Ecole Graduée
MADIS

Université de Lille

Laboratoire
Paul Painlevé

Habilitation à diriger des recherches

Discipline : Mathématiques

présentée par

Arthur Renaudineau

Structures réelles phasées sur les variétés tropicales et patchwork combinatoire.

Soutenue le 25 septembre 2025

COMPOSITION DU JURY :

M.	Charles Favre	Ecole polytechnique
M.	Mark Gross	Université de Cambridge
Mme	Hannah Markwig	Université de Tuebingen
M.	Patrick Popescu-Pampu	Université de Lille
M.	Jean-Yves Welschinger	Université de Lyon 1

A Daphné, Elena, Luna et Valentina.

"Black
then
white are
all I see
in my infancy.
Red and yellow then came to be,
reaching out to me.
Lets me see."
Tool, Lateralus.

REMERCIEMENTS

Je souhaite tout d'abord exprimer ma profonde gratitude à Mark Gross, Hannah Markwig et Jean-Yves Welschinger d'avoir accepté d'être rapporteurs de ce mémoire. Un immense merci également à Patrick Popescu-Pampu, qui a accepté d'en être mon garant et qui m'a accompagné avec bienveillance dans toutes les étapes administratives. Je suis aussi très reconnaissant à Charles Favre d'avoir accepté de faire partie du jury.

Le contenu de ce mémoire doit énormément à mes collaborateurs. Merci pour toutes ces heures passées ensemble à faire des mathématiques, pour l'accueil toujours chaleureux que j'ai reçu dans vos universités respectives, et pour vos visites à Lille. Je pense en particulier à Erwan Brugallé, Benoît Bertrand, Lionel Lang, Diego Matessi, Grisha Mikhalkin, Johannes Rau et Kris Shaw. Un grand merci également à Ilia Itenberg pour l'intérêt qu'il a toujours porté à mes recherches.

Depuis 2018, j'ai le plaisir d'être Maître de Conférences à Lille, au Laboratoire Paul Painlevé. Je tiens à remercier l'ensemble de ses membres, scientifiques comme administratifs, pour leur accueil et pour l'ambiance détendue qui règne aussi bien dans les couloirs que dans les salles de convivialité. J'ai une pensée particulière pour Paolo Aceto, Amaël Broustet, Jules Chenal, Samuel Dentan, Youssef Hantout, Patrick Popescu-Pampu et Antoine Touzé pour toutes les discussions mathématiques que nous avons partagées. Je remercie chaleureusement Patrick Popescu-Pampu pour l'organisation hebdomadaire du séminaire « Géométrie des espaces singuliers », ainsi que pour les nombreuses occasions qu'il m'a offertes d'y présenter mes travaux et pour son soutien constant, en invitant notamment plusieurs de mes collaborateurs afin qu'ils puissent également y exposer leurs recherches.

Table des matières

Chapitre 1

Introduction

1.1 French version

Ce mémoire est dédié à certains aspects du *patchwork*, une méthode découverte par Oleg Viro aux débuts des années 80 permettant de construire des variétés algébriques réelles en recollant certains patches, qui sont des morceaux de variétés algébriques réelles plus simples. On s'intéressera dans ce mémoire au cas où les patches sont les plus simples possibles : des morceaux d'hyperplans (ou éventuellement des espaces linéaires en codimension plus élevée). Dans ce cadre, la méthode prend le nom de "patchwork combinatoire primitif", ou plus simplement "patchwork primitif". Elle devient en effet un jeu purement combinatoire qui permet de construire des variétés algébriques réelles. La question suivante est encore largement ouverte :

Question. *Quelle peut-être la topologie d'une variété algébrique réelle obtenue par patchwork primitif ?*

Résultats principaux de l'article [RS23] avec Kris Shaw et de l'article [ARS21] avec Charles Arnal et Kris Shaw

Un des résultats principal de ce mémoire est le suivant. Les nombres de Betti b_q sont considérés à coefficients dans le corps à deux éléments $\mathbb{F}_2$.

Théorème 1. *Soit V une hypersurface algébrique réelle obtenue par un patchwork primitif dans une variété torique lisse et projective de dimension n. Alors pour tout $q \geq 0$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \ \textit{si}\ q = (n-1)/2, \\ h^{q,n-1-q}(\mathbb{C}V) + h^{q,q}(\mathbb{C}V) \ \textit{sinon.} \end{cases}$$

et le corollaire ci-dessous répond à une conjecture d'Itenberg [Ite17] :

Corollaire 1. *Soit V une hypersurface algébrique réelle de $\mathbb{RP}^n$ obtenue par un patchwork primitif. Alors pour tout entier $0 \leq q \leq n-1$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \ \textit{si}\ q = (n-1)/2, \\ h^{q,n-1-q}(\mathbb{C}V) + 1 \ \textit{sinon.} \end{cases}$$

Notre preuve du Théorème 1 utilise une description tropicale du patchwork (voir Section 3.5) en termes de *structures réelles phasées.* Nous introduisons alors une suite spectrale $E^r_{q,p}$ qui converge vers l'homologie de $\mathbb{R}V$, ce qui implique que les nombres de Betti de $\mathbb{R}V$

sont égaux à la somme des dimensions des $\mathbb{F}_2$-espaces vectoriels apparaissant à la page infinie :

$$b_q(\mathbb{R}V) = \sum_{p=0}^{n} \dim E^{\infty}_{q,p}.$$

La première page de cette suite spectrale est isomorphe à l'homologie tropicale à coefficients dans $\mathbb{F}_2$ d'une hypersurface tropicale X duale à la triangulation utilisée pour le patchwork :

$$E^1_{q,p} \simeq H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

Cette homologie tropicale est une homologie à coefficients non-constants sur des complexes polyhédraux. Les coefficients $\mathcal{F}_p(x)$ sont des $\mathbb{Z}$-modules qui ne dépendent que de la face à laquelle appartient le point x (voir Section 3.7 pour plus de détails). Nous obtenons alors le théorème suivant :

Théorème 2. *Soit V une hypersurface algébrique réelle obtenue par un patchwork primitif à partir d'une hypersurface tropicale X dans une variété lisse et projective de dimension n. Alors*

$$b_q(\mathbb{R}V) \leq \sum_{p=0}^{n-1} \dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

Le cadre naturel de ce théorème va au delà des hypersurfaces : il s'énonce naturellement pour les sous-variétés tropicales non-singulières de variétés toriques tropicales et munies d'une structure réelle phasée (voir Théorème 9). Nous démontrons ensuite que dans le cas des hypersurfaces les dimensions des groupes d'homologies tropicales de X sont égales aux nombres de Hodge de V.

$$\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2) = h^{p,q}(\mathbb{C}V).$$

Pour ce faire nous procédons en trois étapes. Premièrement, nous démontrons que si $p+q < n-1$, alors les groupes d'homologies tropicales de X sont égaux à ceux de la variété torique tropicale ambiante.

Théorème 3. *Soit X une hypersurface tropicale duale à une triangulation primitive convexe d'un polytope régulier Δ de dimension n. Alors pour tout $p + q < n - 1$*

$$H_q(X, \mathcal{F}_p) \simeq H_q(\mathbb{T}\Sigma_\Delta, \mathcal{F}_p)$$

La notation $\mathbb{T}\Sigma_\Delta$ indique la variété torique tropicale associée à Δ (voir Section 3.1). Nous utilisons ensuite ce théorème et la dualité de Poincaré tropicale [JRS18] pour montrer que les groupes d'homologie tropicale sont sans torsion et nous déterminons leurs dimensions :

Théorème 4. *Soit X une hypersurface tropicale duale à une triangulation primitive d'un polytope régulier Δ de dimension n. Alors pour tout p et q, les $\mathbb{Z}$-modules $H_q(X, \mathcal{F}_p)$ sont sans torsions, et de plus pour $p + q \neq n - 1$, on a*

$$\mathrm{rk} H_q(X, \mathcal{F}_p) = \mathrm{rk} H_q(\mathbb{T}\Sigma_\Delta, \mathcal{F}_p) = h^{p,q}(\mathbb{C}\Sigma_\Delta).$$

La notation $\mathbb{C}\Sigma_\Delta$ indique la variété torique complexe associée à Δ. La dernière étape consiste à comparer le genre de Hirzebruch tropical au genre de Hirzebruch complexe :

Théorème 5. *Soit X une hypersurface tropicale duale à une triangulation primitive d'un polytope régulier Δ de dimension n, et soit $\mathbb{C}V$ une hypersurface générique de polytope de Newton Δ compactifiée dans $\mathbb{C}\Sigma_\Delta$. Alors*

$$\chi(C_\bullet(X, \mathcal{F}_p \otimes \mathbb{F}_2)) = \sum_{q=0}^{n-1} (-1)^q h^{p,q}(\mathbb{C}V).$$

La preuve de ces théorèmes est présentée en détails dans les chapitres 6 et 7. Les égalités entre dimensions des groupes d'homologies tropicales et nombres de Hodge ont été étudiées au delà du cadre des hypersurfaces (sur $\mathbb{Q}$) dans [IKMZ19]. Notre approche est spécifiques aux hypersurfaces mais est valable sur $\mathbb{Z}$ et se généralise au delà du cadre projectif, et même complet. Dans les chapitres 6 et 7, nous ne supposons pas que la variété torique soit complète, et nous énonçons les théorèmes à la fois pour l'homologie standard mais aussi pour l'homologie de Borel-Moore (et la cohomologie à support compacte). Nous noterons par $H_\bullet^{\mathrm{BM}}$ l'homologie de Borel-Moore. Par soucis de simplicité, nous présenterons en détails dans ce mémoire une autre situation ou notre approche fonctionne, et je laisse le lecteur consulter éventuellement [ARS21] pour d'autres énoncés plus généraux.

Théorème 6. *Soit V une hypersurface algébrique réelle de $(\mathbb{R}^*)^n$ obtenue par un patchwork primitif. Alors pour tout entier $0 \leq q \leq n-1$,*

$$b_q^{BM}(\mathbb{R}V) \leq \begin{cases} \sum_{l=0}^{q} h^{n-1-q,l}(H_c^{n-1}(\mathbb{C}V)) & \text{si } q < n-1, \\ \sum_{l=0}^{n-1} h^{0,l}(H_c^{n-1}(\mathbb{C}V)) + 2^n - 1 & \text{sinon.} \end{cases}$$

Dans cet énoncé, le membre de gauche b_q^{BM} indique la dimension du q-ième groupe d'homologie de Borel-Moore, et dans le membre de droite nous trouvons les nombres de Hodge-Deligne de la cohomologie à support compacte (voir par exemple [DK]).

Remarque 1.1. Une hypothèse cruciale pour qu'un patchwork produise une variété algébrique réelle est l'hypothèse de *convexité* : la triangulation utilisée pour la construction combinatoire doit être convexe (on dit aussi régulière). En général, on ne sait pas quelle type de structure on obtient sur un patchwork obtenu à partir d'une triangulation non-convexe. Néanmoins, dans [BdMR22], les auteurs ont démontré que cette hypothèse de convexité n'est pas nécessaire pour obtenir les bornes du Théorème 1 : si un patchwork A est obtenu à partir d'une triangulation primitive (non nécessairement convexe) d'un polytope Δ, alors $b_q(A) \leq \sum_{p\geq 0} h_\Delta^{p,q}$, où $h_\Delta^{p,q}$ sont les nombres de Hodges d'une hypersurface complexe non-singulière dans le système linéaire associée à Δ. Pour ce faire, les auteurs interprètent l'homologie tropicale comme une certaine homologie à coefficients non constants sur un ensemble partiellement ordonné (*partially ordered set* ou *poset*) sous-jacent à la triangulation, et démontrent tous les résultats nécessaires pour le Théorème 1 dans ce cadre. Nous adopterons ce langage dans le dernier chapitre de cette thèse, voir Chapitre 8.

Résultats principaux de l'article [RRS22] et [RRS23] avec Johannes Rau et Kris Shaw

La géométrie tropicale, apparue autour des années 2000, donne un nouveau point de vu sur le patchwork, et en permet a priori l'extension en toute codimension (au delà des intersections complètes, cadre où le patchwork avait déjà été étendu par Sturmfels [Stu94]). La primitivité se traduit en géométrie tropicale par la notion de non-singularité. On dit qu'une variété tropicale est *non-singulière* (ou localement de degré 1) si elle est donnée localement par l'éventail de Bergman d'un matroide. Pour faire du patchwork à partir d'une variété tropicale, il faut la munir d'une structure supplémentaire, appelée *structure réelle phasée*. Nous montrons le résultat suivant :

Théorème 7. *Si M est un matroide, il existe une bijection naturelle entre les structures réelles phasées sur l'éventail de Bergman Σ_M et les orientations de M.*

Nous généralisons alors la méthode du patchwork combinatoire pour les variétés tropicales non-singulières dans les variétés toriques. Nous montrons notamment que le résultat d'un patchwork est une variété topologique :

Théorème 8. *Soit X une variété tropicale non-singulière dans une variété torique tropicale, et soit $\mathcal{E}$ une structure réelle phasée sur X. Alors le patchwork $\mathbb{R}_{\mathcal{E}}X$ est une variété topologique.*

De plus, nous transposons notre résultat de [RS23] à ce cadre.

Théorème 9. *Soit X une variété tropicale non-singulière de dimension d dans une variété torique tropicale, et soit $\mathcal{E}$ une structure réelle phasée sur X. Alors*

$$b_q(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{p=0}^{d} \dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

et

$$b_q^{BM}(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{p=0}^{d} \dim H_q^{BM}(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

Nous montrons aussi dans [RRS23] que dans le cas où la variété tropicale est *approximable* (on parle de *limite tropicale* d'une famille analytique, voir [RRS23]), alors la topologie de la partie réelle d'un membre de la famille proche de la limite est déterminée par la variété tropicale avec sa structure réelle phasée.

Théorème 10. *Soit X une variété tropicale non-singulière dans une variété torique tropicale (associé à un éventail Σ), et soit $\mathbf{V} = (V_t)_{0<|t|<1}$ une famille analytique réelle qui admet X pour limite tropicale. La structure réelle sur $\mathbf{V}$ induit une structure réelle phasée $\mathcal{E}$ sur X et de plus, pour t assez petit et positif, on a un homeomorphisme des paires $(\mathbb{R}\Sigma, \mathbb{R}V_t)$ et $(\mathbb{R}\Sigma, \mathbb{R}_{\mathcal{E}}X)$*

Je ne développerai pas dans ce mémoire ce dernier point.

Résultats principaux de l'article [MR25] avec Diego Matessi

Nous étudions dans cet article les patchworks primitifs dans les polytopes réflexifs. Nous démontrons tout d'abord une version combinatoire de la symétrie miroir "à la Batyrev". Soient Δ et Δ° deux polytopes réflexifs duaux de dimension $n+1$, et soient T et T° des triangulations primitives centrales (tout simplexe contient l'origine) et non nécessairement convexes. On associe à T et T° des ensembles partiellement ordonnés X_{T,T° et $X_{T^\circ,T}$ (voir Chapitre 8 pour plus de détails). Ces deux posets sont isomorphes aux posets sous-jacent à des hypersurfaces tropicales (qu'on appelle "miroirs") dans le cas ou T et T° sont des triangulations convexes (voir aussi la remarque 1.1). On démontre alors le résultat suivant :

Théorème 11. *Soit A un anneau commutatif. Pour tout $p \geq 0$ et tout $q \geq 0$ on a des isomorphismes canoniques*

$$H_q(X_{T,T^\circ}, \mathcal{F}_p \otimes A) \simeq H_q(X_{T^\circ,T}, \mathcal{F}_{n-p} \otimes A).$$

Nous remarquons ensuite qu'un patchwork primitif d'une triangulation centrale T dans un polytope réflexif Δ correspond à un diviseur sur $\mathbb{F}_2$ dans la variété torique $\mathbb{C}\Sigma_T$ associé à l'éventail construit sur T. Nous montrons que si deux diviseurs sont linéairement équivalents (sur $\mathbb{F}_2$), alors les patchworks sont isomorphes via un automorphisme torique. On dénote par $\mathbb{R}X_D$ le résultat du patchwork associé au diviseur D. Le diviseur D induit par restriction une classe dans $H_{n-1}(X_{T,T^\circ}; \mathcal{F}_{n-1} \otimes \mathbb{F}_2)$ qu'on note simplement $D|_{X_{T,T^\circ}}$. Nous démontrons alors le résultat suivant :

Théorème 12. *Soit D un $\mathbb{F}_2$-diviseur de $\mathbb{C}\Sigma_T$. Supposons que les $H_n(X_{T^\circ,T}, \mathcal{F}_k \otimes \mathbb{F}_2)$ s'annulent pour tout $0 < k < n$. Alors $\mathbb{R}X_D$ est connexe si et seulement si $D|_{X_{T,T^\circ}} \neq 0$.*

1.2 English version

This dissertation is dedicated to certain aspects of *patchworking*, a method discovered by Oleg Viro in the early 80s for constructing real algebraic varieties by gluing together certain patches, which are pieces of simpler real algebraic varieties. In this text we will focus on the case where the patches are the simplest possible : pieces of hyperplanes (or possibly higher codimensional linear spaces). In this context, the method is called "primitive combinatorial patchworking", or more simply "primitive patchworking". In fact, it becomes a purely combinatorial procedure for constructing real algebraic varieties. The following question is still wide open :

Question. *What is the topology of a real algebraic variety obtained by primitive patchwork ?*

Main results from the article [RS23] with Kris Shaw and the article [ARS21] with Charles Arnal and Kris Shaw

One of the main results of this thesis is the following. Betti numbers b_q are considered with coefficients in the field $\mathbb{F}_2$ with two elements.

Theorem 1.2.1. *Let V be a real algebraic hypersurface obtained by a primitive patchworking in a non-singular projective toric variety of dimension n. Then for all $q \geq 0$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \text{ if } q = (n-1)/2, \\ h^{q,n-1-q}(\mathbb{C}V) + h^{q,q}(\mathbb{C}V) \text{ otherwise.} \end{cases}$$

The following corollary answers a conjecture due to Itenberg [Ite17] :

Corollary 1.2.2. *Let V be a real algebraic hypersurface of $\mathbb{RP}^n$ obtained by a primitive patchworking. Then for any integer $0 \leq q \leq n-1$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) \text{ if } q = (n-1)/2, \\ h^{q,n-1-q}(\mathbb{C}V) + 1 \text{ otherwise.} \end{cases}$$

Our proof of Theorem 1.2.1 uses a tropical description of patchworking in terms of *phased real structures* (see Section 3.5). We introduce then a spectral sequence $E^r_{q,p}$ that converges to the homology of $\mathbb{R}V$, which implies that the Betti numbers of $\mathbb{R}V$ are equal to the sum of the dimensions of the $\mathbb{F}_2$-vector spaces appearing on the infinite page :

$$b_q(\mathbb{R}V) = \sum_{p=0}^{n} \dim E^\infty_{q,p}.$$

The first page of this spectral sequence is isomorphic to the tropical homology with $\mathbb{F}_2$-coefficients of a tropical hypersurface X dual to the triangulation used for the patchworking :

$$E^1_{q,p} \simeq H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

Tropical homology is a non-constant coefficient homology theory on polyhedral complexes. The coefficients $\mathcal{F}_p(x)$ are $\mathbb{Z}$-modules which depend only on the face to which the point x belongs (see Section 3.7 for more details). We then obtain the following theorem :

Theorem 1.2.3. *Let V be a real algebraic hypersurface obtained by a primitive patchworking from a tropical hypersurface X in a smooth projective variety of dimension n. Then*

$$b_q(\mathbb{R}V) \leq \sum_{p=0}^{n-1} \dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

The natural framework for this theorem goes beyond the case of hypersurfaces : it holds for non-singular tropical subvarieties of tropical toric varieties, equipped with a real phase structure (see Theorem 1.2.11). We show then that the dimensions of the tropical homology groups of X are equal to the Hodge numbers of V.

$$\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2) = h^{p,q}(\mathbb{C}V).$$

To do this, we proceed in three steps. First, we show that if $p+q < n-1$, then the tropical homology groups of X are equal to those of the ambient tropical toric variety.

Theorem 1.2.4. *Let X be a tropical hypersurface dual to a convex primitive triangulation of Δ a regular polytope of dimension n. Then for all $p+q < n-1$*

$$H_q(X, \mathcal{F}_p) \simeq H_q(\mathbb{T}\Sigma_\Delta, \mathcal{F}_p)$$

The notation $\mathbb{T}\Sigma_\Delta$ denotes the tropical toric variety associated with Δ (see Section 3.1). We then use this theorem and the tropical Poincaré duality [JRS18] to show that tropical homology groups are torsion-free, and determine their dimensions :

Theorem 1.2.5. *Let X be a tropical hypersurface dual to a primitive triangulation of Δ a regular polytope of dimension n. Then for all p and q, the $\mathbb{Z}$-modules $H_q(X, \mathcal{F}_p)$ are torsion-free, and furthermore for $p+q \neq n-1$, we have*

$$\mathrm{rk} H_q(X, \mathcal{F}_p) = \mathrm{rk} H_q(\mathbb{T}\Sigma_\Delta, \mathcal{F}_p) = h^{p,q}(\mathbb{C}\Sigma_\Delta).$$

The notation $\mathbb{C}\Sigma_\Delta$ indicates the complex toric variety associated with Δ. The final step is to compare the tropical Hirzebruch genus with the complex Hirzebruch genus :

Theorem 1.2.6. *Let X be a tropical hypersurface dual to a primitive triangulation of Δ a regular polytope of dimension n, and let $\mathbb{C}V$ be a generic Δ Newton polytope hypersurface compactified in $\mathbb{C}\Sigma_\Delta$. Then*

$$\chi(C_\bullet(X, \mathcal{F}_p \otimes \mathbb{F}_2)) = \sum_{q=0}^{n-1} (-1)^q h^{p,q}(\mathbb{C}V).$$

The proof of these theorems is presented in detail in Chapters 6 and 7. Equalities between dimensions of tropical homology groups (over $\mathbb{Q}$) and Hodge numbers have been studied beyond the cases of hypersurfaces in [IKMZ19]. Our approach is specific to hypersurfaces but it holds over $\mathbb{Z}$ and it generalizes beyond the projective and complete cases. In chapters 6 and 7, we don't assume that the toric variety is complete, and we state the theorems both for standard homology and for Borel-Moore homology. We will denote by $H_\bullet^{\mathrm{BM}}$ the Borel-Moore homology. For simplicity's sake, I'll present another situation where our approach works, and leave the reader to consult the references for other, more general situations.

Theorem 1.2.7. *Let V be a real algebraic hypersurface of $(\mathbb{R}^*)^n$ obtained by a primitive patchworking. Then for any integer $0 \leq q \leq n-1$,*

$$b_q^{BM}(\mathbb{R}V) \leq \begin{cases} \sum_{l=0}^{q} h^{n-1-q,l}(H_c^{n-1}(\mathbb{C}V)) \text{ if } q < n-1, \\ \sum_{l=0}^{n-1} h^{0,l}(H_c^{n-1}(\mathbb{C}V)) + 2^n - 1 \text{ otherwise.} \end{cases}$$

In this statement, the left-hand side b_q^{BM} indicates the dimension of the q-th Borel-Moore homology group, and in the right-hand side we find the Hodge-Deligne numbers of the compactly supported cohomology (see for example [DK]).

Remark 1.2.8. *A crucial assumption for a patchworking to produce a real algebraic variety is the* convexity *assumption : the triangulation used for combinatorial construction must be convex (we also say regular). In general, we don't know what kind of structure we get on a patchworking obtained from a non-convex triangulation. Nevertheless, in [BdMR22], the authors have shown that this convexity assumption is not necessary to obtain the bounds of Theorem 1.2.1 : if a patchworking A is obtained from a primitive triangulation (not necessarily convex) of a polytope Δ, then $b_q(A) \leq \sum_{p\geq 0} h_\Delta^{p,q}$, where $h_\Delta^{p,q}$ are the Hodge numbers of a non-singular hypersurface in the linear system associated with Δ. To this end, the authors interpret tropical homology as a certain homology with non-constant coefficients on a partially ordered set (*partially ordered set *or* poset*) underlying the triangulation, and prove all the results necessary for Theorem 1.2.1 in this framework. In Chapter 8, we will also adopt this framework.*

Main results of the article [RRS22] and [RRS23] with Johannes Rau and Kris Shaw

Tropical geometry, which appeared around 2000, gives a new point of view on patchworking, and allows a priori its extension in any codimension (beyond complete intersections, a framework where patchworking had already been extended by Sturmfels [Stu94]). In tropical geometry, primitivity translates into non-singularity. A tropical variety is said to be *non-singular* (or locally of degree 1) if it is locally given by the Bergman fan of a matroid. To patchwork a tropical variety, we need to enhance it with an additional structure, called *real phase structure*. We show the following result :

Theorem 1.2.9. *If M is a matroid, there is a natural bijection between the real phase structures on the Bergman fan Σ_M and the orientations of M.*

We then generalize the combinatorial patchworking method for non-singular tropical varieties in toric varieties. In particular, we show that the result of a patchworking is a topological variety :

Theorem 1.2.10. *Let X be a tropical non-singular variety in a toric tropical variety, and let $\mathcal{E}$ be a real phase structure on X. Then the patchworking $\mathbb{R}_{\mathcal{E}}X$ is a topological variety.*

Moreover, we transpose our result from [RS23] to this setting.

Theorem 1.2.11. *Let X be a non-singular tropical variety of dimension d in a toric tropical variety, and let $\mathcal{E}$ be a real phase structure on X. Then*

$$b_q(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{p=0}^{d} \dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

and

$$b_q^{BM}(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{p=0}^{d} \dim H_q^{BM}(X, \mathcal{F}_p \otimes \mathbb{F}_2).$$

We also show in [RRS23] that in the case where the tropical variety is *approximable* (we speak of *tropical limit* of an analytic family, see [RRS23]), then the topology of the real part of a member of the family close to the limit is determined by the tropical variety with its real phase structure.

Theorem 1.2.12. *Let X be a non-singular tropical variety in a toric tropical variety (associated with a Σ fan), and let $\mathbf{V} = (V_t)_{0<|t|<1}$ be a real analytic family that admits X as its tropical limit. The real structure on $\mathbf{V}$ induces a phased real structure $\mathcal{E}$ on X and, moreover, for t small enough and positive, we have an homeomorphism of pairs $(\mathbb{R}\Sigma, \mathbb{R}V_t)$ and $(\mathbb{R}\Sigma, \mathbb{R}_{\mathcal{E}}X)$.*

I won't expand on the latter point in this dissertation.

Main results of the article [MR25] with Diego Matessi

In this article, we study primitive patchworking in reflexive polytopes. We first prove a combinatorial version of mirror symmetry "à la Batyrev". Let Δ and Δ° be two dual reflexive polytopes of dimension $n+1$, and let T and T° be central primitive triangulations (not necessarily convex). Associated with T and T° are partially ordered sets X_{T,T° and $X_{T^\circ,T}$ (see Chapter 8 for details). These two posets are isomorphic to the posets underlying "mirror" tropical hypersurfaces in the case where T and T° are convex triangulations (see also the remark 1.2.8). We prove the following result :

Theorem 1.2.13. *Let A be a commutative ring. For all $p \geq 0$ and all $q \geq 0$ we have canonical isomorphisms*

$$H_q(X_{T,T^\circ}, \mathcal{F}_p \otimes A) \simeq H_q(X_{T^\circ,T}, \mathcal{F}_{n-p} \otimes A).$$

We note that a primitive patchworking of a central triangulation T in a reflexive polytope Δ corresponds to a divisor over $\mathbb{F}_2$ in the toric variety $\mathbb{C}\Sigma_T$ associated with the fan constructed on T. We show that if two divisors are linearly equivalent (over $\mathbb{F}_2$), then the patchworking are isomorphic via a toric automorphism. We denote by $\mathbb{R}X_D$ the patchworking associated with divisor D. The divisor D induces by restriction a class in $H_{n-1}(X_{T,T^\circ}; \mathcal{F}_{n-1} \otimes \mathbb{F}_2)$ which we simply denote $D|_{X_{T,T^\circ}}$. We prove the following result :

Theorem 1.2.14. *Let D be a divisor over $\mathbb{F}_2$ on $\mathbb{C}\Sigma_T$. Suppose that the groups $H_q(X_{T^{circ},T}, \mathcal{F}_p \otimes \mathbb{F}_2)$ vanish for all $0 < k < n$. Then $\mathbb{R}X_D$ is connected if and only if $D|_{X_{T,T^\circ}} \neq 0$.*

Chapitre 2

List of publications

Papers I present in this dissertation :

- ◇ Rau, J., Renaudineau, A. and Shaw, K. Real phase structures on matroid fans and matroid orientations. *J. London Math. Soc.*, 106 : 3687-3710, 2022.
- ◇ Rau, J., Renaudineau, A. and Shaw, K. Real phase structures on tropical manifolds and patchworks in higher codimension. https ://arxiv.org/abs/2310.08313, 2023.
- ◇ Renaudineau, A. and Shaw, K. Bounding the Betti numbers of real hypersurfaces near the tropical limit. *Annales Scientifiques de l'Ecole Normale Supérieure*, volume 56, pages 945-980, 2023.
- ◇ Arnal, C., Renaudineau, A. and Shaw, K. Lefschetz section theorems for tropical hypersurfaces. *Annales Henri Lebesgue*, Volume 4, pages 1347-1387, 2021.
- ◇ Matessi D, Renaudineau A. Mirror symmetry for tropical hypersurfaces and patchworking. Journal of the Institute of Mathematics of Jussieu. Published online 2025 :1-39. doi :10.1017/S1474748025101011

Other papers :

- ◇ Lang, L. and Renaudineau, A. Patchworking the Log-critical locus of planar curves. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 2022, no. 792, pp. 115-143, 2022.
- ◇ Mikhalkin, G. and Renaudineau, A. Tropical limit of log-inflection points for planar curves, *Sb. Math* 209 1273, 2017.
- ◇ Bertrand, B, Brugallé, E and Renaudineau, A. Haas' theorem revisited. *Epijournal de Géométrie Algébrique*, Volume 1, 2017.
- ◇ Renaudineau, A. Real Algebraic Surfaces with Many Handles in $(\mathbb{CP}^1)^3$. *International Mathematics Research Notices*, Volume 2018, Issue 2, Pages 432–469, 2018.
- ◇ Renaudineau, A. A tropical construction of a family of real reducible curves. *Journal of Symbolic Computation*, Volume 80, Part 2, Pages 251-272, 2017.
- ◇ Renaudineau, A. A real sextic surface with 45 handles. *Math. Z.* 281, 241–256, 2015.

Chapitre 3

Preliminaries

3.1 Complex, real and tropical toric varieties

Let $M \cong \mathbb{Z}^n$ be a lattice, let $N = \text{Hom}(M, \mathbb{Z})$ be its dual lattice and define as usual $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = N \otimes \mathbb{R}$. A lattice polytope Δ in $M_\mathbb{R}$ is the convex hull of a finite number of points of M. We will denote by greek letters (σ, τ, etc.) the faces of Δ. The normal fan Σ_Δ of Δ is the collection of convex rational cones $\{C_\sigma\}_{\sigma \preceq \Delta}$ in $N_\mathbb{R}$ given by

$$C_\sigma = \{v \in N_\mathbb{R} \,|\, \langle v, w - u \rangle \geq 0, \quad \forall u \in \sigma \text{ and } w \in \Delta\}.$$

We will work with complex, real and tropical toric varieties. Let us briefly recall how they are constructed. Let Σ be a rational strongly convex polyhedral fan in $N_\mathbb{R}$. For any cone σ of Σ consider the dual cone

$$\sigma^\vee = \{u \in M_\mathbb{R} \,|\, \langle u, v \rangle \geq 0, \quad \forall v \in \sigma\}.$$

By Gordon's Lemma, the intersection $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup. For any semigroup S, one can consider the set $X_\sigma^S := \text{Hom}(S_\sigma, S)$ of semigroup homomorphisms. If moreover the semigroup S has a topology, we equipp the set X_σ^S with the coarsest topology such that for any $\lambda \in S_\sigma$, the corresponding evaluation map $X_\sigma^S \to S$ is continuous. In fact, it is enough to do it for a set of generators of S_σ (say of cardinality N), and the topology coincides with the induced topology from S^N. If $\sigma \subset \tau$, there is a continuous restriction map from X_σ^S to X_τ^S.

