Continuous Random Variables

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Contents

1 Some generalities

First recall that:

Definition 1.1. *A random variable denoted by X (one may choose another letter Y, Z, and so on) is a numerical quantity related with a random experiment; for instance X is a taxpayer's income chosen at random. More precisely, a random variable is a map from a probability space Ω into the set of the real number $\mathbb{R}$.*

Remark 1.1. *Throughout this chapter one restricts to continuous random variables; that is to random variables whose possible values may be all real numbers in some intervals. When X is such a random variable, in contrast with a discrete random variable, it is irrelevant to focus on $\mathbb{P}(X = t)$, the probability that X be equal to some value t, since one has in general $\mathbb{P}(X = t) = 0$. It is much more relevant to determine $\mathbb{P}(a \leq X \leq b)$, the probability that X lies between two arbitrary real numbers a and b such that $a < b$. Throughout this chapter one always assumes that this probability is given by the integral:*

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\, dx, \tag{1.1}$$

where f_X is a fonction (depending on the random variable X), **with nonnegative values**, *called* **density** *(function) of the continuous random variable X, and satisfying*

$$\int_{-\infty}^{+\infty} f_X(x)\, dx = 1.$$

One always assumes that the two integrals

$$\int_{-\infty}^{+\infty} x f_X(x)\, dx \quad \text{and} \quad \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$$

are well-defined and finite. One mentions that the shape of the graph of the density function f_X is rather similar to that of the histogram of the statistical variable which is modeled by the random variable X. Also, one mentions that the fact that X has a density implies that $\mathbb{P}(X = t) = 0$, for all real number t, and consequently that

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X < b)\,. \tag{1.2}$$

Remark 1.2. (Graphical interpretation of the integral of a nonnegative function) *Let f be a function* **with nonnegative values** *defined on $\mathbb{R}$ and let a and b be two arbitrary real numbers such that $a < b$.*

- *The integral $\displaystyle\int_a^b f(x)\,dx$ is the area of the surface which lies between the graph of f, the abscissa axis, and the two vertical lines of equations $x = a$ and $x = b$.*
- *The integral $\displaystyle\int_{-\infty}^a f(x)\,dx$ is the area of the surface which lies between the graph of f and the abscissa axis and is situated at the left side of the vertical line of equation $x = a$.*
- *The integral $\displaystyle\int_b^{+\infty} f(x)\,dx$ is the area of the surface which lies between the graph of f and the abscissa axis and is situated at the right side of the vertical line of equation $x = b$.*
- *The integral $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ is the area of the surface which lies between the graph of f and the abscissa axis .*

Thus, one has:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^b f(x)\,dx + \int_b^{+\infty} f(x)\,dx\,.$$

Definition 1.2. (Cumulative function) *The* **cumulative function** *of a random variable X is the function with values in the interval $[0\,;1]$ denoted by F_X and defined [1], for all real number t, by*

$$F_X(t) = \mathbb{P}(X < t)\,. \tag{1.3}$$

Remark 1.3. (Important properties of cumulative function)

(i) The probability $\mathbb{P}(a \leq X \leq b)$ is connected to the cumulative function F_X through the important formula:

$$\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a)\,; \tag{1.4}$$

notice that, in view of the equalities (1.2), the important equality (1.4) remains valid when $\mathbb{P}(a \leq X \leq b)$ is replaced by $\mathbb{P}(a < X \leq b)$, $\mathbb{P}(a \leq X < b)$ or $\mathbb{P}(a < X < b)$.

(ii) The cumulative function F_X is the primitive of the density function f_X given, for all real number t, by the integral

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx\,.$$

Thus, for any real number t at which the cumulative function F_X is derivable one has

$$F'_X(t) = f_X(t)\,.$$

[1] Many other authors define the cumulative function F_X as $F_X(t) = \mathbb{P}(X \leq t)$, for all real number t; this slight difference with respect to our definition is not really important.

