# Counting and Calculus of Probabilities 

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## 1 Basic notions of set theory

Definition 1.1. A set consists in a collection of elements.
A very usual way for describing a set consists in listing its elements between two braces. For example the possible outcomes resulting from an experiment which consists in rolling a dice form the set of the six numbers $\{1,2,3,4,5,6\}$ which can more concisely be denoted $\{1, \ldots, 6\}$.

Remark 1.1. The empty set is defined to be the set having no elements; it is denoted by $\emptyset$. For example the set consisting in the French cities with more than 30 millions inhabitants is the empty set.

## Operations on sets

One denotes by $A$ and $B$ two subsets of a set $\Omega$. For example one can consider that:

$$
\begin{equation*}
\Omega=\{a, b, c, d, e, f, g\} \quad A=\{a, c, d, g\} \quad \text { and } \quad B=\{a, b, c, f, g\} \tag{1.1}
\end{equation*}
$$

- The union of $A$ and $B$ is denoted by $A \cup B$ and defined as the subset of $\Omega$ whose elements belong to $A$ or $B$. Thus, when $\Omega, A$ and $B$ are as in (1.1) one has $A \cup B=\{a, b, c, d, f, g\}$.
- The intersection of $A$ and $B$ is denoted by $A \cap B$ and defined as the subset of $\Omega$ whose elements belong to $A$ and $B$. Thus, when $\Omega, A$ and $B$ are as in (1.1) one has $A \cap B=\{a, c, g\}$.
- The complement of $A$ is denoted by $\bar{A}$ and defined as the subset of $\Omega$ whose elements do not belong to $A$. Thus, when $\Omega$ and $A$ are as in (1.1) one has $\bar{A}=\{b, e, f\}$.
Problem 1.1. Let $\Omega, A$ and $B$ be as in (1.1). Determine $\bar{B}, \overline{A \cup B}$ and $\overline{A \cap B}$. What do you notice?

Definition 1.2. A set is said to be infinite if it contains infinitely many elements.
Examples 1.1. The set $\mathbb{N}$ of the non-negative integer and the set $\mathbb{R}$ of the real numbers are two very natural examples of infinite sets. Let us also give an example of an infinite set related with a waiting time: A coin is flipped repeatedly until the first time a "Head" appears; one is interested in the random number of times it is thrown. The set of the possible outcomes of this random experiment is infinite, namely it is $\mathbb{N}^{*}=\{1,2,3, \ldots\}$ the set of all the (strictly) positive integers.

Definition 1.3. A set denoted by $S$ is said to be finite if it only contains finitely many elements; the number of these elements is called the cardinality of $S$ and denoted by $\operatorname{Card}(S)$. For instance when $S=\{a, b, c\}$ one has $\operatorname{Card}(S)=3$. By convention the cardinality of the empty set $\emptyset$ is assumed to be equal to 0 (zero).

Theorem 1.1. Let $A$ and $B$ be two subsets of a finite set $\Omega$. One has

$$
\begin{equation*}
\operatorname{Card}(A \cup B)=\operatorname{Card}(A)+\operatorname{Card}(B)-\operatorname{Card}(A \cap B) \tag{1.2}
\end{equation*}
$$

It results from (1.2) that:
Proposition 1.1. Let $A$ be a subset of a finite set $\Omega$ and let $\bar{A}$ be the complement of A. One has

$$
\begin{equation*}
\operatorname{Card}(\bar{A})=\operatorname{Card}(\Omega)-\operatorname{Card}(A) . \tag{1.3}
\end{equation*}
$$

## 2 Counting

Fundamental Principle of Counting: When an experiment $\mathcal{E}$ is composed of $k$ experiments $\mathcal{E}_{1}, \mathcal{E}_{2}, \ldots, \mathcal{E}_{k}$ which respectively have $N_{1}, N_{2}, \ldots, N_{k}$ possible outcomes, then the global number $N$ of the possible outcomes of the experiment $\mathcal{E}$ is $N=N_{1} \times N_{2} \times \ldots \times N_{k}$.

Example 2.1. You will have a diner in a restaurant which is composed of a starter, a main dish and a dessert. In the menu there are $N_{1}=15$ possibilities for the starter, $N_{2}=12$ possibilities for the main dish and $N_{3}=10$ for the dessert. Thus, the global number $N$ of choices for your diner is $N=N_{1} \times N_{2} \times N_{3}=$ $15 \times 12 \times 10=1800$.