Definition 3.1.1. *The toric variety associated to the fan Σ over the semigroup S is the direct limit*

$$S\Sigma := \varinjlim X_\sigma^S.$$

Classically, the most common semigroups are either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{R}_+$ (for multiplication). We will also consider the tropical semigroup $\mathbb{T} := \mathbb{R} \cup -\infty$ with the tropical multiplication being the addition. The neutral element $1_\mathbb{T}$ is equal to 0, while the zero element $0_\mathbb{T}$ is $-\infty$. All these semigroups are equipped with the euclidean topology. The torus $N \otimes S^*$ is open in $S\Sigma$, acts on each X_σ^S and this action extends to all of $S\Sigma$. The orbits are in correspondence with the cones of Σ via the map $\sigma \to \mathcal{O}_\sigma^S$, where $\mathcal{O}_\sigma^S$ is the orbit of the point $x_\sigma \in X_\sigma^S$ defined by

$$x_\sigma(m) = \begin{cases} 1_S \text{ if } m \in \sigma^\perp, \\ 0_S \text{ otherwise }. \end{cases}$$

The orbit $\mathcal{O}_\sigma^S$ can also be described as $\text{Hom}(\sigma^\perp \cap M, S)$. Given a point $y \in \mathcal{O}_\sigma^S$, we call σ the *sedentarity* of y. The *order of sedentarity* of y, denoted by $\text{sed}(y)$, is set to be $\dim(\sigma)$.

For $S = \mathbb{C}$, we obtain the classical construction of complex toric varieties, see Section 1.3 of [Ful93a]. If $\Sigma = \Sigma_\Delta$ is the normal fan of a lattice polytope, the variety $\mathbb{C}\Sigma$ is projective. A fan Σ is said to be *regular* if every cone can be generated by a subset of a basis of N. The toric variety $\mathbb{C}\Sigma$ is smooth if and only if Σ is regular. A polytope in $M_\mathbb{R}$ is also called *regular* if its normal fan is regular. We say Σ is *complete* if its support $|\Sigma|$, i.e. the union of its cones, is equal to $N_\mathbb{R}$. Completeness is necessary and sufficient for the toric variety $\mathbb{C}\Sigma$ to be compact. Let us illustrate in the following example two very important tropical toric varieties : affine and projective tropical spaces.

Example 3.1.2. *The tropical affine space is*

$$\mathbb{T}^n := (\mathbb{R} \cup -\infty)^n.$$

The tropical projective space $\mathbb{TP}^n$ *is*

$$\mathbb{TP}^n = \frac{\mathbb{T}^n \backslash (-\infty, \dots, -\infty)}{(x_0, \dots, x_n) \sim (a + x_0, \dots, a + x_n)},$$

where $a \in \mathbb{T} \backslash -\infty$.

Tropical affine space $\mathbb{T}^n$ *is the tropical toric variety associated to the fan that consists of the negative orthant in* $\mathbb{R}^n$ *and all its faces. Given a subset* A *of* $E = \{1, \dots, n\}$, *we denote by* ρ_A *the cone generated by* $\{-e_i, i \in A\}$, *where the* e_i *are the vectors of the standard basis of* $\mathbb{R}^n$. *Then the torus orbit* $\mathbb{R}^n_A := \mathcal{O}^{\mathbb{T}}_{\rho_A}$ *is equal to* $\mathbb{R}^{E \backslash A}$, *while its closure in* $\mathbb{T}^n$, *denoted by* $\mathbb{T}^n_A$, *can be identified with* $\mathbb{T}^{E \backslash A}$.
Similarly, we set $e_0 = -e_1 - \cdots - e_n$ *and for any* proper *subset* $A \subset \{0, \dots, n\}$ *we let* σ_A *be the cone generated by* $\{-e_i, i \in A\}$. *The collection of these cones forms a complete fan defining the tropical toric variety* $\mathbb{TP}^n$. *We denote the closure of* $\mathcal{O}^{\mathbb{T}}_{\rho_A}$ *in* $\mathbb{TP}^n$ *by* $\mathbb{TP}^n_A$. *It can be identified with a projective space* $\mathbb{TP}^{n-|A|}$ *(whose homogeneous coordinates are labelled by* $E \setminus A$*).*

For any fan Σ, there is a natural map from $\mathbb{C}\Sigma$ to $\mathbb{R}_+\Sigma$ induced by the absolute value from $\mathbb{C}$ to $\mathbb{R}_+$ (composition with the logarithm will identify $\mathbb{R}_+\Sigma$ with $\mathbb{T}\Sigma$). In the case where $\Sigma = \Sigma_\Delta$, one has $\mathbb{R}_+\Sigma_\Delta = \Delta$ and the map $\mathbb{C}\Sigma_\Delta$ to Δ is the moment map. In general, the real toric variety $\mathbb{R}\Sigma$ can be reconstructed combinatorially directly form the fan Σ as follows (see for example [GKZ08]). For any group homomorphism $\xi : M \to \mathbb{F}_2$, consider $\mathbb{R}_+\Sigma(\xi)$ a copy of $\mathbb{R}_+\Sigma$. Then

$$\mathbb{R}\Sigma \simeq \bigsqcup_{\xi \in \mathrm{Hom}(M, \mathbb{F}_2)} \mathbb{R}_+\Sigma(\xi) / \sim, \tag{3.1}$$

where $(p, \xi) \sim (p', \xi')$ if and only if $p = p'$ and $\xi|_{\sigma^\perp} = \xi'|_{\sigma^\perp}$ for the unique cone σ such that $p \in \mathcal{O}^{\mathbb{R}_+}_\sigma$.

Remark 3.1.3. *The logarithm identifies* $\mathbb{R}_+\Sigma$ *with* $\mathbb{T}\Sigma$ *and one can obtain a similar description of the real part as (3.1) using tropical toric varieties.*

Example 3.1.4. *We explicitly describe how to obtain* $\mathbb{RP}^n$ *by glueing together multiple copies of* $\mathbb{TP}^n$. *For every* $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \mathbb{F}_2^n$, *let* $\mathbb{TP}^n(\varepsilon)$ *denote a copy of* $\mathbb{TP}^n$ *indexed by* ε. *Then*

$$\mathbb{R}P^n \cong \bigsqcup_{\varepsilon \in \mathbb{F}_2^n} \mathbb{TP}^n(\varepsilon) / \sim,$$

where $\sim$ is the equivalence relation generated by identifying $x \in \mathbb{TP}^n(\varepsilon)$ and $x' \in \mathbb{TP}^n(\varepsilon')$ for $\varepsilon \neq \varepsilon'$, such that $[x_0, \cdots, x_n] = [x'_0, \cdots, x'_n]$
and

— *if $x_0 \neq -\infty$, then there exist a unique $1 \leq j \leq n$ such that $\varepsilon_j \neq \varepsilon'_j$. Moreover, we must have $x_j = x'_j = -\infty$.*

— *if $x_0 = -\infty$, then we must have $\varepsilon_i \neq \varepsilon'_i$ for all $1 \leq i \leq n$.*

3.2 Polyhedral complexes in tropical toric varieties

A rational polyhedron in a tropical toric variety $\mathbb{T}\Sigma$ is the closure in $\mathbb{T}\Sigma$ of a rational polyhedron in some stratum $\mathcal{O}_\rho^{\mathbb{T}}$. For example, the closure of $\mathcal{O}_\rho^{\mathbb{T}}$ itself is a rational polyhedron that we will denote by $\mathbb{T}\Sigma_\rho$. For a polyhedron σ in $\mathbb{T}\Sigma$, the intersection $\sigma \cap \mathcal{O}_\eta^{\mathbb{T}}$, if non-empty, is a polyhedron in $\mathcal{O}_\eta^{\mathbb{T}}$. A *face* τ of σ is the closure of a face of any of these intersections (hence τ is a polyhedron in $\mathbb{T}\Sigma$). The *relative interior* $\mathrm{relint}(\sigma)$ is the set of points in σ not contained in a proper face. We set $\mathrm{sed}(\sigma) := \mathrm{sed}(y)$ for an arbitrary $y \in \mathrm{relint}(\sigma)$.

A *rational polyhedral complex* X in $\mathbb{T}\Sigma$ is a collection of rational polyhedra in $\mathbb{T}\Sigma$ such that

— if $\sigma \in X$ and $\tau \subset \sigma$ is a face, then $\tau \in X$,

— if $\sigma_1, \sigma_2 \in X$ and $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1 and σ_2.

We refer to the polyhedra $\sigma \in X$ as *faces* of X. We will abuse notations and will identity a polyhedral complex and its support $\cup_{\sigma \in X} \sigma$. The maximal faces of X are called *facets*. A rational polyhedral complex $X \subset \mathbb{T}\Sigma$ is called of *pure sedentarity* $\rho \in \Sigma$ if $X = \overline{X \cap \mathcal{O}_\rho^{\mathbb{T}}}$. A polyhedral complex X is *of pure dimension* d if all its facets have dimension d. In the following, polyhedral complexes X are *always* assumed to be rational, of pure dimension, and of pure sedentarity. Given a polyhedron σ in $\mathbb{T}\Sigma$ of sedentarity ρ, recall that its *tangent space* is defined by $T(\sigma) := T(\sigma \cap \mathcal{O}_\rho^{\mathbb{T}}) \subset \mathcal{O}_\rho^{\mathbb{T}}$. Its integer and binary versions are

$$T_{\mathbb{Z}}(\sigma) := T(\sigma) \cap \mathrm{Hom}(\rho^\perp \cap M, \mathbb{Z})$$
$$T_{\mathbb{F}_2}(\sigma) := T_{\mathbb{Z}}(\sigma) \otimes \mathbb{F}_2 \subset \mathbb{Z}_2^n / \langle \rho \rangle \otimes \mathbb{F}_2.$$

For simplicity, we will denote $\mathbb{Z}(\rho) = \mathrm{Hom}(\rho^\perp \cap M, \mathbb{Z})$ and $\mathbb{F}_2(\rho) = \mathbb{Z}(\rho) \otimes \mathbb{F}_2$. Following this notation, we will denote sometime by $\mathbb{R}(\rho)$ the tropical orbit $\mathcal{O}_\rho^{\mathbb{T}} = \mathrm{Hom}(\rho^\perp \cap M, \mathbb{T})$.

3.3 Primitive patchworking : classical version

Let Δ be a regular lattice polytope in $M_{\mathbb{R}}$ and T a primitive lattice triangulation of Δ. Primitive (or unimodular) means here that all facets of T have lattice volume 1. Let $\varepsilon : \Delta \cap M \to \mathbb{F}_2$ be a sign distribution on the lattice points of Δ (since the triangulation is primitive, note that the lattice points of Δ coincide with the vertices of the triangulation). When we say signs in the context of Viro's patchworking, we will identify $+$ with $0 \in \mathbb{F}_2$ and $-$ with 1.

Recall that the real part of the toric variety associated to Σ_Δ is constructed by considering a certain quotient of copies $\Delta(\xi)$ of Δ for any group homomophism $\xi : M \to \mathbb{F}_2$, see Equation (3.1). Denote by $T(\xi)$ the triangulation of $\Delta(\xi)$ induced by T. Given a lattice point $v \in \Delta$, the sign of its copy $v(\xi)$ in $\Delta(\xi)$ is defined by $\varepsilon(v(\xi)) = \xi(v) + \varepsilon(v) \in \mathbb{F}_2$. For any ξ and any n-simplex σ of $\Delta(\xi)$, separate the vertices having different signs by taking

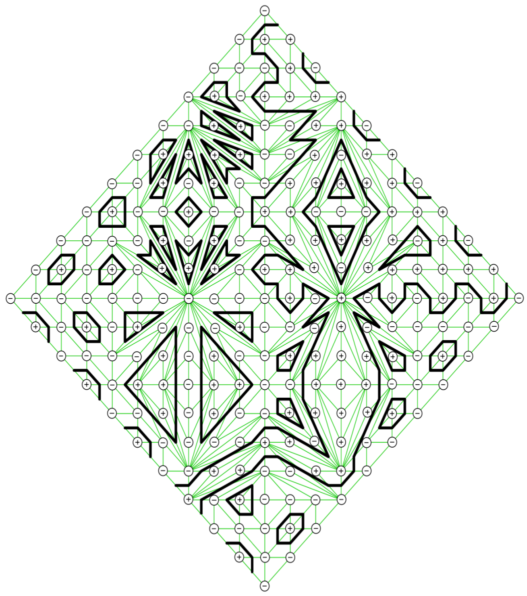

FIGURE 3.1 – Patchworking of a curve dispoving Ragsdale conjecture

the union of convex hulls of barycenters of simplices $\sigma_1 \subset \cdots \subset \sigma$ such that σ_1 is an edge with different signs. This gives a simplicial complex defining a smooth PL-hypersurface in the quotient $\mathbb{R}\Sigma_\Delta$, denoted by $X_{T,\varepsilon}$.

Definition 3.3.1. *The triangulation T is called* convex *or* regular *if it exists a convex function $\mu : \Delta \to \mathbb{R}$ such that $\mu(\mathbb{Z}^n) \subset \mathbb{Z}$ and that the lower faces of the graph of μ project to the triangulation T.*

If μ is a convex function certifying the regularity of T, Viro's combinatorial patchworking theorem ([Vir84] for the original statement, but see also [Vir01] [Ite95] [Ite97] or [Ris93]) states that the PL-hypersurface $X_{T,\varepsilon}$ is ambient isotopic (in $\mathbb{R}\Sigma_\Delta$) to the closure (in $\mathbb{R}\Sigma_\Delta$) of the hypersurface in $(\mathbb{R}^*)^{n+1}$ defined by the polynomial

$$\sum_{v \in \Delta \cap M} (-1)^{\varepsilon(v)} t^{\mu(v)} x^v, \tag{3.2}$$

for t small enough and positive.

Example 3.3.2. *Itenberg used Viro's combinatorial patchworking to construct a real algebraic plane curve of degree* 10 *disproving Ragsdale's conjecture [Ite95]. We included here a picture of this construction in Figure 3.1.*

3.4 Tropical hypersurfaces

Let again Δ be a lattice polytope in $M_\mathbb{R}$ and T a primitive lattice triangulation of Δ. Assume that moreover T is convex and let $\mu : \Delta \cap M \to \mathbb{Z}$ be a convex function ensuring the convexity of T. Consider the Legendre transform of μ (defined over $N_\mathbb{R}$) :

$$\mu^*(x) = \max_{y \in \Delta \cap M} (\langle x, y\rangle - \mu(y)). \tag{3.3}$$

Define the set

$$X_\mu := \{x \in N_\mathbb{R} \mid \exists y_1 \neq y_2,\ \mu^*(x) = \langle x, y_1\rangle - \mu(y_1) = \langle x, y_2\rangle - \mu(y_2)\}.$$

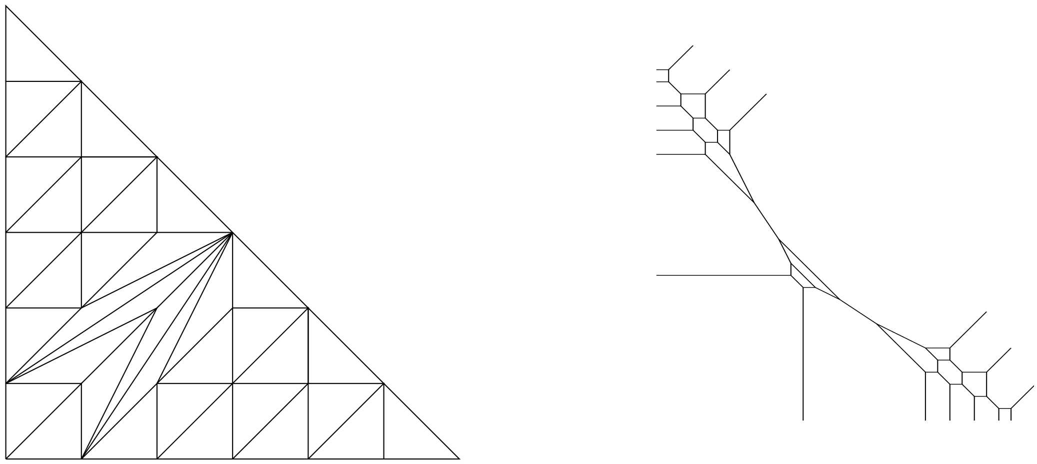

Figure 3.2 – A convex primitive triangulation of the size 6 triangle and a corresponding tropical sextic.

In other words, the set X_μ consists of points where the maximum in (3.3) is attainted at least twice. The function μ^* is called a *tropical polynomial* since it is a polynomial over the tropical semiring ($\mathbb{R}$, "$+$", "$\times$"), where the tropical addition "$+$" denotes the maximum, and the tropical multiplication "$\times$" denotes the standard addition. The set X_μ is called a *non-singular tropical hypersurface* : it is a rational balanced polyhedral complex, inducing a polyhedral subdivision of $N_\mathbb{R}$ which is dual (as a poset) to the triangulation T (see for example [BIMS] for more details). Moreover, the compactification of X_μ in the tropical toric variety $\mathbb{T}\Sigma_\Delta$ induces a subdivision of $\mathbb{T}\Sigma_\Delta$ with underlying poset the subposet of $\Sigma_\Delta \times T$:

$$\Phi := \left\{(\rho, \sigma) \in \Sigma_\Delta \times T \mid \sigma \subset \rho^\vee\right\}, \tag{3.4}$$

where $\rho^\vee$ denotes here the face of the polytope dual to the cone ρ. The order on the poset is the inverse of the inclusion, meaning that $(\rho, \sigma) \leq (\rho', \sigma')$ iff $\rho' \subset \rho$ and $\sigma' \subset \sigma$ (see [BdMR22] Remark 2.3).

Example 3.4.1. *In Figure 3.2 we drew a convex triangulation of the triangle of size* 6*. On the right hand side we drew the tropical curve dual to this subdivision (the tropical curve is unique up to the choice of the convex function which means up to changing the length of the edges and the position of the vertices in* $\mathbb{R}^2$*).*

3.5 Primitive patchworking : tropical version

Let $X_\mu \subset N_\mathbb{R}$ be a non-singular tropical hypersurface.

Definition 3.5.1. *A* real phase structure *on* X_μ *is a collection* $\mathcal{E} = \{\mathcal{E}_\sigma\}_{\sigma \in Facet(X_\mu)}$ *where* $\mathcal{E}_\sigma \subset N \otimes \mathbb{F}_2$ *is an* $(n-1)$*-dimensional affine subspace whose direction is the* $\mathbb{F}_2$*-tangent space of* σ*. The collection* $\mathcal{E}$ *must satisfy the following property :*

If τ *is a face of* X_μ *of codimension* 1*, then for any facet* σ *adjacent to* τ *and any element* $\varepsilon \in \mathcal{E}_\sigma$*, there exists a unique facet* $\sigma' \neq \sigma$ *adjacent to* τ *such that* $\varepsilon \in \mathcal{E}_{\sigma'}$*.*

A non-singular tropical hypersurface equipped with a real phase structure is called a *non-singular real tropical hypersurface.*

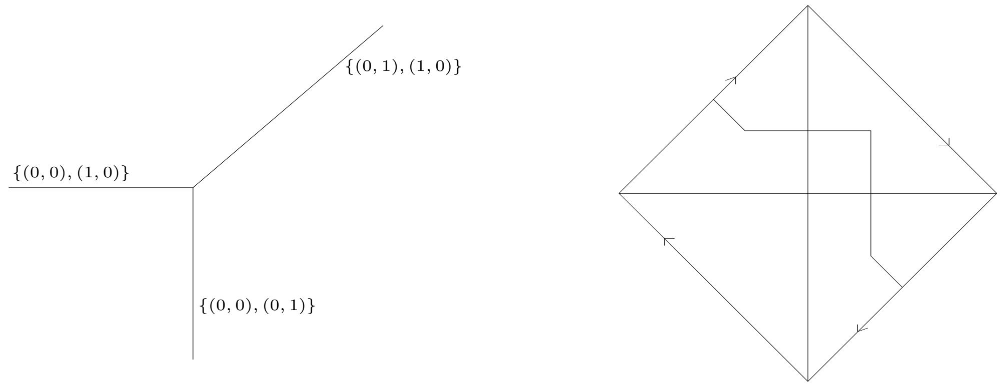

FIGURE 3.3 – On the left is the real tropical line $L \subset \mathbb{R}^2$ with a real phase structure $\mathcal{E}$ from Example 3.5.2. On the right hand side the compactification of its real part $\mathbb{R}L_{\mathcal{E}}$ in $\mathbb{R}P^2$.

Example 3.5.2. *Figure 3.3 depicts a real tropical line L in $\mathbb{R}^2$. On each edge $\sigma_0, \sigma_1, \sigma_2$ of the line there is a set of vectors in $\mathbb{F}_2^2$. These vectors indicate all the points in the affine subspace $\mathcal{E}_{\sigma_i}$ for a real phase structure $\mathcal{E}$.*

The vertex of the tropical line is the only codimension one face. For $(0,0) \in \mathcal{E}_{\sigma_1}$, we have that $(0,0) \in \mathcal{E}_{\sigma_2}$ and $(0,0) \notin \mathcal{E}_{\sigma_0}$. This is the condition in Definition 3.5.1 for the face σ_1 and the element $(0,0)$.

Example 3.5.3. *Let e_1, e_2, e_3 denote the canonical basis of $\mathbb{F}_2^3$, and let $e_0 = e_1 + e_2 + e_3$. The following collection of affine spaces forms a real phase structure on the tropical plane X in $\mathbb{R}^3$ defined by $\max(x, y, z, 1)$:*

$$\mathcal{E}_{\sigma_{12}} = \langle e_1, e_2 \rangle, \quad \mathcal{E}_{\sigma_{13}} = \langle e_1, e_3 \rangle, \quad \mathcal{E}_{\sigma_{23}} = \langle e_2, e_3 \rangle,$$

$$\mathcal{E}_{\sigma_{01}} = \langle e_0, e_1 \rangle + e_3, \quad \mathcal{E}_{\sigma_{02}} = \langle e_0, e_2 \rangle + e_1, \text{ and } \mathcal{E}_{\sigma_{03}} = \langle e_0, e_3 \rangle + e_2.$$

Given a plane $P \subset \mathbb{R}P^3$, the intersection of L with the coordinate hyperplanes of $\mathbb{R}P^3$ defines an arrangement of real hyperplanes on $P \cong \mathbb{R}P^2$. Such is the picture on the left hand side of Figure 3.4. Each region of the complement of this hyperplane arrangement on $P \subset \mathbb{R}P^3$ lives in a single orthant of $\mathbb{R}^3 = \mathbb{R}P^3 \backslash \{x_0 = 0\}$. In Figure 3.4, each connected component of the complement of the line arrangement is labelled with the vector in $\mathbb{F}_2^3$ corresponding to this orthant. Let $L_i = \{x_i = 0\} \cap P \subset \mathbb{R}P^3$ and set $p_{ij} = L_i \cap L_j$. Notice that the points contained in the affine space $\mathcal{E}_{\sigma_{ij}}$ of the real phase structure on X coincide with the collection of signs of the regions of the complement of the line arrangement which are adjacent to the point p_{ij}.

Remark 3.5.4. *The data of a real phase structure on X_μ is equivalent to the data of a sign distribution on the vertices of the triangulation T : if σ is a facet of X_μ dual to an edge e of T, the affine hypersurface $\mathcal{E}_\sigma$ of direction $\sigma^\perp$ contains 0 if and only if the signs of the two vertices of e are distinct. See [Ren17] for more details.*

Definition 3.5.5. *Let X_μ be a non-singular tropical hypersurface in $N_\mathbb{R}$ obtained from an primitive regular subdivision of a lattice polytope Δ, and let $\overline{X_\mu}$ be its compactification in $\mathbb{T}\Sigma_\Delta$. Let $\mathcal{E}$ be a real phase structure on X_μ. Given a polyhedron σ in $\mathbb{T}\Sigma_\Delta$ and $\varepsilon \in N \otimes \mathbb{F}_2$ we let σ^ε denote its copy in $\mathbb{T}\Sigma_\Delta(\varepsilon)$.*

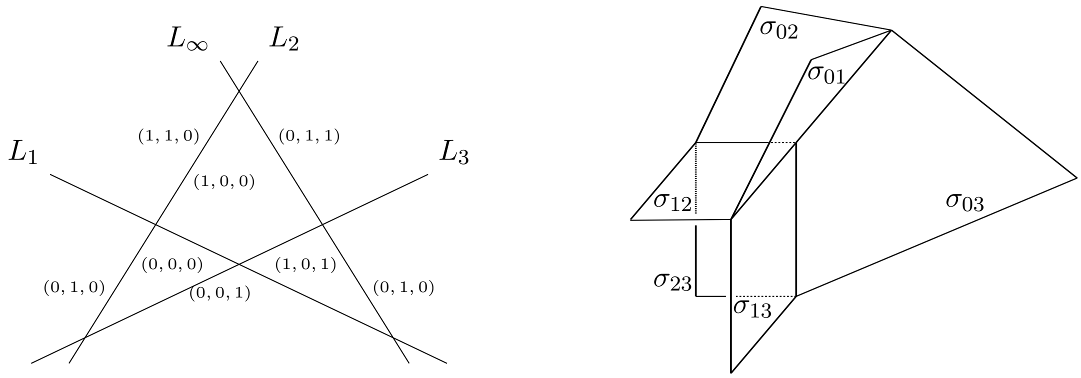

Figure 3.4 – The left hand side depicts a real line arrangement in $\mathbb{R}P^2$ arising from a linear embedding $\mathbb{R}P^2 \to \mathbb{R}P^3$. On the right is a tropical plane X in $\mathbb{T}P^3$. A real phase structure on X is described in Example 3.5.3.

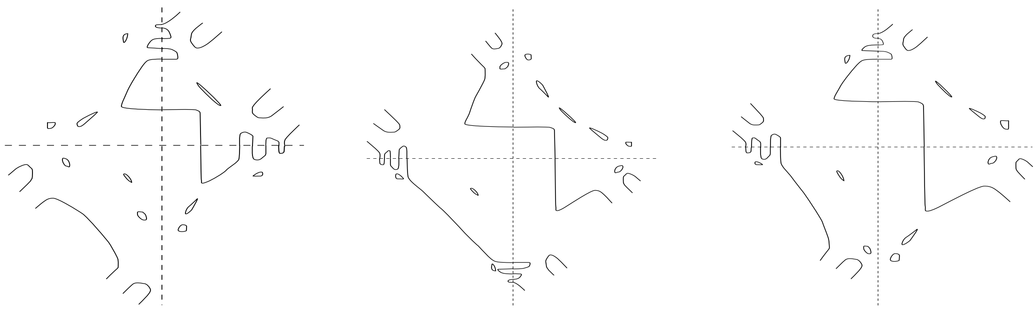

Figure 3.5 – The 3 maximal sextic curves up to isotopy

The real part of $\overline{X_\mu}$ *with respect to the real phase structure* $\mathcal{E}$ *is denoted* $\mathbb{R}_\mathcal{E} X_\mu$ *and is the image in* $\mathbb{R}\Sigma_\Delta$ *of*

$$\bigcup_{\substack{\text{facets of } \sigma \subset X_\mu \\ \varepsilon \in \mathcal{E}_\sigma}} \overline{\sigma}^\varepsilon.$$

where $\overline{\sigma}^\varepsilon$ *denotes the closure in* $\mathbb{T}\Sigma_\Delta(\varepsilon)$.

Remark 3.5.6. *The classical and tropical versions of primitive patchworking are equivalent. If* X_μ *is a a non-singular tropical hypersurface obtained from a convex unimodular triangulation* T *of a lattice polytope* Δ*, and if* ε *in a sign distribution corresponding to a real phase structure* $\mathcal{E}$*, then the pairs* $(X_{T,\varepsilon}, \mathbb{R}\Sigma_\Delta)$ *and* $(\mathbb{R}_\mathcal{E} X_\mu, \mathbb{R}\Sigma_\Delta)$ *are homeomorphic.*

Example 3.5.7. *In Figure 3.5 we drew* 3 *possible patchworks that one can obtain from the tropical sextic curve depicted in Figure 3.2. Remark that in this example, we recover all possible isotopy types of maximal sextic curves in the plane (constructed by Harnack, Hilbert and Gudkov respectively).*

3.6 Matroid fans and non-singular tropical subvarieties

The local models for non-singular tropical varieties are matroid fans. Let's begin by recalling what they are.

A matroid M is a finite set E together with a function rk : $2^E \to \mathbb{Z}_{\geq 0}$, where 2^E denotes the power set of E. The set E is called the ground set of M and rk the rank function. The rank function is subject to the axioms :

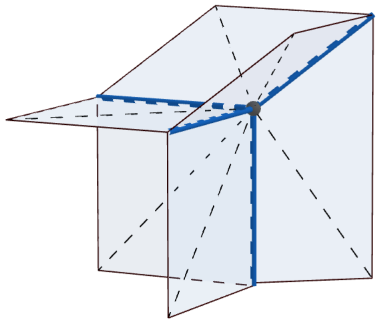

FIGURE 3.6 – The projective fan of the matroid $U_{3,4}$ drawn in $\mathbb{R}^4/(1,\dots,1)$ as described in Example 3.6.1.

1. $0 \le \mathrm{rk}(A) \le |A|$ for all $A \subseteq E$
2. if $A \subseteq B \subseteq E$, then $\mathrm{rk}(A) \le \mathrm{rk}(B)$
3. if $A, B \subseteq E$, then $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \le \mathrm{rk}(A) + \mathrm{rk}(B)$.

The rank function defines a closure operator on subsets by

$$\mathrm{cl}(A) = \{i \in E \mid \mathrm{rk}(A) = \mathrm{rk}(A \cup i)\} \supseteq A.$$

A subset $F \subseteq E$ is a flat of M if it is closed with respect to this operator, namely $\mathrm{cl}(F) = F$. The flats of a matroid M ordered by inclusion form a lattice, known as the lattice of flats, which we denote by $\mathcal{L}$.

A loop is an element of the ground set for which $\mathrm{rk}(i) = 0$. Parallel elements are pairs of non-loop elements for which $\mathrm{rk}(ij) = 1$. A matroid is simple if it contains no loops or parallel elements. A circuit is any set $A \subset E$ such that $|A| = \mathrm{rk}(A)+1$ and $\mathrm{rk}(A) = \mathrm{rk}(A \backslash i)$ for any $i \in A$. A coloop is an element that does not belong to any circuit.

Given a loopfree matroid M on the base set E, we denote by Σ_M the *affine* matroid fan in $\mathbb{R}^E$ and by $\mathbb{P}\Sigma_M = \Sigma_M/\langle(1,\dots,1)\rangle$ the *projective* matroid fan in $\mathbb{R}^E/\langle(1,\dots,1)\rangle$. We now describe how to construct both of these fans following Ardila and Klivans [AK06]. Fix the vectors $v_i = -e_i$ where $\{e_1,\dots,e_n\}$ is the standard basis of $\mathbb{R}^E$ for $E = \{1,\dots,n\}$ and set $v_I = \sum_{i \in I} v_i$ for any subset $I \subset E$. For a chain of flats

$$\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$$

in the lattice of flats $\mathcal{L}$, define the $k+1$-dimensional cone

$$\sigma_{\mathcal{F}} = \langle v_{F_1}, \dots, v_{F_k}, \pm v_E\rangle_{\ge 0}.$$

The affine matroid fan Σ_M is the collection of all such cones ranging over the chains in $\mathcal{L}$. In particular, the top dimensional faces of Σ_M are in one to one correspondence with the maximal chains in the lattice of flats of M. The projective matroid fan $\mathbb{P}\Sigma_M$ is the image of Σ_M in the quotient $\mathbb{R}^E/(1,\dots,1)$.

Example 3.6.1. *The uniform matroid of rank $k+1$ on n elements will be denoted $U_{k+1,n}$. The flats of the matroid $M = U_{k+1,n}$ are all subsets of $\{1,\dots,n\}$ of size less than or equal to k and $\{1,\dots,n\}$. Therefore, the faces of top dimension of Σ_M are in bijection with* *ordered subsets of size k. For example, the ordered set $\{i_1 < i_2 < \cdots < i_k\}$ corresponds to a chain of flats*

$$\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$$

where $F_1 = \{i_1\}$ and $F_{i+1} = F_i \cup i_{i+1}$, which in turn corresponds to the cone

$$\sigma_{\mathcal{F}} = \langle v_{F_1}, \dots, v_{F_k}, \pm v_E \rangle_{\geq 0}.$$

The projective fan for $U_{3,4}$ is shown in Figure 3.6. The dotted rays in this fan correspond to the 6 rank 2 flats.