(iii) F_X *is a non-decreasing function whose limit at* $-\infty$ *equals to* 0 *(zero), and whose limit at* $+\infty$ *equals to* 1 *(one).*

Definition 1.3. (Expectation) *The expectation of the random variable* X *is denoted by* $\mathbb{E}(X)$ *and defined as:*

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx\,. \tag{1.5}$$

Roughly speaking $\mathbb{E}(X)$ *can be viewed as the average value of the random variable* X*.*

Definition 1.4. (Variance and Standard Deviation) *The variance of the random variable* X *is denoted by* $\mathrm{Var}(X)$ *and defined as:*

$$\mathrm{Var}(X) = \mathbb{E}\Big[\big(X - \mathbb{E}(X)\big)^2\Big] = \mathbb{E}(X^2) - \big(\mathbb{E}(X)\big)^2 = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx - \left(\int_{-\infty}^{+\infty} x f_X(x)\, dx\right)^2. \tag{1.6}$$

The standard deviation of X *is denoted by* $\sigma(X)$ *and defined as*

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}\,. \tag{1.7}$$

Roughly speaking $\sigma(X)$ *can be viewed as the average distance between the values of* X *and its expectation* $\mathbb{E}(X)$*.*

Definition 1.5. (Independent random variables)

Two random variables X_1 *et* X_2 *are said to be* **independent***, if for all real numbers* a_1, b_1 *and* a_2, b_2 *satisfying* $a_1 < b_1$ *and* $a_2 < b_2$ *the two events* $\{a_1 \le X_1 \le b_1\}$ *and* $\{a_2 \le X_2 \le b_2\}$ *are independent.*

More generally, for any integer $n \ge 2$*,* n *independent random variables* $X_1, X_2, \ldots, X_n$ *are said to be* **mutually independent***, if for all real numbers* a_1, b_1 *and* a_2, b_2 *and* $\ldots$ *and* a_n, b_n *satisfying* $a_1 < b_1$ *and* $a_2 < b_2$ *and* $\ldots$ *and* $a_n < b_n$*, the* n *events*

$$\{a_1 \le X_1 \le b_1\} \text{ and } \{a_2 \le X_2 \le b_2\} \text{ and } \ldots \text{ and } \{a_n \le X_n \le b_n\}$$

are mutually independent.

2 Exponential distribution and Weibull distribution

Definition 2.1. (Exponential distribution) *Let* λ *be a strictly positive real number. One says that a continuous random variable* X *has an exponential distribution of parameter* λ*, if its density function* f_X *satisfies*

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x)\,, & \text{for all real number } x > 0, \\ \\ 0 \quad \text{for all real number } x \le 0. \end{cases} \tag{2.1}$$

Proposition 2.1. *When a random variable* X *has an exponential distribution of parameter* λ *then its expectation and variance are given by the formulas:*

$$\mathbb{E}(X) = 1/\lambda \quad \text{and} \quad \mathrm{Var}(X) = 1/\lambda^2 \tag{2.2}$$

Weibull distribution is a generalization of exponential distribution:

Definition 2.2. (Weibull distribution) *Let α and λ be two strictly positive real numbers. One says that a continuous random variable X has a Weibull distribution with parameters α et λ, if its density function f_X satisfies*

$$f_X(x) = \begin{cases} \lambda \alpha x^{\alpha-1} \exp(-\lambda x^{\alpha})\,, & \textit{for all real number } x > 0, \\ 0 \quad \textit{for all real number } x \leq 0. \end{cases} \tag{2.3}$$

Remark 2.1. *Exponential distribution and more generally Weibull distribution are very useful in modeling of lifetime (human being, appliance, and so on).*

When a random variable X has an exponential distribution, or more generally a Weibull distribution, then X is with positive values, more precisely $\mathbb{P}(X > 0) = 1$. Actually, this is a consequence of the fact that the density function of X vanishes on the set of the negative real numbers.