Definition 2.1 (arrangement). An arrangement is an ordered sequence of $p$ elements which are chosen among $n$ available elements.

Example 2.2. The sequence of the four digits which is the code of a credit card.
Theorem 2.1.
(a) When repetitions of the same choice of an element are allowed, the number of the arrangements of $p$ elements chosen among $n$ elements is $n^{p}$; notice that in this case one may have $p>n$.
(b) When repetitions of the same choice of an element are not allowed, the number of the arrangements of $p$ elements chosen among $n$ elements is denoted by $A_{n}^{p}$ and satisfies

$$
A_{n}^{p}=\frac{n!}{(n-p)!}=n \times(n-1) \times(n-2) \times \ldots \times(n-p+1)
$$

notice that in this case one necessarily has $p \leq n$.

## Remarks 2.1.

(i) When $k$ is an arbitrary strictly positive integer, the number $k$ !, called "factorial $k^{\prime \prime}$, is the strictly positive integer defined as $k!=1 \times 2 \times \ldots \times k$. Moreover 0! "factorial zero" is defined as $0!=1$.
(ii) For all non-negative integer $n$, one has $A_{n}^{0}=1$.
(iii) When repetitions are not allowed, an arrangement of $n$ elements chosen among $n$ elements is more commonly called a permutation of $n$ elements. In this case one has $A_{n}^{n}=n$ !.

Example 2.3. A building consists in a ground floor and of 8 other floors, 5 persons take together its elevator at the ground floor and then each one of them chooses the floor at which he will go.

1) If each one of the 8 floors can be chosen by several persons, then the number of the possible ways for the five persons of choosing the floors equals to $8^{5}=32768$.
2) If each one the 8 floors can be chosen by at most one person, then the number of the possible ways for the five persons of choosing the floors equals to $A_{8}^{5}=$ $8 \times 7 \times 6 \times 5 \times 4=6720$. (much less than in the previous case!)

Definition 2.2 (combination). A combination is a collection of non-ordered $p$ elements which are chosen among $n$ available elements without repeating the same choice of an element (thus one necessarily has $p \leq n$ ).

Theorem 2.2. The number of the combinations of $p$ elements chosen among $n$ elements is denoted by $C_{n}^{p}$ and satisfies

$$
C_{n}^{p}=\frac{A_{n}^{p}}{p!}=\frac{n!}{p!\times(n-p)!} .
$$

Example 2.4. One has to choose 3 representatives in a group of 100 students; each representative will have the same role as the two others. The number of the possible choices is:

$$
C_{100}^{3}=\frac{100!}{3!\times 97!}=\frac{100 \times 99 \times 98}{6}=50 \times 33 \times 98=161700 .
$$

Remark 2.1. For the sake of convenience for all non-negative integers $n$ and $p$ satisfying $p>n$ one sets $C_{n}^{p}=0$.

## Proposition 2.1 (two important properties of combinations).

(a) For all $p=0,1, \ldots, n$, one has

$$
C_{n}^{p}=C_{n}^{n-p} .
$$

(b) For all integers $n \geq 1$ and $p=1, \ldots, n$, one has

$$
C_{n}^{p}=C_{n-1}^{p-1}+C_{n-1}^{p} .
$$

## Remarks 2.2.

(i) For all $n \geq 0$, one has $C_{n}^{0}=C_{n}^{n}=1$.
(ii) For all $n \geq 1$, one has $C_{n}^{1}=C_{n}^{n-1}=n$.
(iii) For all $n \geq 2$, one has $C_{n}^{2}=C_{n}^{n-2}=n(n-1) / 2$.
(iv) Thanks to the property (b) in Proposition 2.1 the so called "Pascal's triangle" can be obtained. It consists in a step by step method for computing the values of the combinations for all non-negative integers $n$ and $p$ such that $p \leq n$ :

|  | $p=0$ | $p=1$ | $p=2$ | $p=3$ | $p=4$ | $p=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n=0$ | 1 | 0 | 0 | 0 | 0 | 0 |
| $n=1$ | 1 | 1 | 0 | 0 | 0 | 0 |
| $n=2$ | 1 | 2 | 1 | 0 | 0 | 0 |
| $n=3$ | 1 | 3 | 3 | 1 | 0 | 0 |
| $n=4$ | 1 | 4 | 6 | 4 | 1 | 0 |
| $n=5$ | 1 | 5 | 10 | 10 | 5 | 1 |

