\alpha : \Gamma(V) \to \Gamma(V)$ such that $\alpha = \pi \partial_\alpha$.*

Assume that the morphism $\alpha : \Gamma(V) \to V$ is homogeneous of degree -1. The sum $\delta + \partial_\alpha$ defines a differential on $\Gamma(V)$ so that $\Gamma = (\Gamma(V), \partial_\alpha)$ defines a quasi-cofree coalgebra if and only if we have the relation

$$\delta(\alpha) + \alpha \partial_\alpha = 0$$

in $Hom(\Gamma(V), V)$. *Furthermore, the morphism of graded coalgebras* $\nabla_f : K \to \Gamma(V)$ *induced by a homogeneous morphism* $f : K \to V$ *of degree* 0 *defines a morphism of differential graded coalgebras* $\nabla_f : K \to \Gamma$, *where* $\Gamma = (\Gamma(V), \partial_\alpha)$, *if and only if we have the relation*

$$\delta(f) + \alpha\nabla_f = 0$$

in $Hom(K, V)$.

Proof. The construction of ∂_α is similar to the construction of the coalgebra morphism $\nabla_f : K \to \Gamma(V)$ associated to a morphism of dg-modules $f : K \to \Gamma(V)$. In particular, we deduce the existence of ∂_α from our realization of the cofree coalgebra $\Gamma(V) = \Gamma_\infty(V)$. Explicitly, we consider the morphism $\widehat{\partial}_\alpha : \Gamma(V) \to \prod_n V^{\otimes n}$ which maps an element $\gamma \in \Gamma(V)$ to the collection

$$\{\pi^{\otimes i-1} \otimes \alpha \otimes \pi^{\otimes n-i} \cdot \Delta^n(\gamma)\}.$$

One checks readily that this morphism fits a commutative diagram

$$\begin{tikzcd}
\Gamma(V) \arrow[r] \arrow[d, "\widehat{\partial}_\alpha"] & \Gamma(V) \otimes \Gamma(V) \arrow[d, "\nabla_\Pi \cdot (\widehat{\partial}_\alpha \otimes 1 + 1 \otimes \widehat{\partial}_\alpha)"] \\
\prod_N V^{\otimes N} \arrow[r, "\Delta_\Pi"] & \prod_{m,n} \{V^{\otimes m} \otimes V^{\otimes n}\}
\end{tikzcd}$$

and, as in the proof of lemma §3.2.5, we deduce from this property that $\widehat{\partial}_\alpha$ restricts to morphisms

$$\begin{tikzcd}
\Gamma(V) \arrow[d, "\partial_\infty"] \arrow[rr, "\partial_{r+1}"] \arrow[rrr, "\partial_r"] \arrow[rrrrr, "\partial_1 = \widehat{\partial}_\alpha"] & & & & & \\
\Gamma_\infty(V) \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_{r+1}(V) \arrow[r, hook] & \Gamma_r(V) \arrow[r, hook] & \cdots \arrow[r, hook] & \Gamma_1(V) = \prod_n V^{\otimes n}
\end{tikzcd}$$

such that $\partial_\alpha = \partial_\infty$ defines a coderivation of $\Gamma(V) = \Gamma_\infty(V)$. In fact, for a coderivation, the relation $\alpha = \pi\partial_\alpha$ implies that the composite of ∂_α with the embedding $\iota_\infty : \Gamma_\infty(V) \hookrightarrow \prod_n V^{\otimes n}$ agrees with the map $\widehat{\partial}_\alpha$. Therefore the coderivation ∂_α is uniquely characterized by this relation $\alpha = \pi\partial_\alpha$. The verification of the other assertions of the lemma is similar and straighforward. ☐

As explained in the introduction of this section, we aim to determine the structure of morphism of quasi-cofree coalgebras $\nabla_f : (\Gamma(V), \partial_\alpha) \to (\Gamma(W), \partial_\beta)$ that fit a pullback diagram of the form

$$\begin{tikzcd}
(\Gamma(V), \partial_\alpha) \arrow[r] \arrow[d] & \Gamma(\Delta^1 \wedge V) \arrow[d] \\
(\Gamma(W), \partial_\beta) \arrow[r] & \Gamma(\Delta^1 \wedge W) \times_{\Gamma(S^1 \wedge W)} \Gamma(S^1 \wedge V)
\end{tikzcd} .$$

First, we define precisely the dg-modules $\Delta^1 \wedge E$ and $S^1 \wedge E$ that occur in this construction. In fact, we consider nothing but the classical cone and suspension functors in the category of dg-modules.

§3.3.2. *The cone sequence of a dg-module.* Explicitly, for a dg-module E (possibly $\mathbb{Z}$-graded), we let $\Delta^1 \wedge E$, respectively $S^1 \wedge E$, denote the quotient $\mathbb{I} \otimes E / 0 \otimes E$, respectively $\mathbb{I} \otimes E / 0 \otimes E \oplus 1 \otimes E$, of the tensor product of E with the classical interval $\mathbb{I}$ of the category of dg-modules. Recall that this dg-module $\mathbb{I}$ is spanned by homogeneous elements $0, 1, 01$ of degree $\deg(0) = \deg(1) = 0$ and $\deg(01) = 1$ respectively and the differential $\partial : \mathbb{I} \to \mathbb{I}$ is defined by $\partial(01) = 1 - 0$ (see §2.2.1).

Equivalently, the dg-module $\Delta^1 \wedge E$ can be defined by
$$\Delta^1 \wedge E = 01 \otimes E \oplus 1 \otimes E.$$
The differential of $\Delta^1 \wedge E$ can be decomposed into a natural differential $\delta : \Delta^1 \wedge E \to \Delta^1 \wedge E$ induced by the internal differential of E and an extra term $\partial : \Delta^1 \wedge E \to \Delta^1 \wedge E$ induced by the differential of $\mathbb{I}$. By definition, this differential ∂ maps a tensor $01 \otimes x \in 01 \otimes E$ to a corresponding element $1 \otimes x \in 1 \otimes E$ and vanishes on the other component of $\Delta^1 \wedge E$. We have similarly
$$S^1 \wedge E = 01 \otimes E$$
so that $S^1 \wedge E$ can be identified with the suspension of E.

Clearly, we have a natural morphism of dg-modules $\sigma \wedge E : \Delta^1 \wedge E \to S^1 \wedge E$ which can be identified with the projection onto the component $01 \otimes E$ of $\Delta^1 \wedge E = 01 \otimes E \oplus 1 \otimes E$. We have also a natural embedding $d^1 : E \hookrightarrow \Delta^1 \wedge E$ which identifies the dg-module E with the component $1 \otimes E$ of $\Delta^1 \wedge E$.

We begin our constructions with the following simple observation:

§3.3.3. **Observation.** *Any quasi-cofree coalgebra $\Gamma = (\Gamma(V), \partial_\alpha)$ is endowed with a morphism of dg-modules $\pi_\alpha : \Gamma \to \Delta^1 \wedge V$ such that $\pi_\alpha(\gamma) = 01 \otimes \alpha(\gamma) + 1 \otimes \sigma(\gamma)$, for $\gamma \in \Gamma(V)$.*

Proof. One can observe precisely that the commutation of α with differentials, given by the commutativity of the square
$$\begin{tikzcd} \Gamma(V) \arrow[r, "\pi_\alpha"] \arrow[d, "\delta+\partial_\alpha"'] & \Delta^1 \wedge V \arrow[d, "\delta+\partial"'] \\ \Gamma(V) \arrow[r, "\pi_\alpha"] & \Delta^1 \wedge V \end{tikzcd},$$
is equivalent to the relation $\delta(\alpha) + \alpha\partial_\alpha = 0$ of lemma §3.3.1. □

In fact, we have the following general assertion:

§3.3.4. **Fact.** *A map $\tilde{e} : U \to \Delta^1 \wedge E$, where U is a dg-module, defines a dg-module morphism if and only if we have $\tilde{e}(u) = -01 \otimes \delta(e)(u) + 1 \otimes e(u)$, for a homogeneous map $e : U \to E$ of degree 0.*

In the previous observation the differential of $U = (\Gamma(V), \partial_\alpha)$ includes the coderivation ∂_α. Hence the differential of the homogeneous map $\pi_\alpha : \Gamma(V) \to V$ is given by $\delta\pi_\alpha - \pi_\alpha\delta - \pi_\alpha\partial_\alpha = 0 - \alpha = -\alpha$.

These assertions yield also to the following useful observation:

§3.3.5. **Observation.** *Suppose given a dg-module morphism $\tilde{v} : K \to \Delta^1 \wedge V$, where K is a dg-coalgebra, and consider the equivalent homogeneous map $v : K \to V$. The coalgebra map $\nabla_v : K \to \Gamma(V)$ induced by $v : K \to V$ defines a morphism to the quasi-cofree coalgebra $\Gamma = (\Gamma(V), \partial_\alpha)$ if and only if it makes commute the diagram*
$$\begin{tikzcd} K \arrow[d, dashed, "\nabla_v"'] \arrow[dr, "\tilde{v}"] & \\ \Gamma(V) \arrow[r, "\pi_\alpha"'] & \Delta^1 \wedge V \end{tikzcd}.$$
In addition this assertion holds as soon as the composites of $\tilde{v}$ and $\pi_\alpha\nabla_v$ with $\sigma \wedge V : \Delta^1 \wedge V \to S^1 \wedge V$ agree.

Proof. These claims are immediate: we have by definition $\pi_\alpha\nabla_v(x) = 01 \otimes \alpha\nabla_v(x) + 1 \otimes v(x)$ so that the relation $\pi_\alpha\nabla_v = \tilde{v}$ is equivalent to the relation $\alpha\nabla_v = -\delta(v)$ of lemma §3.3.1. □

§3.3.6. *Coalgebra coextensions.* We consider now a morphism of quasi-cofree coalgebras

$$\nabla_f : (\Gamma(V), \partial_\alpha) \to (\Gamma(W), \partial_\beta)$$

induced by a morphism of dg-modules $f : V \to W$. As this morphism f is supposed to commute with the internal differentials of V and W, the relation of lemma §3.3.1, that gives the commutation of ∇_f with quasi-cofree coalgebra differentials, is equivalent to $f\alpha = \beta\nabla_f$.

Assume that the diagram

$$\begin{tikzcd}
(\Gamma(V), \partial_\beta) \arrow[r, "\pi_\alpha"] \arrow[d, "\nabla_f"'] & \Delta^1 \wedge V \arrow[r, "\sigma\wedge V"] & S^1 \wedge V \arrow[d, "S^1\wedge f"] \\
(\Gamma(W), \partial_\beta) \arrow[r, "\pi_\beta"'] \arrow[urr, dashed, "\rho"] & \Delta^1 \wedge W \arrow[r, "\sigma\wedge W"'] & S^1 \wedge W
\end{tikzcd}$$

admits a lifting $\rho : (\Gamma(W), \partial_\beta) \to S^1 \wedge V$ in the category of dg-modules. Observe that the composite $\rho_\alpha = \sigma \wedge V \cdot \pi_\alpha$ is given by $\rho_\alpha(\gamma) = 01 \otimes \alpha(\gamma)$, for all $\gamma \in \Gamma(V)$, and similarly for $\rho_\beta = \sigma \wedge W \cdot \pi_\beta$. As a consequence, if f is epi, then the existence of ρ is equivalent to the existence of a lifting map $\beta' : \Gamma(V) \to W$ in the diagram

$$\begin{tikzcd}
\Gamma(V) \arrow[r, "\alpha"] \arrow[d, "\nabla_f"'] & V \arrow[d, "f"] \\
\Gamma(W) \arrow[r, "\beta"'] \arrow[ur, dashed] & W
\end{tikzcd}$$

since one can set $\rho(\gamma) = 01 \otimes \beta'(\gamma)$ and the epimorphism assumption implies that ρ commutes automatically with differentials.

Anyway, in this situation, we have a diagram of dg-module morphisms

$$\begin{tikzcd}
(\Gamma(V), \partial_\beta) \arrow[r, "\pi_\alpha"] \arrow[d, "\nabla_f"'] & \Delta^1 \wedge V \arrow[d, "(\Delta^1\wedge f, \sigma\wedge V)"] \\
(\Gamma(W), \partial_\beta) \arrow[r, "(\pi_\beta, \rho)"'] & \Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V
\end{tikzcd}$$

and we can consider the associated diagram of coalgebra morphisms

$$\begin{tikzcd}
(\Gamma(V), \partial_\alpha) \arrow[r, "\nabla_{\pi_\alpha}"] \arrow[d, "\nabla_f"'] & \Gamma(\Delta^1 \wedge V) \arrow[d, "(\nabla_{\Delta^1\wedge f}, \nabla_{\sigma\wedge V})"] \\
(\Gamma(W), \partial_\beta) \arrow[r, "(\nabla_{\pi_\beta}, \nabla_\rho)"'] & \Gamma(\Delta^1 \wedge W) \times_{\Gamma(S^1\wedge W)} \Gamma(S^1 \wedge V)
\end{tikzcd} .$$

We obtain:

§3.3.7. **Claim.** *The diagram above forms a pullback in the category of dg-coalgebras.*

Proof. We suppose given a pair of coalgebra morphisms $\nabla_{\tilde{v}} : K \to \Gamma(\Delta^1 \wedge V)$ and $\nabla_w : K \to (\Gamma(W), \partial_W)$ that fit the commutative diagram

$$\begin{tikzcd}
K \arrow[dr, dashed, "\nabla_v"] \arrow[drr, bend left, "\nabla_{\tilde{v}}"] \arrow[ddr, bend right, "\nabla_w"'] & & \\
& (\Gamma(V), \partial_\alpha) \arrow[r, "\nabla_{\pi_\alpha}"] \arrow[d, "\nabla_f"'] & \Gamma(\Delta^1 \wedge V) \arrow[d, "(\nabla_{\Delta^1 \wedge f}, \nabla_{\sigma \wedge V})"] \\
& (\Gamma(W), \partial_\beta) \arrow[r, "(\nabla_{\pi_\beta}, \nabla_\rho)"'] & \Gamma(\Delta^1 \wedge W) \times_{\Gamma(S^1 \wedge W)} \Gamma(S^1 \wedge V)
\end{tikzcd}$$

and we check the existence of a unique filling morphism $\nabla_v : K \to (\Gamma(V), \partial_\alpha)$ that fits this diagram. According to the fact §3.3.4, the coalgebra morphism $\nabla_{\tilde{v}} : K \to \Gamma(\Delta^1 \wedge V)$ is equivalent to a morphism of dg-modules $\tilde{v} : K \to \Delta^1 \wedge V$ such that $\tilde{v}(x) = -01 \otimes \delta(v)(x) + 1 \otimes v(x)$ for a homogeneous map $v : K \to V$ of degree 0. We check that this map v induces a morphism of coalgebras $\nabla_v : K \to (\Gamma(V), \partial_\alpha)$ that provides a filling morphism in our diagram.

The commutativity of the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "\nabla_{\tilde{v}}"] \arrow[drrr, bend left, "\tilde{v}"] \arrow[ddr, bend right, "\nabla_w"'] & & & \\
& (\Gamma(V), \partial_\alpha) \arrow[r] \arrow[d, "\nabla_f"] & \Gamma(\Delta^1 \wedge V) \arrow[r] \arrow[d, "\nabla_{\Delta^1 \wedge f}"] & \Delta^1 \wedge V \arrow[d, "\Delta^1 \wedge f"] \\
& (\Gamma(W), \partial_\beta) \arrow[r] \arrow[rr, bend right, "\pi_\beta"'] & \Gamma(\Delta^1 \wedge W) \arrow[r] & \Delta^1 \wedge W
\end{tikzcd}$$

gives $\Delta^1 \wedge f \cdot \tilde{v} = \pi_\beta \cdot \nabla_w$ and from this relation we deduce the identity $fv = w$. As a consequence, for the induced coalgebra maps, we obtain $\nabla_f \nabla_v = \nabla_w$. Then the commutativity of the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "\nabla_{\tilde{v}}"] \arrow[drrr, bend left, "\tilde{v}"] \arrow[ddr, bend right, "\nabla_w"'] & & & \\
& (\Gamma(V), \partial_\alpha) \arrow[r] \arrow[d, "\nabla_f"] & \Gamma(\Delta^1 \wedge V) \arrow[r] \arrow[d, "\nabla_{\sigma \wedge V}"] & \Delta^1 \wedge V \arrow[d, "\sigma \wedge V"] \\
& (\Gamma(W), \partial_\beta) \arrow[r] \arrow[rr, bend right, "\rho"'] & \Gamma(S^1 \wedge V) \arrow[r] & S^1 \wedge V
\end{tikzcd}$$

gives $\sigma \wedge V \cdot \tilde{v} = \rho \cdot \nabla_w = \rho \cdot \nabla_f \cdot \nabla_v$. By the very definition of ρ, we have $\rho \cdot \nabla_f = \sigma \wedge V \cdot \pi_\alpha$. Hence we obtain the relation $\sigma \wedge V \cdot \tilde{v} = \sigma \wedge V \cdot \pi_\alpha \cdot \nabla_v$. By observation §3.3.5, this assertion implies that ∇_v defines a coalgebra morphism to the quasi-cofree coalgebra $(\Gamma(V), \partial_\alpha)$ and we have in addition $\nabla_{\tilde{v}} = \nabla_{\pi_\alpha} \nabla_v$. As we obtain also $\nabla_v \nabla_f = \nabla_w$, this observation completes the proof of the existence of a filling morphism ∇_v. As this filling morphism is clearly unique, this achieves the proof of claim §3.3.7. $\square$

Observe that we have the identity

$$\Gamma(\Delta^1 \wedge W) \times_{\Gamma(S^1 \wedge W)} \Gamma(S^1 \wedge V) = \Gamma(\Delta^1 \wedge W \times_{S^1 \wedge W} S^1 \wedge V)$$

in the category of coalgebras. Moreover, the morphism $(\nabla_{\Delta^1\wedge f}, \nabla_{\sigma\wedge V})$ involved in the construction of §3.3.6 can be identified with the morphism of cofree coalgebras induced by the dg-module morphism

$$(\Delta^1 \wedge f, \sigma \wedge V) : \Delta^1 \wedge V \to \Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V.$$

For our purposes we give a more explicit form to this morphism. Namely we have the following simple assertion:

§3.3.8. **Observation.** *The dg-module $\Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V$ can be identified with the direct sum $01 \otimes V \oplus 1 \otimes W$ equipped with a differential given by the sum of the internal differentials of V and W with an extra term ∂ such that $\partial(01 \otimes v) = 1 \otimes v$, for all $v \in V$.*

The morphism $(\Delta^1 \wedge f, \sigma \wedge V) : \Delta^1 \wedge V \to \Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V$ is also identified with the direct sum

$$01 \otimes V \oplus 1 \otimes V \xrightarrow{(01\otimes \mathrm{Id}, 1\otimes f)} 01 \otimes V \oplus 1 \otimes W$$

where we consider the identity map $01 \otimes \mathrm{Id} : 01 \otimes V \to 01 \otimes V$ and the map $1 \otimes f : 1 \otimes V \to 1 \otimes W$ on the components of $\Delta^1 \wedge V$. □

From this observation we obtain immediately:

§3.3.9. **Claim.** *If $f : V \to W$ is a fibration, respectively an acyclic fibration, of dg-modules, then so is $(\Delta^1 \wedge f, \sigma \wedge V) : \Delta^1 \wedge V \to \Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V$.* □

By adjunction, the morphism of cofree coalgebras

$$\nabla_{(\Delta^1\wedge f,\sigma\wedge V)} : \Gamma(\Delta^1 \wedge V) \to \Gamma(\Delta^1 \wedge W \times_{S^1\wedge W} S^1 \wedge V)$$

associated to $(\Delta^1 \wedge f, \sigma \wedge V)$ defines a fibration, respectively an acyclic fibration, in the category of dg-coalgebras if f is so. As a corollary, by standard arguments, we obtain:

§3.3.10. **Lemma.** *In the situation of §3.3.6, if f is a fibration, respectively an acyclic fibration in the category of dg-modules, then ∇_f defines a fibration, respectively an acyclic fibration in the category of dg-coalgebras.* □

§3.3.11. *Functorial coextensions.* For our purposes we need a relative version of the result of lemma §3.3.10. To be precise, we consider now a commutative square of quasi-cofree coalgebra morphisms

$$\begin{tikzcd}
(\Gamma(V), \partial_\alpha) \arrow[r, "\nabla_v"] \arrow[d, "\nabla_f"'] & (\Gamma(V'), \partial_{\alpha'}) \arrow[d, "\nabla_{f'}"'] \\
(\Gamma(W), \partial_\beta) \arrow[r, "\nabla_w"] & (\Gamma(W'), \partial_{\beta'})
\end{tikzcd}$$

yielded by a commutative square of dg-module morphisms

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "f"'] & V' \arrow[d, "f'"'] \\
W \arrow[r, "w"] & W'
\end{tikzcd} .$$

In §3.3.6 we observed that ∇_f satisfies the commutation relation $\beta\nabla_f = f\alpha$ and so does $\nabla_{f'}$ since f, respectively f', is supposed to commute with internal differentials of dg-modules. Similarly, as ∇_v and ∇_w are supposed to be morphisms of dg-coalgebras induced by morphisms of dg-modules v and w, we obtain a commutative cube:

$$\begin{tikzcd}
\Gamma(V) \arrow[rr, "\alpha"] \arrow[dd] \arrow[dr] & & V \arrow[dd] \arrow[dr] & \\
& \Gamma(V') \arrow[rr, "\alpha'"] & & V' \arrow[dd] \\
\Gamma(W) \arrow[rr, "\beta"] \arrow[dr] & & W \arrow[dr] & \\
& \Gamma(W') \arrow[rr, "\beta'"] \arrow[from=uu, crossing over] & & W'
\end{tikzcd}$$

As in §3.3.6, we assume the existence of liftings ρ and ρ' of the coderivation maps. Naturally we assume in addition that these liftings commute with the morphisms ∇_v and ∇_w and hence fit the commutative cube above. Clearly, if f and g are epi, then this functoriality requirement is automatically satisfied. Anyway, in this situation, the maps v and w yield a morphism between the coalgebra pullbacks associated to ∇_f and $\nabla_{f'}$. Explicitly, these coalgebra pullbacks form the back and front square of a commutative cube. By pulling back the front corners of this cube, we obtain a commutative square

$$\begin{tikzcd}
(\Gamma(V),\partial_\alpha) \arrow[rr] \arrow[d, "{(\nabla_f,\nabla_v)}"'] & & \Gamma(\Delta^1\wedge V) \arrow[d] \\
(\Gamma(W),\partial_\beta)\times_{(\Gamma(W'),\partial_{\beta'})}(\Gamma(V'),\partial_{\alpha'}) \arrow[rr] & & \Gamma(\Delta^1\wedge W \times_? S^1\wedge V \times_? \Delta^1\wedge V')
\end{tikzcd}$$

in which the bottom right hand-side corner can be defined precisely by the limit of the corner diagram of cofree coalgebras

$$\begin{tikzcd}
& & \Gamma(S^1\wedge V) \arrow[dd] \arrow[dr] & \\
& \Gamma(\Delta^1\wedge V') \arrow[rr] & & \Gamma(S^1\wedge V') \arrow[dd] \\
\Gamma(\Delta^1\wedge W) \arrow[rr] \arrow[dr] & & \Gamma(S^1\wedge W) \arrow[dr] & \\
& \Gamma(\Delta^1\wedge W') \arrow[rr] \arrow[from=uu, crossing over] & & \Gamma(S^1\wedge W')
\end{tikzcd} .$$

§3.3.12. **Observation.** *The cartesian product* $(\Gamma(W),\partial_\beta)\times_{(\Gamma(W'),\partial_{\beta'})}(\Gamma(V'),\partial_{\alpha'})$ *can be identified with the quasi-cofree coalgebra* $(\Gamma(X),\partial_\gamma)$ *such that* $X = W\times_{W'}V'$ *and where the coderivation* ∂_γ *is induced by the map* $(\beta,\alpha') : \Gamma(W\times_{W'}V') \to W\times_{W'}V'$.

Furthermore, our commutative square can be identified with the pullback diagram

$$\begin{tikzcd}
(\Gamma(V),\partial_\alpha) \arrow[r, "\nabla_{\pi_\alpha}"] \arrow[d, "\nabla_{(f,v)}"'] & \Gamma(\Delta^1\wedge V) \arrow[d, "{(\nabla_{\Delta^1\wedge(f,v)},\nabla_{\sigma\wedge V})}"] \\
(\Gamma(X),\partial_\gamma) \arrow[r, "{(\nabla_{\pi_\gamma},\nabla_\sigma)}"'] & \Gamma(\Delta^1\wedge X)\times_{\Gamma(S^1\wedge X)}\Gamma(S^1\wedge V)
\end{tikzcd}$$

associated to the morphism $(f, v) : V \to W \times_{W'} V'$ *and where* $\sigma : (\Gamma(X), \partial_\gamma) \to S^1 \wedge V$ *is given by the composite*

$$(\Gamma(X), \partial_\gamma) \xrightarrow{\mathrm{pr}_1} (\Gamma(W), \partial_\beta) \xrightarrow{\rho} S^1 \wedge V.$$

Proof. This assertion follows from straightforward verifications. Observe simply that we have $\Delta^1 \wedge (W \times_{W'} V') = \Delta^1 \wedge W \times_{\Delta^1 \wedge W'} \Delta^1 \wedge V'$ in view of our definition of the functor $\Delta^1 \wedge -$ and similarly for $S^1 \wedge (W \times_{W'} V')$. □

§3.3.13. **Lemma.** *In the situation of §3.3.11, if the morphism*

$$\nabla_w : (\Gamma(W), \partial_\beta) \to (\Gamma(W'), \partial_{\beta'})$$

is a fibration of dg-coalgebras and $(f, v) : V \to W \times_{W'} V'$ *is a fibration of dg-modules, then*

$$\nabla_v : (\Gamma(V), \partial_\alpha) \to (\Gamma(V'), \partial_{\alpha'})$$

is fibration of dg-coalgebras as well.

If we assume furthermore that ∇_w *and* (f, v) *are weak-equivalences so that* ∇_w *forms an acyclic fibration of dg-coalgebras and* (f, v) *forms an acyclic fibration of dg-modules, then* ∇_v *forms an acyclic fibration of dg-coalgebras as well.*

For (f, v), one can observe that the weak-equivalence property is satisfied as soon as we assume that w forms an acyclic fibration and v is a weak-equivalence. In fact, consider the diagram

$$\begin{tikzcd}
V \arrow[d, "(f,v)"] \arrow[dd, bend right=60, "v"'] & \\
W \times_{W'} V' \arrow[r] \arrow[d, "\sim"'] & W \arrow[d, "w"', "\sim"] \\
V' \arrow[r] & W'
\end{tikzcd}.$$

The pullback of $w : W \to W'$ is an acyclic fibration by standard model category arguments. If we assume that v is a weak-equivalence, then, by the two out of three axiom, we conclude that (f, v) is a weak-equivalence as well, as claimed.

Proof. According to lemma §3.3.10, the morphism $\nabla_{(f,v)}$ forms a fibration, respectively an acyclic fibration, if (f, v) is so. On the other hand, by the general model category argument recalled just above this proof, in the pullback diagram

$$\begin{tikzcd}
(\Gamma(X), \partial_\gamma) \arrow[r] \arrow[d] & (\Gamma(W), \partial_\beta) \arrow[d, "\nabla_w"] \\
(\Gamma(V'), \partial_{\alpha'}) \arrow[r] & (\Gamma(W'), \partial_{\beta'})
\end{tikzcd}$$

the left-hand side morphism forms a fibration, respectively an acyclic fibration, if ∇_w is so. As a conclusion, under the assumptions of the lemma, the morphism ∇_v can be decomposed into a composite of fibrations, respectively acyclic fibrations, and hence forms a fibration, respectively an acyclic fibration, as well. □

§3.4. **Quasi-cofree Hopf Λ_*-modules.** The purpose of this subsection is to extend the constructions of the previous section to the category of Hopf Λ_*-modules. First, observe that the category of Hopf Λ_*-modules is endowed with a cofree object functor $\Gamma : \Lambda_*^{\mathrm{op}}\,\mathrm{Mod} \to \Lambda_*^{\mathrm{op}}\,\mathrm{HopfMod}$. In fact, the *cofree Hopf Λ_*-module* $\Gamma(M)$ associated to a Λ_*-module M can be defined explicitly by the cofree coalgebras

$$\Gamma(M)(r) = \Gamma(M(r))$$

on the dg-modules $M(r)$. Furthermore, the operations $\partial_i : \Gamma(M)(r) \to \Gamma(M)(r-1)$ can be identified with the morphisms of cofree coalgebras $\nabla_{\partial_i} : \Gamma(M(r)) \to \Gamma(M(r-1))$ induced by the operations $\partial_i : M(r) \to M(r-1)$ on $M(r)$.

For our purposes we extend the notion of a quasi-cofree object to Hopf Λ_*-modules. Then we observe that the categorical results of the previous section hold as well for Hopf Λ_*-modules. Finally, we obtain useful sufficient condition for a map to be a fibrations and acyclic fibrations in that category.

§3.4.1. *Quasi-cofree Hopf Λ_*-modules.* As usual, in the differential graded framework, the cofree Hopf Λ_*-module $\Gamma(M)$ is equipped with a natural differential $\delta : \Gamma(M) \to \Gamma(M)$ induced by the internal differential of M.

The structure of a *quasi-cofree Hopf Λ_*-module* is defined by a pair

$$\Gamma = (\Gamma(M), \partial_\alpha)$$

where M is a Λ_*-module, $\Gamma(M)$ denotes the associated cofree Hopf Λ_*-module and $\partial_\alpha : \Gamma(M) \to \Gamma(M)$ denotes a homogeneous morphism of Λ_*-modules which forms a coderivation with respect to the coalgebra structure so that the sum $\delta + \partial$ defines the differential of the Hopf Λ_*-module Γ.