For any random variable X with positive values, the survival function S_X is defined as:

$$S_X(t) = \mathbb{P}(X \geq t)\,, \quad \textit{for all real number } t \geq 0. \tag{2.4}$$

Thus, for all real number $t \geq 0$, one has $S_X(t) = 1 - F_X(t)$, where F_X is the cumulative function of X. When a random variable X has a Weibull distribution with parameters α and λ, then its survival function S_X satisfies

$$S_X(t) = \exp(-\lambda t^{\alpha})\,, \quad \textit{for all real number } t \geq 0. \tag{2.5}$$

Thus, in the particular case of the exponential distribution (where $\alpha = 1$) one has

$$S_X(t) = \exp(-\lambda t)\,, \quad \textit{for all real number } t \geq 0. \tag{2.6}$$

3 Normal distribution

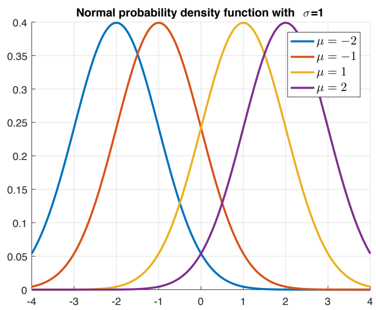
When a random variable X results from many independent causes such that none of them dominates the others, then one can consider that X has a normal distribution. This is for instance the case when X is the total weight of a large packet of sweets since this total weight mainly results from the addition of the very small independent weights of many sweets.

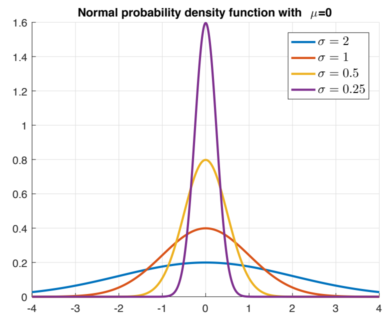
Definition 3.1. (normal distribution) *Let μ (pronounce " mu ") be an arbitrary real number. Let σ (pronounce " sigma ") be an arbitrary strictly positive real number. One says that a continuous random variable X has* **a normal distribution with mean (that is expectation) μ and standard deviation** *σ (or variance σ^2), if its density function f_X satisfies*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right), \quad \textit{for all real number } x\,. \tag{3.1}$$

Observe that the notation $X \sim \mathcal{N}(\mu, \sigma^2)$ means that the random variable X has a normal distribution with mean μ and standard deviation σ. Also observe that in the very important particular case where $\mu = 0$ and $\sigma = 1$ the normal distribution is said to be **standard**.

Remark 3.1. *The following two figures show that the graph of the density f_X of a normal distribution looks like a bell curve which is symmetric with respect to the vertical line of equation $x = \mu$. The larger is the parameter σ the more flat is the curve, and the smaller it is the sharper is the curve. The function f_X reaches its maximum when $x = \mu$ and then one has $f_X(\mu) = \frac{1}{\sigma\sqrt{2\pi}}$.*





Remark 3.2. *When a random variable X has a normal distribution with mean μ and standard deviation $\sigma > 0$, then the random variable*

$$Z = \frac{X - \mu}{\sigma} \tag{3.2}$$

has a standard normal distribution. Moreover F_X and F_Z, the cumulative functions of X and Z, are closely connected through the equality

$$F_X(t) = F_Z\left(\frac{t-\mu}{\sigma}\right), \quad \text{for all real number } t\,. \tag{3.3}$$

On another hand, when a random variable T has a standard normal distribution, then, for all real number μ and for all strictly positive real number σ, the random variable

$$Y = \sigma T + \mu \tag{3.4}$$

has a normal distribution with mean μ and standard deviation σ.

Remark 3.3. *Let Z be a random variable having a standard normal distribution then its cumulative function F_Z satisfies, for all real number t,*

$$F_Z(t) = 1 - F_Z(-t)\,. \tag{3.5}$$

As a consequence, one has $F_Z(0) = 1/2$. Let us point out that the equality (3.5) is very useful. In fact it is derived from symmetry with respect to the ordinate axis (that is the vertical ligne with equation $x = 0$) of the graph of f_Z the density function of Z.