There are coarser fan structures on the support of the fan Σ_M. For example, the fan with maximal cones $\sigma_I = \langle v_{i_1}, \dots, v_{i_k}, \pm v_E \rangle$ where $I = \{i_1, \dots, i_k\}$ ranges over all subsets I of E of size k, has the same support as Σ_M. This in known as the coarse matroid fan of M [AK06].

If a matroid M has loops $L = \mathrm{cl}(\emptyset)$, then we set $\Sigma_M := \Sigma_{M/L} \subset \mathbb{R}^{E\setminus L}$ and $\mathbb{P}\Sigma_M := \mathbb{P}\Sigma_{M/L} \subset \mathbb{R}^{E\setminus L}/(1, \dots, 1)$. We will often assume $E = \{0, \dots, n\}$. Note that by Example 3.1.2 we can consider $\mathbb{R}^{E\setminus L} = \mathcal{O}^{\mathbb{T}}_{\rho_L}$ as a torus orbit of $\mathbb{T}^{n+1}$, and $\mathbb{R}^{E\setminus L}/\langle(1, \dots, 1)\rangle = \mathcal{O}^{\mathbb{T}}_{\rho_L}$ as a torus orbit of $\mathbb{TP}^n$ (excluding the trivial case $E = L$). In this sense, in the following we will regard Σ_M and $\mathbb{P}\Sigma_M$ as subsets of $\mathbb{T}^{n+1}$ and $\mathbb{TP}^n$, respectively.

Definition 3.6.2. *[MR, Chapter 6] Let Σ be a fan and $\mathbb{T}\Sigma$ the associated tropical toric variety. Let $X \subset \mathbb{T}\Sigma$ be a polyhedral subspace and consider $p \in X$ of sedentarity $\rho \in \Sigma$ and $\mathrm{sed}(p) = \dim(\rho) = k$.*

Then X is non-singular *at p if there exist*

— *a toric isomorphism $\psi \colon \mathbb{T}U_\rho \to \mathbb{T}^k \times \mathbb{R}^{n-k} = \mathbb{T}U_{\rho_{\{1,\dots,k\}}} \subset \mathbb{TP}^n$,*

— *an open neighbourhood $p \in U \subset \mathbb{T}U_\rho$,*

— *a matroid M on $E = \{0, 1, \dots, n\}$,*

such that $\psi(p) = (-\infty, 0)$ and $\psi(X \cap U) = \overline{\mathbb{P}\Sigma}_M \cap \psi(U)$.

The polyhedral subspace X is a non-singular tropical subvariety *of $\mathbb{T}\Sigma$ if it is non-singular at all its points.*

3.7 Cellular cosheaves, tropical and poset (co)homology

Let Σ be a fan, let X be a polyhedral complex in $\mathbb{T}\Sigma$ and R be a commutative ring. A *cellular cosheaf $\mathcal{F}$ on X* is a collection of R-modules $\mathcal{F}(\tau)$ for every face τ of X together with linear maps $i_{\sigma\tau} \colon \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ for every inclusion of faces $\tau \subset \sigma$ such that $i_{\sigma\sigma} = \mathrm{id}$ and $i_{\tau\upsilon} \circ i_{\sigma\tau} = i_{\sigma\upsilon}$ for every chain $\upsilon \subset \tau \subset \sigma$.

Given a cellular cosheaf $\mathcal{F}$ on X, the groups of *cellular (Borel-Moore) q-chains with coefficients in $\mathcal{F}$* are

$$C_q(X; \mathcal{F}) = \bigoplus_{\substack{\dim \sigma = q \\ \sigma \text{ compact}}} \mathcal{F}(\sigma) \quad \text{and} \quad C_q^{BM}(X; \mathcal{F}) = \bigoplus_{\dim \sigma = q} \mathcal{F}(\sigma).$$

The boundary maps $\partial \colon C_q(X; \mathcal{F}) \to C_{q-1}(X; \mathcal{F})$ are the usual cellular boundary maps combined with the cosheaf maps $i_{\sigma\tau}$.

The *q-th (Borel-Moore) homology groups of $\mathcal{F}$ on X* are

$$H_q(X; \mathcal{F}) = H_q(C_\bullet(X; \mathcal{F})) \quad \text{and} \quad H_q^{BM}(X; \mathcal{F}) = H_q(C_\bullet^{BM}(X; \mathcal{F})).$$

A *morphism* $\mathcal{F} \to \mathcal{G}$ of cellular cosheaves on a polyhedral complex X is a collection of morphisms $\mathcal{F}(\tau) \to \mathcal{G}(\tau)$ for every face τ of X which commute with the restriction maps of both $\mathcal{F}$ and $\mathcal{G}$. We get the induced notions of *kernels, cokernels, and exact sequences* of morphisms of cellular cosheaves on X.

Let X be a non-singular tropical subvariety of a tropical toric variety $\mathbb{T}\Sigma$ defined by a fan Σ. If $\mathcal{O}_\rho^{\mathbb{T}}$ and $\mathcal{O}_\eta^{\mathbb{T}}$ are two strata corresponding to cones $\rho \subset \eta$ of Σ then there is a canonical projection map of tangent spaces producing the map

$$\pi_{\rho\eta} \colon T_{\mathbb{Z}}(\mathcal{O}_\rho^{\mathbb{T}}) \to T_{\mathbb{Z}}(\mathcal{O}_\eta^{\mathbb{T}}). \tag{3.5}$$

If σ is a polyhedron of sedentarity ρ such that $\sigma \cap \mathcal{O}_\eta^{\mathbb{T}} \neq \emptyset$, then $\pi_{\rho\eta}$ maps $T_{\mathbb{Z}}(\sigma)$ onto $T_{\mathbb{Z}}(\sigma \cap \mathcal{O}_\rho^{\mathbb{T}})$.

Definition 3.7.1. *Let X be a polyhedral complex in $\mathbb{T}\Sigma$. The p-*multi-tangent spaces *of X are cellular cosheaves $\mathcal{F}_p$ on X defined as follows. For a face $\tau \in X$ of sedentarity ρ, we set*

$$\mathcal{F}_p(\tau) = \sum_{\tau \subset \sigma \subset \mathbb{T}\Sigma_\rho} \bigwedge^p T_{\mathbb{Z}}(\sigma). \tag{3.6}$$

Given $\tau' \subset \tau$ of sedentarity η, the maps of the cellular cosheaf $i_{\tau\tau'} \colon \mathcal{F}_p(\tau) \to \mathcal{F}_p(\tau')$ are given by composing $\wedge^p \pi_{\rho\eta} : \bigwedge^p T_{\mathbb{Z}}(\sigma) \to \bigwedge^p T_{\mathbb{Z}}(\sigma \cap \mathbb{T}\Sigma_\eta)$ from (3.5) with the inclusion maps $\bigwedge^p T_{\mathbb{Z}}(\sigma \cap \mathbb{T}\Sigma_\eta) \to \mathcal{F}_p(\tau')$.

The groups

$$H_q(X; \mathcal{F}_p) \quad \text{and} \quad H_q^{BM}(X; \mathcal{F}_p)$$

are called the (p, q)*-th (Borel-Moore) tropical homology group.*

We define the (Borel-Moore) tropical signature of X as

$$\sigma^\diamond(X) := \sum_{p,q} (-1)^q H_q^\diamond(X; \mathcal{F}_p),$$

where $\diamond$ is either empty, denoting usual homology, or $\diamond$ is BM, denoting Borel-Moore homology. Notice that $\sigma^\diamond(X)$ is also equal to $\sum_{p,q}(-1)^q H_q^\diamond(X; \mathcal{F}_p \otimes \mathbb{Q})$.

Let's go back to the case of hypersurfaces. It is shown in [BdMR22] that one can still define an analogue of tropical homology over non-convex triangulation. It is done by means of so-called poset homology. We recall briefly this theory here. Let P be a poset. One can view P as a category, whose objects are its elments and whose morphisms are the ordered pairs $x \leq_P y$. For any ring R, an R-cosheaf $\mathcal{F}$ on P is a contravariant functor $\iota_{\mathcal{F}}$ from P to the category of R-modules. Given additional assumptions on P as explain below, one can associate a differential complex $(C_\bullet(P; \mathcal{F}), \partial)$ to any cosheaf $\mathcal{F}$. A *cover relation* in P is a pair $x \lessdot y$ such that there exists no $z \in P$ with $x < z < y$. A *grading* on P is a function $\dim : P \to \mathbb{Z}$ such that $\dim(y) - \dim(x) = 1$ for any cover relation $x \lessdot y$. An interval of length 2 is an interval $[x, y]$ such that $\dim(y) - \dim(x) = 2$. We say that P is *thin* if every interval of length 2 contains exactly 4 elements. A *signature* is a map s from the set of all cover relations of P to $\{\pm 1\}$, and it is called *balanced* if any interval of length 2 contains an odd number of -1's. Given a graded, thin poset P with a balanced signature and a cosheaf $\mathcal{F}$ on P, its differential complex is defined by

$$C_q(P; \mathcal{F}) = \bigoplus_{\dim(x) = q} \mathcal{F}(x), \quad \partial : C_q(P; \mathcal{F}) \to C_{q-1}(P; \mathcal{F}),$$

where for all $x \in P$ of dimension q, one has

$$\partial|_{\mathcal{F}(x)}(a) = \sum_{y \lessdot x} s(y \lessdot x)\iota(y \leq_P x)(a).$$

Since the poset is thin, we have $\partial^2 = 0$. The homology groups of $(C_\bullet(P;\mathcal{F}), \partial)$ are denoted as usual by $H_\bullet(P;\mathcal{F})$. If a subset U of P is closed under taking larger elements (sometimes U is called open), one can restrict the differential complex and the homology groups to U. We denote this restriction by $C_\bullet(U;\mathcal{F})$ and $H_\bullet(U;\mathcal{F})$. Also maps of cosheaves induce maps between homology groups, and short exact sequences of cosheaves induce long exact sequences of homology groups.

3.8 Reflexive polytopes and Batyrev mirror symmetry

A lattice polytope Δ is called reflexive if it is defined by a system of inequalities $\langle v, u\rangle \leq 1$ with $v \in N$ and $u \in M_{\mathbb{R}}$. In particular, it implies that $\text{Int}(\Delta) \cap M = \{0\}$, i.e. 0 is the unique interior lattice point of Δ. The dual polytope $\Delta^\circ \subset N_{\mathbb{R}}$ is defined as

$$\Delta^\circ = \{u \in N_{\mathbb{R}} \mid \langle u, v\rangle \leq 1, \quad \forall v \in \Delta\}$$

If Δ is reflexive, then Δ° is a lattice polytope which is also reflexive and $(\Delta^\circ)^\circ = \Delta$.

A reflexive polytope $\Delta \subset M_{\mathbb{R}}$ defines another fan, in the same space $M_{\mathbb{R}}$ called the face fan of Δ denoted by Ξ_Δ whose cones are the cones over the faces of Δ. We have the following relations

$$\Xi_\Delta = \Sigma_{\Delta^\circ} \quad \text{and} \quad \Xi_{\Delta^\circ} = \Sigma_\Delta.$$

If Δ is a regular reflexive polytope, then the cones of Ξ_{Δ° are also regular. So all faces of Δ° are simplices with no integer points in the interior. Moreover, if σ is a facet of Δ°, then the convex hull of the origin in N with σ is a unimodular simplex.

Given a pair of dual polytopes Δ and Δ°, we can consider anticanonical subvarieties X and X° inside $\mathbb{C}\Sigma_\Delta$ and $\mathbb{C}\Sigma_{\Delta^\circ}$ respectively. These are both Calabi-Yau varieties and they constitute a so called *mirror pair*. Typically these varieties, as well as the ambient toric varieties, will be singular. Sometimes singularities can be resolved in the realm of Calabi-Yau varieties by a crepant resolution. The easiest way to do this, if possible, is to consider unimodular convex central subdivisions.

Definition 3.8.1. *Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope. A triangulation of Δ with vertices in M is called central if it is obtained by taking convex hulls of the origin in M and any element of a triangulation of the facets of Δ. We will denote by ∂T the simplices of T which are contained in the boundary $\partial\Delta$.*

Let T be a unimodular central triangulation of Δ. Denote by Σ_T the fan constructed by taking the cones from the origin of M over the simplices in the subdivision of Δ. This fan is a refinement of the fan $\Xi_\Delta = \Sigma_{\Delta^\circ}$. So it defines a desingularization $\mathbb{C}\Sigma_T$ of $\mathbb{C}\Sigma_{\Delta^\circ}$.

In this article we assume there exist unimodular triangulations T and T° of Δ and Δ° respectively, we do not always assume they are convex. In this case generic anti-canonical hypersurfaces in $\mathbb{C}\Sigma_{T^\circ}$ or $\mathbb{C}\Sigma_T$ are smooth Calabi-Yau varieties. From now on we will denote by X and X° smooth anticanonical hypersurfaces in $\mathbb{C}\Sigma_{T^\circ}$ and $\mathbb{C}\Sigma_T$ respectively, instead of the singular ones in $\mathbb{C}\Sigma_\Delta$ and $\mathbb{C}\Sigma_{\Delta^\circ}$. When the triangulations are not convex these may be non-projective. In the projective case, i.e. when T and T° are both convex, Batyrev and Borisov [BB96] proved that the Hodge numbers of X and X° satisfy the following mirror identity (see also [Cox99])

$$h^{p,q}(X) = h^{n-p,q}(X^\circ).$$

This relation is in fact proved in Theorem 4.15 of [BB96] in the case where the Hodge numbers are replaced by the so called stringy-Hodge numbers, which only depend on the singular Calabi-Yau. Then, Proposition 1.1 of [BB96] states that the stringy Hodge numbers coincide with the Hodge numbers of the smooth, projective crepant resolution given by T and T°.

Chapitre 4

Real phase structures on matroidal fans

In this chapter, we introduce the notion of real phase structure on matroidal fans and we explain the main steps of the proof of the following theorem from [RRS22].

Theorem 4.0.1. *Given a fixed matroid M, there is a natural bijection between orientations $\mathcal{M}$ of M and real phase structure on the matroid fan Σ_M. In other words, oriented matroids and real phase structures on matroid fans are cryptomorphic concepts.*

4.1 The definition of real phase structures

Definition 4.1.1. *A collection of subsets $S_1, \dots, S_k \subset V$ is called an* even covering *if every element in V is contained in an even number of the sets. Equivalently,*

$$S_1 \triangle \dots \triangle S_k = \emptyset,$$

where $S \triangle T := (S \cup T) \setminus (S \cap T)$ is the symmetric difference.

For a vector space V over $\mathbb{F}_2$, we denote by $\mathrm{Aff}_d(V)$ the set of all affine subspaces in V of dimension d.

Definition 4.1.2. *Let Σ be a rational polyhedral fan of pure dimension d in $\mathbb{R}^n$. A* real phase structure *$\mathcal{E}$ on Σ is a map*

$$\mathcal{E} \colon \mathrm{Facets}(\Sigma) \to \mathrm{Aff}_d(\mathbb{F}_2^n)$$

such that

1. *for every facet σ of Σ, the set $\mathcal{E}(\sigma)$ is an affine subspace of $\mathbb{F}_2^n$ parallel to σ, in formulas, $T(\mathcal{E}(\sigma)) = T_{\mathbb{F}_2}(\sigma)$;*

2. *for every codimension one face τ of Σ with facets $\sigma_1, \dots, \sigma_k$ adjacent to it, the sets $\mathcal{E}(\sigma_1), \dots, \mathcal{E}(\sigma_k)$ are an even covering.*

Definition 4.1.3. *Let $\mathcal{E}$ be a real phase structure on Σ. A* reorientation of $\mathcal{E}$ *is a real phase structure $\mathcal{E}'$ obtained by translating all affine subspaces in a real phase structure $\mathcal{E}$ by a fixed vector $\varepsilon \in \mathbb{F}_2^n$. In other words $\mathcal{E}'(\sigma) = \mathcal{E}(\sigma) + \varepsilon$ for all $\sigma \in \Sigma$.*

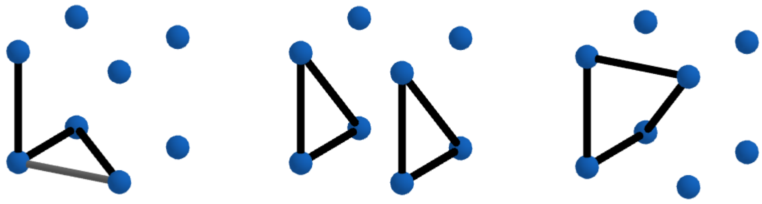

FIGURE 4.1 – Three line arrangements in $\mathbb{F}_2^3$ from Example 4.1.6.

We now give an equivalent reformulation of Condition (2) in Definition 4.1.2 in the case of matroid fans. Notice first that there is a one to one correspondence between real phase structures on Σ_M and $\mathbb{P}\Sigma_M$ induced by the projection $\mathbb{F}_2^E \to \mathbb{F}_2^E/(1,\dots,1)$.

Suppose $\mathcal{E}$ is a real phase structure on a d-dimensional rational polyhedral fan Σ and let τ be a codimension 1 face of Σ. Then the affine subspaces $\mathcal{E}(\sigma_i)$ where σ_i are the facets adjacent to τ all contain the direction of the $(d-1)$-dimensional linear space $T_{\mathbb{F}_2}(\tau)$. The even covering property at the codimension one face τ can be equivalently checked on the lines in $\mathbb{F}_2^n/T_{\mathbb{F}_2}(\tau)$ obtained as projections of the $\mathcal{E}(\sigma_i)$'s.

Given an arrangement of lines $L_1,\dots,L_k \in \mathrm{Aff}_1(V)$, its *intersection complex* is the simplicial complex that consists of a vertex for every line and a simplex on the vertices $i_1,\dots,i_q$ for every point in $L_{i_1}\cap\cdots\cap L_{i_q}$.

Definition 4.1.4. *An arrangement of lines $L_1,\dots,L_k \in \mathrm{Aff}_1(V)$ is a* necklace of lines *if its intersection complex is a cycle graph. An arrangement of subspaces $E_1,\dots,E_k \in \mathrm{Aff}_d(V)$ whose tangent spaces share a $d-1$-dimensional linear space W is called a* necklace arrangement *if the projection to V/W yields lines $L_1,\dots,L_k$ forming a necklace of lines.*

Remark 4.1.5. *We will use this definition exclusively for vector spaces over $\mathbb{F}_2$. Under this assumption, two lines L_1, L_2 form a necklace if and only if $L_1 = L_2$. If a necklace arrangement consists of more than two lines, then these lines must be pairwise distinct.*

For subspace arrangements of higher dimension, note that the definition of necklace arrangement is independent of the choice of W. Indeed, if this choice is not unique, then the affine spaces are all parallel and hence form a necklace if and only if $k = 2$ and $E_1 = E_2$.

Example 4.1.6. *Figure 4.1 shows three different line arrangements in $\mathbb{F}_2^3$. The points in $\mathbb{F}_2^3$ are represented as vertices of a cube. Lines in $\mathbb{F}_2^n$ are in correspondence with pairs of points. So in the figure a line is represented by an edge joining two points. The first line arrangement consists of 4 lines, and the 4 lines considered as subsets of $\mathbb{F}_2^3$ do not form an even cover in the sense of Definition 4.1.1. In the second example, there are 6 lines, which form an even cover but do not form a necklace arrangement. The third example is collection of 4 lines forming a necklace arrangement.*

We can establish the following alternative for condition (2) in Definition 4.1.2 in the case of matroid fans.

(2') For every codimension one face τ of Σ with facets $\sigma_1,\dots,\sigma_k$ adjacent to it, the subspaces $\mathcal{E}(\sigma_1),\dots,\mathcal{E}(\sigma_k)$ form a necklace arrangement.

The proof of the following lemma can be found in [RRS22].

Lemma 4.1.7. *Let $\mathcal{E}\colon$ Facets$(\Sigma) \to \mathrm{Aff}_d(\mathbb{F}_2^n)$ be a map satisfying condition (1) from Definition 4.1.2. If $\mathcal{E}$ satisfies condition (2'), then it also satisfies condition (2). Moreover, if $\Sigma = \Sigma_M$ (or $\Sigma = \mathbb{P}\Sigma_M$) is an affine (or projective) matroid fan, then the two conditions are equivalent.*

Definition 4.1.8. *Given a finite set S, a* necklace ordering *of S is an equivalence class of two cyclic orderings of S, which are related by reversing the order. For example, a cycle graph defines a necklace ordering of its vertices.*

Remark 4.1.9. *A real phase structure on a matroid fan Σ_M determines at every codimension one face of Σ_M a necklace ordering of the facets of Σ_M adjacent to the codimension one face. The necklace ordering is defined by the cycle graph of the necklace line arrangement at each codimension one face. A reorientation of a real phase structure in the sense of Definition 4.1.3 induces the same necklace ordering of facets adjacent to codimension one faces as the original real phase structure.*

Example 4.1.10. *Consider the projective fan of the uniform matroid $M = U_{2,n}$. The fan $\mathbb{P}\Sigma_M \subset \mathbb{R}^{n-1}$ has n edges generated by the images of the vectors $v_1 = -e_1, \dots, v_n = -e_n$ in $\mathbb{R}^n/\langle(1,\cdots,1)\rangle$. Denote by ρ_i the image of the vector v_i. Note that $\sum v_i = 0$. Choosing a real phase structure on $\mathbb{P}\Sigma_M$ amounts to choosing the following ingredients :*

1. *A necklace ordering of the n edges corresponding to $\rho_{i_1}, \dots, \rho_{i_n}$;*
2. *A point $p \in \mathbb{F}_2^{n-1} \simeq \mathbb{F}_2^n/\langle(1,\cdots,1)\rangle$ that serves as the intersection point of $\mathcal{E}(\rho_{i_1})$ and $\mathcal{E}(\rho_{i_2})$.*

From this information, a collection of affine lines $\mathcal{E}(\rho_i)$ satisfying the conditions of Definition 4.1.2 can be uniquely recovered. For example, the choice of point p determines both $\mathcal{E}(\rho_{i_1})$ and $\mathcal{E}(\rho_{i_2})$, since their tangent spaces are fixed. By the necklace arrangement property, the point $p + \sum_{k=2}^{j-1} v_{i_k}$ is in the affine line $\mathcal{E}(\rho_{i_j})$ for $j \geq 3$, where the vector sum is considered mod 2. This determines all of the affine lines $\mathcal{E}(\rho_{i_j})$.

Figure 4.2 shows the fan $\mathbb{P}\Sigma_M \subset \mathbb{R}^4/\langle(1,1,1,1)\rangle \cong \mathbb{R}^3$ for $M = U_{2,4}$ together with an assignment of affine spaces along its edges that determine a real phase structure. The induced necklace ordering of the facets is $\sigma_2, \sigma_3, \sigma_1, \sigma_4$. From Figure 4.2, we see that the point p is contained in the intersection $\mathcal{E}(\sigma_2) \cap \mathcal{E}(\sigma_3)$. If we set $p = (0,0,0,0) \in \mathbb{F}_2^4/\langle(1,1,1,1)\rangle$, then the corresponding necklace of lines is the last of the three arrangements in $\mathbb{F}_2^3 \cong \mathbb{F}_2^4/\langle(1,1,1,1)\rangle$ depicted in Figure 4.1.

Example 4.1.11. *For the matroid $M = U_{n-1,n}$ we will show that there is a unique real phase structure on $\mathbb{P}\Sigma_M$ up to reorientation. Such real phase structures were considered in [RS23].*

By [RS23, Lemma 3.14], a real phase structure $\mathcal{E}$ on $\mathbb{P}\Sigma_M$ satisfies

$$|\bigcup_\sigma \mathcal{E}(\sigma)| = 2^{n-1} - 1.$$

Therefore there is exactly one element ε in the complement $\mathbb{F}_2^{n-1} \setminus \cup_\sigma \mathcal{E}(\sigma)$. Up to reorientation we can suppose that $\varepsilon = (0,\dots,0)$.

Since $\mathbb{P}\Sigma_M$ is of codimension one in $\mathbb{R}^{n-1}$, for each facet σ of $\mathbb{P}\Sigma_M$ there is a choice of exactly two affine subspaces of $\mathbb{F}_2^{n-1}$ which are parallel to the reduction of the span of σ in $\mathbb{F}_2^{n-1}$. One of these spaces is an honest vector subspace and hence contains $(0,\dots,0)$. Therefore, if we are to associate to each σ an affine subspace $\mathcal{E}(\sigma)$ and wish to avoid that it contains $(0,\dots,0)$, then the choice of affine space at each top dimensional face is determined. This proves that there is at most one real phase structure on $\mathbb{P}\Sigma_M$, up to reorientation in the sense of Definition 4.1.3.

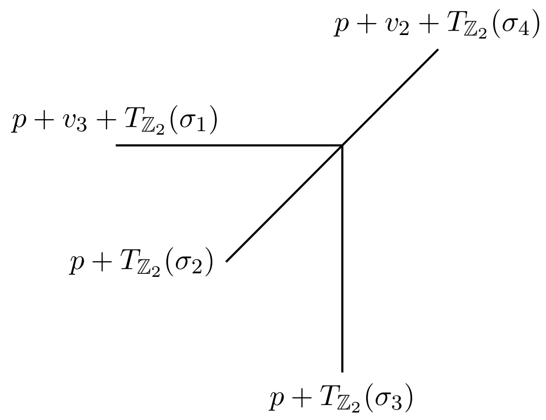

FIGURE 4.2 – The projective fan of the matroid $U_{2,4}$ from Example 4.1.10 drawn in $\mathbb{R}^4/\langle(1,\dots,1)\rangle$ with the labelling of the faces indicating the assignment of an affine space $\mathcal{E}(\sigma_i)$ parallel to $T_{\mathbb{Z}_2}(\sigma_i)$.

4.2 Deletion and contraction of real phase structures

We briefly recall the notion of minors of a matroid. Let M be a matroid with ground set E and rank function rk. For $S \subset E$, then the *deletion* of S is the matroid $M\backslash S$ with ground set $E\backslash S$ and rank function $\mathrm{rk}_{M\backslash S}(A) = \mathrm{rk}_M(A)$. The *contraction* of M by S is the matroid M/S whose ground set is again $E \setminus S$ and rank function $\mathrm{rk}_{M/S}(A) = \mathrm{rk}_M(A \cup S) - \mathrm{rk}_M(A)$. Lastly, the *restriction* of M to S is the matroid $M|_S$ whose ground set is S and rank function $\mathrm{rk}_{M|_S}$ is the restriction of rk_M. Notice that $M|_S = M\backslash S^c$, where $S^c = E \setminus S$. A *minor* of a matroid M is any matroid obtained from M by a sequence of deletions and contractions.

For any subset $A \subset E$, we denote by $p_A \colon \mathbb{R}^E \to \mathbb{R}^{E\backslash A}$ the projection which forgets the coordinates x_i for all $i \in A$. If the matroid M has loops $L \subseteq E$, we use the same notation for the projection $p_A \colon \mathbb{R}^{E\backslash L} \to \mathbb{R}^{E\backslash(L\cup A)}$. We also use the shorthand p_i in the case $A = \{i\}$.

If i is a loop or coloop of M, then $M\backslash i = M/i$, so deletion and contraction are equivalent. The support of $\Sigma_{M\backslash i}$ is the image of the projection of the matroid fan Σ_M under the projection p_i. Note that this is also true if i is a loop, in which case, according to our conventions, $\Sigma_{M\backslash i} = \Sigma_M$ and $p_i = \mathrm{id}$. If i is not a coloop, then the facets of $\Sigma_{M\backslash i}$ are the projections of facets of Σ_M whose dimensions are preserved under p_i.

Suppose that i is not a loop of M. Note that by our convention regarding loops, we have $\Sigma_{M/i} = \Sigma_{M/\mathrm{cl}(i)}$. The support of the matroid fan of M/i is the set $\{x \in \mathbb{R}^{E\backslash \mathrm{cl}(i)} \mid |p^{-1}_{\mathrm{cl}(i)}(x)| > 1\}$. The facets of $\Sigma_{M/i}$ are the images of facets of Σ_M whose dimensions are *not* preserved under the projection by $p_{\mathrm{cl}(i)}$. More details on the geometry of Σ_M, $\Sigma_{M\backslash i}$, and $\Sigma_{M/i}$ and their relations under p_i can be found in [Sha13, Section 2] and also [FR13, Section 3].

A real phase structure on the fan of a matroid M induces canonical real phase structures on the fans of all minors of M. We will describe the induced real phase structures for elementary deletions and contractions of a matroid M. The geometric idea is very simple : Given a facet σ of the matroid fan of a minor, we pick a facet $\tilde{\sigma}$ of Σ_M that projects to σ. Then the affine space associated to σ is the projection of $\mathcal{E}(\tilde{\sigma})$. Given that we work with the fine subdivision of Σ_M induced by the lattice of flats, the choice of $\tilde{\sigma}$ is in general not unique. To simplify the proofs in the following sections, we make a specific choice for $\tilde{\sigma}$. However, Definition 4.2.1 is independent of this choice, as discussed after the definition. For σ a facet of $\Sigma_{M\backslash i}$, let $p_i^*(\sigma)$ be the facet of Σ_M which is obtained by taking the closure in M of all the flats of $M\backslash i$ occurring in the chain of flats describing σ when i is not a coloop of M. If i is a coloop of M, prolongate the chain by one piece by adding i everywhere. Note that $p_i(p_i^*(\sigma)) = \sigma$. Moreover, if i is not a coloop $p_i^*(\sigma)$ is the unique facet that projects to σ. For σ a facet in $\Sigma_{M/i}$ corresponding to the chain of flats

$$\emptyset =\subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E \setminus \mathrm{cl}(i),$$

set $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$ to be the facet of Σ_M given by the chain

$$\mathrm{cl}(\emptyset) \subseteq \mathrm{cl}(i) \subsetneq F_1 \sqcup \mathrm{cl}(i) \subsetneq \cdots \subsetneq F_l \sqcup \mathrm{cl}(i) \subsetneq E.$$

Note that $p_{\mathrm{cl}(i)}(p^{\diamond}_{\mathrm{cl}(i)}(\sigma)) = \sigma$. By abuse of notation, we use the same letter p_A for the reduction mod 2 counterpart $p_A\colon \mathbb{F}_2^E \to \mathbb{F}_2^{E\backslash A}$.

Definition 4.2.1. *Let $\mathcal{E}$ be a real phase structure for the matroid fan Σ_M and choose $i \in E$. The* deletion *$\mathcal{E}\backslash i$ is the real phase structure on $\Sigma_{M\backslash i}$ given by*

$$(\mathcal{E}\backslash i)(\sigma) = p_i(\mathcal{E}(p_i^*(\sigma)))$$

for any facet σ of $\Sigma_{M\backslash i}$. The contraction *$\mathcal{E}/i$ is the real phase structure on $\Sigma_{M/i}$ given by*

$$(\mathcal{E}/i)(\sigma) = p_{\mathrm{cl}(i)}(\mathcal{E}(p^{\diamond}_{\mathrm{cl}(i)}(\sigma)))$$

for any facet σ of $\Sigma_{M/i}$.

As previously mentioned, for σ a facet of either $\Sigma_{M\backslash i}$ or $\Sigma_{M/i}$, we are free to replace $p_i^*(\sigma)$ or $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$ in the above definition with any facet of Σ_M which projects onto σ under p_i or $p_{\mathrm{cl}(i)}$, respectively. Any such facet will be contained in the same facet of the coarsest subdivision of the support of Σ_M as the facets $p_i^*(\sigma)$ or $p^{\diamond}_{\mathrm{cl}(i)}(\sigma)$, respectively. Therefore, Conditions (1) and (2) of a real phase structure imply that $\mathcal{E}$ must assign the same affine space to any such choice of face. Both propositions are proved in [RRS22].