## 3 Calculus of Probabilities

### 3.1 Probabilities on a finite set

General framework: One performs a random experiment having a finite number $n$ of possible (different) outcomes (two of them can not happen at the same time). Usually, they are called the elementary events and denoted by $\omega_{1}, \ldots, \omega_{n}$. The finite set $\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ of all possible outcomes is called the probability space related with the random experiment, and denoted by $\Omega$. The subsets of $\Omega$, for instance $A=\left\{\omega_{2}, \omega_{3}, \omega_{5}\right\}$, are called the events.

Example 3.1. A large group of tourists is composed of people from 6 different countries: Germany, England, France, India, Japan and Morocco (no one of them can be related to two different countries). One of these tourists is randomly selected and his country is the outcome of this random experiment.
$\rightarrow$ The elementary events associated to this random experiment are $\omega_{1}=$ "Germany", $\omega_{2}=$ "England", $\omega_{3}=$ "France", $\omega_{4}=$ "India", $\omega_{5}=$ "Japan" and $\omega_{6}=" M o r o c c o "$.
$\rightarrow$ One respectively denotes by $U, F$ and $G$ the events $U=$ "the selected tourist is from Europa", $F=$ "the selected tourist is from Asia", and $G=$ "the selected tourist is from Africa". These three events can be expressed in terms of elementary events in the following ways: $U=\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}, F=\left\{\omega_{4}, \omega_{5}\right\}$ and $G=\left\{\omega_{6}\right\}$.
Definition 3.1. Let $\Omega=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ be the probability space associated with some random experiment. For all $k=1, \ldots, n$, one can associate to the elementary event $\omega_{k}$ a weight called probability of $\omega_{k}$ and denoted by $\mathbb{P}\left(\left\{\omega_{k}\right\}\right)$. This number $\mathbb{P}\left(\left\{\omega_{k}\right\}\right)$, which lies between 0 and 1 , provides an estimate of the level of plausibility that one associates to $\omega_{k}$ (when $\mathbb{P}\left(\left\{\omega_{k}\right\}\right)$ is close to 0 this means that there is little possibility that $\omega_{k}$ be realized, when $\mathbb{P}\left(\left\{\omega_{k}\right\}\right)$ is close to 1 this means that there is strong possibility that $\omega_{k}$ be realized). Having weighted the elementary events, one can then associate to any event $A \subseteq \Omega$ a probability given by the formula

$$
\begin{equation*}
\mathbb{P}(A)=\sum_{\omega_{k} \in A} \mathbb{P}\left(\left\{\omega_{k}\right\}\right) . \tag{3.1}
\end{equation*}
$$

$\rightarrow$ Since one knows that the event $\Omega$ will certainly happen whatever the outcome of the random experiment might be, one says that $\Omega$ is a sure event, and one always associates to it the probability $\mathbb{P}(\Omega)=1$. This means that one always imposes to the weights $\mathbb{P}\left(\left\{\omega_{1}\right\}\right), \ldots, \mathbb{P}\left(\left\{\omega_{n}\right\}\right)$ to satisfy the condition:

$$
\begin{equation*}
\sum_{k=1}^{n} \mathbb{P}\left(\left\{\omega_{k}\right\}\right)=\mathbb{P}\left(\left\{\omega_{1}\right\}\right)+\ldots+\mathbb{P}\left(\left\{\omega_{n}\right\}\right)=1 \tag{3.2}
\end{equation*}
$$

$\rightarrow$ The empty set $\emptyset$ is called an impossible event, and one always associates to it the probability $\mathbb{P}(\emptyset)=0$.

Example 3.1. (revisited) The following table provides the proportion of tourist from each country

| Country | Germany | England | France | India | Japan | Morocco |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Proportion | $30 \%$ | $25 \%$ | $5 \%$ | $15 \%$ | $23 \%$ | $2 \%$ |

1) Determine $\mathbb{P}\left(\left\{\omega_{1}\right\}\right), \ldots, \mathbb{P}\left(\left\{\omega_{6}\right\}\right)$.
2) Compute $\mathbb{P}(U), \mathbb{P}(F)$ and $\mathbb{P}(G)$ the probabilities of the events $U, F$ and $G$.

