Homology of groups

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This is not a book...

This document is not a book, but the lecture notes of a Master 2 course on the homology of groups, taught in spring 2023 at the university of Lille. The reader will find here all the definitions and statements of the theorems given in the course, without proofs.

The student is assumed to have a knowledge of general topology, fundamental groups and covering spaces (although some recollections are done on these subjects). The first part of the course covers singular homology of topological spaces (including simplicial homology and cellular homology). The second part of the course deals with the definition of Ext and Tor over rings. Finally, the third part of the course introduces the homology of groups in a purely algebraic fashion (Tor over group rings) and explains the equivalent topological definition (Homology of $K(G,1)$ spaces).

A few "vista" sections were explained without proofs, they are just intended as a brief introduction to more advanced topics and they are not part of the material for the exam.

The whole course can be seen as an introduction to homological algebra. Indeed all the most basic concepts of homological algebra are introduced during the course: complexes, quasi-isomorphisms, homotopies, long exact sequences, the fundamental theorem of homological algebra, bicomplexes, universal coefficient theorems, the Künneth formula (over fields), the bar construction.

Contents

The course and the exercise sheets are mainly based on the following excellent textbooks. The curious reader may consult these references for the proofs and further developments.

References

[Félix-Tanré] Y. Felix, D. Tanré, Topologie Algébrique, Dunod.

[Hatcher] A. Hatcher, Algebraic Topology, CUP, available online:
`https://pi.math.cornell.edu/~hatcher/AT/ATpage.html`

[Weibel] C. Weibel, An introduction to homological algebra, CUP

[Rotman] J. Rotman, An introduction to homological algebra, Springer

[Brown] K. S. Brown, Cohomology of groups, Springer

To be more specific, part I on topology and homology is based on chapters 5 and 6 of [Félix-Tanré], with a few additional facts borrowed from [Hatcher], namely the formula for the differential in cellular homology, and some facts relative to cohomology.

Part II on algebra and homology is based on chapters 1 and 3 of [Weibel], with a few additional facts taken from [Rotman]. A nice account of the universal coefficient theorem can be also found in [Félix-Tanré].

Part III on the (co)homology of groups is mainly based on chapter 6 of [Weibel], and on chapters 1–3 of [Brown]. The part relative to Δ-sets and the general construction of $K(G,1)$ spaces is taken from the begining of chapter 2 and the additional topics of chapter 1 of [Hatcher].

Part I
Topology and homology

1 Some algebraic topology

1.1 Homotopy

Definition 1. Given two continuous maps $f, g : X \to Y$, we say that f *is homotopic to* g if there is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a *homotopy* between f and g. If f is homotopic to g, we write $f \sim g$, or $f \sim_H g$ if we want to specify the homotopy H.

Lemma 2. *1. The binary relation $\sim$ is an equivalence relation on the set $\mathcal{C}(X,Y)$ of continuous maps from X to Y.*

2. If $f_1, g_1 : X \to Y$ are homotopic and $f_2, g_2 : Y \to Z$ are homotopic then $f_2 \circ f_1, g_2 \circ g_1 : X \to Z$ are homotopic.

Definition 3. A continuous map $f : X \to Y$ is a *homotopy equivalence* if there is a map $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. Then g is called *a homotopy inverse* of f.

A topological space X is *homotopy equivalent* to Y (or *has the same homotopy type as* Y) if there is a homotopy equivalence $f : X \to Y$. If X is *homotopy equivalent* to Y, we write $X \sim Y$.

A space with the same homotopy type as the one point space $\{*\}$ is called *contractible.*

Lemma 4. *The binary relation $\sim$ defines an equivalence relation on the collection of topological spaces.*

Examples 5. 1. Homeomorphisms are homotopy equivalences.

2. Let D^n denote the n-dimensional disk, i.e. the unit ball in $\mathbb{R}^n$, and S^{n-1} the unit sphere. The inclusion $D^n \setminus S^{n-1} \hookrightarrow D^n$ is a homotopy equivalence, but not an homeomorphism.

3. If A is a subspace of X, a *deformation retraction* of X onto A is a continuous map:
$$r : X \times [0,1] \to X$$
such that $r(x,0) = \mathrm{id}_X$, $r(x,1) \in A$ for all $x \in X$, and $r(a,t) = a$ for all $a \in A$ and all $t \in [0,1]$. If there is a deformation retraction of X onto A, the inclusion $A \subset X$ is a homotopy equivalence.

Remark 6. Deformation retractions are a very visual way of producing homotopy equivalences. In fact, one can prove that every homotopy equivalence can be realized by deformation retractions: X has the same homotopy type as Y if and only if there is a third space Z which contains X and Y as deformation retracts, see e.g. Hatcher, cor 0.21.

1.2 Two basic homotopy invariants

Informally speaking, a "homotopy invariant" is a quantity defined for all topological spaces, and which is equal for two topological spaces of the same homotopy type.

A first homotopy invariant that you already know is the *fundamental group* of a (pointed) topological space (X, x). The fundamental group is usually denoted by $\pi_1(X, x)$. We will use it later, but for the moment we only recall the following general facts.

1. Every continuous map $f : X \to Y$ induces a morphism of groups
$$\pi_1(f, x) : \pi_1(X, x) \to \pi_1(Y, f(x))$$
and if f is a homotopy equivalence, then $\pi_1(f, x)$ is an isomorphism. (This point shows that the fundamental group is a homotopy invariant).

2. Basic calculation: for all $x \in S^n$, $\pi_1(S^n, x) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ \{1\} & \text{if } n \neq 1 \end{cases}$.

Another (easier) homotopy invariant is the set of arcwise connected components of a space. We study it into details now.

Definition 7. For all topological spaces X we let $\pi_0(X)$ denote the set of arcwise connected components of X. We let $\overline{x}$ denote the arcwise component of $x \in X$. For all continuous maps $f : X \to Y$ we define a map
$$\pi_0(f) : \begin{array}{ccc} \pi_0(X) & \to & \pi_0(Y) \\ \overline{x} & \mapsto & \overline{f(x)} \end{array} .$$

Proposition 8. *The following holds:*

1. *a)* $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$,
   b) $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$,
2. *if* $f \sim g$, *then* $\pi_0(f) = \pi_0(g)$.

1.3 The language of categories

The language of categories is convenient to give compact descriptions of the properties of topological invariants.

Definition 9. A *category* is the data of:

- a collection $\mathrm{Ob}(\mathcal{C})$ of objects $X, Y, \dots$,
- for all pairs of objects X, Y a set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
- a composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

  which is associative, and which has units: for all objects X there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Examples 10. We have the following examples of categories. In each example, the composition law is given by the usual composition.

Set**:** objects: the sets, morphisms: the maps.

Top**:** objects: the topological spaces, morphisms: the continuous maps.

Gp**:** objects: the groups, morphisms: the morphisms of groups.

$R-$Mod**:** objects: the R-modules, morphisms: the R-linear maps.

Definition 11. A functor $F : \mathcal{C} \to \mathcal{D}$ is the data:

- for each object X of $\mathcal{C}$, an object $F(X)$ of $\mathcal{D}$.
- for each morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$, such that F preserves composition and identity morphisms.

Exercise 12. An *isomorphism* in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Show that every functor preserves the isomorphisms.

The definition and properties of π_0 can then be reformulated in a compact way by saying that π_0 is a functor $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ which sends homotopic maps to the same map.

Proposition 13. *Let $F : \mathrm{Top} \to \mathcal{C}$ be a functor. The following assertions are equivalent: (i) for all continuous maps f, g if $f \sim g$ then $F(f) = F(g)$, (ii) F sends the homotopy equivalences to isomorphisms in $\mathcal{C}$.*

Corollary 14. *If f is a homotopy equivalence, then $\pi_0(f)$ is a bijection.*

Exercise 15. Describe the fundamental group as a functor.

1.4 Singular homology

Definition 16. Given a set S and a ring R, we denote by RS the *free R-module on S*. This is the set of formal sums $\sum_{s\in S} r_s b_s$ with only finitely many nonzero coefficients $r_s \in R$, with R-linear module structure given by

$$\rho \cdot \left(\sum_{s\in S} r_s b_s\right) + \rho' \cdot \left(\sum_{s\in S} r'_s b_s\right) := \sum_{s\in S} (\rho r_s + \rho' r'_s) b_s \ .$$

It is a free R-module with basis given by the symbols b_s, $s \in S$. Every map $\phi : S \to T$ induces an R-linear map $R\phi : RS \to RT$ such that $\phi(b_s) = b_{\phi(s)}$.

Remark 17. The free R-module construction defines a functor: $R- : \mathrm{Set} \to R-\mathrm{Mod}$.

Definition-Theorem 18. Let R be a ring. For all $i \geq 0$ we can associate to each topological space X a R-module $H_i(X)$, and to each continuous map $f : X \to Y$ an R-linear morphism $H_i(f) : H_i(X) \to H_i(Y)$ in such a way that the following properties are satisfied.

1. **Degree zero:** we have $H_0(X) = R\pi_0(X)$, and for all continuous maps $f : X \to Y$ there $H_0(f) = R\pi_0(f)$.
2. **Functoriality:**
   a) $H_i(\mathrm{id}_X) = \mathrm{id}_{H_i(X)}$,
   b) $H_i(f \circ g) = H_i(f) \circ H_i(g)$.
3. **Homotopy:** if $f \sim g$ then $H_i(f) = H_i(g)$.
4. **Homology of the point:** $H_i(\{*\}) = \begin{cases} 0 & \text{if } i > 0 \\ R & \text{if } i = 0 \end{cases}$.
5. **Additivity:** if the X_α are the arcwise connected components of X the inclusions $X_\alpha \hookrightarrow X$ induce an isomorphism:
$$\bigoplus_\alpha H_i(X_\alpha) \simeq H_i(X)$$
6. **Mayer-Vietoris:** If U and V are two open subsets of X such that $X = U \cup V$, we have a long exact sequence of R-modules (i.e. a sequence of R-modules in which the image of each morphism is equal to the kernel of the following morphism):
$$\dots \xrightarrow{\partial} H_i(U\cap V) \xrightarrow{(1)} H_i(U) \oplus H_i(V) \xrightarrow{(2)} H_i(X) \xrightarrow{\partial} H_{i-1}(U \cap V) \to \cdots$$
$$\cdots \xrightarrow{\partial} H_0(U\cap V) \xrightarrow{(1)} H_0(U) \oplus H_0(V) \xrightarrow{(2)} H_0(X) \to 0 \ .$$

where the coordinates of the maps (1) are equal to $H_i(U \cap V \hookrightarrow U)$ and $H_i(U \cap V \hookrightarrow V)$, and where the components of the maps (2) are equal to $-H_i(U \hookrightarrow X)$ and $H_i(V \hookrightarrow X)$.

Bonus: if $X' = U' \cup V'$ is another decomposition of a space into two open subsets, and if $f : X \to X'$ is a continuous map such that $f(U) \subset U'$ and $f(V) \subset V'$, then the following squares commute:

$$\begin{tikzcd} H_i(X) \arrow[r, "\partial"] \arrow[d, "H_i(f)"] & H_{i-1}(U \cap V) \arrow[d, "H_{i-1}(f)"] \\ H_i(X) \arrow[r, "\partial"] & H_{i-1}(U \cap V) \end{tikzcd} .$$

Proof. The proof will be given later, see section 2. □

Let us comment on the properties listed in definition-theorem 18.

1. **Dependence on R.** The definition of singular homology depends on a ring R. Most often, we take R a field of $R = \mathbb{Z}$. We shall write $H_i(X; R)$ and $H_i(f; R)$ if we want to emphasize the dependence on R, but usually R is clear from the context and we do not use these heavier notations.

2. **Homotopy functors.** The module H_0 contains essentially the same information as π_0. All the H_i are homotopy functors like π_0. In particular, if f is a homotopy equivalence, then all the R-linear maps $H_i(f)$ are R-module isomorphisms.

3. **Contactible spaces.** All contractible spaces (such as D^n, $\mathbb{R}^n$, $[0, 1]$, etc.) have the same homology as the point. They are the space with the simplest possible homology.

4. **Arcwise connected spaces.** Because of the additivity property, we can always restrict ourselves to computing the homology of arcwise connected spaces.

5. **Cut and paste.** The Mayer-Vietoris property is the analogue for singular homology of the Van Kampen theorem for the fundamental group. It allows to compute the homology of a space inductively by cutting it into simpler (ideally contractible) pieces.

   Note that, in constrast to the Van Kampen theorem, we do not assume that $\mathcal{U} \cap V$ is arcwise connected. In particular, in the next section we will compute $H_i(S^1)$ with the Mayer-Vietoris property, whereas it is impossible to compute $\pi_1(S^1, *)$ using the Van Kampen theorem.

1.5 Singular homology of spheres and a few applications

Theorem 19. *Recall that S^n is the unit sphere in the euclidean space* R^{n+1}. *We have:*

$$H_i(S^0) = \begin{cases} R^2 & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}, \qquad \text{and for } n > 0,\ H_i(S^n) = \begin{cases} R & \text{if } i = 0, n \\ 0 & \text{if } i \neq 0, n \end{cases}.$$

Corollary 20 (Invariance of dimension)**.** *If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ then $n = m$.*

Corollary 21 (Brouwer fixed point)**.** *Every continuous $f : D^n \to D^n$ has a fixed point.*

Degree theory.

Definition 22. Let $n > 0$. Since $H_n(S^n; \mathbb{Z}) = \mathbb{Z}$, for all continuous maps $f : S^n \to S^n$ the map $H_n(f; \mathbb{Z})$ is given by multiplication by an integer which is called the *degree of f* and denoted by $\deg(f)$.

Proposition 23. *Let r be the restriction of a linear reflexion to S^n, $n > 1$. Then* $\deg(r) = -1$.

Corollary 24. *Let $\iota : S^n \to S^n$ be the antipodal map (i.e. $\iota(x) = -x$). Then* $\deg(f) = (-1)^{n+1}$

Corollary 25 (Hairy ball)**.** *The sphere S^n has a nonsingular tangent vector field if and only if n is odd.*

2 Complexes and singular homology

Complexes of R-modules are "algebraic models" of topological spaces. In this section, we first develop the basic definitions and properties of complexes, and then we define the homology of complexes. To finish, we will explicitly contruct for each topological space X a complex $C^{\mathrm{sing}}(X)$ which is a algebraic model of X. The singular homology of X is defined as $H_i(X) := H_i(C^{\mathrm{sing}}(X))$.

2.1 The world of complexes

Definition 26 (Complexes and their morphisms)**.** Let R be a ring.

A *complex (of R-modules) C* is a the data of a family of R-modules $\{C_i\}_{i\in\mathbb{Z}}$ together with maps R-linear maps $d_i : C_i \to C_{i-1}$, such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

The elements of C_i are called *homogeneous of degree i*. The maps d_i are called the *differentials*, or the *boundary maps* of C. Usually the index is dropped and all the d_i are simply denoted by the same letter 'd', so that the equality above becomes $d \circ d = 0$.

A *morphism of complexes $f : C \to D$* is the data of a family of R-linear morphisms $f_i : C_i \to D_i$, $i \in \mathbb{Z}$ which preserves the differentials: $d \circ f_i = f_{i-1} \circ d$ for all i.

Morphisms of complexes can be composed : $f \circ g$ is given by the family of R-linear maps $f_i \circ g_i$, $i \in \mathbb{Z}$. We denote by $\mathrm{Ch}(R)$ the category of complexes and morphisms of complexes. A morphism $f : C \to D$ is an isomorphism if there is a morphism $g : D \to C$ such that $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$.

Exercise 27. Show that f is an isomorphism if and only if all its components f_i are R-linear isomorphisms.

Definition 28. Two morphisms $f, g : C \to D$ are *homotopic* is there is a family of maps $h_i : C_i \to D_{i+1}$ such that $f_i - g_i = d \circ s_i + s_{i-1} \circ d$ for all i. The family $h = (h_i)_{i\in\mathbb{Z}}$ is called a *homotopy* between f and g. If f is homotopic to g, we write $f \sim g$, or $f \sim_h g$ if we want to specify the homotopy.

Exercise 29.

1. The binary relation $\sim$ is an equivalence relation on the set $\mathrm{Hom}_{\mathrm{Ch}(R)}(C, D)$ of morphisms of chain complexes from C to D.

2. If $f_1, g_1 : C \to D$ are homotopic and $f_2, g_2 : D \to E$ are homotopic then $f_2 \circ f_1, g_2 \circ g_1 : C \to E$ are homotopic.

Definition 30. A morphism $f : C \to D$ is a *homotopy equivalence* if there is a map $g : D \to C$ such that $g \circ f \sim \mathrm{id}_C$ and $f \circ g \sim \mathrm{id}_D$. Then g is called *a homotopy inverse* of f.

A complex C is *homotopy equivalent* to D (or *has the same homotopy type as* D) if there is a homotopy equivalence $f : C \to D$. If C is *homotopy equivalent* to D, we write $C \sim D$.

Exercise 31. The binary relation $\sim$ defines an equivalence relation on the collection of complexes.

Examples 32.
1. Isomorphisms of complexes are homotopy equivalences.
2. Let $C = R \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \underbrace{R \oplus R}_{\text{deg } 0}$, $\quad D = \underbrace{R}_{\text{deg } 0}$. The morphism $\pi : C \to D$ such that $\pi_0 = [0\,1]$ is a homotopy equivalence (but not an isomorphism!).

2.2 Homology of complexes

Definition 33 (Homology)**.** Given a complex of R-modules C, we define two submodules of C_i:

$$Z_i(C) = \mathrm{Ker}\left(C_i \xrightarrow{d} C_{i-1}\right)$$
$$B_i(C) = \mathrm{Im}\left(C_{i+1} \xrightarrow{d} C_i\right)$$

The elements of $Z_i(C)$ are called the *cycles of degree* i, the elements of $B_i(C)$ are called the *boundaries of degree* i. We have $B_i(C) \subset Z_i(C)$ and the quotient

$$H_i(C) := Z_i(C)/B_i(C)$$

is called *the homology of degree* i of the complex. Every morphism $f : C \to D$ restricts to R-linear maps $f_i : Z_i(C) \to Z_i(D)$ and $f_i : B_i(C) \to B_i(D)$, hence induces a morphism on the level of homology

$$H_i(f) : H_i(C) \to H_i(D) .$$

Explicitly, if z is a cycle of C_i we have $H_i(f) : [z] \mapsto [f_i(z)]$ where the brackets denote the class of a cycle in homology. A morphism of chain complexes is a *quasi-isomorphism* if it induces an isomorphism in homology.

Example 34. Let $r \in R$ and let $C = R \xrightarrow{r\cdot} \underbrace{R}_{\text{deg } 0}$ and $D = \underbrace{R/rR}_{\text{deg } 0}$. Then the morphism $q : C \to D$ such that $q_0 : R \to R/rR$ is the quotient map is a quasi-isomorphism.

Exercise 35. Show that each H_i defines a functor from $\mathrm{Ch}(R)$ to $R{-}\mathrm{Mod}$.

Proposition 36. *If* $f \sim g$ *then* $H_i(f) = H_i(g)$ *for all* $i \in \mathbb{Z}$.

Corollary 37. *If f is a homotopy equivalence, then $H_i(f)$ is an isomorphism for all $i \in \mathbb{Z}$. In other words, homotopy equivalences are quasi-isomorphisms.*

Definition 38. A *short exact sequence of complexes* is a sequence of complexes $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ such that for all i,

$$0 \to C_i \xrightarrow{f_i} D_i \xrightarrow{g_i} E_i \to 0$$

is a short exact sequence of R-modules (i.e. f_i injective, $\operatorname{Ker} g_i = \operatorname{Im} f_i$ and g_i surjective).

Theorem 39. *Every short exact sequence of complexes $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ yields a long exact sequence of R-modules in homology:*

$$\begin{gathered} \dots \xrightarrow{\partial} H_i(C) \xrightarrow{H_i(f)} H_i(D) \xrightarrow{H_i(g)} H_i(E) \xrightarrow{\partial} H_{i-1}(C) \to \cdots \\ \cdots \xrightarrow{\partial} H_0(C) \xrightarrow{H_0(f)} H_0(D) \xrightarrow{H_0(g)} H_0(D) \xrightarrow{\partial} H_{-1}(C) \to \cdots . \end{gathered}$$

Bonus: *if $0 \to C' \xrightarrow{f'} D' \xrightarrow{g'} E' \to 0$ is another short exact sequence of complexes and if we have a commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "f"] \arrow[d, "\alpha_C"] & D \arrow[r, "g"] \arrow[d, "\alpha_D"] & E \arrow[r] \arrow[d, "\alpha_E"] & 0 \\ 0 \arrow[r] & C' \arrow[r, "f'"] & D' \arrow[r, "g'"] & E' \arrow[r] & 0 \end{tikzcd}$$

then the following squares commute:

$$\begin{tikzcd} H_i(E) \arrow[r, "\partial"] \arrow[d, "H_i(\alpha_E)"] & H_{i-1}(C) \arrow[d, "H_{i-1}(\alpha_C)"] \\ H_i(E') \arrow[r, "\partial'"] & H_{i-1}(C') \end{tikzcd} .$$

2.3 Construction of singular homology

We are going to construct a complex $C^{\mathrm{sing}}(X)$ of R-modules for each topological space X, which contains some geometrical information relative to X.

Definition 40. Let $(e_0, \dots, e_n)$ denote the canonical basis of $\mathbb{R}^{n+1}$.

- The *standard n-simplex* Δ^n is the convex hull of the vectors of the canonical basis of $\mathbb{R}^{n+1}$:

$$\Delta^n := < e_0, \dots, e_n > .$$

- The simplex $< e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n >$ is denoted by $\partial_i \Delta^n$. We call it the *i-th face* of Δ^n.

- If $0 \le i \le n$, there is a unique affine map $\epsilon^i : \Delta^{n-1} \to \Delta^n$ such that

$$\epsilon^i(e_k) = \begin{cases} e_k & \text{si } k \le i-1 \\ e_{k+1} & \text{si } k \ge i+1 \end{cases}$$

  This map d^i induces an affine isomorphism of Δ^{n-1} onto $\partial_i \Delta^n$.

Lemma 41. *If $i < j$ the composite maps $\Delta^{n-1} \xrightarrow{\epsilon^i} \Delta^n \xrightarrow{\epsilon^j} \Delta^{n+1}$ and $\Delta^{n-1} \xrightarrow{\epsilon^{j-1}} \Delta^n \xrightarrow{\epsilon^i} \Delta^{n+1}$ are equal.*

Definition-Theorem 42. Let X be a topological space and let R be a ring. The singular complex $C^{\text{sing}}(X)$ is defined by:

(0) For $n < 0$, $C_n^{\text{sing}}(X) = 0$.

(1) For $n \ge 0$, $C_n^{\text{sing}}(X)$ is the free R-module with basis the continuous maps $\sigma : \Delta^n \to X$. These maps σ are called the *singular simplices* of X.

(2) For $n \ge 1$, the boundary map $d_n : C_n^{\text{sing}}(X) \to C_{n-1}^{\text{sing}}(X)$ sends each $\sigma : \Delta^n \to X$ to
$$d_n(\sigma) := \sum_{i=0}^{n} (-1)^i (\sigma \circ \epsilon^i) \ .$$

For all continuous map $f : X \to Y$, we let $C^{\text{sing}}(f) : C^{\text{sing}}(X) \to C^{\text{sing}}(Y)$ be the morphism of complexes which sends each singular simplex $\sigma \in C_n^{\text{sing}}(X)$ to $f \circ \sigma \in C_n^{\text{sing}}(Y)$. Thus, the singular complex defines a functor:

$$C^{\text{sing}} : \text{Top} \to \text{Ch}(R) \ .$$

Examples 43. 1. $C^{\text{sing}}(\emptyset) = 0$.

2. $C^{\text{sing}}(\{*\})$ is a free R-module of rank 1 in each degree, generated by the unique map $\Delta^n \to \{pt\}$. The boundary map d_n equals id_R if n is even and 0 if n is odd.

3. In general $C^{\text{sing}}(X) = \bigoplus_\alpha C^{\text{sing}}(X_\alpha)$ where the X_α are the arcwise connected components of X.

Definition 44. We define the *singular homology* of a topological space X by $H_i(X) = H_i(C^{\text{sing}}(X))$ and for all continuous map $f : X \to Y$ we let $H_i(f) = H_i(C^{\text{sing}}(f))$. This yields functors:

$$H_i : \text{Top} \to R\text{-Mod} \ .$$

Proposition 45. *We have* $H_0(X) = R\pi_0(X)$*.*

Proposition 46. *If* $f \sim_H g$ *then* $C^{\mathrm{sing}}(f) \sim C^{\mathrm{sing}}(g)$*.*

Corollary 47. *If* $f \sim_H g$ *then* $H_i(f) = H_i(g)$ *for all* $i \in \mathbb{Z}$*.*

Theorem 48 ($\mathcal{U}$-small chains)**.** *Let* $\mathcal{U}$ *denote an open covering of* X*, and let* $C_n^{\mathrm{sing},\mathcal{U}}(X)$ *denote the submodule of* $C_n^{\mathrm{sing}}(X)$ *generated by the singular simplices whose image is contained in* V*, for some* $V \in \mathcal{U}$*. Then:*

1. *the* R*-modules* $C_n^{\mathrm{sing},\mathcal{U}}(X)$ *yield a subcomplex* $C^{\mathrm{sing},\mathcal{U}}(X)$ *of* $C^{\mathrm{sing}}(X)$*,*
2. *the inclusion* $C^{\mathrm{sing},\mathcal{U}}(X) \hookrightarrow C^{\mathrm{sing}}(X)$ *is a quasi-isomorphism (in fact, it is even a homotopy equivalence).*

Corollary 49. *The Mayer-Vietoris long exact sequence of theorem 18 holds.*

3 Homology of pairs

Definition 50. Let X be a topological space and let $A \subset X$ be a subspace. The *singular homology of the pair of* (X, A) is the homology of the quotient complex $C^{\mathrm{sing}}(X)/C^{\mathrm{sing}}(A)$. We denote by $H_i(X, A)$ the i-th homology module of the pair (X, A).

If (Y, B) is another pair, every continuous map $f : X \to Y$ such that $f(A) \subset B$ induces a morphism in homology:

$$H_i(f) : H_i(X, A) \to H_i(Y, B) .$$

Examples 51 (two extreme examples.)**.** 1. If $A = \emptyset$, then $C^{\mathrm{sing}}(A) = 0$ hence we have $H_i(X, \emptyset) = H_i(X)$ in all degrees i.

2. If $A = X$ then $C^{\mathrm{sing}}(X) = C^{\mathrm{sing}}(A)$ hence $H_i(X, X) = 0$ for all i.

The homology of (X, A) is somehow like the homology of X modulo A. We will give a clearer topological interpretation in theorem 59. Before this, we give two major tools to compute the homology of a pair.

Theorem 52 (Long exact sequence of a pair)**.** *Every pair* (X, A) *gives rise to a long exact sequence in homology:*

$$\begin{aligned} \dots &\xrightarrow{\partial} H_i(A) \xrightarrow{H_i(\mathrm{incl})} H_i(X) \xrightarrow{H_i(\mathrm{incl})} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \to \cdots \\ \cdots &\xrightarrow{\partial} H_0(A) \xrightarrow{H_0(\mathrm{incl})} H_0(X) \xrightarrow{H_0(\mathrm{incl})} H_0(X, A) \to 0 . \end{aligned}$$

Bonus: *if* (Y, B) *is another pair, and if* $f : X \to Y$ *is a continuous map such that* $f(A) \subset B$*, then the following squares commute:*

$$\begin{tikzcd} H_i(X, A) \arrow[r, "\partial"] \arrow[d, "H_i(f)"] & H_{i-1}(A) \arrow[d, "H_{i-1}(f)"] \\ H_i(Y, B) \arrow[r, "\partial"] & H_{i-1}(B) \end{tikzcd} .$$

Examples 53. 1. Let $x \in X$. For all $i > 0$ the inclusion $(X, \emptyset) \hookrightarrow (X, \{x\})$ induces an isomorphism $H_i(X) \xrightarrow{\simeq} H_i(X, \{x\})$, and in degree zero there is a short exact sequence:

$$0 \to H_0(\{x\}) \to H_0(X) \to H_0(X, \{x\}) \to 0 .$$

2. For $n > 0$ we have $H_i(D^n, D^n \setminus \{0\}) = \begin{cases} R & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$

The next proposition is useful to produce further examples. The proof relies on the five lemma.

Proposition 54. *Let (X, A) and (Y, B) be two pairs, and let $f : X \to Y$ be a continuous map such that $f(A) \subset B$. Assume that $f : X \to Y$ and $f : A \to B$ both induce an isomorphism in singular homology. Then*

$$H_i(f) : H_i(X, A) \to H_i(Y, B)$$

is an isomorphism for all i.

Example 55. If $A \subset B \subset X$ and if A is a deformation retract of B, then $H_i(X, A) \simeq H_i(X, B)$ for all i.

Theorem 56 (Excision)**.** *Let $B \subset A \subset X$ be topological spaces. Assume that $\overline{B} \subset \mathrm{int}(A)$. Then the inclusion $X \setminus B \hookrightarrow X$ induces an isomorphism:*

$$H_i(X \setminus B, A \setminus B) \xrightarrow{\simeq} H_i(X, A) \; .$$

Example 57. If U is an open subset of $\mathbb{R}^n$, and $x \in U$ then

$$H_i(U, U \setminus \{x\}) = \begin{cases} R & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

If X is a topological space and $x \in X$ then $H_i(X, X \setminus \{x\})$ is usually called the *local homology of X at x*. The previous example shows that if U is an open subset of R^n, then the local homology of the topological space U remembers that U can be embedded as an open subset of $\mathbb{R}^n$. The next result generalizes the invariance of dimension.

Corollary 58 (Invariance of domain)**.** *Let U be an open subset of $\mathbb{R}^n$ and let V be an open subset of $\mathbb{R}^m$. If there is a homeomorphism between U and V, then $n = m$.*

We finish by a topological interpretation of the homology of a pair. Note that by example 53.1, if $* = A/A$ denotes a point in the quotient X/A, then the homology $H_i(X/A, *)$ is isomorphic to $H_i(X/A)$ if $i \neq 0$.

Theorem 59. *Let X be a topological space, let A be a nonempty closed subset of X and assume that there is a neighborhood V of A such that A is a deformation retract of V. Then for all i, the quotient map $q : X \to X/A$ induces an isomorphism:*

$$H_i(X, A) \xrightarrow[H_i(q)]{\simeq} H_i(X/A, A/A) \; .$$

4 Homology of combinatorial topological spaces

4.1 Geometric simplicial complexes

Definition 60. A set of $n+1$ points $S = \{x_0, \dots, x_n\}$ of $\mathbb{R}^N$ is *affinely independent* if the affine space generated by S has dimension n.

