Reading seminar on universal characterization of K-theory

Antoine Touzé and Julia Semikina

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Algebraic K-theory is a fundamental algebro-geometric invariant, capturing information in arithmetic, algebraic geometry, and topology. In contrast with the Grothendieck group, the classical construction of the algebraic K-theory does not provide a universal characterization. The crucial result towards such a universality of K-theory is the additivity theorem. Roughly, it says that the S-construction forces the chosen cofiber sequences to split.

The absence of a framework in which to systematically express homotopical constructions in the category of categories was an obstruction for precise universal characterization of algebraic K-theory. The development of higher category theory made it possible to overcome the issue. The goal of the seminar is to understand the infinity-categorical framework for the construction and the universal characterization of K-theory. We will mostly follow the foundational paper [BGT13].

1 Talks

Talk 1: Review of classical algebraic K-theory.

Date: 17 janvier et 24 janvier; Speaker: Théo

Present the construction of a K-theory space using Waldhausen's $S_\bullet$-construction following Weibel [Wei13, Chapter 4, p. 66] or Rognes' lecture notes [Rog10, Section 8.3]. Sketch the proof of the additivity theorem and Morita equivalence (for the category of projective R-modules). Show that K-theory preserves filtered colimits.

Talk 2: Stable ∞-categories

Date: 24 janvier et 31 janvier; Speaker: Quentin

The speaker may take for granted the basic notions on ∞-categories (which were introduced in the reading group of last year).

Give the definition of a stable ∞-category, and explain that its homotopy category is a triangulated category. The references are [Lur17, Chap 1, 1.1] or [Lur09a, section 2 and 3] or [Gro20, section 5.1].

An algebraic example: present the stable category of chain complexes on an additive category, and show that for an abelian category, the associated homotopy category is the usual derived category, following [Lur09a, section 13].

Talk 3: K-theory of ∞-categories

Date: 14 février et 21 mars; Speaker: Matthew

Give the definition of the simplicial category of ∞-categories Cat_∞^Δ, cf. [Lur09b, Definition 3.0.0.1]. Define the (non full) simplicial subcategory with objects the ∞-categories with finite colimits, and simplicial sets of morphisms the right exact functors.

Give the definition of the simplicial category of symmetric spectra Sp^Σ cf. [HSS00, Sections 1.2, 1.3].

Present the construction of a K-theory of stable ∞-categories [BGT13, Section 7.1]. (The construction yields a simplicial functor from the category of ∞-categories with finite colimits and simplicial sets of morphisms the right exact funtors to the simplicial category of symmetric spectra).

Talk 4: The S' construction

Date: 13 juin; Speaker: Julia

This talk will deal with homotopy (co)limits and classical K-theory. We will explain the S' construction, which is a version of Waldhausen's construction using homotopy diagrams instead of strict diagrams. The main reference is [BM08].

Talk 5: simplicial model categories versus ∞-categories

Date: 20 juin; Speaker: Antoine

This talk will explain some fundamental constructions of ∞-categories. Namely, we will recall the ∞-category of ∞-categories. We will review the classical notion of a simplicial model category, and how it compares to ∞-categories. The main reference here is [Lur09b].

Talk 5: Comparison of K-theory

Date: 27 juin; Speaker: Antoine/Julia

In this talk we will explain why the classical K-theory is a special case of the K-theory of ∞-categories, after [BGT13].

References

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