Proposition 3.1 (main property of a probability). Let $A$ and $B$ be two events in a probability space $\Omega$. One has

$$
\begin{equation*}
\mathbb{P}(A \cup B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A \cap B) . \tag{3.3}
\end{equation*}
$$

## Remarks 3.1.

- When the events $A$ and $B$ are incompatible (that is $A$ and $B$ can not be realized simultaneously which can be translated by $A \cap B=\emptyset$ ), then one has

$$
\begin{equation*}
\mathbb{P}(A \cup B)=\mathbb{P}(A)+\mathbb{P}(B) \tag{3.4}
\end{equation*}
$$

- Let $A$ be an event and $\bar{A}$ the opposite event (that is $\bar{A}$ is the complement of A), then one has

$$
\begin{equation*}
\mathbb{P}(\bar{A})=1-\mathbb{P}(A) ; \tag{3.5}
\end{equation*}
$$

in fact (3.5) is a consequence of (3.4).
Definition 3.2. Let $\Omega=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ be the probability space associated with some random experiment. One says that the elementary events $\omega_{1}, \ldots, \omega_{n}$ are equiprobable when they all have same probability to be realized that is one has

$$
\mathbb{P}\left(\left\{\omega_{1}\right\}\right)=\mathbb{P}\left(\left\{\omega_{2}\right\}\right)=\ldots=\mathbb{P}\left(\left\{\omega_{n}\right\}\right)=\frac{1}{n} .
$$

Example 3.2. If the experiment consists in rolling a fair dice. Then one has $\Omega=\{1,2,3,4,5,6\}=\{1, \ldots, 6\}$ and, for all $k=1, \ldots, 6, \mathbb{P}(\{k\})=1 / 6$.

Proposition 3.2. Let $\Omega$ be a finite probability space that is the set of the possible outcomes of some random experiment. When the outcomes are equiprobable, then the probability of any event $A \subseteq \Omega$ is given by the formula:

$$
\begin{equation*}
\mathbb{P}(A)=\frac{\operatorname{Card}(A)}{\operatorname{Card}(\Omega)} . \tag{3.6}
\end{equation*}
$$

Thus, computing probability $\mathbb{P}(A)$ can be done by counting the number of outcomes in the event $A$ and then dividing it by the total number of outcomes in the probability space $\Omega$.

Example 3.2. (revisited) The probability of the event

$$
A=\text { "the outcome is an even number" }=\{2,4,6\}
$$

equals to $\mathbb{P}(A)=3 / 6=1 / 2$.
Definition 3.3. One says that two events $A$ and $B$ in the same probability space $\Omega$ are independent if one has

$$
\mathbb{P}(A \cap B)=\mathbb{P}(A) \mathbb{P}(B)
$$

Roughly speaking this means that the realization of $A$ has no influence on that of $B$ and vice versa.

Problem 3.1. In a population of young people having less than 25 years old, $80 \%$ of the individuals are students, $70 \%$ of the individuals are vaccinated against the hepatitis B, and $10 \%$ of the individuals have motorcycles. Moreover, $56 \%$ of the individuals are students who are vaccinated against the hepatitis B, and $8 \%$ of the individuals have motorcycles and are vaccinated against the hepatitis $B$.

One randomly selects an individual in this population and one denotes by $E, V$ and $M$ the following events: $E=$ "the selected individual is a student", $V=" t h e$ selected individual is vaccinated against the hepatitis $B^{\prime \prime}$ and $M=" t h e ~ s e l e c t e d$ individual has a motocycle".

1) Are the events $E$ and $V$ independent?
2) Are the events $M$ and $V$ independent?

Definitions 3.1. Let $A_{1}, A_{2}$ and $A_{3}$ be 3 events in the same probability space. One says that they are pairwise independent if one has: $\mathbb{P}\left(A_{1} \cap A_{2}\right)=\mathbb{P}\left(A_{1}\right) \mathbb{P}\left(A_{2}\right)$, $\mathbb{P}\left(A_{1} \cap A_{3}\right)=\mathbb{P}\left(A_{1}\right) \mathbb{P}\left(A_{3}\right)$ and $\mathbb{P}\left(A_{2} \cap A_{3}\right)=\mathbb{P}\left(A_{2}\right) \mathbb{P}\left(A_{3}\right)$. Moreover, when they satisfy in addition to these three equalities the fourth equality $\mathbb{P}\left(A_{1} \cap A_{2} \cap A_{3}\right)=$ $\mathbb{P}\left(A_{1}\right) \mathbb{P}\left(A_{2}\right) \mathbb{P}\left(A_{3}\right)$ then one says that they are mutually independent.