Proposition 4.2.2. *The maps $\mathcal{E}\backslash i$ and $\mathcal{E}/i$ from Definition 4.2.1 define real phase structures on $\Sigma_{M\backslash i}$ and $\Sigma_{M/i}$ respectively.*

Proposition 4.2.3. *Let $\mathcal{E}$ be a real phase structure for the matroid fan Σ_M of the matroid M and choose $i \neq j \in E$. Then*

$$\mathcal{E}\backslash i\backslash j = \mathcal{E}\backslash j\backslash i,$$
$$\mathcal{E}\backslash i/j = \mathcal{E}/j\backslash i,$$
$$\mathcal{E}/i/j = \mathcal{E}/j/i.$$

This proposition shows that one can reorder the sequence of deletions and contractions without changing the result. So we can iterate the operations of deletion and contraction to construct general *minors* $\mathcal{E}\backslash A/B$.

4.3 Matroid orientations and real phase structures

Here we will produce a real phase structure on a matroid fan from an oriented matroid. We will use the covector description of oriented matroids. For an oriented matroid $\mathcal{M}$, on ground set E, the covectors of $\mathcal{M}$ are a subset $\mathcal{C} \subseteq \{0, +1, -1\}^E$. Let $X \in \mathcal{C}$. For $i \in E$, the i-th coordinate of X is denoted by X_i. The positive and negative parts of X are respectively

$$X^+ := \{i \in E \mid X_i = +1\},$$

and

$$X^- := \{i \in E \mid X_i = -1\}.$$

The support of X is

$$\text{Supp}(X) := \{i \in E \mid X_i \neq 0\}.$$

The composition operation $\circ$ on covectors X and Y is defined by

$$(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{if } X_i = 0. \end{cases}$$

The separation set $S(X, Y)$ is defined by

$$S(X, Y) := \{i \in E \mid X_i = -Y_i \neq 0\}.$$

The covectors of an oriented matroid satisfy the following axioms :

1. $0 \in \mathcal{C}$
2. $X \in \mathcal{C}$ if and only if $-X \in \mathcal{C}$
3. $X, Y \in \mathcal{C}$ implies that $X \circ Y \in \mathcal{C}$
4. If $X, Y \in \mathcal{C}$ and $i \in S(X, Y)$ then there exists a $Z \in \mathcal{C}$ such that $Z_i = 0$ and $Z_j = (X \circ Y)_j = (Y \circ X)_j$ for all $j \notin S(X, Y)$.

The set of covectors $\mathcal{C}$ forms a lattice under the partial order $0 < +1, -1$ considered coordinatewise. There is a forgetful map ϕ from oriented matroids to matroids which preserves rank and the size of the ground set. Given an oriented matroid $\mathcal{M}$, we let $\underline{\mathcal{M}} = \phi(\mathcal{M})$ denote its underlying matroid. We can describe the forgetful map on the level of the covector lattice $\mathcal{C}$ of $\mathcal{M}$ and the lattice of flats $\mathcal{L}$ of M. Given a covector $X \in \mathcal{C}$ the forgetful map assigns $\phi(X) = \text{Supp}(X)^c \in \mathcal{L}$ where A^c denotes $E \setminus A$. The image of a covector of the oriented matroid under the forgetful map is a flat of the underlying matroid [BLVS+99, Proposition 4.1.13].

The set of *topes* $\mathcal{T}$ are the maximal covectors with respect to the partial order on $\mathcal{C}$. If the underlying matroid of $\mathcal{M}$ has no loops we have $\mathcal{T} \subseteq \{+1, -1\}^{|E|}$. Let $\mathcal{M}$ be an oriented matroid with collection of topes $\mathcal{T}$ and underlying lattice of flats $\mathcal{L}$. For $F \in \mathcal{L}$ and $T \in \mathcal{T}$, we denote by $T \setminus F \in \{0, +1, -1\}^E$ the vector obtained by setting all coordinates in F to 0. We say F is *adjacent to* T if $T \setminus F \in \mathcal{C}$. More generally, given a flag $\mathcal{F} := F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of flats in $\mathcal{L}$, we define the set of topes adjacent to $\mathcal{F}$ by

$$\mathcal{T}(\mathcal{F}) = \{T \in \mathcal{T} \mid T \setminus F_i \in \mathcal{C} \text{ for all } i = 0, \dots, k\}.$$

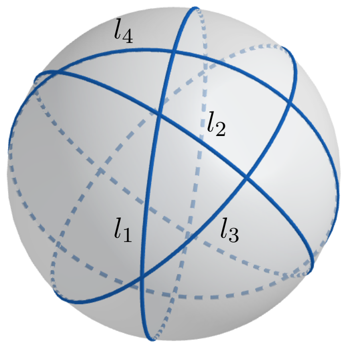

Figure 4.3 – The intersection of a real arrangement of 4 generic planes in $\mathbb{R}^3$ with the unit sphere. Assigning the covector $(+,+,+,+)$ to the region bounded by the spherical triangle facing the viewer formed by l_1, l_2, l_3 determines the covectors of all regions.

Example 4.3.1. *A set $H_1, \dots, H_n$ of hyperplanes in $\mathbb{R}^r$ defined by linear forms $l_1, \dots, l_n$ produces an oriented matroid on $\{1, \dots, n\}$. The underlying matroid on $\{1, \dots, n\}$ is given by the rank function $\mathrm{rk}(A) = codim(\cap_{i \in A} H_i)$. A covector corresponds to a cell of the decomposition of $\mathbb{R}^r$ induced by the positive regions $H_i^+ = \{l_i(x) \geq 0\}$ and negative regions $H_i^- = \{l_i(x) \leq 0\}$. Assuming that none of the linear forms are identically equal to zero, the topes are in bijection with the cells in the complement of the arrangement. The flat associated to a covector is in bijection with the set of hyperplanes containing the corresponding cell. Figure 4.3, shows the intersection of an arrangement of four planes in $\mathbb{R}^3$ with a sphere. The underlying matroid of this arrangement is the uniform matroid $U_{3,4}$. There are 14 cells of dimension two in the subdivision of the sphere induced by the intersections of the four planes. These are the topes of the oriented matroid. Each cell of the complement is labelled by a tuple $\{+, -\}^4$ corresponding to the sign of the linear forms $l_1, \dots, l_4$ evaluated at a point in the open cell.*

Remark 4.3.2. *If T is a tope then $T \circ X = T$ for any X and $X \circ T$ is always another tope, which is distinct from T if and only if $X^+ \not\subseteq T^+$ or $X^- \not\subseteq T^-$. Given a subset $F \subset E$, the* reflection *$r_F(X)$ of a covector X in F is given by flipping the signs for all $e \in F$ while keeping the signs for $e \in E \setminus F$. The reflection of a covector X in a flat F is not always a covector of the oriented matroid. However, note that if F is adjacent to the tope T, then $r_F(T)$ is also a tope. Indeed, setting $X = T \setminus F$, note that we can rewrite $r_F(T) = X \circ (-T)$, hence the statement.*

Recall that given any flat F of M, there is a vector in Σ_M defined by $v_F := \sum_{i \in F} v_i$, where $v_i = -e_i$, see Section 3.6. We denote by ε_F the reduction of v_F modulo 2. Note that $r_F((-1)^{\varepsilon}) = (-1)^{\varepsilon + \varepsilon_F}$.

Remark 4.3.3. *We would like to make the following remark on our choice of conventions. In this paper, we use both the multiplicative and additive notation on the group of two elements $(\{0,1\}, +)$ and $(\{1,-1\}, \cdot)$. When speaking of real phase structures we work with vector spaces over $\mathbb{F}_2$, therefore it is preferable to use the additive notation and denote the field of two elements by $\{0,1\}$. On the other hand it is tradition that the covectors of oriented matroids take values in $\{0,+,-\}$ and we also make use of the group structure on $\{+,-\}$. We routinely use the notation ε to denote elements of the field $\{0,1\}$ or of vector spaces over this field. We use uppercase roman letters, for example, X, Y, T, to denote covectors. Covectors can be multiplied entry by entry and this operation is denoted by $T \cdot T'$.*

To go from the additive group notation to the multiplicative group notation for vectors, we use $(-1)^\varepsilon = ((-1)^{\varepsilon_1}, \dots, (-1)^{\varepsilon_n})$*, where* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$*. This defines a bijection*

$$\begin{aligned} \mathbb{F}_2^E &\to \{+1, -1\}^E, \\ \varepsilon &\mapsto (-1)^\varepsilon. \end{aligned} \tag{4.1}$$

Definition 4.3.4. *Let* $\mathcal{M}$ *be a loopfree oriented matroid on the ground set* E *and let* $\Sigma_M \subseteq \mathbb{R}^E$ *the fan of the underlying matroid* $M = \underline{\mathcal{M}}$*. For every facet* $\sigma_{\mathcal{F}}$ *of* Σ_M *corresponding to the maximal flag of flats* $\mathcal{F}$*, we set*

$$\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}}) = \{\varepsilon \mid (-1)^\varepsilon \in \mathcal{T}(\mathcal{F})\} \subseteq \mathbb{F}_2^E.$$

If $\mathcal{M}$ *has loops* $L = \mathrm{cl}(\emptyset)$*, we set* $\mathcal{E}_{\mathcal{M}}(\sigma) := \mathcal{E}_{\mathcal{M}\setminus L}(\sigma) \subseteq \mathbb{F}_2^{E\setminus L}$*.*

The following proposition is proved in [RRS22].

Proposition 4.3.5. *The map* $\mathcal{E}_{\mathcal{M}}\colon \mathit{Facets}(\Sigma_M) \to \mathrm{Aff}_d(\mathbb{F}_2^{E\setminus \mathrm{cl}(\emptyset)})$ *from Definition 4.3.4 defines a real phase structure on* Σ_M*.*

To summarise, we have constructed a map from orientations of M (that is, oriented matroids $\mathcal{M}$ such that $\underline{\mathcal{M}} = M$) to real phase structures on Σ_M,

$$\{\text{Orientation of } M\} \xrightarrow{\mathbf{E}} \{\text{Real phase structure on } \Sigma_M\}, \tag{4.2}$$

which is given by $\mathbf{E}(\mathcal{M}) = \mathcal{E}_{\mathcal{M}}$. The main result of this paper claims that this map is bijective. We note that the map is injective, since the topes of $\mathcal{M}$ can be recovered from $\mathcal{E}$ as

$$\mathcal{T} = \{(-1)^\varepsilon \mid \varepsilon \in \bigcup_\sigma \mathcal{E}(\sigma)\},$$

where σ runs through all facets of Σ_M and moreover, an oriented matroid is determined by its collection of topes [dS95].

Remark 4.3.6. *Given an oriented matroid* $\mathcal{M}$ *and a subset* $S \subset E$*, the reorientation of* $\mathcal{M}$ *along* S *is the oriented matroid* $\mathcal{M}'$ *whose topes are the covectors* $r_S(T)$ *for any tope* T *of* $\mathcal{M}$*. Clearly, in this case* $\mathcal{E}_{\mathcal{M}'}$ *is a reorientation of* $\mathcal{E}_{\mathcal{M}}$ *(in the sense of Definition 4.1.3) with translation vector* ε_S*. Hence the map* $\mathbf{E}$ *from (4.2) descends to a map modulo reorientations on both sides.*

Example 4.3.7. *Here we consider the matroid* $M = U_{2,n}$ *and describe how we can construct an inverse to the map in (4.2). By Example 4.1.10, a real phase structure* $\mathcal{E}$ *on* Σ_M *determines a necklace ordering of* $\{1, \dots, n\}$*. Note that Example 4.1.10 described the projective matroid fan which can be obtained from* Σ_M *by quotienting by* $(1, \dots, 1)$*. To determine an orientation of* M *from the real phase structure* $\mathcal{E}$*, let* $H_1, \dots, H_n \subset \mathbb{R}^2$ *be a collection of pairwise distinct lines passing through the origin in* $\mathbb{R}^2$*, arranged so that the clockwise/anticlockwise appearance of these lines when making a turn around the origin defines the same necklace ordering on* $\{1, \dots, n\}$ *as the real phase structure* $\mathcal{E}$*. Two lines* H_i, H_j *border a chamber of* $\mathbb{R}^2$ *if and only if* $\mathcal{E}(\sigma_i) \cap \mathcal{E}(\sigma_j) \neq \emptyset$*. Moreover, if this is satisfied then* H_i, H_j *border exactly two chambers which are related by the antipodal map. If the intersection of the two affine spaces* $\mathcal{E}(\sigma_i)$ *and* $\mathcal{E}(\sigma_j)$ *is non-empty, then it consists of two points* $\varepsilon, \varepsilon' = \varepsilon + (1, \dots, 1)$*. Assigning* $(-1)^\varepsilon$ *to one of the chambers bordered by* H_i, H_j *and* $(-1)^{\varepsilon'}$ *to the other chamber determines an orientation of* $U_{2,n}$*. See Figure 4.4 for the example of the 4 lines in* $\mathbb{R}^2$ *corresponding to the real phase structure on the projective fan of* $U_{2,4}$ *from Figure 4.2. This procedure associates to any real phase structure on* Σ_M*, an orientation of* M *and it is easily checked that it provides an inverse map to* $\mathbf{E}$ *from (4.2).*

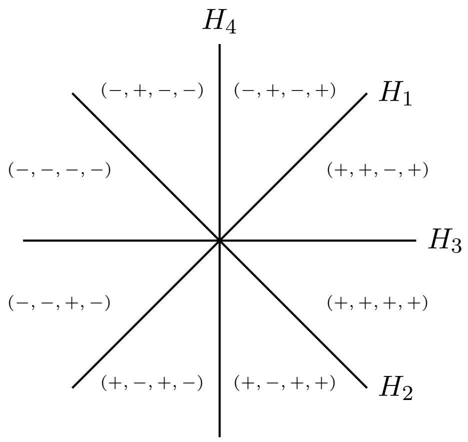

Figure 4.4 – An arrangement of 4 lines through the origin in $\mathbb{R}^2$ as used in Example 4.3.7 to go from real phase structures to oriented matroids. The necklace ordering of the lines corresponds to the necklace ordering of the facets of $\mathbb{P}\Sigma_M$ defined by the real phase structure in Figure 4.2.

Example 4.3.8. *Here we consider the matroid $M = U_{n-1,n}$. Orientations of M can be constructed from generic real hyperplane arrangements like in Example 4.3.1. Moreover, in Example 4.1.11 we showed that Σ_M carries a unique real phase structure up to reorientation. By Remark 4.3.6, the map (4.2) is surjective and hence bijective. An analogous discussion shows that (4.2) is bijective for $M = U_{1,n}$.*

Our main theorem is to prove the equivalence of real phase structures on matroid fans and matroid orientations. In the more general framework of matroids over hyperfields, oriented matroids are identical to matroids over the sign hyperfield $\mathbb{S} = \{0, +, -\}$ [BB18]. More precisely, a chirotope describing an oriented matroid can be interpreted as Grassmann-Plücker function on E^r with values in $\mathbb{S}$. Correspondingly, the affine subspaces provided by real phase structures live in a vector space over $\mathbb{F}_2 \cong \mathbb{S}^*$. For a general hyperfield $\mathbb{H}$ the non-zero elements form a group $\mathbb{H}^*$ under multiplication and we can exchange the role of affine subspaces of $\mathbb{F}_2^n$ for cosets of subgroups of $(\mathbb{H}^*)^n$.

Question. *Can matroids over a general hyperfield $\mathbb{H}$ be equivalently formulated by specifying cosets of the group $(\mathbb{H}^*)^n$ on top dimensional faces of a matroid fan $\Sigma \subseteq \mathbb{R}^n$?*

4.3.1 Real subfans and oriented matroid quotients

In this subsection, we introduce the concept of *real subfans* and study their relationship to oriented matroid quotients. Besides the intrinsic importance of these constructions, they will be used in the following subsection in the induction step in the proof of Theorem 4.0.1.

Given a real phase structure $\mathcal{E}$ on a polyhedral fan Σ, we extend the definition of $\mathcal{E}$ to non-maximal cones $\tau \in \Sigma$ by setting

$$\mathcal{E}(\tau) = \bigcup_{\substack{\sigma \text{ facet} \\ \tau \subset \sigma}} \mathcal{E}(\sigma).$$

As an example, note that if $\mathcal{E} = \mathcal{E}_{\mathcal{M}}$ is induced by an oriented matroid $\mathcal{M}$, the description of $\mathcal{E}(\sigma_{\mathcal{F}})$ as the set of elements ε such that $(-1)^\varepsilon$ is adjacent to $\mathcal{F}$ extends to non-maximal cones $\sigma_{\mathcal{F}}$, that is,

$$\mathcal{E}_{\mathcal{M}}(\sigma_{\mathcal{F}}) = \{\varepsilon \mid (-1)^{\varepsilon} \in \mathcal{T}(\mathcal{F})\} \subseteq \mathbb{F}_2^E. \tag{4.3}$$

The set above no longer has the structure of an affine subspace over $\mathbb{F}_2^E$ when $\sigma_{\mathcal{F}}$ is not a top dimensional face of the fan.

Definition 4.3.9. *Given a real phase structure $\mathcal{E}'$ on a fan Σ', we say that $(\Sigma', \mathcal{E}')$ is a* real subfan *of $(\Sigma, \mathcal{E})$ if $\Sigma' \subseteq \Sigma$ and for any $\tau \in \Sigma'$ we have $\mathcal{E}'(\tau) \subseteq \mathcal{E}(\tau)$.*

Real subfans occur naturally when considering the pair of contraction and deletion of a real phase structure along the same subset, as the following proposition shows (its proof can be found in [RRS22]).

Proposition 4.3.10. *Let M be a matroid on the ground set E and let $\mathcal{E}$ be real phase structure on Σ_M. Let F be a flat of M. Then $(\Sigma_{M/F}, \mathcal{E}/F)$ is a real subfan of $(\Sigma_{M\backslash F}, \mathcal{E}\backslash F)$.*

We denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ two oriented matroids on the same ground set E. For $i = 1, 2$, we denote by M_i, Σ_i, $\mathcal{E}_i$ the underlying matroids, associated matroid fans, and associated real phase structures, respectively.

We call $\mathcal{M}_1$ a *quotient* of $\mathcal{M}_2$ if all covectors of $\mathcal{M}_1$ are covectors of $\mathcal{M}_2$, see [BLVS+99, Section 7.7]. In this case, M_1 is a quotient of M_2, that is, all flats of M_1 are flats of M_2, see [BLVS+99, Corollary 7.7.3].

Proposition 4.3.11. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two loopfree oriented matroids on E. Then $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$ if and only if $(\Sigma_1, \mathcal{E}_1)$ is a real subfan of $(\Sigma_2, \mathcal{E}_2)$.*

For a proof, we refer to [RRS22].

Example 4.3.12. *For $M = U_{3,4}$, the choice of a real phase structure on $\Sigma_M \subset \mathbb{R}^4$ is equivalent to the choice of $\varepsilon \in \mathbb{F}_2^4$ such that $\varepsilon, \varepsilon' \notin \cup_\sigma \mathcal{E}(\sigma)$, where $\varepsilon' = \varepsilon + (1, \dots, 1)$.*

The uniform matroid $M' = U_{2,4}$ is an ordinary matroid quotient of $U_{3,4}$. Given a real phase structure $\mathcal{E}$ on M, there are 12 real phase structures on M' such that $\Sigma_{M'}$ equipped with one of these real phase structures produces a real subfan of $(\Sigma_M, \mathcal{E})$. We now describe them. For every $i \in \{1, \dots, 4\}$ consider the chain of flats

$$\mathcal{F}_i := \{\emptyset \subset \{i\} \subset \{1, 2, 3, 4\}\}.$$

The corresponding cone σ_i is in both Σ_M and $\Sigma_{M'}$. The set $\mathcal{E}(\sigma_i)$ consists of three affine spaces of dimension 3 in $\mathbb{F}_2^4$ which form a necklace arrangement. Moreover, the complement $\mathbb{F}_2^4 \backslash \mathcal{E}(\sigma_i)$ is the unique affine space of dimension two parallel to $T_{\mathbb{F}_2}(\sigma_i)$ containing the points $\varepsilon, \varepsilon'$.

Contained in $\mathcal{E}(\sigma_i)$ there are precisely 3 affine spaces of dimension 2 parallel to $T_{\mathbb{F}_2}(\sigma_i)$. These three affine spaces arise as the intersections of the 3-dimensional affine spaces in the necklace arrangement at σ_i. It follows from this that if $\mathcal{E}'(0) \subset \mathcal{E}(0)$, then $(\Sigma_{M'}, \mathcal{E}')$ is a real subfan of $(\Sigma_M, \mathcal{E})$. This is because if $\mathcal{E}'(0) \subset \mathcal{E}(0)$ then for each face σ_i the affine space $\mathcal{E}'(\sigma_i)$ cannot be equal to the 2-dimensional affine space $\mathbb{F}_2^4 \backslash \mathcal{E}(\sigma_i)$, and hence $\mathcal{E}'(\sigma_i) \subset \mathcal{E}(\sigma_i)$.

Following the description from Example 4.1.10, there are a total of 24 real phase structures on $\Sigma_{M'}$ given by combining the choice of 3 necklace orderings and a choice of the points in the intersection $\mathcal{E}'(\sigma_{i_1}) \cap \mathcal{E}'(\sigma_{i_2})$ where i_1 and i_2 are consecutive faces in the necklace ordering. Of the 24 real phase structures there are exactly 12 which contain $\varepsilon, \varepsilon'$ and 12 which do not contain them. Therefore, for $M' = U_{2,4}$ there are a total of 12 real subfans $(\Sigma_{M'}, \mathcal{E}')$ of $(\Sigma_M, \mathcal{E})$ for a fixed real phase structure $\mathcal{E}$.

The uniform matroid $M'' = U_{1,4}$ is also an ordinary matroid quotient of $U_{3,4}$. The matroid fan of $M'' = U_{1,4}$ is just a point which we denote by 0. A real phase structure $\mathcal{E}'$ on $\Sigma_{M''}$ produces a real subfan of $(\Sigma_M, \mathcal{E})$ if and only if $\mathcal{E}''(0) \neq \{\varepsilon, \varepsilon'\}$.

Following Theorem 4.0.1, the containment $\mathcal{E}'(0) \subset \mathcal{E}(0)$ corresponds to containment of topes of the corresponding oriented matroids. This relation between oriented matroids corresponds to weak maps, [BLVS+99, Proposition 7.7.5]. The condition of being a real phase subfan is not always equivalent to having $\mathcal{E}'(0) \subset \mathcal{E}(0)$ as the next example shows.

Example 4.3.13. *Consider again the uniform matroid $M = U_{3,4}$. The rank 2 matroid N on $E = \{1, \dots, 4\}$ where $1, 2$ are parallel and $3, 4$ are parallel is also an ordinary matroid quotient of $U_{3,4}$. The matroid fan of N is an affine space of dimension 2 in $\mathbb{R}^4$ and hence there are 4 possible real phase structures on Σ_N. However, only one of these possible real phase structures produces a real subfan of $(\Sigma_M, \mathcal{E})$. Indeed, if ρ_1 and ρ_2 denote the two half spaces which are top dimensional cones of Σ_M, then $\mathcal{E}(\rho_1)$ and $\mathcal{E}(\rho_2)$ are transversely intersecting affine subpaces of dimension 3 in $\mathbb{F}_2^4$, and hence their intersection $\mathcal{E}(\rho_1) \cap \mathcal{E}(\rho_2)$ gives the unique real phase structure on Σ_N yielding a real subfan. Yet three out of the four real phase structures on Σ_N satisfy $\mathcal{E}'(0) \subset \mathcal{E}(0)$ and hence correspond to weak maps of oriented matroids.*

4.3.2 The proof of Theorem 4.0.1

In this section we indicate how to prove Theorem 4.0.1. The idea of the proof is a follows : We use double induction on rank and corank of M. The induction step is governed by corank 1 quotients of oriented matroids and (real) tropical modifications, respectively. The crucial ingredient on the oriented matroid side is the positive answer to the *factorization problem* in corank 1. We start by recalling the related facts.

Let $\mathcal{M}_1$ be a quotient of $\mathcal{M}_2$. The factorization problem asks the question whether there exists an oriented matroid $\mathcal{M}$ on a larger ground set $E' \supset E$ such that $\mathrm{rk}_{\mathcal{M}}(E^c) = \mathrm{rk}(M_2) - \mathrm{rk}(M_1)$, $\mathcal{M}_1 = \mathcal{M}/E^c$ and $\mathcal{M}_2 = \mathcal{M} \backslash E^c = \mathcal{M}|E$.

Interestingly, the general answer to this question is no, see [Ric93]. For our purposes, however, it sufficient to consider $\mathrm{rk}(M_2) - \mathrm{rk}(M_1) = 1$, in which case the answer is positive [RZ94]. Here we present a slight generalization of the statement, which allows for parallel elements. The proofs can be found in [RRS22].

Lemma 4.3.14. *Let $E' = E \sqcup F$ be a finite set. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids on E such that $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$ and $\mathrm{rk}(\mathcal{M}_2) - \mathrm{rk}(\mathcal{M}_1) = 1$. Then there exists an oriented matroid $\mathcal{M}$ on E' such that $r_{\mathcal{M}}(F) = 1$, F contains no loops of $\mathcal{M}$, $\mathcal{M}/F = \mathcal{M}_1$ and $\mathcal{M} \backslash F = \mathcal{M}_2$. Moreover, the oriented matroid $\mathcal{M}$ is unique up to reorientation of elements in F.*

Lemma 4.3.15. *Let M be a matroid on E and let F be a flat of rank 1. Let $\mathcal{E}$ and $\mathcal{E}'$ be two real phase structures on Σ_M such that $\mathcal{E}/F = \mathcal{E}'/F$ and $\mathcal{E} \backslash F = \mathcal{E}' \backslash F$. Then $\mathcal{E}$ and $\mathcal{E}'$ agree up to reorientation by an element ε in the kernel of p_F.*

Lemma 4.3.15 and Lemma 4.3.10 tell us that deletion and contraction of real phase structures behave as expected in relation to oriented matroid quotients and real subfans in Proposition 4.3.11. Combining this with the existence of rank 1 extensions proved in Lemma 4.3.14, we are able to prove Theorem 4.0.1.

Proof of Theorem 4.0.1. We need to show that any real phase structure $\mathcal{E}$ on a matroid fan Σ_M can be represented as $\mathcal{E} = \mathcal{E}_{\mathcal{M}}$ for some oriented matroid $\mathcal{M}$. We proceed by double induction on rank and corank. The base cases for the induction are $U_{0,n}$ and $U_{n,n}$ and they are trivial.

In the general case, let F be an arbitrary flat of M of rank 1. Without loss of generality we may assume that M is loopfree. By the induction assumption, the real phase structures $\mathcal{E}/F$ and $\mathcal{E}\backslash F$ are represented by oriented matroids, say $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. By Lemma 4.3.10 and Proposition 4.3.11 we know that $\mathcal{M}_1$ is a quotient of $\mathcal{M}_2$. If F is a connected component of M, then the claim follows from the induction assumption applied to each connected component. Otherwise, we have $\mathrm{rk}(\mathcal{M}_2) - \mathrm{rk}(\mathcal{M}_1) = 1$. By Lemma 4.3.14, there exists an oriented matroid $\mathcal{M}$ on E such that $r_{\mathcal{M}}(F) = 1$, the flat F contains no loops of $\mathcal{M}$, $\mathcal{M}/F = \mathcal{M}_1$ and $\mathcal{M}\backslash F = \mathcal{M}_2$. Since $\underline{\mathcal{M}}/F = M/F$ and $\underline{\mathcal{M}}\backslash F = M\backslash F$, the uniqueness of this extension for ordinary matroids implies that $\underline{\mathcal{M}} = M$, see [Whi86, Proposition 8.3.1]. It follows that $\mathcal{E}' = \mathcal{E}_{\mathcal{M}}$ and $\mathcal{E}$ are two real phase structures on Σ_M whose deletion and contraction along F agree. By Lemma 4.3.15, they agree up to reorientation along some ε. Since reorientations of oriented matroids and real phase structures are compatible, this shows that the corresponding reorientation of $\mathcal{M}$ represents $\mathcal{E}$, which proves the claim. □

Remark 4.3.16. *The previous discussion shows that oriented matroid quotients (or real subfans of codimension* 1 *of matroid fans) play a special role. In general, if* $(\Sigma_1, \mathcal{E}_1)$ *is a real subfan of* $(\Sigma_2, \mathcal{E}_2)$*, we may ask whether this inclusion can be completed to a chain of real subfans whose dimensions increase by one in each step. Interestingly, there is a counter-example of Richter-Gebert [Ric93, Corollary 3.4] which shows that this is in general not the case. It tropical language, it gives rise to a pair of real matroid subfans* $(\Sigma_1, \mathcal{E}_1) \subset (\Sigma_2, \mathcal{E}_2)$ *with* $\dim \Sigma_1 = 1$ *and* $\dim \Sigma_2 = 3$ *such that there exists no real matroid fan* $(\Sigma, \mathcal{E})$ *such that* $(\Sigma_1, \mathcal{E}_1) \subsetneq (\Sigma, \mathcal{E}) \subsetneq (\Sigma_2, \mathcal{E}_2)$*.*

This is in contrast to non-oriented matroids (equivalently, matroid fans without real phase structures), where the factorization problem can be answered affirmatively and hence such chains always exist [Whi86, Chapter 8.2]

Chapitre 5

Real phase structures on non-singular tropical varieties

5.1 Non-singular patchworks : the definition

Let M be a n-dimensional lattice and let $\Sigma \subset N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$ be a rational strongly convex polyhedral fan. Let $X \subset \mathbb{T}\Sigma$ be a non-singular tropical variety. Assume for simpicity that the relative interior of any facet of X is of order of sedentarity 0 (meaning that it is contained in the tropical orbit $\mathcal{O}_0^{\mathbb{T}} \cong \mathbb{R}^n$). We refer to [RRS23] for the more general situation. Assume also that X is of pure dimension, and denote it by d.

Definition 5.1.1. *A* real phase structure *on X is a map*

$$\mathcal{E} \colon \mathrm{Facets}(X) \to \mathrm{Aff}_d(N \otimes \mathbb{F}_2)$$

satisfying

1. *for every facet σ of X, the set $\mathcal{E}(\sigma)$ is an affine subspace of $N \otimes \mathbb{F}_2$ parallel to σ.*

2. *for every codimension one face τ of X with facets $\sigma_1, \dots, \sigma_k$ adjacent to it, the sets $\mathcal{E}(\sigma_1), \dots, \mathcal{E}(\sigma_k)$ are an even covering.*

Remark 5.1.2. *The definition is exactly the same as the real phase structure of a fan, and in fact we could define the same way a real phase structure on any rational polyhedral complex in $\mathbb{T}\Sigma$. We restrict ourself to the case of a tropical variety because it is the case we are interested in here.*

We extend the real phase structure to faces of X of any dimension. Remind some notations from Section 3.2. If ρ is a cone of a fan Σ, we denote $\mathbb{Z}(\rho) = \mathrm{Hom}(\rho^{\perp} \cap M, \mathbb{Z})$ and $\mathbb{F}_2(\rho) = \mathbb{Z}(\rho) \otimes \mathbb{F}_2$ and also $\mathbb{R}(\rho)$ the tropical orbit $\mathcal{O}_\rho^{\mathbb{T}} = \mathrm{Hom}(\rho^{\perp} \cap M, \mathbb{T})$. Given an arbitrary face $\tau \subset X$ of sedentarity η, we set

$$\mathcal{E}(\tau) = \bigcup_{\substack{\sigma \in \mathrm{Facets}(\mathcal{X}) \\ \tau \subset \sigma}} \pi(\mathcal{E}(\sigma)),$$

where π denotes the canonical projection $\pi \colon N \otimes \mathbb{F}_2 \to \mathbb{F}_2(\eta)$. Note that in general $\mathcal{E}(\tau)$ is not an affine subspace of $\mathbb{F}_2(\eta)$.