The coderivation $\partial_\alpha : \Gamma(M) \to \Gamma(M)$ is defined by a collection of coderivations of cofree coalgebras $\partial_\alpha : \Gamma(M(r)) \to \Gamma(M(r))$ which in turn can be specified by a collection of homogeneous maps $\alpha : \Gamma(M(r)) \to M(r)$. As specified in the definition, the coderivation ∂_α is supposed to define a homogeneous morphism of Λ_*-modules $\partial_\alpha : \Gamma(M) \to \Gamma(M)$. Thus we assume explicitly that ∂_α commutes with the action of Σ_r and with the operations $\partial_i : \Gamma(M(r)) \to \Gamma(M(r-1))$. One can assume equivalently that the permutations $w \in \Sigma_r$ induce morphisms of quasi-cofree coalgebras $w : (\Gamma(M(r)), \partial_\alpha) \to (\Gamma(M(r)), \partial_\alpha)$ and similarly for the operations $\partial_i : M(r) \to M(r-1)$.

Observe that the matching objects of a quasi-cofree Hopf Λ_*-module $\Gamma = (\Gamma(M), \partial_\alpha)$ can be defined by the quasi-cofree coalgebras

$$\mathbb{M}\Gamma(r) = (\Gamma(\mathbb{M}M(r)), \partial_\alpha),$$

where ∂_α is induced by the homogeneous map $\alpha : \Gamma(\mathbb{M}M(r)) \to \mathbb{M}M(r)$ that fits the commutative diagrams

$$\begin{tikzcd}
\Gamma(\mathbb{M}M(r)) \arrow[r, "\alpha"] \arrow[d, "\nabla_{\partial_i}"'] & \mathbb{M}M(r) \arrow[d, "\partial_i"] \\
\Gamma(\mathbb{M}M(r-1)) \arrow[r, "\alpha"] & \mathbb{M}M(r)
\end{tikzcd} ,$$

for $i = 1, \ldots, r$. Similarly, if a morphism of quasi-cofree Hopf Λ_*-modules $\nabla_f : \Gamma \to \Gamma'$ where $\Gamma = (\Gamma(M), \partial_\alpha)$ and $\Gamma' = (\Gamma(M'), \partial_{\alpha'})$ is induced by a morphism of Λ_*-modules $f : M \to M'$, then we have

$$\mathbb{M}\Gamma(r) \times_{\mathbb{M}\Gamma'(r)} \Gamma'(r) = (\Gamma(\mathbb{M}M(r) \times_{\mathbb{M}M'(r)} M'(r)), \partial_{(\alpha,\alpha')}).$$

Moreover, the morphism $(\mu, \nabla_f) : \Gamma(r) \to \mathbb{M}\Gamma(r) \times_{\mathbb{M}\Gamma'(r)} \Gamma'(r)$ can be identified with the morphism of quasi-cofree coalgebras induced by the morphism of dg-modules $(\mu, f) : M(r) \to \mathbb{M}M(r) \times_{\mathbb{M}M'(r)} M'(r)$.

§3.4.2. *Coalgebra coextensions.* As in the case of coalgebras, we consider a morphism of quasi-cofree Hopf Λ_*-modules

$$\nabla_f : (\Gamma(M), \partial_\alpha) \to (\Gamma(N), \partial_\beta)$$

induced by a morphism of dg-modules $f : M \to N$. Furthermore, we assume the existence of a lifting $\rho : (\Gamma(M), \partial_\beta) \to S^1 \wedge N$ in a diagram of the form

$$\begin{tikzcd}
(\Gamma(M), \partial_\beta) \arrow[r, "\pi_\alpha"] \arrow[d, "\nabla_f"'] \arrow[rrd, phantom] & \Delta^1 \wedge M \arrow[r, "\sigma \wedge M"] & S^1 \wedge N \arrow[d, "S^1 \wedge f"] \\
(\Gamma(M), \partial_\beta) \arrow[r, "\pi_\beta"'] \arrow[rru, dashed, "\rho"] & \Delta^1 \wedge N \arrow[r, "\sigma \wedge N"'] & S^1 \wedge N
\end{tikzcd} ,$$

where the Λ_*-modules $\Delta^1 \wedge E$ are defined by $(\Delta^1 \wedge E)(r) = \Delta^1 \wedge E(r)$ and similarly for $S^1 \wedge E$. To be precise, we assume that ρ defines a lifting in the category of dg-modules and hence commutes with the action of permutations and with the operations ∂_i.

Clearly, if f is epi, then the existence of ρ is equivalent to the existence of a lifting map $\beta' : \Gamma(M(r)) \to N(r)$ in the diagrams

$$\begin{tikzcd}
\Gamma(M(r)) \arrow[r, "\alpha"] \arrow[d, "\nabla_f"'] & M(r) \arrow[d, "f"] \\
\Gamma(N(r)) \arrow[r, "\beta"'] \arrow[ru, dashed] & N(r)
\end{tikzcd} ,$$

for $r \in \mathbb{N}$. The epimorphism assumption implies that ρ commutes automatically with differentials and with Λ_*-module operations.

Anyway, as in the case of coalgebras, we have a diagram of Hopf Λ_*-modules

$$\begin{tikzcd}
(\Gamma(M), \partial_\alpha) \arrow[r, "\nabla_{\pi_\alpha}"] \arrow[d, "\nabla_f"'] & \Gamma(\Delta^1 \wedge M) \arrow[d, "(\nabla_{\Delta^1 \wedge f}, \nabla_{\sigma \wedge M})"] \\
(\Gamma(N), \partial_\beta) \arrow[r, "(\nabla_{\pi_\beta}, \nabla_\rho)"'] & \Gamma(\Delta^1 \wedge N) \times_{\Gamma(S^1 \wedge N)} \Gamma(S^1 \wedge M)
\end{tikzcd}$$

and, furthermore, we obtain:

§3.4.3. **Claim.** *The diagram above forms a pullback in the category of Hopf Λ_*-modules.* □

Then:

§3.4.4. **Claim.** *If $f : M \to N$ is a Reedy fibration, respectively an acyclic Reedy fibration, of Λ_*-modules, then so is $(\Delta^1 \wedge f, \sigma \wedge M) : \Delta^1 \wedge M \to \Delta^1 \wedge N \times_{S^1 \wedge N} S^1 \wedge M$.*

Proof. This assertion can be deduced readily from the componentwise expansion of $(\Delta^1 \wedge f, \sigma \wedge M)$ given in observation §3.3.8. Observe simply that the matching functor commutes with the cone construction $\Delta^1 \wedge -$ so that

$$\mathbb{M}(\Delta^1 \wedge M)(r) = \Delta^1 \wedge \mathbb{M}M(r) = 01 \otimes \mathbb{M}M(r) \oplus 1 \otimes \mathbb{M}M(r)$$

and similarly for $\mathbb{M}(S^1 \wedge M)$. □

And, by the usual categorical arguments, we obtain:

§3.4.5. **Lemma.** *In the situation of §3.4.2, if f is a Reedy fibration, respectively an acyclic Reedy fibration, in the category of Λ_*-modules, then ∇_f defines a Reedy fibration, respectively an acyclic Reedy fibration, in the category of Hopf Λ_*-modules.* □

§3.4.6. *Functorial coextensions.* Finally, we give also a relative version of this result in the context of Hopf Λ_*-modules. We consider a commutative square of morphisms of quasi-cofree Hopf Λ_*-modules

$$\begin{tikzcd} (\Gamma(M),\partial_\alpha) \arrow[r, "\nabla_v"] \arrow[d, "\nabla_f"'] & (\Gamma(M'),\partial_{\alpha'}) \arrow[d, "\nabla_{f'}"'] \\ (\Gamma(N),\partial_\beta) \arrow[r, "\nabla_w"] & (\Gamma(N'),\partial_{\beta'}) \end{tikzcd}$$

yielded by a commutative square of Λ_*-module morphisms

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "f"'] & M' \arrow[d, "f'"'] \\ N \arrow[r, "w"] & N' \end{tikzcd} .$$

As in the case of coalgebras, we assume the existence of functorial liftings ρ and ρ' of the coderivation maps. Naturally we still assume that ρ and ρ' form Λ_*-module morphisms. Furthermore, if f and g are epi, then these functoriality requirements are automatically satisfied.

§3.4.7. **Lemma.** *In the situation of §3.4.6, if the morphism*

$$\nabla_w : (\Gamma(N),\partial_\beta) \to (\Gamma(N'),\partial_{\beta'})$$

is a Reedy fibration of Hopf Λ_*-modules and $(f,v) : M \to N \times_{N'} M'$ is a Reedy fibration of Λ_*-modules, then

$$\nabla_v : (\Gamma(M),\partial_\alpha) \to (\Gamma(M'),\partial_{\alpha'})$$

is also a Reedy fibration of Hopf Λ_*-modules.

If we assume furthermore that ∇_w and (f,v) are weak-equivalences so that ∇_w forms an acyclic Reedy fibration of Hopf Λ_*-modules and (f,v) forms an acyclic Reedy fibration of Λ_*-modules, then ∇_v forms an acyclic Reedy fibration of Hopf Λ_*-modules as well. □

As in the case of coalgebras, one can observe that the weak-equivalence property for (f,v) is satisfied as soon as we assume that w forms an acyclic Reedy fibration of Λ_*-modules and v is a weak-equivalence.

Proof. One can prove this lemma by a tedious but straightforward generalization of the arguments involved in the case of coalgebras. On the other hand, by definition, the morphism $\nabla_v : (\Gamma(M),\partial_\alpha) \to (\Gamma(M'),\partial_{\alpha'})$ forms a Reedy fibration, respectively an acyclic Reedy fibration, in the category of Hopf Λ_*-modules if and only if the morphism

$$\nabla_{(\mu,v)} : (\Gamma(M(r)),\partial_\alpha) \to (\Gamma(\mathbb{M}M(r) \times_{\mathbb{M}M'(r)} M'(r)),\partial_{(\alpha,\alpha')})$$

forms a fibration in the category of dg-coalgebras. Accordingly, our claims can also be deduced from the statement of lemma §3.3.13 applied to the coalgebra morphisms

$$\begin{tikzcd} (\Gamma(M(r)),\partial_\alpha) \arrow[r, "\nabla_{(\mu,v)}"] \arrow[d, "\nabla_f"'] & (\Gamma(\mathbb{M}M(r) \times_{\mathbb{M}M'(r)} M'(r)),\partial_{(\alpha,\alpha')}) \arrow[d, "\nabla_{f'}"'] \\ (\Gamma(N(r)),\partial_\beta) \arrow[r, "\nabla_{(\mu,w)}"] & (\Gamma(\mathbb{M}N(r) \times_{\mathbb{M}N'(r)} N'(r)),\partial_{(\beta,\beta')}) \end{tikzcd} .$$

This observation achieves the proof of lemma §3.4.7. □

§4. Hopf endomorphism operads

§4.1. **Introduction and sketch of the section results.** The main results of this memoir are proved in this section. For that reason we give in this introduction a detailed summary of the content of this section. Thus we give precisions on the memoir objectives recalled from the memoir introduction.

As in [B1], we let $\mathcal{K}$ denote the chain A_∞-operad of Stasheff's associahedra. We consider operads $\mathcal{P}$ equipped with a fixed operad morphism $\mathcal{K} \to \mathcal{P}$ so that we can extend the bar construction $A \mapsto B(A)$ to the category of $\mathcal{P}$-algebras. To be more precise, as explained in the introduction of §1, it is natural to assume that A is a non-unitary algebra and hence to consider the non-unital version of the A_∞-operad $\mathcal{K}$ and a non-unital operad $\mathcal{P}$ under $\mathcal{K}$. Otherwise we have to replace the algebra A by its augmentation ideal $\overline{A}$ and the operad $\mathcal{P}$ by the associated reduced operad $\overline{\mathcal{P}}$.

Recall that $B(A) = T^c(\Sigma A)$ the *tensor coalgebra* cogenerated by the suspended dg-module ΣA. The differential of $B(A)$ is the sum of the internal differential of A with an additional component $\partial : T^c(\Sigma A) \to T^c(\Sigma A)$ defined by a coalgebra coderivation such that

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{r=2}^{n} \sum_{k=1}^{n-r+1} \pm a_1 \otimes \cdots \otimes \mu_r(a_k, \dots, a_{k+r-1}) \otimes \cdots \otimes a_n,$$

where the μ_r's are the standard generators of Stasheff's operad $\mathcal{K}$. These operations can abusively be identified with their image under the operad morphism $\mathcal{K} \to \mathcal{P}$. The bar complex $B(A)$ is equipped with a canonical unit $\mathbb{F} \to B(A)$ and forms a connected augmented unitary coalgebra.

The Hopf endomorphism operad of the bar complex. The first goal of this section is to extend the endomorphism operad construction of [B1] in the context of Hopf operads. More explicitly, we prove the following theorem:

Theorem §4.A. *Let $\mathcal{P}$ denote a (non-unital) operad in dg-modules equipped with an operad morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ denotes Staheff's chain operad. There is a universal unital Hopf operad $\mathcal{Q} = \mathrm{HopfEnd}_B^{\mathcal{P}}$ such that the bar complex of a $\mathcal{P}$-algebra $B(A)$ is equipped with the structure of a Hopf algebra over $\mathcal{Q}$, functorially in $A \in \mathcal{P}\,\mathrm{Alg}$.*

More precisely, the Hopf operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ operates functorially on the coalgebra $B(A)$ and so that the unital operation $* : \mathbb{F} \to B(A)$ agrees with the unit of $B(A)$. Furthermore we have a one-to-one correspondance between such Hopf operad actions and morphisms of unital Hopf operads $\rho : \mathcal{Q} \to \mathrm{HopfEnd}_B^{\mathcal{P}}$.

The proof of this statement relies on the existence of internal hom-objects in the category of coalgebras. To be precise, as we consider on one hand the bar complex, which forms an augmented unitary coalgebra, and on the other hand the underlying coalgebras of a unital Hopf operad, which are supposed to form augmented but non-unitary coalgebras, we have to adapt our definitions. Explicitly, to any pair of augmented unitary coalgebras L and M, we associate an augmented coalgebra *HopfHom*(L, M) such that a morphism $\phi^\sharp : K \to$ *HopfHom*(L, M) is equivalent to a morphism of augmented coalgebras $\phi : K \otimes L \to M$ that makes the diagram

$$\begin{tikzcd} K \otimes \mathbb{F} \arrow[r, "\epsilon"] \arrow[d, "K\otimes\eta"'] & \mathbb{F} \arrow[d, "\eta"] \\ K \otimes L \arrow[r, "\phi"] & M \end{tikzcd}$$

commute. Then, for any coalgebra Γ, the objects $\operatorname{HopfEnd}_\Gamma(r) = \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$ form clearly a unital Hopf operad and a morphism of unital Hopf operads $\rho : \mathcal{Q} \to \operatorname{HopfEnd}_\Gamma$ is equivalent to a Hopf operad action of $\mathcal{Q}$ on Γ such that the unital operation $* : \mathbb{F} \to \Gamma$ agrees with the internal unit of the coalgebra Γ.

Formally, the endomorphism operad $\operatorname{HopfEnd}^{\mathcal{P}}_B$ is defined by the end

$$\operatorname{HopfEnd}^{\mathcal{P}}_B(r) = \int^{A \in \mathcal{P}\,\mathrm{Alg}} \mathit{HopfHom}(B(A)^{\otimes r}, B(A)),$$

also denoted by

$$\operatorname{HopfEnd}^{\mathcal{P}}_B(r) = \mathit{HopfHom}_{A \in \mathcal{P}\,\mathrm{Alg}}(B(A)^{\otimes r}, B(A)),$$

and where A ranges over the category of $\mathcal{P}$-algebras. Roughly, the idea is to replace morphisms by natural transformations. The end construction permits simply to extend this process to internal hom-objects. According to this construction, we have a morphism

$$\operatorname{HopfEnd}^{\mathcal{P}}_B(r) \to \mathit{HopfHom}(B(A)^{\otimes r}, B(A)),$$

for all $A \in \mathcal{P}\,\mathrm{Alg}$, so that $\operatorname{HopfEnd}^{\mathcal{P}}_B$ operates on the bar complex $B(A)$ functorially in $A \in \mathcal{P}\,\mathrm{Alg}$. Thus the existence assertion of theorem §4.A is a formal consequence of categorical properties of the category of coalgebras.

Fibration properties and contruction of operad actions. The second goal of this section is to give more insights into the structure of $\operatorname{HopfEnd}^{\mathcal{P}}_B$ so that we can prove the existence and uniqueness results stated in the introduction of this memoir. Let us outline our constructions more precisely.

First, we observe that $\operatorname{HopfEnd}^{\mathcal{P}}_B$ forms a quasi-cofree object in the category of Hopf Λ_*-modules. Recall that a quasi-cofree Hopf Λ_*-module $\Gamma = (\Gamma(M), \partial)$ is defined explicitly by a collection of cofree coalgebras $\Gamma(M(r))$ associated to a Λ_*-module M and equipped with coderivations $\partial : \Gamma(M(r)) \to \Gamma(M(r))$, which preserve the Λ_*-module structure, such that the differential of Γ is given by the sum $\delta + \partial$, where δ is the natural differential of the cofree coalgebra induced by the internal differential of M. For the Hopf endomorphism operad $\operatorname{HopfEnd}^{\mathcal{P}}_B$, we obtain precisely

$$\operatorname{HopfEnd}^{\mathcal{P}}_B = (\Gamma(\operatorname{PrimEnd}^{\mathcal{P}}_B), \partial)$$

for the Λ_*-module

$$\operatorname{PrimEnd}^{\mathcal{P}}_B(r) = \mathit{Hom}_{A \in \mathcal{P}\,\mathrm{Alg}} \operatorname{Hom}(T^c(\Sigma A)^{\otimes r}, \Sigma A)$$

formed by all homogeneous morphisms $\theta_A : T^c(\Sigma A)^{\otimes r} \to \Sigma A$ which are functorial in A.

Then, as in [B1], we introduce a smaller operad $\operatorname{HopfOp}^{\mathcal{P}}_B$, the *Hopf operad of universal bar operations*, that behaves better than the endomorphism operad $\operatorname{HopfEnd}^{\mathcal{P}}_B$. As in the context of dg-modules, this operad is equipped with a split injective morphism $\nabla_\Theta : \operatorname{HopfOp}^{\mathcal{P}}_B \hookrightarrow \operatorname{HopfEnd}^{\mathcal{P}}_B$ which becomes an isomorphism if the operad $\mathcal{P}$ is Σ_*-cofibrant or if the ground field $\mathbb{F}$ is infinite. Explicitly, one observes that the homogeneous morphisms $p_A : \Sigma A^{\otimes n_1} \otimes \cdots \otimes \Sigma A^{\otimes n_r} \to \Sigma A$ associated to operations $p \in \mathcal{P}(n_1 + \cdots + n_r)$ span a submodule $\operatorname{PrimOp}^{\mathcal{P}}_B(r)$ of $\operatorname{PrimEnd}^{\mathcal{P}}_B(r)$. The operad $\operatorname{HopfOp}^{\mathcal{P}}_B$ consists of quasi-cofree subcoalgebras of $\operatorname{HopfEnd}^{\mathcal{P}}_B(r)$ such that

$$\operatorname{HopfOp}^{\mathcal{P}}_B(r) = (\Gamma(\operatorname{PrimOp}^{\mathcal{P}}_B(r)), \partial).$$

One checks also that $\operatorname{PrimOp}^{\mathcal{P}}_B$ forms also a Λ_*-submodule of $\operatorname{PrimEnd}^{\mathcal{P}}_B$ so that $\operatorname{HopfOp}^{\mathcal{P}}_B$ forms a quasi-cofree Hopf Λ_*-module.

The endomorphism operad $\operatorname{HopfEnd}^{\mathcal{P}}_B$, as well as $\operatorname{HopfOp}^{\mathcal{P}}_B$, is composed of $\mathbb{Z}$-graded coalgebras unlike the Hopf operads considered in §1 since a dg-module of homogeneous morphisms is naturally a $\mathbb{Z}$-graded object. Similarly, one can observe that these operads are unital but non-connected. On the other hand, we consider only morphisms $\mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{P}}$ where $\mathcal{Q}$ is a non-negatively graded unital Hopf operad. One can check that the coalgebra truncation functor $\mathrm{s}_+^{\mathrm{dg}}$ defined in §3.2 induces a truncation functor on operad categories so that a universal $\mathbb{N}$-graded unital Hopf operad is associated to any $\mathbb{Z}$-graded unital Hopf operad. Accordingly, any morphism as above admits a factorization

$$\begin{tikzcd} \mathcal{Q} \arrow[rr] \arrow[dr, dashed] & & \mathrm{HopfOp}_B^{\mathcal{P}} \\ & \mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{P}}) \arrow[ur] & \end{tikzcd}$$

and in the applications one can replace $\mathrm{HopfOp}_B^{\mathcal{P}}$ by the non-negatively graded operad $\mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{P}})$. If $\mathcal{Q}$ is connected, then we can consider a further factorization by a morphism $\mathcal{Q} \to \mathrm{s}_*^1\, \mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{P}})$ where $\mathrm{s}_*^1\, \mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{P}})$ denotes the connected unital Hopf operad associated to $\mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{P}})$ by the construction of §1.5.8. For the sake of completness, recall that this functor s_*^1 preserves fibrations, acyclic fibrations and all weak-equivalences between fibrant objects.

As in the previous section, we let a morphism of $\mathbb{Z}$-graded unital Hopf operads $\phi : \mathcal{P} \to \mathcal{Q}$ be a Reedy fibration, respectively an acyclic Reedy fibration, if the associated morphism $\mathrm{s}_+^{\mathrm{dg}}(\phi) : \mathrm{s}_+^{\mathrm{dg}}(\mathcal{P}) \to \mathrm{s}_+^{\mathrm{dg}}(\mathcal{Q})$ is so in the model category of non-negatively graded unital Hopf operads. Equivalently, the morphism $\phi : \mathcal{P} \to \mathcal{Q}$ is a Reedy fibration, respectively an acyclic Reedy fibration, if it has the left lifting property with respect to acyclic Reedy cofibrations, respectively Reedy cofibrations, of non-negatively graded unital Hopf operads.

We prove that the Hopf operad of bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$ is endowed with the following property:

Theorem §4.B. *The functor $\mathcal{P} \mapsto \mathrm{HopfOp}_B^{\mathcal{P}}$ maps a fibration, respectively an acyclic fibration, of non-unital operads under $\mathcal{K}$ to a Reedy fibration, respectively an acyclic Reedy fibration, of unital Hopf operads. In particular, the Hopf operad $\mathrm{HopfOp}_B^{\mathcal{P}}$ defines a fibrant object in the category of Hopf operads, for any operad $\mathcal{P}$ under $\mathcal{K}$.*

One can observe that the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ preserve fibrations like $\mathrm{HopfOp}_B^{\mathcal{P}}$ but not acyclic fibrations.

Recall that the commutative operad $\mathcal{C}$ forms the final object of the category of unital Hopf operads. In the introduction of this memoir we mention that the classical shuffle product of tensors corresponds to a morphism $\nabla_c : \mathcal{C} \to \mathrm{HopfEnd}_B^{\mathcal{C}}$. In the final part of this section we check that this morphism admits a factorization

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\nabla_c"] \arrow[dr, dashed, "\nabla_\gamma"'] & \mathrm{HopfEnd}_B^{\mathcal{C}} \\ & \mathrm{HopfOp}_B^{\mathcal{C}} \arrow[u, hook, "\nabla_\Theta"'] \end{tikzcd} .$$

In addition we prove the following result:

Theorem §4.C. *Any morphism of unital Hopf operads $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{C}}$, where $\mathcal{Q}$ is connected and non-negatively graded, makes commute the diagram*

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "\nabla_\rho"] \arrow[dr, "\epsilon"'] & & \mathrm{HopfOp}_B^{\mathcal{C}} \\ & \mathcal{C} \arrow[ur, "\nabla_\gamma"'] & \end{tikzcd} .$$

Accordingly, the morphism ∇_γ induces an isomorphism

$$\nabla_\gamma : \mathcal{C} \xrightarrow{\simeq} \mathrm{s}^1_* \, \mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}_B^{\mathcal{C}})$$

that identifies $\mathrm{s}^1_* \, \mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}_B^{\mathcal{C}})$ with the commutative operad.

Finally, our main existence and uniqueness result, theorem A of the introduction, arises as a formal consequence of theorems §4.B and §4.C. We recall our statement for the sake of completeness. In fact, the claim of theorem A is valid for any unital Hopf operad (not necessarily E_∞):

Theorem §4.D. *Let $\mathcal{E}$ be a Σ_*-cofibrant non-unital E_∞-operad. Let $\mathcal{Q}$ be a unital Hopf operad.*

(a) *The bar complex of an $\mathcal{E}$-algebra $B(A)$ can be equipped with the structure of a Hopf $\mathcal{Q}$-algebra, functorially in A, and so that the unital operation $\mathcal{Q}(0) \to B(A)$ agrees with the natural unit of the bar complex $\mathbb{F} \to B(A)$ provided that $\mathcal{Q}$ is a Reedy cofibrant object in the category of unital Hopf operads.*

(b) *Any such $\mathcal{Q}$-algebra structure where $\mathcal{Q}$ is connected and non-negatively graded satisfies the requirement of the uniqueness theorem of [B1]. More explicitly, if the unit condition of claim (a) is satisfied and the operad $\mathcal{Q}$ is connected and non-negatively graded, then, for a commutative algebra A, the $\mathcal{Q}$-algebra structure of $B(A)$ reduces automatically to the classical commutative algebra structure of $B(A)$, the one given by the shuffle product of tensors.*

We give the proof of these claims assuming theorems §4.B and §4.C. As explained in the introduction of this memoir, for an E_∞-operad $\mathcal{E}$, we consider the lifting problem

$$\begin{tikzcd} & & \mathrm{HopfOp}_B^{\mathcal{E}} \arrow[r] \arrow[d, "\sim"] & \mathrm{HopfEnd}_B^{\mathcal{E}} \arrow[d] \\ \mathcal{Q} \arrow[r] \arrow[urr, dashed, "\exists\nabla_\rho"] & \mathcal{C} \arrow[r, "\nabla_\gamma"] & \mathrm{HopfOp}_B^{\mathcal{C}} \arrow[r] & \mathrm{HopfEnd}_B^{\mathcal{C}} \end{tikzcd}$$

which has automatically a solution if $\mathcal{Q}$ is cofibrant since, by theorem §4.B, the augmentation of an E_∞-operad induces an acyclic fibration $\epsilon_* : \mathrm{HopfOp}_B^{\mathcal{E}} \xrightarrow{\sim} \mathrm{HopfOp}_B^{\mathcal{C}}$. Then the composite morphism

$$\mathcal{Q} \xrightarrow{\nabla_\rho} \mathrm{HopfOp}_B^{\mathcal{E}} \xrightarrow{\nabla_\Theta} \mathrm{HopfEnd}_B^{\mathcal{E}}$$

provides an operad action on the bar complex that fulfils the existence assertions of theorem §4.D.

Conversely, by the universal definition of the Hopf operad $\mathrm{HopfEnd}_B^{\mathcal{E}}$, any operad action on $B(A)$ that satisfies our unit requirement is determined by an operad morphism $\nabla : \mathcal{Q} \to \mathrm{HopfEnd}_B^{\mathcal{E}}$. If $\mathcal{E}$ is Σ_*-cofibrant, then any such morphism factors through $\mathrm{HopfOp}_B^{\mathcal{E}}$ since the embedding $\nabla_\Theta : \mathrm{HopfOp}_B^{\mathcal{E}} \hookrightarrow \mathrm{HopfEnd}_B^{\mathcal{E}}$ is an isomorphism. Then we deduce from theorem §4.C that the diagram

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "\nabla"] \arrow[d] & \mathrm{HopfOp}_B^{\mathcal{E}} \arrow[d] \\ \mathcal{C} \arrow[r, "\nabla_\gamma"] & \mathrm{HopfOp}_B^{\mathcal{C}} \end{tikzcd}$$

commutes automatically. For a commutative algebra A, it follows that the action of $\mathcal{Q}$ on $B(A)$ reduces to the classical commutative operad action as claimed by theorem §4.D. □

The lifting process can be simplified if we assume that $\mathcal{Q}$ is a connected operad. Namely, if we apply the truncation functors, then, by theorem §4.C, our lifting problem becomes equivalent to

$$\begin{tikzcd} & \mathrm{s}_*^1\,\mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{E}}) \arrow[d, "\sim"] \\ \mathcal{Q} \arrow[ur, dashed, "\exists\nabla_\rho"] \arrow[r] & \mathcal{C} \end{tikzcd}.$$

Recall also that the truncation functors preserve acyclic fibrations. Therefore we obtain that the augmentation of an E_∞-operad induces an acyclic fibration $\epsilon_* : \mathrm{s}_*^1\,\mathrm{s}_+^{\mathrm{dg}}(\mathrm{HopfOp}_B^{\mathcal{E}}) \xrightarrow{\sim} \mathcal{C}$ since this morphism represents the truncation of the morphism $\epsilon_* : \mathrm{HopfOp}_B^{\mathcal{E}} \xrightarrow{\sim} \mathrm{HopfOp}_B^{\mathcal{C}}$ which forms an acyclic fibration by theorem §4.B. Finally, we obtain that any morphism $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$ where $\mathcal{Q}$ is a non-negatively graded connected unital Hopf operad fits this lifting diagram since $\mathcal{C}$ forms the terminal object in the category of unital Hopf operads. As a corollary, by usual model category arguments, we obtain:

Theorem §4.E. *Let $\mathcal{E}$ be a non-unital E_∞-operad. Any pair of morphisms of unital Hopf operads $\nabla_0, \nabla_1 : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$ where $\mathcal{Q}$ is a connected and non-negatively graded unital Hopf operads are left-homotopic.*

As in [B1], one could give an interpretation of this uniqueness assertion at the algebra level. Namely suppose given a pair of morphisms of unital Hopf operads

$$\nabla_0, \nabla_1 : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$$

which provide the chain complex $B(A)$ with the structure of a Hopf $\mathcal{Q}$-algebra. Then the Hopf $\mathcal{Q}$-algebras $(B(A), \nabla_0)$ and $(B(A), \nabla_1)$ can be connected by weak-equivalences of Hopf $\mathcal{Q}$-algebras

$$(B(A), \nabla_0) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B(A), \nabla_1).$$

To be precise, we have not checked this claim in full generality. But if we assume that A is a non-negatively graded dg-algebras, then the bar complex $B(A)$ belongs to the category of non-negatively graded dg-coalgebra for which we have a model structure by [13]. In this context the claim can be deduced from the results of [23] extended to the ground model category of dg-coalgebras.