Problem 3.2. 1) Let $A_{1}, A_{2}, A_{3}$ and $A_{4}$ be 4 events in the same probability space. According to you when are they said to be pairwise independent? According to you when are they said to be mutually independent?
2) Similarly to what has be done in Problem 3.1,
(a) provide an example of 3 events which are pairwise independent,
(b) provide an example of 3 events which are mutually independent,
(c) provide an example of 3 events which are pairwise independent without being mutually independent.

### 3.2 Probabilities on an infinite set

A random experiment may have an infinity of possible outcomes. Imagine that one is interested in a random experiment whose outcome is a waiting time; as for instance the number of days someone has to wait for finding a buyer for his house he has just offered for sale. There are infinitely many possible outcomes for such an experiment since one can not ensure with absolute certainty that the house will be sold even after a very long period of time. More precisely the set of the possible outcomes is $\Omega=\{1,2,3, \ldots\}=\mathbb{N}^{*}=$ the set of the strictly positive integers.

The notions we have presented in the frameworks of finite probability spaces can be generalized to infinite probability spaces. Let us mention the following points.

Let $\left(A_{l}\right)_{l \in \mathbb{N}}$ be an infinite sequence of events in a (infinite) probability space $\Omega$.

- One denotes by $\bigcap_{l=1}^{+\infty} A_{l}$ the intersection of the $A_{l}$ 's that is the set of the possible outcomes which belong to all the events $A_{l}$.
- One denotes by $\bigcup_{l=1}^{+\infty} A_{l}$ the union of the $A_{l}$ 's that is the set of the possible outcomes which belong to at least one of the events $A_{l}$.

Proposition 3.3. When the events $A_{l}$ are pairwise incompatible, that is they satisfy $A_{i} \cap A_{j}=\emptyset$ as soon as $i \neq j$, then one has

$$
\begin{equation*}
\mathbb{P}\left(\bigcup_{l=1}^{+\infty} A_{l}\right)=\sum_{l=1}^{+\infty} \mathbb{P}\left(A_{l}\right)=\lim _{L \rightarrow+\infty} \sum_{l=1}^{L} \mathbb{P}\left(A_{l}\right) . \tag{3.7}
\end{equation*}
$$

### 3.3 Conditional probabilities

How should one modify the probability associated to an event when an additional information related to it is available ? The concept of conditional probability allows to answer to this question.

Definition 3.4. Let $\Omega$ be a probability space and let $B \subseteq \Omega$ be an event such that $\mathbb{P}(B) \neq 0$. For any event $A$ in $\Omega$, the conditional probability of $A$ given $B$ is denoted by $\mathbb{P}(A / B)$ and defined as:

$$
\begin{equation*}
\mathbb{P}(A / B)=\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \tag{3.8}
\end{equation*}
$$

What makes the concept of conditional probability to be useful is that often one can associate a value to $\mathbb{P}(A / B)$ in a direct way, and then deduce from it the value of $\mathbb{P}(A \cap B)$ thanks to the formula

$$
\begin{equation*}
\mathbb{P}(A \cap B)=\mathbb{P}(A / B) \mathbb{P}(B) \tag{3.9}
\end{equation*}
$$

Example 3.3. 10 keys are almost identical but only one of them allows to open some gate. Somebody who can not identify this good key wants to open the gate. He decides to try the keys one after the other, each key which does not work is set aside just after its trial.

Let us calculate the probability that he opens the door at the second attempt. For $i=1, \ldots, 10$, let the event $A_{i}=$ "he opens the gate at the $i$-th attempt". Thus, the probability we want to compute can be expressed as $\mathbb{P}\left(\bar{A}_{1} \cap A_{2}\right)$. One has $\mathbb{P}\left(\bar{A}_{1}\right)=9 / 10$ since at the beginning of the first attempt, only one among the 10 keys allows to open the gate. Moreover, one has $\mathbb{P}\left(A_{2} / \bar{A}_{1}\right)=1 / 9$ since at the beginning of the second attempt there remain 9 keys and only one among them allows to open the gate. Finally, using (3.9) one gets that $\mathbb{P}\left(A_{1} \cap A_{2}\right)=$ $\mathbb{P}\left(A_{2} / \bar{A}_{1}\right) \mathbb{P}\left(\bar{A}_{1}\right)=1 / 9 \times 9 / 10=1 / 10=0,1$.