STANDARD NORMAL DISTRIBUTION TABLE

Entries represent Pr($Z \leq z$). The value of z to the first decimal is given in the left column. The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr($Z \leq z$)							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr($Z \leq z$)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

Example 3.1. *The random variable X denotes the profit (expressed in thousands of euros) which will be realized the next month by a small entreprise. One assumes that X has a normal distribution with mean 10 and standard deviation 4. By using the standard normal distribution table calculate the following probabilities: $\mathbb{P}(X \leq 12)$, $\mathbb{P}(X < 9)$, $\mathbb{P}(X > 11)$, $\mathbb{P}(X \geq 6)$ and $\mathbb{P}(-2 < X < 13)$.*

One knows from Remark 3.2 that the random variable

$$Z = \frac{X - 10}{4}$$

has a standard normal distribution, and that its cumulative function F_Z satisfies, for all real number t,

$$F_X(t) = F_Z\left(\frac{t - 10}{4}\right), \tag{3.6}$$

where F_X denotes the cumulative function of X. Using (1.3), the equality $\mathbb{P}(X = 12) = 0$, (3.6) and the standard normal distribution table, one obtains that

$$\mathbb{P}(X \leq 12) = F_X(12) = F_Z\left(\frac{12 - 10}{4}\right) = F_Z(0,5) = 0,6915\,.$$

Using (1.3), (3.6), (3.5) and the standard normal distribution table, one gets that

$$\mathbb{P}(X < 9) = F_X(9) = F_Z\left(\frac{9 - 10}{4}\right) = F_Z(-0,25) = 1 - F_Z(0,25) = 1 - 0,5987 = 0,4013\,.$$

Using $\{X \leq 11\} = \overline{\{X > 11\}}$ (that is the fact that $\{X \leq 11\}$ is the opposite event of $\{X > 11\}$), the equality $\mathbb{P}(X = 11) = 0$, (1.3), (3.6) and the standard normal distribution table, one obtains that

$$\mathbb{P}(X > 11) = 1 - F_X(11) = 1 - F_Z\left(\frac{11 - 10}{4}\right) = 1 - F_Z(0,25) = 0,4013\,.$$

Using the equality $\{X < 6\} = \overline{\{X \geq 6\}}$, (1.3), (3.6), (3.5) and the standard normal distribution table, one gets that

$$\mathbb{P}(X \geq 6) = 1 - \mathbb{P}(X < 6) = 1 - F_X(6) = 1 - F_Z\left(\frac{6 - 10}{4}\right) = 1 - F_Z(-1) = F_Z(1) = 0,8413\,.$$

Using (1.2), (1.4), (3.6), (3.5) and the standard normal distribution table, one obtains that

$$\begin{aligned}\mathbb{P}(-2 < X < 13) &= F_X(13) - F_X(-2) = F_Z\left(\frac{13 - 10}{4}\right) - F_Z\left(\frac{-2 - 10}{4}\right)\\ &= F_Z(0,75) - F_Z(-3) = F_Z(0,75) - 1 + F_Z(3) = 0,7734 - 1 + 0,9987 = 0,7721\,.\end{aligned}$$

Proposition 3.1. (Approximation of a binomial distribution by a normal distribution) *Let Y be a discrete random variable having a binomial distribution of parameters n and p. Assume that at least one of the following three conditions hold:*

(i) $n \geq 30$ *and* p *close to* $0,5$,

(ii) $np > 15$ *and* $n(1 - p) > 15$,

(iii) $np(1 - p) > 10$.

Then one can approximate the binomial distribution of Y by a normal distribution with mean np et and standard deviation $\sqrt{np(1-p)}$.