An *affine n-simplex* is the convex hull of a set of $n+1$ affinely independent points. These points $x_0, \dots, x_n$ are called the *vertices* of the affine simplex, and the affine simplex is denoted by $< x_0, \dots, x_n >$.

If $\{x_{i_0}, \dots, x_{i_k}\}$ is a subset of $\{x_0, \dots, x_n\}$, then the affine k-simplex $< x_{i_0}, \dots, x_{i_k} >$ is called a *k-face* of $< x_0, \dots, x_n >$. Thus the vertices are the 0-faces. The $(n-1)$-faces are often denoted by $< x_0, \dots, \widehat{x_k}, \dots, x_n >$ where $\widehat{x_k}$ means that x_k is deleted.

Example 61. The standard n-simplex Δ^n is an affine n-simplex of $\mathbb{R}^{n+1}$. In fact, Δ is an affine n-simplex of $\mathbb{R}^N$ if and only if there is an affine map $\mathbb{R}^{n+1} \to \mathbb{R}^N$ which restricts to a bijection $\Delta^n \xrightarrow{\simeq} \Delta$.

Definition 62. A *geometric simplicial complex* (of finite dimension) is a nonempty set K of affine simplices of $\mathbb{R}^N$ satisfying the following axioms.

(i) If $\Delta \in K$ then all the faces of Δ are also in K.

(ii) If Δ and Δ' are in K, then either $\Delta \cap \Delta' = \emptyset$, or $\Delta \cap \Delta'$ is a face of both Δ and Δ'.

(iii) Every ball in $\mathbb{R}^n$ meets a finite number of simplicies of K.

The *dimension* of K is the maximal n such that there is an n-simplex in K. The 0-simplices of K are called the *vertices* of K. We denote by $|K|$ the *realization* of K, or the *polyedron* associated to K. It is the topological subspace of $\mathbb{R}^N$:

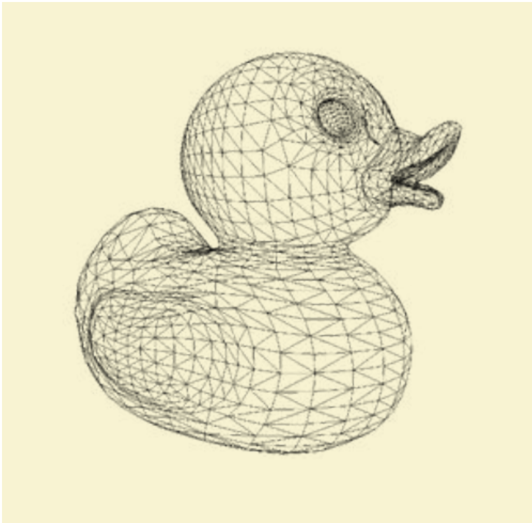
$$|K| = \bigcup_{\Delta \in K} \Delta\ .$$

Remark 63. Because of the local finiteness axiom (iii), $|K|$ is compact if and only if K has a finite number of simplices. The last axiom also guaranties that every geometric simplicial complex has a countable number of simplicies.

Remark 64. A *triangulation of a topological space X* is the data of a geometric simplicial complex K and a homeomorphism

$$h : |K| \xrightarrow{\approx} X\ .$$

Polyedra of $\mathbb{R}^N$ admit a triangulation (by definition), but there are many other examples, such as differentiable manifolds or semi-algebraic subsets of $\mathbb{R}^n$. Thus a wide variety of geometric objects can be described as geometric simplicial complexes. For example, here is a beautiful image of a triangulated duck (image taken from a course on simplicial homology on M. Mandell's page):



For further beautiful explanations relative to triangulations, I recommend to visit the website "Analysis Situs".

From a geometric simplicial complex K, we can cook up a complex $C^{\mathrm{simpl}}(K)$ called the *simplicial complex of* K, as follows.

1. Choose a total order on the vertices of K. Thus every simplex of K is of the form $< x_0, \dots, x_n >$ where $x_0 < x_1 < \cdots < x_n$.

2. Define $C_n^{\mathrm{simpl}}(K)$ as the free R-module with basis the n-simplices of K.

3. Define the boundary maps $d : C_n^{\mathrm{simpl}}(K) \to C_{n-1}^{\mathrm{simpl}}(K)$ by the formula:

$$d < x_0, \cdots, x_n > = \sum_{k=0}^{n} (-1)^k < x_0, \cdots, \widehat{x_k}, \cdots, x_n > \ .$$

Theorem 65. *The homology of the complex $C^{\mathrm{simpl}}(K)$ is isomorphic to the singular homology of $|K|$.*

Proof. This will be a special case of theorem 73 for CW-complexes proved later. A direct proof for finite geometric simplicial complexes may be found in the exercises (exercise 30) $\square$

4.2 CW-complexes

The geometric realization $X = |K|$ of a geometric simplicial complex can be constructed step by step: consider first the 0-simplicies (this is called the 0-skeleton of X, and usually denoted by X_0) , then paste the 1-simplicies on the 0 skeleton (this produces the 1-skeleton $X_1 \subset X$), then paste the 2-simplicies onto the 1-skeleton to obtain the 2-skeleton X_2 and so on.

The notion of a CW-complex generalize this step-by-step construction of spaces. To formulate the definition of a CW-complex, we must first formalize the operation of pasting a space onto another one.

Definition 66 (Pushout squares)**.** Let $\mathcal{C}$ be a category. A commutative square in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is called a *pushout square* if it satisfies the following universal property. For all morphisms ϕ_B and ϕ_C making the outer square (ABZC) commute, there is a unique map ϕ making the two triangle commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \arrow[ddr, bend left, "\phi_B"] & \\ C \arrow[r] \arrow[drr, bend right, "\phi_C"'] & D \arrow[dr, dashed, "\exists ! \phi"] & \\ & & Z \end{tikzcd}$$

Interpretation 67 (in the category Top)**.** In Top, every diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

can be completed into a pushout square. Indeed, if $A = \emptyset$, we may take $D = B \sqcup C$. If $A \neq \emptyset$ then we let $\sim$ be the smallest equivalence relation such that for all $a \in A$, $f(a) \in B$ is identified with $g(a) \in C$. Then we may take $D = B \sqcup D/ \sim$. Thus, in a pushout square, the space D is the glueing of the spaces B and C 'along A' (in a way prescribed by the maps $f : A \to B$ and $g : A \to C$).

Definition 68. Let X be a topological space. A CW-complex structure on X is the following data.

1. An increasing sequence of subspaces $X_0 \subset \cdots \subset X_k \subset \ldots$ such that (i) X_0 is a discrete space, (ii) $X = \bigcup_{k\geq 0} X_k$, and (iii) a subset U is open in $X \Leftrightarrow$ for all k $U \cap X_k$ is an open subset of X_k.

2. For k positive, a family of maps $f_\alpha : D^k_\alpha \to X_k$, $\alpha \in A$ which restrict to maps $f_{\partial\alpha} : S^{k-1}_\alpha \to X_{k-1}$, (where D^k_α denote a copy of the k-disk and S^{k-1}_α is its boundary) such that the following square is a pushout square.

$$\begin{tikzcd} \bigsqcup_{\alpha\in A} S^{k-1}_\alpha \arrow[r, "\sqcup f_{\partial\alpha}"] \arrow[d, hook] & X_{k-1} \arrow[d, hook] \\ \bigsqcup_{\alpha\in A} D^k_\alpha \arrow[r, "\sqcup f_\alpha"] & X_k \end{tikzcd} .$$

Terminology.

- The subspace X_k is called the k-skeleton of X. If $X = X_k$ then we say that X has dimension k.

- We let $e^k_\alpha = f_\alpha(D^k_\alpha \setminus S^{k-1}_\alpha)$. These subspaces of X are called the cells (of dimension k). The cells form a partition of X.

- The map $f_\alpha : D^k_\alpha \to X_k \subset X$ is the *characteristic map* of the cell e^k_α, and its restriction $f_{\partial\alpha} : S^{k-1}_\alpha \to X_{k-1}$ is the *attaching map* of e^k_α.

Remark 69. A CW-complex X is constructed inductively by attaching **C**ells. Condition 1(iii) says that X has the **W**eak topology with respect to the X_k. These two facts explain the letters CW in "CW-complexes".

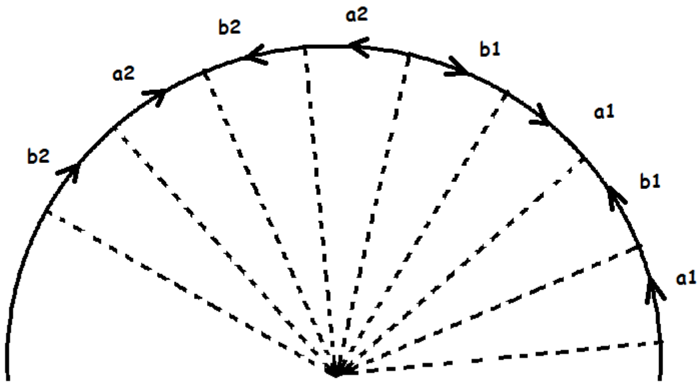
Examples 70. 1. Polyedra have a natural structure of a CW-complex (Yes, polyedra are obtained by glueing affine simplicies, but an affine simplex of dimension n is homeomorphic to D^n.)

2. The n-sphere S^n can be considered as an n-dimensional CW-complex with one 0-cell (the south pole $*$), and an n-cell attached on it via the constant map $S^{n-1} \to \{*\}$. In this cell structure, $X_0 = X_1 = \cdots = X_{n-1} = \{*\}$ and $X_n = S^n$.

3. The n-sphere S^n can be considered as an n-dimensional CW-complex, with i-skeleton $S^i := S^n \cap \{(x_0, \ldots, x_i, 0, \ldots, 0) \subset \mathbb{R}^{n+1}$, and each S^{i+1} is obtained by pasting two copies of D^{i+1} via the identity maps $S^i \overset{=}{\rightrightarrows} S^i$.

4. A *topological graph* is a CW-complex of dimension 1. The *vertices* are the 0-cells, the *edges* are the 1-cells.

5. Let $g \in \mathbb{N}^*$. We denote by S_g the quotient of D^2 (the unit disc of $\mathbb{C}$ by the identification depicted by the following drawing.



The quotient space S_g is the compact orientable surface of genus g:

Then S_g has the structure of a CW-complex of dimension 2, with one 0-cell (the image of the extremities of the segments a_1, b_1, etc, which are all identified in the quotient), with 1-skeleton the image of the boundary of D_2 in S_g (this 1-skeleton is a bouquet of $2g$ circles $a_1,\dots,a_g$, $b_1,\dots,b_g$).

Proposition 71 (Topology of CW complexes)**.** *Let X be a CW-complex.*

(i) *$A \subset X$ is open (resp. closed) $\Leftrightarrow$ for all α, $f_\alpha^{-1}(A)$ is open (resp. closed) in D^n_α.*

(ii) *X is Hausdorff.*

(iii) *$A \subset X$ is compact $\Leftrightarrow$ A is closed and it is contained in a finite union of cells.*

Remark 72. The last fact of the proposition has many consequences:
- A CW complex is compact iff it has a finite number of cells,
- the closure of every cell e^n_α meets only a finite number of cells,
- If a subset A of X is compact, then it is contained in X_n for some n.

From a CW-complex X we can cook up a complex $C^{\mathrm{cell}}(X)$ called the *cellular complex of* X, as follows.

1. For each n, $C^{\mathrm{cell}}(X)$ is the free R-module with basis the n-cells of X.
2. Define the boundary map $d : C_1^{\mathrm{cell}}(X) \to C_0^{\mathrm{simpl}}(X)$ by sending each 1-cell e^1_α with attaching map $f_\alpha : D^1_\alpha = [-1,1] \to X_1$ to
$$d_1(e^1_\alpha) = f_\alpha(1) - f_\alpha(-1)\ .$$
3. For $n \geq 1$, define the boundary map $d : C_{n+1}^{\mathrm{cell}}(X) \to C_n^{\mathrm{simpl}}(X)$ by sending each $n+1$-cell e^{n+1}_α with attaching map $f_\alpha : D^{n+1}_\alpha \to X_{n+1}$ to
$$d(e^{n+1}_\alpha) = \sum_\beta d_{\alpha,\beta} e^n_\beta$$
where $d_{\alpha\beta} \in \mathbb{Z}$ is the degree of the composition
$$S^n_\alpha \xrightarrow{f_{\partial\alpha}} X_n \twoheadrightarrow X_n/(X_n \setminus e^n_\beta) \xleftarrow[\simeq]{f_\beta} D^n_\beta/S^{n-1}_\beta = S^n\ .$$

Theorem 73. *The homology of the complex $C^{\mathrm{cell}}(X)$ is isomorphic to the singular homology of X.*

5 A word on the ground ring R of homology

The homology of the spheres does not really depend on the ring R. But this phenomenon is in fact quite exceptional.

Example 74. The real projective plane $\mathbb{R}P^2 = \mathbb{R}^3/\mathbb{R}^*$ has a CW structure with one 0-cell, with one 1-cell and with one 2-cell. To be more specific,

$$(\mathbb{R}P^2)_1 = (\mathbb{R}^2 \times \{0\})/\mathbb{R}^* = \mathbb{R}P^1 \simeq S^1$$

and the attaching map of 2-cell has degree 2. Thus the associated cellular complex of $\mathbb{R}P^2$ has the form:

$$Re^2 \xrightarrow{2} Re^1 \xrightarrow{0} Re^0 \ .$$

In particular, $H_i(\mathbb{R}P^2)$ is zero if $i \geq 3$ and we have:

$$H_0(\mathbb{R}P^2) = R \ , \quad H_1(\mathbb{R}P^2) = R/2R \ , \quad H_2(\mathbb{R}P^2) = \mathrm{Ker}(R \xrightarrow{2} R) \ .$$

Thus we see that the homology of $\mathbb{R}P^2$ strongly depends on the ring R:

- if R is a field of characteristic $\neq 2$, then $\mathbb{R}P^2$ has the same homology as a point (zero everywhere except in degree zero),
- if R is a field of characteristic 2, then $\mathbb{R}P^2$ has homology equal to R in degrees 0, 1, and 2, and it is zero elsewhere,
- if $R = \mathbb{Z}$, then $\mathbb{R}P^2$ has nonzero homology in degrees 0 and 1 only.

Natural question 75. given a topological space X and a ring R, if we know $H_i(X, R)$, is it possible to predict what $H_i(X, S)$ will be for other rings S ?

We shall give a partial answer to this question here. If F is a functor such that $F(0) = 0$ then for all complexes C of R-modules, we can apply F to obtain a complex[1] $F(C)$ of S-modules

$$F(C) := \quad \dots \xrightarrow{F(d)} F(C_i) \xrightarrow{F(d)} F(C_{i-1}) \xrightarrow{F(d)} F(C_{i-2}) \xrightarrow{F(d)} \dots \ .$$

Definition 76. A functor $F : R\mathrm{-Mod} \to S\mathrm{-Mod}$ is called *exact* if for all short exact sequence of R-modules $0 \to M \underset{f}{\to} N \underset{g}{\to} P \to 0$, the diagram:

$$0 \to F(M) \xrightarrow[F(f)]{} F(N) \xrightarrow[F(g)]{} F(P) \to 0$$

is also an exact sequence.

[1] The hypothesis $F(0) = 0$ guaranties that $F(d) \circ F(d) = F(d \circ d) = F(0) = 0$.

Proposition 77. *If F is exact, then it preserves surjective morphisms, injective morphisms, kernels, cokernels, images, and $F(0) = 0$.*

Proposition 78. *If $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ is exact, then it preserves homology. To be more specific, for all i we have*

$$H_i(F(C)) \simeq F(H_i(C)) \; .$$

The previous proposition applies in particular if $F = -\otimes_R S : R-\mathrm{Mod} \to S-\mathrm{Mod}$ is the *extension of scalars* along a morphism of rings $R \to S$, such that S is flat as an R-module. We will recall flatness in details later, but we mention two elementary cases in the next corollary.

Corollary 79. *Assume that R is a field and that S is an R-algebra, or that $R = \mathbb{Z}$ and that $S = \mathbb{Q}$ or a field of characteristic zero. Then there are isomorphisms for all i:*

$$H_i(C \otimes_R S) \simeq H_i(C) \otimes_R S \; .$$

In particular, if X is a topological space, there are isomorphisms

$$H_i(X, S) \simeq H_i(X, R) \otimes_R S \; .$$

6 An overview of cohomology

6.1 Cochain complexes

The complexes that we have used until now are also called *chain complexes*. We are now going to introduce an analogous notion, in which the differentials (=boundary morphisms) raise the degree by one instead of lowering it by one.

Definition 80. A *cochain complex* C is a family of R-modules $(C^i)_{i\in\mathbb{Z}}$ equipped with morphisms:

$$d : C^i \to C^{i+1}$$

satisfying $d \circ d = 0$. These morphisms are called the *differentials* of C, and C^i is called the R-module of *homogeneous elements of cohomological degree* i of C. A *morphism of cochain complexes* $f : C \to D$ is a family of R-linear morphisms $f^i : C^i \to D^i$, $i \in \mathbb{Z}$ which preserve the differentials, that is: $d \circ f^i = f^{i+1} \circ i$ for all i. Morphisms can be composed: $f \circ g$ is the family $(f_i \circ g_i)_{i\in\mathbb{Z}}$. We denote by $\mathrm{Ch}^\bullet(R)$ the category of cochain complexes.

Example 81. If M is a manifold, the De Rham complex of M (see the differential geometry course) is a cochain complex of $\mathbb{R}$-vector spaces.

> **Important Observation**
>
> Cochain complexes are nothing but chain complexes with a different notation. To be more specific, one passes from cochain complexes to chain complexes (and from morphisms of chain complexes to morphisms of cochain complexes) by the formulas:
>
> $$C_i = C^{-i} \text{ and } f_i = f^{-i} .$$

In particular, all the concepts and all the results relative to chain complexes can be converted into concepts and results relative to cochain complexes, simply by changing the notations.

1. The *cohomology of degree* i a cochain complex is the subquotient of C^i defined by:
$$H^i(C) = Z^i/B^i$$
where $Z^i = \mathrm{Ker}\big(d : C^i \to C^{i+1}\big)$ and $B^i = \mathrm{Im}\big(d : C^{i-1} \to C^i\big)$. Cohomology defines functors:
$$H^i : \mathrm{Ch}^\bullet(R) \to R\text{–Mod} .$$

A morphism $f : C \to D$ is called a *quasi-isomorphism* if $H^i(f)$ is an isomorphism for all i.

Theorem 82. *If $0 \to C \to D \to E \to 0$ is a short exact sequence of cochain complexes, there is a long exact sequence in cohomology:*

$$\dots \xrightarrow{\partial} H^i(C) \to H^i(D) \to H^i(E) \xrightarrow{\partial} H^{i+1}(C) \to \dots .$$

Bonus: *every morphism of short exact sequence of complexes induces a morphism of cohomology long exact sequences.*

2. Two morphisms of cochain complexes $f, g : C \to D$ are *homotopic* if there exists a family of maps $h^i : C^i \to D^{i-1}$ such that $f^i - g^i = d \circ h^i + h^{i+1} \circ d$. Notation : $f \sim g$. We say that $f : C \to D$ is a *homotopy equivalence* if there is a morphism $g : D \to C$ such that $f \circ g \sim \mathrm{id}_D$ and $g \circ f \sim \mathrm{id}_C$.

   Proposition 83. *Two homotopic morphisms induce the same morphism in cohomology. In particular, every homotopy equivalence is a quasi-isomorphism.*

6.2 Duality

If R is a commutative ring, every R-module M has a *dual module* $M^\vee = \mathrm{Hom}_R(M, R)$. Every R-linear map $f : M \to N$ induces an R-linear map $f^\vee : N^\vee \to M^\vee$, such that $f^\vee(u) = u \circ f$. The map $f^\vee$ is also called the *transpose map* of f. Duality does not define a functor, but rather a contravariant functor in the following sense.

Definition 84. Let $\mathcal{C}$ and $\mathcal{D}$ two categories. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the data

- for all objects X of $\mathcal{C}$, of an object $F(X)$ of $\mathcal{D}$,
- for all morphisms $f : X \to Y$, of a 'backward morphism' $F(f) : F(Y) \to F(X)$, such that this assignment is associative: $F(f \circ g) = F(g) \circ F(f)$, and preserves identities: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

For all chain complexes C of R-modules, we can use duality to construct a cochain complex of R-modules:

$$C^\vee = \cdots \xrightarrow{d^\vee} \underbrace{C_0^\vee}_{=:(C^\vee)^0} \xrightarrow{d^\vee} \underbrace{C_1^\vee}_{=:(C^\vee)^1} \xrightarrow{d^\vee} \cdots \to \xrightarrow{d^\vee} \underbrace{C_n^\vee}_{=:(C^\vee)^n} \xrightarrow{d^\vee} \underbrace{C_{n+1}^\vee}_{=:(C^\vee)^{n+1}} \xrightarrow{d^\vee} \dots$$

One may expect that the cohomology of $C^\vee$ can be computed from the homology of C by a nice formula. This is indeed the case. The next proposition gives such a formula when R is a field. We will see later a formula for more general rings R.

Proposition 85. *Assume that R is a field. Then for all i we have an isomorphism:*

$$H^i(C^\vee) \simeq H_i(C)^\vee$$

If R is not a field, the following example shows that the formula must be more complicated than that of proposition 85

Example 86. Let $R = \mathbb{Z}$. Let C denote the complex with only nonzero objects $C_0 = C_1 = \mathbb{Z}$, and with differential $d : C_1 \to C_0$ given by multiplication by a positive integer n. Then $H_0(C) = \mathbb{Z}/p$ and $H^1(C) = 0$, hence

$$H_0(C)^\vee = H_1(C)^\vee = 0 \ .$$

But $C^\vee$ is the complex $\mathbb{Z} \to \mathbb{Z}$ with differential given by multiplication by n, hence

$$H_0(C^\vee) = 0 \text{ and } H_1(C^\vee) = \mathbb{Z}/n \ .$$

6.3 Singular cohomology

Definition 87. Let R be a commutative ring and let X be a topological space. The dual $C^{\mathrm{sing}}(X)^\vee$ of the singular chain complex is a cochain complex and for all $i \geq 0$ we define the cohomology module of degree i by:

$$H^i(X) := H^i(C^{\mathrm{sing}}(X)^\vee) \ .$$

Every continuous map $f : X \to Y$ induces a 'backward map':

$$H^i(f) := H^i(C^{\mathrm{sing}}(f)^\vee) : H^i(Y) \to H^i(X) \ .$$

In particular, singular cohomology of degree i defines a contravariant functor:

$$H^i : \mathrm{Top} \to R\mathrm{-Mod} \ .$$

By applying duality $^\vee$ to $C^{\mathrm{sing}}(X)$, one can prove that cohomology satisfies properties which are dual to the properties of homology. In particular:

- Two homotopic continuous maps $f, g : X \to Y$ induce the same morphism in cohomology. Hence, homotopy equivalences yield isomorphisms in cohomology.
- If U and V are two open subsets of X such that $U \cup V = X$ there is a long exact sequence:

$$0 \to H^0(X) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \xrightarrow{\partial} H^1(X) \to \dots$$
$$\dots \to H^{i-1}(U \cap V) \xrightarrow{\partial} H^i(X) \to H^i(U) \oplus H^i(V) \to H^i(U \cap V) \xrightarrow{\partial} \dots$$

Natural question 88. Since cohomology has the 'same' formal properties as homology, and since cohomology can even be computed from homology by a nice formula, what is the point in introducing cohomology ?

The answer to this will be given in the next section.

6.4 The cup product

Definition 89. Let R be a commutative ring. A *graded R-algebra A* is a R-algebra A equipped with an R-module decomposition:

$$A = \bigoplus_{i\geq 0} A^i$$

and such that the product is compatible with this decomposition, that is, if $a \in A^i$ and $b \in A^j$ then $ab \in A^{i+j}$. An element of A is *homogeneous* if it belongs to an A^i. In this case, i is called the degree of a and denoted by $|a|$

Example 90. Tensor algebras, symmetric algebras, exterior algebras, can be equipped with the structure of graded algebras.

Definition 91. A *differential graded R-algebra* is a graded algebra A, equipped with differentials $d : A^i \to A^{i+1}$ satisfying $d \circ d = 0$, and the derivation rule:

$$d(ab) = d(a)b + (-1)^{|a|} a d(b) \;.$$

Since a differential graded algebra A is a complex of R modules, we can take its homology.

Proposition 92. *Let A be a differential graded R-algebra. Set $H^*(A) = \bigoplus_{i\geq 0} H^i(A)$. Then $H^*(A)$ is a graded algebra with product:*

$$\begin{array}{ccc} H^i(A) \times H^j(A) & \to & H^{i+j}(A) \\ ([y],[z]) & \mapsto & [yz] \end{array} .$$

Now let us go back to singular cohomology. An element of degree i of the cochain complex $C^{\text{sing}}(X)^\vee$ is an R-linear function $\phi : C_i^{\text{sing}}(X) \to R$. Since $C_i^{\text{sing}}(X)$ is the free R-module with basis the set of singular i-simplices of X,such a function is uniquely determined by its linearity and its values on the i-simplices.

Definition 93. We define the *cup product*

$$\begin{array}{ccc} C_i^{\text{sing}}(X)^\vee \times C_j^{\text{sing}}(X)^\vee & \to & C_{i+j}^{\text{sing}}(X)^\vee \\ (\phi,\psi) & \mapsto & \phi \smile \psi \end{array}$$

by the formula:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma_{|<e_0,\dots,e_i>})\psi(\sigma_{|<e_i,\dots,e_{i+j}>}) \;.$$

Theorem 94. *The cup product endows $C^{\text{sing}}(X)^\vee$ with the structure of a differential graded algebra. In particular, the cohomology $H^*(X)$ is a graded R-algebra. Moreover, every continuous map $f : X \to Y$ induces a morphism of graded R-algebras $H^*(f) : H^*(Y) \to H^*(X)$.*

6.5 Vista : some applications of singular cohomology

Our purpose here is just to give a glance at what one can do with singular cohomology. We first give three elementary ways of using cup products.

1. **Distinguish spaces with the same (co)homology.** The torus T^2 and the wedge of spheres $S^1 \vee S^1 \vee S^2$ have the same homology and cohomology modules, namely

$$H^i(T^2) = H^i(S^1 \vee S^2 \vee S^1) = \begin{cases} R & \text{if } i = 0, 2, \\ R^2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   But they have different cohomology rings, namely $H^*(T^2)$ is an exterior algebra on two generators a and b of degree 1. In particular the product $a \smile b$ is nonzero. On the contrary, if $x \smile y = 0$ for all elements x, y in $H^1(S^1 \vee S^2 \vee S^1)$.

2. **Give a better control of maps in homology.** One computes that

$$H^*(\mathbb{R}P^n, \mathbb{F}_2) = \mathbb{F}_2[x]/x^{n+1} \text{ with } \deg(x) = 1 \ .$$

   Algebraically, every morphism of $\mathbb{F}_2$-algebras $\phi : \mathbb{F}_2[x]/x^{n+1} \to A$ is completely determined by the element $\phi(x) \in A$. Thus for all continuous map $f : X \to \mathbb{R}P^n$, if we know the ring $H^*(X, \mathbb{F}_2)$ then it suffices to compute the map $H^1(f, \mathbb{F}_2)$ to completely determine the ring morphism $H^*(f, \mathbb{F}_2)$.

   Here is a slightly different way of using this ring description. If $N > n$, then every morphism of rings

$$\phi : \mathbb{F}_2[x]/x^{n+1} \to \mathbb{F}_2[x]/x^{N+1}$$

   is zero in positive degrees. We deduce that for all continuous maps $f : \mathbb{R}P^N \to \mathbb{R}P^n$ we have $H^i(f, \mathbb{F}_2) = 0$ for all positive i.

   Similar considerations can be adapted to complex projective spaces since for all rings R, $H^*(\mathbb{C}P^n, R) = R[x]/x^{n+1}$ where $\deg x = 2$.

3. **Study coverings by contractible sets.** If X can be covered by n contractible sets then for all cohomology classes $c_1, \dots, c_n$ we have

$$c_1 \smile \cdots \smile c_n = 0$$

   For example $\mathbb{R}P^n$ and $\mathbb{C}P^n$ can't be covered by less than $(n + 1)$ contractible sets.

We also have a comparison theorem with De Rham cohomology.

Theorem 95 (De Rham)**.** *Let M be a manifold. There is an isomorphism of $\mathbb{R}$-algebras, natural with respect to M (the right hand side is the De Rham cohomology of M):*

$$H^*(M,\mathbb{R}) \simeq H^*_{\mathrm{DR}}(M)\ .$$

This theorem tells us that despite its definition relying on differential geometry, de Rham cohomology only measures properties of the underlying topological spaces of differentiable manifolds. Thus we can use topological methods to determine De Rham cohomology.

Part II
Algebra and homology

7 Projective modules and Ext

7.1 Projective modules

Definition 96. A functor $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ is *left exact* if for all short exact sequences $0 \to M' \underset{f}{\to} M \underset{g}{\to} M'' \to 0$, the diagram

$$0 \to F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'')$$

is an exact sequence. It is *exact* if in addition $F(g)$ is surjective.

Let R be a ring and let M be an R-module. We consider the functor[2]

$$\mathrm{Hom}_R(M, -) : R-\mathrm{Mod} \to \mathbb{Z}-\mathrm{Mod}$$

which sends an R-module M to the abelian group $\mathrm{Hom}_R(M, N)$, and a morphism $f : N \to N'$ to the morphism $f \circ - : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$. This functor is *left exact* but not exact in general.

Example 97. Let $R = \mathbb{Z}$, and let $M = \mathbb{Z}/n$. Then $\mathrm{Hom}_{\mathbb{Z}}(M, -)$ is not exact, as one can see it by considering the image of $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$.

Definition 98. An R-module M is called *projective* if the functor $\mathrm{Hom}_R(M, -)$ is exact.