Section outline. Here is the plan of this section: in §4.2 we prove the existence of an internal hom in the category of coalgebras; in subsections §4.3-§4.4 we define the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$, the related Hopf operad of bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$ and we make explicit the internal structure of these operads; we prove the fibration properties asserted by theorem §4.B in §4.5 and we give the proof of theorem §4.C in §4.6.

§4.2. **Morphism coalgebras.** The purpose of this subsection is to construct *morphism coalgebras* $\mathit{HopfHom}(L, M)$ that satisfy the adjunction property specified in the section introduction. Before we shall give precisions on the coalgebra categories that occur in the definition of $\mathit{HopfHom}(L, M)$.

§4.2.1. *Augmented and unitary coalgebras.* In general we work within the category of *augmented coassociative coalgebras* denoted by CoAlg^a_+. But, as explained above, we consider also coalgebras in the category of *non-augmented coalgebras* CoAlg^a, in the category of *augmented unitary coalgebras* CoAlg^a_* and in the category of *connected coalgebras* CoAlg^a_0.

To be precise, an object $K \in \mathrm{CoAlg}^a_+$ denotes a coassociative coalgebra equipped with an augmentation defined by a morphism of coalgebras $\epsilon : K \to \mathbb{F}$; an object $L \in \mathrm{CoAlg}^a_*$ denotes a coassociative coalgebra equipped with an augmentation and a coalgebra unit defined by a morphism of augmented coalgebra $\eta : \mathbb{F} \to L$. The unit cokernel of a unitary coalgebra $\overline{L} = \operatorname{coker}(\eta : \mathbb{F} \to L)$ defines an object of CoAlg^a. Clearly, the map $L \mapsto \overline{L}$ defines an equivalence between the category of augmented unitary coalgebras CoAlg^a_* and the category of non-augmented coalgebras CoAlg^a since any augmented unitary coalgebra L has a natural decomposition $L = \mathbb{F} \oplus \overline{L}$.

§4.2.2. *Cofree unitary coalgebras.* The cofree coalgebra $\Gamma(V)$ defined in the previous subsection is characterized by the adjunction relation

$$\mathrm{Hom}_{\mathrm{dg\,Mod}}(K, V) = \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \Gamma(V)),$$

for any augmented coalgebra $K \in \mathrm{CoAlg}^a_+$. Notice that $\Gamma(V)$ is equipped with a canonical unit $\eta : \mathbb{F} \to \Gamma(V)$ induced by the null morphism $0 : 0 \to V$. The unit cokernel of $\Gamma(V)$ is denoted by $\overline{\Gamma}(V)$. One observes readily that the following adjunction relations hold

$$\mathrm{Hom}_{\mathrm{dg\,Mod}}(\overline{L}, V) = \mathrm{Hom}_{\mathrm{CoAlg}^a}(\overline{L}, \overline{\Gamma}(V)) = \mathrm{Hom}_{\mathrm{CoAlg}^a_*}(L, \Gamma(V)),$$

for any augmented unitary coalgebra $L \in \mathrm{CoAlg}^a_*$. Accordingly, the cofree coalgebra $\Gamma(V)$ forms also a cofree object in the category of augmented unitary coalgebras and its unit cokernel $\overline{\Gamma}(V)$ forms a cofree object in CoAlg^a. Clearly, a morphism of cofree coalgebras $\nabla_f : \Gamma(V) \to \Gamma(W)$ preserves units if and only if the associated map $f : \Gamma(V) \to W$ cancels the unit of $\Gamma(V)$ and hence is equivalent to a map $f : \overline{\Gamma}(V) \to W$.

We consider also quasi-cofree objects $\Gamma = (\Gamma(V), \partial_\alpha)$ in the category of augmented unitary coalgebras CoAlg^a_*. We assume in this case that the coderivation ∂_α cancels the unit of $\Gamma(V)$ so that the unit morphism of the cofree coalgebra $\eta : \mathbb{F} \to \Gamma(V)$ defines a morphism of dg-coalgebras $\eta : \mathbb{F} \to \Gamma$. As in the context of coalgebra morphisms, it is equivalent to assume that the coderivation ∂_α is induced by a map $\alpha : \overline{\Gamma}(V) \to V$.

Recall that a tensor product of coassociative coalgebras $K \otimes L$ is equipped with a natural coalgebra structure so that the coalgebra categories considered in this memoir are symmetric monoidal. In the language of monoidal categories, our internal morphism coalgebra $\mathit{HopfHom}(L, M)$ can be characterized by a closure property. In fact, we define $\mathit{HopfHom}(L, M)$ by the following assertion:

§4.2.3. **Proposition.** *We have a bifunctor*

$$\mathit{HopfHom} : \mathrm{CoAlg}^{a\,\mathrm{op}}_* \times \mathrm{CoAlg}^a_* \to \mathrm{CoAlg}^a_+$$

that satisfies the adjunction relation

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{M}) = \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \mathit{HopfHom}(L, M)),$$

for all augmented coalgebras $K \in \mathrm{CoAlg}^a_+$ and all augmented unitary coalgebras $L, M \in \mathrm{CoAlg}^a_*$.

Recall that the forgetful functor from a coalgebra category to the category of dg-modules creates colimits. This assertion implies immediately that the functor $K \mapsto K \otimes \overline{L}$ preserves colimits. On the other hand, we mention in §3.2 that the category of coalgebras is equipped with a set of generating objects. Consequently, the proposition can be deduced from the special adjoint functor theorem.

Nevertheless we prefer to give another proof of proposition §4.2.3 so that we can obtain an explicit construction of $\mathit{HopfHom}(L, M)$ in the case of a quasi-cofree coalgebra $M = (\Gamma(V), \partial)$. Before observe that the adjunction relation of the proposition is equivalent to the adjunction property specified in the section introduction. More explicitly, we have the following assertion:

§4.2.4. **Fact.** *Let $K \in \mathrm{CoAlg}_+^a$ denote an augmented coalgebra. Let $L, M \in \mathrm{CoAlg}_*^a$ be augmented unitary coalgebras as in proposition §4.2.3. We have a one-to-one correspondance bewteen morphisms of augmented coalgebras $\phi : K \otimes L \to M$ that make the diagram*

$$\begin{tikzcd} K \otimes \mathbb{F} \arrow[r, "\epsilon"] \arrow[d, "K\otimes\eta"'] & \mathbb{F} \arrow[d, "\eta"] \\ K \otimes L \arrow[r, "\phi"] & M \end{tikzcd}$$

commute and morphisms of non-augmented coalgebras $\overline{\phi} : K \otimes \overline{L} \to \overline{M}$ such that $\overline{\phi}$ is defined by the restriction of $\phi : K \otimes L \to M$ to the summand $K \otimes \overline{L}$ of the tensor product $K \otimes L$.

Hence a morphism of augmented coalgebras $\phi^\sharp : K \to \mathit{HopfHom}(L, M)$ is equivalent to a morphism of augmented coalgebras $\phi : K \otimes L \to M$ that makes the diagram above commute.

As explained previously, we aim to prove the existence of morphism coalgebras $\mathit{HopfHom}(L, M)$ by an effective construction. In fact, our arguments rely on a classical proof of the existence of adjoint functors in a category of coalgebras over a comonad.

To begin with, we have the following immediate observation:

§4.2.5. **Observation.** *For a cofree coalgebra $M = \Gamma(V)$, the required morphism coalgebra is given by the cofree coalgebra*

$$\mathit{HopfHom}(L, \Gamma(V)) = \Gamma(\mathit{Hom}(\overline{L}, V))$$

since we have adjunction relations

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{\Gamma}(V)) \\ &\qquad = \mathrm{Hom}_{\mathrm{dg\,Mod}}(K \otimes \overline{L}, V) = \mathrm{Hom}_{\mathrm{dg\,Mod}}(K, \mathit{Hom}(\overline{L}, V)) \\ &\qquad\qquad = \mathrm{Hom}_{\mathrm{CoAlg}_+^a}(K, \Gamma(\mathit{Hom}(\overline{L}, V))). \quad \square \end{aligned}$$

These adjunction relations are functorial in $K \in \mathrm{CoAlg}_+^a$ and $L \in \mathrm{CoAlg}_*^a$ but the middle terms are not functors in $M = \Gamma(V) \in \mathrm{CoAlg}_*^a$. Therefore we prove directly that $\mathit{HopfHom}(L, \Gamma(V)) = \Gamma(\mathit{Hom}(\overline{L}, V))$ defines a functor on the full subcategory of CoAlg_*^a generated by cofree coalgebras $M = \Gamma(V)$. For this purpose we consider the augmentation morphism of the adjunction of observation §4.2.5:

$$\mathrm{ev}_\Gamma : \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \to \Gamma(V).$$

An explicit definition of this morphism can be obtained by going through our adjunction relations. We obtain precisely:

§4.2.6. **Observation.** *In the adjunction of observation §4.2.5*

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{\Gamma}(V)) = \mathrm{Hom}_{\mathrm{CoAlg}_+^a}(K, \Gamma(\mathit{Hom}(\overline{L}, V))),$$

the augmentation morphism can be identified with the coalgebra morphism

$$\mathrm{ev}_\Gamma : \Gamma(Hom(\overline{L}, V)) \otimes \overline{L} \to \overline{\Gamma}(V)$$

induced by the composite morphism of dg-modules

$$\Gamma(Hom(\overline{L}, V)) \otimes \overline{L} \to Hom(\overline{L}, V) \otimes \overline{L} \xrightarrow{\mathrm{ev}} V,$$

where we consider the universal morphism of the cofree coalgebra

$$\Gamma(Hom(\overline{L}, V)) \to Hom(\overline{L}, V),$$

and the augmentation morphism

$$\mathrm{ev} : Hom(\overline{L}, V) \otimes \overline{L} \to V$$

of the adjunction relation

$$\mathrm{Hom}_{\mathrm{dg\,Mod}}(K \otimes \overline{L}, V) = \mathrm{Hom}_{\mathrm{dg\,Mod}}(K, Hom(\overline{L}, V))$$

in the category of dg-modules. □

Then let

$$\mathrm{ev}_\Gamma^\sharp : \Gamma(Hom(\overline{L}, V)) \to Hom(\overline{L}, \overline{\Gamma}(V)).$$

denote the adjoint morphism of $\mathrm{ev}_\Gamma : \Gamma(Hom(\overline{L}, V)) \otimes \overline{L} \to \overline{\Gamma}(V)$ in the category of dg-modules. We obtain:

§4.2.7. **Claim.** *We suppose given a morphism of cofree coalgebras $\nabla_f : \Gamma(V) \to \Gamma(W)$ induced by a morphism of dg-modules $f : \overline{\Gamma}(V) \to W$ so that ∇_f preserves the unit of the cofree coalgebra. We consider the morphism of cofree coalgebras*

$$\nabla_{f_* \mathrm{ev}_\Gamma^\sharp} : \Gamma(Hom(\overline{L}, V)) \to \Gamma(Hom(\overline{L}, W)).$$

induced by the composite

$$\Gamma(Hom(\overline{L}, V)) \xrightarrow{\mathrm{ev}_\Gamma^\sharp} Hom(\overline{L}, \overline{\Gamma}(V)) \xrightarrow{f_*} Hom(\overline{L}, W).$$

We claim that this morphism fits a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{\Gamma}(V)) \arrow[r, "="] \arrow[d, "(\nabla_f)_*"] & \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \Gamma(Hom(\overline{L}, V))) \arrow[d, "(\nabla_{f_* \mathrm{ev}_\Gamma^\sharp})_*"] \\
\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{\Gamma}(W)) \arrow[r, "="] & \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \Gamma(Hom(\overline{L}, W)))
\end{tikzcd}$$

and hence makes the adjunction relation of observation §4.2.5 functorial with respect to the coalgebra morphism $\nabla_f : \Gamma(V) \to \Gamma(W)$.

Proof. One can check that $\nabla_{f_* \mathrm{ev}_\Gamma^\sharp}$ makes the diagram

$$\begin{tikzcd}
\Gamma(Hom(\overline{L}, V)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] \arrow[d, "\nabla_{f_* \mathrm{ev}_\Gamma^\sharp} \otimes \overline{L}"'] & \overline{\Gamma}(V) \arrow[d, "\nabla_f"] \\
\Gamma(Hom(\overline{L}, W)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] & \overline{\Gamma}(W)
\end{tikzcd}$$

commute. For this purpose it suffices to compare the composite

$$\Gamma(Hom(\overline{L}, V)) \otimes \overline{L} \xrightarrow{\nabla_{f_* \mathrm{ev}_\Gamma^\sharp} \otimes \overline{L}} \Gamma(Hom(\overline{L}, W)) \otimes \overline{L} \xrightarrow{\mathrm{ev}_\Gamma} \overline{\Gamma}(W) \xrightarrow{\pi} W$$

with $\pi \cdot \nabla_f \cdot \mathrm{ev}_\Gamma = f \cdot \mathrm{ev}_\Gamma$. The identity of these morphisms follows from formal verifications involving essentially adjunctions in the category of dg-modules.

According to observation §4.2.5, the adjoint of a coalgebra morphism $\nabla_u : K \to \Gamma(\mathit{Hom}(\overline{L}, V))$ is obtained by the composite

$$K \otimes \overline{L} \xrightarrow{\nabla_u \otimes \overline{L}} \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \xrightarrow{\mathrm{ev}_\Gamma} \overline{\Gamma}(V).$$

Therefore our adjunction claim follows from the commutativity of the diagram

$$\begin{tikzcd} K \otimes \overline{L} \arrow[r, "\nabla_u \otimes \overline{L}"] & \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] \arrow[d, "\nabla_{f_* \mathrm{ev}_\Gamma^\sharp}"'] & \overline{\Gamma}(V) \arrow[d, "\nabla_f"'] \\ & \mathit{Hom}(\overline{L}, W)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] & \overline{\Gamma}(W) \end{tikzcd}$$

which is proved above. □

Finally, claim §4.2.7 gives the following result:

§4.2.8. **Lemma.** *The map $V \mapsto \Gamma(\mathit{Hom}(\overline{L}, V))$ extends to a functor on the full subcategory of CoAlg^a_* formed by cofree coalgebras $\Gamma(V) \in \mathrm{CoAlg}^a_*$ so that the adjunction relation*

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{\Gamma}(V)) = \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \Gamma(\mathit{Hom}(\overline{L}, V)))$$

is functorial in K, L and $\Gamma(V) \in \mathrm{CoAlg}^a_*$.

The crux of our construction is supplied by this statement. In fact, according to general categorical constructions, any coalgebra M is the equalizer of a natural pair of cofree coalgebra morphisms

$$\begin{tikzcd} M \arrow[r] & \Gamma(V^0) \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Gamma(V^1) \end{tikzcd} .$$

As a consequence:

§4.2.9. **Fact.** *The coalgebra $\mathit{HopfHom}(L, M)$ can be defined by the equalizer diagram*

$$\begin{tikzcd} \mathit{HopfHom}(L, M) \arrow[r] & \Gamma(\mathit{Hom}(\overline{L}, V^0)) \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \Gamma(\mathit{Hom}(\overline{L}, V^1)) \end{tikzcd} ,$$

where the morphisms d^0, d^1 are deduced from claim §4.2.7. The adjunction relation

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, \overline{M}) = \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, \mathit{HopfHom}(L, M))$$

follows then from the case of cofree coalgebras stated in lemma §4.2.8 by an immediate and classical exactness argument.

This assertion achieves the proof of proposition §4.2.3. □

As usual for internal hom-objects, we have a composition product

$$\mathit{HopfHom}(M, N) \otimes \mathit{HopfHom}(L, M) \xrightarrow{\circ} \mathit{HopfHom}(L, N)$$

equivalent to the composite evaluation morphism

$$\mathit{HopfHom}(M, N) \otimes \mathit{HopfHom}(L, M) \otimes L \xrightarrow{\mathrm{Id} \otimes \mathrm{ev}} \mathit{HopfHom}(M, N) \otimes M \xrightarrow{\mathrm{ev}} N.$$

For cofree coalgebras $M = \Gamma(V)$ and $N = \Gamma(W)$, we obtain readily:

§4.2.10. **Observation.** *For cofree coalgebras $M = \Gamma(V)$ and $N = \Gamma(W)$, the composition product*

$$\mathit{HopfHom}(\Gamma(V), \Gamma(W)) \otimes \mathit{HopfHom}(L, \Gamma(V)) \xrightarrow{\circ} \mathit{HopfHom}(L, \Gamma(W))$$

can be identified with the coalgebra morphism

$$\Gamma(\mathit{Hom}(\overline{\Gamma}(V), W)) \otimes \Gamma(\mathit{Hom}(\overline{L}, V)) \xrightarrow{\circ_\Gamma} \Gamma(\mathit{Hom}(\overline{L}, W))$$

induced by the composite

$$\Gamma(Hom(\overline{\Gamma}(V),W))\otimes\Gamma(Hom(\overline{L},V)) \xrightarrow{\pi_*\otimes\mathrm{ev}_\Gamma^\sharp} Hom(\overline{\Gamma}(V),W)\otimes Hom(\overline{L},\overline{\Gamma}(V)) \xrightarrow{\circ} Hom(\overline{L},W)$$

where we consider the universal projection

$$\Gamma(Hom(\overline{\Gamma}(V),W)) \xrightarrow{\pi} Hom(\overline{\Gamma}(V),W),$$

the morphism

$$\Gamma(Hom(\overline{L},V)) \xrightarrow{\mathrm{ev}_\Gamma^\sharp} Hom(\overline{L},\overline{\Gamma}(V))$$

and the composition product of homogeneous maps of dg-modules

$$Hom(\overline{\Gamma}(V),W)\otimes Hom(\overline{L},\overline{\Gamma}(V)) \xrightarrow{\circ} Hom(\overline{L},W). \quad \square$$

We observe that a morphism of augmented unitary coalgebras $\phi: L\to M$ is equivalent to a group-like element $\phi\in HopfHom(L,M)$. Formally, the module $\mathbb{F}[X]$ spanned by a set X is usually equipped with the structure of an augmented coalgebra in which the basis elements $x\in X$ are group-like. This coalgebra satisfies the adjunction relation $\mathrm{Hom}_{\mathrm{CoAlg}_+^a}(\mathbb{F}[X],\Gamma)=\mathrm{Hom}_{\mathrm{Set}}(X,\mathrm{Gr}(\Gamma))$, where $\mathrm{Gr}(\Gamma)$ denotes the set of group-like elements in an augmented coalgebra $\Gamma\in\mathrm{CoAlg}_+^a$. One checks readily that the evaluation of morphisms yields a coalgebra morphism

$$\mathbb{F}[\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,M)]\otimes L\to M$$

that satisfies the requirement of fact §4.2.4. Accordingly, this morphism is equivalent to a morphism of augmented coalgebras

$$\mathbb{F}[\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,M)]\to HopfHom(L,M).$$

We obtain:

§4.2.11. **Claim.** *The coalgebra morphism above yields a canonical bijection*

$$\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,M)\xrightarrow{\simeq}\mathrm{Gr}(HopfHom(L,M)).$$

Proof. This bijection property is stated is a remark. Therefore we just sketch the proof of this statement. By left exactness, it is sufficient to check the assertion for a cofree coalgebra $M=\Gamma(V)$ for which we have $HopfHom(L,\Gamma(V))=\Gamma(Hom(\overline{L},V))$. In this case, by adjunction, we have natural bijections

$$\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,\Gamma(V))\xrightarrow{\simeq}\mathrm{Hom}_{\mathrm{dg\,Mod}}(L,\Gamma(V))\xrightarrow{\simeq}\mathrm{Gr}(\Gamma(Hom(\overline{L},V)))$$

whose composite can be identified with the map of the claim. $\square$

The last assertion of the proof can be deduced from the following observation:

§4.2.12. **Observation.** *For a cofree coalgebra $M=\Gamma(V)$, the map*

$$\mathbb{F}[\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,\Gamma(V))]\to HopfHom(L,\Gamma(V)).$$

can be identified with the coalgebra morphism

$$\mathbb{F}[\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,\Gamma(V))]\to\Gamma(Hom(\overline{L},V))$$

induced by the canonical morphism of dg-modules

$$\mathbb{F}[\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(L,\Gamma(V))] \xrightarrow{\simeq}\mathbb{F}[\mathrm{Hom}_{\mathrm{dg\,Mod}}(\overline{L},V)]\to\mathrm{Hom}_{\mathrm{dg\,Mod}}(\overline{L},V) \hookrightarrow Hom(\overline{L},V). \quad \square$$

This assertion arises as a straightforward consequence of our constructions. To conclude this set of observations, we can form an enriched category of augmented unitary coalgebras in which morphism objects are given by the morphism coalgebras $\mathit{HopfHom}(L, M)$. The bijection $\mathrm{Hom}_{\mathrm{CoAlg}_+^a}(L, M) \xrightarrow{\simeq} \mathrm{Gr}(\mathit{HopfHom}(L, M))$ yields an embedding from the ground category to the enriched category of augmented unitary coalgebras. The morphism coalgebra $\mathit{HopfHom}(L, M)$ extends clearly to a bifunctor on this enriched category. Observe also that for a group-like element $\nabla_f \in \mathit{HopfHom}(\Gamma(V), \Gamma(W)$, equivalent to a morphism $\nabla_f : \Gamma(V) \to \Gamma(W)$, the composition process of observation §4.2.10 extends the definition of claim §4.2.7.

We claim that our construction permits to obtain an explicit realization of the morphism coalgebra $\mathit{HopfHom}(L, M)$ for a quasi-cofree coalgebra $M = (\Gamma(V), \partial_\alpha)$. We obtain precisely the following result:

§4.2.13. **Lemma.** *If M is a quasi-cofree coalgebra, then so is the morphism coalgebra $\mathit{HopfHom}(L, M)$. To be more explicit, suppose given a quasi-cofree unitary coalgebra $M = (\Gamma(V), \partial_\alpha)$, where the coderivation $\partial_\alpha : \Gamma(V) \to \Gamma(V)$ is induced by a homogeneous map $\alpha : \overline{\Gamma}(V) \to V$. Then we have:*

$$\mathit{HopfHom}(L, M) = (\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}),$$

where the coderivation $\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp} : \Gamma(\mathit{Hom}(\overline{L}, V)) \to \Gamma(\mathit{Hom}(\overline{L}, V))$ is induced by the composite map

$$\Gamma(\mathit{Hom}(\overline{L}, V)) \xrightarrow{\mathrm{ev}_\Gamma^\sharp} \mathit{Hom}(\overline{L}, \overline{\Gamma}(V)) \xrightarrow{\alpha_*} \mathit{Hom}(\overline{L}, V).$$

Furthermore, the adjunction augmentation

$$\mathrm{ev} : \mathit{HopfHom}(L, M) \otimes \overline{L} \to \overline{M}$$

can be identified with a coalgebra morphism

$$\mathrm{ev}_\Gamma : (\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}) \otimes \overline{L} \to (\overline{\Gamma}(V), \partial_\alpha)$$

supplied by observation §4.2.6.

Roughly, we check that the equations of lemma §3.3.1 involved in quasi-cofree coalgebra structures hold for the pair $(\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp})$ and that the adjunction relation of observation §4.2.5 extends to quasi-cofree coalgebras. Basically, our results are consequences of the following commutation relation between the coderivations and the adjunction augmentation:

§4.2.14. **Claim.** *The coderivation $\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}$ makes the diagram*

$$\begin{tikzcd}
\Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] \arrow[d, "\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp} \otimes \overline{L}"'] & \overline{\Gamma}(V) \arrow[d, "\partial_\alpha"] \\
\Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \arrow[r, "\mathrm{ev}_\Gamma"] & \overline{\Gamma}(V)
\end{tikzcd}$$

commute.

Proof. In the diagram both composites define coderivations $\partial : L' \to \overline{\Gamma}(V)$, where $L' = \Gamma(\mathit{Hom}(\overline{L}, V))$. One can extend the correspondence of lemma §3.3.1 to this relative context. Hence these coderivations agree if and only if their composite with the universal morphism $\pi : \Gamma(V) \to V$ agree. Thus, as in the proof of claim §4.2.7, it suffices to compare the composite

$$\Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \xrightarrow{\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp} \otimes \overline{L}} \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \xrightarrow{\mathrm{ev}_\Gamma} \overline{\Gamma}(V) \xrightarrow{\pi} V$$

with $\pi \cdot \alpha \cdot \mathrm{ev}_\Gamma = \alpha \cdot \mathrm{ev}_\Gamma$. The identity of these morphisms follows also from formal verifications. □

As a corollary, we obtain:

§4.2.15. **Claim.** *The relation $\delta(\alpha) + \alpha \partial_\alpha = 0$ implies the same relation $\delta(\alpha_* \mathrm{ev}_\Gamma^\sharp) + \alpha_* \mathrm{ev}_\Gamma^\sharp \cdot \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp} = 0$ for the coderivation $\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}$. Hence the pair*

$$(\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp})$$

defines a quasi-cofree coalgebra.

Proof. By adjunction, we deduce from claim §4.2.14 that the diagram

$$\begin{tikzcd}
\Gamma(\mathit{Hom}(\overline{L}, V)) \arrow[r, "\mathrm{ev}_\Gamma^\sharp"] \arrow[d, "\partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}"'] & \mathit{Hom}(\overline{L}, \Gamma(V)) \arrow[d, "(\partial_\alpha)_*"] \\
\Gamma(\mathit{Hom}(\overline{L}, V)) \arrow[r, "\mathrm{ev}_\Gamma^\sharp"] & \mathit{Hom}(\overline{L}, \Gamma(V)) \arrow[d, "\alpha_*"] \\
 & \mathit{Hom}(\overline{L}, V)
\end{tikzcd}$$

commute. Hence the relation $\delta(\alpha) + \alpha \partial_\alpha = 0$ implies $\delta(\alpha)_* \mathrm{ev}_\Gamma^\sharp + \alpha \partial_\alpha \mathrm{ev}_\Gamma^\sharp = \delta(\alpha_* \mathrm{ev}_\Gamma^\sharp) + \alpha_* \mathrm{ev}_\Gamma^\sharp \cdot \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp} = 0$. □

Then lemma §4.2.13 is a consequence of the following claim:

§4.2.16. **Claim.** *A dg-module map $u : K \otimes \overline{L} \to V$ induces a morphism of dg-coalgebras*

$$\nabla_u : K \otimes \overline{L} \to (\overline{\Gamma}(V), \partial_\alpha)$$

to the quasi-cofree coalgebra $M = (\overline{\Gamma}(V), \partial_\alpha)$ if and only if the adjoint map $u^\sharp : K \to \mathit{Hom}(\overline{L}, V)$ induces a morphism of dg-coalgebras

$$\nabla_{u^\sharp} : K \to (\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp})$$

to the quasi-cofree coalgebra $(\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp})$.

Accordingly, the adjunction relation $\nabla_u \mapsto \nabla_{u^\sharp}$ yields an adjunction relation

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K \otimes \overline{L}, (\overline{\Gamma}(V), \partial_\alpha)) = \mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K, (\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp}))$$

for the quasi-cofree coalgebra $M = (\overline{\Gamma}(V), \partial_\alpha)$.

Proof. The map $u : K \otimes \overline{L} \to V$ is not assumed to commute with internal differentials. Accordingly, the induced coalgebra morphism $\nabla_u : K \otimes \overline{L} \to \overline{\Gamma}(V)$ forms only a morphism of graded coalgebras. Hence, in the correspondence $\nabla_u \mapsto \nabla_{u^\sharp}$, we assume implicitly that we extend the adjunction relations of observation §4.2.5 to morphisms of graded coalgebras. The morphism ∇_u can also be obtained from $\nabla_{u^\sharp}$ by the composite

$$K \otimes \overline{L} \xrightarrow{\nabla_{u^\sharp} \otimes \overline{L}} \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \xrightarrow{\mathrm{ev}_\Gamma} \overline{\Gamma}(V)$$

as in the case of an actual morphism of dg-coalgebras.