Proposition 3.2. (Approximation of a Poisson distribution by a normal distribution) *Let Y be a discrete random variable having a Poisson distribution of parameter λ. When $\lambda > 10$, one can approximate the Poisson distribution of Y by the normal distribution with mean λ and standard deviation $\sqrt{\lambda}$.*

Remark 3.4. (Sum of independent random variables with normal distributions) *Let X_1 and X_2 be two independent random variables with normal distributions $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$. Let a_1, a_2 and b be 3 real numbers. Then, the random variable*

$$Y = a_1X_1 + a_2X_2 + b$$

has a normal distribution with mean $\mu = a_1\mu_1 + a_2\mu_2 + b$ and variance $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

More generally, let X_1, $X_2, \ldots,$ X_n be n mutually independent random variables with normal distributions $\mathcal{N}(\mu_1, \sigma_1^2)$, $\mathcal{N}(\mu_2, \sigma_2^2), \ldots, \mathcal{N}(\mu_n, \sigma_n^2)$. Let $a_1, a_2, \ldots, a_n$ and b be $n+1$ real numbers. Then, the random variable

$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n + b$$

has a normal distribution with mean

$$\mu = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n + b$$

and variance

$$\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2 \, .$$

4 Problems

Problem 4.1. *A light bulb has been selected at random. The random variable X denotes its lifetime measured in decimal number of days. One assumes that X has an exponential distribution of parameter the strictly positive real number λ.*
1) Calculate the expected lifetime of this light bulb in the particular case where $\lambda = 1/1000$.
2) In the general case where the λ is an arbitrary strictly positive real number, show that, for any nonnegative real numbers t and s the conditional probability $\mathbb{P}(X \geq t+s/X \geq t)$ satisfies

$$\mathbb{P}(X \geq t+s/X \geq t) = \mathbb{P}(X \geq s) \, .$$

3) Do you think that this result remains true in the more general case where X has a Weibull distribution whose parameters α and λ are arbitrary strictly positive real numbers ?

Problem 4.2. *In some country a study realized by the National Highway Traffic Safety has shown that the random number of the fatal road traffic accidents caused by sleep deprivation is a random variable Y having a normal distribution with mean equals to 1550 and standard deviation equals to 300.*
1) By using the standard normal distribution table, calculate in terms of percentages the following two probabilities: $\mathbb{P}(Y < 1000)$ and $\mathbb{P}(1000 \leq Y \leq 2000)$.
2) By using the results obtained in your answer to question 1), calculate in terms of percentage the probability $\mathbb{P}(Y > 2000)$.
3) By using the standard normal distribution table, find the number a such that $\mathbb{P}(Y \geq a) = 0,025$.

Problem 4.3. *A fair coin is tossed 1000 times. One denotes by Y the total number of heads find approximative values for the probabilities:* $\mathbb{P}(480 \leq Y \leq 520)$, $\mathbb{P}(Y > 510)$ *and* $\mathbb{P}(Y \leq 470)$.

Problem 4.4. *Mister Smith is the owner of* 3 *cinemas A, B, and C in* 3 *different small cities. The total fixed cost per week he has to support for making the* 3 *cinemas work is* 5500 *euros. While the price of a ticket for seeing a movie in any one of them is* 8 *euros. The independent random variables X_1, X_2 and X_3 respectively denote the numbers of tickets per week sold by the cinemas A, B and C. One assumes they have Poisson distributions of parameters* $\lambda_1 = 223$, $\lambda_2 = 254$ *and* $\lambda_3 = 279$.

The random variable B denotes Mister Smith's total profit per week due to the 3 *cinemas. Find approximative values for the probabilities:* $\mathbb{P}(B > 500)$, $\mathbb{P}(B \leq 800)$ *and* $\mathbb{P}(350 < B \leq 650)$.