The formula (3.9) can be generalized in the following way:
Proposition 3.4. Let an integer $m \geq 2$ and let $A_{1}, A_{2}, \ldots, A_{m}$ be $m$ events in the same probability space $\Omega$ which satisfy $\mathbb{P}\left(A_{1} \cap A_{2} \cap \ldots \cap A_{m-1}\right) \neq 0$. Then, one has

$$
\begin{align*}
& \mathbb{P}\left(A_{1} \cap A_{2} \cap \ldots \cap A_{m}\right)  \tag{3.10}\\
& =\mathbb{P}\left(A_{1}\right) \times \mathbb{P}\left(A_{2} / A_{1}\right) \times \mathbb{P}\left(A_{3} / A_{1} \cap A_{2}\right) \times \ldots \times \mathbb{P}\left(A_{m} / A_{1} \cap A_{2} \cap \ldots \cap A_{m-1}\right)
\end{align*}
$$

Example 3.3. (revisited) Let us calculate the probability that he opens the door at the 5 -th attempt. The probability we want to compute is in fact

$$
\mathbb{P}\left(\bar{A}_{1} \cap \bar{A}_{2} \cap \bar{A}_{3} \cap \bar{A}_{4} \cap A_{5}\right) .
$$

It follows from (3.10) that the latter probability is equal to

$$
\begin{aligned}
& \mathbb{P}\left(\bar{A}_{1}\right) \times \mathbb{P}\left(\bar{A}_{2} / \bar{A}_{1}\right) \times \mathbb{P}\left(\bar{A}_{3} / \bar{A}_{1} \cap \bar{A}_{2}\right) \times \mathbb{P}\left(\bar{A}_{4} / \bar{A}_{1} \cap \bar{A}_{2} \cap \bar{A}_{3}\right) \times \mathbb{P}\left(A_{5} / \bar{A}_{1} \cap \bar{A}_{2} \cap \bar{A}_{3} \cap \bar{A}_{4}\right) \\
& \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8} \times \frac{6}{7} \times \frac{1}{6}=\frac{1}{10}=0,1
\end{aligned}
$$

Remark 3.1. Let $A$ and $B$ be two events in the same probability space. One assumes that $\mathbb{P}(B) \neq 0$. When $A$ and $B$ are independent then one has that $\mathbb{P}(A / B)=\mathbb{P}(A)$. Conversely, when $\mathbb{P}(A / B)=\mathbb{P}(A)$ then $A$ and $B$ are independent.

Definition 3.5. One says that events $B_{1}, \ldots, B_{m}$ form a partition of a probability space $\Omega$ when they satisfy the following 3 properties.
(i) None of the events $B_{1}, \ldots, B_{m}$ is an impossible event, that is one has $B_{i} \neq \emptyset$ for all $i=1, \ldots, m$.
(ii) The events $B_{1}, \ldots, B_{m}$ are pairwise incompatible, that is one has $B_{i} \cap B_{i^{\prime}}=\emptyset$ as soon as $i \neq i^{\prime}$.
(iii) One of the events $B_{1}, \ldots, B_{m}$ will necessarily be realized, that is one has $\Omega=B_{1} \cup \ldots \cup B_{m}=\bigcup_{i=1}^{m} B_{i}$.

Example 3.4. The probability space $\Omega$ consists in the video cards available to a laptop manufacturer. $20 \%$ of these cards come from the supplier $1,40 \%$ from the supplier 2 and $40 \%$ from the supplier 3. For $i=1,2,3$ the event $B_{i}$ consists in the video cards coming from the supplier $i$. The three events $B_{1}, B_{2}$ and $B_{3}$ form a partition of $\Omega$.

A statistical survey has shown that the proportions of defective cards are 2\% for those coming from the supplier 1, $5 \%$ for those coming from the supplier 2, and $7 \%$ for those coming from the supplier 3.

A quality inspection is made on the assembly line of this laptop manufacturer by randomly selecting a laptop.

1) Calculate the probability that the video card of the selected laptop be defective.
2) If it turns out that this card is defective, what is then the probability that it comes from the supplier 1?
3) Same question as 2) but for the supplier 2.
4) Same question as 2) but for the supplier 3.