Explicitly, we check that the coalgebra map $\nabla_u : K \otimes \overline{L} \to \overline{\Gamma}(V)$ induced by $u : K \otimes \overline{L} \to V$ satisfies the commutation relation $(\delta + \partial_\alpha)\nabla_u = \nabla_u \delta$ for the differential $\delta + \partial_\alpha$ of the quasi-cofree coalgebra $M = (\overline{\Gamma}(V), \partial_\alpha)$ if and only if the adjoint coalgebra map $\nabla_{u^\sharp} : K \to \Gamma(\mathit{Hom}(\overline{L}, V))$ satisfies the commutation relation $(\delta + \partial_{\alpha_* \mathrm{ev}_\Gamma^\sharp})\nabla_{u^\sharp} = \nabla_{u^\sharp}\delta$. This claim is a formal consequence of the observation above and of the commutation assertion of claim §4.2.14. □

This claim achieves the proof of lemma §4.2.13. □

One can observe further that the morphism

$$(\nabla_f)_* : \mathit{HopfHom}(L, (\Gamma(V), \partial_\alpha)) \to \mathit{HopfHom}(L, (\Gamma(W), \partial_\beta))$$

induced by a morphism of quasi-cofree coalgebras $\nabla_f : (\Gamma(V), \partial_\alpha) \to (\Gamma(W), \partial_\beta)$ is obtained by the same construction as in the case of cofree coalgebras. Namely this morphism can be identified with a morphism of quasi-cofree coalgebras

$$\nabla_{f_* \operatorname{ev}_\Gamma^\sharp} : (\Gamma(\mathit{Hom}(\overline{L}, V)), \partial_{\alpha_* \operatorname{ev}_\Gamma^\sharp}) \to (\Gamma(\mathit{Hom}(\overline{L}, W)), \partial_{\beta_* \operatorname{ev}_\Gamma^\sharp})$$

which is induced by the composite map

$$\Gamma(\mathit{Hom}(\overline{L}, V)) \xrightarrow{\operatorname{ev}_\Gamma^\sharp} \mathit{Hom}(\overline{L}, \overline{\Gamma}(V)) \xrightarrow{f_*} \mathit{Hom}(\overline{L}, W).$$

This observation can be extended to the composition product

$$\mathit{HopfHom}((\Gamma(V), \partial_\alpha), (\Gamma(W), \partial_\beta)) \otimes \mathit{HopfHom}(L, (\Gamma(V), \partial_\alpha)) \xrightarrow{\circ} \mathit{HopfHom}(L, (\Gamma(W), \partial_\beta))$$

which is also given by the construction of observation §4.2.10.

In the next subsections, we consider morphism coalgebras $\mathit{HopfHom}(L, M)$ for connected coalgebras L and M. Therefore, for our needs, we state connected variants of our previous assertions.

§4.2.17. *Connected coalgebras.* First, recall that a unitary coalgebra L is *connected* if the iterated coproduct of the associated non-unitary coalgebra $\Delta^n : \overline{L} \to \overline{L}^{\otimes n}$ vanishes for n sufficiently large. The full subcategory of CoAlg_*^a formed by connected coalgebras is denoted by CoAlg_0^a. This category is equipped with cofree object like CoAlg_*^a. One observes precisely that the cofree object cogenerated by V in CoAlg_0^a is realized by the *tensor coalgebra* $T^c(V)$ defined by the direct sum

$$T^c(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

and equipped with the diagonal $\Delta_{\mathrm{II}} : T^c(V) \to T^c(V) \otimes T^c(V)$ induced by the deconcatenation of tensors. In fact, we have clearly an adjunction relation

$$\mathrm{Hom}_{\mathrm{dg\,Mod}}(\overline{L}, V) = \mathrm{Hom}_{\mathrm{CoAlg}_0^a}(L, T^c(V)),$$

for any connected coalgebra $L \in \mathrm{CoAlg}_0^a$. The unit cokernel of the tensor coalgebra is also denoted by $\overline{T}^c(V)$.

Our results on the morphism coalgebra $\mathit{HopfHom}(L, M)$ associated to connected coalgebras L and M are consequences of the following observation:

§4.2.18. **Observation.** *If L is a connected coalgebra, then the iterated coproduct $\Delta^n : K \otimes \overline{L} \to (K \otimes \overline{L})^{\otimes n}$ vanishes for n large, for any coalgebra K. Accordingly, the tensor product $K \otimes \overline{L}$ forms also a connected (but non-augmented) coalgebra.* □

Accordingly, for a connected coalgebra L, The map $K \mapsto K \otimes \overline{L}$ defines a functor from the category of augmented coalgebras to the category of connected coalgebras. We have clearly:

§4.2.19. **Fact.** *The restriction of the functor $M \mapsto \mathit{HopfHom}(L, M)$ to connected coalgebras M defines a right adjoint of the functor $K \mapsto K \otimes \overline{L}$ considered above.*

On the other hand, one can deduce from observation §4.2.18 that the relations of observation §4.2.5 hold for a connected cofree coalgebra $M = T^c(V)$. Explicitly, we have the following assertion:

§4.2.20. **Observation.** *If L is a connected coalgebra, then we have adjunction relations*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CoAlg}^a}(K\otimes\overline{L},\overline{T}^c(V))&\\ &=\mathrm{Hom}_{\mathrm{dg\,Mod}}(K\otimes\overline{L},V)=\mathrm{Hom}_{\mathrm{dg\,Mod}}(K,\mathit{Hom}(\overline{L},V))\\ &\qquad=\mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K,\Gamma(\mathit{Hom}(\overline{L},V)))\end{aligned}$$

Accordingly, for a connected coalgebra L and a cofree connected coalgebra $M=T^c(V)$, we obtain:

$$\mathit{HopfHom}(L,T^c(V))=\Gamma(\mathit{Hom}(\overline{L},V)).\quad\square$$

In fact, one can adapt the previous constructions in order to obtain explicit realizations of the morphism coalgebras $\mathit{HopfHom}(L,M)$ associated to connected coalgebras as in the non-connected context. First, we have a tractable realization of the adjunction augmentation

$$\mathrm{ev}_T:\Gamma(\mathit{Hom}(\overline{L},V))\otimes\overline{L}\to\overline{T}^c(V).$$

Namely:

§4.2.21. **Observation.** *In the adjunction of observation §4.2.20*

$$\mathrm{Hom}_{\mathrm{CoAlg}^a}(K\otimes\overline{L},\overline{T}^c(V))=\mathrm{Hom}_{\mathrm{CoAlg}^a_+}(K,\Gamma(\mathit{Hom}(\overline{L},V))),$$

the augmentation morphism can be identified with the morphism of connected coalgebras

$$\mathrm{ev}_T:\Gamma(\mathit{Hom}(\overline{L},V))\otimes\overline{L}\to\overline{T}^c(V)$$

induced by the composite morphism of dg-modules

$$\Gamma(\mathit{Hom}(\overline{L},V))\otimes\overline{L}\to\mathit{Hom}(\overline{L},V)\otimes\overline{L}\xrightarrow{\mathrm{ev}}V$$

as in the case of non-connected coalgebras. $\square$

Then, for quasi-cofree connected coalgebras, we obtain:

§4.2.22. **Lemma.** *If L is a connected coalgebra and M is a quasi-cofree connected coalgebra, then the morphism coalgebra $\mathit{HopfHom}(L,M)$ is a quasi-cofree (non-connected) coalgebra. More precisely, if $M=(T^c(V),\partial_\alpha)$, for a coderivation $\partial_\alpha:T^c(V)\to T^c(V)$ induced by a homogeneous map $\alpha:\overline{T}^c(V)\to V$, we have:*

$$\mathit{HopfHom}(L,M)=(\Gamma(\mathit{Hom}(\overline{L},V)),\partial_{\alpha_*\,\mathrm{ev}^\sharp_T}),$$

where the coderivation $\partial_{\alpha_*\,\mathrm{ev}^\sharp_T}:\Gamma(\mathit{Hom}(\overline{L},V))\to\Gamma(\mathit{Hom}(\overline{L},V))$ is induced by the composite map

$$\Gamma(\mathit{Hom}(\overline{L},V))\xrightarrow{\mathrm{ev}^\sharp_T}\mathit{Hom}(\overline{L},\overline{T}^c(V))\xrightarrow{\alpha_*}\mathit{Hom}(\overline{L},V).$$

Furthermore, the adjunction augmentation

$$\mathrm{ev}:\mathit{HopfHom}(L,M)\otimes\overline{L}\to\overline{M}$$

can be identified with the coalgebra morphism

$$\mathrm{ev}_T:(\Gamma(\mathit{Hom}(\overline{L},V)),\partial)\otimes\overline{L}\to(\overline{T}^c(V),\partial)$$

supplied by observation §4.2.21.

Proof. The proof relies on the same verifications as in the case of non-connected quasi-cofree coalgebras stated in lemma §4.2.13. $\square$

For our needs we give the construction of the composition product

$$\mathit{HopfHom}(M,N)\otimes\mathit{HopfHom}(L,M)\xrightarrow{\circ}\mathit{HopfHom}(L,N)$$

in the case of quasi-cofree connected coalgebras $M=(T^c(V),\partial_\alpha)$ and $N=(T^c(W),\partial_\beta)$. By definition, we have simply to determine explicitly the adjoint map of the composite evaluation morphism

$$\mathit{HopfHom}(M,N)\otimes\mathit{HopfHom}(L,M)\otimes L\xrightarrow{\mathrm{Id}\otimes\mathrm{ev}}\mathit{HopfHom}(M,N)\otimes M\xrightarrow{\mathrm{ev}}N.$$

From the assertions of lemma §4.2.22 we obtain:

§4.2.23. **Observation.** *For quasi-cofree connected coalgebras $M=(T^c(V),\partial_\alpha)$ and $N=(T^c(W),\partial_\beta)$, the composition product*

$$\mathit{HopfHom}(M,N)\otimes\mathit{HopfHom}(L,M)\xrightarrow{\circ}\mathit{HopfHom}(L,N)$$

can be identified with the morphism of quasi-cofree coalgebras

$$\big(\Gamma(\mathit{Hom}(\overline{T}^c(V),W)),\partial\big)\otimes\big(\Gamma(\mathit{Hom}(\overline{L},V)),\partial\big)\xrightarrow{\circ_\Gamma}\big(\Gamma(\mathit{Hom}(\overline{L},W)),\partial\big)$$

induced by the composite

$$\Gamma(\mathit{Hom}(\overline{T}^c(V),W))\otimes\Gamma(\mathit{Hom}(\overline{L},V))$$
$$\xrightarrow{\pi\otimes\mathrm{ev}_T^\sharp}\mathit{Hom}(\overline{T}^c(V),W)\otimes\mathit{Hom}(\overline{L},\overline{T}^c(V))\xrightarrow{\circ}\mathit{Hom}(\overline{L},W)$$

where we consider the universal projection

$$\Gamma(\mathit{Hom}(\overline{T}^c(V),W))\xrightarrow{\pi}\mathit{Hom}(\overline{T}^c(V),W),$$

the morphism

$$\Gamma(\mathit{Hom}(\overline{L},V))\xrightarrow{\mathrm{ev}_T^\sharp}\mathit{Hom}(\overline{L},\overline{T}^c(V))$$

and the composition product of homogeneous maps of dg-modules

$$\mathit{Hom}(\overline{T}^c(V),W)\otimes\mathit{Hom}(\overline{L},\overline{T}^c(V))\xrightarrow{\circ}\mathit{Hom}(\overline{L},W).\quad\square$$

As explained before for cofree coalgebras, this construction covers the definition of the coalgebra morphism

$$(\nabla_f)_*:\mathit{HopfHom}(L,(T^c(V),\partial_\alpha))\to\mathit{HopfHom}(L,(T^c(W),\partial_\beta))$$

induced by a morphism of quasi-cofree connected coalgebras $\nabla_f:(T^c(V),\partial_\alpha)\to(T^c(W),\partial_\beta)$ since ∇_f can be identified with a group-like element in the morphism coalgebra $\mathit{HopfHom}((T^c(V),\partial_\alpha),(T^c(W),\partial_\beta))$.

The relation $\mathit{HopfHom}(L,M)=(\Gamma(\mathit{Hom}(\overline{L},V)),\partial)$ is also trivially functorial in $L\in\mathrm{CoAlg}_0^a$. To be explicit, if $\phi:K\to L$ is a morphism of connected coalgebras, then the associated morphism

$$\phi^*:\mathit{HopfHom}(L,(T^c(V),\partial_\alpha))\to\mathit{HopfHom}(K,(T^c(V),\partial_\alpha))$$

can be identified with the morphism of quasi-cofree coalgebras

$$\phi^*:\big(\Gamma(\mathit{Hom}(\overline{L},V)),\partial_{\alpha_*\,\mathrm{ev}_T^\sharp}\big)\to\big(\Gamma(\mathit{Hom}(\overline{K},V)),\partial_{\alpha_*\,\mathrm{ev}_T^\sharp}\big)$$

induced by the map

$$\mathit{Hom}(\overline{L},V)\xrightarrow{\phi^*}\mathit{Hom}(\overline{K},V).$$

Observe that the adjunction augmentations ev_Γ and ev_T have also an explicit description with respect to our construction of the cofree coalgebra. Namely we have the following straightforward assertion:

§4.2.24. **Observation.** *The adjunction augmentation*

$$\mathrm{ev}_\Gamma : \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \to \overline{\Gamma}(V)$$

defined in observation §4.2.6 can also be obtained by a restriction of the natural morphism

$$\mathrm{ev}_\Pi : \Big\{\prod_n \mathit{Hom}(\overline{L}, V)^{\otimes n}\Big\} \otimes \overline{L} \to \prod_{n>0} V^{\otimes n}$$

defined componentwise by the composite

$$\Big\{\prod_n \mathit{Hom}(\overline{L}, V)^{\otimes n}\Big\} \otimes \overline{L}$$
$$\xrightarrow{\mathrm{pr}_n \otimes \mathrm{Id}} \mathit{Hom}(\overline{L}, V)^{\otimes n} \otimes \overline{L} \xrightarrow{\mathrm{Id} \otimes \Delta^n} \mathit{Hom}(\overline{L}, V)^{\otimes n} \otimes \overline{L}^{\otimes n} \xrightarrow{\mathrm{ev}^{\otimes n}} V^{\otimes n}.$$

If the coalgebra L is connected, then this morphism admits a further restriction to the adjunction augmentation

$$\mathrm{ev}_T : \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \to \overline{T}^c(V)$$

of observation §4.2.21.

To summarize, whenever it makes sense, we have a commutative diagram:

$$\begin{tikzcd}
\Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \arrow[r, "="] \arrow[d, dashed, "\mathrm{ev}_T"] & \Gamma(\mathit{Hom}(\overline{L}, V)) \otimes \overline{L} \arrow[r, hook] \arrow[d, dashed, "\mathrm{ev}_\Gamma"] & \Big\{\prod_n \mathit{Hom}(\overline{L}, V)^{\otimes n}\Big\} \otimes \overline{L} \arrow[d, "\mathrm{ev}_\Pi"] \\
\overline{T}^c(V) \arrow[r, hook] & \overline{\Gamma}(V) \arrow[r, hook] & \prod_{n>0} V^{\otimes n}
\end{tikzcd} \qquad \square$$

Recall simply that the cofree coalgebra is endowed with a natural embedding

$$\Gamma(\mathit{Hom}(\overline{L}, M)) \hookrightarrow \prod_n \mathit{Hom}(\overline{L}, M)^{\otimes n}$$

given by the composite

$$\Gamma(\mathit{Hom}(\overline{L}, M)) \xrightarrow{\{\Delta^n\}} \prod_n \Gamma(\mathit{Hom}(\overline{L}, M))^{\otimes n} \xrightarrow{\{\pi^{\otimes n}\}} \prod_n \mathit{Hom}(\overline{L}, M)^{\otimes n},$$

where we consider the n-fold coproducts Δ^n and the canonical projection π of the cofree coalgebra $\Gamma(\mathit{Hom}(\overline{L}, M))$ (see observation §3.2.6).

§4.3. **The Hopf endomorphism operad of the bar complex.** Let Γ be an augmented unitary coalgebra. The *Hopf endomorphism operad* of Γ is the unital Hopf operad $\mathrm{HopfEnd}_\Gamma$ defined by the morphism coalgebras $\mathrm{HopfEnd}_\Gamma(r) = \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$. Observe that $\mathrm{HopfEnd}_\Gamma(0) = \mathbb{F}$. This operad satisfies the following expected property (announced in the section introduction):

§4.3.1. **Proposition.** *The Hopf operad* $\mathrm{HopfEnd}_\Gamma$ *operates on the coalgebra* Γ *so that* Γ *forms a Hopf algebra over* $\mathrm{HopfEnd}_\Gamma$ *and the unital operation* $\mathrm{HopfEnd}_\Gamma(0) \to \Gamma$ *agrees with the unit of* Γ.

Furthermore, the Hopf endomorphism operad $\mathrm{HopfEnd}_\Gamma$ *is the universal Hopf operad with this property. To be more explicit, we have a one-to-one correspondance between Hopf algebra structures as above and morphisms of unital Hopf operads* $\mathcal{P} \to \mathrm{HopfEnd}_\Gamma$.

Proof. This proposition is a direct consequence of the adjunction relation for the internal hom $\mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$. Observe simply that a Hopf operad action such that $\mathcal{P}(0) \to \Gamma$ agrees with the unit of Γ gives rise to a commutative diagram of coalgebras

$$\begin{tikzcd} \mathcal{P}(r) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{F} \arrow[d] \\ \mathcal{P}(r) \otimes \Gamma^{\otimes r} \arrow[r] & \Gamma \end{tikzcd}$$

as in fact §4.2.4. Indeed recall that the composition product $\mathcal{P}(r) \otimes \mathcal{P}(0)^{\otimes r} \to \mathcal{P}(0)$ is supposed to commute with coalgebra augmentations. Consequently, the augmentation $\epsilon : \mathcal{P}(r) \to \mathbb{F}$ is equivalent to a composite with unital operations. More precisely, we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}(r) \arrow[r, "\epsilon"] \arrow[d, "\simeq"] & \mathbb{F} \arrow[d, "="] \\ \mathcal{P}(r) \otimes \mathcal{P}(0)^{\otimes r} \arrow[r] & \mathcal{P}(0) \end{tikzcd}$$

from which we deduce the commutativity of the considered diagram. □

§4.3.2. *The Hopf endomorphism operad of the bar complex.* The *Hopf endomorphism operad of the bar construction in the category of $\mathcal{P}$-algebras* is defined by the formula

$$\mathrm{HopfEnd}^{\mathcal{P}}_B(r) = \mathit{HopfHom}_{A\in\mathcal{P}\,\mathrm{Alg}}(B(A)^{\otimes r}, B(A)),$$

where, according to the conventions of [B1], the right hand-side denotes the end of the bifunctor $\mathit{HopfHom}(B(A)^{\otimes r}, B(A))$ over the category of $\mathcal{P}$-algebras. As explained in the section introduction, this operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$ satisfies the following feature by construction:

§4.3.3. **Fact** (theorem §4.A)**.** *The Hopf endomorphism operad* $\mathrm{HopfEnd}^{\mathcal{P}}_B$ *operates functorially on the coalgebra $B(A)$ so that $B(A)$ forms a Hopf algebra over* $\mathrm{HopfEnd}^{\mathcal{P}}_B$ *and the unital operation* $\mathrm{HopfEnd}^{\mathcal{P}}_B(0) \to B(A)$ *is given by the unit of the bar construction $B(A)$.*

Furthermore, the Hopf endomorphism operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$ *is the universal unital Hopf operad with this property. To be more explicit, we have a one-to-one correspondance between such Hopf actions and morphisms of Hopf operads $\mathcal{Q} \to$* $\mathrm{HopfEnd}^{\mathcal{P}}_B$ *that preserve unital operations.*

The aim of this subsection is to give more insights into the structure of the Hopf endomorphism operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$. For this purpose, we give first structure results for the Hopf endomorphism operad of the bar complex of a fixed $\mathcal{P}$-algebra $\mathrm{HopfEnd}_{B(A)}$. Then we use the relation

$$\mathrm{HopfEnd}^{\mathcal{P}}_B(r) = \int^{A\in\mathcal{P}\,\mathrm{Alg}} \mathrm{HopfEnd}_{B(A)}(r)$$

to extend our results to the Hopf endomorphism operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$.

§4.3.4. *On the coalgebra structure of the Endomorphism operad of the bar complex.* First, observe that the morphism coalgebra $\mathit{HopfHom}(L, B(A))$ is quasi-cofree, for any connected coalgebra L, since the bar complex $B(A)$ is defined by a quasi-cofree connected coalgebra $B(A) = (T^c(\Sigma A), \partial)$. More precisely, according to results of the previous section, we have

$$\mathit{HopfHom}(L, B(A)) = (\Gamma(\mathit{Hom}(\overline{L}, \Sigma A)), \partial),$$

for a coderivation $\partial : \Gamma(\mathit{Hom}(\overline{L}, \Sigma A)) \to \Gamma(\mathit{Hom}(\overline{L}, \Sigma A))$ determined functorially by the differential of $B(A)$. For $L = B(A)^{\otimes r}$, we have identifications

$$\overline{L} = \bigoplus_{m_1+\cdots+m_r>0} \Sigma A^{\otimes m_1+\cdots+m_r}$$
$$\text{and} \quad \mathit{Hom}(\overline{L}, \Sigma A) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A).$$

Consequently, we obtain

$$\mathrm{HopfEnd}_{B(A)}(r) = (\Gamma(\mathrm{PrimEnd}_{B(A)}(r)), \partial)$$
$$\text{where} \quad \mathrm{PrimEnd}_{B(A)}(r) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A).$$

To be precise, the differential of $L = B(A)^{\otimes r}$ is given by the sum of the internal differential of ΣA together with extra terms ∂_i^h given by the bar coderivation $\partial : B(A) \to B(A)$ on the ith factor of L. By convention we assume that the coderivation $\partial : \Gamma(\mathrm{PrimEnd}_{B(A)}(r)) \to \Gamma(\mathrm{PrimEnd}_{B(A)}(r))$ of $\mathrm{HopfEnd}_{B(A)}(r)$ includes these bar coderivations on the source ∂_i^h as well as a coderivation ∂^v determined by the bar coderivation on the target. Accordingly, we assume that the differential $\delta : \Gamma(\mathrm{PrimEnd}_{B(A)}(r)) \to \Gamma(\mathrm{PrimEnd}_{B(A)}(r))$ is determined by the internal differential of A only.

Observe that the action of permutations $w \in \Sigma_r$ on $\mathit{HopfHom}(B(A)^{\otimes r}, B(A))$ as well as the partial composites with unital operations $\partial_i = - \circ_i *$ are induced by coalgebra morphisms on the source $B(A)^{\otimes r}$. But, according to the constructions of the previous section, any operation on the source of $\mathit{HopfHom}(L, B(A))$ can be identified with the morphism of quasi-cofree coalgebras induced the same operation on the source of $\mathit{Hom}(\overline{L}, B(A))$. Therefore we obtain immediately:

§4.3.5. **Proposition.** *The modules* $\mathrm{PrimEnd}_{B(A)}(r)$ *are equipped with a* Λ_**-module structure so that the coalgebras*

$$\mathrm{HopfEnd}_{B(A)}(r) = \big(\Gamma(\mathrm{PrimEnd}_{B(A)}(r)), \partial\big)$$

form a quasi-cofree Hopf Λ_**-module.*

The purpose of the next paragraphs is to make explicit the structure of the endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ with respect to the representation supplied by the assertion above. More specifically, we give an explicit definition of the coderivations ∂_i^h and ∂^v of $\mathrm{HopfEnd}_{B(A)}(r)$ in §4.3.7, of the Λ^*-module structure in §4.3.8 and of the operad composition products in §4.3.9. We obtain our results simply by going through the definition of the endomorphism operad and the constructions of §4.2. More specifically, for the coderivations, we deduce the construction of §4.3.7 from lemma §4.2.22; for Λ_*-module operations and composition products, we deduce the constructions of paragraphs §4.3.8-§4.3.9 from observation §4.2.23 and remarks below this statement.

But first, in the next paragraph, we give an explicit definition of the operad evaluation product $\mathrm{HopfEnd}_{B(A)}(r) \otimes B(A)^{\otimes r} \to B(A)$.

§4.3.6. *On the evaluation product.* By definition, the evaluation product of the Hopf endomorphism operad $\mathrm{HopfEnd}_{B(A)}$ is an instance of the universal evaluation product of a morphism coalgebra

$$\mathit{HopfHom}(L, M) \otimes L \xrightarrow{\mathrm{ev}} M$$

in the case $L = B(A)^{\otimes r}$ and $M = B(A) = (T^c(\Sigma A), \partial)$. We apply the explicit constructions of the previous subsection in order to obtain the expansion of an operation $\mathrm{ev}(\gamma) : B(A)^{\otimes n} \to B(A)$ associated to an element $\gamma \in \mathrm{HopfEnd}_{B(A)}(r)$.

First, let $\theta = \pi(\gamma)$ denotes the image of γ under the canonical projection $\pi : \mathrm{HopfEnd}_{B(A)}(r) \to \mathrm{PrimEnd}_{B(A)}(r)$. Recall that θ represents a natural transformation $\theta : \overline{L} \to \Sigma A$ equivalent to a collection of natural maps

$$\theta = \{\theta_{m_*}\} \in \prod_{m_*} \mathit{Hom}_{A \in \mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

that represent the components of the map θ. According to lemma §4.2.22 and observation §4.2.20, the natural transformation θ determines the composite of the operation $\mathrm{ev}(\gamma) : B(A)^{\otimes n} \to B(A)$ with the natural projection $\pi : B(A) \to \Sigma A$. Equivalently, for each component $\Sigma A^{\otimes m_1+\cdots+m_r} \subset B(A)^{\otimes r}$, we have a commutative diagram:

$$\begin{tikzcd}
\Sigma A^{\otimes m_1+\cdots+m_r} \arrow[r, "\theta_{m_*}"] \arrow[d, hook] & \Sigma A \\
B(A)^{\otimes r} \arrow[r, "\mathrm{ev}(\gamma)"] & B(A) \arrow[u, "\pi"']
\end{tikzcd},$$

where we consider the natural projection $\pi : B(A) \to \Sigma A$ of the bar complex.

Then consider the tensors

$$\pi^{\otimes n} \cdot \Delta^n(\gamma) = \sum \theta^1 \otimes \cdots \otimes \theta^n \in \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n}$$

associated to the n-fold coproduct of γ in $\mathrm{HopfEnd}_{B(A)}(r)$. According to observation §4.2.24, the operation $\mathrm{ev}(\gamma) : B(A)^{\otimes r} \to B(A)$ has an expansion of the form

$$\mathrm{ev}(\gamma)(\alpha_1, \ldots, \alpha_r) = \sum_{n=1}^{\infty} \Big\{ \sum \Big[\underbrace{\theta^1(\alpha_1^1, \ldots, \alpha_r^1) \otimes \cdots \otimes \theta^n(\alpha_1^n, \ldots, \alpha_r^n)}_{\in \Sigma A^{\otimes n}} \Big] \Big\}$$

for all elements in the bar complex $\alpha_1, \ldots, \alpha_r \in B(A)$, where we consider the n-fold diagonals $\sum \alpha_i^1 \otimes \cdots \otimes \alpha_i^n \in B(A)^{\otimes n}$ of the tensors $\alpha_i \in B(A)$.

§4.3.7. *On differentials.* Recall that

$$\mathit{HopfHom}(B(A)^{\otimes r}, B(A)) \xrightarrow{\partial_i^h} \mathit{HopfHom}(B(A)^{\otimes r}, B(A))$$

denotes the coderivation of $\mathit{HopfHom}(B(A)^{\otimes r}, B(A))$ induced by the bar coderivation on the ith factor of the tensor product $B(A)^{\otimes r}$. By construction, these coderivations $\partial_i^h = \partial_{\beta_i^h}$ are induced by maps

$$\mathrm{PrimOp}_{B(A)}(r) \xrightarrow{\beta_i^h} \mathrm{PrimOp}_{B(A)}(r).$$

Explicitly, for a map $\theta : \Sigma A^{\otimes m_1+\cdots+m_r} \to \Sigma A$, the components of $\beta_i^h(\theta)$ are given by the composite of θ with the natural transformations

$$\Sigma A^{\otimes m_1} \otimes \cdots \otimes \Sigma A^{\otimes m_i+n-1} \otimes \cdots \otimes \Sigma A^{\otimes m_r} \xrightarrow{\partial_i^h} \Sigma A^{\otimes m_1} \otimes \cdots \otimes \Sigma A^{\otimes m_i} \otimes \cdots \otimes \Sigma A^{\otimes m_r}$$

induced by the operations $\mu_n \in \mathcal{K}(n)$ on the factors $a_k \otimes \cdots \otimes a_{k+n-1}$ such that $m_1 + \cdots + m_{i-1} + 1 \leq k < k+n-1 \leq m_1 + \cdots + m_{i-1} + m_i + n - 1$. Accordingly, the map β_i^h maps the component $\mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$ of $\mathrm{PrimEnd}_{B(A)}(r)$ to the components $\mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+(m_i+n-1)+m_r}, \Sigma A)$ such that $n = 2, 3, \ldots$.