Problem 4.5. *Stephan is a waiter in a restaurant located in the downtown of Lille; his net monthly income consists in a fixed salary of 1565 euros and in tips whose total amount in euros is a random variable X_1 having a normal distribution with mean equals to 190 and standard deviation equals to 36. His wife Virginia is a waitress in a restaurant located in a suburb of Lille; her monthly income consists in a fixed salary of 1460 euros and in tips whose total amount in euros is a random variable X_2 having a normal distribution with mean equals to 244 and standard deviation equals to 22. The two random variables X_1 and X_2 are independent.*
1) By using the standard normal distribution table, calculate in terms of percentages the following six probabilities:

(a) $\mathbb{P}(X_1 < 125)$, *(b)* $\mathbb{P}(X_1 \geq 265)$, *(c)* $\mathbb{P}(265 > X_1 \geq 125)$,
(d) $\mathbb{P}(211 \geq X_2)$, *(e)* $\mathbb{P}(X_2 \geq 280)$, *(f)* $\mathbb{P}(211 \leq X_2 \leq 280)$.

2) Let Y be the random variable defined as $Y = X_2 - X_1$. Determine the probability distribution of Y and the values of its parameters (justify your answer).
3) By using the standard normal distribution table, calculate in terms of percentage the probability that the net monthly income of Virginia be higher than that of her hunsband.
4) The random variable T denotes the total net monthly income of the couple Virginia and Stephan.

(a) Determine the probability distribution of T and the values of its parameters (justify your answer).

(b) By using the standard normal distribution table, calculate in terms of percentage the probability $\mathbb{P}(T > 3500)$.

Problem 4.6. *A dressmaker works in the ready-to-wear industry. She manufactures in series cooking aprons. One assumes that the manufacturing time in minutes of such an apron is a random variable X having a normal distribution with mean equals to* 22 *and standard deviation equals to* 2.
1) Calculate the probability that the manufacturing time of such an apron be:

a) less than 23 *minutes;*
b) less than 20 *minutes;*
c) more than 25 *minutes;*
d) between 20 *and* 25 *minutes.*

2) The dressmaker has to manufacture 8 *aprons per half-day. Except for the last one, each time she finishes to manufacture one of them she takes a break of* 5 *minutes. The random variable X_i denotes the manufacturing time in minutes of the i-th apron of a half-day. One assumes that thanks to breaks she takes the dressmaker has no fatigue. Thus, the random variables $X_1, X_2, \ldots, X_8$ can be assumed to be mutually independent and with the same normal distribution as X.*

a) Let us consider the random variable $Y = X_1 + X_2 + \ldots + X_8 + 35$. *Give a concrete interpretation of Y.*

b) Determine the probability distribution of Y *(justify your answer).*

c) Tomorrow in the afternoon the dressmaker will begin to work at $2:00$ *pm; calculate the probability that she will still be working after* $5:40$ *pm.*

d) Calculate the probability that the dressmaker will take more than 20 *minutes for manufacturing each one of the* 8 *aprons.*

e) Calculate the probability that there will be a difference of more than 2 *minutes between the time which will be taken by the dressmaker for manufacturing the first apron of the afternoon and the time for manufacturing the last one.*

Problem 4.7. *A batch of sugar packets consists in* 500 *packets placed in a box whose tare weight* [2] *is* 30 *kilograms (kg). The weight in grams (g) of a packet is a random variable having a normal distribution with mean* 1000 *g and standard deviation* 100 *g. One assumes that the weights of the packets are independent of each other. For delivering a batch one needs to use a lift which can not work if the weight of this batch is more than* 535 *kg. Calculate the probability that the lift not work.*

Problem 4.8. *For a basketball player chosen at random the size of a sharp spring measured in centimeters (cm) is a random variable* X *with normal distribution; its mean* $\mu > 0$ *is unknown, yet its standard deviation* σ *is known to be equal to* $2,3$*. One chooses at random independently of each other* 6 *basketball players. For each one of them the size in cm of a sharp spring has been measured. The* 6 *sizes for the* 6 *basketball players are:* $59,4 \quad 57,7 \quad 60,5 \quad 58,2 \quad 58,6 \quad 61,0$. *One can consider that these* 6 *sizes are* 6 *values of* 6 *independent random variables* $X_1, \ldots, X_6$ *having the distribution as* X*.*

Derive from these data a confidence interval for μ *at the confidence level* $\alpha = 98\%$*; the size of this interval should be as small as possible.*

[2] This is the weight of the box when it is empty.