Example 5.1.3. *The tropical line in $\mathbb{T}P^2$ from Example 3.5.2 contains three points of non-zero sedentarity. The projection of the affine vector spaces $\mathcal{E}_{\sigma_1}, \mathcal{E}_{\sigma_2}$ for the horizontal and vertical edges are simply $0 \in \mathcal{F}_1(\mathbb{TP}^{n+1}_{\{1\}}) \otimes \mathbb{F}_2$ and $0 \in \mathcal{F}_1(\mathbb{TP}^{n+1}_{\{2\}}) \otimes \mathbb{F}_2$, respectively. For the diagonal edge we obtain $\mathcal{E}_{\sigma_0} = 1 \in \mathcal{F}_1(\mathbb{TP}^{n+1}_{\{0\}}) \otimes \mathbb{F}_2$.*

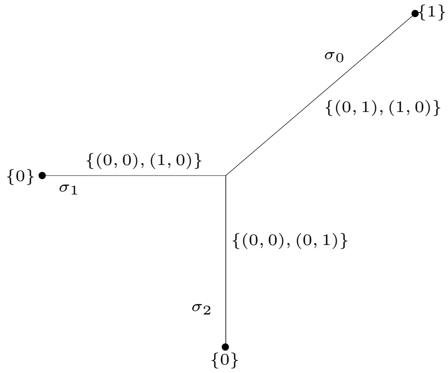

Figure 5.1 – The extension of the real phase structure on the tropical line from Example 3.5.2 to faces of non-zero sedentarity.

Example 5.1.4. *The real phase structure on the tropical plane $X \subset \mathbb{T}P^3$ from Example 3.5.3 can be extended to the facets of all strata X_I for I a proper subset of $\{0, \dots, n+1\}$. If $|I| = 1$ then X_I is a tropical line in as in Example 3.5.2. Consider for example $I = \{1\}$ and the facet $\rho_2 = \overline{\sigma_{12}} \cap X_{\{1\}}$. The projection $\pi_{\sigma_{12}\rho_2}$ has kernel the first coordinate direction and therefore $\mathcal{E}_{\rho_2} = \langle e_1, e_2\rangle / \langle e_1 \rangle \subset \mathcal{F}_1(\mathbb{TP}^3_{\{1\}}) \otimes \mathbb{F}_2$. Furthermore, $X_{\{12\}}$ is a point and $\mathcal{E}_{X_{\{12\}}} = 0$.*

Recall from the preliminaries (see also [GKZ08]) that the real toric variety $\mathbb{R}\Sigma$ is homeomorphic to the following space

$$\bigsqcup_{\xi \in \mathrm{Hom}(M, \mathbb{F}_2)} \mathbb{T}\Sigma(\xi) / \sim, \tag{5.1}$$

where the $\mathbb{T}\Sigma(\xi)$ are disjoint copies of $\mathbb{T}\Sigma$ and $(p, \xi) \sim (p', \xi')$ if and only if $p = p'$ and $\xi|_{\sigma^\perp} = \xi'|_{\sigma^\perp}$ for the unique cone σ such that $p \in \mathcal{O}^{\mathbb{T}}_\sigma$. We will denote the quotient 5.1 by $\mathbb{R}\mathbb{T}\Sigma$. It is a cellular complex homeomorphic to $\mathbb{R}\Sigma$ as a topological spave but having its own cellular structure. For example the spaces $\mathbb{R}\mathbb{T}^n$ and $\mathbb{R}\mathbb{T}\mathbb{P}^n$ are obtained by gluing 2^n copies of $\mathbb{T}^n$ and $\mathbb{TP}^n$ respectively and are homeomorphic to $\mathbb{R}^n$ and $\mathbb{RP}^n$ respectively. To explain the connection to the real toric variety $\mathbb{R}\Sigma$, first note that $\log\colon (\mathbb{R}_{\geq}, \cdot) \to (\mathbb{T}, +)$ (with $\log(0) = -\infty$) is an isomorphism of semigroups. Moreover, $\mathbb{R}$ can be described as the semigroup quotient of $\mathbb{R}_{\geq} \times \mathbb{F}_2$ modulo $\{0\} \times \mathbb{Z}_2$. Hence we get an extended semigroup homomorphism $\log^{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}\mathbb{T}$. These semigroup homomorphisms yield homeomorphisms $\mathrm{Log}\colon \mathbb{R}_{\geq}\Sigma \to \mathbb{T}\Sigma$ and $\mathrm{Log}^{\mathbb{R}}\colon \mathbb{R}\Sigma \to \mathbb{R}\mathbb{T}\Sigma$. Note also that this construction is a tropical reformulation of for example [GKZ08, Theorem 11.5.4].

Example 5.1.5. *We consider the tropical affine space $\mathbb{T}\Sigma = \mathbb{T}^n$. In continuation of the notation introduced in Example 3.1.2, given $A \subset E = \{1, \dots, n\}$ and $\varepsilon \in \mathbb{F}_2^{E \setminus A}$, we denote by $\mathbb{T}^n_A(\varepsilon)$ the corresponding copy of $\mathbb{T}^n_A$ in $\mathbb{R}\mathbb{T}^n$. Instead of tuples (A, ε), we can alternatively label the strata of $\mathbb{R}\mathbb{T}^n$ by the set of* signed vectors *$C \in \{0, +1, -1\}^n$. The connection is given by setting $A = E \setminus supp(C)$ and $C|_{supp(C)} = (-1)^\varepsilon$.*

For $i = 1, \dots, n$, we use the shorthand $H_i := \bigcup_{\varepsilon \in \mathbb{F}_2^{E \setminus \{i\}}} \mathbb{T}^n_{\{i\}}(\varepsilon) \subset \mathbb{R}\mathbb{T}^n$. Under $\mathrm{Log}^{\mathbb{R}}$, this subset corresponds to the coordinate hyperplane $\{x_i = 0\} \subset \mathbb{R}^n$.

Given a polyhedron σ in $\mathcal{O}^{\mathbb{T}}_\rho$ and $\varepsilon \in \mathbb{F}_2(\rho)$, we let $\sigma(\varepsilon)$ denote the copy of σ in $\mathbb{R}\mathbb{T}\Sigma$. We extend the notion of polyhedral complex from $\mathbb{T}\Sigma$ to $\mathbb{R}\mathbb{T}\Sigma$ by declaring the sets $\sigma(\varepsilon)$ to be the polyhedra of $\mathbb{R}\mathbb{T}\Sigma$.

Definition 5.1.6. *Let X be a non-singular tropical variety in $\mathbb{T}\Sigma$ equipped with a real phase structure $\mathcal{E}$. The* patchwork *of $(X,\mathcal{E})$ is the polyhedral complex $\mathbb{R}_{\mathcal{E}}X$ in $\mathbb{R}\mathbb{T}\Sigma$ given by*

$$\mathbb{R}_{\mathcal{E}}X = \{\sigma(\varepsilon) : \sigma \subset X, \varepsilon \in \mathcal{E}(\sigma)\}.$$

5.2 Non-singular patchworks are topological manifold

In this section we give a proof of the following theorem. See [RRS23] for eventually more details.

Theorem 5.2.1. *Let $X \subset \mathbb{T}\Sigma$ be a non-singular tropical subvariety of a tropical toric variety $\mathbb{T}\Sigma$. Let $\mathcal{E}$ be a real phase structure on X. Then the patchwork $\mathbb{R}_{\mathcal{E}}X$ is a topological manifold.*

We start with two essential examples describing the faces of the simplicial complexe coming from patchworking closure of (affine and projective) matroidal fans

Example 5.2.2. *Let $\mathcal{M}$ be an oriented matroid with underlying matroid M and let $\mathcal{E}$ be the associated real phase structure on Σ_M. Hence $\mathcal{E}$ is also a real phase structure on $\overline{\Sigma}_M \subset \mathbb{T}^n$. By combining Example 5.1.5 with the definition of $\mathcal{E}$, we see that the faces of the polyhedral complex $\mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M \subset \mathbb{R}\mathbb{T}^n$ are labelled by the chains of covectors $0 < Z_1 < \cdots < Z_l$ from $\mathcal{M}$. The lattice of covectors is called sometime the big face lattice. See [BLVS+99, Definition 4.1.2]. The final covector Z_l determines the stratum of $\mathbb{R}\mathbb{T}^n$ that contains the face. The associated chain of flats $F_i = \phi(Z_{l-i})$ determines the corresponding face of $\overline{\Sigma}_M$. The map ϕ here denotes the forgetful map from oriented matroids to matroids sending a covector X to the flat $\mathrm{Supp}(X)^c$, see Section 4.3. In particular, the intersection $H_i \cap \mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M$ is the subcomplex corresponding to all chains such that $i \in \phi(Z_l)$.*

Example 5.2.3. *Let $\mathcal{M}$ be an oriented matroid with underlying matroid M and let $\mathcal{E}$ be the associated real phase structure on $\mathbb{P}\Sigma_M \subset \mathbb{R}^{n+1}/(1,\ldots,1) \simeq \mathbb{R}^n$. Again, we denote by the same letter the induced real phase structure on $\overline{\mathbb{P}\Sigma}_M \subset \mathbb{T}\mathbb{P}^n$.*

Pick $\varepsilon \in \mathbb{Z}_2^{n+1}$ such that $T = (-1)^{\varepsilon}$ is a tope of $\mathcal{M}$. We denote by $\mathbb{T}\mathbb{P}^n(\varepsilon) \subset \mathbb{R}\mathbb{T}\mathbb{P}^n$ the copy of $\mathbb{T}\mathbb{P}^n$ in $\mathbb{R}\mathbb{T}\mathbb{P}^n$ obtained as the projection of $\mathbb{T}^{n+1}(\varepsilon)$. We are interested in the polyhedral complex $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M \cap \partial\mathbb{T}\mathbb{P}^n(\varepsilon)$. By Examples 3.1.2 and 5.2.2, it is a simplicial complex whose faces are labelled by chains of covectors $0 < Z_1 < \cdots < Z_l < T$ or, in other words, chains of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_l \subsetneq E$ which are touching T. The lattice of flats touching ε is known as the Las Vergnas lattice *of $\mathcal{M}$ with respect to T. We denote it by $\mathcal{F}_{lv}^{\mathcal{M}}(T)$. See [BLVS+99, Definition 4.1.2]*

The simplicial complexes mentioned in the previous examples have been studied by Folkman and Lawrence in the context of the topological representation theorem for oriented matroids. We translate the results to our setup in the following proposition. To state the results in the conventional way, it is useful to intersect with a sphere $S^{n-1} \subset \mathbb{R}\mathbb{T}^n$. A natural choice for S^{n-1} is given by taking, in each stratum $\mathbb{T}^n(\varepsilon)$ of $\mathbb{R}\mathbb{T}^n$, the symmetric copy of the boundary of the negative orthant, that is

$$S(\varepsilon) = \{x \in \mathbb{T}^n(\varepsilon) \mid \forall\ i\ x_i \leq 0 \text{ and } \exists\ i :\ x_i = 0\}.$$

The union of all of the $S(\varepsilon)$ in $\mathbb{R}\mathbb{T}^n$ is a polyhedral sphere of dimension $n-1$, see 5.2. In fact, it is the boundary of a cube. We do a simple example to help the reader with all the heavy notations.

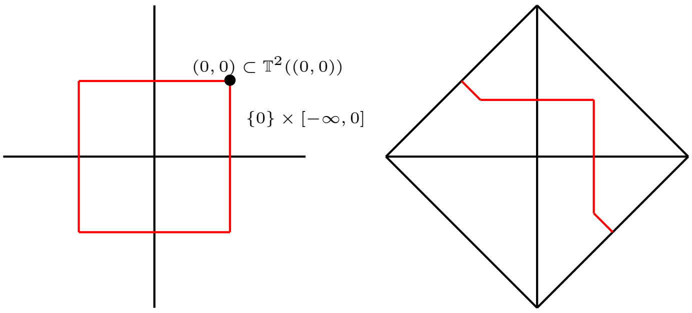

FIGURE 5.2 – On the left is a polyhedral sphere (drawn in red) in $\mathbb{R}\mathbb{T}^2$. On the right is some patchwork $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M$ (drawn in red) in $\mathbb{R}\mathbb{T}\mathbb{P}^2$ for the uniform matroid $M = U_{2,3}$.

Proposition 5.2.4. *Let M be a matroid of rank d on n elements and let $\mathcal{E}$ be a real phase structure on Σ_M.*

1. *The complexes $\mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M \cap S^{n-1}$, $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M$ and $\mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M$ are topological manifolds and homeomorphic to S^{d-1}, $\mathbb{R}\mathbb{P}^{d-1}$ and $\mathbb{R}^d$, respectively.*

2. *For a sign vector $\varepsilon \in \cup_{\sigma \in \Sigma_M} \mathcal{E}(\sigma)$, the complexes $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M \cap \partial\mathbb{T}\mathbb{P}^n(\varepsilon)$ and $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M \cap \mathbb{T}\mathbb{P}^n(\varepsilon)$ are topological manifolds (with boundary in the second case) and homeomorphic to S^{d-2} and D^{d-1}, respectively.*

3. *The subcomplexes $H_i \cap \mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M \cap S^{n-1} \subset \mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M \cap S^{n-1}$, $i = 1, \dots, n$, form a pseudosphere arrangement. The associated oriented matroid is equal to $\mathcal{M}_{\mathcal{E}}$.*

Démonstration. Let $\mathcal{M} = \mathcal{M}_{\mathcal{E}}$ denote the oriented matroid associated to $\mathcal{E}$ by Theorem 4.0.1. By Example 5.2.2, the simplicial complex $\mathbb{R}_{\mathcal{E}}\overline{\Sigma}_M \cap S^{n-1} \subset \mathbb{R}\mathbb{T}^n$ is a geometric realization of the order complex of the poset $\mathcal{C}$ of covectors of $\mathcal{M}$. Analogously, by Example 5.2.3, given ε such that $T = (-1)^{\varepsilon}$ is tope of $\mathcal{M}$, the simplicial complex $\mathbb{R}_{\mathcal{E}}\overline{\mathbb{P}\Sigma}_M \cap \partial\mathbb{T}\mathbb{P}^n(\varepsilon)$ is the order complex of the poset $\mathcal{F}_{lv}^{\mathcal{M}}(T) \setminus \{0, T\}$. The statements then follow from the Folkman-Lawrence topological representation theorem [FL78]. See also [BLVS+99, Theorems 4.3.3, 4.3.5, 5.2.1]. □

Proof of 5.2.1. Fix a point $p \in X$. Let $\psi \colon \mathbb{T}U_\rho \to \mathbb{T}^k \times \mathbb{R}^{n-k}$, $p \in U \subset \mathbb{T}U_\rho$ and M be the objects guaranteed by the definition of X being non-singular at p. Since $(-\infty, 0) \in \overline{\mathbb{P}\Sigma}_M$, the set $F = \{1, \dots, k\}$ is a flat of M. After replacing M by $F \oplus M/F$ (and possibly shrinking U), we may assume that v_F is contained in the lineality space of $\mathbb{P}\Sigma_M$. In particular, any facet of $\mathbb{P}\Sigma_M$ (in the coarse sense) intersects $\psi(U)$. It follows that ψ induces a real phase structure from $X \cap U$ on $\mathbb{P}\Sigma_M$.

Given an open subset of a toric variety $U \subset \mathbb{T}\Sigma$, we denote by $\mathbb{R}U \subset \mathbb{R}\mathbb{T}\Sigma$ the open subset obtained by copying U in each stratum $\mathbb{T}\Sigma(\varepsilon) \subset \mathbb{R}Y$. We consider the open subset $\mathbb{R}X \cap \mathbb{R}U \subset \mathbb{R}X$. By definition, under ψ it is homeomorphic to $\mathbb{R}\overline{\mathbb{P}\Sigma}_M \cap \psi(\mathbb{R}U)$. Since $\mathbb{R}\overline{\mathbb{P}\Sigma}_M$ is a topological manifold by Corollary 5.2.4, it follows that $\mathbb{R}X$ is a topological manifold at all points in $\mathbb{R}X \cap \mathbb{R}U$. Since p was arbitrary, these sets cover $\mathbb{R}X$, hence the claim follows. □

Chapitre 6

A spectral sequence for the homology of a patchwork

In this section, given a non-singular tropical subvariety X of a tropical toric variety $\mathbb{T}\Sigma$ and a real phase structure $\mathcal{E}$ on X, we describe following [RS23] a filtration of $H_*(\mathbb{R}_\mathcal{E}X;\mathbb{F}_2)$ and we give a proof of the following theorem

Theorem 6.0.1. *If X is a d-dimensional non-singular tropical subvariety equipped with a real phase structure $\mathcal{E}$, then*

$$b_q(\mathbb{R}_\mathcal{E}X) \leq \sum_{p=0}^{d} \dim H_q(X;\mathcal{F}_p \otimes \mathbb{F}_2)$$

and

$$b_q^{BM}(\mathbb{R}_\mathcal{E}X) \leq \sum_{p=0}^{d} \dim H_q^{BM}(X;\mathcal{F}_p \otimes \mathbb{F}_2).$$

Moreover, the (Borel-Moore) Euler characteristic of $\mathbb{R}_\mathcal{E}X$ is equal to the (Borel-Moore) tropical signature of X, namely,

$$\chi(\mathbb{R}_\mathcal{E}X) = \sigma(X), \qquad \chi^{BM}(\mathbb{R}_\mathcal{E}X) = \sigma^{BM}(X).$$

6.1 The sign cosheaf

Let X be a polyhedral complex in $\mathbb{T}\Sigma$. Suppose X is equipped with a real phase structure $\mathcal{E}$. We now recall the sign cosheaf defined in [RS23].

We define the abstract vector space $\mathcal{S}_\mathcal{E}(\tau)$ with basis given by the elements of $\mathcal{E}(\tau)$,

$$\mathcal{S}_\mathcal{E}(\tau) = \mathbb{F}_2 \langle w_\varepsilon \mid \varepsilon \in \mathcal{E}(\tau)\rangle . \tag{6.1}$$

The vector space $\mathcal{S}_\mathcal{E}(\tau)$ is a linear subspace of the abstract vector space $\mathbb{F}_2 \langle w_\varepsilon \mid \varepsilon \in T_{\mathbb{F}_2}(\mathbb{T}\Sigma_\rho)\rangle$, where ρ is the sedentarity of τ. It follows from the definition of a real phase structure that

$$i_{\tau\tau'} \colon \mathcal{S}_\mathcal{E}(\tau) \to\colon \mathcal{S}_\mathcal{E}(\tau'), \quad w_\varepsilon \mapsto w_{\pi_{\rho\eta}(\varepsilon)} \tag{6.2}$$

is a well-defined linear map for all $\tau' \subset \tau$ of sedentarity η (resp. ρ). The map $\pi_{\rho\eta}$ are the ones defined in Equation 3.5.

Definition 6.1.1. *The* sign cosheaf $\mathcal{S}_\mathcal{E}$ *on X is the cellular cosheaf on X given by the spaces in (6.1) and the cosheaf maps in (6.2).*

Example 6.1.2. *We describe some of the vector spaces $\mathcal{S}_{\mathcal{E}}(\tau)$ and maps between them for the real phase structure on the tropical plane $X \subset \mathbb{T}P^3$ from Example 3.5.3.*

For the facets σ_{01}, σ_{12}, and σ_{13} of sedentarity 0 from Example 3.5.3 we have,

$$\mathcal{S}_{\mathcal{E}}(\sigma_{01}) = \langle w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3}, w_{\varepsilon_1+\varepsilon_2}, w_{\varepsilon_2}\rangle,$$

$$\mathcal{S}_{\mathcal{E}}(\sigma_{12}) = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2}\rangle, \text{ and}$$

$$\mathcal{S}_{\mathcal{E}}(\sigma_{13}) = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3}\rangle.$$

Consider the one dimensional face τ_1 of sedentarity 0 and in direction e_1. Then we have

$$\mathcal{S}_{\mathcal{E}}(\tau_1) = \langle w_\varepsilon \mid \varepsilon \in \mathcal{E}_{\sigma_{12}} \cup \mathcal{E}_{\sigma_{13}} \cup \mathcal{E}_{\sigma_{01}}\rangle = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2}, w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3}\rangle,$$

and there is an injection $i_{\sigma_{12}\tau_1} \colon \mathcal{S}_{\mathcal{E}}(\sigma_{12}) \to \mathcal{S}_{\mathcal{E}}(\tau_1)$.

For the face ρ_2 from Example 5.1.4 we have

$$\mathcal{S}_{\mathcal{E}}(\rho_2) = \langle w_0, w_{\varepsilon_2}\rangle \subset \mathbb{F}_2 \left\langle w_\varepsilon \mid \varepsilon \in \mathcal{F}_1(\mathbb{TP}^{n+1}_{\{1\}}) \otimes \mathbb{F}_2 \right\rangle.$$

The map $i_{\sigma_{12}\rho_2} \colon \mathcal{S}_{\mathcal{E}}(\sigma_{12}) \to \mathcal{S}_{\mathcal{E}}(\rho_2)$ has kernel generated by w_{ε_1} and $w_{\varepsilon_1+\varepsilon_2}$. These bases elements correspond to the points in the kernel of the map $\pi_{\sigma_{12}\rho_2}$.

The proof of the next proposition is exactly the same as the proof of Proposition 3.17 in [RS23]. We denote by $C_\bullet(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2)$ and $C^{BM}_\bullet(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2)$ the ordinary cellular chain complexes of $\mathbb{R}_{\mathcal{E}}X$ over $\mathbb{F}_2$.

Proposition 6.1.3. *Let X be a polyhedral complex with real phase structure $\mathcal{E}$ and associated sign cosheaf $\mathcal{S}$. Then we have isomorphisms of chain complexes*

$$C_\bullet(X; \mathcal{S}) \cong C_\bullet(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2) \quad \text{and} \quad C^{BM}_\bullet(X; \mathcal{S}) \cong C^{BM}_\bullet(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2)$$

and therefore also isomorphisms

$$H_q(X; \mathcal{S}) \cong H_q(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2) \quad \text{and} \quad H^{BM}_q(X; \mathcal{S}) \cong H^{BM}_q(\mathbb{R}_{\mathcal{E}}X; \mathbb{F}_2).$$

6.2 The filtration of the sign cosheaf

We describe here a filtration of the sign cosheaf $\mathcal{S}_{\mathcal{E}}$. This filtration is a direct generalisation of the one described in [RS23] in the case of hypersurfaces. Given a subset $H \subset T_{\mathbb{F}_2}(\rho)$, we denote by $\mathrm{Aff}_p(H)$ the set of all p-dimensional affine subspaces of $T_{\mathbb{F}_2}(\rho)$ contained in H. For a subset $G \subset T_{\mathbb{F}_2}(\rho)$, set

$$w_G := \sum_{\epsilon \in G} w_\epsilon \in \mathbb{F}_2 \langle w_\varepsilon \mid \varepsilon \in T_{\mathbb{F}_2}(\rho)\rangle.$$

Finally, we define

$$K_p(H) := \langle w_G \mid G \in \mathrm{Aff}_p(H)\rangle.$$

Definition 6.2.1. *Let X be a polyhedral complex of sedentarity ρ in $\mathbb{T}\Sigma$ with real phase structure $\mathcal{E}$. For all p, we define a collection of cosheaves $\mathcal{K}_p$ on X. On a face τ of X of sedentarity η, we set*

$$\mathcal{K}_p(\tau) = \sum_{\sigma \supset \tau} K_p(\pi_{\rho\eta}(\mathcal{E}(\sigma))) \subset \mathcal{S}_{\mathcal{E}}(\tau), \tag{6.3}$$

where the sum runs through facets of X. If $\tau' \subset \tau$, the cosheaf maps $\mathcal{K}_p(\tau) \to \mathcal{K}_p(\tau')$ are the restrictions of the maps $i_{\tau\tau'} \colon \mathcal{S}_{\mathcal{E}}(\tau) \to \mathcal{S}_{\mathcal{E}}(\tau')$.

Remark 6.2.2. *Note that, with the assumption from above,*

$$\mathcal{E}(\tau) = \bigcup_{\sigma \supset \tau} \pi_{\rho\eta}(\mathcal{E}(\sigma)),$$

where the union runs through facets of X. This explains the inclusion $\mathcal{K}_p \subset \mathcal{S}_\mathcal{E}$ mentioned in the above definition. In fact, the cosheaves $\mathcal{K}_p$ form a filtration of $\mathcal{S}_\mathcal{E}$ given by

$$\mathcal{K}_d \subset \cdots \subset \mathcal{K}_2 \subset \mathcal{K}_1 \subset \mathcal{K}_0 = \mathcal{S}_\mathcal{E}. \tag{6.4}$$

Indeed, any affine subspace G of dimension p can be written as disjoint union $G = H_1 \sqcup H_2$ of two affine subpaces of dimension $p-1$. Then $w_G = w_{H_1} + w_{H_2}$, which shows $\mathcal{K}_p \subset \mathcal{K}_{p-1}$.

Example 6.2.3. *For the real tropical plane from Example 3.5.3 we describe the filtration in (6.4) for some faces. For every facet σ_{ij} of X, the vector space $\mathcal{K}_1(\sigma_{ij})$ is of codimension one in $\mathcal{S}_\mathcal{E}(\sigma_{ij})$. Moreover, the vector space $\mathcal{S}_\mathcal{E}(\sigma_{ij})$ is two dimensional. Therefore, the only element in $\mathrm{Aff}_2(\mathcal{S}_\mathcal{E}(\sigma))$ is the whole vector space itself. This implies that $\mathcal{K}_2(\sigma_{ij}) = \langle w_{\mathcal{S}_\mathcal{E}(\sigma_{ij})} \rangle$, in particular it is one dimensional. For instance for σ_{12} we have,*

$$\mathcal{S}_\mathcal{E}(\sigma_{12}) = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2} \rangle,$$

$$\mathcal{K}_1(\sigma_{12}) = \langle w_0 + w_{\varepsilon_1}, w_0 + w_{\varepsilon_2}, w_0 + w_{\varepsilon_1+\varepsilon_2} \rangle, \textit{ and}$$
$$\mathcal{K}_2(\sigma_{12}) = \langle w_0 + w_{\varepsilon_1} + w_{\varepsilon_2} + w_{\varepsilon_1+\varepsilon_2} \rangle.$$

For the face τ_1 from Example 6.1.2, since $\mathcal{K}_1(\tau_1)$ is generated by $\mathcal{K}_1(\sigma_{01})$, $\mathcal{K}_1(\sigma_{12})$, and $\mathcal{K}_1(\sigma_{13})$, we have

$$\mathcal{K}_1(\tau_1) = \langle w_0 + w_{\varepsilon_1}, w_0 + w_{\varepsilon_2}, w_0 + w_{\varepsilon_1+\varepsilon_2}, w_0 + w_{\varepsilon_3}, w_0 + w_{\varepsilon_1+\varepsilon_3} \rangle.$$

For $p = 2$ we have

$$\mathcal{K}_2(\tau_1) = \langle w_{\mathcal{S}_\mathcal{E}(\sigma_{01})}, w_{\mathcal{S}_\mathcal{E}(\sigma_{12})} \rangle,$$

since $w_{\mathcal{S}_\mathcal{E}(\sigma_{01})} + w_{\mathcal{S}_\mathcal{E}(\sigma_{12})} + w_{\mathcal{S}_\mathcal{E}(\sigma_{13})} = 0$.

Assume that $H \subset T_{\mathbb{F}_2}(\mathbb{T}\Sigma_\rho)$ is a affine subspace. Then by general linear algebra we have

$$K_p(H)/K_{p+1}(H) \cong \bigwedge^p T(H), \tag{6.5}$$

induced by mapping w_G to $v_1 \wedge \cdots \wedge v_p$. Here $G \in \mathrm{Aff}_p(M)$ and $v_1, \ldots, v_p$ form a basis of $T(G)$. Furthermore, note that $\mathcal{F}_p$ can be written in the form

$$\mathcal{F}_p(\tau) = \sum_{\sigma \supset \tau} \bigwedge^p \pi_{\rho\eta}(T_{\mathbb{F}_2}(\sigma))$$

analogous to (6.3). Since

$$\pi_{\rho\eta}(T_{\mathbb{F}_2}(\sigma)) = \pi_{\rho\eta}(T_{\mathbb{F}_2}(\mathcal{E}(\sigma))) = T_{\mathbb{F}_2}(\pi_{\rho\eta}(\mathcal{E}(\sigma))),$$

the restriction of the map $\mathcal{K}_p(T_{\mathbb{F}_2}(\mathbb{T}\Sigma_\rho)) \to \bigwedge^p T_{\mathbb{F}_2}(\mathbb{T}\Sigma_\rho)$ from (6.5) to $\mathcal{K}_p(\tau)$ takes image in $\mathcal{F}_p(\tau)$. Hence, it gives rise to a morphism of cellular cosheaves $\mathcal{K}_p \to \mathcal{F}_p$. Applying (6.5) to each summand, we see that this morphism is surjective and $\mathcal{K}_{p+1}$ lies in the kernel. Since the sum is not a direct sum, in general, we cannot conclude that $\mathcal{K}_{p+1}$ is *equal* to the kernel. Equality holds, however, if X represents a non-singular tropical variety. This is the content of the following proposition.

Proposition 6.2.4. *Let X be a non-singular tropical variety in $\mathbb{T}\Sigma$ with a real phase structure $\mathcal{E}$. Then for all p there is an exact sequence of cosheaves*

$$0 \to \mathcal{K}_{p+1} \to \mathcal{K}_p \to \mathcal{F}_p \to 0.$$

Démonstration. The proof follows the same lines as the proof of Proposition 4.10 in [RS23]. As mentioned before, it is true in general that $\mathcal{K}_{p+1} \to \mathcal{K}_p$ (the inclusion map) is injective, $\mathcal{K}_p \to \mathcal{F}_p$ is surjective, and $\mathcal{K}_{p+1} \subset \ker(\mathcal{K}_p \to \mathcal{F}_p)$. We prove equality in the last statement by a comparison of dimensions. More precisely, let X be a polyhedral complex and pick a face $\tau \subset X$. It suffices to show that

$$\sum_{p=0}^{\dim X} \dim \mathcal{F}_p(\tau) = \dim \mathcal{S}_{\mathcal{E}}(\tau).$$

This statement is a local statement and so by Definition 3.6.2 we may assume $X = \overline{\mathbb{P}\Sigma}_M$ for some matroid M. By Theorem 4.0.1 there is an oriented matroid $\mathcal{M}$ representing the real phase structure on X.

The dimension of $\mathcal{S}_{\mathcal{E}}(\tau)$ is equal to $| \cup_{\sigma \subset X} \mathcal{E}(\sigma)|$ and so $2 \dim \mathcal{S}_{\mathcal{E}}(\tau)$ is equal to the number of topes of the oriented matroid of M. By [Zas75], the number of topes of $\mathcal{M}$ is equal to $(-1)^r \chi_M(-1)$, where χ_M is its characteristic polynomial of the underlying matroid M of $\mathcal{M}$, see also [BLVS+99, Theorem 4.6.1]. Thus $\dim \mathcal{S}_{\mathcal{E}}(\tau) = (-1)^r \tilde{\chi}_M(-1)$ where $\tilde{\chi}_M(t)$ is the reduced characteristic polynomial of M. Finally, by [Zha13], we obtain

$$(-1)^r \tilde{\chi}_M(-1) = \sum_{0 \leq p \leq r} \dim \mathcal{F}_p(\Sigma_M),$$

and the proof follows. □

6.3 The spectral sequence

We now establish the spectral sequence relating the homology of the patchwork of a tropical manifold to the tropical homology groups.

We denote the complex of relative chains by

$$C_\bullet^\diamond(X; \mathcal{K}_p, \mathcal{K}_{p+1}) := C_\bullet^\diamond(X; \mathcal{K}_p)/C_\bullet^\diamond(X; \mathcal{K}_{p+1})$$

where the $\diamond$ denotes either normal or Borel-Moore homology. This is the 0-th page of the spectral sequence for the chain complex $C_\bullet^\diamond(X; \mathcal{S}_{\mathcal{E}})$ obtained by the filtration by the $\mathcal{K}_p$ cellular cosheaves. Thanks to Proposition 6.2.4, we obtain the following statement, analogous to [RS23, Proposition 4.12].