Proposition 99. *Let P be an R-module. The following are equivalent.*

1. *P is projective.*
2. **Lifting property.** *For all surjective morphisms $g : N \twoheadrightarrow N'$ and all $u : P \to N'$ there is a morphism $\overline{u} : P \to N$ such that the following diagram commutes (we say that $\overline{u}$ is a lifting of u):*

$$\begin{tikzcd} & N \arrow[d, two heads, "g"] \\ P \arrow[ur, dashed, "\overline{u}"] \arrow[r, "u"'] & N' \end{tikzcd} .$$

3. *P is a direct summand of a free R-module.*

[2]If R is a ring, then $\mathrm{Hom}_R(M, N)$ is only an abelian group in general. Indeed, one would like to define the linear combination of R-linear maps by $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$, but if R is not commutative, $(\lambda f + \mu g)(rx) = \lambda r f(x) + \mu r g(x)$ may differ from $r(\lambda f + \mu g)$, so that the linear combination may fail to be an R-linear map.

7.2 Projective resolutions

Not all R-modules are projective, but every R-module can be approximated by projective R-modules. Such approximations are called projective resolutions.

Definition 100. A *resolution* of M is a chain complex of R-modules P (zero in negative degrees) equipped with a map $\epsilon : P_0 \to M$ such that the following complex is exact (i.e. has homology zero in each degree):

$$\dots \xrightarrow{d} P_n \xrightarrow{d} P_{n-1} \xrightarrow{d} \dots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \;.$$

A *projective resolution* is a resolution such that every P_i is a projective R-module.

Proposition 101. *Every R-module M admits a projective resolution.*

The next result is sometimes called the "fundamental theorem of homological algebra".

Theorem 102. *Let P be a complex of projective objects (zero in negative degrees), equipped with a map $\epsilon_M : P_0 \to M$, and let (R, ϵ_N) be a resolution of N. Then for all R-linear maps $f : M \to N$ there is a morphism of chain complexes $\overline{f} : P \to R$ such that the following square commutes:*

$$\begin{tikzcd} P_0 \arrow[r, "\overline{f}_0"] \arrow[d, "\epsilon_M"] & R_0 \arrow[d, "\epsilon_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \;.$$

Such a morphism $\overline{f}$ is unique up to homotopy. (And the morphism $\overline{f}$ is called a lifting of the R-linear map f).

Corollary 103 (Uniqueness of projective resolutions)**.** *If (P, ϵ) and (Q, ϵ) are two projective resolutions of M, there is a homotopy equivalence $P \to Q$ which lifts* id_M*.*

7.3 Ext

Let $\Bbbk$ be a commutative ring. Then a $\Bbbk$-*algebra* is a ring R equipped with a morphism of rings $\Bbbk \to R$ with image contained in the center of R (thus, a ring is nothing but a $\mathbb{Z}$-algebra, and every commutative ring is an algebra over itself via the identity morphism of R). If R is a $\Bbbk$-algebra, then $\mathrm{Hom}_R(M, N)$ is not only an abelian group, but also a $\Bbbk$-module (the action of a scalar $\lambda \in \Bbbk$ is defined by $(\lambda f)(m) = \lambda f(m)$ for all $m \in M$).

Definition 104. Let R be a $\Bbbk$-algebra. Let M be an R-module, and let P be a projective resolution of M. Then for all N, the $\Bbbk$-module extensions of degree i between M and N is defined for all $i \geq 0$ by:

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(P,N)) \;.$$

For all R-linear maps $f : M \to M'$ we denote by

$$\mathrm{Ext}^i_R(f,N) : \mathrm{Ext}^i_R(M',N) \to \mathrm{Ext}^i_R(M,N)$$

the $\Bbbk$-linear map induced in cohomology by the morphism of complexes $\mathrm{Hom}_R(\overline{f},N) : \mathrm{Hom}_R(P',N) \to \mathrm{Hom}_R(P,N)$, where $\overline{f}$ is a lifting of f on the level of the projective resolutions. For all R-linear maps $g : N \to N'$ we denote by

$$\mathrm{Ext}^i_R(M,g) : \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M,N')$$

the $\Bbbk$-linear map induced in cohomology of degree i by the morphism of complexes $\mathrm{Hom}_R(P,\overline{g}) : \mathrm{Hom}_R(P,N) \to \mathrm{Hom}_R(P,N)$.

Remark 105. The Ext-modules do not depend on the choice of the projective resolution P. If P' another choice, there is a homotopy equivalence $f : P \to P'$ which induces a canonical isomorphism between the two versions of Ext. Similarly, the map $\mathrm{Ext}^i_R(f,N)$ does not depend on the choice of the lifting $\overline{f}$. Indeed, two such liftings $\overline{f}$ and $\overline{f}'$ are homotopic, hence the morphisms of complexes $\mathrm{Hom}_R(\overline{f},N)$ and $\mathrm{Hom}_R(\overline{f}',N)$ are homotopic, hence they induce the same map in homology.

Theorem 106 (fundamental properties of Ext)**.**

1. **Bifunctoriality.** *For all M and N, and all i, the $\Bbbk$-modules of degree i extensions define a functor*

$$\mathrm{Ext}^i_R(M,-) : R\text{−Mod} \to \Bbbk\text{−Mod} \;,$$

and a contravariant functor

$$\mathrm{Ext}^i_R(-,N) : R\text{−Mod} \to \Bbbk\text{−Mod} \;.$$

Moreover for all R-linear maps $f : M \to M'$ and $g : N \to N'$ there is a commutative square:

$$\begin{tikzcd}
\mathrm{Ext}^i_R(M',N) \arrow[r, "\mathrm{Ext}^i_R(M',g)"] \arrow[d, "\mathrm{Ext}^i_R(f,N)"] & \mathrm{Ext}^i_R(M',N') \arrow[d, "\mathrm{Ext}^i_R(f,N')"] \\
\mathrm{Ext}^i_R(M,N) \arrow[r, "\mathrm{Ext}^i_R(M,g)"] & \mathrm{Ext}^i_R(M,N') \;.
\end{tikzcd}$$

2. **Degree zero.** *There is an isomorphism, natural with respect to M and N:*
$$\mathrm{Ext}^0_R(M,N)\simeq \mathrm{Hom}_R(M,N)\;.$$

3. **Direct sums and products.** *There are isomorphisms, natural with respect to M, M_α, N and N_β:*
$$\mathrm{Ext}^i_R(\bigoplus_{\alpha\in A} M_\alpha,N)\simeq \prod_{\alpha\in A}\mathrm{Ext}^i_R(M_\alpha,N)\;,$$
$$\mathrm{Ext}^i_R(M,\prod_{\beta\in B} N_\beta)\simeq \prod_{\beta\in B}\mathrm{Ext}^i_R(M,N_\beta)\;.$$

4. **Projectives.** *The following assertions are equivalent.*
   (a) M *is projective,*
   (b) $\mathrm{Ext}^1_R(M,N)=0$ *for all R-modules N,*
   (c) $\mathrm{Ext}^i_R(M,N)=0$ *for all positive i and all N.*

5. **Long exact sequences.** *Every short exact sequence $0\to N'\xrightarrow{f}N\xrightarrow{g}N''\to 0$ induces a long exact sequence:*
$$0\to \mathrm{Ext}^0_R(M,N')\xrightarrow{\mathrm{Ext}^0_R(M,f)}\mathrm{Ext}^0_R(M,N)\xrightarrow{\mathrm{Ext}^0_R(M,g)}\dots$$
$$\dots\xrightarrow{\mathrm{Ext}^i_R(M,g)}\mathrm{Ext}^i_R(M,N'')\xrightarrow{\partial}\mathrm{Ext}^{i+1}_R(M,N')\xrightarrow{\mathrm{Ext}^{i+1}_R(M,f)}\dots$$

   Bonus. *Moreover, every morphism $f:M_1\to M_2$ and every morphism of short exact sequences*
$$\begin{tikzcd} 0 \arrow[r] & N_1' \arrow[r]\arrow[d,"\alpha'"] & N_1 \arrow[r]\arrow[d,"\alpha"] & N_1'' \arrow[r]\arrow[d,"\alpha''"] & 0 \\ 0 \arrow[r] & N_2' \arrow[r] & N_2 \arrow[r] & N_2'' \arrow[r] & 0 \end{tikzcd}$$
   induces a morphism between the corresponding exact sequences. In particular we have commutative squares:
$$\begin{tikzcd} \mathrm{Ext}^i_R(M,N_1'') \arrow[r,"\partial"]\arrow[d,"\mathrm{Ext}^i_R(M,\alpha'')"] & \mathrm{Ext}^{i+1}_R(M,N_1') \arrow[d,"\mathrm{Ext}^{i+1}_R(M,\alpha')"] \\ \mathrm{Ext}^i_R(M,N_2'') \arrow[r,"\partial"] & \mathrm{Ext}^{i+1}_R(M,N_2') \end{tikzcd}\;,$$
$$\begin{tikzcd} \mathrm{Ext}^i_R(M_2,N'') \arrow[r,"\partial"]\arrow[d,"\mathrm{Ext}^i_R(f,N'')"] & \mathrm{Ext}^{i+1}_R(M_2,N') \arrow[d,"\mathrm{Ext}^{i+1}_R(f,N')"] \\ \mathrm{Ext}^i_R(M_1,N'') \arrow[r,"\partial"] & \mathrm{Ext}^{i+1}_R(M_1,N') \end{tikzcd}\;.$$

Now we justify the name Ext for the homological notion introduced here.

Definition 107. Let M and N be two R-modules. An *extension of N by M* is a short exact sequence:

$$0 \to M \xrightarrow{u} E \xrightarrow{v} N \to 0 \;.$$

Two extensions of N by M are called *equivalent* is there is an isomorphism ϕ making the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "u"] \arrow[d, equal] & E \arrow[r, "v"] \arrow[d, "\phi", "\simeq"'] & N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "u'"] & E' \arrow[r, "v'"] & N \arrow[r] & 0 \end{tikzcd} \;.$$

An extension is called *split* if it is equivalent to the trivial extension (here ι_M is the canonical injection and π_M the canonical projection):

$$0 \to M \xrightarrow{\iota_M} M \oplus N \xrightarrow{\pi_N} N \to 0 \;.$$

We let $\mathcal{E}^1(M,N)$ be the set of equivalent classes of extensions of N by M.

Theorem 108. *There is a bijection, which sends* 0 *to the class of the split extension:*

$$\mathrm{Ext}^1_R(M,N) \simeq \mathcal{E}^1(M,N) \;.$$

Example 109. Take $R = \mathbb{Z}$ and let p be a prime integer. Every extension of $\mathbb{Z}/p$ by $\mathbb{Z}$ is either trivial, or equivalent to the following exact sequence, for a uniquely determined $a \in \mathbb{Z}/p \setminus \{0\}$, where π_a denotes the unique map such that $\pi(1) = a$:

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\pi_a} \mathbb{Z}/p \to 0 \;.$$

8 Injective modules and Ext

8.1 Injective modules

Definition 110. A contravariant functor $F : R\text{−Mod} \to S\text{−Mod}$ is *left exact* if for all short exact sequences $0 \to M' \underset{f}{\to} M \underset{g}{\to} M'' \to 0$, the diagram

$$0 \to F(M'') \xrightarrow{F(g)} F(M) \xrightarrow{F(f)} F(M')$$

is an exact sequence. It is *exact* if in addition $F(f)$ is surjective.

Let $\Bbbk$ be a commutative ring, let R be a $\Bbbk$-algebra, and let M be an R-module. The contravariant functor

$$\mathrm{Hom}_R(-, M) : R\text{−Mod} \to \Bbbk\text{−Mod}$$

is *left exact* but not exact in general.

Example 111. Let $R = \Bbbk = \mathbb{Z}$, and let $M = \mathbb{Z}/n$. Then $\mathrm{Hom}_{\mathbb{Z}}(-, M)$ is not exact, as one can see it by considering the image of $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$.

Definition 112. An R-module M is called *injective* if the functor $\mathrm{Hom}_R(-, M)$ is exact.

Proposition 113. *Let J be an R-module. The following are equivalent.*

1. *J is injective.*
2. **Extension property.** *For all injective morphisms $g : N \hookrightarrow N'$ and all $u : N \to J$ there is a morphism $\overline{u} : N \to J$ such that the following diagram commutes (we say that $\overline{u}$ is an extension of u to N'):*

$$\begin{tikzcd} N' \arrow[dr, dashed, "\overline{u}"] & \\ N \arrow[u, hook, "g"] \arrow[r, "u"] & J \end{tikzcd} .$$

3. **Restricted extension property.** *For all left ideal $I \subset R$, every map $u : I \to J$ extends to a map $\overline{u} : R \to J$.*

Remarks 114. It is clear from the extension property that a product of injective R-modules is again injective.

The last criterion is known as "Baer's criterion". It says that instead of testing the extension property for all submodules of all modules, one may test the extension property for all submodules of *one* module, namely R.

We now describe in more details the injective modules for various classes of rings.

8.1.1 Injective modules for fields

Proposition 115. *If R is a field, then every R-module is injective.*

8.1.2 Injective modules for PIDs

Lemma 116 ("Injective = divisible" for PID)**.** *Let R be a PID. Then J is injective if and only if it is divisible, that is, if and only if for all nonzero r in R and all $a \in J$ there is an $b \in J$ such that $rb = a$.*

Proposition 117. *Let R be a PID, let F denote its fraction field, and assume that $R \neq F$ (i.e. R is not a field).*

(i) The R-modules F and F/R are injective.

(ii) An R-module M is injective if and only if it is a direct summand of a product of copies of F/R.

(iii) Every R-module is a submodule of a product of copies of F/R.

8.1.3 Injective modules for general rings

Definition 118. One says that J is an *injective cogenerator* of R-modules if J is injective and if every R-module embeds into a product of copies of J.

Lemma 119. *If J is an injective cogenerator of R-modules, then every injective R-module is a direct summand of a product of copies of J.*

Example 120. If R is a field, then R is an injective cogenerator. If R is a PID with fraction field $F \neq R$, then F/R is an injective cogenerator.

Now we are going to prove that if $R \to S$ is a morphism of rings, then one can construct an injective cogenerator of S-modules from an injective cogenerator of R-modules. The main tool for this is coinduction.

Definition 121. Let $S \to R$ be a morphism of rings. If M is an S-module, we define an R-module structure on the abelian group $\mathrm{Hom}_S(R, M)$ by letting $(\lambda f)(x) := f(x\lambda)$. This defines a*coinduction functor*:

$$\mathrm{Hom}_S(R, -) : S\mathrm{-Mod} \to R\mathrm{-Mod}\ .$$

Proposition 122 (Properties of coinduction.)**.**

1. **Left exactness.** *Coinduction is a left exact functor.*
2. **Adjunction.** *Let* $\mathrm{res}^R_S : R\mathrm{-Mod} \to S\mathrm{-Mod}$ *denote the functor which sends every R-module M to the underlying S-module. There is an isomorphism, natural with respect to the R-module N and the S-module N:*

$$\mathrm{Hom}_S(\mathrm{res}^R_S N, M) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_S(R, M))\ .$$

3. **Injectives.** *If J is injective, then* $\mathrm{Hom}_S(R,J)$ *is injective as well.*

4. **Injective cogenerators.** *If J is an injective cogenerator of S-modules, then* $\mathrm{Hom}_S(R,J)$ *is an injective cogenerator of R-modules.*

Corollary 123. *1. For all rings R, the coinduced module* $\mathrm{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})$ *is an injective cogenerator of R-modules.*

2. For all $\Bbbk$-algebras R over a field $\Bbbk$, the coinduced module $\mathrm{Hom}_{\Bbbk}(R,\Bbbk)$ *is an injective cogenerator of R-modules.*

8.2 Injective coresolutions

The treatment of injective coresolutions is parallel to that of projective resolutions.

Definition 124. A *coresolution* of M is a cochain complex of R-modules J (zero in negative degrees) equipped with a map $\eta : M \to J^0$ such that the following complex is exact (i.e. has homology zero in each degree):

$$M \xrightarrow{\eta} J^0 \xrightarrow{d} J^1 \xrightarrow{d} \dots \xrightarrow{d} J^n \xrightarrow{d} J^{n+1} \xrightarrow{d} \dots .$$

An *injective coresolution* is a coresolution such that every J^k is an injective R-module.

Proposition 125. *Every R-module M admits an injective coresolution.*

Theorem 126. *Let J be a complex of injective objects (zero in negative degrees), equipped with a map $\eta_N : N \to J^0$, and let (R,ϵ_M) be a coresolution of M. Then for all R-linear maps $f : M \to N$ there is a morphism of chain complexes $\overline{f} : P \to R$ such that the following square commutes:*

$$\begin{array}{ccc} R^0 & \xrightarrow{\overline{f}^0} & J^0 \\ {\scriptstyle \eta_M}\uparrow & & {\scriptstyle \eta_N}\uparrow \\ M & \xrightarrow{f} & N \end{array} .$$

Such a morphism $\overline{f}$ is unique up to homotopy. (And the morphism $\overline{f}$ is called a lifting of the R-linear map f).

Corollary 127 (Uniqueness of injective coresolutions)**.** *If (J,η) and (Q,η) are two injective coresolutions of M, there is a homotopy equivalence $P \to Q$ which lifts* id_M*.*

8.3 Ext with injective coresolutions

Theorem 128. *Let M and N be R-modules, and let (Q, η) be an injective coresolution of N. Then for all i, $\mathrm{Ext}^i_R(M, N)$ is isomorphic to the cohomology of the cochain complex $\mathrm{Hom}_R(M, Q)$.*

Moreover, for all R-linear maps $f : M \to M'$, the map $\mathrm{Ext}^i_R(f, N)$ is the map induced in cohomology by the morphism of complexes $\mathrm{Hom}_R(f, Q)$. Similarly, for all R-linear maps $g : N \to N'$, the map $\mathrm{Ext}^i_R(M, g)$ is the map induced in cohomology by the morphism of complexes $\mathrm{Hom}_R(M, \overline{g})$, where $\overline{g}$ denotes a lifting of g to the level of the injective coresolutions.

The proof of theorem 128 will be given later (see section 9.2.)

Corollary 129. *Every short exact sequence of modules $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence:*

$$0 \to \mathrm{Ext}^0_R(M'', N) \to \mathrm{Ext}^0_R(M, , N) \to \mathrm{Ext}^0(M', N) \xrightarrow{\partial} \mathrm{Ext}^1_R(M'', N) \to \dots$$

$$\dots \to \mathrm{Ext}^i_R(M'', N) \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M', N) \xrightarrow{\partial} \mathrm{Ext}^{i+1}_R(M'', N) \to \dots$$

Proof. We use the definition of Ext from an injective coresolution (Q, η) of N. We have a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to 0 ,$$

and the associated long exact sequence gives the result. □

8.4 An application to semi-simplicity

Proposition 130 (Homological characterization of semi-simplicity.)**.** *Let R be an algebra over a field $\Bbbk$. Then the following statements are equivalent.*

(1) Every R-module M of finite dimension over $\Bbbk$ is a sum of simple R-modules (i.e. the category of finite dimensional R-modules is semi-simple).

(2) For all R-modules M, N of finite dimension over $\Bbbk$, $\mathrm{Ext}^1_R(M, N) = 0$.

(3) For all simple R-modules M, N of finite dimension over $\Bbbk$, $\mathrm{Ext}^1_R(M, N) = 0$.

9 Bicomplexes and their applications

9.1 Cochain bicomplexes

Definition 131. A *(cochain) bicomplex of R-modules* is a family of R-modules $(C^{i,j})_{(i,j)\in\mathbb{Z}^2}$ equipped with R-linear maps $d : C^{i,j} \to C^{i+1,j}$ and $\delta : C^{i,j} \to C^{i,j+1}$ which satisfy:

$$d\circ d = 0 = \delta\circ\delta\;, \qquad d\circ\delta = \delta\circ d\;.$$

The elements of $C^{i,j}$ are called the *homogeneous elements of bidegree* (i,j), the maps d and δ are the differentials of the bicomplex. Cochain bicomplexes are depicted as commutative diagrams:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\dots \arrow[r, "d"] & C^{i,j+1} \arrow[r, "d"] \arrow[u, "\delta"] & C^{i+1,j+1} \arrow[r, "d"] \arrow[u, "\delta"] & \dots \\
\dots \arrow[r, "d"] & C^{i,j} \arrow[r, "d"] \arrow[u, "\delta"] & C^{i+1,j} \arrow[r, "d"] \arrow[u, "\delta"] & \dots \\
 & \vdots \arrow[u, "\delta"] & \vdots \arrow[u, "\delta"] &
\end{tikzcd} .$$

A *morphism of bicomplexes* $f : C \to D$ is a family of R-linear maps $f^{i,j} : C^{i,j} \to D^{i,j}$ which preserves the differentials, i.e.

$$d\circ f^{i,j} = f^{i+1,j}\circ d\;, \qquad \delta\circ f^{i,j} = f^{i,j+1}\circ\delta\;.$$

Morphims of bicomplexes can be composed degreewise (i.e. $(f\circ g)^{i,j} = f^{i,j}\circ g^{i,j}$), so that cochain bicomplexes and morphisms of cochain complexes form a category $\mathrm{Ch}^{\bullet,\bullet}(R)$.

Examples 132.

1. Every cochain complex can be regarded as a bicomplex concentrated in the 0-th row, i.e. as a cochain bicomplex with $C^{i,0} := C^i$ and $C^{i,j} = 0$ for $j \neq 0$, and with only nonzero differential given by the differential of the complex.

2. Every cochain complex can be regarded as a bicomplex concentrated in the 0-th column, i.e. as a cochain bicomplex with $C^{0,j} := C^j$ and $C^{i,j} = 0$ for $i \neq 0$, and with only nonzero differential given by the differential of the complex.

3. If C is a chain complex and D is a cochain complex, we can form a cochain bicomplex $\mathrm{Hom}_R(C,D)$ with $\mathrm{Hom}_R(C_i, D^j)$ placed in bidegree (i,j), and with differentials induced by those of C and D:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\dots \arrow[r] & \mathrm{Hom}_R(C_i, D^{j+1}) \arrow[r, "\mathrm{Hom}_R(d,D^{j+1})"] \arrow[u] & \mathrm{Hom}_R(C_{i+1}, D^{j+1}) \arrow[r] \arrow[u] & \dots \\
\dots \arrow[r] & \mathrm{Hom}_R(C_i, D^{j}) \arrow[r, "\mathrm{Hom}_R(d,D^{j})"] \arrow[u, "\mathrm{Hom}_R(C_i,d)"] & \mathrm{Hom}_R(C_{i+1}, D^{j}) \arrow[r] \arrow[u, "\mathrm{Hom}_R(C_{i+1},d)"] & \dots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} .$$

Definition 133. The *total complex* associated to a bicomplex (C, d, δ) is the complex $\mathrm{Tot}\, C$ defined in each degree by:

$$(\mathrm{Tot}\, C)^k = \bigoplus_{i+j=k} C^{i,j}$$

and whose differential $d^{\mathrm{Tot}} : (\mathrm{Tot}\, C)^k \to (\mathrm{Tot}\, C)^{k+1}$ sends $x \in C^{i,j}$ to

$$d^{\mathrm{Tot}}(x) = d(x) + (-1)^i \delta(x) .$$

Every morphism of bicomplexes $f : C \to D$ induces a morphism of complexes $\mathrm{Tot}\, f : \mathrm{Tot}\, C \to \mathrm{Tot}\, D$, namely, if $x \in C^{i,j}$ then $(\mathrm{Tot}\, f)(x) = f^{i,j}(x)$.

Example 134. If C is a cochain complex viewed as a bicomplex concentrated in the 0-th row or the 0-th column, the total complex associated to this bicomplex is isomorphic to the cochain complex C.

Theorem 135. *Let $f : C \to D$ be a morphism of bicomplexes. Assume that for all i, the morphism of chain complexes given by restricting f to the i-th column*

$$f^{i,\bullet} : (C^{i,\bullet}, \delta) \to (D^{i,\bullet}, \delta)$$

is a quasi-isomorphism. Assume furthermore that C and D are such that for all k the direct sums

$$\bigoplus_{i+j=k} C^{i,j} \quad \text{and} \quad \bigoplus_{i+j=k} D^{i,j}$$

are finite. Then $\mathrm{Tot}\, f$ *is a quasi-isomorphism.*

Remark 136. The hypothesis that f is a column-wise quasi-isomorphism can be replaced by the hypothesis that f is a row-wise quasi-isomorphism (i.e for all fixed j the morphism $f^{\bullet,j} : (C^{\bullet,j}, d) \to (D^{\bullet,j}, d)$ between the j-th rows is a quasi-isomorphism), with the same conclusion that $\mathrm{Tot}\, f$ is a quasi-isomorphism (and a similar proof).

9.2 The proof of theorem 128

Theorem 135 can be used to prove theorem 128 in the following way. Let (P, ϵ) be a projective resolution of M and (Q, η) be an injective coresolution of N. Then we have three cochain bicomplexes C, C' and C'' whose objects are given by

$$\begin{aligned} C^{i,j} &= \mathrm{Hom}(P_i, Q^j), \\ (C')^{i,j} &= \begin{cases} \mathrm{Hom}_R(P_i, N) & \text{if } j = 0, \\ 0 & \text{if } j \neq 0, \end{cases} \\ (C'')^{i,j} &= \begin{cases} \mathrm{Hom}_R(M, Q^j) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \end{aligned}$$

The map ϵ and η induce morphisms of bicomplexes

$$C' \xrightarrow{\mathrm{Hom}_R(\eta, P)} C \xleftarrow{\mathrm{Hom}_R(\epsilon, Q)} C'' \;.$$

And theorem 135 shows that these morphisms induces quasi-isomorphisms on the level of the associated total complexes. Now the cohomology of $\mathrm{Tot}\, C'$ is the definition of Ext via projective resolutions while the cohomology of $\mathrm{Tot}\, C''$ is the definition of Ext via injective resolutions.

9.3 The Künneth formula

Definition 137. If (C, d^C) and (D, d^D) are two cochain complexes of R-modules over a commutative ring R, their *tensor product* is the cochain complex $(C \otimes D, d)$ such that

$$\begin{aligned} (C \otimes D)^k &= \bigoplus_{i+j=k} C^i \otimes D^j \;, \\ d(x \otimes y) &= d^C(x) \otimes y + (-1)^i x \otimes d^D(y) \quad \text{ if } x \otimes y \in C^i \otimes D^j. \end{aligned}$$

If z is a cycle of degree i in C and z' is a cycle of degree j in D, then $z \otimes z'$ is a cycle of degree $i+j$ of $C \otimes D$. Moreover, if z or z' is a boundary, then $z \otimes z'$ is a boundary. Hence we have a map, called the *Künneth morphism*:

$$\begin{array}{cccc} \kappa : & \bigoplus_{i+j=k} H^i(C) \otimes H^j(D) & \to & H^k(C \otimes D) \\ & [z] \otimes [z'] & \mapsto & [z \otimes z'] \end{array} .$$

The following theorem is a consequence of theorem 135.

Theorem 138 (Künneth formula)**.** *Assume that R is a field. Then κ is an isomorphism for all k.*

Remark 139. The theorem fails if R is not a field. For example, take $R = \mathbb{Z}$, $C = \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$ (viewed as a complex in degrees 0 and 1) and $D = \mathbb{Z}/n\mathbb{Z}$ (viewed as a complex concentrated in degree 0). Then in degree 0, the domain of the Künneth morphism is zero, whereas its codomain is $\mathbb{Z}/n\mathbb{Z}$.

9.4 Chain bicomplexes

There is an obvious notion of a chain bicomplex, where degrees are denoted by indices and differentials lower the degree. Chain complexes form a category $\mathrm{Ch}_{\bullet,\bullet}(R)$. Just as in the case of complexes, the difference between cochain bicomplexes and chain bicomplexes is just a matter of notation. One can convert cochain bicomplexes and their morphisms into chain complexes and their morphisms by the formulas

$$C^{i,j} = C_{-i,-j} \;, \qquad f^{i,j} = f_{-i,-j} \;,$$

so that we have an equality $\mathrm{Ch}^{\bullet,\bullet}(R) = \mathrm{Ch}_{\bullet,\bullet}(R)$.

If we use the chain complex notation instead of the cochain complex notation, then theorems 135 and 138 translate as follows.

- Let $f : C \to D$ be a morphism of bicomplexes, such that f is column-wise a quasi-isomorphism of complexes, or such that f is row-wise a quasi-isomorphism of complexes. Assume furthermore that for all k the direct sums
$$\bigoplus_{i+j=k} C_{i,j} \quad \text{and} \quad \bigoplus_{i+j=k} D_{i,j}$$
are finite. Then $\mathrm{Tot}\, f$ is a quasi-isomorphism.

- If C and D are chain complexes of R-modules over a field R, the Künneth morphism induces an isomorphism of R-modules:
$$\begin{array}{rccc} \kappa : & \bigoplus_{i+j=k} H_i(C) \otimes H_j(D) & \to & H_k(C \otimes D) \\ & [z] \otimes [z'] & \mapsto & [z \otimes z'] \end{array} .$$

9.5 Vista: the singular homology of products of spaces

Let us briefly mention an application of the Künneth theorem 138 in the context of singular homology of topological spaces.

If X and Y are two topological spaces and R is a commutative ring, one can prove that there is an homotopy equivalence:

$$C^{\mathrm{sing}}(C) \otimes C^{\mathrm{sing}}(Y) \to C^{\mathrm{sing}}(X \times Y) \;.$$

If R is a field, the homology of the left-hand-side can be computed by the Künneth theorem. Hence we obtain the following statement.

Theorem 140 (Topological Künneth theorem)**.** *Assume that R is a field. Then for all topological spaces X and Y and all integers k there is an isomorphism:*

$$\bigoplus_{i+j=k} H_i(X) \otimes H_j(Y) \simeq H_k(X \times Y) \;.$$

Example 141. The torus T^2 is homeomorphic to $S^1 \times S^1$. The topological Künneth theorem allows one to compute its homology from that of S^1:

$$\begin{aligned} H_0(T^2) =& H_0(S^1) \otimes H_0(S^1) = R \;, \\ H_1(T^2) =& H_1(S^1) \otimes H_0(S^1) \;\oplus\; H_0(S^1) \otimes H_1(S^1) = R^2 \;, \\ H_2(T^2) =& H_1(S^1) \otimes H_1(S^1) = R \;. \end{aligned}$$

More material on this topic can be found e.g. in [Hatcher, Section 3.B].

10 Flat modules and Tor

10.1 Tensor products over noncommutative rings

I assume that you are familiar with the tensor product of modules over a commutative ring. Here is the more general definition when the ring is not commutative.