The other coderivation of $\mathrm{HopfEnd}_{B(A)}(r)$, denoted by

$$\mathit{HopfHom}(B(A)^{\otimes r}, B(A)) \xrightarrow{\partial^v} \mathit{HopfHom}(B(A)^{\otimes r}, B(A)),$$

is yielded by the bar coderivation of the target. According to lemma §4.2.22 and observation §4.2.24, this coderivation can be identified with a coderivation $\partial^v = \partial_{\beta^v}$ of the cofree coalgebra $\Gamma(\mathrm{PrimEnd}_{B(A)}(r))$ induced by a homogeneous morphism

$$\begin{tikzcd} \Gamma(\mathrm{PrimEnd}_{B(A)}(r)) \arrow[r, hook] \arrow[rr, bend right, "\beta^v"'] & \prod_n \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n} \arrow[r, "\mu_*"] & \mathrm{PrimEnd}_{B(A)}(r) \end{tikzcd},$$

where μ_* has a component

$$\bigotimes_{j=1}^{n} \mathit{Hom}(\Sigma A^{\otimes m_1^j+\cdots+m_r^j}, \Sigma A) \xrightarrow{\mu_*} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A),$$

for all collections such that $m_i = m_i^1+\cdots+m_i^n$, for $i = 1, \dots, r$. Explicitly, the morphism μ_* maps a tensor product of homogeneous morphisms $\theta_j : \Sigma A^{\otimes m_1^j+\cdots+m_r^j} \to \Sigma A$ to the composite

$$\Sigma A^{\otimes m_1+\cdots+m_r} \xrightarrow{Sh^*}$$
$$\Sigma A^{\otimes m_1^1+\cdots+m_r^1} \otimes \cdots \otimes \Sigma A^{\otimes m_1^n+\cdots+m_r^n} \xrightarrow{\theta_1\otimes\cdots\otimes\theta_n} \Sigma A \otimes \cdots \otimes \Sigma A \xrightarrow{\mu_n} \Sigma A$$

where $Sh^* = Sh(m_i^j) \in \Sigma_{m_1+\cdots+m_r}$ denotes the bloc permutation which shuffles the rn tensor groupings $\Sigma A^{\otimes m_i^j}$ according to the permutation $Sh \in \Sigma_{rn}$ such that

$$Sh((j-1)r+i) = (i-1)n+j, \quad \text{for } i = 1, \dots, r \text{ and } j = 1, \dots, n.$$

§4.3.8. *On Λ_*-module structures.* By definition, the action of the symmetric group Σ_r on $\mathrm{HopfEnd}_{B(A)}(r) = \mathit{HopfHom}(B(A)^{\otimes r}, B(A))$ is given by tensor permutations on the source. Then we observe in §4.3.4 that this action is induced by an action of Σ_r on the dg-module $\mathrm{PrimEnd}_{B(A)}(r)$. In fact, a permutation $w \in \Sigma_r$ gives rise to a natural isomorphism

$$\Sigma A^{\otimes m_{w^{-1}(1)}+\cdots+m_{w^{-1}(r)}} \xrightarrow{\simeq} \Sigma A^{\otimes m_1+\cdots+m_r}$$

given by the permutation of the r blocs of tensors $A^{\otimes m_i}$ specified by w. The composite of a homogeneous map $\theta : \Sigma A^{m_1+\cdots+m_r} \to \Sigma A$ with this isomorphism yields a map

$$w\theta : \Sigma A^{m_{w^{-1}(1)}+\cdots+m_{w^{-1}(r)}} \to \Sigma A$$

and this process defines the action of Σ_r on the dg-module

$$\mathrm{PrimEnd}_{B(A)}(r) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A).$$

Similarly, the morphisms $\partial_i : \mathrm{HopfEnd}_{B(A)}(r) \to \mathrm{HopfEnd}_{B(A)}(r-1)$ are induced by the canonical morphisms

$$B(A)^{\otimes r-1} \simeq B(A) \otimes \cdots \otimes \mathbb{F} \otimes \cdots \otimes B(A) \hookrightarrow B(A)^{\otimes r}$$

given by the insertion of a unit at the ith position of the tensor product. Consequently, this operation on $\mathrm{HopfEnd}_{B(A)}(r)$ is induced by an operation on $\mathrm{PrimEnd}_{B(A)}(r)$:

$$\partial_i : \mathrm{HopfEnd}_{B(A)}(r) \to \mathrm{HopfEnd}_{B(A)}(r-1).$$

Explicitly, this operation is given by the projection onto the components

$$\mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+0+\cdots+m_r}, \Sigma A)$$

for which $m_i = 0$. Observe that these components of $\mathrm{PrimEnd}_{B(A)}(r)$ are naturally identified with components of $\mathrm{PrimEnd}_{B(A)}(r-1)$.

§4.3.9. *On composition products.* By definition, and according to the result of observation §4.2.23, the partial composition product

$$\circ_i : \mathrm{HopfEnd}_{B(A)}(s) \otimes \mathrm{HopfEnd}_{B(A)}(t) \to \mathrm{HopfEnd}_{B(A)}(s+t-1)$$

can be identified with the morphism of cofree coalgebras induced by a morphism of dg-modules

$$\mathrm{PrimEnd}_{B(A)}(s) \otimes \Gamma(\mathrm{PrimEnd}_{B(A)}(t)) \hookrightarrow \mathrm{PrimEnd}_{B(A)}(s) \otimes \prod_m \mathrm{PrimEnd}_{B(A)}(t)^{\otimes m} \xrightarrow{\gamma_i} \mathrm{PrimEnd}_{B(A)}(s+t-1)$$

which has components

$$\mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_s}, \Sigma A) \otimes \Big\{ \bigotimes_{k=1}^m \mathit{Hom}(\Sigma A^{n_1^k+\cdots+n_t^k}, \Sigma A) \Big\} \xrightarrow{\gamma_i} \mathit{Hom}(\Sigma A^{m_1+\cdots+(n_1+\cdots+n_t)+\cdots+m_s}, \Sigma A),$$

for all collections such that $m = m_i$ and $n_j = n_j^1 + \cdots + n_j^m$. Explicitly, for homogeneous maps $\phi : \Sigma A^{\otimes m_1+\cdots+m_s} \to \Sigma A$ and $\psi_k : \Sigma A^{\otimes n_1^k+\cdots+n_t^k} \to \Sigma A$, $k = 1,\ldots,m$, the map $\gamma_i(\phi \otimes \{\psi_1 \otimes \cdots \otimes \psi_m\})$ is defined by the composite

$$\begin{aligned} &\Sigma A^{\otimes m_1+\cdots+n_1+\cdots+n_t+\cdots+m_s} \\ &\xrightarrow{Sh_i^*} \Sigma A^{\otimes m_1} \otimes \cdots \otimes \Big\{ \bigotimes_{k=1}^m \Sigma A^{\otimes n_1^k+\cdots+n_t^k} \Big\} \otimes \cdots \otimes \Sigma A^{\otimes m_s} \\ &\xrightarrow{\mathrm{Id} \otimes \cdots \otimes \{\bigotimes_{k=1}^m \psi_k\} \otimes \cdots \otimes \mathrm{Id}} \Sigma A^{\otimes m_1} \otimes \cdots \otimes \Sigma A^{\otimes m} \otimes \cdots \otimes \Sigma A^{\otimes m_s} \\ &\xrightarrow{\phi} \Sigma A, \end{aligned}$$

where $Sh_i^* = Sh_i(n_j^k) \in \Sigma_{n_1+\cdots+n_t}$ denotes the bloc permutation which shuffles the tm tensor groupings $\Sigma A^{\otimes n_j^k}$ according to the permutation $Sh \in \Sigma_{tm}$ defined in §4.3.7.

§4.3.10. *On operad units.* The operad unit of the Hopf endomorphism operad is represented by the collection

$$\{\mathrm{Id}^{\otimes n}\} \in \prod_n \mathit{Hom}(\Sigma A, \Sigma A)^{\otimes n},$$

where $\mathrm{Id} \in \mathit{Hom}(\Sigma A, \Sigma A)$ is the identity morphism. In fact, this collection represents the image of the identity morphism $\mathrm{Id} : B(A) \to B(A)$ under the bijection

$$\mathrm{Hom}_{\mathrm{CoAlg}_*^a}(B(A), B(A)) \xrightarrow{\simeq} \mathrm{Gr}(\mathit{HopfHom}(B(A), B(A)))$$

defined by claim §4.2.11.

Observe that the identity morphism $\mathrm{Id} \in \mathit{Hom}(\Sigma A, \Sigma A)$ specifies also a unit element in $\mathrm{PrimEnd}_{B(A)}(1)$. Accordingly, the module $\mathrm{PrimEnd}_{B(A)}$ forms a unitary Λ_*-module.

As announced, we deduce structure results on the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ from the assertions obtained in the previous paragraphs on $\mathrm{HopfEnd}_{B(A)}$. This pre-statement follows from the following formal assertion:

§4.3.11. **Fact.** *Consider quasi-cofree coalgebras of the form*

$$(\Gamma(G(X,Y), \partial_{X,Y}),$$

where $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{dg}\,\mathrm{Mod}$ *is a bifunctor on an essentially small category and such that the coderivation* $\partial_{X,Y}$ *is natural in* $X, Y \in \mathcal{C}$. *Let* $G_{X\in\mathcal{C}}(X,X) = \int^{X\in\mathcal{C}} G(X,X)$. *The morphisms of quasi-cofree coalgebras*

$$(\Gamma(G_{X\in\mathcal{C}}(X,X), \partial_{X,X}) \to (\Gamma(G(X,X), \partial_{X,X})$$

induced by the natural morphism of dg-modules $G_{X\in\mathcal{C}}(X,X) \to G(X,X)$ *yields an end isomorphism*

$$(\Gamma(G_{X\in\mathcal{C}}(X,X), \partial_{X,X}) \xrightarrow{\equiv} \int^{X\in\mathcal{C}} (\Gamma(G(X,X), \partial_{X,X}).$$

Then we deduce from proposition §4.3.5:

§4.3.12. **Proposition.** *The Hopf endomorphism operad* $\mathrm{HopfEnd}_B^{\mathcal{P}}$ *forms a quasi-cofree Hopf* Λ_**-module such that*

$$\mathrm{HopfEnd}_B^{\mathcal{P}}(r) = \big(\Gamma(\mathrm{PrimEnd}_B^{\mathcal{P}}(r)), \partial\big),$$

where

$$\begin{aligned}\mathrm{PrimEnd}_B^{\mathcal{P}}(r) = \int^{A\in\mathcal{P}\,\mathrm{Alg}} & \mathrm{PrimEnd}_{B(A)}(r) \\ & = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A).\end{aligned}$$

The differential of $\mathrm{HopfEnd}_B^{\mathcal{P}}$, the Λ_*-module structure and the operad structure can also be deduced from the constructions of paragraphs §4.3.7-§4.3.9 extended to natural transformations.

§4.4. **The Hopf operad of universal bar operations.** We use ideas of [B1, Section 1.2] in order to reduce the structure of the operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$. Precisely, for an operad $\mathcal{P}$, we consider the natural morphism

$$\mathcal{P}(m) \to \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A)$$

which identifies an operad element $p \in \mathcal{P}(m)$ with the associated operation $p : A^{\otimes m} \to A$ defined for A a $\mathcal{P}$-algebra.

§4.4.1. **Fact** (See *loc. cit.*)**.** *The morphism*

$$\mathcal{P}(m) \to \mathrm{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A)$$

is split injective in general and defines an isomorphism if the operad $\mathcal{P}$ *is* Σ_**-projective or if the ground field* $\mathbb{F}$ *is infinite.*

For our purpose we consider the module

$$\mathrm{PrimOp}_B^{\mathcal{P}}(r) = \prod_{m_1+\cdots+m_r>0} \Lambda\mathcal{P}(m_1+\cdots+m_r),$$

where $\Lambda\mathcal{P}$ denotes the operadic suspension of $\mathcal{P}$ (see [14]), and the associated cofree coalgebra

$$\mathrm{HopfOp}_B^{\mathcal{P}}(r) = \Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r)).$$

Recall that the suspension $\Lambda\mathcal{P}$ of an operad $\mathcal{P}$ is the operad whose algebras are suspensions ΣA of $\mathcal{P}$-algebras A. This operad satisfies the relation $\Lambda\mathcal{P}(r) = \Sigma^{1-r}\mathcal{P}(r) \otimes \mathrm{sgn}(r)$, where $\mathrm{sgn}(r)$ denotes the signature representation of Σ_r.

We have canonical embeddings

$$\Theta_{m_*} : \Lambda\mathcal{P}(m_1+\cdots+m_r) \hookrightarrow \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

which induce embeddings of dg-modules $\Theta : \mathrm{PrimOp}_B^{\mathcal{P}}(r) \to \mathrm{PrimEnd}_B^{\mathcal{P}}(r)$ and embeddings of coalgebras $\nabla_\Theta : \mathrm{HopfOp}_B^{\mathcal{P}}(r) \to \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$. These embeddings form an isomorphism if the operad $\mathcal{P}$ is Σ_*-projective or if the ground field is infinite. As a consequence, the module $\mathrm{HopfOp}_B^{\mathcal{P}}$ is equipped with the structure of a differential graded operad that reflects the structure of the Hopf endomorphism operad of the bar construction. Precisely, we have the following assertion:

§4.4.2. **Proposition.** *The coalgebras* $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *are equipped with the structure of a differential graded Hopf operad so that* $\mathrm{HopfOp}_B^{\mathcal{P}}$ *forms a quasi-cofree Hopf* Λ_**-module and the canonical embeddings*

$$\nabla_\Theta : \mathrm{HopfOp}_B^{\mathcal{P}}(r) \to \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$$

form a morphism of differential graded Hopf operads.

We check this proposition simply by going through the constructions of §4.3. In fact, we perform analogous constructions for $\mathrm{HopfOp}_B^{\mathcal{P}}$ so that we provide $\mathrm{HopfOp}_B^{\mathcal{P}}$ with the structure of a suboperad of $\mathrm{HopfEnd}_B^{\mathcal{P}}$. Then we obtain explicit definitions for the structure of the Hopf operad $\mathrm{HopfOp}_B^{\mathcal{P}}$.

First, for the differential, we have:

§4.4.3. **Claim.**

(a) *The dg-coalgebra* $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *can be equipped with coderivations* ∂_i^h *that correspond to the terms* ∂_i^h *of the differential of* $\mathrm{HopfEnd}_B^{\mathcal{P}}(r)$ *under the embedding* $\mathrm{HopfOp}_B^{\mathcal{P}}(r) \hookrightarrow \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$.

(b) *The dg-coalgebra* $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *can be equipped with a coderivation* ∂^v *that corresponds to the term* ∂^v *of the differential of* $\mathrm{HopfEnd}_B^{\mathcal{P}}(r)$ *under the embedding* $\mathrm{HopfOp}_B^{\mathcal{P}}(r) \hookrightarrow \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$.

Consequently, the dg-module $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *forms a quasi-cofree subcoalgebra of* $\mathrm{HopfEnd}_B^{\mathcal{P}}(r)$.

Proof. The construction of the differentials ∂_i^h and ∂^v of $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ are immediate consequences of the assertions of §4.3.7.

For $i = 1, \dots, r$, the coderivation $\partial_i^h : \mathrm{HopfOp}_B^{\mathcal{P}}(r) \to \mathrm{HopfOp}_B^{\mathcal{P}}(r)$ is induced by a map $\beta_i^h : \mathrm{PrimOp}_B^{\mathcal{P}}(r) \to \mathrm{PrimOp}_B^{\mathcal{P}}(r)$ given componentwise by maps

$$\Lambda\mathcal{P}(m_1 + \dots + m_i + \dots + m_r) \xrightarrow{\beta_i^h} \Lambda\mathcal{P}(m_1 + \dots + (m_i + n - 1) + \dots + m_r)$$

such that $\beta_i^h(p) = \sum_t p \circ_t \mu_n$, where the summation ranges over the interval $t = m_1 + \dots + m_{i-1} + 1, \dots, m_1 + \dots + m_{i-1} + m_i$.

The term $\partial^v : \mathrm{HopfOp}_B^{\mathcal{P}}(r) \to \mathrm{HopfOp}_B^{\mathcal{P}}(r)$ of the differential is defined by a coderivation $\partial^v = \partial_{\beta^v}$ of the cofree coalgebra $\mathrm{HopfOp}_B^{\mathcal{P}}(r) = \Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r))$ determined by a homogeneous morphism

$$\begin{tikzcd} \Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r)) \arrow[r, hook] \arrow[rr, bend right, "\beta^v"'] & \prod_n \mathrm{PrimOp}_B^{\mathcal{P}}(r)^{\otimes n} \arrow[r, "\mu_*"] & \mathrm{PrimOp}_B^{\mathcal{P}}(r) \end{tikzcd},$$

where μ_* has a component

$$\bigotimes_{j=1}^{n} \Lambda\mathcal{P}(m_1^j + \dots + m_r^j) \xrightarrow{\mu_*} \Lambda\mathcal{P}(m_1 + \dots + m_r),$$

for all collections such that $m_i = m_i^1 + \dots + m_i^n$. This morphism μ_* is defined explicitly by the formula $\mu_*(p_1 \otimes \dots \otimes p_n) = Sh(m_i^j) \cdot \mu_n(p_1, \dots, p_n)$, where $Sh(m_i^j) \in \Sigma_{m_1 + \dots + m_r}$ denotes the bloc permutation introduced in §4.3.7. ☐

Then for the Λ_*-module structure:

§4.4.4. **Claim.** *The modules* $\mathrm{PrimOp}^{\mathcal{P}}_B(r)$ *can be equipped with an action of the symmetric group* Σ_r *and with operations* $\partial_i : \mathrm{PrimOp}^{\mathcal{P}}_B(r) \to \mathrm{PrimOp}^{\mathcal{P}}_B(r-1)$ *so that* $\mathrm{PrimOp}^{\mathcal{P}}_B$ *forms a* Λ_**-submodule of* $\mathrm{PrimEnd}^{\mathcal{P}}_B$.

The module $\mathrm{PrimOp}^{\mathcal{P}}_B(1)$ *is also equipped with a unit element* $1 \in \mathrm{PrimOp}^{\mathcal{P}}_B(1)$ *that corresponds to the unit element of* $\mathrm{PrimEnd}^{\mathcal{P}}_B(1)$ *(see §4.3.10). Accordingly, the module* $\mathrm{PrimOp}^{\mathcal{P}}_B$ *forms a unitary* Λ_**-submodule of* $\mathrm{PrimEnd}^{\mathcal{P}}_B$.

Proof. This result is also an immediate consequence of the explicit definition of the Λ_*-module structure of $\mathrm{PrimEnd}^{\mathcal{P}}_B$ given in §4.3.8

Explicitly, the action of a permutation $w \in \Sigma_r$ on

$$\mathrm{PrimOp}^{\mathcal{P}}_B(r) = \prod_{m_1+\cdots+m_r>0} \Lambda\mathcal{P}(m_1+\cdots+m_r)$$

is given componentwise by the action of the bloc permutations

$$\Lambda\mathcal{P}(m_1+\cdots+m_r) \xrightarrow{w(m_{w^{-1}(1)},\dots,m_{w^{-1}(r)})} \Lambda\mathcal{P}(m_{w^{-1}(1)}+\cdots+m_{w^{-1}(r)})$$

and the operation

$$\partial_i : \mathrm{PrimOp}^{\mathcal{P}}_B(r) \to \mathrm{PrimOp}^{\mathcal{P}}_B(r-1)$$

is given by the projection onto the components of $\mathrm{PrimOp}^{\mathcal{P}}_B(r)$ such that $m_i = 0$ which can be identified with components of $\mathrm{PrimOp}^{\mathcal{P}}_B(r-1)$ as in the case of $\mathrm{PrimEnd}^{\mathcal{P}}_B(r)$.

The unit operation $1 \in \Lambda\mathcal{P}(1)$ corresponds tautologically to the identity morphism $\mathrm{Id} : \Sigma A \to \Sigma A$ that defines the unit element of the module $\mathrm{PrimEnd}^{\mathcal{P}}_B(1)$ and hence specifies an appropriate unit element in $\mathrm{PrimOp}^{\mathcal{P}}_B(1)$. ☐

Finally, for the operad structure:

§4.4.5. **Claim.** *The dg-coalgebras* $\mathrm{HopfOp}^{\mathcal{P}}_B(r) = \Gamma(\mathrm{PrimOp}^{\mathcal{P}}_B(r))$ *are equipped with an action of the symmetric group induced by the action of* Σ_r *on* $\mathrm{PrimOp}^{\mathcal{P}}_B(r)$. *We have also composition products*

$$\circ_i : \mathrm{HopfOp}^{\mathcal{P}}_B(s) \otimes \mathrm{HopfOp}^{\mathcal{P}}_B(t) \to \mathrm{HopfOp}^{\mathcal{P}}_B(s+t-1)$$

so that $\mathrm{HopfOp}^{\mathcal{P}}_B$ *forms a Hopf suboperad of* $\mathrm{HopfEnd}^{\mathcal{P}}_B$.

Proof. As in the case of the Hopf endomorphism operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$, the partial composition product

$$\circ_i : \mathrm{HopfOp}^{\mathcal{P}}_B(s) \otimes \mathrm{HopfOp}^{\mathcal{P}}_B(t) \to \mathrm{HopfOp}^{\mathcal{P}}_B(s+t-1)$$

is the morphism of cofree coalgebras induced by a morphism of dg-modules

$$\begin{aligned} &\mathrm{PrimOp}^{\mathcal{P}}_B(s) \otimes \Gamma(\mathrm{PrimOp}^{\mathcal{P}}_B(t)) \\ &\qquad \hookrightarrow \mathrm{PrimOp}^{\mathcal{P}}_B(s) \otimes \prod_m \mathrm{PrimOp}^{\mathcal{P}}_B(t)^{\otimes m} \xrightarrow{\gamma_i} \mathrm{PrimOp}^{\mathcal{P}}_B(s+t-1)\end{aligned}$$

which has components

$$\begin{aligned} &\Lambda\mathcal{P}(m_1+\cdots+m_s) \otimes \Bigl\{\bigotimes_{k=1}^m \Lambda\mathcal{P}(n_1^k+\cdots+n_t^k)\Bigr\} \\ &\qquad \xrightarrow{\gamma_i} \Lambda\mathcal{P}(m_1+\cdots+n_1+\cdots+n_t+\cdots+m_s),\end{aligned}$$

for all collections such that $m = m_i$ and $n_j = n_j^1 + \cdots + n_j^m$. We have explicitly $\gamma_i(p \otimes \{q_1 \otimes \cdots \otimes q_m\}) = Sh_i(n_j^k) \cdot p(1,\dots,q_1,\dots,q_m,\dots,1)$, where the operation q_k is substituted to the entry $k = m_1 + \cdots + m_{i-1} + k$ of p and where $Sh_i(n_j^k)$ denotes the bloc permutation introduced in §4.3.9.

Observe also that the operad unit of $\mathrm{HopfEnd}_B^{\mathcal{P}}$ corresponds to an element of $\mathrm{HopfOp}_B^{\mathcal{P}}$. Namely this element can be represented by the collection

$$\{1^{\otimes n}\} \in \prod_n \mathrm{PrimOp}_B^{\mathcal{P}}(1)^{\otimes n},$$

where $1 \in \Lambda\mathcal{P}(1)$ is the unit of $\mathcal{P}$. □

This claim achieves the proof of proposition §4.4.2. To recapitulate, we have the following result:

§4.4.6. **Lemma.** *The dg-coalgebras* $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *equipped with the differentials supplied by claim §4.4.3 and the structure specified by claim §4.4.5 form a differential graded unital Hopf operad. Moreover, the canonical embeddings*

$$\nabla_\Theta : \mathrm{HopfOp}_B^{\mathcal{P}}(r) \hookrightarrow \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$$

define a natural morphism of unital Hopf operads which is an isomorphism if the operad $\mathcal{P}$ *is* Σ_**-projective or if the ground field* $\mathbb{F}$ *is infinite.*

In addition we obtain the following assertion:

§4.4.7. **Observation.** *The Hopf operad* $\mathrm{HopfOp}_B^{\mathcal{P}}(r)$ *forms a quasi-cofree Hopf* Λ_**-module such that*

$$\mathrm{HopfOp}_B^{\mathcal{P}}(r) = \bigl(\Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r)), \partial\bigr),$$

where ∂ *is composed of the coderivations* ∂^v *and* ∂_i^h *specified in §4.4.3.* □

§4.5. **Fibration properties.** One can deduce from formal properties of monoidal model categories that a Hopf endomorphism operad $\mathrm{HopfEnd}_\Gamma$ is a Reedy fibrant object if Γ is a fibrant unitary dg-coalgebras. One can extend this assertion to connected unitary coalgebras because the adjoint definition of morphism coalgebras holds in the connected context as well by fact §4.2.19. As any quasi-cofree connected coalgebra $\Gamma = (T^c(V), \partial_\alpha)$ forms a fibrant object in the category of connected coalgebras, we obtain that the Hopf endomorphism operad of the bar complex $\mathrm{HopfEnd}_{B(A)}$ is Reedy fibrant.

In the next paragraphs, we give another more effective proof of this assertion in order to extend our results to the Hopf operad of bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$. Explicitly, as mentioned in the introduction of §3, we prove that the augmentation morphism of the Hopf endomorphism operad $\mathrm{HopfEnd}_{B(A)}$ splits up into a sequence

$$\begin{aligned}\mathrm{HopfEnd}_{B(A)} = \lim_m \mathrm{ck}_m \mathrm{HopfEnd}_{B(A)} \to \dots \\ \dots \to \mathrm{ck}_m \mathrm{HopfEnd}_{B(A)} \to \mathrm{ck}_{m-1} \mathrm{HopfEnd}_{B(A)} \to \dots \\ \dots \to \mathrm{ck}_0 \mathrm{HopfEnd}_{B(A)} = \mathcal{C},\end{aligned}$$

where $\mathrm{ck}_m \mathrm{HopfEnd}_{B(A)} \to \mathrm{ck}_{m-1} \mathrm{HopfEnd}_{B(A)}$ is obtained by a coextension of a cofree Hopf Λ_*-modules morphism. Then we extend this result to the Hopf endomorphism operad of the bar complex $\mathrm{HopfEnd}_B^{\mathcal{P}}$ and to the Hopf operad of bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$.

§4.5.1. *On* Λ_**-module structures and the canonical filtration of the bar complex.* Recall that the Λ_*-module structure of a Hopf endomorphism operad $\mathrm{HopfEnd}_\Gamma(r) = \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$ is deduced from operations on the source. To be more precise, one can observe that the tensor powers of a unitary coalgebra $\Gamma^{\otimes r}$ form a left Hopf Λ^*-module. The symmetric group operates on $\Gamma^{\otimes r}$ by tensor permutations and we have operations $\partial^i : \Gamma^{\otimes r-1} \to \Gamma^{\otimes r}$ given by the insertion of a coalgebra unit $* : \mathbb{F} \to \Gamma$ at the ith place. The induced operations $w^* : \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma) \to \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$ and $(\partial^i)^* : \mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma) \to \mathit{HopfHom}(\Gamma^{\otimes r-1}, \Gamma)$ determine the Λ^*-module structure of $\mathit{HopfHom}(\Gamma^{\otimes r}, \Gamma)$.

The quotient Hopf Λ_*-modules $\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}$ are associated to a sequence of Hopf Λ_*-submodules of $B(A)^{\otimes r}$. Explicitly, for $m \in \mathbb{N}$, we let $\operatorname{sk}_m(B(A)^{\otimes r})$ denote the submodule of $B(A)^{\otimes r}$ such that:

$$\operatorname{sk}_m(B(A)^{\otimes r}) = \bigoplus_{m \geq m_1 + \cdots + m_r \geq 0} \Sigma A^{\otimes m_1 + \cdots + m_r}.$$

Recall that $B(A)^{\otimes r}$ is equipped with coderivations denoted by ∂_i^h and given by the bar coderivation of the ith factor of $B(A)^{\otimes r}$, for $i = 1, \dots, r$. Clearly, the submodule $\operatorname{sk}_m B(A)^{\otimes r}$ is preserved by the bar coderivations ∂_i^h and hence form a dg-submodule of $B(A)$. In fact, we have more precisely:

§4.5.2. **Observation.** *We have $\partial_i^h(\operatorname{sk}_m(B(A)^{\otimes r})) \subset \operatorname{sk}_{m-1}(B(A)^{\otimes r})$.*

Proof. By definition, the bar coderivation ∂_i^h maps the module $\Sigma A^{\otimes m_i} \subset B(A)$ into components $\Sigma A^{\otimes m_i - n + 1} \subset B(A)$ such that $n \geq 2$. Hence the assertion is immediate. □

Then, as expected, we have clearly:

§4.5.3. **Fact.** *The dg-modules*

$$\operatorname{sk}_m(B(A)^{\otimes r}) \subset B(A)^{\otimes r}$$

are preserved by the diagonal and by the differential of $B(A)^{\otimes r}$, by the action of the symmetric group and by the operations $\partial^i : B(A)^{\otimes r} \to B(A)^{\otimes r-1}$. Hence these dg-modules define a nested sequence of Hopf Λ_*-submodules of $B(A)^{\otimes r}$ such that

$$B(A)^{\otimes r} = \operatorname{colim}_m \operatorname{sk}_m(B(A)^{\otimes r}).$$

As a corollary we obtain:

§4.5.4. **Fact.** *The coalgebras*

$$\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}(r) = \mathit{HopfHom}(\operatorname{sk}_m B(A)^{\otimes r}, B(A))$$

define a tower of quotient Hopf Λ_*-modules of $\operatorname{HopfEnd}_{B(A)}$ *such that*

$$\operatorname{HopfEnd}_{B(A)} = \lim_m \operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}.$$

Furthermore, we have

$$\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)} = (\Gamma(\operatorname{ck}_m \operatorname{PrimEnd}_{B(A)}), \partial),$$

where $\operatorname{ck}_m \operatorname{PrimEnd}_{B(A)}$ *denotes the quotient Λ_*-module of* $\operatorname{PrimEnd}_{B(A)}$ *defined by*

$$\operatorname{ck}_m \operatorname{PrimEnd}_{B(A)}(r) = \prod_{m \geq m_1 + \cdots + m_r > 0} \mathit{Hom}(\Sigma A^{\otimes m_1 + \cdots + m_r}, \Sigma A).$$

Observe that $\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}$ forms also a unital unitary quotient Hopf Λ_*-module of $\operatorname{HopfEnd}_{B(A)}$. Explicitly, we have clearly $\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}(0) = \mathbb{F}$. Furthermore, the collection

$$\{\operatorname{Id}^{\otimes n}\} \in \prod_n \mathit{Hom}(\Sigma A, \Sigma A)^{\otimes n}$$

that represents the unit element of $\operatorname{HopfEnd}_{B(A)}$ specifies clearly a unit element in $\operatorname{ck}_m \operatorname{HopfEnd}_{B(A)}$ for all $m \geq 1$ and projects tautologically to the unit element of the commutative operad for $m = 0$.