Corollaire 6.1. *Let X be a non-singular tropical variety in $\mathbb{T}\Sigma$ with a real phase structure $\mathcal{E}$. The first page of the spectral sequence associated to the filtration of the chain complex $C_\bullet^\diamond(X; \mathcal{S}_{\mathcal{E}})$ by the chain complexes $C_\bullet^\diamond(X; \mathcal{K}_p)$ has terms*

$$E^{1,\diamond}_{q,p} \cong H_q^\diamond(X; \mathcal{F}_p).$$

Proof of Theorem 6.0.1. The pages of a spectral sequence satisfy $\dim E^\infty_{q,p} \leq \dim E^r_{q,p}$ for all r. By 6.1, and the convergence of the spectral sequence associated to the filtration we obtain

$$\dim H_q^\diamond(X; \mathcal{S}_{\mathcal{E}}) = \sum_{p=0}^{\dim X} E^{\infty,\diamond}_{q,p} \leq \sum_{p=0}^{\dim X} E^{1,\diamond}_{q,p} = \sum_{p=0}^{\dim X} H_q^\diamond(X; \mathcal{F}_p).$$

The first part of the theorem now follows since $\dim H_q^\diamond(\mathbb{R}_\mathcal{E}X;\mathbb{F}_2) = \dim H_q^\diamond(X;\mathcal{S}_\mathcal{E})$ by 6.1.3.

For the relation between the tropical signature and the Euler characteristic of $\mathbb{R}_\mathcal{E}X$, using again 6.1.3 we have $\chi^\diamond(\mathbb{R}_\mathcal{E}X) = \chi^\diamond(\mathcal{S}_\mathcal{E})$, and moreover,

$$\chi^\diamond(\mathcal{S}_\mathcal{E}) = \sum_{p,q}(-1)^q E_{q,p}^{\infty,\diamond} = \sum_{p,q}(-1)^q E_{q,p}^{1,\diamond} = \sum_p \chi^\diamond(\mathcal{F}_p) = \sigma^\diamond(X).$$

This completes the proof. □

6.4 Case of plane curves

In this section we explicitly describe the only possibly non-zero differential map in the spectral sequence in the case of curves. In this case, Viro's primitive patchworking construction, equivalently, the real phase structures on tropical curves, can be reformulated in terms of *admissible twists*. For definitions and examples of tropical curves, tropical toric surfaces, the twist description of Viro's patchworking, and the tropical homology of tropical curves we refer to [BIMS].

Given a compact non-singular tropical curve C in a tropical toric surface there is another equivalent way of describing a real phase structure on C in terms choosing a subset of *twisted edges* of the bounded edges of $C \cap \mathbb{R}^2$ satisfying an admissibility condition. A collection T of bounded edges of a tropical curve $C\cap\mathbb{R}^2$ is *admissible* if for all $\gamma \in H_1(C;\mathcal{F}_0)$ we have

$$\sum_{e\in T\cap \mathrm{Supp}(\gamma)} v_e = 0 \in \mathbb{F}_2^2,$$

where v_e is the primitive integer direction of the edge e. The edges in T are called twisted edges because of how a real algebraic curve near a tropical limit C with a real phase structure given by the twisted edges behaves under the logarithm map. See the right hand-side of Figure 6.1.

Let C be a non-singular compact tropical curve with a real phase structure $\mathcal{E}$. For a bounded edge e of C_0, its symmetric copy, e^ε in $\mathbb{R}C_\mathcal{E}$, is adjacent to two other edges e_1^ε, e_2^ε of $\mathbb{R}C_\mathcal{E}$ which are also contained in the quadrant corresponding to ε. The twisted edges for a real phase structure $\mathcal{E}$ correspond to those edges e of C for which e_1, e_2 are not contained in a closed half space of $\mathbb{R}^2(\varepsilon)$ whose boundary contains e^ε. A detailed description of this approach can be found in [BIMS, Section 3.2].

Using the twist formulation we describe explicitly the map

$$\partial_1 : H_1(C;\mathcal{F}_0) \to H_0(C;\mathcal{F}_1)$$

arising from the spectral sequence on the chain level when the curve C is compact. In this case, both of the above homology groups are isomorphic to $\mathbb{F}_2^g$, where g is the first Betti number of C. For explicit calculations of the tropical (co)homology of tropical curves see [BIMS, Section 7.8].

Example 6.4.1. *Figure 6.1, shows a non-singular plane tropical cubic with a twist-admissible set of edges, and the image by coordinatewise logarithm map* Log *of the real part* $\mathbb{R}V$ *of the curve* V *which is defined by a Viro polynomial. Figure 6.2 depicts* $\mathbb{R}C_T$. *Notice that this curve is maximal in the sense of Harnack's inequality, namely* $b_0(\mathbb{R}V) = g(\mathbb{C}V) + 1$.

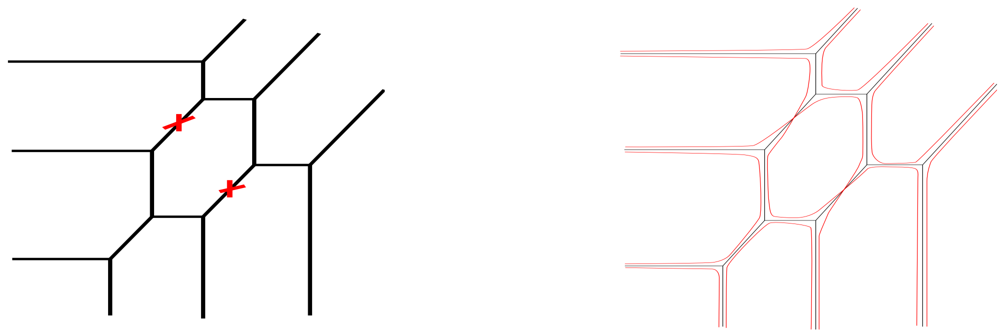

FIGURE 6.1 – On the left is a non-singular cubic with a twist-admissible set of edges. On the right hand side is the image by the coordinatewise logarithm map of $\mathbb{R}V$.

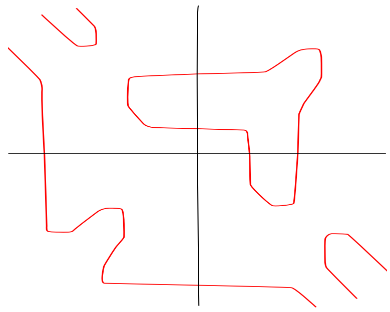

FIGURE 6.2 – The real part $\mathbb{R}V$ for the real cubic from Figure 6.1 and Example 6.4.1.

Let $\tilde{C}$ denote the first barycentric subdivision of C, which results in adding a vertex in the middle of each edge. Then the vertices of $\tilde{C}$ are the vertices of C together with additional vertices v_e for each edge of C. For every edge e of C there are now two edges e' and e'' of $\tilde{C}$, moreover v_e is in the boundary of each of these edges.

We can extend any cellular cosheaf $\mathcal{G}$, in particular, $\mathcal{F}_0, \mathcal{F}_1$, or $\mathcal{S}_\mathcal{E}$, to a cellular cosheaf on $\tilde{C}$ in the following way. Set $\mathcal{G}(\tilde{e}') = \mathcal{G}(\tilde{e}'') = \mathcal{G}(e)$. If v_e is the midpoint of an edge e then define $\mathcal{G}(v_e) = \mathcal{G}(e)$. The cosheaf morphisms $\mathcal{G}(\tilde{e}') \to \mathcal{G}(v_e)$ are the identity maps. Changing the cellular structure does not change the homology groups of the cosheaves $\mathcal{F}_0, \mathcal{F}_1$, and $\mathcal{S}_\mathcal{E}$. Namely, $H_i(\tilde{C}; \mathcal{F}_0) \cong H_i(C; \mathcal{F}_0)$, $H_i(\tilde{C}; \mathcal{F}_1) \cong H_i(C; \mathcal{F}_1)$, and $H_i(\tilde{C}; \mathcal{S}_\mathcal{E}) \cong H_i(C; \mathcal{S}_\mathcal{E})$.

For a cellular homology class $\gamma \in H_1(C; \mathcal{F}_0)$, we denote by $\mathrm{Supp}(\gamma)$ the collection of edges of C appearing in some chain representing γ. This is well defined since we are working with $\mathbb{F}_2$-coefficients.

Theorem 6.4.2. *Let C be a non-singular compact tropical curve in a tropical toric surface. Suppose C is equipped with a real phase structure corresponding to a collection of twists T of edges of C. Then the boundary map of the spectral sequence $\partial_1 \colon H_1(\tilde{C}; \mathcal{F}_0) \to H_0(\tilde{C}; \mathcal{F}_1)$ is given by*

$$\partial_1(\gamma) = \sum_{e \in T \cap Supp(\gamma)} v_e \otimes s_e,$$

where s_e is the generator of $\mathcal{F}_1(v_e)$. In particular, the number of connected components of $\mathbb{R}C$ is equal to $\dim \mathrm{Ker}(\partial_1) + 1$

Démonstration. It is enough to prove the statement for cycles in C which are boundaries of bounded connected components of the complement $\mathbb{R}^2 \backslash C$ since they form a basis of $H_1(C; \mathcal{F}_0)$. Given such a cycle $\gamma \in C_1(\tilde{C}; \mathcal{F}_0)$, we first choose a lift $\tilde{\gamma} \in C_1(C; \mathcal{S}_\mathcal{E})$ as

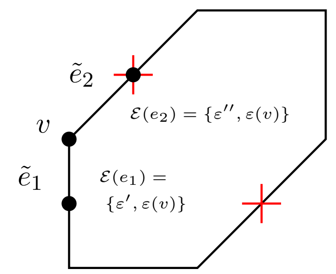

Figure 6.3 – The cycle γ of the cubic from Figure 6.1 and the lift around a vertex.

follows. Let v be a trivalent vertex of C and suppose that v is in the cycle γ. Let $\tilde{e}_1$ and $\tilde{e}_2$ be the two edges of $\tilde{C}$ (or half edges in C) which share the endpoint v and are contained in γ, see Figure 6.3. Let $\varepsilon(v)$ denote the unique element in $\mathcal{E}(\tilde{e}_1) \cap \mathcal{E}(\tilde{e}_2)$.

We set

$$\tilde{\gamma} = \sum_{\tilde{e} \in \gamma \cap \tilde{C}} \tilde{e} \otimes w_{\varepsilon(v)} \in C_1(C; \mathcal{S}_{\mathcal{E}}),$$

where in the sum above v, is the unique trivalent vertex of $\tilde{C}$ adjacent to the edge $\tilde{e}$.

If $e \in \text{Edge}(C \cap \gamma) \cap T$ and v, v' are the two adjacent vertices of e, then $w_{\varepsilon(v)}$ and $w_{\varepsilon(v')}$ are different and the image of the sum in $\mathcal{F}_1(e)$ is equal to w_e. If e is not twisted, then $w_{\varepsilon(v)} = w_{\varepsilon(v')}$. This proves that $\partial\tilde{\gamma} \in C_0(C; \mathcal{K}_1)$ is supported by the midpoints of twisted edges and that the image in $\mathcal{F}_1(e)$ of the coefficient over e is exactly the generator of $\mathcal{F}_1(e)$. This proves the lemma. □

Example 6.4.3. *Consider the tropical curve C in both sides of Figure 6.4. The red markings on the edges denote collections of twisted edges T_1 on the left and T_2 and the right. It can be verified that both collections of twists are admissible.*

Consider the basis $\gamma_1, \gamma_2, \gamma_3$ of $H_1(C; \mathcal{F}_0)$ where γ_i's are the boundaries of the three bounded connected components of $\mathbb{R}^2 \backslash C$. Let $\gamma_1^*, \gamma_2^*, \gamma_3^*$ denote the dual basis of $H_0(C; \mathcal{F}_1)$. We can represent the map from ∂_1 from Theorem 6.4.2 by a matrix using these two ordered bases and we obtain the matrices

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

for the twists T_1 and T_2, respectively. The matrix on the left has a 1-dimensional null space, and therefore a real algebraic curve produced from the collection of twists on the left of Figure 6.4 has two connected components. On the right the matrix has a 2-dimensional null space and the curve from the twists on the right of Figure 6.4 has 3 connected components.

6.4.1 M-curves and Haas theorem

Haas in his thesis [Haa97] studied maximal curves obtained by primtive patchworking. In particular, he found a necessary and sufficient criterion for maximality (see also [BIMS, Section 3.3] and [BBR17]). Here as an example we reformulate and reprove Haas criterion for maximality using the techniques of the last section.

Definition 6.4.4. *An edge e of a tropical curve C in a 2-dimensional non-singular tropical toric variety $\mathbb{T}Y$ is called* exposed *if e is in the closure of an unbounded connected*

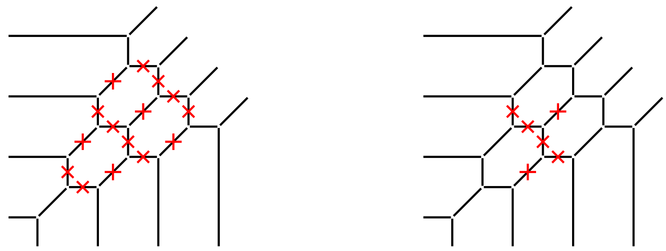

FIGURE 6.4 – The curve C with two collections of twists T_1 and T_2 from Example 6.4.3.

component of $\mathbb{T}Y_0 \backslash C_0$. *The set of exposed edges is denoted by* $\mathrm{Ex}(C)$. *Denote by* $\mathrm{Ex}^c(C)$ *the complement of* $\mathrm{Ex}(C)$ *in the set of bounded edges of* $C_0 \cap \mathbb{T}Y_0$.

The following theorem is a reformulation of Haas maximality condition reproved using the description of the map in the spectral sequence from Theorem 6.4.2.

Theorem 6.4.5 (Haas maximality condition [Haa97])**.** *Let C be a non-singular compact tropical curve in a tropical toric variety equipped with a real phase structure corresponding to a collection of twisted edges $T \subset Edges(C)$. Then $\mathbb{R}C$ is a maximal curve if and only if* $T \cap \mathrm{Ex}^c(C) = \emptyset$ *and for every cycle $\gamma \in H_1(C; \mathbb{Z}_2)$ the intersection $\gamma \cap T$ consists of an even number of edges.*

Démonstration. By the spectral sequence, the curve $\mathbb{R}C$ is maximal if and only if $\partial_1 = 0$. Cycles in $C_1(C; \mathcal{F}_0)$ which are boundaries of connected components of the complement $\mathbb{R}^2 \backslash C$ form a basis of $H_1(C; \mathcal{F}_0)$. There are $g := b_1(C)$ such cycles and we denote them by $\gamma_1, \dots, \gamma_g$. Therefore, it suffices to show that $\partial_1(\gamma_i) = 0$ for all i.

For C a non-singular tropical curve there is a non-degenerate pairing :

$$\langle \ , \ \rangle \colon H_0(C; \mathcal{F}_1) \times H_1(C; \mathcal{F}_0) \to \mathbb{F}_2$$

induced from the pairing on integral homology groups for non-sinuglar tropical curves in [Sha11]. A similar non-degenerate pairing defined between tropical homology and cohomology groups is also defined in [BIMS, Section 7.8] and [MZ14, Section 3.2]. On the chain level this pairing is :

$$\langle \beta, \gamma \rangle = |\mathrm{EdgeSupp}(\beta') \cap \gamma| \mod 2,$$

where $\beta' \sim \beta$ and $\beta' \in C_0(\tilde{C}; \mathcal{F}_1)$ is supported on the midpoints of edges of C. The set $\mathrm{EdgeSupp}(\beta')$ consists of the edges of C whose midpoint is in the support of β'. Therefore, it suffices to show that for all pairs of such cycles γ_i and γ_j, the non-degenerate pairing $\langle \partial_1(\gamma_i), \gamma_j \rangle$ is zero.

The intersection $\gamma_i \cap T$ is even if and only if $\langle \partial_1(\gamma_i), \gamma_i \rangle = 0$. Secondly, the pairing $\langle \partial_1(\gamma_i), \gamma_j \rangle = 0$ if and only if $\gamma_i \cap \gamma_j \cap T$ is a set of even cardinality. Since γ_i and γ_j are boundaries of convex regions in $\mathbb{R}^2$ they can only intersect in at most one edge of C. Therefore, the intersection $\gamma_i \cap \gamma_j \cap T$ must be empty and the statement is proved. $\square$

Chapitre 7

Bounds on Betti numbers for hypersurfaces

In this chapter, we indicate how to find the bounds in Theorem 1 and Theorem 6 from Theorem 1.2.3. We proceed in three steps. First we prove that the tropical homology groups of a non-singular hypersurface is given under the horizontal line of the Hodge diamond $p+q=n$ by the tropical homology of the ambient space. Then e apply tropical Poincaré duality to show that the tropical homology groups are torsion free and to compute their rank for all p,q such that $p+q \neq n$. Finally, we use the tropical Hodge genus to deduce the rank in the remaining case $p+q=n$. Everything in this chapter extend to non-primitve triangulation by [BdMR22].

7.1 Heredity theorem for tropical hypersurfaces

Let Δ be a non-singular lattice polytope in $M_{\mathbb{R}}$ (of dimension n) and T be a convex primitive lattice triangulation of Δ. Let $\mu : \Delta \cap M \to \mathbb{Z}_{\geq 0}$ be a convex function ensuring the convexity of T. Denote as in Section 3.4 by X_μ the associated tropical hypersurface in $N_{\mathbb{R}}$. Let Σ be a subfan of Σ_Δ, and denote simply by X the compactification of X_μ in $\mathbb{T}\Sigma$. We indicate here how to prove the following theorem and refer to [ARS21] for eventually more details.

Theorem 7.1.1. *The map induced by inclusion*

$$i_*\colon H_q^{\diamond}(X; \mathcal{F}_p) \to H_q^{\diamond}(\mathbb{T}\Sigma; \mathcal{F}_p)$$

is an isomorphism when $p+q < n-1$ and a surjection when $p+q = n-1$.

Recall that the $\diamond$ notation denotes either the standard or Borel-Moore homology. The tropical hypersurface X induces a polyhedral structure on $\mathbb{T}\Sigma$. We will use this polyhedral structure on $\mathbb{T}\Sigma$ to compute its cellular tropical homology groups. If $Z' \subset Z$ is a rational subpolyhedral complex and $\mathcal{G}$ is a cosheaf on Z, then the restriction cosheaf $\mathcal{G}|_{Z'}$ is a cosheaf on Z' which assigns the $\mathbb{Z}$-module $\mathcal{G}(\sigma)$ for σ a face of Z'. The cosheaf $\mathcal{G}|_{Z'}$ can also be considered as a cosheaf on Z. In this case, it assigns $\mathcal{G}(\sigma)$ if σ is a face of Z' and 0 otherwise.

Since we consider the polyhedral structure on $\mathbb{T}\Sigma$ induced by X, the tropical hypersurface X is a rational subpolyhedral complex of $\mathbb{T}\Sigma$ and we have the cosheaves $\mathcal{F}_p^{\mathbb{T}\Sigma}|_X$, which can be considered on X or $\mathbb{T}\Sigma$ as described above.

To prove Theorem 7.1.1, we consider two exact sequences of cosheaves. The first is the exact sequence of cosheaves on $\mathbb{T}\Sigma$ given by,

$$0 \to \mathcal{F}_p^{\mathbb{T}\Sigma}|_X \to \mathcal{F}_p^{\mathbb{T}\Sigma} \to \mathcal{Q}_p \to 0. \tag{7.1}$$

The second one consists of cosheaves on X and is given by,

$$0 \to \mathcal{F}_p^X \to \mathcal{F}_p^{\mathbb{T}\Sigma}|_X \to \mathcal{N}_p \to 0. \tag{7.2}$$

The injective maps on the left hand side of both cosheaf sequences are both natural inclusions on the stalks over faces. The cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ are defined as the cokernel cosheaves in both short exact sequences. The cosheaves $\mathcal{F}_p^{\mathbb{T}\Sigma}|_X$, $\mathcal{F}_p^{\mathbb{T}\Sigma}$, and $\mathcal{F}_p^X$ are all free $\mathbb{Z}$-modules. Moreover, we proved in [ARS21] that since X is a non-singular tropical hypersurface, the cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ are also cosheaves of free $\mathbb{Z}$-modules.

Proposition 7.1.2. *If X is a non-singular tropical hypersurface in Y, the cosheaves $\mathcal{Q}_p$ and $\mathcal{N}_p$ from (7.1) and (7.2) are cosheaves of free $\mathbb{Z}$-modules.*

Example 7.1.3. *Consider again the tropical line X in $\mathbb{TP}^2$. Then the cosheaf $\mathcal{Q}_p$ on $\mathbb{TP}^2$ assigns the trivial $\mathbb{Z}$-module to any face of $\mathbb{TP}^2$ which is also a face of X. For σ a face of $\mathbb{TP}^2$ and not a face of X, then $\mathcal{Q}_p(\sigma) = \mathcal{F}_p^{\mathbb{TP}^2}(\sigma)$. The inclusion maps $\mathcal{Q}_p(\sigma) \to \mathcal{Q}_p(\tau)$ are either 0 or equal to $\iota_{\sigma\tau} \colon \mathcal{F}_p^{\mathbb{TP}^2}(\sigma) \to \mathcal{F}_p^{\mathbb{TP}^2}(\tau)$.*

For x the unique vertex of sedentarity 0 of X, the cosheaf $\mathcal{N}_p$ assigns $\mathcal{N}_p(x) = 0$, for all $p < 2$. When $p = 2$, we have $\mathcal{N}_p(x) = \bigwedge^2 \mathbb{Z}^2$.

For an edge σ_i of X the $\mathbb{Z}$-module $\mathcal{N}_p(\sigma_i)$ is a free module of rank 1, similarly for the three other vertices τ_i of X that have non-zero sedentarity.

We proved in [ARS21] the following two vanishing propositions.

Proposition 7.1.4. *For all $q < n-1$, one has $H_q^{\diamond}(\mathbb{T}\Sigma; \mathcal{Q}_p) = 0$ and therefore the map*

$$H_q^{\diamond}(X; \mathcal{F}_p^{\mathbb{T}\Sigma}|_X) \to H_q^{\diamond}(\mathbb{T}\Sigma; \mathcal{F}_p^{\mathbb{T}\Sigma})$$

is an isomorphism when $q < n-1$ and a surjection when $q = n-1$.

Proposition 7.1.5. *For all $p + q \leq n-1$, one has*

$$H_q^{\diamond}(X; \mathcal{N}_p) = 0$$

and therefore the map

$$H_q^{\diamond}(X; \mathcal{F}_p^X) \to H_q^{\diamond}(X; \mathcal{F}_p^{\mathbb{T}\Sigma}|_X)$$

is an isomorphism when $p + q < n-1$ and a surjection when $p + q = n-1$.

Now the proof of Theorem 7.1.1 follows by combining the statements in Propositions 7.1.4 and 7.1.5.

7.2 Tropical homology of non-singular hypersurfaces are torsion free

In this section we prove the following theorem as in [ARS21]. Again X denotes the compactification of a non-singular tropical hypersurface X_μ inside $\mathbb{T}\Sigma$, where μ is a convex function certifying the convexity of a primitive triangulation of Δ and Σ is a subfan of Σ_Δ.

Theorem 7.2.1. *If the standard (resp. Borel-Moore) tropical homology groups of $\mathbb{T}\Sigma$ are torsion free then also the standard (resp. Borel-Moore) tropical homology groups of X are torsion free.*

We indicate the proof which is a standard application of the universal coefficient theorem :

Démonstration. By the universal coefficient theorem for cohomology for every p and q we have the following short exact sequence :

$$0 \to \mathrm{Ext}(H_{n-q-2}(X;\mathcal{F}^X_{n-1-p}),\mathbb{Z}) \to H^{n-1-q}(X;\mathcal{F}_X^{n-1-p}) \to \mathrm{Hom}(H_{n-1-q}(X;\mathcal{F}^X_{n-1-p}),\mathbb{Z}) \to 0.$$

Notice that the cohomology of the sheaf $\mathcal{F}_X^{n-1-p}$ appears in the middle term because

$$C^q(X;\mathcal{F}_X^p) \cong \mathrm{Hom}(C_q(X;\mathcal{F}^X_q),\mathbb{Z}).$$

If $p+q \geq n-1$, then $2n-2-p-q < n$, and it follows from Theorem 7.1.1 that

$$H_{n-q-2}(X;\mathcal{F}^X_{n-1-p}) \cong H_{n-q-2}(Y,\mathcal{F}^Y_{n-1-p}).$$

Since $H_{n-q-2}(Y;\mathcal{F}^Y_{n-1-p})$ is a free $\mathbb{Z}$-module by hypothesis, we conclude that

$$\mathrm{Ext}(H_{n-q-2}(X;\mathcal{F}^X_{n-1-p}),\mathbb{Z}) = 0.$$

Also the $\mathbb{Z}$-module $\mathrm{Hom}(H_{n-1-q}(X;\mathcal{F}^X_{n-1-p}),\mathbb{Z})$ is free since it consists of maps to a free module. Therefore, for all $p+q \geq n-1$ we have

$$H^{n-1-q}(X;\mathcal{F}_X^{n-1-p}) \cong \mathrm{Hom}(H_{n-1-q}(X;\mathcal{F}^X_{n-1-p}),\mathbb{Z})$$

and $H^{n-1-q}(X;\mathcal{F}_X^{n-1-p})$ is torsion free. The tropical hypersurface X is a non-singular tropical manifold, so by Poincaré duality for tropical homology with integral coefficients from [JRS18] we have

$$H^{n-1-q}(X;\mathcal{F}_X^{n-1-p}) \cong H_q^{BM}(X;\mathcal{F}^X_p)$$

for all p,q. This, combined with the torsion freeness of $H^{n-1-q}(X;\mathcal{F}_X^{n-1-p})$ established above, proves that $H_q^{BM}(X;\mathcal{F}^X_p)$ is torsion free for all $p+q \geq n-1$.

Notice that applying the above argument to the tropical homology of $\mathbb{T}\Sigma$ shows that if the groups $H_q(\mathbb{T}\Sigma,\mathcal{F}_p)$ are torsion free for all p and q, then $H_q^{BM}(\mathbb{T}\Sigma,\mathcal{F}_p)$ are also torsion free for all p and q. It follows from this and Theorem 7.1.1, that $H_q^{BM}(X;\mathcal{F}^X_p)$ is torsion free for $p+q<n$, so the Borel-Moore tropical homology groups of X are all torsion free.

To prove that the standard tropical homology groups of X are torsion free, we use the universal coefficient theorem for cohomology with compact support. For every p and q we have the following short exact sequence

$$0 \to \mathrm{Ext}(H^{BM}_{q-1}(X;\mathcal{F}^X_p),\mathbb{Z}) \to H^q_c(X;\mathcal{F}^p_X) \to \mathrm{Hom}(H_q^{BM}(X;\mathcal{F}^X_p),\mathbb{Z}) \to 0.$$

Since $H_q^{BM}(\mathbb{T}\Sigma,\mathcal{F}_p)$ are torsion free for all p and q, it follows from Theorem 7.1.1 that $H_q^{BM}(X,\mathcal{F}_p)$ are torsion free for all $p+q<n-1$. Then the $\mathbb{Z}$-modules $H_q^{BM}(X;\mathcal{F}^X_p)$ are torsion free for all p and q, and the $\mathbb{Z}$-modules $H^q_c(X;\mathcal{F}^p_X)$ are also torsion free for all p and q. Applying again Poincaré duality, we have

$$H^q_c(X;\mathcal{F}^p_X) \cong H_{n-1-q}(X;\mathcal{F}^X_{n-1-p}),$$

and $H_q(X;\mathcal{F}^X_p)$ are also torsion free for all p and q. □

We now establish that the integral tropical homology groups of a projective tropical toric variety are torsion free. For a non-singular projective complex toric variety $\mathbb{C}\Sigma_\Delta$, we let $h^{p,q}(\mathbb{C}\Sigma_\Delta)$ denote its (p,q)-th Hodge number. Recall that $h^{p,q}(\mathbb{C}\Sigma_\Delta) = 0$ if $p \neq q$ and the numbers $h^{p,p}(\mathbb{C}\Sigma_\Delta)$ form the toric h-vector of Δ [Ful93b, Section 5.2].

Proposition 7.2.2. *The integral tropical homology groups of a non-singular projective tropical toric variety $\mathbb{T}\Sigma_\Delta$ are torsion free. Moreover, we have*

$$\mathrm{rank} H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = h^{p,q}(\mathbb{C}\Sigma_\Delta).$$

In particular, we have $H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = 0$ unless $p = q$.

Démonstration. We now switch to computing the cellular homology groups of $\mathbb{T}\Sigma_\Delta$ using the polyhedral structure on $\mathbb{T}\Sigma_\Delta$ which is dual to the polyhedral structure on the defining fan Σ_Δ. Let us first show that $H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = 0$ for all $p > q$. A face σ of dimension q of $\mathbb{T}\Sigma_\Delta$ has sedentarity order $n - q$. By Definition of the $\mathcal{F}_p$, we have that $\mathcal{F}_p(\sigma) = \bigwedge^p \mathcal{F}_1(\sigma)$ where $\dim \mathcal{F}_1(\sigma) = q$. Therefore, we have $\mathcal{F}_p(\sigma) = 0$ if $p > q$. Hence the chain groups $C_q(Y; \mathcal{F}_p)$ are equal to zero for any $q < p$, which implies that $H_q(Y; \mathcal{F}_p) = 0$ for $q < p$.

The tropical cohomology groups are the cohomology of the complex dual to the tropical cellular complexes. Therefore we can apply the universal coefficient theorem for cohomology [Hat02, Theorem 3.2] to get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}(H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p), \mathbb{Z}) \to H^{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}^p) \to \\ \mathrm{Hom}(H_{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p), \mathbb{Z}) \to 0. \end{aligned} \tag{7.3}$$

When $q < p$ we have $H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = 0$, so there is the isomorphism

$$H^{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}^p) \cong \mathrm{Hom}(H_{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p), \mathbb{Z}).$$

The tropical toric variety Y is a tropical manifold, thus Poincaré duality for tropical homology with integral coefficients from [JRS18] states that

$$H^{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}^p) \cong H_{n-q}(\mathbb{T}\Sigma_\Delta; \mathcal{F}_{n+1-p}).$$

If $q \geq p$, then $n - 1 - q < n - p$ and applying the isomorphism above we obtain

$$H^{q+1}(\mathbb{T}\Sigma_\Delta; \mathcal{F}^p) = H_{n-1-q}(\mathbb{T}\Sigma_\Delta; \mathcal{F}_{n-p}) = 0.$$

This means that

$$\mathrm{Tor}(H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p)) = \mathrm{Ext}(H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p), \mathbb{Z}) = 0,$$

and so $H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p)$ is torsion-free for all $q \geq p$ and thus for all p, q. We also see from the sequence in (7.3) that $H_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = 0$ for all $q \neq p$.

All of the chain groups for the cellular tropical homology of $\mathbb{T}\Sigma_\Delta$ are also free so we have

$$\chi(C_\bullet(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p)) := \sum_{q=0}^{n} (-1)^q \mathrm{rank} C_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = (-1)^p \mathrm{rank} H_p(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p).$$

Let f_q denote the number of strata of $\mathbb{T}\Sigma_\Delta$ of dimension q. Then $(f_0, \dots, f_n)$ is the f-vector of Δ. Then for every p and q we have $\mathrm{rank} C_q(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p) = \binom{q}{p} f_q$. Therefore,

$$\chi(C_\bullet(\mathbb{T}\Sigma_\Delta; \mathcal{F}_p)) := \sum_{q=0}^{n} (-1)^q \binom{q}{p} f_q = (-1)^p h_p,$$

where $(h_0, \dots, h_n)$ is the h-vector of the simple polytope Δ. By [Ful93b, Section 5.2], we have $h_p = \dim H^{2p}(\mathbb{C}\Sigma_\Delta) = h_{p,p}(\mathbb{C}\Sigma_\Delta)$ which completes the proof. □

7.3 Hirzebruch polynomials

So far, using Theorem 7.1.1, Theorem 7.2.1 and Proposition 7.2.2, one can prove that if $\Sigma = \Sigma_\Delta$, then for any p, q such that $p + q \neq n - 1$, one has

$$\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2) = h^{p,q}(\mathbb{C}\Sigma_\Delta).$$

To determine the remaining groups $\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2)$ when $p + q = n$, we will consider tropical and complex Hirzebruch polynomials. Again, let Δ be a lattice polytope in $M_\mathbb{R}$ (of dimension n) and T be a convex primitive lattice triangulation of Δ. Let $\mu : \Delta \cap M \to \mathbb{Z}_{\geq 0}$ be a convex function ensuring the convexity of T, and denote by X_μ the associated tropical hypersurface in $N_\mathbb{R}$. Let Σ be a subfan of Σ_Δ, and denote simply by X the compactification of X_μ in $\mathbb{T}\Sigma$. Let $X_\mathbb{C}$ be a torically non-degenerate complex hypersurface of $\mathbb{C}\Sigma$ of Newton polytope Δ. Recall that the Hirzebruch genus of $X_\mathbb{C}$ is defined by

$$\chi_y(X_\mathbb{C}) = \sum_{p,q} e_c^{p,q}(X_\mathbb{C}) y^p, \tag{7.4}$$

where $e_c^{p,q}(X_\mathbb{C})$ is defined via the mixed Hodge structure on $X_\mathbb{C}$ by

$$e_c^{p,q}(X_\mathbb{C}) := \sum_k (-1)^k h^{p,q}(H_c^k(X_\mathbb{C})).$$

See for instance [DK]. We define the $\mathbb{F}_2$-*tropical Hirzebruch genus* of X by

$$\chi_y^{trop}(X) := \sum_{p,q=0}^{d} (-1)^{p+q} \dim H_q^{\mathrm{BM}}(X, \mathcal{F}_p \otimes \mathbb{F}_2) y^p.$$

Note that using the universal coefficient theorem for the field $\mathbb{F}_2$, one obtains that

$$\chi_y(X) = \sum_{p,q=0}^{d} (-1)^{p+q} \dim H_c^q(X; \mathcal{F}^p \otimes \mathbb{F}_2) y^p,$$

in accordance with the classical definition (7.4). We recall the proof of the next proposition which can be found also in [ARS21] or [RRS23] (where it is more general).