Definition 142. Let R be a $\Bbbk$-algebra over a commutative ring $\Bbbk$. For all right R-modules M and all left R-modules N, the tensor product $M \otimes_R N$ is the $\Bbbk$-module $M \otimes_R N$ defined by:

$$M \otimes_R N = \frac{M \otimes_{\Bbbk} N}{\langle mr \otimes n - m \otimes rn \mid (m,n,r) \in M \times N \times R \rangle} \;.$$

If $f : M \to M'$ and $g : N \to N'$ are R-linear, they define a $\Bbbk$-linear map:

$$f \otimes g : M \otimes_R N \to M' \otimes_R N'$$

such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$. The composition $(f' \otimes g') \circ (f \otimes g)$ equals $(f' \circ f) \otimes (g' \circ g)$. In particular, we have functors:

$$M \otimes_R - : R\text{−Mod} \to \Bbbk\text{−Mod} \;,$$
$$- \otimes_R N : \text{Mod} - R \to \Bbbk\text{−Mod} \;.$$

Remark 143. If R is a $\Bbbk$-algebra, then it is also a ring, hence a $\mathbb{Z}$-algebra. Thus we have two tensor products $M \otimes_R N$: the one defined using the $\Bbbk$-algebra structure and the one defined using the $\mathbb{Z}$-algebra structure. One can show that there is in fact a canonical isomorphism of abelian groups between these two objects (see exercise 55 in exercise sheet 8).

This construction has the same basic properties as the tensor product over commutative rings, namely:

1. The universal property,
2. The adjunction property,
3. Additivity and unitality,
4. Right exactness.

We now review this properties in detail.

Definition 144. Let M be a right R-module, let N be a left R-module, and let P be a $\Bbbk$-module. A map $f : M \times N \to P$ is $(R, \Bbbk)$-*bilinear* if it is $\Bbbk$-linear with respect to each variable, and if moreover for all $r \in R$ we have:

$$f(mr, n) = f(m, rn).$$

Example 145. The map $\begin{array}{rccc}\pi : & M\times N & \to & M\otimes_R N\\ & (m,n) & \mapsto & m\otimes n\end{array}$ is $(R,\Bbbk)$-bilinear.

Proposition 146 (Universal property)**.** *For all $(R,\Bbbk)$-bilinear maps $f : M\times N\to P$ there is a unique $\Bbbk$-linear map $\overline{f} : M\otimes_R N\to P$ such that $f=\overline{f}\circ\pi$:*

$$\begin{tikzcd} M\times N \arrow[r, "f"] \arrow[d, "\pi"] & P \\ M\otimes_R N \arrow[ur, dashed, "\exists!\overline{f}"'] & \end{tikzcd} .$$

Proposition 147 (Adjunction)**.** *There are isomorphisms of $\Bbbk$-modules, natural with respect to M, N, P:*

$$\mathrm{Hom}_R(M,\mathrm{Hom}_\Bbbk(N,P))\simeq \mathrm{Hom}_\Bbbk(M\otimes_R N,P)\simeq \mathrm{Hom}_R(N,\mathrm{Hom}_\Bbbk(M,P))\;.$$

Proposition 148 (Additivity and unitality)**.** *We have isomorphisms, natural with respect to the modules M, M_α, N, N_β:*

$$\begin{aligned}&\bigoplus_{\alpha\in A}(M_\alpha\otimes_R N)\simeq(\bigoplus_{\alpha\in A}M_\alpha)\otimes_R N\;,\\ &\bigoplus_{\beta\in B}(M\otimes_R N_\beta)\simeq M\otimes_R(\bigoplus_{\beta\in B}N_\beta)\;,\\ &R\otimes_R N\simeq N\;,\\ &M\otimes_R R\simeq M\;.\end{aligned}$$

One can prove the isomorphisms of proposition 148 by using the adjunction property together with the following useful lemma.

Lemma 149. *Let S be a ring, and let $\theta : \mathrm{Hom}_S(M,P)\to\mathrm{Hom}_S(M',P)$ be a morphism of abelian groups which is natural with respect to P. Then there exists a unique S-linear morphism $f : M'\to M$ such that $\theta(g)=g\circ f$ for all g. Moreover, θ is an isomorphism if and only if f is an isomorphism.*

Definition 150. A functor $F : R-\mathrm{Mod}\to S-\mathrm{Mod}$ is *right exact* if for all short exact sequences $0\to M'\xrightarrow[f]{} M\xrightarrow[g]{} M''\to 0$, the diagram

$$F(M')\xrightarrow[F(f)]{} F(M)\xrightarrow[F(g)]{} F(M'')\to 0$$

is exact. It is *exact* if in addition $F(f)$ is injective.

In the previous definition we are considering *left* modules, but the definition applies as well to right modules, since $\mathrm{Mod}-R = R^{\mathrm{op}}-\mathrm{Mod}$ where R^{op} is the opposite ring of R.

Proposition 151 (Right exactness)**.** *The functors $M\otimes_R -$ and $-\otimes_R N$ are right exact.*

One can prove proposition 151 by using the adjunction property together with the following useful lemma.

Lemma 152. *A sequence of S-modules $M' \to M \to M'' \to 0$ is exact if and only if for all S-modules P, the following sequence of modules is exact:*

$$0 \to \mathrm{Hom}_S(M'', P) \to \mathrm{Hom}_S(M, P) \to \mathrm{Hom}_S(M', P)\ .$$

10.2 Flatness

Definition 153. One says that a left R-module M is *flat* if the functor $-\otimes_R M : \mathrm{Mod}-R \to \mathbb{Z}-\mathrm{Mod}$ is exact.

Proposition 154 (First properties of flatness)**.**

1. *If M is projective, then M is flat.*
2. *If $(M_\alpha)_{\alpha\in A}$ is a family of flat R-modules, then $\bigoplus_{\alpha\in A} M_\alpha$ is flat.*

Proposition 155 (Localizations)**.** *If R is a commutative ring and if S is a multiplicative part of R, then $S^{-1}R$ is a flat R-module.*

Example 156. 1. If R is a field, every R-module is free, hence flat.

2. If $R = \mathbb{Z}$, then $\mathbb{Z}/n\mathbb{Z}$ is not flat and $\mathbb{Q}$ is flat but not projective.

10.3 Tor

Definition 157. Let R be a $\Bbbk$-algebra. Let M be a right R-module, and let P be a projective resolution of M. Then for all N, the $\Bbbk$-module $\mathrm{Tor}_i^R(M,N)$ is defined for all $i \geq 0$ by:

$$\mathrm{Tor}_i^R(M,N) = H_i(P \otimes_R N)\ .$$

For all R-linear maps $f : M \to M$ we denote by

$$\mathrm{Tor}_i^R(f,N) : \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M',N)$$

the $\Bbbk$-linear map induced in homology by the morphism of complexes $\overline{f} \otimes_R \mathrm{id}_N : P \otimes_R N \to P' \otimes_R N$, where $\overline{f}$ is a lifting of f on the level of the projective resolutions. For all R-linear maps $g : N \to N'$ we denote by

$$\mathrm{Tor}_i^R(M,g) : \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M,N')$$

the $\Bbbk$-linear map induced on homology of degree i by the morphism of complexes $P \otimes_R g : P \otimes_R N \to P \otimes_R N'$.

Remark 158. The definition of Tor depends on choices. First one chooses a projective resolution P. However, since two projective resolutions are homotopy equivalent, changing the projective resolution only changes the complex $P \otimes_R N$ up to a homotopy equivalence, hence it only changes $\mathrm{Tor}_i^R(M,N)$ up to an isomorphism. Second, the definition of $\mathrm{Tor}_i^R(f,N)$ depends on the choice of a lifting $\overline{f}$, but since any two liftings are homotopic, they actually define the same map $\mathrm{Tor}_i^R(f,N)$.

The following result is the Tor-analogue of theorem 128, and it has an analogous proof.

Theorem 159. *Let M be a right R-module, let N be a left R-module and let (P,ϵ) be a projective resolution of N. There is an isomorphism*

$$\mathrm{Tor}_i^R(M,N) \simeq H_i(M \otimes_R P)\;.$$

Moreover, for all R-linear maps $f : M \to M'$, the map $\mathrm{Tor}_i^R(f,N)$ is the map induced in homology by the morphism of complexes $f \otimes_R \mathrm{id}$. Similarly, for all R-linear maps $g : N \to N'$, the map $\mathrm{Tor}_i^R(M,g)$ is the map induced in homology by the morphism of complexes $M \otimes_R \overline{g}$, where $\overline{g}$ denotes a lifting of g to the level of the projective resolutions.

All the basic properties of Ext have a counterpart for Tor. We gather these properties in the following long theorem.

Theorem 160. *1.* **Bifunctoriality.** *For all M and N, and all i, the* Tor*-modules define functors*

$$\mathrm{Tor}_i^R(M,-) : R\mathrm{-Mod} \to \Bbbk\mathrm{-Mod}\;,$$
$$\mathrm{Tor}_i^R(-,N) : \mathrm{Mod}-R \to \Bbbk\mathrm{-Mod}\;,$$

and moreover for all R-linear maps $f : M \to M'$ and $g : N \to N'$ there is a commutative square:

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M,N) & \xrightarrow{\mathrm{Tor}_i^R(M,g)} & \mathrm{Tor}_i^R(M,N') \\ \downarrow{\scriptstyle \mathrm{Tor}_i^R(f,N)} & & \downarrow{\scriptstyle \mathrm{Tor}_i^R(f,N')} \\ \mathrm{Tor}_i^R(M',N) & \xrightarrow{\mathrm{Tor}_i^R(M',g)} & \mathrm{Tor}_i^R(M',N')\;. \end{array}$$

2. **Degree zero.** *There is an isomorphism of $\Bbbk$-modules, natural with respect to M and N:*

$$\mathrm{Tor}_0^R(M,N) \simeq M \otimes_R N\;.$$

3. **Additivity.** *There are isomorphisms of $\Bbbk$-modules, natural with respect to M, M_α, N, N_β:*

$$\begin{aligned}\mathrm{Tor}_i^R(\bigoplus_{\alpha\in A} M_\alpha, N) &\simeq \bigoplus_{\alpha\in A}\mathrm{Tor}_i^R(M_\alpha,N)\;,\\ \mathrm{Tor}_i^R(M,\bigoplus_{\beta\in B} N_\beta) &\simeq \bigoplus_{\beta\in B}\mathrm{Tor}_i^R(M,N_\beta)\;.\end{aligned}$$

4. **Long exact sequences (first variable).** *Every short exact sequence $0\to M'\xrightarrow{f} M\xrightarrow{g} M''\to 0$ induces a long exact sequence in* Tor*:*

$$\begin{aligned}&\cdots\to\mathrm{Tor}_i^R(M,N)\xrightarrow{\mathrm{Tor}_i^R(g,N)}\mathrm{Tor}_i^R(M'',N)\xrightarrow{\partial}\mathrm{Tor}_{i-1}^R(M',N)\to\dots\\ &\cdots\to\mathrm{Tor}_0^R(M',N)\xrightarrow{\mathrm{Tor}_0^R(f,N)}\mathrm{Tor}_0^R(M,N)\xrightarrow{\mathrm{Tor}_0^R(g,N)}\mathrm{Tor}_0^R(M'',N)\to 0.\end{aligned}$$

Bonus. *Every morphism $f:N_1\to N_2$ and every morphism of short exact sequences:*

$$\begin{tikzcd} 0 \arrow[r] & M_1' \arrow[r]\arrow[d,"\alpha'"] & M_1 \arrow[r]\arrow[d,"\alpha"] & M_1'' \arrow[r]\arrow[d,"\alpha''"] & 0\\ 0 \arrow[r] & M_2' \arrow[r] & M_2 \arrow[r] & M_2'' \arrow[r] & 0 \end{tikzcd}$$

induces morphisms between the corresponding long exact sequences. In particular the following squares commute:

$$\begin{tikzcd} \mathrm{Tor}_i^R(M_1'',N) \arrow[r,"\partial"]\arrow[d,"\mathrm{Tor}_i^R(\alpha'',N)"] & \mathrm{Tor}_{i-1}^R(M_1',N)\arrow[d,"\mathrm{Tor}_{i-1}^R(\alpha',N)"]\\ \mathrm{Tor}_i^R(M_2'',N) \arrow[r,"\partial"] & \mathrm{Tor}_{i-1}^R(M_2',N) \end{tikzcd}\;,$$

$$\begin{tikzcd} \mathrm{Tor}_i^R(M'',N_1) \arrow[r,"\partial"]\arrow[d,"\mathrm{Tor}_i^R(M'',f)"] & \mathrm{Tor}_{i-1}^R(M',N_1)\arrow[d,"\mathrm{Tor}_{i-1}^R(M',f)"]\\ \mathrm{Tor}_i^R(M'',N_2) \arrow[r,"\partial"] & \mathrm{Tor}_{i-1}^R(M',N_2) \end{tikzcd}\;.$$

5. **Long exact sequences (second variable).** *Every short exact sequence $0\to N'\xrightarrow{f} N\xrightarrow{g} N''\to 0$ induces a long exact sequence in* Tor*:*

$$\begin{aligned}&\cdots\to\mathrm{Tor}_i^R(M,N)\xrightarrow{\mathrm{Tor}_i^R(M,g)}\mathrm{Tor}_i^R(M,N'')\xrightarrow{\partial}\mathrm{Tor}_{i-1}^R(M,N')\to\dots\\ &\cdots\to\mathrm{Tor}_0^R(M,N')\xrightarrow{\mathrm{Tor}_0^R(M,f)}\mathrm{Tor}_0^R(M,N)\xrightarrow{\mathrm{Tor}_0^R(M,g)}\mathrm{Tor}_0^R(M,N'')\to 0.\end{aligned}$$

Bonus. *As in the case of the first variable, these long exact sequences are natural with respect to M and with respect to the short exact sequence $0 \to N' \to N \to N'' \to 0$.*

6. **Flatness.** *The following assertions are equivalent.*

   (a) M *is flat,*
   (b) for all N, $\mathrm{Tor}_1^R(M,N) = 0$,
   (c) for all N, $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i > 0$.

   Similarly, N *is flat iff* $\mathrm{Tor}_1^R(M,N) = 0$ *for all* M *iff* $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i > 0$ *and all* M.

The name 'Tor' comes from the following relation with torsion elements of R-modules.

Definition 161. Let $r \in R \setminus \{0\}$. We denote by ${}_rN$ the module of r-torsion elements of N:

$$ {}_rM = \{m \in M \mid rm = 0\}\ . $$

If R is a domain, the subset $\bigcup_{r\in R\setminus\{0\}} {}_rN$ of all torsion elements of N is a submodule of N that we denote by $t(N)$.

Remark 162. If R is not a domain, then the subset of torsion elementsof M may fail to be a submodule of M. For example, take $R = \mathbb{Z}/6\mathbb{Z}$ and let $M = R$. Then $\{0,2,3,4\}$ is the set of all torsion elements of M, but this set is not stable under addition : $2+3=5$ is not a torsion element of M!

Proposition 163. *For all rings R, and for all elements $r \in R \setminus \{0\}$ which is not a zero divisor, there is an isomorphism:*

$$ \mathrm{Tor}_1^R(R/rR, N) \simeq {}_rN\ . $$

If R is a domain with fraction field F there is an isomorphism:

$$ \mathrm{Tor}_1^R(F/R, N) \simeq t(N)\ . $$

11 Universal coefficient formulas

Theorem 164. *Assume that R is a PID, and let $R \to S$ be a morphism of rings. For all complexes C of free R-modules, we have short exact sequences:*

$$0 \to H_i(C) \otimes_R S \to H_i(C \otimes_R S) \to \mathrm{Tor}_1^R(H_{i-1}(C), S) \to 0 .$$

The first map in the short exact sequence is given by sending $[z] \otimes s$ to $[z \otimes s]$. This short exact sequence is natural with respect to C. It splits, non-naturally with respect to C.

Example 165. Assume that $R = \mathbb{Z}$, and $S = \mathbb{Z}/n\mathbb{Z}$. If C is the complex $\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}$ concentrated in homological degrees 0 and 1 then $H_1(C) = 0$ and $H_0(C) = \mathbb{Z}/n\mathbb{Z}$. The universal coefficient theorem gives isomorphisms:

$$\begin{aligned} &\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq H_0(C \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) , \\ &H_1(C \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} . \end{aligned}$$

Yhis agrees with the direct computation of the homology of $C \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ which is nothing but the complex $\mathbb{Z}/n\mathbb{Z} \xrightarrow{0} \mathbb{Z}/n\mathbb{Z}$ concentrated in homological degrees 0 and 1.

The universal coefficient theorem has the following application for singular homology, which shows that homology with integral coefficients allows to compute the homology with any coefficient ring S.

Corollary 166. *Let X be a topological space. For all rings S, there is a split exact sequence:*

$$0 \to H_i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} S \to H_i(X, S) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{i-1}(X, \mathbb{Z}), S) \to 0 .$$

There is an analogous statement involving Hom_R instead of $\otimes_R$.

Theorem 167. *Assume that R is a PID, and let $R \to S$ be a morphism of rings. For all complexes C of free R-modules, we have short exact sequences:*

$$0 \to \mathrm{Ext}_R^1(H_{i-1}(C), S) \to H^i(\mathrm{Hom}_R(C, S)) \to \mathrm{Hom}_R(H_i(C), S) \to 0 .$$

This short exact sequence is natural with respect to C. It splits, non-naturally with respect to C.

Corollary 168. *Let X be a topological space. For all rings S, there is a split short exact sequence:*

$$0 \to \mathrm{Ext}_R^1(H_{i-1}(X, \mathbb{Z}), S) \to H^i(X, S) \to \mathrm{Hom}_R(H_i(X, \mathbb{Z}), S) \to 0 .$$

Part III
Homology of groups

12 Basic notions on representations

12.1 Group algebras and representations

Definition 169. Let G be a group and let $\Bbbk$ be a commutative ring.

The *group algebra* $\Bbbk G$ is the free $\Bbbk$-module with basis $(e_g)_{g\in G}$. Thus an element of $\Bbbk G$ is a sum $\sum_{g\in G}\lambda_g e_g$ where only a finite number of λ_g are nonzero. The product of two elements of $\Bbbk G$ is defined by:

$$\left(\sum_{g\in G}\lambda_g e_g\right)\left(\sum_{h\in G}\lambda'_h e_h\right) = \sum_{(g,h)\in G^2}\lambda_g\lambda'_h e_{gh}\;.$$

The category $\Bbbk G-\mathrm{Mod}$ is called the *category of $\Bbbk$-linear representations of G*.

Proposition 170. *A left $\Bbbk G$-module is the same as a $\Bbbk$-module M together with a left action of G on M by $\Bbbk$-linear maps. A morphism of left $\Bbbk G$ modules is the same as a $\Bbbk$-linear map $f : M \to M'$ which is G-equivariant, that is, $f(gm) = gf(m)$ for all $m \in M$ and all $g \in G$.*

Examples 171. 1. **The (left) regular representation of G : $\Bbbk G$.**

The *(left) regular representation* is the $\Bbbk$-module $\Bbbk G$, with G-action:

$$h\cdot\left(\sum_{g\in G}\lambda_g e_g\right) = \sum_{g\in G}\lambda_g e_{hg}\;.$$

Lemma 172. *The (left) regular representation $\Bbbk G$ is a projective generator of* $\Bbbk G-\mathrm{Mod}$.

2. **The trivial representations.**

   A *trivial representation* is a $\Bbbk$-module M, together with the action given by $gm = m$ for all $g \in G$ and all $m \in M$.

   A $\Bbbk$-module M with trivial action may be denoted simply by M or by M^{triv} if we want to emphasize that the action is trivial.

3. **The dual representation: $V^\sharp$.**

   The dual of a representation V is the $\Bbbk$-module $V^\sharp = \mathrm{Hom}_\Bbbk(V,\Bbbk)$ with G-action given by:
   $$(gf)(v) := f(g^{-1}v)\;.$$

4. **The representation** $\mathrm{Map}(X, M)$**.**

   If X is a set with left action of G and if W is a $\Bbbk$-module, then we can endow the $\Bbbk$-module $\mathrm{Map}(X, W)$ of maps from X to M with an action of G by:
$$(gf)(x) = f(g^{-1}x) \ .$$

   Lemma 173. *Consider G as a set with G-action given by left translations. Let J be an injective cogenerator of $\Bbbk-\mathrm{Mod}$. Then $\mathrm{Map}(G, J^{\mathrm{triv}})$ is an injective cogenerator of $\Bbbk G-\mathrm{Mod}$.*

Remark 174. Without further notification, our representations will be *left* representations, i.e. left $\Bbbk G$-modules. Sometimes we will also need *right* representations, i.e. right $\Bbbk G$-modules. Observe that every left representation V gives rise to a right representation with the same underlying $\Bbbk$-module, and with right action given by $vg := g^{-1}v$. A $\Bbbk$-linear morphism $f : V \to W$ is equivariant for the left action of G if and only if it is equivariant for the right action of G.

12.2 Tensor product and Hom

Definition 175. Let V and W be two $\Bbbk$-linear representations of G. The $\Bbbk$-modules $V \otimes_\Bbbk W$ and $\mathrm{Hom}_\Bbbk(V, W)$ are representations of G, with G-action:
$$g(v \otimes w) := (gv) \otimes (gw) \ , \qquad\qquad (gf)(v) := gf(g^{-1}v) \ .$$

Example 176. Consider the $\Bbbk$-module $M_n(\Bbbk)$ of $n \times n$-matrices as a $GL_n(\Bbbk)$ representation, with $GL_n(\Bbbk)$ acting by conjugation. Then $M_n(\Bbbk) \simeq \mathrm{Hom}_\Bbbk(\Bbbk^n, \Bbbk^n)$ where $\Bbbk^n$ is the standard representation of $GL_n(\Bbbk)$, i.e. $gv := g(v)$.

Proposition 177. *There is a canonical morphism of representations, which is an isomorphism if V or W are free of finite rank as $\Bbbk$-modules :*
$$\begin{array}{ccc} V^\sharp \otimes_\Bbbk W & \to & \mathrm{Hom}_\Bbbk(V, W) \ . \\ f \otimes w & \mapsto & (v \mapsto f(v)w) \end{array}$$

Proposition 178. *For all representations V there is an isomorphism of representations:*
$$V \otimes_\Bbbk \Bbbk G \simeq V^{\mathrm{triv}} \otimes_\Bbbk \Bbbk G$$

Corollary 179. *If V is projective as a $\Bbbk$-module and P is a projective representation of G, then $V \otimes_\Bbbk P$ is a projective representation. In particular, the tensor product of projective representations is projective.*

12.3 Invariants and coinvariants

Definition 180. Let V be a $\Bbbk$-linear representation of G. The $\Bbbk$-*module of invariants* is the submodule $V^G \subset V$ defined by:

$$V^G = \{v \in V \mid gv = v \quad \forall g \in G\}\ .$$

The $\Bbbk$-*module of coinvariants* is the quotient module V_G defined by:

$$V_G = \frac{V}{\langle gm - m\rangle}\ .$$

Example 181. Let $\Bbbk$ be a commutative ring and let $M_n(\Bbbk)$ denote the $\Bbbk$-module of $n \times n$-matrices with conjugation action of $GL_n(\Bbbk)$. Then $M_n(\Bbbk)^{GL_n(\Bbbk)}$ is a free $\Bbbk$-module of rank one, generated by the identity matrix, and $(M_n(\Bbbk)^\sharp)^{GL_n(\Bbbk)}$ is a free $\Bbbk$-module of rank one, generated by the trace map.

Proposition 182. *We have equalities:*

$$\mathrm{Hom}_\Bbbk(V,W)^G = \mathrm{Hom}_{\Bbbk G}(V,W)\ , \quad (V \otimes_\Bbbk W)_G = \overline{V} \otimes_{\Bbbk G} W\ ,$$

where $\overline{V}$ denotes the right representation associated to the left representation V (i.e. the right representation with the same underlying $\Bbbk$-module, and with right action defined by $vg := g^{-1}v$). As a consequence, we have isomorphisms, natural with respect to V:

$$V^G \simeq \mathrm{Hom}_{\Bbbk G}(\Bbbk^{\mathrm{triv}}, V)\ , \quad V_G \simeq \Bbbk^{\mathrm{triv}} \otimes_{\Bbbk G} V\ .$$

12.4 Induction and coinduction

Let H be a subgroup of G. We have a restriction functor:

$$\mathrm{res}^G_H : \Bbbk G{-}\mathrm{Mod} \to \Bbbk H{-}\mathrm{Mod}$$

which sends a representation of G to the representation of H obtained by restricting the action to H. There are two functors going in the other direction, namely the induction and the coinduction:

$$\mathrm{ind}^G_H = \Bbbk G \otimes_{\Bbbk H} - : \Bbbk H{-}\mathrm{Mod} \to \Bbbk G{-}\mathrm{Mod}$$
$$\mathrm{coind}^G_H = \mathrm{Hom}_{\Bbbk H}(\Bbbk[G], -) : \Bbbk H{-}\mathrm{Mod} \to \Bbbk G{-}\mathrm{Mod}$$

Proposition 183. *Let H be a subgroup of G. Induction is exact, preserves projectives, and satisfies the following adjunction isomorphism, natural with respect to V and W:*

$$\mathrm{Hom}_{\Bbbk G}(\mathrm{ind}^G_H V, W) \simeq \mathrm{Hom}_{\Bbbk H}(V, \mathrm{res}^G_H W)\ .$$

Coinduction is exact, preserves injectives, and satisfies the following adjunction isomorphism, natural with respect to V and W:

$$\mathrm{Hom}_{\Bbbk H}(\mathrm{res}^G_H V, W) \simeq \mathrm{Hom}_{\Bbbk G}(V, \mathrm{coind}^G_H W)\ .$$

Proposition 184. *If H has finite index in G, there is an isomorphism natural with respect to V:*

$$\mathrm{ind}_H^G V \simeq \mathrm{coind}_H^G V \ .$$

13 Homology of groups

13.1 Homology of groups and Tor

Definition 185. Let G be a group, and let V be a (left) $\Bbbk$-linear representation of G. Far all nonnegative degree i, the homology of G with coefficients in V is defined by:

$$H_i(G,V) = \mathrm{Tor}_i^{\Bbbk G}(\Bbbk^{\mathrm{triv}}, V)\ .$$

The following properties are simply properties of Tor that we restate in our context.

1. There is a natural isomorphism: $H_0(G,V) \simeq V_G$.
2. Homology is additive: $H_i(G, \bigoplus_\alpha V_\alpha) = \bigoplus_\alpha H_i(G, V_\alpha)$
3. $H_i(G, \Bbbk G) = 0$ if $i > 0$,
4. Every short exact sequence of representations $0 \to V' \to V \to V'' \to 0$ induces a short exact sequence in homology:

$$\cdots \to H_i(G,V) \to H_i(G,V'') \xrightarrow{\partial} H_{i-1}(G,V') \to H_{i-1}(G,V) \to \dots\ .$$

Definition 185 shows homology of a group as a special case of Tor. Conversely Tor can be presented as a special case of homology.

Proposition 186. *Let G be a group, let V and W be two left representations of G and let $\overline{V}$ denote the right representation associated to V. Assume that V is flat as a $\Bbbk$-module. Then for all i, there is an isomorphism, natural with respect to V and W:*

$$H_i(G, V \otimes_\Bbbk W) \simeq \mathrm{Tor}_i^{\Bbbk G}(\overline{V}, W)\ .$$

Example 187 (groups with cardinal invertible in $\Bbbk$)**.** Let G be a finite group with cardinal $|G|$ invertible in $\Bbbk$. Then $\Bbbk^{\mathrm{triv}}$ is a direct summand of $\Bbbk G$. Consequently:

$$H_i(G,V) = \begin{cases} V_G & \text{if } i=0\ , \\ 0 & \text{if } i>0 \end{cases}$$

Example 188 (infinite cyclic groups)**.** Let C denote the infinite cyclic group, with generator g. There is a projective resolution:

$$0 \to \Bbbk C \xrightarrow{1-g} \Bbbk C \xrightarrow{\epsilon} \Bbbk$$

In particular for all representations V we have

$$H_i(G,V) = \begin{cases} V_G & \text{if } i=0\ , \\ V^G & \text{if } i=1\ , \\ 0 & \text{if } i>1\ . \end{cases}$$

Example 189 (finite cyclic groups)**.** Let C_n denote the finite cyclic group of order n, with generator g. There is a projective resolution:

$$\dots \xrightarrow{1+g+\dots+g^{n-1}} \Bbbk C_n \xrightarrow{1-g} \Bbbk C_n \xrightarrow{1+g+\dots+g^{n-1}} \Bbbk C_n \xrightarrow{1-g} \Bbbk C_n \xrightarrow{\epsilon} \Bbbk$$

In particular for all representations V we have

$$H_i(G,V) = \begin{cases} V_G & \text{if } i=0\ , \\ \mathrm{Ker}(N : V_G \to V^G) & \text{if } i \text{ is even and positive} \\ \mathrm{Coker}(N : V_G \to V^G) & \text{if } i \text{ is odd} \end{cases} ,$$

where N denotes the norm map, which sends $[v]$ to $\sum_{g\in G} gv$. For example, if V is a trivial representation, then $V_G = V^G = V$ and N is multiplication by $|G|$. Consequently:

$$H_i(G,V) = \begin{cases} V & \text{if } i=0\ , \\ V/|G|V & \text{if } i \text{ is even and positive}\ , \\ {}_{|G|}V & \text{if } i \text{ is odd}\ . \end{cases}$$

13.2 The standard complex

Definition 190. Let R be a $\Bbbk$-algebra, let M be a right R-module and let N be a left R-module. The *bar construction* $B(M,R,N)$ is the complex of $\Bbbk$-modules whose term of degree k is the $\Bbbk$-module $B_k(M,R,N)$ defined by:

$$B_k(M,R,N) = \begin{cases} 0 & \text{if } k<0 \\ M \otimes_\Bbbk \underbrace{R \otimes_\Bbbk \cdots \otimes_\Bbbk R}_{k \text{ terms}} \otimes_\Bbbk N & \text{if } k \geq 0 \end{cases}$$

and with differential given by the formula:

$$d(m\otimes r_1 \otimes\cdots\otimes r_k\otimes n) = \sum_{i=0}^{k} (-1)^k d_k(m\otimes r_1 \otimes\cdots\otimes r_k\otimes n)$$

where

$$d_i(m\otimes r_1 \otimes\cdots\otimes r_k\otimes n) = \begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_k \otimes n & \text{if } i=0, \\ m\otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_k \otimes n & \text{if } 0<i<k, \\ r_1 \otimes \cdots \otimes r_{k-1} \otimes r_k n & \text{if } i=k\ . \end{cases}$$

Remark 191. To check that $B(M,R,N)$ is indeed a complex, one checks that for $i<j$ we have $d_i d_j = d_{j-1} d_i$. Then the proof is similar to the proof that the singular chain complex of a topological space is indeed a chain complex. There is actually a notion hidden behind this similarity, namely the notion of a simplicial object. You may have a look e.g. at [Weibel, Chap 8] for more details relative to simplicial objects. The closely related notion of a semi-simplicial set is explained in section 14.4.