There are two successive levels of randomness in the experiment related with quality inspection described in Example 3.4; the first level consists in the choice of a supplier and the second level in that of a video card. For dealing with such type of random experiment one often has to make use of conditional probabilities; more precisely one needs to use the following fundamental theorem:

Theorem 3.1. Let $\Omega$ be a probability space and $B_{1}, B_{2}, \ldots, B_{m}$ events forming a partition of $\Omega$. The following fundamental two results hold.
(a) (Formula of total probability) For each event $A \subseteq \Omega$,

$$
\begin{align*}
\mathbb{P}(A) & =\mathbb{P}\left(A / B_{1}\right) \mathbb{P}\left(B_{1}\right)+\mathbb{P}\left(A / B_{2}\right) \mathbb{P}\left(B_{2}\right)+\ldots+\mathbb{P}\left(A / B_{m}\right) \mathbb{P}\left(B_{m}\right) \\
& =\sum_{i=1}^{m} \mathbb{P}\left(A / B_{i}\right) \mathbb{P}\left(B_{i}\right) \tag{3.11}
\end{align*}
$$

(b) (Bayes Formula) For each event $A \subseteq \Omega$ such that $\mathbb{P}(A) \neq 0$, one has, for all $j=1,2, \ldots, m$,

$$
\begin{align*}
\mathbb{P}\left(B_{j} / A\right) & =\frac{\mathbb{P}\left(A / B_{j}\right) \mathbb{P}\left(B_{j}\right)}{\mathbb{P}(A)} \\
& =\frac{\mathbb{P}\left(A / B_{j}\right) \mathbb{P}\left(B_{j}\right)}{\sum_{i=1}^{m} \mathbb{P}\left(A / B_{i}\right) \mathbb{P}\left(B_{i}\right)} . \tag{3.12}
\end{align*}
$$

Answers to the questions in Example 3.4 1) Let the event $A=$ "the video card of the selected laptop is defective". It follows from the formula of total probability given in (3.11) that

$$
\begin{aligned}
\mathbb{P}(A) & =\mathbb{P}\left(A / B_{1}\right) \mathbb{P}\left(B_{1}\right)+\mathbb{P}\left(A / B_{2}\right) \mathbb{P}\left(B_{2}\right)+\mathbb{P}\left(A / B_{3}\right) \mathbb{P}\left(B_{3}\right) \\
& =0,02 \times 0,2+0,05 \times 0,4+0,07 \times 0,4 \\
& =0,052 \quad(5,2 \%)
\end{aligned}
$$

2), 3) and 4) Using the Bayes formula given in (3.12) one gets that

$$
\begin{align*}
& \mathbb{P}\left(B_{1} / A\right)=\frac{\mathbb{P}\left(A / B_{1}\right) \mathbb{P}\left(B_{1}\right)}{\mathbb{P}(A)}=\frac{0,02 \times 0,2}{0,052} \simeq 0,077 \\
& \mathbb{P}\left(B_{2} / A\right)=\frac{\mathbb{P}\left(A / B_{2}\right) \mathbb{P}\left(B_{2}\right)}{\mathbb{P}(A)}=\frac{0,05 \times 0,4}{0,052} \simeq 0,385 \\
& \mathbb{P}\left(B_{3} / A\right)=\frac{\mathbb{P}\left(A / B_{3}\right) \mathbb{P}\left(B_{3}\right)}{\mathbb{P}(A)}=\frac{0,07 \times 0,4}{0,052} \simeq 0,538
\end{align*}
$$

## 4 Problems on counting

Problem 4.1. 1) In how many ways can 7 differently colored balls be arranged in a row ?
2) In how many ways can 12 people be seated on a bench if only 4 seats are available ?
3) A committee of 3 members is to be formed consisting of one representative each from labor, management, and the public. If there are 2 possible representatives from labor, 4 from management, and 6 from the public, determine how many different committees can be formed.
4) It is required to seat 7 men and 6 women in a row so that the women occupy the even places. How many such arrangements are possible?

Problem 4.2. Calculate how many 4-digit numbers can be formed with the 10 digits $0,1,2, \ldots, 9$ in each one of the following three cases.
(a) The first digit must not be zero and repetitions are allowed.
(b) The last digit must be even and repetitions are not allowed.
(c) The first digit must be odd, the last digit must be even and repetitions are not allowed.