Clearly, the Hopf Λ_*-module

$$\operatorname{HopfEnd}_B^{\mathcal{P}}(r) = \mathit{HopfHom}_{A \in \mathcal{P} \operatorname{Alg}}(B(A)^{\otimes r}, B(A))$$

is equipped with a similar decomposition induced by the decomposition of $\mathrm{HopfEnd}_{B(A)}$. Explicitly, we have $\mathrm{HopfEnd}_B^{\mathcal{P}} = \lim_{m\in\mathbb{N}} \mathrm{ck}_m \mathrm{HopfEnd}_B^{\mathcal{P}}$ for unital unitary Hopf Λ_*-modules such that:

$$\mathrm{ck}_m \mathrm{HopfEnd}_B^{\mathcal{P}}(r) = \mathit{HopfHom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\mathrm{sk}_m B(A)^{\otimes r}, B(A)).$$

Furthermore, we have

$$\mathrm{ck}_m \mathrm{HopfEnd}_B^{\mathcal{P}} = (\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_B^{\mathcal{P}}), \partial),$$

for unitary quotient Λ_*-modules of $\mathrm{PrimEnd}_B^{\mathcal{P}}$, that can be defined by

$$\mathrm{ck}_m \mathrm{PrimEnd}_{B(A)}(r) = \prod_{m\geq m_1+\cdots+m_r>0} \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A).$$

We check that the requirements of §3.4.2 are satisfied for the Hopf Λ_*-modules $(\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_{B(A)}), \partial)$ so that the morphisms

$$\mathrm{p}_m : (\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_{B(A)}), \partial) \to (\Gamma(\mathrm{ck}_{m-1} \mathrm{PrimEnd}_{B(A)}), \partial)$$

have the structure specified in §3.4 and similarly for the functorial Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$. First, for $\mathrm{HopfEnd}_{B(A)}$, we have explicitly:

§4.5.5. **Claim.** *Let* $K = \mathrm{PrimEnd}_{B(A)}$. *The homogeneous maps* $\beta = \beta_i^h, \beta^v : \Gamma(K) \to K$ *that determine the coderivations* $\partial = \partial_i^h, \partial^v$ *of* $\mathrm{HopfEnd}_{B(A)}$ *admit factorizations*

$$\begin{tikzcd} \Gamma(K) \arrow[r, "\beta"] \arrow[d] & \Pi \arrow[d] \\ \Gamma(\mathrm{ck}_m K) \arrow[r, dashed, "\beta"] & \mathrm{ck}_{m-1} K \end{tikzcd}.$$

As a corollary, the projection morphisms

$$\mathrm{p}_m : (\Gamma(\mathrm{ck}_m K), \partial) \to (\Gamma(\mathrm{ck}_{m-1} K), \partial)$$

fit coextension diagrams of the form

$$\begin{tikzcd} (\Gamma(\mathrm{ck}_m K), \partial) \arrow[r] \arrow[d] & \Gamma(\Delta^1 \wedge \mathrm{ck}_m K) \arrow[d] \\ (\Gamma(\mathrm{ck}_{m-1} K), \partial) \arrow[r] & \Gamma(\Delta^1 \wedge \mathrm{ck}_{m-1} K \times_{S^1\wedge \mathrm{ck}_{m-1} K} S^1 \wedge \mathrm{ck}_m K) \end{tikzcd}.$$

Proof. For the coderivations $\partial = \partial_i^h$, the assertion is a corollary of observation §4.5.2 since these coderivations are induced by the components ∂_i^h of the differentials of $B(A)^{\otimes r}$. Explicitly, by observation §4.5.2, the maps ∂_i^h have factorizations

$$\begin{tikzcd} B(A)^{\otimes r} \arrow[r, "\partial_i^h"] & B(A)^{\otimes r} \\ \mathrm{sk}_m B(A)^{\otimes r} \arrow[u] \arrow[r, dashed] & \mathrm{sk}_{m-1} B(A)^{\otimes r} \arrow[u] \end{tikzcd}$$

which yield a factorization at the level of $\mathrm{PrimEnd}_{B(A)}$:

$$\begin{tikzcd} \mathrm{PrimEnd}_{B(A)} \arrow[r, "\beta_i^h"] & \mathrm{PrimEnd}_{B(A)} \\ \mathrm{ck}_m \mathrm{PrimEnd}_{B(A)} \arrow[u] \arrow[r, dashed] & \mathrm{ck}_{m-1} \mathrm{PrimEnd}_{B(A)} \arrow[u] \end{tikzcd}.$$

Recall that the other differential $\partial = \partial^v$ is induced by homogeneous morphisms

$$\Gamma(\mathrm{PrimEnd}_{B(A)}(r)) \hookrightarrow \prod_n \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n} \xrightarrow{\mu_*} \mathrm{PrimEnd}_{B(A)}(r)$$

which admit a component

$$\bigotimes_{j=1}^n \mathit{Hom}(\Sigma A^{\otimes m_1^j+\cdots+m_r^j}, \Sigma A) \xrightarrow{\mu_*} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

for all $n \geq 2$ and for all collections (m_i^j) such that $m_i = m_i^1 + \cdots + m_i^n$. Clearly, if $m_1^j + \cdots + n_r^j > m-1$ for some j, as we assume $m_1^k + \cdots + n_r^k > 0$ for all k, we obtain $m_1 + \cdots + m_r = \sum_{i,j} m_i^j > m_1^j + \cdots + n_r^j > m-1$.

Accordingly, for a given tensor $u_1 \otimes \cdots \otimes u_n \in \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n}$, if we have $u_j \in \prod_{m_1^j+\cdots+m_r^j>m-1} \mathit{Hom}(\Sigma A^{m_1^j+\cdots+m_r^j}, \Sigma A)$ for some j, then we obtain $\mu_*(u_1 \otimes \cdots \otimes u_n) \in \prod_{m_1+\cdots+m_r>m} \mathit{Hom}(\Sigma A^{m_1^j+\cdots+m_r^j}, \Sigma A)$ so that μ_* admits a factorization

$$\begin{tikzcd}
\prod_n \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n} \arrow[r, "\mu_*"] \arrow[d] & \mathrm{PrimEnd}_{B(A)}(r) \arrow[d] \\
\mathrm{ck}_{m-1} \mathrm{PrimEnd}_{B(A)}(r)^{\otimes n} \arrow[r, dashed, "\mu_*"] & \mathrm{ck}_m \mathrm{PrimEnd}_{B(A)}(r)
\end{tikzcd}$$

and this assertion implies our claim for the differential ∂^v. $\square$

Observe that the lifting construction of the claim is natural in A. To give a more proper assertion we should extend this structure result to Hopf Λ_*-modules of morphisms $\mathrm{HopfEnd}_{B(A),B(A')}$, which are defined by

$$\mathrm{HopfEnd}_{B(A),B(A')}(r) = \mathit{HopfHom}(B(A)^{\otimes r}, B(A')).$$

One checks precisely that the coderivation liftings

$$\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_{B(A),B(A')}) \overset{\beta}{\dashrightarrow} \mathrm{ck}_{m-1} \mathrm{PrimEnd}_{B(A),B(A')}$$

are functorial in A and A'. In fact, this assertion holds simply because the modules $\mathrm{ck}_m \mathrm{PrimEnd}_{B(A),B(A')}$ are quotient of $\mathrm{PrimEnd}_{B(A),B(A')}$. As a consequence, we obtain:

§4.5.6. **Fact.** *The coderivation liftings of claim §4.5.5 induce a coderivation lifting*

$$\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_B^{\mathcal{P}}) \overset{\beta}{\dashrightarrow} \mathrm{ck}_{m-1} \mathrm{PrimEnd}_B^{\mathcal{P}}$$

on the end of the Λ_*-modules $\mathrm{ck}_m \mathrm{PrimEnd}_{B(A)}$. As a consequence, the results of claim §4.5.5 hold for the Λ_*-module $K = \mathrm{PrimEnd}_B^{\mathcal{P}}$ and the quotient Hopf Λ_*-modules

$$\mathrm{ck}_m \mathrm{HopfEnd}_B^{\mathcal{P}} = (\Gamma(\mathrm{ck}_m \mathrm{PrimEnd}_B^{\mathcal{P}}), \partial)$$

of the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$.

We prove now that the Hopf operad of universal operations $\mathrm{HopfOp}_B^{\mathcal{P}}$ is equipped with the same decomposition as the Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$. We consider the dg-modules

$$\mathrm{ck}_m \mathrm{PrimOp}_B^{\mathcal{P}}(r) = \prod_{m \geq m_1+\cdots+m_r>0} \Lambda\mathcal{P}(m_1 + \cdots + m_r)$$

which come equipped with a canonical embedding

$$\mathrm{ck}_m \Theta : \mathrm{ck}_m \mathrm{PrimOp}_B^{\mathcal{P}}(r) \hookrightarrow \mathrm{ck}_m \mathrm{PrimEnd}_B^{\mathcal{P}}(r).$$

These modules equipped with the canonical projections

$$\mathrm{p}_m : \mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}(r) \to \mathrm{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}}(r)$$

form clearly a subtower of $\mathrm{ck}_m \operatorname{PrimEnd}_B^{\mathcal{P}}(r)$ such that

$$\operatorname{PrimOp}_B^{\mathcal{P}}(r) = \lim_m \mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}(r).$$

We have in addition:

§4.5.7. **Fact.** *The dg-modules* $\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}(r)$ *defined above can be identified with the image of* $\operatorname{PrimOp}_B^{\mathcal{P}}(r)$ *under the composite map*

$$\operatorname{PrimOp}_B^{\mathcal{P}}(r) \hookrightarrow \operatorname{PrimEnd}_B^{\mathcal{P}}(r) \to \mathrm{ck}_m \operatorname{PrimEnd}_B^{\mathcal{P}}(r)$$

As a corollary, we obtain the following assertions:

§4.5.8. **Fact.**

(a) *The dg-modules* $\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}(r)$ *form unitary quotient* Λ_**-modules of* $\operatorname{PrimOp}_B^{\mathcal{P}}$ *so that the canonical embedding* $\Theta : \operatorname{PrimOp}_B^{\mathcal{P}} \hookrightarrow \operatorname{PrimEnd}_B^{\mathcal{P}}$ *splits up into an embedding of* Λ_**-module towers*

$$\{\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}\}_m \hookrightarrow \{\mathrm{ck}_m \operatorname{PrimEnd}_B^{\mathcal{P}}\}_m.$$

(b) *The coalgebras* $\Gamma(\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}(r))$ *are preserved by the differential of* $\operatorname{HopfOp}_B^{\mathcal{P}}$ *and define unital unitary quotient Hopf* Λ_**-modules of* $\operatorname{HopfOp}_B^{\mathcal{P}}$. *Accordingly, the canonical morphism* $\nabla_\Theta : \operatorname{HopfOp}_B^{\mathcal{P}} \hookrightarrow \operatorname{HopfEnd}_B^{\mathcal{P}}$ *splits up into a morphism of Hopf* Λ_**-module towers*

$$\{\mathrm{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}}\}_m \hookrightarrow \{\mathrm{ck}_m \operatorname{HopfEnd}_B^{\mathcal{P}}\}_m,$$

where $\mathrm{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}} = (\Gamma(\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}), \partial)$.

We have in addition:

§4.5.9. **Fact.** *The assertion of claim §4.5.5 holds for* $K = \operatorname{PrimOp}_B^{\mathcal{P}}$. *To be more precise, let* $K' = \operatorname{PrimEnd}_B^{\mathcal{P}}$. *The factorization of the homogeneous maps* $\beta = \beta_i^h, \beta^v : \Gamma(K') \to K'$ *that determine the coderivations* $\partial = \partial_i^h, \partial^v$ *of the quasi-cofree Hopf* Λ_**-module* $\operatorname{HopfEnd}_B^{\mathcal{P}} = (\Gamma(K'), \partial)$ *admit a restriction to* $\Gamma(\mathrm{ck}_m K)$.

As a corollary, the projection morphisms

$$\mathrm{p}_m : (\Gamma(\mathrm{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}), \partial) \to (\Gamma(\mathrm{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}}), \partial)$$

can be obtained by coextensions of the form specified in §3.4 like the projection morphism of Hopf endomorphism operads $\operatorname{HopfEnd}_B^{\mathcal{P}}$ *and* $\operatorname{HopfEnd}_{B(A)}$.

This assertion can either be deduced from the relationship between the Hopf operad of bar operations $\operatorname{HopfOp}_B^{\mathcal{P}}$ and the Hopf endomorphism operad $\operatorname{HopfEnd}_B^{\mathcal{P}}$ or can be checked directly as in the proof of claim §4.5.5 for the Hopf endomorphism operad $\operatorname{HopfEnd}_{B(A)}$.

We use the decomposition $\operatorname{HopfOp}_B^{\mathcal{P}} = \lim_m \mathrm{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}}$ and the results of §3 in order to prove the fibration properties asserted by theorem §4.B. Namely we prove:

§4.5.10. **Claim** (theorem §4.B)**.** *The morphism*

$$\phi_* : \operatorname{HopfOp}_B^{\mathcal{P}} \to \operatorname{HopfOp}_B^{\mathcal{P}'}$$

induced by a fibration, respectively an acyclic fibration, of non-unital operads $\phi : \mathcal{P} \to \mathcal{P}'$ *forms a Reedy fibration, respectively an acyclic Reedy fibration, of Hopf* Λ_**-modules.*

For this purpose we check first the following statement:

§4.5.11. **Observation.** *If $\phi : \mathcal{P} \to \mathcal{P}'$ is a fibration, respectively an acyclic fibration, of non-unital operads, then the map*

$$\operatorname{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}} \xrightarrow{(\mathrm{p}_m, \operatorname{ck}_m \phi_*)} \operatorname{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}} \times_{\operatorname{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}'}} \operatorname{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}'}$$

is a Reedy fibration, respectively an acyclic Reedy fibration, of Λ_*-modules.

Proof. Recall that a morphism of Λ_*-modules $f : M \to N$ forms a Reedy fibration, respectively an acyclic Reedy fibration, if the morphisms $(\mu, f) : M(r) \to \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r)$ are fibrations, respectively acyclic fibrations, of dg-modules, for all $r \in \mathbb{N}$.

The matching object of the Λ_*-module $M = \operatorname{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}$ can clearly be identified with the restricted product

$$\mathbb{M} \operatorname{PrimOp}_B^{\mathcal{P}}(r) = \prod_{m_*}{}' \Lambda\mathcal{P}(m_1 + \cdots + m_r)$$

which ranges over the collections $m \geq m_1 + \cdots + m_r > 0$ such that $m_i = 0$ for some i. For $M = \operatorname{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}}$ and

$$N = \operatorname{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}} \times_{\operatorname{ck}_{m-1} \operatorname{PrimOp}_B^{\mathcal{P}'}} \operatorname{ck}_m \operatorname{PrimOp}_B^{\mathcal{P}'},$$

the relative matching object $\mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r)$ can be identified with the direct product

$$\prod_{m_*}{}' \Lambda\mathcal{P}(m_1 + \cdots + m_r) \times \prod_{m_*}{}'' \Lambda\mathcal{P}'(m_1 + \cdots + m_r),$$

where $\prod'_{m_*}$ ranges over all collections $m \geq m_1 + \cdots + m_r > 0$ such that $m_i = 0$ for some i or $m - 1 \geq m_1 + \cdots + m_r > 0$ and $\prod''_{m_*}$ ranges over all collections $m \geq m_1 + \cdots + m_r > 0$ such that $m_i > 0$ for $i = 1, \ldots, r$. Moreover, the relative matching morphism (ϕ, μ) is given componentwise by the identity of $\Lambda\mathcal{P}(m_1 + \cdots + m_r)$ or by the morphism $\phi : \Lambda\mathcal{P}(m_1 + \cdots + m_r) \to \Lambda\mathcal{P}'(m_1 + \cdots + m_r)$. Accordingly, this morphism forms clearly a fibration, respectively an acyclic fibration, if ϕ is so. $\square$

Then we deduce an inductive proof of the properties of claim §4.5.10 from the results of §3. Explicitly, we prove the following statement:

§4.5.12. **Claim.** *Under the assumptions of claim §4.5.10, the morphism*

$$\operatorname{ck}_m \phi_* : \operatorname{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}} \to \operatorname{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}'}$$

is a Reedy fibration, respectively an acyclic Reedy fibration of Λ_*-modules, for all $m \in \mathbb{N}$.

Proof. We apply lemma §3.4.7 to the commutative square

$$\begin{tikzcd}
\operatorname{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}} \arrow[r, "\operatorname{ck}_m \phi_*"] \arrow[d, "\mathrm{p}_m"] & \operatorname{ck}_m \operatorname{HopfOp}_B^{\mathcal{P}'} \arrow[d, "\mathrm{p}_m"] \\
\operatorname{ck}_{m-1} \operatorname{HopfOp}_B^{\mathcal{P}} \arrow[r, "\operatorname{ck}_{m-1} \phi_*"] & \operatorname{ck}_{m-1} \operatorname{HopfOp}_B^{\mathcal{P}'}
\end{tikzcd} .$$

Under the assertion of observation §4.5.11, we obtain that $\operatorname{ck}_m \phi_*$ forms a fibration, respectively an acyclic fibration if $\operatorname{ck}_{m-1} \phi_*$ is so. Hence the claim follows by induction. $\square$

This result achieves the proof of claim §4.5.10 and theorem §4.B.

§4.6. **Universal bar operations for the commutative operad.** As recalled in the memoir and section introductions, the bar complex $B(A)$ of a commutative algebra A is equipped with an associative and commutative product $\smile: B(A) \otimes B(A) \to B(A)$, the shuffle product of tensors, that provides $B(A)$ with the structure of an associative and commutative differential graded Hopf algebra. This product is equivalent to a morphism of unital Hopf operads $\nabla_c : \mathcal{C} \to \mathrm{HopfEnd}_B^{\mathcal{C}}$.

In the introduction we claim also that:

§4.6.1. **Lemma.** *The morphism $\nabla_c : \mathcal{C} \to \mathrm{HopfEnd}_B^{\mathcal{C}}$ admits a factorization*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\nabla_c"] \arrow[dr, dashed, "\nabla_\gamma"'] & \mathrm{HopfEnd}_B^{\mathcal{C}} \\ & \mathrm{HopfOp}_B^{\mathcal{C}} \arrow[u, hook, "\nabla_\Theta"'] \end{tikzcd} .$$

Here is the proof of this lemma.

Proof. The r-fold shuffle product yields the operad evaluation product

$$\mathrm{ev} : \mathcal{C}(r) \otimes B(A)^{\otimes r} \to B(A).$$

To make explicit the associated morphism $\nabla_c : \mathcal{C}(r) \to \mathrm{HopfEnd}_B^{\mathcal{C}}(r)$, we consider the composite of this evaluation product with the projection $B(A) \to \Sigma A$ and the associated adjoint morphism which gives a map

$$c : \mathcal{C}(r) \to \mathrm{PrimEnd}_B^{\mathcal{C}}(r).$$

According to the construction of lemma §4.2.22, the morphism ∇_c is the coalgebra morphism

$$\nabla_c : \mathcal{C}(r) \to (\Gamma(\mathrm{PrimEnd}_B^{\mathcal{C}}(r)), \partial)$$

induced by this map c.

We apply this construction. First, the components of $c(1_r) \in \mathrm{PrimEnd}_B^{\mathcal{C}}(r)$, where $1_r \in \mathcal{C}(r)$ denotes the generator of $\mathcal{C}(r)$, are given by composites

$$\Sigma A^{\otimes m_1} \otimes \cdots \otimes \Sigma A^{\otimes m_r} \hookrightarrow B(A)^{\otimes r} \xrightarrow{\smile} B(A) \to \Sigma A.$$

By definition of the shuffle product, these composites are given by the identical morphism of ΣA on components such that

$$m_k = \begin{cases} 0 & \text{for } k \neq i, \\ 1 & \text{for } k = i \end{cases}$$

and vanish otherwise. As a consequence, the maps

$$c(1_r)_{m_*} \in \mathit{Hom}_{A \in \mathcal{C}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1 + \cdots + m_r}, \Sigma A)$$

are identified with the natural transformations $\Theta(\gamma_{m_*})$ associated to elements $\gamma_{m_*} \in \Lambda\,\mathcal{C}(m_1 + \cdots + m_r)$ given by operad units $1 \in \Lambda\,\mathcal{C}(0 + \cdots + 1 + \cdots + 0)$. Accordingly, the maps $c : \mathcal{C}(r) \to \mathrm{PrimEnd}_B^{\mathcal{C}}(r)$ admit factorizations

$$\begin{tikzcd} \mathcal{C}(r) \arrow[r, "\nabla_c"] \arrow[dr, dashed, "\nabla_\gamma"'] & \mathrm{PrimEnd}_B^{\mathcal{C}}(r) \\ & \mathrm{PrimOp}_B^{\mathcal{C}}(r) \arrow[u, hook, "\Theta"'] \end{tikzcd} .$$

and the induced coalgebra morphisms

$$\begin{tikzcd} \mathcal{C}(r) \arrow[r, "\nabla_c"] \arrow[dr, dashed, "\nabla_\gamma"'] & (\Gamma(\mathrm{PrimEnd}^{\mathcal{C}}_B(r)), \partial) \\ & (\Gamma(\mathrm{PrimOp}^{\mathcal{C}}_B(r)), \partial) \arrow[u, hook, "\Theta"'] \end{tikzcd}$$

return a factorization $\nabla_\gamma : \mathcal{C} \to \mathrm{HopfOp}^{\mathcal{C}}_B$ of the given operad morphism ∇_c.

Observe that this factorization defines automatically an operad morphism since $\nabla_\Theta : \mathrm{HopfOp}^{\mathcal{C}}_B \hookrightarrow \mathrm{HopfEnd}^{\mathcal{C}}_B$ is an injection. This assertion can also be checked directly from the explicit form of the map ∇_γ. □

The goal of the next paragraphs is to prove that any morphism of unital Hopf operads

$$\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}^{\mathcal{C}}_B,$$

where $\mathcal{Q}$ is connected and non-negatively graded, matches the same construction. This gives the result of theorem §4.C stated in the introduction of this section. In fact, our arguments uses only the unital unitary Hopf Λ_*-module structure of a unital Hopf operad. Therefore our claim holds naturally for connected unital unitary Hopf Λ_*-modules and not only for connected unital Hopf operads. Finally, we check the following statement:

§4.6.2. **Claim** (Compare with theorem §4.C)**.** *Any morphism of unital unitary Hopf Λ_*-modules $\nabla_\rho : M \to \mathrm{HopfOp}^{\mathcal{C}}_B$, where M is connected and non-negatively graded, makes commute the diagram*

$$\begin{tikzcd} M \arrow[rr, "\nabla_\rho"] \arrow[dr, "\epsilon"'] & & \mathrm{HopfOp}^{\mathcal{C}}_B \\ & \mathcal{C} \arrow[ur, "\nabla_\gamma"'] & \end{tikzcd}$$

Recall that a unital unitary Λ_*-module refers to a Λ_*-module equipped with a distinguished element $* \in M(0)$ that spans $M(0)$ and with a unit element $1 \in M(1)$ such that $\partial_1(1) = *$. Furthermore, a unital unitary Λ_*-module is connected if $M(1) = \mathbb{F}$ as in the case of a unital operad. These objects are equipped with a canonical augmentation $\epsilon : M \to \mathcal{C}$ given by the Λ_*-module operation $\eta_0^* : M(r) \to M(0) = \mathbb{F}$ associated to the initial map $\eta_0 : \emptyset \to \{1, \ldots, r\}$. In the case of an operad, the augmentation can be identified with the operadic composite with unital operations $\epsilon(p) = p(*, \ldots, *)$.

Any morphism of Hopf Λ_*-modules $\nabla_\rho : M \to \mathrm{HopfOp}^{\mathcal{C}}_B$ is determined by a collection of dg-module maps $\rho : M(r) \to \mathrm{PrimOp}^{\mathcal{C}}_B(r)$ as in the case of the commutative operad $M = \mathcal{C}$ since $\mathrm{HopfOp}^{\mathcal{C}}_B$ consists of quasi-cofree Hopf coalgebras. We check that these maps agree with the maps $\gamma : \mathcal{C}(r) \to \mathrm{PrimOp}^{\mathcal{C}}_B(r)$ considered in the proof of lemma §4.6.1. Accordingly, we have automatically $\rho = \gamma\epsilon$ so that $\nabla_\rho = \nabla_\gamma \cdot \epsilon$.

For our purpose we determine the components of $\mathrm{PrimOp}^{\mathcal{C}}_B(r)$ of degree $* \geq 0$. To begin with, we have the following easy observation:

§4.6.3. **Observation.** *In degree $* > 0$, we have $\Lambda\,\mathcal{C}(m)_* = 0$, for all $m > 0$.*

In degree $* = 0$, we have $\Lambda\,\mathcal{C}(m)_0 = \mathbb{F}$ if $m = 1$ and $\Lambda\,\mathcal{C}(m)_0 = 0$ for all $m > 1$. □

Proof. By definition, we have

$$\Lambda\,\mathcal{C}(m)_* = \Sigma^{1-m}\,\mathcal{C}(m)_* = \mathcal{C}(m)_{*+m-1}$$

(up to signs). Consequently, the module $\Lambda\mathcal{C}(m)_*$ is concentrated in degree $* = 1 - m$. The observation follows. $\square$

As a consequence, we obtain immediately:

§4.6.4. **Observation.** *In degree $* > 0$, we have $\mathrm{PrimOp}_B^{\mathcal{C}}(r)_* = 0$. In degree $* = 0$, we have $\mathrm{PrimOp}_B^{\mathcal{C}}(1)_0 = \mathbb{F}$ and $\mathrm{PrimOp}_B^{\mathcal{C}}(r)_0 = \prod_{m_*} \Lambda\mathcal{C}(m_1+\cdots+m_r)_0$ is reduced to components of the form $\Lambda\mathcal{C}(0+\cdots+1+\cdots+0)_0 = \mathbb{F}$, where $m_k = 0$ except for one index $k = i$ for which we have $m_i = 1$.* $\square$

In order to determine the components of a morphism $\mathcal{Q}(r) \to \mathrm{PrimOp}_B^{\mathcal{C}}(r)$, we consider again the Λ_*-module operation $\eta_i^* : M(r) \to M(1)$ associated to the map $\eta_i : \{1\} \to \{1,\dots,r\}$ such that $\eta_i(1) = i$. In fact, we observe that these operations isolate the degree 0 components of $\mathrm{PrimOp}_B^{\mathcal{C}}$. More formally, we have the following assertion:

§4.6.5. **Observation.** *In degree 0, we obtain $\mathrm{PrimOp}_B^{\mathcal{C}}(1)_0 = \Lambda\mathcal{C}(1) = \mathbb{F}$ and the operations*

$$\eta_i^* : \mathrm{PrimOp}_B^{\mathcal{C}}(r) \to \mathrm{PrimOp}_B^{\mathcal{C}}(1)$$

yield an isomorphism

$$(\eta_i^*)_i : \mathrm{PrimOp}_B^{\mathcal{C}}(r)_0 \xrightarrow{\simeq} \mathbb{F}^{\times r}.$$

Proof. By definition of the Λ_*-module structure of $\mathrm{PrimOp}_B^{\mathcal{C}}$ (see claim §4.4.4) the morphism

$$\eta_i^* : \prod_{m_*} \Lambda\mathcal{C}(m_1+\cdots+m_r)_0 \to \mathbb{F}$$

is an identical morphism on the component $\Lambda\mathcal{C}(m_1+\cdots+m_r)_0$ such that

$$m_k = \begin{cases} 0 & \text{for } k \neq i, \\ 1 & \text{for } k = i \end{cases}$$

and vanishes otherwise.

The claim follows immediately since the module $\mathrm{PrimOp}_B^{\mathcal{C}}(r)_0$ is the product of these components $\Lambda\mathcal{C}(0+\cdots+1+\cdots+0)_0$ according to the previous observation §4.6.4. $\square$

Then claim §4.6.2 arises as a corollary of the following result:

§4.6.6. **Observation.** *Suppose given a morphism of unital unitary Hopf Λ_*-modules $\nabla_\rho : M \to \mathrm{HopfOp}_B^{\mathcal{C}}$ induced by homogeneous maps of degree 0:*

$$\rho : M(r) \to \mathrm{PrimOp}_B^{\mathcal{C}}(r).$$

If we assume that M is connected, so that $M(1) = \mathbb{F}$, then the maps ρ are determined in degree $* = 0$ by the commutative triangle

$$\begin{tikzcd} M(r)_0 \arrow[rr] \arrow[dr, "(\eta_i^*)_i"'] & & \mathrm{PrimOp}_B^{\mathcal{C}}(r)_0 \arrow[dl, "\simeq", "(\eta_i^*)_i"'] \\ & \mathbb{F}^{\times r} & \end{tikzcd}.$$

If we assume that M is non-negatively graded, then the other components of ρ vanish since we observe that $\mathrm{PrimOp}_B^{\mathcal{C}}(r)_* = 0$ in degree $* > 0$.