Proposition 192. *Assume that R and M are projective as $\Bbbk$-modules. Then for all R-modules M and N and all nonnegative integers i, we have*

$$\mathrm{Tor}_i^R(M,N) = H_i(B(M,R,N))\ .$$

Corollary 193. *For all $\Bbbk$-linear representations V, $H_i(G,V)$ is the homology of the standard complex $B(\Bbbk,\Bbbk G,V)$.*

Corollary 194. *For all groups G we have $H_1(G,\mathbb{Z}) = G_{\mathrm{ab}}$.*

13.3 Functoriality with respect to G

Let V be a left representation of G and let U be a right representation of G. Every morphism of groups $\alpha : H \to G$ induces a surjective morphism $\alpha_0 : U \otimes_{\Bbbk H} V \to U \otimes_{\Bbbk G} V$ such that $\alpha_0(u\otimes v) = u\otimes v$.

Definition 195. If (P,ϵ) is a projective resolution of $\Bbbk$ as a $\Bbbk H$-module, and if (Q,ϵ') is a projective resolution of $\Bbbk$ as a $\Bbbk G$-module, then for all representations V of G and for all morphisms $\alpha : H \to G$ there is a morphism of complexes

$$P\otimes_{\Bbbk H} V \xrightarrow{\overline{\mathrm{id}}\otimes \mathrm{id}_V} Q\otimes_{\Bbbk H} V \xrightarrow{\alpha_0} Q\otimes_{\Bbbk G} V$$

and we let $\alpha_i : H_i(H,V) \to H_i(G,V)$ be the induced application in homology of degree i. (Here $\overline{\mathrm{id}} : P \to Q$ is a lift of the identity map of $\Bbbk$ to the level of the resolutions).

Lemma 196. *Given morphisms of groups $K \xrightarrow{\alpha} H \xrightarrow{\beta} G$ we have $(\beta\circ\alpha)_i = \beta_i\circ\alpha_i$ for all i.*

Remark 197. In the definition of α_i, we have again denoted by V the restriction to H of the representation of G. It is an abuse of notation that we will often make. To be perfectly correct we should have used the heavy notation $\mathrm{res}^G_H V$, so that α_i is actually a map $H_i(H,\mathrm{res}^G_H V) \to H_i(G,V)$. In fact, even the heavy notation $\mathrm{res}^G_H V$ is not completely precise, because α does not appear whereas there may be many morphisms of groups $\alpha : H \to G$. In the situations where we want to be very precise, we will denote by $\alpha^* V$ the representation of H obtained by restriction along α. Here is a situation where we *need* to be precise to avoid confusion: if $\alpha : G \to G$ is a morphism of groups, definition 195 yields morphisms of $\Bbbk$-modules:

$$\alpha_i : H_i(G,\alpha^* V) \to H_i(G,V)\ .$$

Functoriality with respect to G is often used to compute the homology of G from the homology of its subgroups (which is simpler to understand). In the remainder of this section we explain two basic examples of this approach.

13.3.1 Schapiro's lemma

Let G be a group, let H be a subgroup, and let V be a representation of H. There is an H-equivariant morphism, natural with respect to the H-representation V

$$\begin{array}{rccc}\eta_V: & V & \to & \mathrm{ind}_H^G V = \Bbbk G\otimes_{\Bbbk H} V \\ & v & \mapsto & 1\otimes v\end{array}$$

Theorem 198 (Schapiro's lemma)**.** *Let G be a group, let H be a subgroup, and let V be a representation of H. The following composition is an isomorphism for all i:*

$$H_i(H,V)\xrightarrow{H_i(H,\eta_V)} H_i(H,\mathrm{ind}_H^G V)\xrightarrow{\mathrm{incl}_i} H_i(G,\mathrm{ind}_H^G V)\;.$$

One says that a ring R has *finite (left) homological dimension* if there is an integer d such that every left R-module has a projective resolution of the form

$$0\to P_d\to P_{d-1}\to\cdots\to P_0\xrightarrow{\epsilon} M\to 0\;.$$

If such a d does not exist, then R has *infinite (left) homological dimension.* The following result is a direct consequence of Schapiro's lemma.

Corollary 199. *Let $\Bbbk$ be a field of characteristic $p>0$. If G has an element of order p, then $\Bbbk G$ has infinite homological dimension.*

13.3.2 The Cartan-Eilenberg formula for normal subgroups

Let H be a normal subgroup of G. Then for all $g\in G$, conjugation by g defines an isomorphism of groups:

$$\begin{array}{rccc}c_g: & H & \to & H \\ & h & \mapsto & c_g(h)=ghg^{-1}\end{array}\;.$$

Now if V is a representation of G, then for all $g\in G$, the action of g on V defines an equivariant isomorphism:

$$\begin{array}{rccc}g\cdot: & V & \to & c_g^*V \\ & v & \mapsto & gv\end{array}\;.$$

For all integers i, we denote by $g\star: H_i(H,V)\to H_i(H,V)$ the isomorphism of $\Bbbk$-modules given by the composition:

$$g_\star: H_i(H,V)\xrightarrow[\simeq]{H_i(H,g\cdot)} H_i(H,c_g^*V)\xrightarrow[\simeq]{(c_g)_i} H_i(H,V)\;.$$

Proposition 200. *The map $x\mapsto g\star x$ defines a $\Bbbk$-linear action of G on $H_i(H,V)$. Moreover the elements $h\in H$ act trivially through this action, so that the action factors as an action of the quotient group G/H on $H_i(H,V)$. Moreover, the map $H_i(H,V)\xrightarrow{\mathrm{incl}_i} H_i(G,V)$ passes to the quotient and induces a map*

$$\overline{\mathrm{incl}}_i: H_i(H,V)_{G/H}\to H_i(G,V)\;.$$

Proof. Let (P,ϵ) be a projective resolution of $\Bbbk$ as a (right) representation of H. Then by its definition $g\star$ is induced by the morphism of chain complexes

$$P\otimes_{\Bbbk H}V\xrightarrow{\overline{\mathrm{id}}\otimes(g\cdot)}c_g^*P\otimes_{\Bbbk H}c_g^*V\xrightarrow[=]{(c_g)_0}P\otimes_{\Bbbk H}V\;,\qquad(*)$$

where $\overline{\mathrm{id}}:P\to c_g^*P$ is a lift of the identity of $\Bbbk$ to the level of the resolutions. If we choose P to be the restriction to H of a projective resolution of $\Bbbk$ as a (right) representation of G, then we may take $\overline{\mathrm{id}}$ to be given by the right action of g^{-1}, i.e. $\overline{\mathrm{id}}(x)=xg^{-1}$. With this choice, the morphism $g\star$ is simply given on the level of chain complexes by the map

$$\begin{array}{ccc}P\otimes_{\Bbbk H}V&\to&P\otimes_{\Bbbk H}V\\x\otimes v&\mapsto&xg^{-1}\otimes gv\end{array}\;.$$

with this expression the statement of the proposition is easy to prove. □

Theorem 201 (Cartan-Eilenberg formula)**.** *Assume that V is a representation of G, and that H is a normal subgroup of G such that $|G:H|$ is invertible in $\Bbbk$. Then the morphism*

$$\overline{\mathrm{incl}}_i:H_i(H,V)_{G/H}\to H_i(G,V)\;.$$

is an isomorphism of $\Bbbk$-modules.

The proof of the Cartan-Eilenberg formula relies on the following lemma (which is a more precise version of proposition 78).

Lemma 202. *Let K be a group, let C be a complex of $\Bbbk K$-modules. Then C_K is a quotient complex of C considered as a complex of $\Bbbk$-modules, and the morphism $H_i(\pi):H_i(C)\to H_i(C_K)$ induces a comparison morphism*

$$H_i(C)_K\to H_i(C_K)\;.$$

If $|K|$ is invertible in $\Bbbk$, this comparison morphism is an isomorphism.

Example 203. Let us consider the symmetric group $\mathfrak{S}_3$ on three letters, with normal subgroup C_3 and quotient $\mathfrak{S}_3/C_3\simeq C_2$. The Cartan-Eilenberg formula tells us that

$$H_i(\mathfrak{S}_3,\mathbb{F}_3)\simeq H_i(C_3,\mathbb{F}_3)_{C_2}\;.$$

We know that $H_i(C_3,\mathbb{F}_3)\simeq\mathbb{F}_3$, so in order to compute the homology of $\mathfrak{S}_3$, we only have to determine the action of C_2, hence of a transposition $\tau\in\mathfrak{S}_3$. For this purpose, we use the expression $(*)$ of the action of τ, together with the resolution of example 189. We have a diagram:

$$\begin{tikzcd}[column sep=small] P= & \Bbbk \arrow[d, "="] & \Bbbk C_3 \arrow[l, "\epsilon"'] \arrow[d, dashed, "="] & \Bbbk C_3 \arrow[l, "1-c"'] \arrow[d, dashed, "-c"] & \Bbbk C_3 \arrow[l, "N"'] \arrow[d, dashed, "-c"] & \Bbbk C_3 \arrow[l, "1-c"'] \arrow[d, dashed, "c^2"] & \Bbbk C_3 \arrow[l, "N"'] \arrow[d, dashed, "c^2"] \\ c_\tau^*P= & \Bbbk & \Bbbk C_3 \arrow[l, "\epsilon"'] & \Bbbk C_3 \arrow[l, "1-c^2"'] & \Bbbk C_3 \arrow[l, "N"'] & \Bbbk C_3 \arrow[l, "1-c^2"'] & \Bbbk C_3 \arrow[l, "N"'] \end{tikzcd}\;.$$

Hence we see that τ acts as $(-1)^i$ on $H_{2i-1}(C_3,\Bbbk)$ and $H_{2i}(C_3,\Bbbk)$. Therefore we obtain that

$i=$	0	1	2	3	4	5	6	7	8	9	$\dots$
$H_i(\mathfrak{S}_3,\mathbb{F}_3)=$	$\mathbb{F}_3$	0	0	$\mathbb{F}_3$	$\mathbb{F}_3$	0	0	$\mathbb{F}_3$	$\mathbb{F}_3$	0	$\dots$

Remark 204. The Cartan-Eilenberg formula has a generalization when the subgroup H is not normal in G. The proof of this generalized statement is not much harder, but this generalized statement is a bit more complicated to state. Indeed, the conjugation map $c_g : G \to G$ now takes H to the conjugate subgroup gHg^{-1}. Hence $g \in G$ does not act on $H_i(H,V)$ anymore, but induces an isomorphism between $H_i(H,V)$ and $H_i(gHg^{-1},V)$. So one has to generalize the meaning of coinvariants. With this new meaning, the Cartan-Eilenberg formula stays valid, i.e. there is an isomorphism

$$H_i(H,V)_{G/H} \simeq H_i(G,V)\ .$$

14 Homology of groups and topology

14.1 Recollections of covering spaces

Definition 205. A *covering map* is a continuous map $p : E \to X$ such that every $x \in X$ is contained in an open set U such that $p^{-1}U$ is a disjoint union of open sets U_i and the restriction maps $p_{|U_i} : U_i \to U$ are homeomorphisms. The space E is called a *covering space* of X, but the term covering space often also refers to the map p.

Examples 206.
1. The protopype of a covering map is the map $\mathbb{R} \to S^1$, $t \mapsto e^{i2\pi t}$.
2. Let Y be a topological space endowed with a free action by homeomorphisms of a group G. Assume that every $y \in Y$ is contained in an open set U such that $gU \cap U \neq \emptyset$ for all $g \in G$. Then the quotient map
$$q : Y \to Y/G$$
is a covering map. (Covering maps constructed in this way are called *Galois covering maps*)

One of the most important theorem of covering theory if the following lifting theorem. Given a continuous map $f : Y \to X$ and a covering $p : E \to X$, a *lifting of f* is a continuous map $\overline{f}$ such that the following diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\overline{f}"] \arrow[r, "f"'] & X \end{tikzcd} .$$

Recall also that a space Y is *locally arcwise connected* if for all $y \in Y$ and all open subset U containing y, one can find an arcwise connected neighborhood V of y which is contained in U. For a covering map $p : E \to X$ the base space X is locally arcwise connected if and only if the total space E is locally arcwise connected.

Theorem 207. *Let $p : E \to X$ be a covering map, let $f : Y \to X$ be a continuous map. Assume that Y is arcwise connected, and locally arcwise connected. Let us fix basepoints e, x, y such that $p(e) = x$ and $f(y) = x$.*

Then f admits a lifting $\overline{f}$ such that $\overline{f}(y) = e$ if and only if the image of $\pi_1(f) : \pi_1(Y, y) \to \pi_1(X, x)$ is contained in the image of $\pi_1(p) : \pi_1(E, e) \to \pi_1(X, x)$. Moreover, if the lifting $\overline{f}$ exists, then it is unique.

Definition 208. A covering $p : \widetilde{X} \to X$ is called *universal* if $\widetilde{X}$ is locally arcwise connected and simply connected.

If $p : (\widetilde{X}, \widetilde{x}) \to (X, x)$ is a universal covering, it follows from the lifting theorem 207 that it satisfies the following universal property. For all coverings $q : (E, e) \to (X, x)$ there is a unique continuous map $p' : (\widetilde{X}, \widetilde{x}) \to (E, e)$ making the following triangle commute:

$$\begin{tikzcd} (\widetilde{X}, \widetilde{x}) \arrow[rr, dashed, "p'"] \arrow[dr, "p"'] & & (E, e) \arrow[dl, "q"] \\ & (X, x) & \end{tikzcd}$$

One can deduce from the universal property that a universal covering is unique up to homeomorphism. One also deduces from the universal property the following structure result.

Proposition 209. *If $p : \widetilde{X} \to X$ is a universal covering, then it has the form of example 206 2), and $G \simeq \pi_1(X, x)$.*

14.2 $K(G, 1)$ and group homology

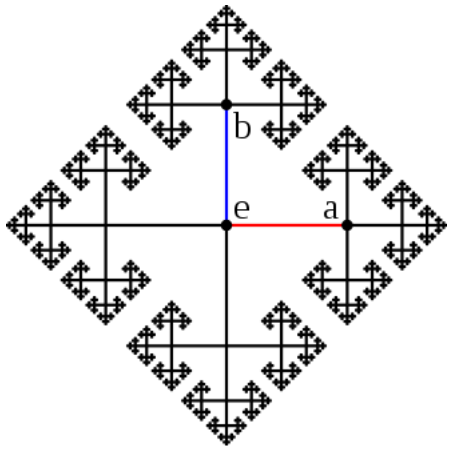
Definition 210. Let G be a group. A topological space X is called a $K(G, 1)$ space if (i) it is arcwise connected and locally arcwise connected, (ii) $\pi_1(X, x) \simeq G$ and (iii) X admits a universal covering $p : \widetilde{X} \to X$ such that $\widetilde{X}$ is contractible.

Remark 211. The "1" in the name "$K(G, 1)$" refers to the fact that $\pi_1(X, x)$ is involved in the definition. See section 15 for more details.

Examples 212.

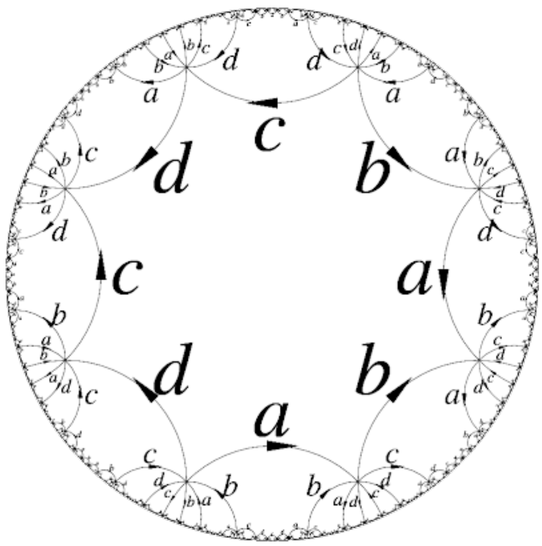
1. The circle S^1 is a $K(\mathbb{Z}, 1)$. Indeed the universal covering is $\mathbb{R} \to S^1$, $t \mapsto e^{2i\pi t}$ and $\mathbb{R}$ is contractible.
2. If $n \neq 1$, the sphere S^n is not a $K(G, 1)$.
3. The wedge of two circles $S^1 \vee S^1$ is a $K(F_2, 1)$. Indeed, the universal covering is $p : C \to S^1 \vee S^1$ where C is the Cayley graph of F_2, which is contractible (it is a tree).

   Here is a picture (taken from wikipedia) of the Cayley graph of F_2. It is the topological graph whose vertices are labelled by the elements of F_2, and which has one edge between vertex x and vertex y if there is $\ell \in \{a, b, a^{-1}, b^{-1}\}$ such that $y = x\ell$.



4. The torus $T^2 \simeq S^1 \times S^1$ is a $K(\mathbb{Z}^2, 1)$. Indeed the universal covering is $\mathbb{R}^2 \to S^1 \times S^1$, $(s,t) \mapsto (e^{2i\pi s}, e^{2i\pi t})$ and $\mathbb{R}^2$ is contractible.

5. More generally the oriented compact surface S_g of positive genus g is a $K(G,1)$ with $G = F_{2g}/[a_1,b_1] \cdot \cdots \cdot [a_g,b_g]$. Indeed, its universal covering space is the Poincaré disk D^2:



6. Let $f : S^1 \to S^3$ be a knot (i.e. a smooth embedding). Then $S^3 \setminus \operatorname{Im} f$ is a $K(G,1)$. (This is a nontrivial theorem of 3-manifold theory).

Theorem 213. *Let X be a $K(G,1)$. Then $C^{\mathrm{sing}}(\widetilde{X}, \Bbbk)$ is a projective resolution of $\Bbbk^{\mathrm{triv}}$ as a $\Bbbk G$-module, and there is an isomorphism of $\Bbbk$-modules:*

$$H_i^{\mathrm{sing}}(X, \Bbbk) \simeq H_i(G, \Bbbk) \ .$$

Comment 214 (Singular homology with local coefficients)**.** Assume that X is an arcwise connected space, with $\pi_1(X,x)\simeq G$ and which admits a universal covering. Then X is maybe not a $K(G,1)$. However, $C^{\mathrm{sing}}(\widetilde{X},\Bbbk)$ is still a complex of projective $\Bbbk G$-modules, and for all left representations V of G, one may consider the complex of $\Bbbk$-modules

$$C^{\mathrm{sing}}(\widetilde{X},\Bbbk)\otimes_{\Bbbk G}V\;.$$

The homology of this complex is called the *singular homology of X with local coefficients V*, and denoted by $H_i^{\mathrm{sing}}(X,V)$. This generalization of singular homology appears in several places of algebraic topology.

The previous theorem extends to the setting of singular homology with local coefficients. Namely, if X is a $K(G,1)$ then for all degrees i there is an isomorphism

$$H_i^{\mathrm{sing}}(X,V)\simeq H_i(G,V)\;.$$

14.3 Projective resolutions of $\Bbbk^{\mathrm{triv}}$ via G-complexes

Definition 215. A *free G-complex* is a CW-complex Y, equipped with a free action of the group G which freely permutes the open cells of Y.

Theorem 216. *If Y is a* free G-complex*, then $C^{\mathrm{cell}}(Y)$ is a complex of projective $\Bbbk G$-modules. Moreover, assume that Y is arcwise connected, and that there is an $n\in]0,+\infty]$ such that $H_i(Y,\Bbbk)=0$ for $0<i<n$. Then $C^{\mathrm{cell}}(Y)$ is a projective resolution of $\Bbbk^{\mathrm{triv}}$ up to degree $n-1$, i.e. we have an exact complex:*

$$C_n^{\mathrm{cell}}(Y,\Bbbk)\to\cdots\to C_1^{\mathrm{cell}}(Y,\Bbbk)\to C_0^{\mathrm{cell}}(Y,\Bbbk)\xrightarrow{\epsilon}\Bbbk^{\mathrm{triv}}\to 0,$$

and in addition, for all $i<n$ we have

$$H_i(Y/G)\simeq H_i(G,\Bbbk)\;.$$

Examples 217.

1. $Y=S^n$ with 2 cells in each dimension. This is a free G-complex, for $G=C_2$ with antipodal action. The associated cellular complex coincides with the projective resolution given in example 189 up to degree $n-1$.

2. $Y=\mathbb{R}$ with action of C by translation. This is a free G-complex. The associated cellular complex coincides with the projective resolution given in example 188.

Corollary 218. *Let X be an arcwise connected CW-complex of dimension n with contractible universal covering. Then $\pi_1(X,*)$ contains no element of finite order. In particular, the group $\pi_1(X,*)$ is either trivial or infinite.*

(This corollary applies in particular to arcwise connected manifolds of dimension n. It shows that all the nice examples of $K(G,1)$ coming from geometry have infinite groups G in a strong sense.)

14.4 Semi-simplicial sets

In this section, we briefly introduce *semi-simplicial sets*[3], which we will use later to construct $K(G,1)$ spaces. Semi-simplicial sets are a generalization of geometric simplicial complexes. They can be thought of as combinatorial instructions to build CW-complexes by pasting standard simplices together.

Definition 219. A *semi-simplicial set S* is a family of sets $(S_n)_{n\geq 0}$ equiped with maps $d_i : S_n \to S_{n-1}$, $0 \leq i \leq n$ such that:

$$d_i \circ d_j = d_{j-1} \circ d_i \qquad \text{if } i < j.$$

A *morphism of semi-simplicial sets* $f : S \to T$ is a collection of maps $f_n : S_n \to T_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ for all i and all n.

Example 220.
1. If K is a geometric simplicial complex (see definition 62) gives rise to a semi-simplicial set $\Delta(K)$ such that $\Delta(K)_n$ is the set of n-simplices of K and $d_i : \Delta(K)_n \to \Delta(K)_{n-1}$ sends every n-simplex onto its i-th face.

2. There is a semi-simplicial set such that $S_i = \emptyset$ if $i > 1$, S_1 and S_0 have exactly one element and the d_0 and d_1 are equal to the unique map $S_1 \to S_0$.

Definition 221. The *realization of a semi-simplicial set S* is the topological space

$$|S| = \frac{\bigsqcup_{n\geq 0} S_n \times \Delta^n}{(d_i s, t) \sim (s, \epsilon^i t)}$$

where S_n is considered as a discrete set (i.e. $S_n \times \Delta^n$ is homeomorphic to the disjoint union of copies of the standard n-simplex Δ^n indexed by the elements of S_n) and where $\epsilon^i : \Delta^{n-1} \to \Delta^n$ is the map which identifies the standard $(n-1)$-simplex with the i-th face of the standard n-simplex (see definition 40).

If $f : S \to T$ is a morphism of semi-simplicial sets, the continuous map

$$\bigsqcup_{n\geq 0} f_n \times \mathrm{id}_{\Delta^n} : \bigsqcup_{n\geq 0} S_n \times \Delta^n \to \bigsqcup_{n\geq 0} T_n \times \Delta^n$$

induces a continuous map $|f| : |S| \to |T|$ on the quotients, which is called the *realization of f*.

Example 222.
1. Let K be a geometric simplicial complex. Then there is a homeomorphism:

$$|\Delta(K)| \simeq |K| = \bigcup_{\sigma\in K} \sigma\ .$$

[3]The terminology and notation varies a lot in the literature. Semi-simplicial sets are also called Δ-sets, Δ_+-sets, Δ_{inj}-sets...

2. If S is the semi-simplicial set such that $S_i = \emptyset$ for $i > 1$ and S_1 and S_0 have one element, there is a homeomorphism $|S| \simeq S^1$.

The realization of a semi-simplicial set produces a CW-complex. To prove this, we need to introduce characteristic maps.

Definition 223. For all $s \in S_n$, we define the *characteristic map* $f_s : \Delta^n \to |S|$ be the continuous map given by the composition

$$f_s := \Delta_n \xrightarrow{t \mapsto (s,t)} \bigsqcup_{n \geq 0} S_n \times \Delta_n \twoheadrightarrow |S| \ .$$

By definition of the disjoint union topology and of the quotient topology, a subset $U \subset |S|$ is open if and only if for all s, $f_s^{-1}(U)$ is opened in Δ^n. Also, the equivalence relation defining $|S|$ only involves the boundary of Δ^n. Hence the restriction of f_s to $\Delta^n \setminus \partial\Delta^n$ is injective.

Proposition 224. *Let S be a semi-simplicial set and let $S_{\leq n}$ denote the semi-simplicial set with*

$$(S_{\leq n})_i = \begin{cases} S_i & \text{if } i \leq n, \\ \emptyset & \text{if } > n \end{cases}$$

Then $|S|$ has a CW-complex structure with n-th skeleton $|S_{\leq n}|$, and with characteristic maps f_s, i.e. for all positive n we have a pushout square:

$$\begin{tikzcd} \sqcup_{s \in S_n} \partial\Delta^n \arrow[r, "\sqcup f_{s|\partial\Delta^n}"] \arrow[d, hook] & {|S_{\leq n-1}|} \arrow[d, hook] \\ \sqcup_{s \in S_n} \Delta^n \arrow[r, "\sqcup f_s"] & {|S_{\leq n}|} \end{tikzcd} .$$

14.5 Construction of $K(G,1)$-spaces

Definition 225. For all groups G, we can define a semi-simplicial set EG with $EG_n = G^{n+1}$ and $d_i : EG_n \to EG_{n-1}$ is defined by

$$d_i(g_0, g_1, \dots, g_n) = (g_0, \dots, g_{i-1}, g_{i+1}, \dots g_n) \ .$$

Every $g \in G$ defines a morphism of simplicial sets $EG \to EG$ which sends the n-simplex $(g_0, g_1, \dots, g_n)$ to $(gg_0, gg_1, \dots, gg_n)$.

Theorem 226. *The topological space $|EG|$ is a contractible free G-CW-complex. In particular, $|EG|/G$ is a $K(G,1)$ space.*

15 Vista: homotopy groups and EML spaces

We defined $K(G,1)$-spaces as spaces with fundamental group G and contractible universal cover. In this section, we give an equivalent definition of $K(G,1)$-spaces, relying on the notion of homotopy groups.

15.1 Homotopy groups

Given two pairs of topological spaces (X,A) and (Y,B), we denote by $\mathrm{Map}(X,A;Y,B)$ the set of maps $f : X \to Y$ such that $f(A) \subset B$. We define an equivalence relation $\sim$ on this set by $f \sim g$ if and only if there is a homotopy $H : X \times [0,1] \to Y$ between f and g such that $H(A,t) \subset B$ for all $t \in [0,1]$. We use the following notation for the quotient set:

$$[(X,A),(Y,B)] := \frac{\mathrm{Map}(X,A;Y,B)}{\sim}\,.$$

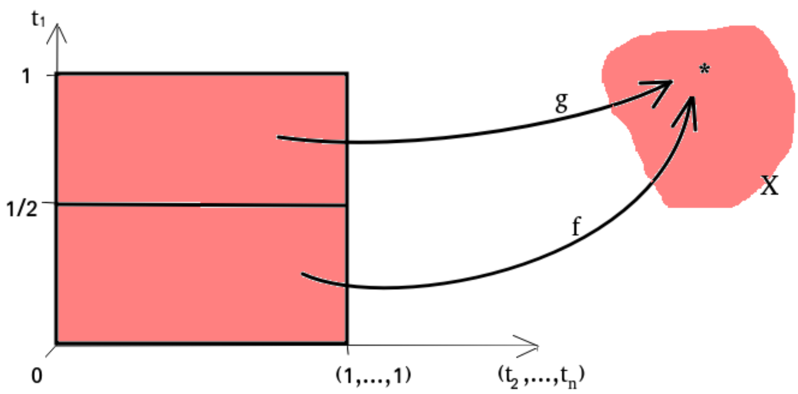
The equivalence class of a map f is denoted by $[f]$. If $A = B = \emptyset$ then we recover the homotopy classes of maps from X to Y.

Definition 227. Let $(X,*)$ be a pointed topological space. For all $n \geq 1$, let $C^n = [0,1]^n$ denote the n-th dimensional cube. The n-th homotopy group $\pi_n(X,*)$ is the set:

$$\pi_n(X,*) = [\,(C^n, \partial C^n)\,,\,(X,*)\,]$$

equipped with the multiplication $[f]\cdot[g] = [fg]$, where $fg : C^n \to Y$ is the concatenation of f and g along the first coordinate:

$$(fg)(t_1,\dots,t_n) = \begin{cases} f(2t_1,t_2,\dots,t_n) & \text{if } t \leq \frac{1}{2}, \\ g(2t_1-1,t_2,\dots,t_n) & \text{if } t \geq \frac{1}{2}. \end{cases}$$



One should make small verifications to ensure definition 227 is consitent, that is, the group operation is well-defined, associative, with a unit. The unit element is the class of the constant map $C^n \to X$ with value $*$.

If $n = 1$, definition 227 is the usual definition of the fundamental group. Hence for $n \geq 2$, the homotopy groups can be thought of as higher dimensional analogues of the fundamental group. The basic properties of the fundamental group generalize to higher homotopy groups:

1. If X is arcwise connected, the group $\pi_n(X, *)$ does not depend on the basepoint: different basepoints yield isomorphic groups.

2. Every pointed map $\phi : (X, *) \to (Y, *)$ induces a group morphism $\pi_n(\phi) : \pi_n(X, *) \to \pi_n(Y, *)$ such that $\phi([f]) = [\phi \circ f]$.

3. If $\phi : X \to Y$ is a homotopy equivalence, then for all $* \in X$ the map $\pi_n(\phi) : \pi_n(X, *) \to \pi_n(Y, \phi(*))$ is an isomorphism of groups.

A small surprise with higher homotopy groups is the following additional basic fact, which is of course not true if $n = 1$.

Proposition 228. *For all $n \geq 2$ and for all pointed spaces $(X, *)$, the group $\pi_n(X, *)$ is abelian.*

In view of proposition 228, one may think that higher homotopy groups are nicer and easier to compute than the fundamental group. But this is not the case. The reason is that the powerful Van Kampen theorem, which is used to compute the fundamental groups of many spaces, has no equivalent in higher dimensions. As an illustration of the difficulty of computing homotopy groups, at the current time we don't know all the homotopy groups of the spheres S^d, $d \geq 2$, although we computations in low dimension n, as well as good qualitative information in all dimensions. We refer the reader e.g. to chap 0 of Neisendorfer's book "Algebraic methods in unstable homotopy theory" for more details on this subject.