Problem 4.3. Mister Smith has to arrange on a shelf 5 different science books, 6 different history books, and 2 different statistics books. Calculate how many different dispositions are possible in each one of the following two cases.
(a) The books in each particular subject must all stand together.
(b) Only the science books must stand together.

Problem 4.4. One intends to arrange 5 red balls, 2 white balls, and 3 blue balls in a row. How many different dispositions are possible, knowing that all the balls of the same color are not distinguishable from each other ?

Problem 4.5. Compute the number of possible ways that 7 people can be seated at a round table in each one of the following three cases.
(a) They can sit anywhere.
(b) Two particular people must sit next to each other.
(c) Two particular people must not sit next to each other.

## 5 Problems on calculus of probabilities

Problem 5.1. A box contains 12 marbles: 5 red and 7 blue. Consider an experiment that consists in taking one marble from the box then replacing it in the box and drawing a second marble from the box.

1) What is the probability that the first marble which is taken be red and the second one be blue?
2) What is the probability that the two marbles which are successively taken be of the same color?

Problem 5.2. Repeat Problem 5.1 when the second marble is drawn without replacing the first marble.

Problem 5.3. $M \& M$ sweets are of varying colors and the different colors occur in different proportions. The table below gives the probability that a randomly chosen M\&M has each color, but the value for tan candies is missing.

| Color | Brown | Red | Yellow | Green | Orange | Tan |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Probability | 0,3 | 0,2 | 0,2 | 0,1 | 0,1 | $?$ |

1) What value must the missing probability be ?
2) You draw an M\&M at random from a packet. What is the probability of each of the following events?
(i) You get a brown one or a red one.
(ii) You don't get a yellow one.
(iii) You don't get either an orange one or a tan one.
(iv) You get one that is brown or red or yellow or green or orange or tan.

Problem 5.4. Two sisters maintain that they can communicate telepathically. To test this assertion, you place the sisters in separate rooms and show the sister A three cards one after the other. Each card is equally likely to depict either a circle or a star or a square. For each card presented to the sister A, the sister $B$ writes down "circle", or "star" or "square", depending on what she believes the sister $A$ to be looking at. Under the assumptions that the two sisters can not communicate telepathically and that successive attempts at matching cards are mutually independent, calculate the probabilities of the following events.
(i) The sister $B$ does not correctly match any one of the 3 cards.
(ii) The sister $B$ correctly matches at least 1 of the 3 cards.
(iii) The sister $B$ correctly matches only 1 of the 3 cards.
(iv) The sister B correctly matches at least 2 of the 3 cards.

Problem 5.5. Stores A, B, and C have 50, 75, and 100 employees, and respectively 50, 60, and 70 percent of these are women. Resignations are equally likely among all employees, regardless of sex. One employe resigns and this is a woman. What is the probability that she works in store $C$ ?

Problem 5.6. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 2 piles of 26 cards each. What is the probability that each pile has exactly 2 aces?

Problem 5.7. An examination consists of multiple-choice questions, each having five possible answers. Suppose you are a student taking the exam. Also suppose that you reckon you have probability 0,75 of knowing the answer to any question that may be asked and that, if you do not know, you intend to guess an answer with probability $1 / 5$ of being correct. What is the probability you will give the correct answer to a question?

Problem 5.8. I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads. I take one of the coins at random from my pocket and toss it one time.

1) What is the probability that this coin comes up tail?
2) If this coin comes up head, what is then the probability that it be the coin with two heads?
3) If I toss this coin two times (instead of one time), what is the probability that it comes up head the first time and tail the second time?

Problem 5.9. Bill and George go to target shooting together. Both shoot at a target at the same time. Suppose Bill hits the target with probability 0,7, whereas George, independently, hits the target with probability 0,4.

1) What is the probability that no shot hits the target?
2) What is the probability that at least one shot hits the target?
3) What is the probability that exactly one shot hits the target?
4) Given that one shot hit the target, what is the probability that it was George's shot?
5) Given that the target is hit, what is the probability that George hit it ?

Problem 5.10. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be free, claiming that their would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from $1 / 3$ to $1 / 2$, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

Problem 5.11. 60\% of the families in a certain community own their own car, $30 \%$ own their own home, and $20 \%$ own both their own car and their own home. If a family is randomly chosen, what is the probability that this family own a car or a house but not both?