Proof. Recall that the maps

$$\rho : M(r) \to \mathrm{PrimOp}_B^{\mathcal{C}}(r)$$

are given by the composite of the associated coalgebra morphisms

$$\nabla_\rho : M(r) \to (\Gamma(\mathrm{PrimOp}_B^{\mathcal{C}}(r)), \partial)$$

with the projection $\Gamma(\mathrm{PrimOp}^{\mathcal{C}}_B(r)) \to \mathrm{PrimOp}^{\mathcal{C}}_B(r)$. These projections commute with Λ_*-module operations by construction since $\mathrm{HopfOp}^{\mathcal{C}}_B = (\Gamma(\mathrm{PrimOp}^{\mathcal{C}}_B), \partial)$ is defined as a quasi-cofree Hopf Λ_*-module. Accordingly, so do the maps $\rho : M(r) \to \mathrm{PrimOp}^{\mathcal{C}}_B(r)$.

Recall also that the operad unit of $\mathrm{HopfOp}^{\mathcal{C}}_B$ is represented by the collection

$$1 = \{1^{\otimes n}\} \in \prod_n \mathrm{PrimOp}^{\mathcal{C}}_B(1).$$

Consequently, the projection $\mathrm{HopfOp}^{\mathcal{C}}_B(1) \to \mathrm{PrimOp}^{\mathcal{C}}_B(1)$ maps this unit to the element $1 \in \Lambda\,\mathcal{C}(1)$ that generates the degree 0 component of $\mathrm{PrimOp}^{\mathcal{C}}_B(1)$. Since a morphism of unitary Λ_*-modules is supposed to preserve units, we have $\nabla_\rho(1) = \{1^{\otimes n}\}$. As a consequence, if M is connected, then the component $\rho : M(1) \to \mathrm{PrimOp}^{\mathcal{C}}_B(1)$ of the map ρ is given by the identical morphism of $M(1) = \mathrm{PrimOp}^{\mathcal{C}}_B(1)_0 = \mathbb{F}$.

Then our claim follows from the commutativity of the diagrams

$$\begin{tikzcd} M(r) \arrow[r] \arrow[d, "\eta_i^*"] & \mathrm{PrimOp}^{\mathcal{C}}_B(r) \arrow[d, "\eta_i^*"] \\ M(1) \arrow[r] & \mathrm{PrimOp}^{\mathcal{C}}_B(1) \end{tikzcd} ,$$

for $r \geq 1$ and $i = 1, \dots, r$, and from the previous observation. $\square$

These verifications achieve the proof of claim §4.6.2 and hence of theorem §4.C. $\square$

As asserted in the section introduction, claim §4.6.2 implies that the morphism $\nabla_\gamma : \mathcal{C} \to \mathrm{HopfOp}^{\mathcal{C}}_B$ induces an isomorphism

$$\nabla_\gamma : \mathcal{C} \xrightarrow{\simeq} \mathrm{s}^1_* \, \mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{C}}_B)$$

that identifies the image of $\mathrm{HopfOp}^{\mathcal{C}}_B$ under the truncation functors with the commutative operad. As a consequence, we obtain that the augmentation morphism of an E_∞-operad induces an acyclic fibration of connected unital Hopf operads

$$\epsilon_* : \mathrm{s}^1_* \, \mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{E}}_B) \xrightarrow{\sim} \mathcal{C}$$

since truncation functors preserve fibrations and acyclic fibrations. As a corollary, for a connected unital Hopf operad $\mathcal{Q}$, the lifting problem considered in the section introduction is equivalent to adjoint lifting problems

$$\begin{tikzcd} & \mathrm{s}^1_* \, \mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{E}}_B) \arrow[d, "\sim"] \\ \mathcal{Q} \arrow[r] \arrow[ur, dashed, "\exists\nabla_\rho"] & \mathcal{C} \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} & \mathrm{HopfOp}^{\mathcal{E}}_B \arrow[d] \\ \mathcal{Q} \arrow[r] \arrow[ur, dashed, "\exists\nabla_\rho"] & \mathcal{C} \end{tikzcd}$$

which have automatically a solution if $\mathcal{Q}$ is cofibrant.

§4.7. **Prospects: actions of cellular operads.** Thinking about it, in our construction, we consider a natural cocellular decomposition of the Hopf operad of universal bar operations $\mathrm{HopfOp}^{\mathcal{P}}_B$ which arises from the degreewise filtration of the bar complex $B(A)$. In subsequent work, we plan to study operadic cellular structures giving rise to refinements of this cocellular decomposition. As alluded to in the memoir introduction, this might shed light on the algebraic structure of the bar complex for subclasses of particular algebras. This prospect motivates in part our presentation choices and the detailed accounts of §4.3-§4.5.

Toward effective constructions

§5. The explicit equations of Hopf operad actions

§5.1. **Introduction.** In this section we address the issue of constructing effectively Hopf operad actions on the bar construction. In fact, our results can supply explicit recursive constructions for the operations $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ associated to operad elements $q \in \mathcal{Q}(r)$. Our purpose is to give this elementary construction either as an illustration of our techniques or for a direct application.

To be explicit, we consider again the universal Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$, defined in §4.3, the operad of universal bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}$, defined in §4.4, and the canonical morphism $\nabla_\Theta : \mathrm{HopfOp}_B^{\mathcal{P}} \to \mathrm{HopfEnd}_B^{\mathcal{P}}$. According to theorem §4.A, the action of a unital Hopf operad $\mathcal{Q}$ on the bar complex is equivalent to a morphism $\nabla_\theta : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{P}}$. In §4.4 we observe that the natural morphism ∇_Θ is an isomorphism. Consequently:

Fact §5.A. *For a Σ_*-projective operad $\mathcal{P}$, any morphism $\nabla_\theta : \mathcal{Q} \to \mathrm{HopfEnd}_B^{\mathcal{P}}$ is equivalent to a composite*

$$\begin{tikzcd} \mathcal{Q} \arrow[rr, "\nabla_\theta"] \arrow[dr, dashed, "\nabla_\rho"'] & & \mathrm{HopfEnd}_B^{\mathcal{P}} \\ & \mathrm{HopfOp}_B^{\mathcal{P}} \arrow[ur, "\simeq", "\nabla_\Theta"'] & \end{tikzcd}.$$

As a byproduct, the action of a unital Hopf operad $\mathcal{Q}$ on the bar complex is actually determined by a morphism to the Hopf operad of universal bar operations $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{P}}$ such that $\nabla_\theta = \nabla_\Theta \nabla_\rho$.

Then, as an application of the structure results of §4, we obtain that the operations $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ satisfy the following characteristic properties:

Fact §5.B. *The components*

$$\begin{tikzcd} \Sigma A^{\otimes m_1+\cdots+m_r} \arrow[r, "\theta_{m_1,\dots,m_r}(q)"] \arrow[d, hook] & \Sigma A \\ B(A)^{\otimes r} \arrow[r, "\nabla_\theta(q)"] & B(A) \arrow[u] \end{tikzcd}$$

of an operation $\nabla_\theta(q)$ associated to an element $q \in \mathcal{Q}(r)$ are given by the evaluation of operations

$$\rho_{m_1,\dots,m_r}(q) \in \Lambda\mathcal{P}(m_1 + \cdots + m_r)$$

associated to q.

Recall that $\Lambda\mathcal{P}$ denotes the operadic suspension of $\mathcal{P}$: the operad whose algebras are suspensions ΣA of $\mathcal{P}$-algebras A.

Fact §5.C. *The operation $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ associated to an element $q \in \mathcal{Q}(r)$ has an expansion of the form*

$$\nabla_\theta(q)(\alpha_1,\dots,\alpha_r) = \sum_{n=1}^{\infty} \Big\{ \sum \Big[\underbrace{\theta(q^1)(\alpha_1^1,\dots,\alpha_r^1) \otimes \cdots \otimes \theta(q^n)(\alpha_1^n,\dots,\alpha_r^n)}_{\in \Sigma A^{\otimes n}} \Big] \Big\},$$

for elements in the bar complex $\alpha_1,\dots,\alpha_r \in B(A)$, where we consider the n-fold diagonals $\sum \alpha_i^1 \otimes \cdots \otimes \alpha_i^n \in B(A)^{\otimes n}$ of the tensors $\alpha_i \in B(A)$, the n-fold diagonals $\sum q^1 \otimes \cdots \otimes q^n \in \mathcal{Q}(r)^{\otimes n}$ of the operation $q \in \mathcal{Q}(r)$ and the homogeneous transformations

$$\theta_{m_*^j}(q^j) : \Sigma A^{\otimes m_1^j} \otimes \cdots \otimes \Sigma A^{\otimes m_r^j} \to \Sigma A$$

defined by the operations $\rho_{m^j_*}(q^j) \in \Lambda\mathcal{P}(m^j_1 + \cdots + m^j_r)$ *associated to the elements* $q^j \in \mathcal{Q}(r)$.

The arguments are surveyed in §5.2. According to these statements, our problem is reduced to the construction of appropriate maps $\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\mathcal{P}(m_1 + \cdots + m_r)$. In §5.2 we recall briefly the structure of the Hopf operad of natural bar operations $\mathrm{HopfOp}^{\mathcal{P}}_B$ and we make explicit the equations satisfied by the maps ρ_{m_*} associated to an operad morphism $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}^{\mathcal{P}}_B$. Our results are recorded in theorem §5.D in this subsection.

One can observe that these equations give rise to a recursive definition for the operations $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ associated to elements of a cellular cofibrant operad $\mathcal{Q}$. For the sake of precision, we state this recursive definition in a theorem, namely theorem §5.E in §5.3. Then we survey the abstract lifting arguments set in sections §3-§4 in order to prove this theorem and to give thorough justifications for this recursive construction.

To recapitulate: in §5.2, we give an elementary interpretation, in term of operations, of our abstract structure results; in §5.3, we survey our lifting arguments and we give a recursive construction of the action of an operad on the bar complex.

§5.2. **The expansion of operations on the bar complex.** As stated in the section introduction, the goal of this subsection is to make explicit the equations satisfied by maps $\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\mathcal{P}(m_1+\cdots+m_r)$ in order to obtain an elementary and effective characterization of the action of a unital Hopf operad on the bar construction. This result is obtained as a consequence of constructions of the previous section. Namely such a collection of maps is associated to coalgebra morphisms $\nabla_\rho : \mathcal{Q}(r) \to \mathrm{HopfOp}^{\mathcal{P}}_B(r)$ that specify an operad action on the bar complex if and only if they define a morphism of unital Hopf operad. Therefore we relate simply the equations of an operad morphism to equivalent properties for the maps ρ_{m_*}. For this aim we recall briefly the definition of the Hopf operad of universal bar operations $\mathrm{HopfOp}^{\mathcal{P}}_B$.

§5.2.1. *Recalls: the coalgebra structure of the universal Hopf operads.* Precisely, recall that the Hopf endomorphism operad $\mathrm{HopfEnd}^{\mathcal{P}}_B$, respectively the Hopf operad of universal bar operations $\mathrm{HopfOp}^{\mathcal{P}}_B$, is defined by quasi-cofree coalgebras such that

$$\mathrm{HopfEnd}^{\mathcal{P}}_B(r) = (\Gamma(\mathrm{PrimEnd}^{\mathcal{P}}_B(r)), \partial),$$
$$\text{where}\quad \mathrm{PrimEnd}^{\mathcal{P}}_B(r) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A),$$

respectively

$$\mathrm{HopfOp}^{\mathcal{P}}_B(r) = (\Gamma(\mathrm{PrimOp}^{\mathcal{P}}_B(r)), \partial),$$
$$\text{where}\quad \mathrm{PrimOp}^{\mathcal{P}}_B(r) = \prod_{m_1+\cdots+m_r>0} \Lambda\mathcal{P}(m_1+\cdots+m_r).$$

The isomorphism

$$\underbrace{(\Gamma(\mathrm{PrimOp}^{\mathcal{P}}_B), \partial)}_{\mathrm{HopfOp}^{\mathcal{P}}_B} \xrightarrow[\simeq]{\nabla_\Theta} \underbrace{(\Gamma(\mathrm{PrimEnd}^{\mathcal{P}}_B), \partial)}_{\mathrm{HopfEnd}^{\mathcal{P}}_B}$$

is induced by the canonical morphisms

$$\underbrace{\prod_{m_1+\cdots+m_r>0} \Lambda\mathcal{P}(m_1+\cdots+m_r)}_{\mathrm{PrimOp}^{\mathcal{P}}_B(r)} \xrightarrow{\Theta} \underbrace{\mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)}_{\mathrm{HopfEnd}^{\mathcal{P}}_B(r)}$$

which map an operad element $p \in \Lambda\mathcal{P}(m_1 + \cdots + m_r)$ to the associated natural operation $\Theta(p) : \Sigma A^{\otimes m_1+\cdots+m_r} \to \Sigma A$.

According to lemma §3.3.1, a coalgebra morphism $\nabla_\theta : \mathcal{Q}(r) \to \mathrm{HopfEnd}_B^{\mathcal{P}}(r)$ is determined by a homogeneous map of degree 0

$$\theta : \mathcal{Q}(r) \to \mathrm{PrimEnd}_B^{\mathcal{P}}(r)$$

such that $\theta = \pi\nabla_\theta$, where we consider the natural projection

$$\mathrm{HopfEnd}_B^{\mathcal{P}}(r) = (\Gamma(\mathrm{PrimEnd}_B^{\mathcal{P}}(r)), \partial) \xrightarrow{\pi} \mathrm{PrimEnd}_B^{\mathcal{P}}(r).$$

Similarly, a coalgebra morphism $\nabla_\rho : \mathcal{Q}(r) \to \mathrm{HopfOp}_B^{\mathcal{P}}(r)$ is determined by a homogeneous map $\rho : \mathcal{Q}(r) \to \mathrm{PrimOp}_B^{\mathcal{P}}(r)$ such that $\theta = \pi\nabla_\rho$. If we assume $\nabla_\theta = \nabla_\Theta\nabla_\rho$, then we have also the relation $\theta = \Theta\rho$.

The precise purpose of this subsection is to make explicit the equations satisfied by a collection of maps $\rho : \mathcal{Q}(r) \to \mathrm{HopfOp}_B^{\mathcal{P}}(r)$ so that the associated coalgebra morphisms $\nabla_\rho : \mathcal{Q}(r) \to \mathrm{HopfOp}_B^{\mathcal{P}}(r)$ define a morphism of unital Hopf operads.

§5.2.2. *On the expansion of operations.* Before we explain briefly that, for the actual operation $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ associated to an element $q \in \mathcal{Q}(r)$, the maps $\theta_{m_*}(q) : \Sigma A^{\otimes m_1+\cdots+m_r} \to \Sigma A$ considered in fact §5.B are determined by the components

$$\theta_{m_*} : \mathcal{Q}(r) \to \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

of the maps $\theta : \mathcal{Q}(r) \to \mathrm{PrimEnd}_B^{\mathcal{P}}(r)$. In fact, this relationship is obtained in §4.3.6 where we make explicit the evaluation product of a Hopf endomorphism operad

$$\mathrm{HopfEnd}_{B(A)}(r) \otimes B(A)^{\otimes r} \to B(A).$$

The expansion given in fact §5.C for an operation $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ comes also from this paragraph.

Recall simply that the Hopf endomorphism operad $\mathrm{HopfEnd}_{B(A)}$ consists of quasi-cofree coalgebras $\mathrm{HopfEnd}_{B(A)}(r) = (\Gamma(\mathrm{PrimEnd}_{B(A)}(A)), \partial)$ like the universal Hopf endomorphism operad $\mathrm{HopfEnd}_B^{\mathcal{P}}$ except that we replace the modules of natural transformations

$$\mathrm{PrimEnd}_B^{\mathcal{P}}(r) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

by the modules of homogeneous morphisms

$$\mathrm{PrimEnd}_{B(A)}(r) = \prod_{m_1+\cdots+m_r>0} \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

associated to the given $\mathcal{P}$-algebra A. The canonical morphism $\mathrm{HopfEnd}_B^{\mathcal{P}} \to \mathrm{HopfEnd}_{B(A)}$ is induced by the obvious maps

$$\mathit{Hom}_{A\in\mathcal{P}\,\mathrm{Alg}}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A) \to \mathit{Hom}(\Sigma A^{\otimes m_1+\cdots+m_r}, \Sigma A)$$

given by the specialization of a natural transformation to the given algebra A.

Therefore the assertions of §4.3.6 together with fact §5.A give exactly the claim of fact §5.B and the expansion of fact §5.C for an operation $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$.

The operations $\rho_{m_*}(q) \in \Lambda\mathcal{P}(m_1 + \cdots + m_r)$ that occur in facts §5.B-§5.C are also yielded by the components $\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\mathcal{P}(m_1 + \cdots + m_r)$ of the maps $\rho : \mathcal{Q}(r) \to \mathrm{PrimOp}_B^{\mathcal{P}}(r)$. Finally, we have the following theorem that fulfils the objective of this subsection:

Theorem §5.D. *A collection of maps*

$$\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\mathcal{P}(m_1 + \cdots + m_r)$$

determines a morphism of unital operads $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{P}}$ if and only if the following properties are satisfied:

(a) *for the unit element $1 \in \mathcal{Q}(1)$, we have*

$$\rho_{m_1}(1) = \begin{cases} 1 \in \Lambda\mathcal{P}(1) & \text{if } m_1 = 1, \\ 0 \in \Lambda\mathcal{P}(m_1) & \text{otherwise}; \end{cases}$$

(b) *for any element $q \in \mathcal{Q}(r)$, we have the permutation relation*

$$\rho_{m_1,\ldots,m_r}(w \cdot q) = w(m_1,\ldots,m_r) \cdot \rho_{m_{w(1)},\ldots,m_{w(r)}}(q),$$

for all permutations $w \in \Sigma_r$, and the Λ_*-module relation

$$\rho_{m_1,\ldots,m_r}(q) = \rho_{m_1,\ldots,\widehat{m_i},\ldots,m_r}(q \circ_i *),$$

for all collections m_* such that $m_i = 0$;

(c) *for a composite operation $p \circ_i q \in \mathcal{Q}(s+t-1)$, where $p \in \mathcal{Q}(s)$ and $q \in \mathcal{Q}(t)$, we have the composition relation*

$$\rho_{l_*}(p \circ_i q) = \sum_m \Bigl\{ \sum Sh_i(n_j^k) \cdot \rho_{m_*}(p)\bigl(1,\ldots,\rho_{n_*^1}(q^1),\ldots,\rho_{n_*^m}(q^m),\ldots,1\bigr) \Bigr\},$$

for indices $(m_*),(n_*)$ such that

$$\begin{aligned} &m_k = l_k, \quad \text{for } k = 1,\ldots,i-1, \\ &m_i = m \quad \text{ranges over positive integers}, \\ &m_k = l_{k+t-1} \quad \text{for } k = i+1,\ldots,s, \\ &n_k = l_{k+i-1} \quad \text{for } k = 1,\ldots,t \end{aligned}$$

and where we consider the partitions $n_k^1 + \cdots + n_k^m = n_k$ and the operations $\rho_{n_*^j}(q^j)$ associated to the m-fold diagonals $\sum q^1 \otimes \cdots \otimes q^n \in \mathcal{Q}(s)^{\otimes n}$ of the element $q \in \mathcal{Q}(s)$; on the right-hand side, the operations $\rho_{n_*^j}(q^j)$ are substituted to the entries $t = m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m$ of the operations $\rho_{m_*}(p)$;

(d) *for any element $q \in \mathcal{Q}(r)$, we have the differential relation*

$$\begin{aligned} \delta(\rho_{m_*}(q)) &= \rho_{m_*}(\delta(q)) + \rho_{m_*}(\partial(q)) \\ &- \sum_n \Bigl\{ \sum Sh(m_i^j) \cdot \mu_n(\rho_{m_*^1}(q^1),\ldots,\rho_{m_*^n}(q^n)) \Bigr\} \\ &\pm \sum_i \Bigl\{ \sum_{n,t} \rho_{m'_*}(q) \circ_t \mu_n \Bigr\}; \end{aligned}$$

in the first summation we consider the n-fold diagonals $\sum q^1 \otimes \cdots \otimes q^n \in \mathcal{Q}(r)^{\otimes n}$ of q and the partitions $m_i^1 + \cdots + m_i^n = m_i$; in the second summation we consider the collections m'_* such that $m'_* = m_*$ for $* \neq i$ and $m'_i = m_i + n - 1$ and t ranges over the interval $t = m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m_i$.

Proof. This theorem follows from direct applications of lemma §3.3.1 and from the explicit definition of the Hopf operad $\mathrm{HopfOp}_B^{\mathcal{P}}$: in general, as a Hopf operad structure is determined by coalgebra morphisms and since the Hopf operad $\mathrm{HopfOp}_B^{\mathcal{P}}$ consists of quasi-cofree coalgebras, it is sufficient to check relations onto $\mathrm{PrimOp}_B^{\mathcal{P}}$.

The properties (a-c) reflect the equations satisfied by an operad morphism $\nabla_\rho : \mathcal{Q}(r) \to \mathrm{HopfOp}^{\mathcal{P}}_B(r)$. Explicitly, the unit relation $\nabla_\rho(1) = 1$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "\eta"'] \arrow[dr, "\eta"] & \\ \mathcal{Q}(1) \arrow[rr, "\nabla_\rho"'] & & \mathrm{HopfOp}^{\mathcal{P}}_B(1) \end{tikzcd},$$

where we consider the coalgebra morphism $\eta : \mathbb{F} \to \mathcal{Q}(1)$, respectively $\eta : \mathbb{F} \to \mathrm{HopfOp}^{\mathcal{P}}_B(1)$, specified by the unit element $1 \in \mathcal{Q}(1)$, respectively $1 \in \mathrm{HopfOp}^{\mathcal{P}}_B(1)$. As explained above, the identity of the coalgebra morphisms $\nabla_\rho \eta = \eta$ is satisfied if and only if the composite of these morphisms with the projection $\pi : \mathrm{HopfOp}^{\mathcal{P}}_B(1) \to \mathrm{PrimOp}^{\mathcal{P}}_B(1)$ agree. Thus we obtain the equation

$$\pi\nabla_\rho(1) = \pi(1) \Leftrightarrow \rho(1) = \pi(1).$$

If we go back to the definition of the unit element of $\mathrm{HopfOp}^{\mathcal{P}}_B$ given in the proof of claims §4.4.4-§4.4.5, then we obtain exactly the relation (a) of the theorem.

Similarly, the permutation relation $\nabla_\rho(wq) = w\nabla_\rho(q)$ and the Λ_*-module relation $\nabla_\rho(q \circ_i *) = \nabla_\rho(q) \circ_i *$ are equivalent to the commutativity of diagrams of coalgebra morphisms. Namely:

$$\begin{tikzcd} \mathcal{Q}(r) \arrow[r, "\nabla_\rho"] \arrow[d, "w"] & \mathrm{HopfOp}^{\mathcal{P}}_B(r) \arrow[d, "w"] \\ \mathcal{Q}(r) \arrow[r, "\nabla_\rho"] & \mathrm{HopfOp}^{\mathcal{P}}_B(r) \end{tikzcd}, \quad \text{respectively} \quad \begin{tikzcd} \mathcal{Q}(r) \arrow[r, "\nabla_\rho"] \arrow[d, "\partial_i"] & \mathrm{HopfOp}^{\mathcal{P}}_B(r) \arrow[d, "\partial_i"] \\ \mathcal{Q}(r) \arrow[r, "\nabla_\rho"] & \mathrm{HopfOp}^{\mathcal{P}}_B(r) \end{tikzcd}.$$

Then the commutativity of these diagrams is equivalent to the relations

$$\pi(\nabla_\rho(wq)) = \pi(w\nabla_\rho(q)) \Leftrightarrow \rho(wq) = w\rho(q),$$
$$\text{respectively} \quad \pi(\nabla_\rho(q \circ_i *)) = \pi(\nabla_\rho(q) \circ_i *) \Leftrightarrow \rho(q \circ_i *) = \partial_i(\rho(q)).$$

If we go back to the definition of the Λ_*-module structure of $\mathrm{PrimOp}^{\mathcal{P}}_B$ given in the proof of claim §4.4.4, then we obtain immediately the relation (b) of the theorem.

Notice that the statements (a-b) assert exactly that the maps $\rho : \mathcal{Q}(r) \to \mathrm{PrimOp}^{\mathcal{P}}_B(r)$ define a morphism of unitary Λ_*-modules.

The composition relation $\nabla_\rho(p \circ_i q) = \nabla_\rho(p) \circ_i \nabla_\rho(q)$ is given by the diagram

$$\begin{tikzcd} \mathcal{Q}(s) \otimes \mathcal{Q}(t) \arrow[r, "\nabla_\theta \otimes \nabla_\theta"] \arrow[d, "\circ_i"] & \mathrm{HopfOp}^{\mathcal{P}}_B(s) \otimes \mathrm{HopfOp}^{\mathcal{P}}_B(t) \arrow[d, "\circ_i"] \\ \mathcal{Q}(s+t-1) \arrow[r, "\nabla_\theta"] & \mathrm{HopfOp}^{\mathcal{P}}_B(s+t-1) \end{tikzcd}.$$

Again this diagram commutes if and only if the involved morphisms agree on the quotient object $\mathrm{PrimOp}^{\mathcal{P}}_B(s+t-1)$ of $\mathrm{HopfOp}^{\mathcal{P}}_B(s+t-1)$. The quotient of the evaluation product

$$\mathrm{HopfOp}^{\mathcal{P}}_B(s) \otimes \mathrm{HopfOp}^{\mathcal{P}}_B(t) \xrightarrow{\circ_i} \mathrm{HopfOp}^{\mathcal{P}}_B(s+t-1)$$

is made explicit in the proof of claim §4.4.5. If we go back to this definition, then we obtain exactly the relation (c) of the theorem.

Finally, recall that a morphism of quasi-cofree coalgebras, like

$$\mathcal{Q}(r) \xrightarrow{\nabla_\rho} \underbrace{(\Gamma(\mathrm{PrimOp}^{\mathcal{P}}_B(r)), \partial)}_{\mathrm{HopfOp}^{\mathcal{P}}_B(r)},$$

commutes with differentials if and only if the associated map ρ satisfies the equation

$$\delta(\rho) + \beta\nabla_\rho = 0,$$

where $\beta : \Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r)) \to \mathrm{PrimOp}_B^{\mathcal{P}}(r)$ denotes the homogeneous map that determine the coderivation $\partial = \partial_\beta$ of the quasi-cofree coalgebra (see lemma §3.3.1). In the case of the Hopf operad of bar operations $\mathrm{HopfOp}_B^{\mathcal{P}}(r) = (\Gamma(\mathrm{PrimOp}_B^{\mathcal{P}}(r)), \partial)$ this coderivation ∂ is composed of components ∂_i^h, ∂^v associated to maps β_i^h, β^v (see claim §4.4.3). If we apply the formula given in the proof of claim §4.4.3, then we obtain exactly the relation (d) of the theorem. The terms

$$\sum_{n,t} \rho_{m'_*}(q) \circ_t \mu_n, \quad \text{respectively} \quad \sum_n \Big\{ \sum Sh(m_i^j) \cdot \mu_n(\rho_{m_*^1}(q^1), \dots, \rho_{m_*^n}(q^n)) \Big\},$$

represent precisely components of $\beta_h^i(\rho(q)) \in \mathrm{PrimOp}_B^{\mathcal{P}}(r)$, respectively $\beta^v(\rho(q)) \in \mathrm{PrimOp}_B^{\mathcal{P}}(r)$. ☐

§5.3. **Lifting process and effective constructions.** According to results of §4.5, for an E_∞-operad $\mathcal{E}$, a morphism $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$, that supplies effectively an operad action on the bar complex, can be obtained by lifting the morphism $\nabla_\gamma : \mathcal{C} \to \mathrm{HopfOp}_B^{\mathcal{C}}$ associated to the classical shuffle product for the bar complex of commutative algebras. Explicitly, a morphism $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}_B^{\mathcal{E}}$ is defined by a solution of the lifting problem

$$\begin{tikzcd} & & \mathrm{HopfOp}_B^{\mathcal{E}} \arrow[d, "\sim"] \\ \mathcal{Q} \arrow[r] \arrow[urr, dashed, "\nabla_\rho"] & \mathcal{C} \arrow[r, "\nabla_\gamma"'] & \mathrm{HopfOp}_B^{\mathcal{C}} \end{tikzcd} .$$

The purpose of this subsection is to survey our arguments in order to make this lifting process effective. As a byproduct, we obtain a recursive definition of the operation $\nabla_\theta(q) : B(A)^{\otimes r} \to B(A)$ associated to an element $q \in \mathcal{Q}(r)$. We state this result as a theorem in order to motivate the study of this subsection.