To finish our quick tour of homotopy groups, let us mention two fundamental results relating homotopy groups and singular homology. Firstly, for all $n \geq 1$, the integral homology of the pair $(C^n, \partial C^n)$ in degree n is equal to $\mathbb{Z}$. Choose a generator c of this homology group. For all pointed spaces $(X, *)$, the *Hurewicz map* is the map:

$$h_n : \pi_n(X, *) \to H_n(X, *; \mathbb{Z}) = H_n(X; \mathbb{Z})$$

such that $h_n([f]) = H_n(f)(c)$. One checks that this map is a group morphism, natural with respect to X.

Theorem 229 (Hurewicz)**.** *Assume that $(X, *)$ is a pointed, arcwise connected space.*

1. *The map h_1 vanishes on the commutator subgroup of $\pi_1(X,*)$, and induces an isomorphism of abelian groups:*

$$\pi_1(X,*)_{\mathrm{ab}} \simeq H_1(X,\mathbb{Z}) \;.$$

2. *If $n \geq 2$ and if $\pi_i(X,*) = 0$ for $i < n$ then h_i is an isomorphism in degrees $i \leq n$.*

If X is simply connected, the Hurewicz theorem is often used to compute the first nontrivial homotopy group since the homology groups of X are usually easier to compute than its homotopy groups.

Example 230. We have $\pi_n(S^d) = 0$ if $0 < n < d$ and $\pi_d(S^d) = \mathbb{Z}$.

Another fundamental theorem is the following algebraic criterion to recognize homotopy equivalences. In part I of the course, we have introduced homology groups as a tool to show that two spaces have different homotopy types. The Whitehead theorem allows us to use homology groups to prove that two spaces have *the same* homotopy type.

Theorem 231 (Whitehead)**.** *Let $\phi : X \to Y$ be a map between two arcwise connected CW-complexes, and let x be a basepoint in X. The following statements are equivalent.*

1. *For all $n \geq 1$, $\pi_n(\phi) : \pi_n(X,x) \to \pi_n(Y,\phi(x))$ is an isomorphism.*
2. *The map $\pi_1(\phi) : \pi_1(X,x) \to \pi_1(Y,\phi(x))$ is an isomorphism, and for all $n \geq 1$ the map $H_n(\phi) : H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is an isomorphism.*
3. *The map $f : X \to Y$ is a homotopy equivalence.*

Observe that the Whitehead theorem does not say that two spaces with the same homotopy or homology groups are homotopy equivalent: one really needs a map between the two spaces. However, there is one situation where such a map always exists, namely when we want to compare a space X with the point $*$. In that case, the Whitehead theorem applied to the unique map $\phi : X \to *$ gives the following result.

Corollary 232. *Let $(X,*)$ be a pointed arcwise connected CW-complex. Then: $\pi_n(X,*)$ is trivial for all $n \geq 1 \Leftrightarrow \pi_1(X,*)$ and the $H_n(X;\mathbb{Z})$ for all $n \geq 1$ are trivial $\Leftrightarrow X$ is contractible.*

15.2 $K(G,1)$-spaces and EML spaces

We are now going to characterize $K(G,1)$-spaces with homotopy groups. The characterization relies on the following lemma, which is an easy exercise[4] on the lifting properties of coverings.

[4]For this exercise, you may use that $\pi_n(X,*)$ is isomorphic to the set $[\,(S^n,*)\,,\,(X,*)\,]$ of pointed homotopy classes of maps $S^n \to X$, and that $\pi_1(S^n)$ is trivial if $n \geq 2$.

Lemma 233. *Let $p : E \to B$ be a covering map, and let $* \in E$. Then $\pi_n(p) : \pi_n(E,*) \to \pi_n(B,*)$ is an isomorphism of groups for all $n \geq 2$.*

Let X be a $K(G,1)$-space with contractible universal cover $\widetilde{X}$. Then all the homotopy groups of $\widetilde{X}$ are trivial. By applying the previous lemma to the covering map $p : \widetilde{X} \to X$ we obtain

$$\pi_n(X,*) = \begin{cases} G & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Conversely, assume that X is an arcwise connected space with trivial higher homotopy groups and with fundamental group equal to G. If X is a CW-complex then its universal cover $\widetilde{X}$ is a CW-complex, and by applying lemma 233 and corollary 232, we obtain that $\widetilde{X}$ is contractible, hence that X is a $K(G,1)$ space. The next result summarizes this discussion.

Proposition 234. *Let X be an arcwise connected CW-complex. Then X is a $K(G,1)$-space if and only if $\pi_1(X,*) \simeq G$ and $\pi_n(X,*) = 0$ for all $n \geq 2$.*

Remark 235. The circle S^1 is a $K(\mathbb{Z},1)$, hence it has fundamental group isomorphic to $\mathbb{Z}$ and trivial higher homotopy groups. Among the spheres S^d, $d \geq 1$, the circle is the only one whose homotopy groups are completely known.

The characterization given in proposition 234 justifies the notation $K(G,1)$ and lead to the following generalization of the concept.

Definition 236. Let $n \geq 1$ and let G be a group, which is assumed to be abelian if $n \geq 2$. A CW-complex X is called a *$K(G,n)$-space* if it is arcwise connected and if its homotopy groups $\pi_k(X,*)$ are trivial if $k \neq n$ and $\pi_n(X,*) \simeq G$.

The $K(G,n)$-spaces were first studied by Eilenberg and MacLane, and for this reason, they are often refered to as *Eilenberg-MacLane spaces* or *EML spaces* for short.

In classical homotopy theory, there is a kind of symmetry between homotopy groups and homology groups. On the one hand, spheres are the topological spaces with the simplest homology groups, but their homotopy groups are very complicated. On the other hand, Eilenberg-MacLane spaces are the topological spaces with the simplest homotopy groups, but their have complicated homology groups[5]

[5] The symmetry is not perfect though: whereas the homotopy groups of spheres are still mysterious, the homology groups of Eilenberg-Mac Lane spaces $K(G,n)$ for G abelian, were completely computed by Serre and Cartan in the 1950es, see `http://www.numdam.org/volume/SHC_1954-1955__7_1/`.

This symmetry can be pushed further. Spheres play a fundamental role in the study of topological spaces. Indeed, every CW-complex can be constructed as an increasing union of spaces, its skelettons X_n, $n \geq 0$, and the difference between the n-th skeleton and the $(n-1)$-st skeleton behaves like a wedge of spheres of dimension n:

$$X_n/X_{n-1} \simeq \bigvee_{n\text{-cells of } X} S^n \ .$$

This construction behaves particularly well with homology groups, in fact it is at the heart of the definition of the cellular chain complex, which computes the homology of X. But it is not adapted to the computation of homotopy groups, let alone because the homotopy groups of spheres are not known.

Likewise, Eilenberg-MacLane spaces play a fundamental role in the study of topological spaces. Indeed, every arcwise connected CW complex X can be decomposed as a tower of spaces

$$X^0 \twoheadleftarrow X^1 \twoheadleftarrow X^2 \twoheadleftarrow X^3 \twoheadleftarrow \cdots$$

where the maps in the tower are fibrations (in particular surjective maps) and the difference between X^{n-1} and X^n behaves like a $K(G,n)$-space. This is called the *Postnikov decomposition of* X, and we refer the reader to chap 4.3 of Hatcher's book for further details on this. This decomposition is useful to compute homotopy groups, but it is not well-adapted to compute homology groups in general.

16 Vista: the cohomology of groups

So far, we have introduced the homology of groups, studied its basic properties and its relations with singular homology. In this section we give a quick overview of the dual theory, namely of *group cohomology*.

Definition 237. Let M be a $\Bbbk$-linear representation of G. For all $n \geq 0$, the degree n cohomology of G with coefficients in M is defined by

$$H^n(G, M) = \mathrm{Ext}^n_{\Bbbk G}(\Bbbk, M) \;,$$

where $\Bbbk$ is equipped with the trivial action of G.

It follows directly from the definition that cohomology of degree zero is isomorphic to invariants:

$$H^0(G, M) = \mathrm{Hom}_{\Bbbk G}(\Bbbk, M) \simeq M^G \;.$$

In higher degrees, one may try to compute $H^n(G, M)$ by taking a projective resolution P of the trivial $\Bbbk G$-module, and by computing the homology of the complex of $\Bbbk$-modules $\mathrm{Hom}_{\Bbbk G}(P, M)$. If X is a $K(G, 1)$, the singular chain complex $C^{\mathrm{sing}}_*(\widetilde{X}; \Bbbk)$ is a projective resolution of $\Bbbk$ by theorem 213. If in addition $M = \Bbbk$ is the trivial module, then we have isomorphisms of cochain complexes

$$\begin{aligned}\mathrm{Hom}_{\Bbbk G}(C^{\mathrm{sing}}_*(\widetilde{X}; \Bbbk), \Bbbk) &\simeq \mathrm{Hom}_{\Bbbk}(C^{\mathrm{sing}}_*(\widetilde{X}; \Bbbk)_G, \Bbbk)\\ &\simeq \mathrm{Hom}_{\Bbbk}(C^{\mathrm{sing}}_*(X; \Bbbk), \Bbbk)\\ &= C^*_{\mathrm{sing}}(X; \Bbbk) \;,\end{aligned}$$

which proves the following analogue of theorem 213.

Theorem 238. *Assume that X is a $K(G, 1)$. There is an an isomorphism in every degree $i \geq 0$:*

$$H^i_{\mathrm{sing}}(X, \Bbbk) \simeq H^i(G, \Bbbk) \;.$$

All the properties of group homology that we have explained in this course (functoriality, Schapiro's lemma, the Cartan-Eilenberg formula...) have an analogue in cohomology. But, in the same way as singular cohomology of spaces, cohomology of groups has a very useful additional structure, namely is has cup products. To be more specific, if M and N are two $\Bbbk$-linear representations of G, the cup product is a family of $\Bbbk$-linear maps

$$\begin{array}{ccc} H^i(G, M) \otimes_{\Bbbk} H^j(G, N) & \to & H^{i+j}(G, M \otimes_{\Bbbk} N) \\ a \otimes b & \mapsto & a \smile b \end{array}$$

which has nice properties:

- associativity: $(a \smile b) \smile c = a \smile (b \smile c)$,
- unitality: if 1 denotes the unit of $\Bbbk = H^0(G, \Bbbk)$ and if we identify M with $\lambda : \Bbbk \otimes_{\Bbbk} M \simeq M$ and $\lambda' : M \otimes_{\Bbbk} \Bbbk \to M$ are the canonical isomorphisms then for all $a \in H^i(G, M)$ we have
$$H^i(G, \lambda)(a \smile 1) = a = H^i(G, \lambda')(1 \smile a) .$$
- graded commutativity: if $\tau : M \otimes_{\Bbbk} N \simeq N \otimes_{\Bbbk} M$ denotes the isomorphism such that $\tau(x \otimes y) = y \otimes x$, then for all $a \in H^i(G, M)$ and all $b \in H^j(G, N)$ we have
$$b \smile a = (-1)^{ij} H^{i+j}(G, \tau)(a \smile b) .$$

In particular we have the following statement.

Theorem 239. *Let G be a group.*

1. *The cup product makes $H^*(G, \Bbbk) = \bigoplus_{i \geq 0} H^i(G, \Bbbk)$ into a graded commutative $\Bbbk$-algebra.*
2. *Moreover, for all $\Bbbk$-linear representations M, the cup product makes $H^*(G, M) = \bigoplus_{i \geq 0} H^i(G, M)$ a module over $H^*(G, \Bbbk)$.*

Products are of great help in explicit computations of group homology. They are also involved in many qualitative results. For example, a result of Venkov (1959), Evens (1961) and Quillen (1971) shows that if G is a finite group and $\Bbbk$ is a field, then $H^*(G, \Bbbk)$ is a finitely generated algebra, and $H^*(G, M)$ is a finitely generated module over it for all finite dimensional representations M.

This finite generation result plays a role in a deep connection between finite groups and algebraic geometry. Namely, since $H^*(G, \Bbbk)$ is graded commutative and finitely generated, its subalgebra $H^{\mathrm{ev}}(G, \Bbbk) = \bigoplus_{i \geq 0} H^{2i}(G, \Bbbk)$ is commutative and finitely generated. Thus it is the coordinate algebra of an algebraic variety $V(G)$. By its definition, the geometry of this algebraic variety contain significant information on the group G. For example Quillen proved (1971) that if $\Bbbk$ has positive characteristic p, then the dimension of $V(G)$ equals the p-rank of G, i.e. the dimension of the maximal $\mathbb{F}_p$-vector subspace of G. One can push this correspondence between finite groups and algebraic varieties further and associate to every $\Bbbk$-linear representation M a subvariety $V(G)_M \subset V(G)$. This is a starting point of the *theory of support varieties*, which establishes a dictionnary between the geometric properties of the varieties $V(G)_M$ and the algebraic properties of the representations M, and which is very active nowadays.

In a different direction, let us mention a recent progress on the understanding of the cohomology ring $H^*(G, \Bbbk)$. Namely, Peter Symonds has proved (2010) that the generators of $H^*(G,\Bbbk)$ are in degrees less than $|G|$, and the relations in degrees less than $2|G|-1$. In particular, the knowledge of the truncated ring $\bigoplus_{i<2|G|-1} H^i(G,\Bbbk)$ is sufficient to understand the whole cohomology ring. The proof of this spectacular result is spectacular as well. It relies on differential geometry, namely on the study of abelian group actions on stratified manifolds. At the very end, this purely algebraic result relies on the existence of tubular neighborhoods!

Part IV
Exercises

Exercise sheet n°1

Homotopy and categories

1. Homotopy

In the euclidean space $\mathbb{R}^n$, we let S^{n-1} denote the unit sphere and D^n denote the unit ball (= the n-dimensional disc).

Exercice 1. Some explicit homeomorphisms. Show that $[0,1]^n$ is homeomorphic to D^n. Let $(D^n)^+ = \{x = (x_1, \dots, x_n) \mid x_n \geq 0\ ,\ x \in D^n\}$. Show that D^n is homeomorphic to $(D^n)^+$.

Exercice 2. Alphabet. In the following shortened alphabet, we consider the letters as topological subspaces of $\mathbb{R}^2$. Classify the letters into groups according to: 1) their homotopy type, 2) their homeomorphism type.

A B C D E F G H I J K L M N O P

Exercice 3. Half homotopy inverses. We say that $f : X \to Y$ has a left homotopy inverse if there is $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$. We say that f has a right homotopy inverse if there is $h : Y \to X$ such that $f \circ h \sim \mathrm{id}_Y$.

1. Show that if f has a left homotopy inverse g and a right homotopy inverse h, then $g \sim h$.
2. Show that f is a homotopy equivalence if and only if it has a left homotopy inverse and a right homotopy inverse.
3. Show that if f is a homotopy equivalence, then the homotopy inverse of f is unique up to homotopy (i.e. two homotopy inverses are always homotopic).

Exercice 4. The "2 out of 3" property. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two continuous maps. Prove that if any two maps among f, g and $g \circ f$, are homotopy equivalences, then so is the third map.

Exercice 5. Examples of homotopy equivalences.

1. Show that for all positive n, the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence.

2. Let X and Y be two contractible spaces. Show that every map $f : X \to Y$ is a homotopy equivalence.

3. Let C be a contractible space. Show that for all X, the canonical projection $C \times X \to X$ is a homotopy equivalence.

Exercice 6. Convexity and contractibility. Let C be a convex subset of $\mathbb{R}^n$ with the induced topology. Show that C is contractible. Find a contractible subset of $\mathbb{R}^2$ which is not convex.

Exercice 7. Cones. Given a topological space X, the cone CX is defined as the quotient[6] of the cylinder $X \times [0,1]$ in which all the points of $X \times \{1\}$ are identified.

1. Show that CS^{n-1} is homeomorphic[7] to D^n, and that CD^n is homeomorphic to D^{n+1}.

2. Show that for all topological spaces X, CX is contractible[8].

2. The language of categories

Exercice 8. Uniqueness of identity morphisms. Show that in a category $\mathcal{C}$ the identity morphisms id_X are uniquely determined (i.e. any txo identity morphisms of X must be equal).

Exercice 9. Isomorphisms. An *isomorphism* in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. The morphism g is called an *inverse* of f.

[6]Recall the definition of the **quotient topology**. Let X be a space, let $\sim$ be an equivalence relation on X, and let $p : X \to X/\sim$ be the quotient map. The quotient topology on $X/\sim$ is such that U is open in $X/\sim$ if and only if $p^{-1}(U)$ is open in X.

Set $Y = X/\sim$. Then the quotient map $p : X \to Y$ satisfies the following **universal property**: for all continuous $f : X \to Z$ such that f is constant on the subsets $p^{-1}(\{y\})$, $y \in Y$, there is a unique continuous map $\overline{f} : Y \to Z$ such that $f = \overline{f} \circ p$.

In fact every continuous map $p : X \to Y$ satisfying the previous universal property is a quotient map, and Y is then the quotient of X by the equivalence relation whose equivalence classes are the subsets $p^{-1}(\{y\})$, $y \in Y$.

[7]A very convenient way to contruct homeomorphisms in the compact case is the following lemma. **Lemma:** if $f : X \to Y$ is a continuous bijection, if X is compact (i.e. every covering of X by open subsets contains a covering of X by a finite number of open subsets.) and Y is separated, then f is a homeomorphism. (The proof is elementary)

[8]A very convenient way to construct homotopies from a quotient space is the following fact. **Proposition:** Let $\pi : X \to Y$ be a quotient map, and let I denote a locally compact separated space, then $\pi \times \mathrm{id}_I : X \times I \to Y \times I$ is a quotient map. (The proof is not that easy. See e.g. Bredon, Topology and Geometry, Chapter I, prop. 13.19)

1. Show that if f is an isomorphism, then its inverse is unique.
2. Show that $f : X \to Y$ is an isomorphism if and only if there is $g, h : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $h \circ f = \mathrm{id}_X$.
3. The "2 out of 3" property. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two morphisms in a category $\mathcal{C}$. Show that if two morphisms among f, g, $g \circ f$, are isomorphisms, then the third morphism is also an isomorphism.
4. Show that a functor preserves isomorphisms.

Exercice 10. Homotopy functors. Let $F : \mathrm{Top} \to \mathcal{C}$ be a functor. Show that the following assertions are equivalent: (i) for all continuous maps f, g if $f \sim g$ then $F(f) = F(g)$, (ii) F sends the homotopy equivalences to isomorphisms in $\mathcal{C}$. (A functor satisfying these conditions is called a *homotopy functor.*)

Exercice 11. The homotopy category. The homotopy category hTop is the category whose objects are topological spaces, whose set of morphisms $\mathrm{Hom}_{\mathrm{hTop}}(X, Y)$ is the set of homotopy classes of continuous maps $f : X \to Y$, and whose composition is given by $[f] \circ [g] = [f \circ g]$ (the brackets indicate the homotopy class of a map).

1. Check that the composition law in hTop is well-defined and that hTop is indeed a category.
2. Let $\pi : \mathrm{Top} \to \mathrm{hTop}$ be the functor which is the identity on objects and such that $\pi(f) = [f]$. Show that a functor $F : \mathrm{Top} \to \mathcal{C}$ is a homotopy functor if and only if there is a functor $\overline{F} : \mathrm{hTop} \to \mathcal{C}$ such that $F = \overline{F} \circ \pi$.
3. Redo exercises 3 and 4 by using the category hTop and exercise 8.

Exercice 12. Categorical coproducts. Let $(X_\alpha)_{\alpha \in A}$ be a family of objects of a category $\mathcal{C}$. An object X of $\mathcal{C}$, equipped with morphisms $i_\alpha : X_\alpha \to X$, is called a *coproduct* of the X_α if it satisfies the following universal property.

"For all families of morphisms $f_\alpha : X_\alpha \to Y$ there is a unique $f : X \to Y$ such that $f \circ i_\alpha = f_\alpha$."

1. Show that if X equipped with the i_α is a coproduct of the X_α and if X' equipped with the i'_α is another coproduct of the X_α then there is a unique isomorphism $\phi : X \xrightarrow{\simeq} X'$ such that $\phi \circ \iota_\alpha = \iota'_\alpha$ for all α.

2. Show that coproducts always exist (and describe them!) in the following categories: Ens, Top, $R-$Mod.

Exercice 13. Pushouts. A commutative square in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "p"] \\ C \arrow[r, "q"] & D \end{tikzcd}$$

is called a *pushout square* if it satisfies the following universal property. For all morphisms ϕ_B and ϕ_C making the outer square (ABZC) commute, there is a unique map ϕ making the two triangle commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left, "\phi_B"] & \\ C \arrow[r] \arrow[drr, bend right, "\phi_C"'] & D \arrow[dr, dashed, "\exists ! \phi"] & \\ & & Z \end{tikzcd}.$$

1. Show that if $\begin{matrix} A \xrightarrow{f} B \\ \downarrow_g \quad \downarrow_{p'} \\ C \xrightarrow{q'} D' \end{matrix}$ is another pushout with the same maps f and g, then there is a unique isomorphism $\phi : D \xrightarrow{\simeq} D'$, such that $p' \circ \phi = p$ and $q' \circ \phi = q$.

2. Show that if X is a topological space and if F and G are two closed subsets, then the square with $F \cap G = A$, $F = B$, $G = C$, $X = D$ and where the morphisms are given by the inclusions, is a pushout square. Show that this may not be the case if we don't assume that both F and G are closed.

3. Show that every diagram $\begin{matrix} A \xrightarrow{f} B \\ \downarrow_g \quad \\ C \quad \end{matrix}$ in Ens, in Top or in $R-$Mod can be completed into a pushout square.

Exercise sheet n°2

First computations in singular homology

3. First singular homology calculations

Exercice 14. General properties of degrees. Let $n > 1$.

1. Let $f, g : S^n \to S^n$. Show that $\deg(f \circ g) = \deg(f)\deg(g)$.
2. Show that if $f : S^n \to S^n$ is a homotopy equivalence then $\deg(f) = \pm 1$.
3. Show that $(\deg f \neq 0) \Rightarrow (f$ is surjective$)$.

Exercice 15. Degrees of homogeneous polynomials. Let S^1 be the unit circle of $\mathbb{C}$. Let $0 < \epsilon < \frac{1}{2n}$, and let H^+ and H^- be the two open hemispheres of S^1:

$$H^+ = \{e^{i\theta}\,, \theta \in]-\epsilon, \pi+\epsilon[\}\ , \qquad H^- = \{e^{i\theta}\,, \theta \in]\pi-\epsilon, 2\pi+\epsilon[\}\ .$$

Let $n > 1$ and consider the following two open subsets of S^1:

$$U^+ = \bigsqcup_{0\le i<n} \left\{e^{i\theta}\,, \frac{2i\pi}{n} - \frac{\epsilon}{n} < \theta < \frac{(2i+1)\pi}{n} + \frac{\epsilon}{n}\right\}\ ,$$
$$U^- = \bigsqcup_{0\le i<n} \left\{e^{i\theta}\,, \frac{(2i+1)\pi}{n} - \frac{\epsilon}{n} < \theta < \frac{(2i+2)\pi}{n} + \frac{\epsilon}{n}\right\}\ .$$

Let $f : S^1 \to S^1$ be such that $f(z) = z^n$.

By considering the Mayer-Vietoris sequences associated to the decompositions $S^1 = U^+ \cup U^-$ and $S^1 = H^+ \cup H^-$, show that $\deg(f) = n$.

Exercice 16. A proof of the fundamental theorem of algebra. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 0$.

1. Assume that P has no zeroes in the unit disc. Prove that the map $f : S^1 \to S^1$, such that $f(z) = P(z)/|P(z)|$ is homotopic to the constant map with value $P(0)/|P(0)|$, hence that it has degree 0.
2. Assume that P has no zeroes in $\{z \in \mathbb{C}\,, |z| \geq 1\}$. Prove that $f : S^1 \to S^1$, $f(z) = P(z)/|P(z)|$ is homotopic to $z \mapsto z^d$.
3. Use degree theory to prove that P must have at least a zero in $\mathbb{C}$.

Exercice 17. Bouquet. Let $(X_\alpha, x_\alpha)_{\alpha\in A}$ be a family of pointed topological spaces. Their *bouquet* or their *wedge* is the topological space $\bigvee_{\alpha\in A} X_\alpha$ obtained as the quotient of $\coprod_{\alpha\in A} X_\alpha$ by the smallest equivalence relation[9] which identifies all the x_α.

1. Let (X,x) and (Y,y) be two pointed topological spaces. Assume that $\{x\}$ is closed in X and that $\{y\}$ is closed in Y. Show that $X \vee Y$ is homeomorphic to the subspace $X\times\{y\}\cup\{x\}\times Y$ of $X\times Y$.

2. Let (X,x) and (Y,y) be two pointed topological spaces. Assume that $\{x\}$ and $\{y\}$ are strong deformation retracts of open subsets $U\subset X$ and $V\subset Y$. Show that Y, X and $Y\cap X$ (viewed as subspaces of $X\vee Y$) are strong deformation retracts of the open subsets $U\vee Y$, $X\vee V$, and $U\vee V$.

3. Show that for all positive i, the inclusions $X\hookrightarrow X\vee Y$ and $Y\hookrightarrow X\vee Y$ induce an isomorphism $H_i(X)\oplus H_i(Y)\simeq H_i(X\vee Y)$, and that $H_0(X\vee Y)$ is a free R-module of rank $\sharp\pi_0(X)+\sharp\pi_0(Y)-1$.

Exercice 18. The Torus and the Klein Bottle. Let $C = S^1\times[0,1]$ be the cylinder. Let $0<\epsilon<\frac{1}{6}$. View S^1 as the unit circle of $\mathbb{C}$. Let $I =]1/3-\epsilon, 2/3+\epsilon[\subset[0,1]$ and let $J=[0,1/3+\epsilon[\cup]2/3-\epsilon,1]\subset[0,1]$.

1. The Torus is the quotient of C by the smallest equivalence relation such that $(z,0)\equiv(z,1)$. Draw a picture.

   (a) Let U, V be the images in the Torus of the subsets $S^1\times I$, $S^1\times J$ of C. Show that U and V are two open subsets of the Torus. Show furthermore that there is a commutative square:

$$\begin{tikzcd} H_i(U\cap V) \arrow[r] & H_i(U)\oplus H_i(V) \\ R\oplus R \arrow[u, "\simeq"] \arrow[r] & R\oplus R \arrow[u, "\simeq"] \end{tikzcd}$$

   in which the top horizontal map is induced by the inclusions of $U\cap V$ in U and V and the bottom horizontal map is given by the matrix $\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$.

   (b) Use the Mayer-Vietoris property to compute the homology of the Torus.

[9] i.e. by the equivalence relation which identifies all the x_α, and which makes no other identification. In general the smallest equivalence relation means the equivalence relation with the smallest possible equivalent classes.

2. The Klein Bottle is the quotient of C by the smallest equivalence relation such that $(z,0) \equiv (\overline{z},1)$ ($\overline{z}$ is the conjugate of z). Draw a picture.

   (a) Let U and V be the images of $S^1 \times I$ and $S^1 \times J$ in the Klein bottle. Show that U and V are two open subsets of the Torus. Show furthermore that there is a commutative square:

$$\begin{tikzcd} H_i(U\cap V) \arrow[r] & H_i(U)\oplus H_i(V) \\ R\oplus R \arrow[u, "\simeq"] \arrow[r] & R\oplus R \arrow[u, "\simeq"'] \end{tikzcd}$$

   in which the top horizontal map is induced by the inclusions of $U\cap V$ in U and V and the bottom horizontal map is given by the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

   (b) Use the Mayer-Vietoris property to compute the homology of the Klein bottle. Compare this homology for $R = \mathbb{F}_2$, $\mathbb{F}_p$ with p odd, and $\mathbb{Z}$.

Exercice 19. Suspension. The *suspension* ΣX of a topological space X is the quotient of the cylinder $X \times [0,1]$ by the smallest equivalence relation which identifies together the points of $X \times \{0\}$ on the one hand, and which identifies together the points of $X \times \{1\}$ on the other hand. Every map $f : X \to Y$ induces a map $\Sigma f : \Sigma X \to \Sigma Y$.

1. Show that ΣS^n is homeomorphic to S^{n+1} and that ΣD^n is homeomorphic to D^{n+1}.

2. Show that ΣX is always arcwise connected.

3. Show that $H_1(\Sigma X) \simeq \mathrm{Ker}(H_0(X) \to H_0(\{*\}))$, and that for all $i > 0$ there is an isomorphism $H_i(X) \simeq H_{i+1}(\Sigma X)$, which is natural with respect to f, i.e. there are commutative squares

$$\begin{tikzcd} H_i(X) \arrow[r, "\simeq"] \arrow[d, "H_i(f)"] & H_{i+1}(\Sigma X) \arrow[d, "H_{i+1}(\Sigma f)"] \\ H_i(Y) \arrow[r, "\simeq"] & H_{i+1}(\Sigma Y) \end{tikzcd}$$

4. Show that for all $n > 0$ and all $k \in \mathbb{Z}$ there exists a map $f : S^n \to S^n$ of degree k.

Exercise sheet n°3

Complexes

4. Complexes

Exercice 20. Injectivity and surjectivity in homology.

1. Find an explicit complex C and an explicit subcomplex $D \subset C$ such that the map $H_1(C) \to H_1(D)$ induced by the inclusion is neither injective, nor surjective.

2. Find an explicit complex C and an explicit quotient complex $C \twoheadrightarrow E$ such that the map $H_1(C) \to H_1(E)$ induced by the inclusion is neither injective, nor surjective.

Exercice 21. Nullhomotopic complexes. Let C be a complex and assume that $\mathrm{id}_C \sim 0$ (such a complex is called *nullhomotopic*). Show that $H_i(C) = 0$ for all $i \in \mathbb{Z}$.

Exercice 22. Euler characteristic. Assume that k is a field. Let C be a complex with only a finite number of nonzero k-vector spaces C_n, and such that these vector spaces are finite-dimensional. The *Euler characteristic* of C is the number of $\chi(C) = \sum_{n\in\mathbb{Z}}(-1)^n \dim C_n$.

1. Show that $\chi(C) = \sum_{n\in\mathbb{Z}}(-1)^n \dim H_n(C)$. (Use the rank-nullity theorem).

2. Show that homotopy equivalent complexes have the same Euler characteristic.

3. Show that in a short exact sequence of complexes $0 \to C \to D \to E \to 0$ one has $\chi(C) + \chi(E) = \chi(D)$.

Exercice 23. The nine lemma. Consider the commutative diagram in which the rows are short exact sequences and the columns are complexes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Show that if any two of the columns are short exact sequences, then so is the third one.