Proposition 7.1.

$$\chi_y(X_\mathbb{C}) = \chi_y(X).$$

Démonstration. Given $\rho \in \Sigma$, we set $X_\mathbb{C}^\rho := X_\mathbb{C} \cap \mathcal{O}_\rho^\mathbb{C}$ and $X^\rho := X \cap \mathcal{O}_\rho^\mathbb{T}$. We first claim that both sides of the equation are motivic in the sense that

$$\chi_y(X_\mathbb{C}) = \sum_{\rho \in \Sigma} \chi_y(X_\mathbb{C}^\rho), \qquad \chi_y(X) = \sum_{\rho \in \Sigma} \chi_y(X^\rho).$$

The first equation follows from the fact that the Hirzebruch genus is motivic. On the tropical side, it follows from the following computation :

$$\begin{aligned}
\chi_y(X) &= \sum_{p,q} (-1)^{p+q} \dim H_q^{\mathrm{BM}}(X, \mathcal{F}_p) y^p \\
&= \sum_p \sum_{\sigma \subset X} (-1)^{p+\dim\sigma} \dim \mathcal{F}_p(\sigma) y^p \\
&= \sum_{\rho \in \Sigma} \sum_p \sum_{\mathrm{Sed}\sigma = \rho} (-1)^{p+\dim\sigma} \dim \mathcal{F}_p(\sigma) y^p \\
&= \sum_{\rho \in \Sigma} \chi_y(X^\rho).
\end{aligned}$$

It hence suffices to consider the case $\Sigma = \{0\}$, so we now assume that X is in $\mathbb{R}^n$ and $X_{\mathbb{C}}$ is in $(\mathbb{C}^*)^n$. In [KS16, Section 5.2], Katz and Stapeldon give a formula for the χ_y genus of a torically non-degenerate hypersurface in the torus. Their formula utilizes regular subdivisions of polytopes to refine the formula in terms of Newton polytopes of Danilov and Khovanskii [DK]. Note that they use the term schön in exchange for torically non-degenerate. Recall that Δ is the Newton polytope for $X_{\mathbb{C}}$ and T a regular subdivision of Δ. Then the formula is

$$\chi_y(X_{\mathbb{C}}) = \sum_{\substack{F\subset T \\ F\not\subset\partial\Delta}} (-1)^{n-\dim F}\chi_y(X_{\mathbb{C},F}), \tag{7.5}$$

where $X_{\mathbb{C},F}$ is the hypersurface in the torus $(\mathbb{C}^*)^n$ defined by the polynomial obtained by restricting the polynomial defining $X_{\mathbb{C}}$ to the monomials corresponding to the lattice points in the face F of $\tilde{\Delta}$. Again, we show that the analogous statement holds tropically. To do so, we denote by $S(\sigma)$ the star of X at σ :

$$S(\sigma) := \{v \in \mathbb{R}^n \mid \forall\varepsilon \ll 1,\ x + \varepsilon v \in X\},$$

where x is any point in the relative interior of σ. The set $S(\sigma)$ is naturally a fan of dimension $n-1$ in $\mathbb{R}^n$. Applying Poincaré duality [JRS18] we obtain that

$$\begin{aligned}
\chi_y(X) &= \sum_{p,q}(-1)^{p+q}\dim H_q^{\mathrm{BM}}(X,\mathcal{F}_p)y^p \\
&= \sum_{p,q}(-1)^{p+q}\dim H_{n-1-q}(X,\mathcal{F}_{n-1-p})y^p \\
&= \sum_{p}\sum_{\substack{\sigma\subset X \\ \sigma \text{ bounded}}}(-1)^{p+d-\dim\sigma}\dim\mathcal{F}_{n-1-p}(\sigma)y^p \\
&= \sum_{\substack{\sigma\subset X \\ \sigma \text{ bounded}}}(-1)^{\dim\sigma}\sum_p(-1)^{p+n-1}\dim H_0(S(\sigma),\mathcal{F}_{n-1-p})y^p \\
&= \sum_{\substack{\sigma\subset X \\ \sigma \text{ bounded}}}(-1)^{\dim\sigma}\sum_p(-1)^{p+n-1}\dim H_d^{\mathrm{BM}}(S(\sigma),\mathcal{F}_p)y^p \\
&= \sum_{\substack{\sigma\subset X \\ \sigma \text{ bounded}}}(-1)^{\dim\sigma}\chi_y(S(\sigma)).
\end{aligned}$$

It is then sufficient to prove that $\chi_y(S(\sigma)) = \chi_y(X_{\mathbb{C},F})$ if F is the face of Δ dual to σ. This follows from the fact that for any linear subvariety $L \subset (\mathbb{C}^*)^n$ with associated matroidal fan M_L, one has

$$\chi_y(L) = \chi_y(\mathbb{P}\Sigma_M).$$

By [KS12, Lemma 7.5], the Hirzebruch genus of L is equal to the reduced characteristic polynomial $\overline{\mu}_M(y)$ of M. Moreover, by [Zha13] and [OT92], the dimension of $\mathcal{F}_p(\mathbb{P}\Sigma_M)$ is the $(n-1-p)$-th coefficient of $\overline{\mu}_M(y)$. Then we get

$$\begin{aligned}
\chi_y(L) &= \overline{\mu}_M(y) \\
&= \sum_p(-1)^{n-1-p}\dim\mathcal{F}_{n-1-p}(\mathbb{P}\Sigma_M)y^p \\
&= \sum_p(-1)^{p+n-1}\dim H_{n-1}^{\mathrm{BM}}(\mathbb{P}\Sigma_M,\mathcal{F}_p)y^p \\
&= \chi_y(\mathbb{P}\Sigma_M).
\end{aligned}$$

□

Corollary 7.3.1. *For any p, q, one has*

$$\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2) = h^{p,q}(X_{\mathbb{C}}).$$

7.4 Proof of Itenberg's conjecture and beyond

We conclude this chapter by combining the main statements to prove the main Theorem in [RS23]. Recall one more time our notations. Let Δ be a non-singular lattice polytope in $M_{\mathbb{R}}$ (of dimension n) and T be a convex primitive lattice triangulation of Δ. Let $\mu : \Delta \cap M \to \mathbb{Z}_{\geq 0}$ be a convex function ensuring the convexity of T. Denote by X_μ the associated tropical hypersurface in $N_{\mathbb{R}}$, and by X its compactification in $\mathbb{T}\Sigma_\Delta$. Denote by $X_{\mathbb{C}}$ a non-degenerate complex hypersurface in $\mathbb{C}\Sigma_\Delta$ of Newton polytope Δ. Then

Theorem 7.4.1. *Let $\mathcal{E}$ be a real phase structure on X. Then for all $0 \leq q \leq n$,*

$$b_q(\mathbb{R}_{\mathcal{E}} X) \leq \begin{cases} h^{q,q}(X_{\mathbb{C}}) \text{ if } q = (n-1)/2, \\ h^{q,n-1-q}(X_{\mathbb{C}}) + h^{q,q}(X_{\mathbb{C}}) \text{ if not.} \end{cases}$$

In particular, for Δ being the n-simplex $Conv(0, de_1, \cdots, de_n)$ one recovers Itenberg's conjecture.

Démonstration. By Section 6.0.1, we get

$$b_q(\mathbb{R}_{\mathcal{E}} X) \leq \sum_{p=0}^{n-1} \dim H_q(X; \mathcal{F}_p \otimes \mathbb{F}_2).$$

By Corollary 7.3.1, we know that $\dim H_q(X, \mathcal{F}_p \otimes \mathbb{F}_2) = h^{p,q}(X_{\mathbb{C}})$, and the proof follows. □

We would like to present here also the case where we do not compactify X_μ. We will denote $X = X_\mu \subset N_{\mathbb{R}}$ and $X_{\mathbb{C}}$ a non-degenerate complex hypersurface in $N_{\mathbb{C}}$ of Newton polytope Δ.

Theorem 7.4.2. *Let $\mathcal{E}$ be a real phase structure on X. Then for all $0 \leq q \leq n-1$,*

$$b_q^{BM}(\mathbb{R}_{\mathcal{E}} X) \leq \begin{cases} \sum_{l=0}^{q} h^{n-1-q,l}(H_c^{n-1}(X_{\mathbb{C}})) \text{ if } q \neq n-1, \\ \sum_{l=0}^{n-1} h^{0,l}(H_c^{n-1}(X_{\mathbb{C}})) + 2^n - 1 \text{ if not.} \end{cases}$$

Démonstration. Again by 6.0.1, we get

$$b_q^{\mathrm{BM}}(\mathbb{R}_{\mathcal{E}} X) \leq \sum_{p=0}^{n-1} \dim H_q^{\mathrm{BM}}(X; \mathcal{F}_p \otimes \mathbb{F}_2).$$

By Theorem 7.2.1, we know that $\dim H_q^{\mathrm{BM}}(X; \mathcal{F}_p \otimes \mathbb{F}_2) = \mathrm{rank} H_q^{\mathrm{BM}}(X; \mathcal{F}_p \otimes)$. Remind that

$$H_q(\mathbb{R}^n; \mathcal{F}_p) = \begin{cases} \bigwedge^p \mathbb{Z}^n \text{ if } q = 0, \\ 0 \text{ if } q \neq 0 \end{cases} \quad \text{and} \quad H_q^{\mathrm{BM}}(\mathbb{R}^n; \mathcal{F}_p) = \begin{cases} \bigwedge^p \mathbb{Z}^n \text{ if } q = n, \\ 0 \text{ if } q \neq n. \end{cases}$$

By Theorem 7.1.1 and Poincaré duality, one obtains that if $p+q \neq n-1$ then

$$H_q^{\mathrm{BM}}(X;\mathcal{F}_p \otimes \mathbb{F}_2) = \begin{cases} \bigwedge^{n-1-p} \mathbb{F}_2^n \text{ if } q = n-1, \\ 0 \text{ if } q \neq n-1. \end{cases}$$

The hypersurface $X_{\mathbb{C}}$ is a non-singular affine variety, so the Andreotti-Frankel theorem and Poincaré duality imply $H_c^k(X_{\mathbb{C}}) = 0$ if $k < n$. By the Lefschetz-type theorems for the Hodge Deligne numbers on $H_c^n(X_{\mathbb{C}})$ [DK, Section 3], if $k > n$ one has

$$h^{p,q}(H_c^k(X_{\mathbb{C}})) = \begin{cases} \binom{n+1}{p+1} & \text{if } p = q \text{ and } k = n+p \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$e_c^{p,q}(X_{\mathbb{C}}) = \begin{cases} (-1)^n h^{p,q}(H_c^n(X_{\mathbb{C}})) & \text{if } p+q \leq n \text{ and } p \neq q \\ (-1)^n h^{p,q}(H_c^n(X_{\mathbb{C}})) + (-1)^{n+p}\binom{n+1}{p+1} & \text{if } p+q \leq n \text{ and } p = q \\ (-1)^{n+p}\binom{n+1}{p+1} & \text{if } p+q > n \text{ and } p = q \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 7.1, we obtain that if $p \neq 0$ then

$$\dim H_{n-1-p}^{\mathrm{BM}}(X, \mathcal{F}_p \otimes \mathbb{F}_2) = \sum_{q=0}^{n-1-p} h^{p,q}(H_c^{n-1}(X_{\mathbb{C}})),$$

and

$$\dim H_{n-1}^{\mathrm{BM}}(X, \mathcal{F}_0 \otimes \mathbb{F}_2) = \sum_{q=0}^{n-1} h^{0,q}(H_c^{n-1}(X_{\mathbb{C}})) + n.$$

Then if $q \neq n-1$, one has

$$b_q^{\mathrm{BM}}(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{l=0}^{q} h^{n-1-q,l}(H_c^{n-1}(X_{\mathbb{C}})),$$

and for $q = n-1$ we get

$$b_{n-1}^{\mathrm{BM}}(\mathbb{R}_{\mathcal{E}}X) \leq \sum_{l=0}^{n-1} h^{n-1,l}(H_c^{n-1}(X_{\mathbb{C}})) + 2^n - 1.$$

□

Chapitre 8

Patchworking and mirror symmetry

8.1 Combinatorial mirror pairs

Let Δ be a lattice reflexive polytope and let T be an unimodular central triangulation of Δ. Let us introduce first some important notations we will use all along this chapter. Given a simplex $\sigma \in T$

— denote by $C(\sigma) \in \Sigma_T$ the cone over σ ;

— denote by $\sigma^\perp$ the subspace of N orthogonal to the integral tangent space of σ ;

— if $\sigma \neq 0$, let

$$\sigma_\infty = \sigma \cap \partial\Delta.$$

Given a cone $\rho \in \Sigma_T$

— denote by $S(\rho)$ the simplex $\rho \cap \Delta$;

— denote by $\hat{\sigma}$ the simplex $S(C(\sigma))$, i.e. the convex hull of 0 and σ.

In particular, when $0 \in \sigma$, then $\hat{\sigma} = \sigma$ and σ_∞ is the unique facet of σ contained in $\partial\Delta$. If $0 \notin \sigma$, then $\sigma_\infty = \sigma$.

Given two fans Σ and Σ' such that Σ is a refinement of Σ',

— for every cone $\rho \in \Sigma$, denote by $\min(\rho)$ the smallest cone of Σ' containing ρ.

If Σ is the normal fan of a polytope Δ

— denote by $\rho^\vee$ the face of Δ which is normal to ρ, and by $F^\vee$ the cone of Σ which is normal to the face F of Δ.

Definition 8.1.1. *Let T be an unimodular convex central triangulation of Δ and T° be an unimodular convex central triangulation of Δ°, induced respectively by convex functions μ and μ°. We denote by X_{trop} the compactification of V_μ inside $\mathbb{T}\Sigma_{T^\circ}$ and by X°_{trop} the compactification of V_{μ° inside $\mathbb{T}\Sigma_T$.*

The tropical variety X_{trop} induces a subdivision of $\mathbb{T}\Sigma_{T^\circ}$ with underlying poset the subposet of $T^\circ \times T$ defined by

$$\mathcal{P}(T^\circ, T) := \left\{(\tau, \sigma) \in T^\circ \times T \mid 0 \in \tau \text{ and } \sigma \subset \min(C(\tau))^\vee\right\}.$$

Recall that $C(\tau)$ is the cone (in Σ_{T°) over τ. The fan Σ_{T° is a refinement of the fan Ξ_{Δ°, which is the normal fan of Δ. Then by definition $\min(C(\tau))^\vee$ is a face of Δ. The order on the poset $\mathcal{P}(T^\circ, T)$ is also the inverse of the inclusion.

Example 8.1.2. *Let*

$$\Delta = \text{Conv}((-1,-1),(-1,2),(2,-1)).$$

It is a reflexive polygon and its dual is given by

$$\Delta^\circ = \text{Conv}((1,1),(-1,0),(0,-1)).$$

Denote by T and T° their unique primitive central triangulations. They are both convex. We draw in Figure 8.1 the subdivision of $\mathbb{T}\Sigma_{T^\circ}$ induced by a tropical curve dual to T and some examples of cells in the underlying poset $\mathcal{P}(T^\circ, T)$.

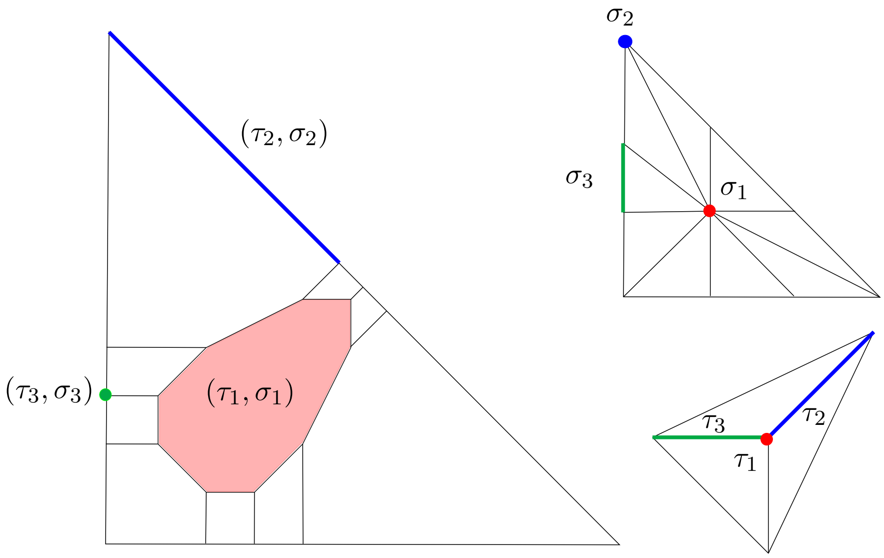

FIGURE 8.1 – The subdivision of $\mathbb{T}\Sigma_{T^\circ}$ induced by a tropical curve dual to T.

In fact in this chapter we do not assume any convexity hypothesis. For any T and T° as above (not necessarily convex), one can still consider the posets $\mathcal{P}(T, T^\circ)$ and $\mathcal{P}(T^\circ, T)$. Recall that to any poset P is associated an abstract simplicial complex, called the order complex, whose vertices are the elements of P and whose simplices are the (finite) nonempty chains of P. Moreover, to any simplicial complex, there is a topological space called the *geometric realization* defined by formal convex combinations of vertices belonging to a simplex. The geometric realization of a poset P will be denoted by $|P|$. By definition, the geometric realization is subivided into simplices : for any chain $x_\bullet = (x_1 < \cdots < x_n)$ in P, we denote the corresponding simplex of $|P|$ by $|x_\bullet|$. We will consider here a coarser subdivision, defined as follows. For any $x \in P$, consider the set of chains whose elements are smaller than x :

$$x_\leq = \{x_\bullet \text{ chains in } P \mid x_k \leq x \ \forall k\},$$

and

$$F(x) = \bigcup_{x_\bullet \in x_\leq} |x_\bullet|.$$

The coarser subdivision of $|P|$ we consider is defined by

$$|P| := \bigcup_{x \in P} F(x). \tag{8.1}$$

Example 8.1.3. *Let Σ be a fan. It defines a poset with the order given by reversing the inclusion of cones. Then $|\Sigma|$ with the coarser subdivision* (8.1) *is isomorphic (as a CW-complex) to the tropical toric variety $\mathbb{T}\Sigma$. The isomorphism is given by $F(\sigma) \to \overline{\mathcal{O}_\sigma^{\mathbb{T}}}$ for any cone σ.*

The cells $F(\tau, \sigma)$ associated to elements (τ, σ) in the poset $\mathcal{P}(T^\circ, T)$, form a regular CW-structure on $\mathbb{T}\Sigma_{T^\circ}$. The following proposition is proved in [MR25].

Proposition 8.1.4. *The poset $\mathcal{P}(T^\circ, T)$ is the face poset of a regular CW-structure on $\mathbb{T}\Sigma_{T^\circ}$ with cells $F(\tau, \sigma)$.*

Again, in the case of convex triangulations, this subdivision is the subdivision induced on $\mathbb{T}\Sigma_{T^\circ}$ by X_{trop}. We will be intersted more specifically in the subposet of $\mathcal{P}(T^\circ, T)$ defined by

$$\mathcal{P}^1(T^\circ, T) := \{(\tau, \sigma) \in \mathcal{P}(T^\circ, T) | \dim(\sigma) \geq 1\} .$$

In the convex case, this poset parametrizes X_{trop}. As it will appear later, this poset will be the support of the cosheaves we will consider. Denote by $X_{T^\circ,T}$ the geometric realization $|\mathcal{P}^1(T^\circ, T)|$ as defined above with the subdivision (8.1). Define the following subspace of $X_{T^\circ,T}$:

$$S_{T^\circ,T} = \{F(0, \sigma) \in X_{T^\circ,T} \mid 0 \in \sigma\} . \tag{8.2}$$

This is topologically a sphere and in the case where $X_{T^\circ,T} = X_{\text{trop}}$, this is the union of all bounded faces of X_{trop} in $N_{\mathbb{R}}$.

Definition 8.1.5. *Let T be a unimodular central triangulation of a reflexive polytope Δ and T° a unimodular central triangulation of its dual Δ°. The pair of CW-complexes*

$$(X_{T^\circ,T}, X_{T,T^\circ})$$

is called a combinatorial mirror pair.

8.2 Combinatorial mirror symmetry

Define $\mathcal{P}^\infty(T^\circ, T)$ to be the subposet of $\mathcal{P}(T^\circ, T)$ consisting of elements whose first coordinate is non zero. Its elements parametrize the boundary of $\mathbb{T}\Sigma_{T^\circ}$. In [MR25] we prove the following lemma.

Lemma 8.2.1. *The map*

$$\begin{aligned} p_{T^\circ,T} : \mathcal{P}^\infty(T^\circ, T) &\longrightarrow \mathcal{P}^\infty(T, T^\circ) \\ (\tau, \sigma) &\mapsto (\hat{\sigma}, \tau_\infty) \end{aligned} \tag{8.3}$$

is an isomorphism of posets.

Example 8.2.2. *In Figure 8.2, we represented the geometric realization of the poset $\mathcal{P}(T, T^\circ)$, where T and T° are defined as in Example 8.1.2, with some examples of cells. Notice that the cells (τ_2, σ_2) and (τ_3, σ_3) appearing in Figure 8.2 correspond, under the map $p_{T^\circ,T}$, to the cells with the same label in Figure 8.1.*

We will next refine the poset $\mathcal{P}(T^\circ, T)$, in order to extend this correspondence to the whole poset minus the origin.

Definition 8.2.3. *The poset $\mathcal{J}(T^\circ, T)$ is the subposet of $T^\circ \times T$ defined by*

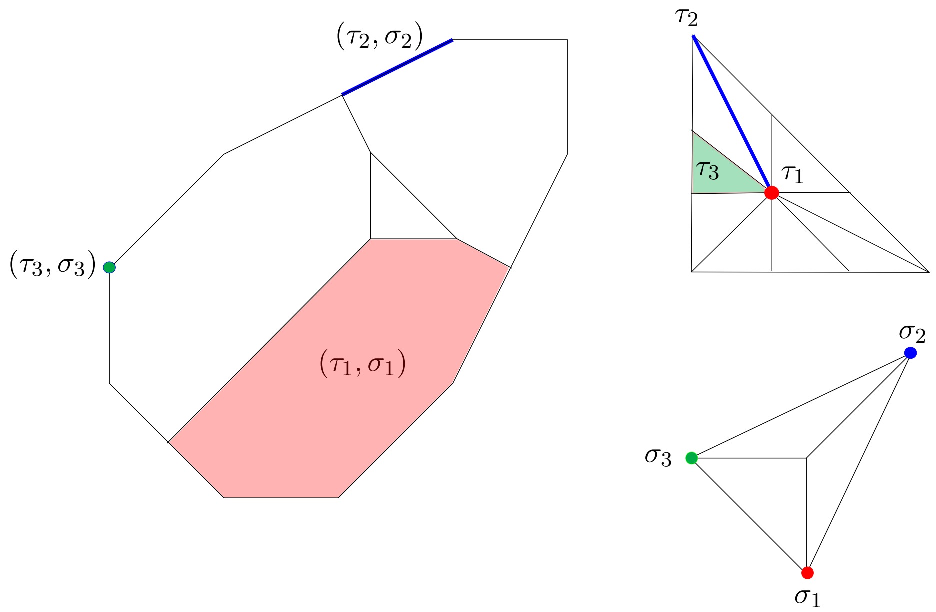

FIGURE 8.2 – The subdivision $\mathcal{P}(T, T^\circ)$

— *If $\sigma \neq 0$ and $0 \in \sigma$, then consider all pairs (τ, σ) such that $\tau \in \partial T^\circ$ and $\sigma_\infty \subset \min(C(\tau))^\vee$.*

— *If $\sigma \in \partial T$, consider all pairs (τ, σ) such that $\tau \neq 0$ and $\sigma \subset \min(C(\tau))^\vee$.*

Define $\mathcal{J}^S(T^\circ, T)$ the subposet of $\mathcal{J}(T^\circ, T)$ consisting of pairs of the first kind, and by $\mathcal{J}^\infty(T^\circ, T)$ the subposet of $\mathcal{J}(T^\circ, T)$ consisting of pairs (τ, σ) such that $0 \in \tau$.

Notice that we do not consider the pair $(0, 0)$ in $\mathcal{J}(T^\circ, T)$. In fact, by definition the pair $(0, 0)$ would be isolated (i.e. not comparable to any other pair inside $\mathcal{J}(T^\circ, T)$).

It follows from the definition that the map from $\mathcal{J}(T^\circ, T)$ to $\mathcal{P}(T^\circ, T) \setminus (0, 0)$ sending (τ, σ) to $(0, \sigma)$ if $0 \notin \tau$, and to itself if $0 \in \tau$ is surjective. The poset $\mathcal{J}(T^\circ, T)$ is still a graded poset (with the same grading as before), thin and still admits a balanced signature. In fact, its geometric realization (as defined above) is a subdivision of the one of $\mathcal{P}(T^\circ, T)$ (minus the cell $F(0, 0)$). Similarly, the geometric realization of $\mathcal{J}^S(T^\circ, T)$ is a subdivision of $S_{T^\circ, T}$, and $\mathcal{J}^\infty(T^\circ, T) = \mathcal{P}^\infty(T^\circ, T)$. As a corollary of Lemma 8.2.1, one obtains the following

Lemma 8.2.4. *The map*

$$j_{T^\circ, T} : \mathcal{J}(T^\circ, T) \longrightarrow \mathcal{J}(T, T^\circ) \tag{8.4}$$

sending

— *(τ, σ) to $(\hat{\sigma}, \tau_\infty)$ if $0 \in \tau$ and $\sigma \in \partial T$,*

— *(τ, σ) to $(\sigma_\infty, \hat{\tau})$ if $\tau \in \partial T^\circ$ and $0 \in \sigma$, and*

— *(τ, σ) to (σ, τ) if $(\tau, \sigma) \in \partial T^\circ \times \partial T$*

is an isomorphism of posets.

Notice that $j_{T^\circ, T}$ sends $\mathcal{J}^S(T^\circ, T)$ to $\mathcal{J}^S(T, T^\circ)$, sends $\mathcal{J}^\infty(T^\circ, T)$ to $\mathcal{J}^\infty(T, T^\circ)$ and preserves the dimensions.

Example 8.2.5. *In Figures 8.3 and 8.4 we represented the geometric realizations of the posets $\mathcal{J}(T, T^\circ)$ and $\mathcal{J}(T^\circ, T)$, where T and T° are defined again as in Example 8.1.2. The cells depicted in the two figures are dual in the sense that they correspond under the map $j_{T^\circ,T}$.*

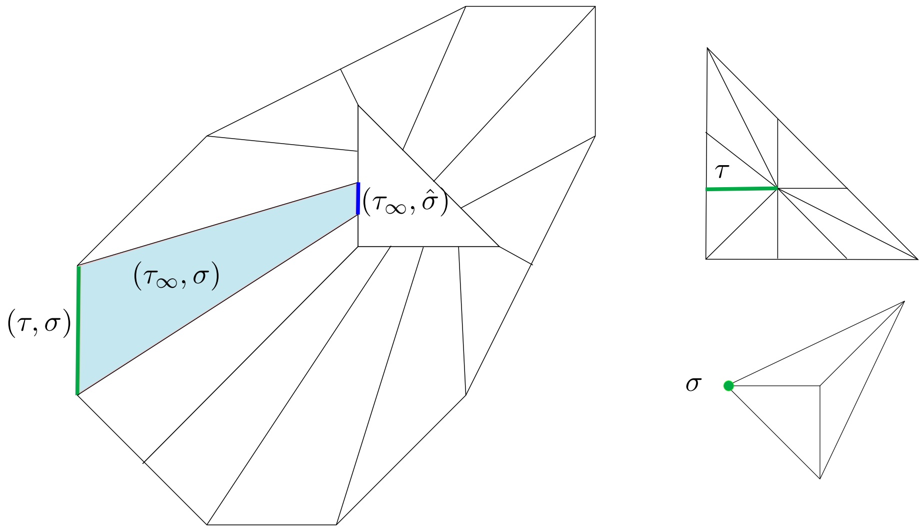

FIGURE 8.3 – The subdivision $\mathcal{J}(T, T^\circ)$

We extend the cosheaves $\mathcal{F}_p(\tau, \sigma)$ to the poset $\mathcal{J}(T^\circ, T)$ by precomposing with the map from above sending (τ, σ) to $(0, \sigma)$ if $0 \notin \tau$ and to itself if $0 \in \tau$. Define as before

$$\mathcal{J}^1(T^\circ, T) := \{(\tau, \sigma) \in \mathcal{J}(T^\circ, T) | \dim(\sigma) \geq 1\}.$$

The geometric realization of $\mathcal{J}^1(T^\circ, T)$ defines a subdivision of $X_{T^\circ,T}$ and it is the support of the cosheaf $\mathcal{F}_p$. One has a canonical isomorphism

$$H_q(\mathcal{J}(T^\circ, T); \mathcal{F}_p) \cong H_q(X_{T^\circ,T}; \mathcal{F}_p).$$

We defined now the mirror cosheaf.

Let $(\tau, \sigma) \in \mathcal{J}^S(T^\circ, T)$. Since $\sigma_\infty \subset \min(C(\tau))^\vee$, in particular the cone $C(\tau)$ is a subset of $\sigma_\infty^\perp$. Then $\langle C(\tau)\rangle \wedge \mathcal{F}_{p-1}(0, \sigma_\infty)$ is a submodule of $\mathcal{F}_p(0, \sigma)$, and we can consider the quotient.

Definition 8.2.6. *Let $p \geq 0$. The p-th mirror cosheaf $\mathcal{M}_p$ is defined over $\mathcal{J}(T^\circ, T)$ as follows :*

— *if $\sigma \neq 0$ and $0 \in \sigma$, then define*

$$\mathcal{M}_p(\tau, \sigma) = \frac{\mathcal{F}_p(0, \sigma)}{\langle C(\tau)\rangle \wedge \mathcal{F}_{p-1}(0, \sigma_\infty)} \tag{8.5}$$

where $\langle C(\tau)\rangle$ is the submodule generated by $C(\tau) \cap N$. When $p = 0$, we define $\mathcal{M}_0(\tau, \sigma) = \mathbb{Z}$.