In order to obtain effective results we need to have an effective model of an E_∞-operad $\mathcal{E}$. Explicitly, we suppose given a computable strong deformation retract

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "\eta"'] & \mathcal{E} \arrow[l, shift right, "\epsilon"'] \arrow[loop right, "\nu"] \end{tikzcd} ,$$

where $\mathcal{E}$ is an E_∞-operad of which $\epsilon : \mathcal{E} \to \mathcal{C}$ is the augmentation. Similarly, we have to consider a cofibrant Hopf operad $\mathcal{Q}$ provided with a manageable cellular structure. To be explicit, we assume that $\mathcal{Q}$ is a connected unital Hopf operad that arises as the colimit of a sequence of Hopf operads

$$* = \mathcal{Q}^{-1} \to \mathcal{Q}^0 \to \cdots \to \mathcal{Q}^d \to \dots \to \mathrm{colim}_d\, \mathcal{Q}^d = \mathcal{Q}$$

obtained by pushouts

$$\begin{tikzcd} \mathcal{F}_*(C^d) \arrow[r, "f^d"] \arrow[d, "i^d"] & \mathcal{Q}^{d-1} \arrow[d, dashed] \\ \mathcal{F}_*(D^d) \arrow[r, dashed] & \mathcal{Q}^d \end{tikzcd} ,$$

where $i^d : \mathcal{F}_*(C^d) \to \mathcal{F}_*(D^d)$ is a morphism of free operads associated to a Reedy cofibration of Hopf Λ_*-modules $i^d : C^d \to D^d$. In addition we shall assume that i^d splits Σ_*-equivariantly degreewise so that D^d is identified as a dg-Σ_*-module with a direct sum $D^d = (C^d \oplus E^d, \partial)$ for a projective Σ_*-module E^d defined effectively as a direct summand of a finitely generated free Σ_*-module. For simplicity we can assume that E^d is a free Σ_*-module equipped with a finite set of generators ξ_κ. Equivalently, the operad $\mathcal{Q}$ is associated to a quasi-free reduced operad $\overline{\mathcal{Q}} = (\mathcal{F}(\bigoplus_{d=0}^\infty E^d), \partial)$ for modules of generators E^d such that $E^d(r) = \bigoplus_\kappa \mathbb{F}[\Sigma_r]\xi_\kappa$. The suboperads $\mathcal{Q}^d$ are determined by $\overline{\mathcal{Q}}^d = (\mathcal{F}(\bigoplus_{i=0}^d E^i), \partial)$. The differential of a basis element $\xi_\alpha \in E^d(r)$ is given by a sum $(\delta + \partial)(\xi_\kappa) \in \mathcal{Q}^d(r)$, where $\delta(\xi_\kappa) \in E^d(r)$ and $\partial(\xi_\kappa) \in \mathcal{Q}^{d-1}(r)$. According to results of the previous section, the Boardman-Vogt construction can supply explicit operads $\mathcal{Q} = W^\square(\mathcal{E})$ that satisfy these requirements.

In the abstract context, the morphism $\nabla_\theta : \mathcal{Q} \to \mathrm{HopfEnd}_B^{\mathcal{E}}$ can be obtained effectively by applications of cellular lifting properties. In an equivalent elementary fashion, the maps $\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\,\mathcal{E}(m_1+\cdots+m_r)$ that determine the action of $\mathcal{Q}$ on the bar construction are obtained inductively as solutions of differential equations. In the effective context, we can use the chain contraction $\nu : \mathcal{E}(m) \to \mathcal{E}(m)$ in order to specify solutions of these equations. Then our result takes the following form:

Theorem §5.E. *In the context set in the previous paragraphs, an appropriate collection of maps*

$$\rho_{m_*} : \mathcal{Q}(r) \to \Lambda\,\mathcal{E}(m_1 + \cdots + m_r)$$

that give rise to an action of $\mathcal{Q}$ on the bar construction can be defined recursively by the following requirements:

(a) *for the unit element $1 \in \mathcal{Q}(1)$, we have*

$$\rho_{m_1}(1) = \begin{cases} 1 \in \Lambda\,\mathcal{E}(1) & \text{if } m_1 = 1, \\ 0 \in \Lambda\,\mathcal{E}(m_1) & \text{otherwise}; \end{cases}$$

(b) *for a composite operation $p \circ_i q \in \mathcal{Q}(s+t-1)$, where $p \in \mathcal{Q}(s)$ and $q \in \mathcal{Q}(t)$, we have*

$$\rho_{l_*}(p \circ_i q) = \sum_m \Big\{ \sum Sh_i(n_j^k) \cdot \rho_{m_*}(p)\big(1, \ldots, \rho_{n_*^1}(q^1), \ldots, \rho_{n_*^m}(q^m), \ldots, 1\big) \Big\},$$

for indices $(m_*), (n_*)$ such that

$$\begin{aligned} &m_k = l_k \quad \text{for } k = 1, \ldots, i-1, \\ &m_i = m \quad \text{ranges over positive integers}, \\ &m_k = l_{k+t-1} \quad \text{for } k = i+1, \ldots, s, \\ &n_k = l_{k+i-1} \quad \text{for } k = 1, \ldots, t \end{aligned}$$

and where we consider the partitions $n_k^1 + \cdots + n_k^m = n_k$ and the operations $\rho_{n_*^j}(q^j)$ associated to the m-fold diagonals $\sum q^1 \otimes \cdots \otimes q^n \in \mathcal{Q}(s)^{\otimes n}$ of the element $q \in \mathcal{Q}(s)$; on the right-hand side, the operations $\rho_{n_*^j}(q^j)$ are substituted to the entries $t = m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m$ of the operations $\rho_{m_*}(p)$;

(c) *for a permutation $w \in \Sigma_r$, we have the relation*

$$\rho_{m_1,\ldots,m_r}(w \cdot q) = w(m_1, \ldots, m_r) \cdot \rho_{m_{w(1)},\ldots,m_{w(r)}}(q),$$

for any operation $q \in \mathcal{Q}(r)$;

(d) *for a generator $\xi_\kappa \in \mathcal{Q}(r)$, we set*

$$\begin{aligned}\rho_{m_1,\dots,m_r}(\xi_\kappa) &= \rho_{m_1,\dots,\widehat{m_i},\dots,m_r}(\xi_\kappa \circ_i *) \quad \textit{if } m_i = 0 \textit{ for some } i,\\ \rho_{m_1,\dots,m_r}(\xi_\kappa) &= \nu\Big\{\rho_{m_*}(\delta(\xi_\kappa)) + \rho_{m_*}(\partial(\xi_\kappa))\\ &\qquad - \sum_n \Big\{\sum Sh(m_i^j)\cdot \mu_n(\rho_{m_*^1}(\xi_\kappa^1),\dots,\rho_{m_*^n}(\xi_\kappa^n))\Big\}\\ &\qquad \pm \sum_i \Big\{\sum_{n,t} \rho_{m'_*}(\xi_\kappa)\circ_t \mu_n\Big\}\Big\} \quad \textit{otherwise};\end{aligned}$$

in the first summation we consider the n-fold diagonals $\sum \xi_\kappa^1 \otimes \cdots \otimes \xi_\kappa^n \in \mathcal{Q}(r)^{\otimes n}$ of the operation ξ_κ and the partitions $m_i^1 + \cdots + m_i^n = m_i$; in the second summation we consider the collections m'_* such that $m'_* = m_*$ for $* \neq i$ and $m'_i = m_i + n - 1$ and t ranges over the interval $t = m_1 + \cdots + m_{i-1} + 1, \dots, m_1 + \cdots + m_{i-1} + m_i$.

Thorough justifications of this recursive constructions can be obtained directly from the assertions of theorem §5.D. One checks essentially that the terms on the right-hand side of the equation of $\rho_{m_1,\dots,m_r}(\xi_\kappa)$ are determined by operations $\rho_{n_1,\dots,n_r}(q)$ which are either associated to elements $q \in \mathcal{Q}(r)$ of lower degree than ξ_κ or such that $n_1 + \cdots + n_r < m_1 + \cdots + m_r$.

In this subsection we give another proof of this theorem in order to illustrate the abstract lifting arguments of sections §3-§4.

§5.3.1. *Connectedness assumptions.* Throughout this subsection we assume that $\mathcal{Q}$ is connected and non-negatively graded. Consequently, our lifting problem can be simplified according to observations of the introduction of §4 (see also §4.6).

Explicitly, recall that we have truncation functors

$$\mathrm{dg}_{\mathbb{Z}}\,\mathrm{Op}^1_* \xrightarrow{\mathrm{s}^{\mathrm{dg}}_+} \mathrm{dg}_{\mathbb{N}}\,\mathrm{Op}^1_* \xrightarrow{\mathrm{s}^1_*} \mathrm{dg}_{\mathbb{N}}\,\mathrm{Op}^*_*$$

that target to the category formed by the non-negatively graded and connected unital Hopf operads. Furthermore, by adjunction, our lifting problem is equivalent to

$$\begin{tikzcd} & \mathrm{s}^1_*\,\mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{E}}_B) \arrow[r, hook] \arrow[d, "\sim"] & \mathrm{HopfOp}^{\mathcal{E}}_B \arrow[d] \\ \mathcal{Q} \arrow[ur, dashed, "\exists\nabla_\rho"] \arrow[r] & \mathcal{C} \arrow[r, "="] & \mathcal{C} \end{tikzcd}$$

since we have $\mathrm{s}^1_*\,\mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{C}}_B) = \mathcal{C}$. In the construction of the next paragraphs we do not need to introduce truncation functors explicitly. Thus we consider only a reduced lifting problem:

$$\begin{tikzcd} & \mathrm{HopfOp}^{\mathcal{E}}_B \arrow[d] \\ \mathcal{Q} \arrow[ur, dashed, "\exists\nabla_\rho"] \arrow[r] & \mathcal{C} \end{tikzcd}.$$

By adjunction, we are ensured that any lifting morphism in this diagram factorizes through $\mathrm{s}^1_*\,\mathrm{s}^{\mathrm{dg}}_+(\mathrm{HopfOp}^{\mathcal{E}}_B)$ and all the lifting problems considered in this paragraph are equivalent.

§5.3.2. *Lifting construction for a cellular cofibrant Hopf operad.* The idea is to define our lifting by induction by using cellular structures. For this aim we assume that $\mathcal{Q}$ arises as the colimit of a sequence of Hopf operads

$$* = \mathcal{Q}^{-1} \to \mathcal{Q}^0 \to \cdots \to \mathcal{Q}^d \to \cdots \to \mathrm{colim}_d\,\mathcal{Q}^d = \mathcal{Q}$$

obtained by pushouts

$$\begin{tikzcd} \mathcal{F}_*(C^d) \arrow[r, "f^d"] \arrow[d, "i^d"] & \mathcal{Q}^{d-1} \arrow[d, dashed] \\ \mathcal{F}_*(D^d) \arrow[r, dashed] & \mathcal{Q}^d \end{tikzcd},$$

where $i^d : \mathcal{F}_*(C^d) \to \mathcal{F}_*(D^d)$ is a morphism of free operads induced by a Reedy cofibration of unitary Λ_*-modules $i^d : C^d \to D^d$.

Then the lifting $\nabla_\rho : \mathcal{Q} \to \mathrm{HopfOp}^{\mathcal{E}}_B$ can be obtained as the colimit of morphisms $\nabla_\rho = \nabla^d_\rho : \mathcal{Q}^d \to \mathrm{HopfOp}^{\mathcal{E}}_B$ constructed by induction on d. Accordingly, we are reduced to specify inductively a lifting in the diagram of unitary Hopf Λ_*-modules

$$\begin{tikzcd} C^d \arrow[r] \arrow[d, "i^d"] & \mathcal{F}_*(C^d) \arrow[r] & \mathcal{Q}^{d-1} \arrow[r] & \mathrm{HopfOp}^{\mathcal{E}}_B \arrow[d] \\ D^d \arrow[rrr] \arrow[urrr, dashed, "\nabla_\rho"] & & & \mathcal{C} \end{tikzcd},$$

for all $d \geq 0$.

§5.3.3. *Cocellular structures and lifting constructions.* In order to obtain these Hopf Λ_*-module liftings, we consider the cocellular decomposition of $\mathrm{HopfOp}^{\mathcal{E}}_B$ introduced in §4.5. Explicitly, recall that $\mathrm{HopfOp}^{\mathcal{E}}_B$ is the limit of a tower of quasi-cofree Hopf Λ_*-modules $\mathrm{ck}_m \mathrm{HopfOp}^{\mathcal{E}}_B$ such that $\mathrm{ck}_m \mathrm{HopfOp}^{\mathcal{E}}_B = (\Gamma(\mathrm{ck}_m \mathrm{PrimOp}^{\mathcal{E}}_B), \partial)$. Furthermore, the projection morphism

$$\mathrm{p}_m : \mathrm{ck}_m \mathrm{HopfOp}^{\mathcal{E}}_B \to \mathrm{ck}_{m-1} \mathrm{HopfOp}^{\mathcal{E}}_B$$

fits in a coextension diagram

$$\begin{tikzcd} (\Gamma(K), \partial) \arrow[r] \arrow[d] & \Gamma(\Delta^1 \wedge K) \arrow[d] \\ (\Gamma(L), \partial) \arrow[r] & \Gamma(S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L) \end{tikzcd},$$

where we let $K = \mathrm{ck}_m \mathrm{PrimOp}^{\mathcal{E}}_B$ and $L = \mathrm{ck}_{m-1} \mathrm{PrimOp}^{\mathcal{E}}_B$.

The lifting $\nabla_\rho : D^d \to \mathrm{HopfOp}^{\mathcal{E}}_B$ can be obtained as the limit of morphisms $\nabla_\rho = \nabla_{\rho_m} : D^d \to \mathrm{ck}_m \mathrm{HopfOp}^{\mathcal{E}}_B$ constructed by induction on m. Accordingly, we are reduced to specify inductively a lifting in the diagram of unitary Λ_*-modules

$$\begin{tikzcd} C^d \arrow[r] \arrow[d, "i^d"] & (\Gamma(K), \partial) \arrow[r] & \Delta^1 \wedge K \arrow[d] \\ D^d \arrow[r] \arrow[urr, dashed, "\tilde{\rho}"] & (\Gamma(L), \partial) \arrow[r] & S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L \end{tikzcd},$$

for all $m \geq 0$.

Recall that $\Delta^1 \wedge K = (01 \otimes K \oplus 1 \otimes K, \partial)$, for homogeneous elements 01 and 1 such that $\partial(01) = 1$. Furthermore, we observe in §3.3 (see more especially fact §3.3.4) that a morphism $\tilde{\rho} : D^d \to \Delta^1 \wedge K$ has the form

$$\tilde{\rho}(\xi) = -01 \otimes \delta(\rho)(\xi) + 1 \otimes \rho(\xi),$$

for a homogeneous map $\rho : D^d \to K$ of degree 0. In fact, the morphisms $\nabla_\rho = \nabla_{\rho_m} : D^d(r) \to \mathrm{ck}_m \mathrm{HopfOp}^{\mathcal{E}}_B(r)$ can be identified with the coalgebra morphisms induced by these maps $\rho = \rho_m : D^d(r) \to \mathrm{ck}_m \mathrm{PrimOp}^{\mathcal{E}}_B(r)$ (see §3.3.6 and claim §3.3.7). Accordingly, in an elementary fashion, the lifting $\tilde{\rho}$ determines the components

$\rho_m(\xi) = \{\rho_{m_*}(\xi)\} \in \prod_{m \geq m_1 + \cdots + m_r > 0} \Lambda\,\mathcal{E}(m_1 + \cdots + m_r)$ of the collection of operations associated to an element $\xi \in E^d(r)$.

In the next paragraph we survey the definition of the coextension diagram above in order to give an elementary interpretation of this lifting construction.

§5.3.4. *The elementary interpretation of the lifting process.* Recall that $\Delta^1 \wedge K = (01 \otimes K \oplus 1 \otimes K, \partial)$. We have similarly

$$S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L = (01 \otimes K \oplus 1 \otimes L, \partial)$$

and the canonical map

$$\Delta^1 \wedge K \xrightarrow{(\sigma \wedge K, \Delta^1 \wedge \mathrm{p}_m)} S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L$$

can be identified with the obvious morphism

$$(01 \otimes K \oplus 1 \otimes K, \partial) \xrightarrow{(01 \otimes \mathrm{Id}, 01 \otimes \mathrm{p}_m)} (01 \otimes K \oplus 1 \otimes L, \partial)$$

induced on one summand by the identity of K and on the other summand by the projection morphism $\mathrm{p}_m : K \to L$ (see §3.3 and more especially observation §3.3.8).

The precise construction of the commutative square

$$\begin{tikzcd} (\Gamma(K), \partial) \arrow[r] \arrow[d] & \Delta^1 \wedge K \arrow[d] \\ (\Gamma(L), \partial) \arrow[r] & S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L \end{tikzcd}$$

is given in §3.3.6. Recall that the morphism

$$(\Gamma(K), \partial) \to 01 \otimes K \oplus 1 \otimes K$$

is defined on the summand $01 \otimes K \subset 01 \otimes K \oplus 1 \otimes K$ by the homogeneous maps $\beta = \beta_i^h, \beta^v$ that determine the coderivations $\partial = \partial_i^h, \partial^v$ of the quotient Hopf operad $\mathrm{ck}_m \operatorname{HopfOp}_B^{\mathcal{E}}$. The other component of this morphism is given simply by the canonical projection of the cofree coalgebra $\Gamma(K)$. The morphism

$$(\Gamma(L), \partial) \to 01 \otimes K \oplus 1 \otimes L$$

is defined similarly once we observe that the homogeneous maps $\beta = \beta_i^h, \beta^v$ that determine the coderivations of the Hopf operad $\operatorname{HopfOp}_B^{\mathcal{E}}$ admit factorizations

$$\begin{tikzcd} \Gamma(\operatorname{PrimOp}_B^{\mathcal{E}}) \arrow[r, "\beta"] \arrow[d] & \operatorname{PrimOp}_B^{\mathcal{E}} \arrow[d] \\ \Gamma(K) \arrow[r, "\beta"] \arrow[d] & K \arrow[d] \\ \Gamma(L) \arrow[r, "\beta"] \arrow[ur, dashed] & L \end{tikzcd} .$$

Accordingly, in the lifting problem considered in the previous paragraph

$$\begin{tikzcd} C^d \arrow[r] \arrow[d] & 01 \otimes K \oplus 1 \otimes K \arrow[d] \\ D^d \arrow[r, "\tilde{\rho}"'] \arrow[ur, dashed, "\tilde{\rho}"] & 01 \otimes K \oplus 1 \otimes L \end{tikzcd} ,$$

the lower horizontal morphism is given by $\tilde{\rho}(\xi) = 01 \otimes \beta\nabla_{\rho_{m-1}}(\xi) + 1 \otimes \rho_{m-1}(\xi)$, where we consider the map $\rho_{m-1} : D^d \to L = \mathrm{ck}_{m-1} \mathrm{PrimOp}_B^{\mathcal{E}}$ specified by induction, the induced coalgebra morphism $\nabla_{\rho_{m-1}} : D^d \to (\Gamma(L), \partial)$ and the map $\beta : \Gamma(L) \to K$ defined by a quotient of the total coderivation of $\mathrm{HopfOp}_B^{\mathcal{E}}$.

Therefore the lifting morphism $\tilde{\rho} : D^d \to \Delta^1 \wedge K$ is associated to a map $\rho_{m-1} : D^d \to K = \mathrm{ck}_m \mathrm{PrimOp}_B^{\mathcal{E}}$ that can be characterized by the relation $\mathrm{p}_m \rho_m = \rho_{m-1}$ and by the equation

$$\delta(\rho_m(\xi)) - \rho_m(\delta(\xi)) - \rho_m(\partial(\xi)) = -\beta\nabla_{\rho_{m-1}}(\xi).$$

In fact, for a component $\rho_{m_*}(\xi)$ of $\rho_m(\xi)$, we recover the equation of theorem §5.D:

$$\begin{aligned}\delta(\rho_{m_*}(\xi)) - \rho_{m_*}(\delta(\xi)) - \rho_{m_*}(\partial(\xi)) = \\ - \sum_n \Big\{ \sum Sh(m_i^j) \cdot \mu_n(\rho_{m_*^1}(\xi^1), \dots, \rho_{m_*^n}(\xi^n)) \Big\} \\ \pm \sum_i \Big\{ \sum_{n,t} \rho_{m'_*}(\xi) \circ_i \mu_n \Big\}.\end{aligned}$$

The important point addressed in this lifting process is that the right-hand side of this equation consists of components $\rho_{n_*}(q)$ such that $n_1 + \dots + n_r \leq m-1$ and that are already specified by the inductive construction. Similarly, as we assume $\partial(\xi) \in C^d$, the term $\rho_{m_*}(\delta(\xi))$ is determined by the map

$$C^d \to \mathcal{Q}^{d-1} \xrightarrow{\nabla_\rho^{d-1}} \mathrm{HopfOp}_B^{\mathcal{E}}$$

specified at a previous stage of this inductive construction.

Finally, the lifting $\tilde{\rho} : D^d \to \Delta^1 \wedge K$ has to be defined as a morphism of unitary Λ_*-modules. Therefore, in the next paragraph, we survey the construction of liftings in the Reedy model category of Λ_*-modules in order to achieve our construction.

§5.3.5. *On Λ_*-module liftings and the Reedy model structure.* In fact, in view of the definition of the Reedy model structure (see more especially the proof of claim §1.3.5 for properties M4.i-ii), the components of the lifting morphism $\widetilde{\rho} : D^d(r) \to \Delta^1 \wedge K(r)$ can be obtained by induction on r as lifting morphisms in the diagrams of Σ_r-modules

$$\begin{tikzcd} C^d(r) \arrow[r] \arrow[d] & M(r) \arrow[d] \\ D^d(r) \arrow[r] \arrow[ur, dashed] & \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r) \end{tikzcd} ,$$

where $M = K \wedge \Delta^1$ and $N = S^1 \wedge K \times_{S^1 \wedge L} \Delta^1 \wedge L$.

We make the cartesian product $\mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r)$ explicit in order to make this lifting process effective. In fact, we have clearly

$$\begin{aligned}\mathbb{M}M(r) = 01 \otimes \mathbb{M}K(r) \oplus 1 \otimes \mathbb{M}K(r) \\ \text{and} \quad \mathbb{M}M(r) \times_{\mathbb{M}N(r)} N(r) = 01 \otimes K(r) \oplus 1 \otimes (\mathbb{M}K(r) \times_{\mathbb{M}L(r)} L(r)).\end{aligned}$$

By definition of the Λ_*-module structure of $\mathrm{PrimOp}_B^{\mathcal{E}}$, we obtain

$$\mathbb{M}K(r) \times_{\mathbb{M}L(r)} L(r) = \prod_{m_*}{}' \Lambda\, \mathcal{E}(m_1 + \dots + m_r),$$

where $\prod'_{m_*}$ ranges over all collections $m \geq m_1 + \dots + m_r > 0$ such that $m_i = 0$ for some i or $m - 1 \geq m_1 + \dots + m_r > 0$ (compare with observation §4.5.11). Furthermore, the matching morphism $K(r) \to \mathbb{M}K(r) \times_{\mathbb{M}L(r)} L(r)$ is given by the obvious projection of $K(r) = \prod_{m_*} \Lambda\,\mathcal{E}(m_1 + \cdots + m_r)$ onto the components of $\mathbb{M}K(r) \times_{\mathbb{M}L(r)} L(r)$.

As a conclusion the maps $\rho_{m_*} : D^d(r) \to \Lambda\,\mathcal{E}(m_1 + \cdots + m_r)$ that determine our lifting $\tilde{\rho} : D^d(r) \to \Delta^1 \wedge K(r)$ can be characterized by the following properties:
– for the unit element $1 \in D^d(1)$, we have

$$\rho_{m_1}(1) = \begin{cases} 1 \in \Lambda\,\mathcal{E}(1) & \text{if } m_1 = 1, \\ 0 \in \Lambda\,\mathcal{E}(m_1) & \text{otherwise} \end{cases}$$

so that $\tilde{\rho}$ defines a morphism of unitary Λ_*-modules;
– for a permutation $w \in \Sigma_r$, we have the equivariance relations

$$\rho_{m_1,\dots,m_r}(w \cdot \xi) = w(m_1, \dots, m_r) \cdot \rho_{m_{w(1)},\dots,m_{w(r)}}(\xi)$$

which imply that $\tilde{\rho}$ commutes the action of w; if $m_i = 0$ for some i, then we have

$$\rho_{m_1,\dots,m_r}(\xi) = \rho_{m_1,\dots,\widehat{m_i},\dots,m_r}(\partial_i \xi)$$

so that the composite of $\tilde{\rho}$ with the matching morphism $K(r) \to \mathbb{M}K(r) \times_{\mathbb{M}L(r)} L(r)$ matches the previously defined components of $\tilde{\rho}$;
– otherwise we have the differential equation

$$\begin{aligned} \delta(\rho_{m_*}(\xi)) - \rho_{m_*}(\delta(\xi)) - \rho_{m_*}(\partial(\xi)) = \\ - \sum_n \Big\{ \sum Sh(m_i^j) \cdot \mu_n(\rho_{m_*^1}(\xi^1), \dots, \rho_{m_*^n}(\xi^n)) \Big\} \\ \pm \sum_i \Big\{ \sum_n \rho_{m'_*}(\xi) \circ_i \mu_{n,t} \Big\}, \end{aligned}$$

where in the first summation we consider the n-fold diagonals $\sum \xi^1 \otimes \cdots \otimes \xi^n \in D^d(r)^{\otimes n}$ of the element ξ and the partitions $m_i^1 + \cdots + m_i^n = m_i$, in the second summation we consider the collections m'_* such that $m'_* = m_*$ for $* \neq i$ and $m'_i = m_i + n - 1$ and t ranges over the interval $t = m_1 + \cdots + m_{i-1} + 1, \dots, m_1 + \cdots + m_{i-1} + m_i$.

Clearly, if we assume that the morphism $i^d : C^d(r) \to D^d(r)$ splits equivariantly so that $D^d(r) = (C^d(r) \oplus E^d(r), \partial)$ for a finitely generated free Σ_r-module $E^d(r) = \bigoplus_\kappa \mathbb{F}[\Sigma_r]\xi_\kappa$, then one can fix easily a map $\rho_m : D^d(r) \to K(r)$ that fulfils our requirements. In fact, it is sufficient to specify solutions of our differential equation for generators $\xi = \xi_\kappa$ of $E^d(r)$. If the operad $\mathcal{E}(r)$ is equipped with a contracting homotopy $\nu : \mathcal{E}(r) \to \mathcal{E}(r)$ as stated in the introduction, then a solution can be specified by the formula of theorem §5.E:

$$\begin{aligned} \rho_{m_1,\dots,m_r}(\xi_\kappa) = \nu\Big\{ \rho_{m_*}(\delta(\xi_\kappa)) + \rho_{m_*}(\partial(\xi_\kappa)) \\ - \sum_n \Big\{ \sum Sh(m_i^j) \cdot \mu_n(\rho_{m_*^1}(\xi_\kappa^1), \dots, \rho_{m_*^n}(\xi_\kappa^n)) \Big\} \\ \pm \sum_i \Big\{ \sum_{n,t} \rho_{m'_*}(\xi_\kappa) \circ_t \mu_n \Big\} \Big\}. \end{aligned}$$

Hence this comprehensive survey of our lifting constructions gives thorough justifications for the recursive construction of theorem §5.E. ☐

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We put in a special reference for the article [B1] which provides a background and motivations for this research project. Otherwise numerical references to the bibliography are given according to the author alphabetical order, as usual.

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Index and notation glossary

Index

Notation glossary

c^1_*: the truncation functor, left adjoint to the category embedding i^1_*, 15, see also 27 for Hopf objects
CoAlg^a: the category of non-augmented coassociative coalgebras, 76
CoAlg^a_+: the category of augmented coassociative coalgebras, 24, 76
CoAlg^a_*: the category of augmented unitary coassociative coalgebras, 25, 76
CoAlg^a_0: the category of connected augmented unitary coassociative coalgebras, 76, 83
$\mathbb{D}^\tau$: the cubical chain complex build on the internal edges of a tree τ, 33
$\Delta^1 \wedge E$: the cone of a dg-module E, 61
∂_i: the operadic composite with a unital operation in an operad or the corresponding operation in a Λ_*-module, 11
∂_α: the coalgebra coderivation induced by a map α, 60, 68
∂^h_i: the coalgebra coderivation induced by the bar coderivations on the source in $\mathrm{HopfEnd}_{B(A)}$, $\mathrm{HopfEnd}^{\mathcal{P}}_B$ and $\mathrm{HopfOp}^{\mathcal{P}}_B$, 88, 89, 93
∂^v: the coalgebra coderivation induced by the bar coderivation on the target in $\mathrm{HopfEnd}_{B(A)}$, $\mathrm{HopfEnd}^{\mathcal{P}}_B$ and $\mathrm{HopfOp}^{\mathcal{P}}_B$, 88, 89, 93
ϵ: the augmentation of an augmented object in a category, 11, of a unital operad, 12, of a coalgebra 24, 76, of the Boardman-Vogt construction, 29, of the chain interbal, 31
$E(\tau)$: the edge set of a tree, 32
$E'(\tau)$: the set of internal edges in a tree, 32
η_0: the initial morphism of Λ_*, 14
η^0: the 0-face of the chain interval, 31
η^1: the 1-face of the chain interval, 31
ev: the evaluation morphism for internal hom objects in general, 78
ev_Γ: the internal evaluation morphism for cofree coalgebras, 77, 86
ev_T: the internal evaluation morphism for tensor coalgebras, 84, 86
$\mathbb{F}$: the ground field, 10
$\mathcal{F}(M)$: the free non-unital operad, 13
$\mathcal{F}_*(M)$: the free unital operad, 14
$\mathcal{F}_*(\Gamma)$: the free unital operad generated by a Hopf Λ_*-module, 25
$\Gamma(V)$: the cofree coalgebra, 56
$\overline{\Gamma}(V)$: the unit cokernel of the cofree coalgebra, 76
$\Gamma(M)$: the cofree Hopf Λ_*-module, 67
$\mathit{HopfHom}(L, M)$: the morphism coalgebra of unitary coalgebras L and M, 76
$\mathrm{HopfEnd}_\Gamma$: the Hopf endomorphism operad of a coalgebra, 86
$\mathrm{HopfEnd}^{\mathcal{P}}_B$: the Hopf endomorphism operad of the bar construction, 86
$\mathrm{HopfOp}^{\mathcal{P}}_B$: the Hopf operad of unversal bar operations, 92
HopfOp^1_*: the category of unitary Hopf operads, 25
HopfOp^*_*: the category of connected unital unitary Hopf operads, 27
$\mathbb{I}$: the chain interval, 31
I_v: the entry set of a vertex in a tree, 32
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