Exercice 24. A universal coefficient theorem. Let C be a complex whose objects are free $\mathbb{Z}$-modules. Fix an integer q, and denote by $C\otimes_{\mathbb{Z}}\mathbb{Z}/q\mathbb{Z}$ the complex which is equal to the free $\mathbb{Z}/q\mathbb{Z}$-module $C_n \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z}$ in degree n, and whose differential is equal to $d \otimes \mathrm{id}_{\mathbb{Z}/q\mathbb{Z}}$.

1. **An example.** Let $C = \mathbb{Z} \xrightarrow{q} \underbrace{\mathbb{Z}}_{\text{degree } 0}$. Compute the homology of $C \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z}$. Compare with the homology of C tensored with $\mathbb{Z}/q\mathbb{Z}$.

2. **The universal coefficient theorem.** Show that for all complexes C and all i, there is a short exact sequence of $\mathbb{Z}$-modules:

$$0 \to H_i(C) \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} \to H_i(C \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z}) \to {}_{(q)}H_{i-1}(C) \to 0 \ .$$

   where ${}_{(q)}A$ denotes the q-torsion subgroup of an abelian group A, that is: ${}_{(q)}A = \{a \in A\ ; qa = 0\}$.

   [Hint: use the short exact sequence $0 \to C \xrightarrow{q} C \to C/qC \to 0$.]

Exercice 25. The five lemma and a typical application of it.

1. Consider the following commutative diagram of R-modules in which the rows are exact sequences.

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d, "f_1"] & H_2 \arrow[r] \arrow[d, "f_2"] & H_3 \arrow[r] \arrow[d, "f_3"] & H_4 \arrow[r] \arrow[d, "f_4"] & H_5 \arrow[d, "f_5"] \\
K_1 \arrow[r] & K_2 \arrow[r] & K_3 \arrow[r] & K_4 \arrow[r] & K_5
\end{tikzcd}$$

   Show that if f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism. (This is the "five lemma")

2. Consider the following commutative diagram of complexes of R-modules in which the rows are short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "f'"] & C'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & D \arrow[r] & D' \arrow[r] & D'' \arrow[r] & 0
\end{tikzcd}$$

Show that if two of the morphisms of chain complexes f f' and f'' are quasi-isomorphisms, then so is the third one. isomorphisms, then f_3 is an isomorphism.

Exercice 26. Homology of posets. Let P be a finite poset. For all $i \geq -1$, an i-chain in P is an increasing sequence of $i+1$ elements in P: $(x_0 < \cdots < x_i)$. (By convention there is only one -1-chain, namely the empty chain ().) We let $C_i(P)$ be the free R-module generated by the i-chains in P, and we define R-linear maps $d_i : C_i(P) \to C_{i-1}(P)$ by

$$d_i(x_0 < \cdots < x_i) = \sum_{k=0}^{j} (-1)^k (x_0 < \cdots < \widehat{x_k} < \cdots < x_i)$$

where $\widehat{x_k}$ means that x_k is deleted.

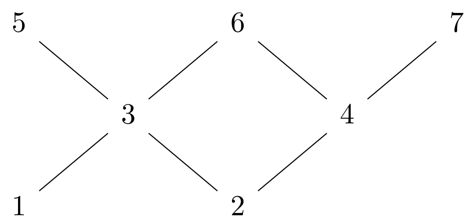
1. Check that the $C_i(P)$ together with the maps d_i yield a complex of R-modules. (The homology of this complex is called the *homology of the poset P*.)

2. Compute the homology of the poset P_1 with one element.

3. A vanishing property. Let P be a finite poset having a least element m (i.e. for all x in P, $x \geq m$).

   Let $h_i : C_i(P) \to C_{i+1}(P)$ be the R-linear map such that $s_i(x_0 < \cdots < x_i) = (m < x_0 < \cdots < x_i)$ if $x_0 > m$ and 0 if $x_0 = m$. Show that h is a homotopy between id_X and 0, and deduce $H_i(P)$ for all i.

4. Consider the poset P with 7 elements and Hasse diagram[10]:



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[10]In the Hasse diagram of a poset P, there is one vertex for each element of P, and there is a line $x - y$ with y above x if $y > x$.

(a) What is the homology of the poset $P \setminus \{1\}$?

(b) The complex $C(P \setminus \{1\})$ identifies with a subcomplex of $C(P)$. Compute the homology of the quotient complex $C(P)/C(P\setminus\{1\})$.

(c) Deduce the homology of P.

Exercise sheet n°4

Complexes

5. Singular and simplicial homology.

Exercice 27. Reduced singular homology[11]**.** Let X be a topological space, and let $p : X \to \{*\}$ denote the unique map to the one-point space. The *reduced singular homology of degree i of X* is the R-module $\overline{H}_i(X)$ defined by

$$\overline{H}_i(X) := \mathrm{Ker}\left(H_i(X) \xrightarrow{H_i(p)} H_i(\{x\}) \right) .$$

1. Show that $\overline{H}_i(X)$ is a functor of the variable X.

2. Show that the composition $\overline{H}_i(X) \to H_i(X) \to H_i(X, \{x\})$ is an isomorphism for all $x \in X$.

3. Show that the Mayer-Vietoris sequence and the long exact sequence of a pair remain valid if the singular homology is replaced by the reduced singular homology.

4. Let $\overline{C}^{\mathrm{sing}}(X)$ be the complex which equals the singular cohomology complex $C^{\mathrm{sing}}(X)$ in degrees ≥ 0, and which has an additional object $\overline{C}^{\mathrm{sing}}_{-1}(X) = R$, and an additional differential $\epsilon : C^{\mathrm{sing}}_0(X) \to R$ such that $\epsilon(\sigma) = 1$ for all 0-singular simplices $\Delta^0 \to X$:

$$\overline{C}^{\mathrm{sing}}(X) := \dots \xrightarrow{d} C^{\mathrm{sing}}_2(X) \xrightarrow{d} C^{\mathrm{sing}}_1(X) \xrightarrow{d} C^{\mathrm{sing}}_0(X) \xrightarrow{\epsilon} R$$

   Show that $\overline{H}_i(X) = H_i(\overline{C}^{\mathrm{sing}}(X))$ for all i.

Exercice 28. The Mayer-Vietoris sequence for simplicial homology. Let K be a geometric simplicial complex. A *simplicial subcomplex* of K is a subset $L \subset K$ which is a geometric simplicial complex. Show that if L and L' are two simplicial subcomplexes of K, then $L \cap L'$ is also a simplicial subcomplex of K. Show moreover that if $L \cup L' = K$ there is a long exact sequence in simplicial homology:

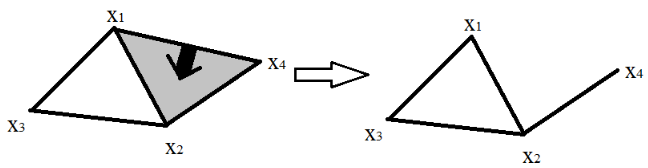
$$\dots \to H^{\mathrm{simpl}}_i(L \cap L') \to H^{\mathrm{simpl}}_i(L) \oplus H^{\mathrm{simpl}}_i(L') \to H^{\mathrm{simpl}}_i(K) \xrightarrow{\partial} H^{\mathrm{simpl}}_{i-1}(L \cap L') \to \dots$$

[11] The reduced singular homology contains the same information as the homology, and it is often more convenient in computations. To understand why, compute the reduced homology of the spheres with the Mayer-Vietoris exact sequence, and compare with the computation of the (unreduced) homology of spheres done in the course!

Exercice 29. Collapsing pairs. Let K be a geometric simplicial complex. A *collapsing pair* of K is a pair (s_k, s_{k+1}) of simplices of K such that (i) s_{k+1} has dimension $k+1 \geq 1$, (ii) s_k is a face of s_{k+1} and of no other simplex of dimension $k+1$ (iii) s_{k+1} is not a face of any simplex of K.

1. Show that $L = K \setminus \{s_{k+1}, s_k\}$ is a geometric simplicial complex.
2. Show that the inclusion $C_*^{\mathrm{simpl}}(L) \subset C_*^{\mathrm{simpl}}(K)$ is a quasi-isomorphism.

   The drawing below represents on the left a geometric simplicial complex K with collapsing pair $(\langle x_1, x_4\rangle, \langle x_1, x_2, x_4\rangle)$ and on the right the geometric simplicial complex L.



3. Application : compute the singular homology of the geometric simplicial complex corresponding to Δ^3.

Exercice 30. Simplicial homology vs singular homology. Recall the morphism of complexes $\phi_K : C^{\mathrm{simpl}}(K) \to C^{\mathrm{sing}}(|K|)$ from the course. In this exercise we give a direct proof that this morphism is a quasi-isomorphism when the geometric simplicial complex K is finite.

1. Show that when K has dimension zero, ϕ_K is a quasi-isomorphism.
2. Assume that ϕ_K is an isomorphism for geometric simplicial complexes of dimension $n-1$. We now turn our attention to the geometric simplicial complexes of degree n.
   (a) Let K the geometric simplicial complex corresponding[12] to an affine n-simplex $\langle a_0, \dots, a_n\rangle$ and let ∂K be the geometric simplicial complex corresponding to its boundary.
      i. Show that ϕ_K induces an isomorphism in degree 0 homology.
      ii. Show that $C^{\mathrm{simpl}}(K)$ is isomorphic to the part of degree ≥ 0 of the complex of the poset $\{a_0 < \cdots < a_n\}$ (see exercise 26) and deduce that $H_i^{\mathrm{simpl}}(K)$ is zero for $i > 0$.
      iii. Deduce that ϕ_K is a quasi-isomorphism.

[12]That is, K is the set of all the simplices which are convex hulls of nonempty subsets of $\{a_0, \dots, a_n\}$, and $\partial K = K \setminus \{\langle a_0, \dots, a_n\rangle\}$. For example, if $n = 2$ then $K = \{\langle a_0, a_1, a_2\rangle, \langle a_0, a_1\rangle, \langle a_0, a_2\rangle, \langle a_1, a_2\rangle, \langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle\}$ and $\partial K = \{\langle a_0, a_1\rangle, \langle a_0, a_2\rangle, \langle a_1, a_2\rangle, \langle a_0\rangle, \langle a_1\rangle, \langle a_2\rangle\}$.

iv. Check that there is a commutative diagram of complexes:

$$\begin{tikzcd} C^{\mathrm{simpl}}(\partial K) \arrow[r, hook] \arrow[d, "\phi_{\partial K}"] & C^{\mathrm{simpl}}(K) \arrow[r, two heads] \arrow[d, "\phi_K"] & R\langle a_0, \dots, a_n\rangle \arrow[d, "\overline{\phi_K}"] \\ C^{\mathrm{sing}}(|\partial K|) \arrow[r, hook] & C^{\mathrm{sing}}(|K|) \arrow[r, two heads] & C^{\mathrm{sing}}(|K|)/C^{\mathrm{sing}}(|\partial K|) \end{tikzcd}$$

where the horizontal morphisms are inclusions and quotient morphisms and where $\overline{\phi_K}$ is obtained by passing ϕ_K to the quotient. Deduce that $\overline{\phi_K}$ is a quasi-isomorphism.

(b) Assume now that ϕ_K is a quasi-isomorphism for geometric simplicial complexes of dimension n with less than N n-simplices. Let M be a geometric simplicial complex with N simplicies, and let L be the geometric simplicial complex obtained from M by removing one n-simplex $K = \langle a_0, \dots, a_n\rangle$.

i. Check that there is a commutative diagram of complexes:

$$\begin{tikzcd} C^{\mathrm{simpl}}(L) \arrow[r, hook] \arrow[d, "\phi_L"] & C^{\mathrm{simpl}}(M) \arrow[r, two heads] \arrow[d, "\phi_M"] & R\langle a_0, \dots, a_n\rangle \arrow[d, "\phi_{M,L}"] \\ C^{\mathrm{sing}}(|L|) \arrow[r, hook] & C^{\mathrm{sing}}(|M|) \arrow[r, two heads] & C^{\mathrm{sing}}(|M|)/C^{\mathrm{sing}}(|L|) \end{tikzcd}$$

where the horizontal morphisms are inclusions and quotient morphisms and where $\overline{\phi_K}$ is obtained by passing ϕ_K to the quotient.

ii. Check that there is a commutative triangle in which ψ is induced by the inclusion $C^{\mathrm{sing}}(|K|) \subset C^{\mathrm{sing}}(|M|)$:

$$\begin{tikzcd} & R\langle a_0, \dots, a_n\rangle \arrow[dl, "\overline{\phi_K}"'] \arrow[d, "\phi_{M,L}"] \\ C^{\mathrm{sing}}(|K|)/C^{\mathrm{sing}}(|\partial K|) \arrow[r, "\psi"] & C^{\mathrm{sing}}(|M|)/C^{\mathrm{sing}}(|L|) \end{tikzcd}$$

Deduce that $\phi_{M,L}$ is a quasi-isomorphism.

iii. Deduce that ϕ_M is a quasi-isomorphism.

3. Conclude the proof that ϕ_K is an isomorphism for all finite geometric simplicial complex.

Exercice 31. The Euler formula. Let K be a geometric simplicial complex of dimension n, such that $|K|$ is homeomorphic to S^n. Let n_i denote the number of i-simplices of K. Show that

$$\sum_{0 \le i \le n} (-1)^i n_i = 1 + (-1)^n .$$

Deduce the Euler Formula, i.e. if P is a regular polyedron of dimension 2, then:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2\ .$$

Exercise sheet n°5

CW-complexes

6. CW-structures

Exercice 32. Real projective spaces. If $\Bbbk$ is a field, we denote by $\Bbbk P^n$ the projective space of dimension n over $\Bbbk$.

As a set, $\Bbbk P^n$ is the set of lines of $\Bbbk^{n+1}$ (passing through the origin 0). Since every line is determined by a nonzero vector, and since two nonzero vectors determine the same line if and only if they are colinear, we can also view $\Bbbk P^n$ as the set of nonzero vectors module the action of the multiplicative group $\Bbbk^*$ by multiplication:

$$\Bbbk P^n = \big(\Bbbk^{n+1} \setminus \{0\}\big)/\Bbbk^* \,.$$

We denote by $[x_0 : \cdots : x_n] \in \Bbbk P^n$ the class of $(x_1, \ldots, x_n) \in \Bbbk^{n+1} \setminus \{0\}$. Then $\Bbbk P^{n-1}$ identifies with the subset of points of $\Bbbk P^n$ such that $x_n = 0$.

If $\Bbbk = \mathbb{R}$ of $\mathbb{C}$, we endow $\Bbbk P^n$ with the quotient topology relative to the canonical quotient map $\Bbbk^{n+1} \setminus \{0\} \to \Bbbk P^n$.

In the sequel of this exercise, we assume that $\Bbbk = \mathbb{R}$.

1. Prove that $\mathbb{R}P^n$ is homeomorphic to the quotient of S^n by the equivalence relation $\cong$ which identifies antipodal points (i.e. $x \cong -x$ for all $x \in S^n$).

2. Let $S^{n-1} = S^n \cap \mathbb{R}^n \times \{0\}$, and let $H^+ = S^n \cap \mathbb{R}^n \times \mathbb{R}^+$. Prove that there is a pushout square (where q denotes the quotient map):

$$\begin{tikzcd} S^{n-1} \arrow[r, "q"] \arrow[d, hook] & S^{n-1}/\cong \arrow[d, hook] \\ H^+ \arrow[r, "q"] & S^n/\cong \end{tikzcd} .$$

3. Deduce that $\mathbb{R}P^n$ has a CW-complex structure with i-skeleton equal to $\mathbb{R}P^i$ for $0 \leq i \leq n$, and with exactly one cell in each dimension i.

4. Show that $\mathbb{R}P^1$ is homeomorphic to S^1, and that the attaching map of the 2-cell $S^1 \to \mathbb{R}P^1 \simeq S^1$ has degree 2.

5. Show that for all $n \geq 2$, $\mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \simeq S^{n-1}$, and that the attaching map of the n-cell has degree $(-1)^n + 1$.

6. Compute the homology of $\mathbb{R}P^n$.

Exercice 33. Complex projective spaces. We view S^{2n+1} as the unit sphere of the euclidean space $\mathbb{C}^{n+1}$. We let U denote the multiplicative group of complex numbers of module 1. Thus U acts on S^{2n+1} by $\lambda\cdot(z_0,\dots,z_n) = (\lambda z_0,\dots,\lambda z_n)$.

1. Prove that $\mathbb{C}P^n$ is homeomorphic to the quotient of S^{2n+1} by the action of U.
2. Let $S^{2n-1} = S^{2n+1}\cap\mathbb{C}^n\times\{0\}$, and let $H^+ = S^{2n+1}\cap(\mathbb{C}^n\times\mathbb{R}^+)$. Prove that there is a pushout square (where q denotes the quotient map):
$$\begin{tikzcd} S^{2n-1} \arrow[r, "q"] \arrow[d, hook] & S^{2n-1}/U \arrow[d, hook] \\ H^+ \arrow[r, "q"] & \mathbb{C}P^n \end{tikzcd}.$$
3. Deduce that $\mathbb{C}P^n$ has a CW-complex structure with i-skeleton equal to $\mathbb{C}P^i$ for $0\le i\le n$, and with exactly one cell in each dimension $2i$ for $0\le i\le n$.
4. Compute the homology of $\mathbb{C}P^n$.

Exercice 34. CW-structures and coverings. Let $p:E\to B$ be a covering space. Assume that B is a CW-complex. Prove that E is a CW-complex as well, with i-skeleton $E_i = p^{-1}(B_i)$.

7. A solution of question 5 of exercise 32.

Let us fix the notations.

- $S^n = \{(\underbrace{x_0,\dots,x_{n-1}}_{=:\underline{x}}, x_n) \mid \|\underline{x}\|^2 + x_n^2 = 1\} \subset \mathbb{R}^{n+1}$
- $H^+ = \{(\underline{x}, x_n)\in S^n \mid x_n\ge 0\}$
- $H^- = \{(\underline{x}, x_n)\in S^n \mid x_n\ge 0\}$
- $S^{n-1} = H^+\cap H^-$
- We denote by $a: S^n\to S^n$ the antipodal map, i.e. $a(x) = -x$.
- $\mathbb{R}P^n = \dfrac{S^n}{a(x)\sim x\ \forall x\in S^n}$
- $\mathbb{R}P^{n-1} = \dfrac{S^{n-1}}{a(x)\sim x\ \forall x\in S^{n-1}} \subset \mathbb{R}P^n$

7.1. Let us show that $\mathbb{R}P^n/\mathbb{R}P^{n-1}$ is homeomorphic to S^n.

We consider the continuous map induced by the inclusion $H^+ \subset S^n$:

$$\frac{H^+}{S^{n-1}} \longrightarrow \frac{S^n}{\begin{array}{c} a(x) \sim x \ \forall x \in S^n \\ + \text{ all points of } S^{n-1} \\ \text{identified} \end{array}} = \frac{\mathbb{R}P^n}{\mathbb{R}P^{n-1}} \quad .$$

It is bijective, its source is compact and its target is Hausdorff hence it is a homeomorphism. Thus, to construct a homeomorphism $h : \mathbb{R}P^n/\mathbb{R}P^{n-1} \xrightarrow{\approx} S^n$, it suffices to construct a homeomorphism $h^+ : H^+/S^{n-1} \xrightarrow{\approx} S^n$.

Let us give an explicit formula for h^+. We first introduce a map $\pi^+ : S^n \to S^n$ by the following formulas:

$$\pi^+(\underline{x}, x) = \begin{cases} \left(\sqrt{\frac{4x_n}{1+x_n}}\underline{x}, 2x_n - 1\right) & \text{if } x_n \geq 0, \\ (\underline{0}, -1) & \text{if } x_n \leq 0. \end{cases}$$

Then π^+ is continuous on H^+ and on H^-, hence on S^n. Moreover,

- the restriction of π^+ to H^- is constant equal to $(\underline{0}, -1)$,
- the restriction of π^+ to $S^n \setminus H^-$ is a bijection [13] onto $S^n \setminus \{\underline{(0, -1)}\}$.

Thus $\pi^+ : H^+ \to S^n$ induces a a continuous bijection:

$$h^+ : H^+/S^{n-1} \to S^n \ .$$

The domain of h^+ is compact, its codomain is Hausdorff, hence h^+ is a homeomorphism.

Remark. We only need to introduce the restriction $\pi^+ : H^+ \to S^n$ to define h^+, but the map π^+ defined on the whole sphere will be useful later in the solution.

7.2 A degree computation.

Now we have to compute $\deg(f)$, where f is the following composition (the first two maps are the quotient maps):

$$f := S^n \twoheadrightarrow \mathbb{R}P^n \twoheadrightarrow \mathbb{R}P^n/\mathbb{R}P^{n-1} \xrightarrow{h} S^n \ .$$

[13] The inverse bijection is the map:

$$(\underline{y}, y_n) \mapsto \left(\sqrt{\frac{y_n + 3}{4(1 + y_n)}}\underline{y}, \frac{1}{2}(y_n + 1)\right) \ .$$

Let us give a concrete formula for f. We start by a formula for the restriction of f to H^+. For this purpose we consider the following commutative diagram, in which the vertical double arrows are the quotient maps and the horizontal arrows are induced by the inclusion $H^+ \subset S^n$:

$$\begin{tikzcd}
H^+ \arrow[r, hook] \arrow[d, two heads] & S^n \arrow[d, two heads] \\
\dfrac{H^+}{a(x) \sim x \;\forall x \in S^{n-1}} \arrow[r] \arrow[d, two heads] & \dfrac{S^n}{a(x) \sim x \;\forall x \in S^n} = \mathbb{R}P^n \arrow[d, two heads] \\
\dfrac{H^+}{S^{n-1}} \arrow[r] \arrow[dr, bend right, "h^+"'] & \dfrac{S^n}{\substack{a(x) \sim x \;\forall x \in S^n \\ + \text{ all points of } S^{n-1} \\ \text{identified}}} = \dfrac{\mathbb{R}P^n}{\mathbb{R}P^{n-1}} \arrow[d, "h"] \\
 & S^n
\end{tikzcd}.$$

By definition, the restriction of f to H^+ is the composition given by the top right corner of the diagram. By commutativity of the diagram, this is equal to the composition given by the bottom left corner of the diagram. By definition of h^+, the latter sends $x \in H^+$ to $\pi^+(x) \in S^n$. Thus we obtain:

$$f(x) = \pi^+(x) \quad \text{ if } x \in H^+.$$

Moreover, we have $f(-x) = f(x)$ for all $x \in S^n$, because x and $-x$ are identified in $\mathbb{R}P^n$. But if $x \in H^-$ then $-x \in H^+$, thus we obtain:

$$f(x) = \pi^+(-x) \quad \text{ if } x \in H^-.$$

We note that $f(x)$ is equal to $(\underline{0}, -1)$ for all $x \in S^{n-1}$.

Let us compute the degree of f. The following lemma allows us to compute the degree of a map which is constant on S^{n-1}, from the restrictions to H^+ and H^-.

Lemma. *Let $\alpha : S^n \to Z$ be a continuous map, such that $\alpha(x) = z$ for all $x \in S^{n-1}$. Define $\alpha^+, \alpha^- : S^n \to Z$ by*

$$\alpha^+(x) = \begin{cases} \alpha(x) & \text{if } x \in H^+, \\ z & \text{if } x \in H^-, \end{cases} \quad \text{and} \quad \alpha^-(x) = \begin{cases} \alpha(x) & \text{if } x \in H^-, \\ z & \text{if } x \in H^+. \end{cases}$$

Then $H_n(\alpha) = H_n(\alpha^+) + H_n(\alpha^-)$. In particular, if $Z = S^n$, then $\deg \alpha = \deg \alpha^+ + \deg \alpha^-$.

Now we apply the previous lemma to our situation. The expression of f that we have computed shows that

$$f^+ = \pi^+ \ , \quad f^- = \pi^+ \circ a \ .$$

Thus we have

$$\begin{aligned}\deg f &= \deg(\pi^+) + \deg(\pi^+ \circ a)\\ &= \deg(\pi^+) + \deg(\pi^+)\deg(a)\\ &= \deg(\pi^+)(1 + (-1)^n) \ .\end{aligned}$$

Now we claim that π^+ is homotopic to the identity. Indeed, $\pi^+(x) \neq -x$ for all $x \in S^n$, hence $0 \notin [x, \pi^+(x)]$, hence $tx + (1-t)\pi^+(x)$ is never zero. Thus the continuous map:

$$H(x,t) = \frac{tx + (1-t)\pi^+(x)}{\|tx + (1-t)\pi^+(x)\|}$$

is well defined an yields a homotopy between π^+ and id. Thus $\deg(\pi^+) = 1$ hence:

$$\deg(f) = 1 + (-1)^n \ .$$

QED.

Proof of the Lemma. We proceed in several steps.

1) Let $\iota^\pm : H^\pm/S^{n-1} \to S^n/S^{n-1}$ be the maps induced by the inclusions $H^\pm \subset S^n$. Then it follows from the Mayer-Vietoris exact sequence that the R-linear map

$$\Phi = \begin{bmatrix} H_n(\iota^+) \\ H_n(\iota^-) \end{bmatrix} : H_n\left(\frac{H^+}{S^{n-1}}\right) \oplus H_n\left(\frac{H^-}{S^{n-1}}\right) \to H_n\left(\frac{S^n}{S^{n-1}}\right)$$

is an isomorphism. (see also exercise 17.)

2) Consider the composition:

$$p^\pm : \frac{S^n}{S^{n-1}} \twoheadrightarrow \frac{S^n}{H^\mp} \xleftarrow{\approx} \frac{H^\pm}{S^{n-1}} \ .$$

Then $p^\pm \circ \iota^\pm = \mathrm{id}$ and $p^\pm \circ \iota^\mp$ is constant. Thus by using the functoriality of H_n, we see that the composition (in dots)

$$\begin{tikzcd} H_n\left(\frac{H^+}{S^{n-1}}\right) \oplus H_n\left(\frac{H^-}{S^{n-1}}\right) \arrow[r, "\Phi"] \arrow[dr, dotted] & H_n\left(\frac{S^n}{S^{n-1}}\right) \arrow[d, "{[H_n(p^+), H_n(p^-)]}"] \\ & H_n\left(\frac{H^+}{S^{n-1}}\right) \oplus H_n\left(\frac{H^-}{S^{n-1}}\right) \end{tikzcd}$$

is equal to the identity, hence $\Phi^{-1} = [H_n(p^+), H_n(p^-)]$.

3) We have a commutative diagram (this diagram commutes because it comes from a commutative diagram on the level of topological spaces. The upper horizontal bars denote the maps induced on the quotients.)

$$\begin{tikzcd} H^n(S^n) \arrow[r, "H_n(\mathrm{quot.})"] \arrow[dr, "H_n(\alpha)"'] & H^n\left(\frac{S^n}{S^{n-1}}\right) \arrow[d, "H_n(\overline{\alpha})"] & H_n\left(\frac{H^+}{S^{n-1}}\right) \oplus H_n\left(\frac{H^-}{S^{n-1}}\right) \arrow[l, "\Phi"'] \arrow[dl, "{[H_n(\overline{\alpha_{|H^+}}), H_n(\overline{\alpha_{|H^-}})]}"] \\ & Z & \end{tikzcd}$$

Let Ψ denote the composition:

$$\Psi = \Phi^{-1} \circ H_n(\mathrm{quot.}) \; .$$

The commutativity of the above diagram implies that $H_n(\alpha)$ equals the composition

$$H^n(S^n) \xrightarrow{\Psi} H_n\left(\frac{H^+}{S^{n-1}}\right) \oplus H_n\left(\frac{H^-}{S^{n-1}}\right) \xrightarrow{[H_n(\overline{\alpha_{|H^+}}), H_n(\overline{\alpha_{|H^-}})]} Z$$

4) We write down explicitly the components of Ψ.

$$\text{We have } \Psi = \begin{bmatrix} H_n(q^+) \\ H_n(q^-) \end{bmatrix} \text{ where } q^\pm = S^n \twoheadrightarrow \frac{S^n}{H^\mp} \xleftarrow{\approx} \frac{H^\pm}{S^{n-1}} \; .$$

Thus we have

$$H_n(\alpha) = H_n(\overline{\alpha_{|H^+}} \circ q^+) + H_n(\overline{\alpha_{|H^-}} \circ q^-)$$

To finish, we observe that $\alpha^\pm = \overline{\alpha_{|H^\pm}} \circ q^\pm$, whence the result.

$\square$

Exercise sheet n°6

Extensions

8. Ext and projectives

Exercice 36. Split exact sequences.

1. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of R-modules. Show that the following assertions are equivalent.

   (i) There exists a morphism $i : M'' \to M$ such that $g \circ i = \mathrm{id}$.

   (ii) There exists a morphism $p : M \to M'$ such that $p \circ f = \mathrm{id}$.

   (iii) The exact sequence is isomorphic to the trivial exact sequence, that is, there is an isomorphism ϕ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\phi"] & M' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M'' \arrow[r, "\left[\begin{smallmatrix}\mathrm{id}\\0\end{smallmatrix}\right]"] & M'' \oplus M' \arrow[r, "{[0,\mathrm{id}]}"] & M' \arrow[r] & 0
\end{tikzcd}$$

   When a short exact sequence satisfies these conditions, one says that it *splits*.

2. Give an example of a short exact sequence which does not split.

3. Show that an R-module P is projective if and only if every short exact sequence $0 \to M \to N \to P \to 0$ splits.

Exercice 37. Some Ext**-computations over a PID.** Let R be a PID.

1. Write an explicit projective resolution of a cyclic module $C = R/rR$.

2. Let $C = R/rR$ and $C' = R/sR$ be two cyclic modules. Compute $\mathrm{Ext}^i(C, C')$ for all i.

3. Let M be a finitely generated R-module. Prove that $\mathrm{Ext}^1_R(M, R) = 0$ if and only if M is free[14].

[14] For $R = \mathbb{Z}$, one can prove that the property still holds for M countable instead of finitely generated. The Whitehead problem asks if this holds for all abelian groups M, without any finiteness or countability hypothesis. It was proved by Shelah in 1974 that the answer to this question depends on the axioms of set theory used. In particular, under the continuum hypothesis (i.e. there is no set cardinal strictly between the cardinal of $\mathbb{N}$ and the cardinal of $\mathbb{R}$) then the equivalence holds for all abelian groups M.

Exercice 38. Some Ext-computations over $\Bbbk[x]/x^{n+1}$. Let $\Bbbk$ be a field and let $R = \Bbbk[x]/x^{n+1}$. There is a morphism of $\Bbbk$-algebras $\epsilon : R \to \Bbbk$ such that $\epsilon(x) = 0$. In this way, we see $\Bbbk$ as an R-module.