— *for all other elements $(\tau, \sigma) \in \mathcal{J}(T^\circ, T)$ set*

$$\mathcal{M}_p(\tau, \sigma) = 0. \tag{8.6}$$

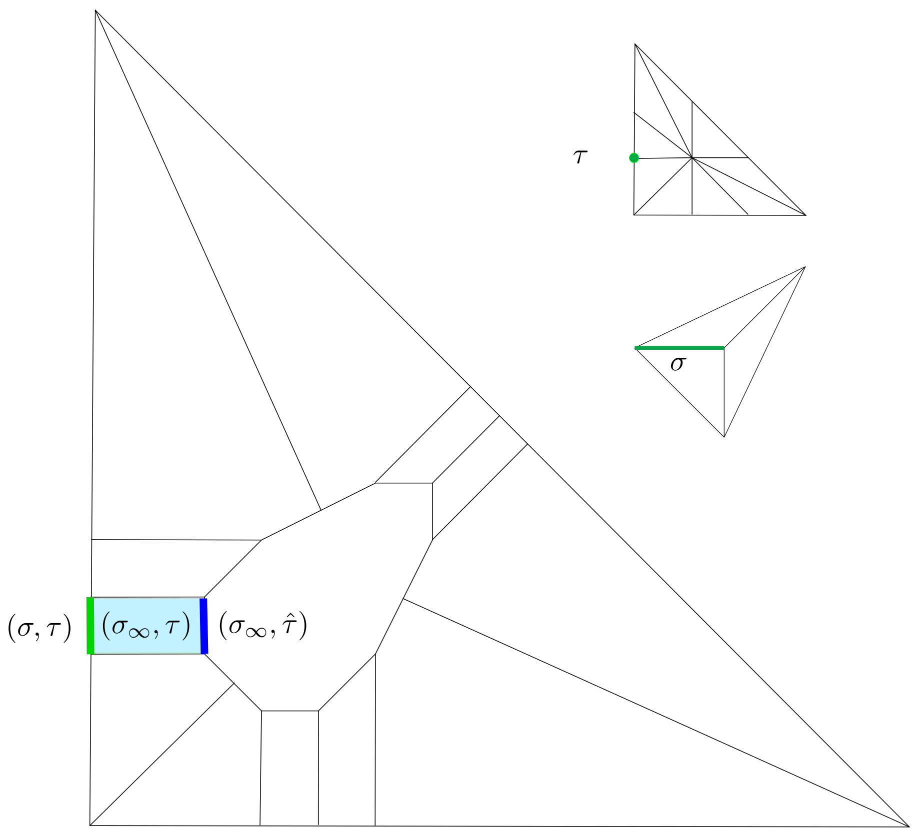

FIGURE 8.4 – The subdivision $\mathcal{J}(T^\circ, T)$

The cosheaf maps are induced by the ones from $\mathcal{F}_p$.

Notice that the support of $\mathcal{M}_p$ is $\mathcal{J}^S(T^\circ, T)$, and that moreover

$$\mathcal{M}_p(\tau, \sigma) = \mathcal{F}_p(0, \sigma) \quad \text{when } 0 \in \sigma \text{ and } \dim \sigma = 1.$$

We proved in [MR25] the following result :

Theorem 8.2.7. *Let A be a commutative ring. For all $q \geq 0$, there are canonical isomorphisms*

$$H_q(\mathcal{J}(T^\circ, T); \mathcal{F}_p \otimes A) \cong H_q(\mathcal{J}(T^\circ, T); \mathcal{M}_p \otimes A), \tag{8.7}$$

$$H_q(\mathcal{J}(T^\circ, T); \mathcal{M}_p \otimes A) \cong H_q(\mathcal{J}(T, T^\circ); \mathcal{M}_{n-p} \otimes A). \tag{8.8}$$

Moreover, they induce a canonical isomorphism

$$H_q(X_{T^\circ, T}; \mathcal{F}_p \otimes A) \cong H_q(X_{T, T^\circ}; \mathcal{F}_{n-p} \otimes A). \tag{8.9}$$

Corollary 8.2.8. *If T and T° are convex subdivisions, and X_{trop} and X°_{trop} are some correponding tropical hypersurfaces as in Definition 8.1.1, then we have the mirror symmetry of the tropical Hodge diamonds*

$$\dim H_q(X_{trop}; \mathcal{F}_p \otimes \mathbb{Q}) = \dim H_q(X^\circ_{trop}; \mathcal{F}_{n-p} \otimes \mathbb{Q}).$$

Definition 8.2.9 (Mirror classes)**.** *Two classes $\gamma \in H_q(\mathcal{P}(T^\circ, T); \mathcal{F}_p \otimes A)$ and $\gamma' \in H_q(\mathcal{P}(T, T^\circ); \mathcal{F}_{n-p} \otimes A)$ are called* mirror classes *if they coincide under isomorphism (8.9).*

8.3 Patchworking and divisor in the mirror

Consider the Picard group of a toric variety $\mathbb{C}\Sigma$ with $\mathbb{F}_2$ coefficients, denoted by $\mathrm{Pic}_{\mathbb{F}_2}(\mathbb{C}\Sigma)$. For a smooth toric variety $\mathbb{C}\Sigma$ with Σ a complete fan in M, this group is computed as follows. Let $\mathrm{Div}_{\mathrm{toric}}\,\mathbb{C}\Sigma$ be the free module generated by toric divisors, i.e.

$$\mathrm{Div}_{\mathrm{toric}}\,\mathbb{C}\Sigma = \sum_{\rho\in\Sigma(1)} \mathbb{F}_2 D_\rho$$

where the sum runs over all rays of the fan (i.e. 1-dimensional cones) and D_ρ is the toric divisor associated to the ray. Then we have the following short exact sequence

$$0 \to N\otimes\mathbb{F}_2 \to \mathrm{Div}_{\mathrm{toric}}\,\mathbb{C}\Sigma \to \mathrm{Pic}_{\mathbb{F}_2}(\mathbb{C}\Sigma) \to 0. \tag{8.10}$$

The second arrow is the map $n \mapsto \sum_\rho \langle n, v_\rho\rangle D_\rho$, where $v_\rho \in M\otimes\mathbb{F}_2$ is the $\mod 2$ reduction of the primitive generator of the ray ρ, see [Ful93b] and [CLS11].

We now go back to the dual reflexive polytopes Δ and Δ° with unimodular central triangulations T and T°. Suppose we do patchworking with T and some sign distribution ε, thus obtaining a PL-hypersurface $X_{T^\circ,T,\varepsilon}$ in $\mathbb{R}\Sigma_{T^\circ}$, which , in the convex case, is isotopic to a real Calabi-Yau variety.

The sign distribution ε also determines a toric divisor $D^\varepsilon \in \mathrm{Div}_{\mathrm{toric}}\,\mathbb{C}\Sigma_T$ as follows. Let $\mathbf{o}\in T$ be the unique interior lattice point, then the other lattice points $v\in\partial T$ are the primitive generators of the rays of the fan Σ_T. Therefore we can define

$$D^\varepsilon = \sum_{\varepsilon(v)=\varepsilon(\mathbf{o})} D_v.$$

It is clear that, up to inverting the sign of every lattice point in T, there is a one to one correspondence between toric divisors with $\mathbb{F}_2$ coefficients on $\mathbb{C}\Sigma_T$ and a choice of patchworking signs on T. Therefore we denote by $X_{T^\circ,T,D}$ the PL-hypersurface in $\mathbb{R}\Sigma_{T^\circ}$ constructed by the patchworking signs determined by $D\in\mathrm{Div}_{\mathrm{toric}}\,\mathbb{C}\Sigma_T$. We prove in [MR25] the following proposition :

Proposition 8.3.1. *If two divisors D and D' in $\mathrm{Div}_{toric}\,\mathbb{C}\Sigma_T$ define the same class in $\mathrm{Pic}_{\mathbb{F}_2}(\mathbb{C}\Sigma_T)$ then $X_{T^\circ,T,D}$ and $X_{T^\circ,T,D'}$ coincide, up to an automorphism of $\mathbb{R}\Sigma_{T^\circ}$.*

Example 8.3.2. *The real cubic curves in $\mathbb{P}^2$ obtained from patchworking with the sign distribution corresponding to $D = D_7 + D_8$ and to $D = D_8$ are depicted in Figure 8.5. Notice that the first one is connected, while the second one is not.*

8.4 Connectedness

In this last section, we will use Theorem 8.2.7 combined with the spectral sequence from Section 6.3 to give a necessary and sufficient condition on a patchworking of a reflexive polytope to be connected in terms of the mirror. Let first recall the reformulation of a real phase structure from [BdMR22] in terms of triangulation. Remind that we consider non-necesseraly convex triangulations.

Definition 8.4.1. *A* real phase structure *on (Δ, T) is the data $\mathcal{E}$ given by a choice, for every edge $\sigma\in T$, of an element $\mathcal{E}_\sigma \in \frac{N\otimes\mathbb{F}_2}{\sigma^\perp}$, such that for any two dimensional simplex $\tau\in T$ the number of edges σ of τ such that $\mathcal{E}_\sigma = 0$ is either* 0 *or* 2*.*

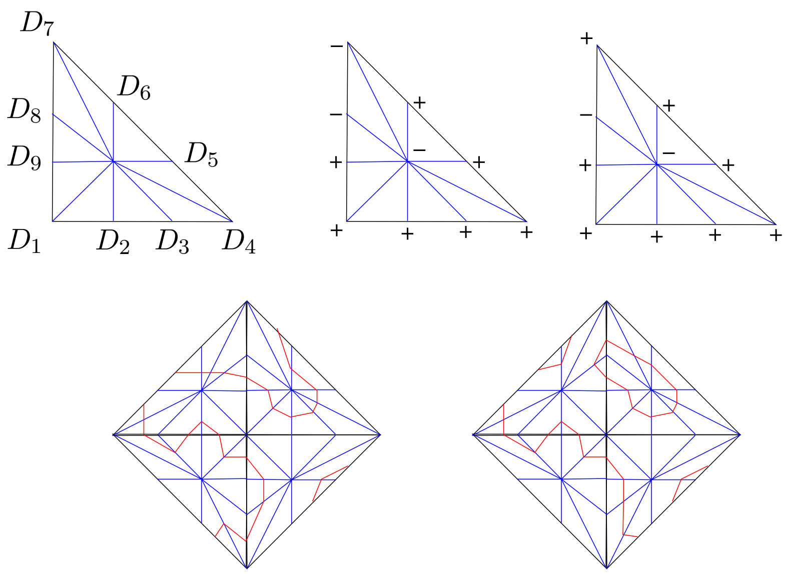

FIGURE 8.5 – The signs corresponding to $D = D_7 + D_8$ and to $D = D_8$ and the corresponding real cubics.

Definition 8.4.2. *Let $\mathcal{E}$ be a real phase structure on (Δ, T). For $(\tau, \sigma) \in \mathcal{P}^1(T^\circ, T)$, define*

$$\mathcal{E}(\tau, \sigma) = \{s \in N \otimes \mathbb{F}_2/\langle C(\tau)\rangle \mid \pi_\gamma(s) = \mathcal{E}_\gamma \text{ for some edge } \gamma \subset \sigma\},$$

where $\pi_\gamma : N \otimes \mathbb{F}_2/\langle C(\tau)\rangle \to N \otimes \mathbb{F}_2/\gamma^\perp$.

Define also

$$\mathcal{P}_\mathbb{R}(T^\circ, T) = \{(\tau, \sigma, s) \mid (\tau, \sigma) \in \mathcal{P}(T^\circ, T), s \in N \otimes \mathbb{F}_2/C(\tau)\}.$$

The set $\mathcal{P}_\mathbb{R}(T^\circ, T)$ is also a poset, where the order is given by $(\tau, \sigma, s) \leq (\tau', \sigma', s')$ if $(\tau, \sigma) \leq_\mathcal{P} (\tau', \sigma')$ and $\pi_{\tau', \tau}(s') = s$, where $\pi_{\tau',\tau} : N \otimes \mathbb{F}_2/\langle C(\tau')\rangle \to N \otimes \mathbb{F}_2/\langle C(\tau)\rangle$ is the $\mathbb{F}_2$ reduction of the projection maps (3.5). Consider the geometric realization of the poset $\mathcal{P}_\mathbb{R}(T^\circ, T)$ and its subdivision given by

$$|\mathcal{P}_\mathbb{R}(T^\circ, T)| = \bigcup_{(\tau,\sigma,s)\in\mathcal{P}_\mathbb{R}(T^\circ,T)} F(\tau, \sigma, s).$$

where

$$F(\tau, \sigma, s) = \bigcup_{(\tau,\sigma,s)_\bullet \in (\tau,\sigma,s)_\leq} |(\tau, \sigma, s)_\bullet|.$$

In [MR25] we proved the two following propositions

Proposition 8.4.3. *The poset $\mathcal{P}_\mathbb{R}(T^\circ, T)$ is the face poset of a regular CW-structure on $\mathbb{R}\Sigma_{T^\circ}$ with cells $F(\tau, \sigma, s)$.*

Given a real phase structure $\mathcal{E}$ on (Δ, T), consider the following subposet of $\mathcal{P}_\mathbb{R}(T^\circ, T)$:

$$\mathcal{P}^1_\mathcal{E}(T^\circ, T) = \{(\tau, \sigma, s) \in \mathcal{P}_\mathbb{R}(T^\circ, T) \mid \dim \sigma \geq 1, \quad s \in \mathcal{E}(\tau, \sigma)\}.$$

Proposition 8.4.4. *Let $\mathcal{E}$ be a real structure on (Δ, T) and let $\varepsilon : \Delta \cap M \to \mathbb{F}_2$ be a corresponding sign distribution. Then $\mathcal{P}^1_\mathcal{E}(T^\circ, T)$ is a regular CW-structure on $X_{T^\circ, T, \varepsilon}$.*

There is an obvious surjective map from $\mathcal{P}_{\mathbb{R}}(T^\circ, T)$ to $\mathcal{P}(T^\circ, T)$ by forgetting the last coordinate, and results from Section 6.3 extend to this case. We state them in this framework for reader convenience. The proofs are similar as in [RS23].

Definition 8.4.5. *The sign cosheaf on $\mathcal{P}(T^\circ, T)$ is given by*

$$\mathcal{S}(\tau, \sigma) := \mathbb{F}_2^{\mathcal{E}(\tau,\sigma)}.$$

For $\varepsilon \in \mathcal{E}(\tau, \sigma)$, we will denote by w_ε the corresponding generator of $\mathcal{S}(\tau, \sigma)$.

Proposition 8.4.6. *For every integer q, the groups $H_q(X_{T^\circ,T}; \mathcal{S})$ and $H_q(X_{T^\circ,T,\varepsilon}; \mathbb{F}_2)$ are isomorphic.*

Proposition 8.4.7. *Let $\mathcal{E}$ be a real structure on (Δ, T). There exists a filtration of cosheaves on $\mathcal{P}(T^\circ, T)$*

$$0 = \mathcal{K}_{n+1} \subset \mathcal{K}_n \subset \mathcal{K}_{n-1} \subset \cdots \subset \mathcal{K}_0 = \mathcal{S}$$

such that

$$\mathcal{K}_p / \mathcal{K}_{p+1} \simeq \mathcal{F}_p.$$

The spectral sequence $(E^k_{p,q}, \delta^{[k]})$ associated to this filtration has first page

$$E^1_{p,q} = H_q(X_{T^\circ,T}; \mathcal{F}_p)$$

and converges to $H_\bullet(X_{T^\circ,T}; \mathcal{S})$ which by Proposition 8.4.6 is the homology of $X_{T^\circ,T,\varepsilon}$.

To simplify a little the notations in this section, for a $\mathbb{Z}$-cosheaf $\mathcal{G}$, we will denote by $\mathcal{G}^{\mathbb{F}_2}$ the tensor product $\mathcal{G} \otimes \mathbb{F}_2$. Let $D = \sum_{\rho \in \Sigma_T(1)} a_\rho D_\rho$ be a toric divisor on $\mathbb{C}\Sigma_T$ with $\mathbb{F}_2$-coefficients. Denote by $\operatorname{supp}(D)$ the support of D, meaning the set of rays in Σ_T with a non-vanishing coefficient. Again, to simplify the notations we will identify the ray ρ with the segment $S(\rho)$ in T. The divisor D defines, by the same formula, a divisor on $\mathbb{T}\Sigma_T$ and when restricted to X_{T,T° it defines a class in $H_{n-1}(X_{T,T^\circ}; \mathcal{F}^{\mathbb{F}_2}_{n-1})$ represented by the cycle

$$D_{|X_{T,T^\circ}} := \sum_{\substack{\rho \,\in\, \operatorname{supp}(D) \\ \dim(\delta) = 1 \text{ and } \delta \subset \min(\rho)^\vee}} (\rho, \delta) \otimes v_{\rho,\delta},$$

where $v_{\rho,\delta}$ is the generator of $\Lambda^{n-1}_{\mathbb{F}_2}(\frac{\delta^\perp}{\rho})$.

Remark 8.4.8. *If T and T° are convex, the Viro polynomial defines the mirror family of complex hypersurfaces X°_t in $\mathbb{C}\Sigma_T$, and the divisor D restricts to a family of divisors $D_{|X^\circ_t} = \sum_{\rho \in \Sigma_T(1)} a_\rho D_\rho \cap X^\circ_t$. Each $D_\rho \cap X^\circ_t$ when restricted to the toric orbit $\mathcal{O}^{\mathbb{C}}_\rho$ defines in turn a Viro polynomial which is the one of the tropical hypersurface $X_{T,T^\circ} \cap \mathcal{O}^{\mathbb{T}}_\rho$. The facets of $X_{T,T^\circ} \cap \mathcal{O}^{\mathbb{T}}_\rho$ are precisely dual to the edges δ of Δ° such that $\delta \subset \min(\rho)^\vee$.*

On the other hand the divisor D also defines a real phase structure $\mathcal{E} = \mathcal{E}^D$ on (Δ, T) and a corresponding patchworking $X_{T^\circ,T,\varepsilon}$. Now we indicate how to prove the following theorem :

Theorem 8.4.9. *Assume that*

$$H_n(X_{T^\circ,T}; \mathcal{F}^{\mathbb{F}_2}_k) = 0 \quad \text{for all} \quad 0 < k < n. \tag{8.11}$$

Then the patchworking $X_{T^\circ,T,\varepsilon}$ is connected if and only if $[D_{|X_{T,T^\circ}}] = 0$ in $H_{n-1}(X_{T,T^\circ}; \mathcal{F}^{\mathbb{F}_2}_{n-1})$.

Remark 8.4.10. *The hypothesis (8.11) is satisfied if Δ is a polytope giving rise to a non-singular toric variety. This follows from Proposition 3.2 in [BdMR22] (see also [ARS21]). We expect that in the convex case, the tropical divisor class $[D_{|X_{T,T^\circ}}]$ vanishes in $H_{n-1}(X^\circ_{trop};\mathcal{F}^{\mathbb{F}_2}_{n-1})$ if and only if, for t small enough, the class of the complex divisor $[D_{|X^\circ_t}]$ vanishes in the Picard group modulo 2 of X°_t.*

Recall that the real phase structure $\mathcal{E}^D$ defines a sign cosheaf $\mathcal{S}_D$ on $\mathcal{P}(T^\circ,T)$ and induces a spectral sequence $(E^k_{p,q},\delta^{[k]})$ converging to $H_*(X_{T^\circ,T};\mathcal{S}_D)$. Obviously the spectral sequence depends on D, but to keep notation simple we remove D from the decorations. In particular the number of connected components of $X_{T^\circ,T,\varepsilon}$ is given by the dimension of the homology group

$$H_n(X_{T^\circ,T,\varepsilon},\mathbb{F}_2)\cong\bigoplus_k E^\infty_{k,n}.$$

Therefore, with assumption (8.11), the number of connected components of $X_{T^\circ,T,\varepsilon}$ can be either one or two. If the differential at the first page

$$\delta^{[1]}:H_n(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_0)\to H_{n-1}(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_1)$$

does not vanish, i.e. it is injective, then $X_{T^\circ,T,\varepsilon}$ is connected. If it vanishes we need to study also higher differntials $\delta^{[k]}$. Recall that $X_{T^\circ,T}$ contains the sphere $S=S_{T^\circ,T}$ (see (8.2)) and it is homotopically equivalent to it. One direction of Theorem 8.4.9 follows from the following

Theorem 8.4.11. *Let $S\in H_n(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_0)$ be the generator. Then $\delta^{[1]}(S)$ and $[D_{|X_{T,T^\circ}}]$ are mirror classes. In particular, if $[D_{|X_{T,T^\circ}}]\neq 0$ then $X_{T^\circ,T,\varepsilon}$ is connected.*

For the proof, we refer to [MR25].

We now investigate what happens when $\delta^{[1]}S=0$, (i.e. if $[D_{|X_{T,T^\circ}}]=0$). In this case the term $E^2_{0,n}$ is equal to $H_n(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_0)$, and thus generated by S. More generally the term $E^k_{0,n}$ is equal to $H_n(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_0)$ as long as $\delta^{[j]}S=0$ for all $j<k$. The next theorem proves the other direction in Theorem 8.4.9.

Theorem 8.4.12. *If $\delta^{[1]}S=0$, then*

$$E^\infty_{0,n}=H_n(X_{T^\circ,T};\mathcal{F}^{\mathbb{F}_2}_0)=\mathbb{F}_2$$

In particular, assuming (8.11), if $[D_{|X_{T,T^\circ}}]=0$ then the patchworking $X_{T^\circ,T,\varepsilon}$ has two connected components.

Again, we refer to [MR25] for the proof.

Example 8.4.13. *Let $D=\sum_{a_\rho}D_\rho$ be supported in the interior of the facets of Δ, then $D_{|X_{T,T^\circ}}=0$, since divisors in the interior of maximal faces are mapped to vertices by the blowdown map $\mathrm{Bl}:\mathbb{T}\Sigma_T\to\mathbb{T}\Sigma_{\Delta^\circ}$. In particular $X_{T^\circ,T,\epsilon}$ will have two connected components.*

Example 8.4.14. *Consider the divisors $D=D_7+D_8$ and $D=D_8$ as in Figure 8.5. In the first case $D_{|X_{T,T^\circ}}$ is one point, in the second $D_{|X_{T,T^\circ}}=0$ (see Figure 8.6). The corresponding patchworkings are depicted in Figure 8.5 and they are respectivley connected and disconnected.*

Example 8.4.15. *Consider the two reflexive polytopes in Figure 8.7, where T is on the left. We have drawn an example of patchworking and the corresponding divisor. The divisor $D_{|X_{T,T^\circ}}$ consists of two points and therefore it is zero ($\bmod 2$). Indeed any patchworking in this example will be disconnected.*

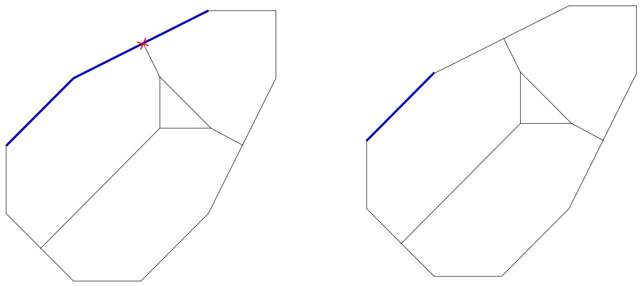

FIGURE 8.6 – The intersection of X_{T,T° with two different divisors.

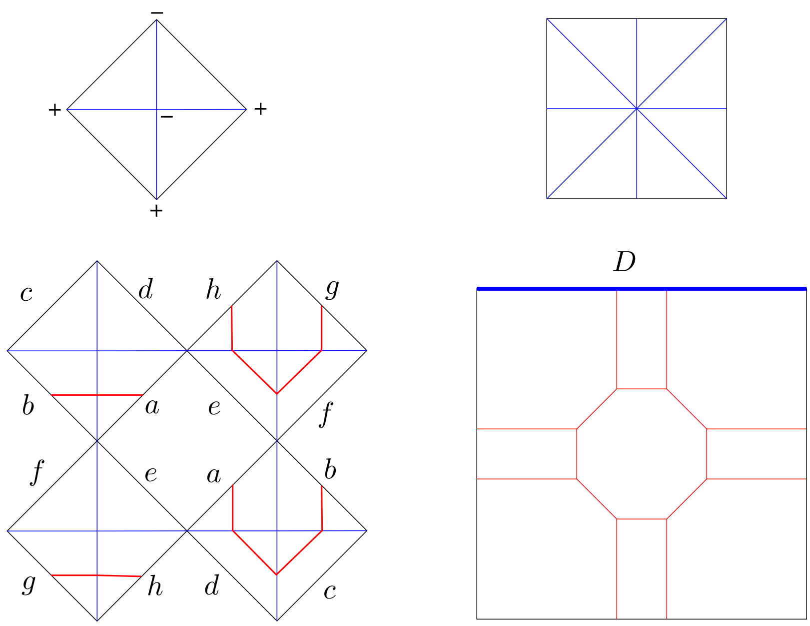

FIGURE 8.7 – Two dual reflexive polygons. A divisor D in the mirror and the corresponding disconnected patchworking. The divisor intersects the mirror curve in two points, i.e. $D_{|X_{T,T^\circ}} = 0$.

Bibliographie

[AK06] Federico Ardila et Caroline J Klivans : The bergman complex of a matroid and phylogenetic trees. *Journal of Combinatorial Theory, Series B*, 96(1):38–49, 2006.

[ARS21] Charles Arnal, Arthur Renaudineau et Kris Shaw : Lefschetz section theorems for tropical hypersurfaces. *Annales Henri Lebesgue*, 4:1347–1387, 2021.

[BB96] V. V. Batyrev et L. A. Borisov : Mirror duality and string-theoretic Hodge numbers. *Invent. Math.* **126**, pages 183–203, 1996.

[BB18] Matt Baker et Nathan Bowler : Matroids over hyperfields. *AIP Conference Proceedings*, 1978(1):340010, 2018.

[BBR17] B. Bertrand, E. Brugallé et A. Renaudineau : Haas' theorem revisited. *Épijournal Geom. Algébrique*, 1:Art. 9, 22, 2017.

[BdMR22] Erwan Brugallé, Lucía López de Medrano et Johannes Rau : Combinatorial patchworking : back from tropical geometry. *arXiv preprint arXiv :2209.14043*, 2022.

[BIMS] E. Brugallé, I. Itenberg, G. Mikhalkin et K. Shaw : Brief introduction to tropical geometry. *To appear in the Proceedings of 21st Gökova Geometry-Topology Conference.*

[BLVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White et Gunter M. Ziegler : *Oriented Matroids.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 édition, 1999.

[CLS11] D. A. Cox, J. B. Little et H. K. Schenck : *Toric varieties*, volume 124 de *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.

[Cox99] Cox, David A. and Katz, Sheldon : *Mirror symmetry and algebraic geometry.* Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

[DK] Vladimir I Danilov et Khovanskiĭ : Newton polyhedra and an algorithm for computing hodge–deligne numbers.

[dS95] Ilda P. F. da Silva : Axioms for maximal vectors of an oriented matroid : a combinatorial characterization of the regions determined by an arrangement of pseudohyperplanes. *European J. Combin.*, 16(2):125–145, 1995.

[FL78] Jon Folkman et Jim Lawrence : Oriented matroids. *J. Combin. Theory Ser. B*, 25(2):199–236, 1978.

[FR13] Georges Francois et Johannes Rau : The diagonal of tropical matroid varieties and cycle intersections. *Collect. Math.*, 64(2):185–210, 2013.

[Ful93a] W. Fulton : *Introduction to toric varieties*, volume 131 de *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Ful93b] William Fulton : *Introduction to toric varieties*, volume 131 de *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[GKZ08] I. M. Gelfand, M. M. Kapranov et A. V. Zelevinsky : *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition.

[Haa97] B. Haas : Real algebraic curves and combinatorial constructions. Thèse doctorale, Université de Strasbourg, 1997.

[Hat02] Allen Hatcher : *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[IKMZ19] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin et Ilia Zharkov : Tropical homology. *Mathematische Annalen*, 374(1):963–1006, 2019.

[Ite95] I. Itenberg : Counter-examples to Ragsdale conjecture and T-curves. *In Real algebraic geometry and topology (East Lansing, MI, 1993)*, volume 182 de *Contemp. Math.*, pages 55–72. Amer. Math. Soc., Providence, RI, 1995.

[Ite97] I. Itenberg : Topology of real algebraic T-surfaces. *Rev. Mat. Univ. Complut. Madrid*, 10(Special Issue, suppl.):131–152, 1997. Real algebraic and analytic geometry (Segovia, 1995).

[Ite17] Ilia Itenberg : Tropical homology and betti numbers of real algebraic varieties, 2017.

[JRS18] Philipp Jell, Johannes Rau et Kristin Shaw : Lefschetz $(1,1)$-theorem in tropical geometry. *Épijournal Geom. Algébrique*, 2:Art. 11, 27, 2018.

[KS12] Eric Katz et Alan Stapledon : Tropical geometry and the motivic nearby fiber. *Compositio Mathematica*, 148(1):269–294, 2012.

[KS16] E. Katz et A. Stapledon : Tropical geometry, the motivic nearby fiber, and limit mixed Hodge numbers of hypersurfaces. *Res Math Sci*, 3, 2016.

[MR] Grigory Mikhalkin et Johannes Rau : Tropical geometry. https://math.uniandes.edu.co/~j.rau.

[MR25] Diego Matessi et Arthur Renaudineau : Mirror symmetry for tropical hypersurfaces and patchworking. *Journal of the Institute of Mathematics of Jussieu*, page 1–39, 2025.

[MZ14] G. Mikhalkin et I. Zharkov : Tropical eigenwave and intermediate Jacobians. *In Homological mirror symmetry and tropical geometry*, volume 15 de *Lect. Notes Unione Mat. Ital.*, pages 309–349. Springer, Cham, 2014.

[OT92] Peter Orlik et Hiroaki Terao : *Arrangements of hyperplanes*, volume 300 de *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Ren17] A. Renaudineau : A tropical construction of a family of real reducible curves. *J. Symbolic Comput.*, 80(part 2):251–272, 2017.

[Ric93] Jürgen Richter-Gebert : Oriented matroids with few mutations. *Discrete Comput. Geom.*, 10(3):251–269, 1993.

[Ris93] J. J. Risler : Construction d'hypersurfaces réelles (d'après Viro). *Astérisque*, (216):Exp. No. 763, 3, 69–86, 1993. Séminaire Bourbaki, Vol. 1992/93.

[RRS22] Johannes Rau, Arthur Renaudineau et Kris Shaw : Real phase structures on matroid fans and matroid orientations. *Journal of the London Mathematical Society*, 106(4):3687–3710, 2022.

[RRS23] Johannes Rau, Arthur Renaudineau et Kris Shaw : Real phase structures on tropical manifolds and patchworks in higher codimension. *arXiv preprint arXiv :2310.08313*, 2023.

[RS23] Arthur Renaudineau et Kris Shaw : Bounding the betti numbers of real hypersurfaces near the tropical limit. *In Annales Scientifiques de l'Ecole Normale Supérieure*, volume 56, pages 945–980. Société Mathématique de France, 2023.

[RZ94] Jürgen Richter-Gebert et Günter M. Ziegler : Zonotopal tilings and the Bohne-Dress theorem. *In Jerusalem combinatorics '93 : an international conference in combinatorics, May 9-17, 1993, Jerusalem, Israel*, pages 211–232. Providence, RI : American Mathematical Society, 1994.

[Sha11] K. Shaw : *Tropical intersection theory and surfaces.* Thèse de doctorat, University of Geneva, 2011.

[Sha13] Kristin Shaw : A tropical intersection product in matroidal fans. *SIAM J. Discrete Math.*, 27(1):459–491, 2013.

[Stu94] Bernd Sturmfels : Viro's theorem for complete intersections. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 21(3):377–386, 1994.

[Vir84] O. Viro : Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. *In Topology (Leningrad, 1982)*, volume 1060 de *Lecture Notes in Math.*, pages 187–200. Springer, Berlin, 1984.

[Vir01] O. Viro : Dequantization of real algebraic geometry on logarithmic paper. *In European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 de *Progr. Math.*, pages 135–146. Birkhäuser, Basel, 2001.

[Whi86] Neil White, éditeur. *Theory of matroids*, volume 26 de *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1986.

[Zas75] Thomas Zaslavsky : Facing up to arrangements : face-count formulas for partitions of space by hyperplanes. *Mem. Amer. Math. Soc.*, 1(issue 1, 154): vii+102, 1975.

[Zha13] Ilia Zharkov : The orlik-solomon algebra and the bergman fan of a matroid. *Journal of Gökova Geometry Topology*, 7:25–31, 2013.