1. Write an explicit projective resolution of $\Bbbk$.
2. Compute $\mathrm{Ext}^i_R(\Bbbk, \Bbbk)$ for all i.

Exercice 39. Some Ext-computations over $\Bbbk[x, y]$. Let $\Bbbk$ be a field and let $R = \Bbbk[x, y]$. There is a morphism of $\Bbbk$-algebras $\epsilon : R \to \Bbbk$ such that $\epsilon(x) = 0 = \epsilon(y)$. In this way, we see $\Bbbk$ as an R-module.

1. We consider the free R-module $F = R \otimes_\Bbbk \Lambda(x, y)$, that is, F is the free R-module with basis given by the symbols $1 \otimes 1, 1 \otimes x, y \otimes 1, 1 \otimes (x \wedge y)$:
$$F = R_{1\otimes 1} \oplus R_{1\otimes x} \oplus R_{1\otimes y} \oplus R_{1\otimes (x\wedge y)} \;.$$
We let $d : F \to F$ be the R-linear map such that
$$\begin{aligned} &d(1 \otimes x \wedge y) = x(1 \otimes y) - y(1 \otimes x) \\ &d(1 \otimes x) = x(1 \otimes 1) \\ &d(1 \otimes y) = y(1 \otimes 1) \\ &d(1 \otimes 1) = 0 \end{aligned}$$
We decide that the summand $R_{1\otimes 1}$ has degree zero, that the summands $R_{1\otimes x}$ and $R_{1\otimes y}$ have degree 1 and that the summand $R_{1\otimes(x\wedge y)}$ has degree 2. Show that (F, d) is a chain complex of R-modules and compute its homology.
2. Compute $\mathrm{Ext}^i_R(\Bbbk, \Bbbk)$ for all i.

Exercice 40. Degree shifting.

1. Assume that there is a short exact sequence $0 \to K \to P \to M \to 0$ with P projective. Show that $\mathrm{Ext}^i_R(M, N) \simeq \mathrm{Ext}^{i+1}_R(K, N)$ for all N and all positive i.
2. Assume that there is an exact sequence
$$0 \to K \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$
in which every P_k is projective. Prove that there is an isomorphism $\mathrm{Ext}^i_R(M, N) \simeq \mathrm{Ext}^{i+n}_R(K, N)$ for all positive i.

Exercice 41. Projective dimension of a module. We say that a module M has *projective dimension less or equal to* n if it admits a projective resolution (P,ϵ) with $P_i = 0$ for $i > n$.

Prove that the following assertions are equivalent (Hint: you may use the previous exercise on degree shifting).

1. M has projective dimension less or equal to n,
2. $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > n$ and all R-modules N,
3. $\mathrm{Ext}^{n+1}_R(M,N) = 0$ for all R-modules N.

Exercice 42. Finiteness of Ext **over noetherian rings.** Let R be a left noetherian ring. Let M and N be finitely generated R-modules. Show that for all i, $\mathrm{Ext}^i_R(M,N)$ is a finitely generated R-module.

Exercice 43. The Horseshoe lemma.

1. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of R-modules. Let (P^X, ϵ_X) be a projective resolution of X and let (P^Z, ϵ_Z) be a projective resolution of Z.

   Prove that there exists a projective resolution (P^Y, ϵ_Y) of Y satisfying the following three properties.

   a) $P^Y_i = P^X_i \oplus P^Z_i$ for all $i \geq 0$.
   b) The maps $\overline{f}_i = \left[\begin{smallmatrix}\mathrm{id}\\ 0\end{smallmatrix}\right] : P^X_i \to P^X_i \oplus P^Z_i$ define a morphism of complexes $\overline{f} : P^X \to P^Y$ which lifts f.
   c) The maps $\overline{g}_i = [0, \mathrm{id}] : P^X_i \oplus P^Z_i \to P^Z_i$ define a morphism of complexes $\overline{g} : P^Y \to P^Z$ which lifts g.

2. Prove that every short exact sequence $0 \to M'' \to M \to M' \to 0$ induces a long exact sequence[15] in Ext:

$$\cdots \to \mathrm{Ext}^{i-1}_R(M'',N) \xrightarrow{\partial} \mathrm{Ext}^i_R(M',N) \to \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M'',N) \xrightarrow{\partial} \dots .$$

[15] In the course, we constructed the long exact sequence by using injective coresolutions - this exercise gives a contruction which uses only projectives.

9. Injectives

Exercice 44. Injectives and splitting.

1. Show that an R-module J is injective if and only if every short exact sequence $0 \to J \to M \to N \to 0$ splits.

2. Let $0 \to J_n \to J_{n-1} \to \cdots \to J_0 \to M \to 0$ be an exact sequence in which all the J_k are injective. Show that M is a direct summand of J_0 (hence M is injective).

Exercice 45. Injectives and domains. Let R be a domain, with fraction field F.

1. Let I be a nonzero ideal of R and let $f : I \to F$ be an R-linear map.
   (a) Show that the map $I \setminus \{0\} \to F$, $a \mapsto f(a)/a$ is constant. Let c denote its value, hence $f(r) = rc$ for all $r \in I$.
   (b) Deduce that F is an injective R-module.

2. Let R be a domain. Prove that R is injective as an R-module if and only if R is a field. (Hint: R cannot be a proper summand of F)

Exercice 46. Direct sums of injectives. We know from the course that a product of injectives is injective. It is a natural question to ask whether a direct sum of injectives is injective. In this exercise we prove that this is indeed the case for noetherian rings[16] Let R be a left noetherian ring and let $(J_\alpha)_{\alpha\in A}$ be a family of injectives R-modules.

1. Let I be an ideal of R and let $f : I \to \bigoplus_{\alpha\in A} J_\alpha$ be an R-linear map. Show that there is a finite subset $B \subset A$ such that $\operatorname{Im} f \subset \bigoplus_{\alpha\in B} J_\alpha$. (Hint: use that J is a finitely generated R-module).

2. Use Baer's criterion to prove that $\bigoplus_{\alpha\in A} J_\alpha$ is injective.

Exercice 47. Self-injective rings. A ring R is called self-injective if the left module R is injective.

1. Show that for $n > 0$, $\mathbb{Z}/n\mathbb{Z}$ is self-injective (use Baer's criterion).

2. Show that if R is self-injective and left noetherian, then every projective R-module is injective. (use the previous exercise on direct sums of injectives)

[16] In fact the converse is also true: a ring R is left noetherian if and only if every direct sum of injectives is injective (Bass-Papp theorem).

3. Let R be a self-injective and noetherian ring. Let M be an R-module. Show that the following assertion are equivalent.
   - (i) M is projective
   - (ii) For all $i > 0$ and all R-modules N, $\mathrm{Ext}^i_R(M,N) = 0$.
   - (iii) There is a positive integer i such that for all R-modules N, $\mathrm{Ext}^i_R(M,N) = 0$.

10. Extensions

Exercice 48. Classification of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}$. Let p be a prime number. Show that every extension of $\mathbb{Z}/p$ by $\mathbb{Z}$ is either trivial, or equivalent to the following exact sequence, for a uniquely determined $a \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$, where π_a denotes the unique map such that $\pi(1) = a$:

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \to 0\ .$$

Exercice 49. Proof of the "Ext^1 = extensions" theorem. Let $\mathcal{E}^1(M,N)$ denote the set of isomorphism classes of extensions of M by N. Let us fix a projective resolution (P,ϵ) of M, and let $\mathrm{Ext}^1_R(M,N) = H^1(\mathrm{Hom}_R(P,N))$.

1. Every extension $0 \to N \to E \xrightarrow{\pi} M \to 0$ can be viewed as a resolution (R,π) of M (with $R_0 = E$, $R_1 = N$ and $R_i = 0$ for $i > 1$), hence by the fundamental theorem of homological algebra there is a lifting $\overline{f} : P \to R$ of id_M.
   - (a) Show that $\overline{f}_1 : P_1 \to N$ is a cycle of the complex $\mathrm{Hom}_R(P,N)$.
   - (b) Show that $\overline{f}, \overline{f}'$ are two different choices of liftings, then $\overline{f}_1 - \overline{f}'_1$ is a boundary of the complex $\mathrm{Hom}_R(P,N)$.
   - (c) Show that if $0 \to N \to E \xrightarrow{\pi} M \to 0$ is the trivial extension, then $\overline{f}_1$ is a boundary of the complex $\mathrm{Hom}_R(P,N)$.
   - (d) Show that the map
   $$\Phi : \mathcal{E}^1(M,N) \to \mathrm{Ext}^1_R(M,N)$$
   which sends the isomorphism class of $0 \to N \to E \to M \to 0$ to $[\overline{f}_1]$ is well-defined, and sends that trivial extension to 0.
2. Let $f : P_1 \to N$ be a cycle of $\mathrm{Hom}_R(P,N)$. Set $\Delta_f = \left[\begin{smallmatrix} d \\ -f \end{smallmatrix}\right] : P_1 \to P_0 \oplus N$.

(a) Show that the map $[\epsilon, 0] : P_0 \oplus N \to M$ induces a morphism $\pi : (P_0 \oplus N)/\operatorname{Im}\Delta_f \to M$.

(b) Show that the diagram

$$0 \to N \xrightarrow{\left[\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix}\right]} P_0 \oplus N/\operatorname{Im}\Delta_f \xrightarrow{\pi} M \to 0 \qquad (*)$$

is a short exact sequence.

(c) Show that if f and f' are two cycles such that $f - f'$ is a boundary then the associated exact sequences are isomorphic.

(d) Show that the map

$$\Psi : \operatorname{Ext}^1_R(M, N) \to \mathcal{E}^1(M, N)$$

which sends $[f]$ to the isomorphism class of $(*)$, is well-defined.

3. Check that Φ and ψ are mutually inverse bijections.

Exercise sheet n°7

Bicomplexes

11. Bicomplexes

Exercice 46. Short exact sequences of bicomplexes. A short exact sequence of bicomplexes is a diagram of bicomplexes $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$ such that for all indices i, j, the following diagram is a short exact sequence of R-modules:

$$0 \to C'^{i,j} \xrightarrow{f^{i,j}} C^{i,j} \xrightarrow{g^{i,j}} C''^{i,j} \to 0$$

1. Show such an exact sequence of bicomplexes induces a long exact sequence in homology:

$$\cdots \to H^i(\operatorname{Tot} C') \to H^i(\operatorname{Tot} C) \to H^i(\operatorname{Tot} C'') \xrightarrow{\partial} H^{i+1}(\operatorname{Tot} C') \to \ldots$$

2. Let C be a bicomplex. A sub-bicomplex of C i.e. a family $C'^{i,j}$ of submodules of the $C^{i,j}$ which is preserved by the differentials of C. Define a quotient bicomplex C/C', in such a way that there is a short exact sequence:

$$0 \to C' \to C \to C/C'' \to 0$$

in which the morphism of bicomplexes $C \to C'$ is given by the inclusions of R-modules $C'^{i,j} \hookrightarrow C^{i,j}$.

Exercice 47. The mapping cone. Let $f : C \to D$ be a morphism of complexes. Let B denote the bicomplex whose -1-rst column is equal to C, whose 0-th column is equal to D, whose other columns are zero, and whose horizontal differential is given by f. The *mapping cone of* f is the complex $C(f) := \operatorname{Tot} B$.

1. Show that there is a long exact sequence in cohomology:

$$\cdots \to H^i(D) \to H^i(C(f)) \to H^{i+1}(C) \xrightarrow{H^{i+1}(f)} H^{i+1}(D) \to \ldots$$

(Hint: show that D is a subcomplex of $C(f)$ and identify the quotient complex. In the associated long exact sequence, use the expression of the connecting morphism to get an explicit expression of it).

2. Show that f is a quasi-isomorphism if and only if its mapping cone has trivial cohomology (i.e. equal to zero in each degree).

Exercice 48. A generalization of the nine lemma. Let C be a *first quadrant* cochain bicomplex, that is a cochain bicomplex such that $C^{i,j} = 0$ if $i < 0$ or $j < 0$.

1. Assume that all the rows of C are complexes with trivial cohomology (i.e. equal to zero in each degree). Show that $\mathrm{Tot}\, C$ has trivial cohomology.

2. Assume that $\mathrm{Tot}\, C$ has trivial cohomology, and that in addition all the columns but one are also cochain complexes with trivial cohomology. Show that the last column also has trivial cohomology.

Exercice 49. Tensor products of algebras and Ext**.** Let R and S be two $\Bbbk$-algebras over a commutative ring $\Bbbk$. Their *tensor product algebra* $R \otimes_{\Bbbk} S$ is the $\Bbbk$-module $R \otimes_{\Bbbk} S$, equipped with the product $(r \otimes s)(r' \otimes s') := (rr') \otimes (ss')$. If M is an R-module and M' is an S-module, the tensor product $M \otimes_{\Bbbk} M'$, equipped with the action $(r \otimes s)(m \otimes m') := (rm) \otimes (sm')$ is naturally an $R \otimes_{\Bbbk} S$-module.

1. Prove that if M is projective as an R-module and if M' is projective as an S-module, then $M \otimes_{\Bbbk} M'$ is projective as an $R \otimes_{\Bbbk} S$-module.

2. Assume that $\Bbbk$ is a field. Prove that if (Q, ϵ) is a projective resolution of an R-module M and if (Q', ϵ') is a projective resolution of an S-module M' then $(Q \otimes_{\Bbbk} Q', \epsilon \otimes \epsilon')$ is a projective resolution of $M \otimes_{\Bbbk} M'$ as a $R \otimes_{\Bbbk} S$-module.

3. Assume that $\Bbbk$ is a field, that R and S are noetherian $\Bbbk$-algebras and that M and M' are finitely generated $\Bbbk$-modules. Show that for all nonnegative k there is an isomorphism
$$\bigoplus_{i+j=k} \mathrm{Ext}^i_R(M, N) \otimes_{\Bbbk} \mathrm{Ext}^j_S(M', N') \simeq \mathrm{Ext}^k_{R \otimes_{\Bbbk} S}(M \otimes_{\Bbbk} M', N \otimes_{\Bbbk} N') \;.$$
Hint: start by proving the case of free finitely generated modules M and M', and then take free resolutions.

Exercice 50. Some Ext**-computations over** $\Bbbk[x_1, \dots, x_n]$**.** Let $\Bbbk$ be a field and let $R = \Bbbk[x_1, \dots, x_n]$. There is a morphism of $\Bbbk$-algebras $\epsilon : R \to \Bbbk$ such that $\epsilon(x_i) = 0$ for all i. In this way, we see $\Bbbk$ as an R-module. Compute $\mathrm{Ext}^i_R(\Bbbk, \Bbbk)$ for all i. (Hint: start with $n = 1$, and for higher n, use the previous exercise)

Exercise sheet n°8

Left modules, right modules and tensor products.

12. Left modules versus right modules: examples.

Exercice 52. Group rings. Let G be a group, and let $\Bbbk$ be a commutative ring. Define $\Bbbk G$ as the free $\Bbbk$-module with basis $(e_g)_{g\in G}$. Thus every element of $\Bbbk G$ can be written as a linear combination $\sum_{g\in G}\lambda_g e_g$ in which only a finite number of scalars λ_g are nonzero. We make this $\Bbbk$-module into a $\Bbbk$-algebra by letting:

$$\left(\sum_{g\in G}\lambda_g e_g\right)\cdot\left(\sum_{h\in G}\lambda'_h e_h\right)=\sum_{g,h\in G^2}\lambda_g\lambda'_h e_{gh}\;.$$

1. Prove that $\Bbbk G$ is a commutative ring if and only if G is abelian.
2. Prove that defining a left $\Bbbk G$-module M is equivalent to specifying a $\Bbbk$-module M together with a left action of G on M by $\Bbbk$-linear morphisms (i.e. an action such that $g\cdot - : M\to M$ is $\Bbbk$-linear).
3. Prove that defining a morphism of $\Bbbk G$-modules $f : M \to N$ is equivalent to specifying a $\Bbbk$-linear map which is G-equivariant, i.e. such that $f(g\cdot m) = g\cdot f(m)$ for all $g\in G$ and all $m\in M$.
4. Let M be a left $\Bbbk G$-module. Thus M is a $\Bbbk$-module equipped with a left $\Bbbk$-linear action of G "$\cdot$".

   i) Prove that the map:

   $$M\times G\to M\;,(m,g)\mapsto g^{-1}\cdot m$$

   defines a right action of G on M, and in fact a right $\Bbbk G$-module structure on M. We let M^{right} denote the $\Bbbk$-module M considered as a right module in this way.

   ii) Prove that sending M to M^{right} defines an isomorphism of categories $\Bbbk G-\mathrm{Mod}\simeq\mathrm{Mod}-\Bbbk G$ (i.e. a functor which is a bijection between the objects of the two categories and which is a bijection on the morphisms.)

Exercice 53. Rings of matrices. Let R be the ring of (2×2)-matrices with coefficients in a field $\Bbbk$. Let $C_1\subset R$ denote the subset of matrices of the form $\left[\begin{smallmatrix} * & 0\\ * & 0\end{smallmatrix}\right]$, and let C_2 denote the subset of matrices of the form $\left[\begin{smallmatrix} 0 & *\\ 0 & *\end{smallmatrix}\right]$

1. Show that C_1 and C_2 are left R-submodules of the left R-module R. Are they right R-submodules of the right R-module R ?

2. Show that C_1 and C_2 are simple and projective as left R-modules.

3. Show that C_1 and C_2 are isomorphic, and that C_1 is, up to isomorphism, the only simple left R-module.

4. Show that every finite dimensional left R-module is semi-simple, i.e. is isomorphic to a direct sum of copies of C_1. (Hint: an easy way to prove this is by using the Ext-criterion for semi-simplicity)

5. If M is a left R-module, we may define a right action of R on it by the map:
$$M \times R \to M \ , (m, r) \mapsto {}^t r \cdot m$$
where ^{t} refers to the transpose of a matrix. We let M^{right} denote the right R-module obtained from M in this way. Prove that sending M to M^{right} defines an isomorphism of categories $R-\mathrm{Mod} \simeq \mathrm{Mod} - R$.

Exercice 54. A ring with different left and right modules. Let R be the ring of (3×3)-matrices of the form

$$\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

with coefficients in a field $\Bbbk$. Let $C_i \subset R$ denote the subset of R whose elements are the matrices of R with entries zero in the columns of index $\neq i$. Let $R_i \subset R$ denote the subset of matrices of R whose entries are zero in the rows of index different from i.

1. Show that $R = C_1 \oplus C_2 \oplus C_3$ as left R-modules and that $R = R_1 \oplus R_2 \oplus R_3$ as right R-modules.

2. Show that the left R-modules C_1 and C_2 are simple, non-isomorphic and projective.

3. Show that every simple projective right R-module is isomorphic to R_3. (Hint: show that there is up to isomorphism only 3 simple R modules, which are the simple quotients of the R_i, and show that only one of them is projective.)

4. Show that there is no isomorphism of categories between $R-\mathrm{Mod}$ and $\mathrm{Mod} - R$.

13. Tensor products.

Exercice 55. Two constructions of the same tensor product. Let R be a $\Bbbk$-algebra, let M be a right R-module and let N be a left R-module. Let T be the abelian group obtained by tensoring M and N over the ring R and let T' denote the $\Bbbk$-module obtained by tensoring M and N over the $\Bbbk$-algebra R:

$$T = \frac{M \otimes_{\mathbb{Z}} N}{\langle mr \otimes n - m \otimes rn \rangle}\ , \quad T' = \frac{M \otimes_{\Bbbk} N}{\langle mr \otimes n - m \otimes rn \rangle}\ .$$

1. Show that there exists a unique map $\alpha : \Bbbk \times T \to T$ such that $\alpha(\lambda, m \otimes n) = m \otimes \lambda n$, and that this map endows T with the structure of a $\Bbbk$-module.

2. Show that the map $\pi : M \times N \to T$ such that $\pi(m, n) = m \otimes n$ is $(R, \Bbbk)$-bilinear, and that for all $\Bbbk$-modules P and all $(R, \Bbbk)$-bilinear map $f : M \times N \to P$ there is a unique $\Bbbk$-linear map $\overline{f} : T \to P$ such that $f = \overline{f} \circ \pi$.

3. Deduce that there is a $\Bbbk$-linear isomorphism $\phi : T \xrightarrow{\simeq} T'$ such that $\phi(m \otimes n) = m \otimes n$.

Exercice 56. Extension of scalars. Let $R \to S$ be a morphism of rings. We view S as a right R-module via this morphism.

1. Let M be a left R-module. Show that there exists a unique map $\alpha : S \times S \otimes_R M \to S \otimes_R M$ such that $\alpha(s, s' \otimes m) = ss' \otimes m$. Show that this map defines a left S-module structure on $S \otimes_R M$. Show moreover that if $f : M \to M'$ is R-linear then $\mathrm{id}_S \otimes f$ is S-linear.

2. Show that there is an isomorphism of abelian groups, natural with respect to the R-module M and the S-module N:

$$\mathrm{Hom}_S(S \otimes_R M, N) \simeq \mathrm{Hom}_R(M, \mathrm{res}^S_R N)\ .$$

3. Show that if M is a projective R-module, then $S \otimes_R M$ is a projective S-module. Show that if M is a flat R-module then $S \otimes_R M$ is a flat S-module.

Exercise sheet n°9

Tor.

14. Tensor products and Tor.

Exercice 57. Computation of Tor over $\mathbb{Z}$. Compute $\mathrm{Tor}_i^{\mathbb{Z}}(A,B)$ for all i, when A and B belong to the following list of abelian groups: $\{$finite cyclic groups , $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}\}$.

Exercice 58. Left exactness of Tor over $\mathbb{Z}$. Show that for all abelian groups A, Tor_1 defines a left exact functor:

$$\mathrm{Tor}_1^{\mathbb{Z}}(A,-) : \mathbb{Z}-\mathrm{Mod} \to \mathbb{Z}-\mathrm{Mod}\ .$$

Exercice 59. Computation of Tor over $\Bbbk[x]$. Let $\Bbbk$ be a commutative ring. Let n and m be positive integers. Compute $\mathrm{Tor}_i^{\Bbbk[x]}(\Bbbk[x]/x^n, \Bbbk[x]/x^m)$ for all i.

Exercice 60. Computation of Tor over $k[x]/x^d$. Let $\Bbbk$ be a commutative ring, and let d be a positive integer. Let n and m be two positive integers less or equal to d. Compute $\mathrm{Tor}_i^{\Bbbk[x]/x^d}(\Bbbk[x]/x^n, \Bbbk[x]/x^m)$ for all i.

Exercice 61. Commutativity of Tor. Let R be a $\Bbbk$-algebra and let R^{op} denote the opposite $\Bbbk$-algebra (i.e. the same $\Bbbk$-module with product $\cdot_{\mathrm{op}} : R \times R \to R$ given by $r \cdot_{\mathrm{op}} s := sr$). Every left R-module N can be considered as a right R^{op}-module, and every right R-module M can be considered as a left R^{op}-module.

1. Show that the isomorphism $M \otimes_\Bbbk N \simeq N \otimes_\Bbbk M$, $m \otimes n \mapsto n \otimes m$ induces an isomorphism: $M \otimes_R N \simeq N \otimes_{R^{\mathrm{op}}} M$.
2. Show that for all i, there is an isomorphism

$$\mathrm{Tor}_i^R(M,N) \simeq \mathrm{Tor}_i^{R^{\mathrm{op}}}(N,M)\ .$$

Exercice 62. Flatness versus torsion-freeness.

1. Let R be a domain. Show that a module M is flat, then M is torsion-free (i.e. 0 is its only torsion element).

2. Assume now that R is a PID.

   (a) Show that every finitely generated torsion-free module is flat.

   (b) Let M be a torsion free-module, let N be a module with free resolution $P_1 \xrightarrow{d} P_0$. Let

$$x \in \mathrm{Tor}_1^R(M,N) = \mathrm{Ker}\,(M \otimes P_1 \xrightarrow{\mathrm{id}\otimes d} M \otimes P_0).$$

      i. Show that there is a submodule $S \subset M$ such that x is in the image of the map $\mathrm{id} \otimes \mathrm{incl} : S \otimes P_1 \to M \otimes P_1$.

      ii. In the following commutative square, show that the maps $(*)$ and $(**)$ are injective.

$$\begin{tikzcd} S \otimes_R P_1 \arrow[r, "\mathrm{incl}\otimes\mathrm{id}"] \arrow[d, "(*)"', "\mathrm{id}\otimes d"] & M \otimes_R P_1 \arrow[d, "\mathrm{id}\otimes d"] \\ S \otimes_R P_0 \arrow[r, "\mathrm{incl}\otimes\mathrm{id}", "(**)"'] & M \otimes_R P_0 \end{tikzcd} .$$

      iii. Deduce that $x = 0$.

   (c) Show that over a PID, being flat is equivalent to being torsion-free.

3. Take $R = \Bbbk[x,y]$ and let I be the ideal generated by x and y.

   (a) Show that $\mathrm{Tor}_2^{k[x,y]}(\Bbbk,\Bbbk) \neq 0$. (Hint: you may use the projective resolution of $\Bbbk$ given in exercise 35 in the exercise sheet 6)

   (b) Use the short exact sequence $0 \to I \to R \to \Bbbk \to 0$ to deduce that I is not flat (although I is torsion-free).

Exercise sheet n°10

Homology of groups

15. Homology of groups

Exercice 63. Independence from the ground ring $\Bbbk$. Let $\Bbbk$ be a commutative ring.

1. Let C be a complex of projective $\Bbbk$-modules, such that $C_i = 0$ for $i < 0$.
   (a) Prove that $H_i(C) = 0$ for all i if and only if the identity morphism $\mathrm{id}_C : C \to C$ is homotopic to the zero morphism.
   (b) Assume that $H_i(C) = 0$ for all i. Prove that for all $\Bbbk$-modules M, the homology of the complex $C \otimes_{\Bbbk} M$ is zero in all degrees.
2. Let V be a $\Bbbk$-linear representation of a group G. Then we can forget the action of $\Bbbk$, and see V as a $\mathbb{Z}$-linear representation of G.
   (a) Show that for all free $\mathbb{Z}G$-modules F there is an an isomorphism of abelian groups, natural with respect to F:
   $$F \otimes_{\mathbb{Z}G} V \simeq (F \otimes_{\mathbb{Z}} \Bbbk) \otimes_{\Bbbk G} V \; .$$
   (b) Show that there is an isomorphism of abelian groups:
   $$\mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}^{\mathrm{triv}}, V) \simeq \mathrm{Tor}_i^{\Bbbk G}(\Bbbk^{\mathrm{triv}}, V) \; .$$
   [Hint: consider first a free resolution (P, ϵ) of $\mathbb{Z}^{\mathrm{triv}}$, and show that $(P \otimes_{\mathbb{Z}} \Bbbk, \epsilon \otimes \mathrm{id}_{\Bbbk})$ is a free resolution of $\Bbbk^{\mathrm{triv}}$. Use the latter to compute $\mathrm{Tor}_i^{\Bbbk G}(\Bbbk^{\mathrm{triv}}, V)$.]

Exercice 64. An exact sequence involving tensor products. Let $\Bbbk$ be a commutative ring, let V and W be two $\Bbbk$-modules and let $I \subset V$ and $J \subset W$ be two submodules.

1. Let B denote the bicomplex whose rows are given by the Tor-exact sequences, and whose vertical morphisms are given by functoriality with respect to the canonical inclusion $\iota : J \hookrightarrow W$:
$$\begin{array}{ccccccccccc}
\dots \xrightarrow{\partial} & \mathrm{Tor}_1^{\Bbbk}(I,J) & \longrightarrow & \mathrm{Tor}_1^{\Bbbk}(V,J) & \longrightarrow & \mathrm{Tor}_1^{\Bbbk}(V/I,J) & \xrightarrow{\partial} & I \otimes J & \longrightarrow & V \otimes J & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \xrightarrow{\partial} & \mathrm{Tor}_1^{\Bbbk}(I,W) & \to & \mathrm{Tor}_1^{\Bbbk}(V,W) & \to & \mathrm{Tor}_1^{\Bbbk}(V/I,W) & \xrightarrow{\partial} & I \otimes W & \to & V \otimes W & \to 0
\end{array}$$
   Show that $\mathrm{Tot}B$ is quasi-isomorphic to the complex $V/I \otimes J \xrightarrow{\mathrm{id}\otimes\iota} V/I \otimes W$.

2. Show that $H_0(\mathrm{Tot}B) \simeq V/I \otimes W/J$ and deduce that there is a short exact sequence, natural with respect to V and W

$$V \otimes J \oplus I \otimes W \to V \otimes W \to V/I \otimes W/J \to 0\ .$$

Exercice 65. The Künneth theorem. If V is a $\Bbbk$-linear representation of G and W is a $\Bbbk$-linear representation of H, then $V \otimes_{\Bbbk} W$ is a $\Bbbk$-linear representation of $G \times H$, with action given by $(g,h) \cdot (v \otimes w) := (gv, hw)$.

1. Show that we have an isomorphism $(V \otimes W)_{G\times H} \simeq V_G \otimes W_H$ natural with respect to V and W.

   [Hint: use the previous exercise, with $I = \langle gv - v \mid (g,v) \in G \times V\rangle$ and $J = \langle hw - w \mid (h,w) \in H \times W\rangle$.]

2. Assume that $\Bbbk$ is a field. Show that

$$H_i(G \times H, V \otimes W) \simeq \bigoplus_{k+\ell=i} H_k(G,V) \otimes H_\ell(H,W)\ .$$

Exercice 66. Morphisms between cyclic groups. Let $f : C_{qr} \to C_q$ be a surjection of cyclic groups, which sends the generator g of C_{qr} to the generator g' of C_q. Compute the map $f_i : H_i(C_{qr}, \Bbbk) \to H_i(C_q, \Bbbk)$ induced in homology by f.

Exercice 67. Homology of dihedral groups. Let $D_{2q} = C_q \rtimes C_2$ denote the dihedral group with $2q$ elements, q odd. Use the Cartan-Eilenberg stable element formula to compute the homology of D_{2q} with coefficients in $\mathbb{F}_p$, p odd.

Exercice 68. The universal coefficient theorem. Show that for all i there is a short exact sequence natural with respect to G:

$$0 \to H_i(G,\mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk \to H_i(G,\Bbbk) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{i-1}(G,\mathbb{Z}),\Bbbk) \to 0$$

which splits (not naturally with resepct to G). Deduce that $H_1(G,\Bbbk) \simeq G_{\mathrm{ab}} \otimes_{\mathbb{Z}} \Bbbk$.