From Multifractional Brownian Motion to Multifractional Process with Random Exponent

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1-Introduction and motivation

Is possible to replace the Hurst parameter of FBM by a stochastic process $\{S(t)\}_{t\in\mathbb{R}}$ with values in fixed interval $[a,b]\subset(0,1)$?
$\rightarrow$ Clearly, this is possible when $\{S(t)\}_{t\in\mathbb{R}}$ is **independent on** the white noise $d\widehat{W}$ (Papanicolaou and K. Solna): in this case, the stochastic integral

$$Z(t)=\int_{\mathbb{R}}\frac{e^{it\cdot\xi}-1}{|\xi|^{h(t)+1/2}}\,d\widehat{W}(\xi), \tag{1}$$

is well-defined and the main results on MBM can be readily extended to the process $\{Z(t)\}_{t\in\mathbb{R}}$.
$\rightarrow$ The more general case where $\{S(t)\}_{t\in\mathbb{R}}$ can be **dependent on** the white noise $d\widehat{W}$ is more tricky, since the stochastic integral (1) is no longer defined.

This is why, in the more general case, Taqqu and Ayache have proposed to use the standard random wavelet series representation of FBM:

$$B(x,H)=\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}}2^{-jH}\epsilon_{j,k}\big(\Psi(2^jx-k,H)-\Psi(-k,H)\big), \tag{2}$$

instead of a stochastic integral representation of FBM. Note that:

- $\{\epsilon_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ is a sequence of independent $\mathcal{N}(0,1)$ Gaussian random variables;
- Ψ is a C^∞ function on $\mathbb{R}\times(0,1)$, moreover it is well-localized in the first variable, uniformly in the second one, which means that, for all $(n,p)\in\mathbb{N}^2$,

$$\sup\Big\{\big(1+|x|\big)^p\big|(\partial_x^{(n)}\Psi)(x,H)\big|\,:\,(x,H)\in\mathbb{R}\times(0,1)\Big\}<+\infty. \tag{3}$$

The random wavelet series representation of B is, with probability 1, unifomly convergent in (x, H) on each compact subset of $\mathbb{R} \times (0,1)$; this is why it completely makes sense to replace in it, (x, H) by $(t, S(t))$. Thus, we obtain the non Gaussian process $\{Z(t)\}_{t\in\mathbb{R}}$ defined as

$$Z(t) = B(t, S(t)) = \sum_{j=-\infty}^{+\infty} \sum_{k\in\mathbb{Z}} 2^{-jS(t)} \epsilon_{j,k} \big(\Psi(2^j t - k, S(t)) - \Psi(-k, S(t)) \big), \tag{4}$$

where the series is with probability 1, uniformly convergent in t on each compact subset of $\mathbb{R}$.

$\rightarrow$ The process $\{Z(t)\}_{t\in\mathbb{R}}$ is called **Multifractional Process with Random Exponent (MPRE)**.

2-On the local Hölder regularity of MPRE

One of the main interests of MBM is that its pointwise Hölder exponent can be prescribed via its deterministic functional parameter h. **This nice property can be extended to MPRE:**

Theorem 1 (Jaffard, Taqqu and Ayache)

Assume that, with probability 1*, the paths of* $\{S(t)\}_{t\in\mathbb{R}}$ *are, on each compact interval* K*, Hölder functions of order* $\beta(K) > \max_{t\in K} S(t)$*. Then* $\{\alpha_Z(t)\}_{t\in\mathbb{R}}$ *the pointwise Hölder exponent of the process* $\{Z(t)\}_{t\in\mathbb{R}}$ *satisfies:*

$$\mathbb{P}\big\{\forall t\in\mathbb{R} : \alpha_Z(t) = S(t)\big\} = 1. \tag{5}$$

We will only give the proof of the fact that

$$\mathbb{P}\big\{\forall t\in\mathbb{R} : \alpha_Z(t) \le S(t)\big\} = 1, \tag{6}$$

since the fact $\mathbb{P}\big\{\forall t\in\mathbb{R} : \alpha_Z(t) \ge S(t)\big\} = 1$, is less difficult to show.

The following two lemmas allow to obtain the result we want to prove.

Lemma 1

For all compact $K \subset \mathbb{R}$ *and all reals* $0 < a < b < 1$, *there is a random variable* $C > 0$ *of finite moment of any order, such that one has almost surely for all* $H_1, H_2 \in [a, b]$,

$$\sup_{t \in K} |B(x, H_1) - B(x, H_2)| \leq C|H_1 - H_2|. \tag{7}$$

Lemma 2

For every fixed $H \in (0, 1)$, *we denote by* $\{\alpha_{B_H}(t)\}_{t \in \mathbb{R}}$ *the pointwise Hölder exponent of the FBM* $\{B(t, H)\}_{t \in \mathbb{R}}$. *One has*

$$\mathbb{P}\{\alpha_{B_H}(t) \leq H \,:\, \forall\, t \in \mathbb{R} \text{ and } \forall\, H \in (0, 1)\} = 1. \tag{8}$$

Proof of the result: Since we are interested in a local problem, we can restrict to a compact interval K. Using the fact, for all $u \in K$, one has $Z(u) = B(u, S(u))$, as well as the triangle inequality, it follows that,

$$\begin{aligned} |Z(t+s) - Z(t)| \;\; &\geq \;\; \Big|B(t+s, S(t)) - B(t, S(t))\Big| \\ &\quad - \sup_{x \in K} \Big|B(x, S(t+h)) - B(x, S(t))\Big|. \end{aligned} \tag{9}$$

Moreover, Lemma 1 and the fact that $\{S(t)\}_{t \in K}$ paths are with probability 1, β-Hölder functions, imply that

$$\sup_{x \in K} \Big|B(x, S(t+h)) - B(x, S(t))\Big| \leq C\Big|S(t+s) - S(t)\Big| \leq C'|s|^{\beta}. \tag{10}$$

where $\beta > S(t)$. Putting together (9), (10) and Lemma 2, it follows that almost surely, for all $\theta \in \big(S(t), \beta\big)$,

$$\limsup_{s \to 0} \frac{|Z(t+s) - Z(t)|}{|s|^{\theta}} = +\infty.$$

$\square$

Proof of Lemma 1: For simplicity, we suppose that $K = [0, 1]$, thus we have to show that a.s. for all $H_1, H_2 \in [a, b] \subset (0, 1)$ one has

$$\sup_{x \in K} |B(x, H_1) - B(x, H_2)| \leq C|H_1 - H_2|. \tag{11}$$

To this end, we will the wavelet represention,

$$B(x, H) = \sum_{j=-\infty}^{+\infty} \sum_{k \in \mathbb{Z}} 2^{-jH} \epsilon_{j,k} \big(\Psi(2^j x - k, H) - \Psi(-k, H) \big), \tag{12}$$

and make, the heuristical assumption, that for all $H \in (0, 1)$,

$$\operatorname{Supp} \Psi(\cdot, H) \subseteq [0, 1]. \tag{13}$$

Of course, (13) is not true, yet it clarifies the main intuition behind the fact that the function Ψ is well-localized function in the first variable, uniformly in the second one.

Thus, one has for all $x \in [0,1]$ and $H_1, H_2 \in [a,b]$,

$$B(x,H_1) - B(x,H_2) = \sum_{j=-\infty}^{-1} \epsilon_{j,0}\big(2^{-jH_1}\Psi(2^j x, H_1) - 2^{-jH_2}\Psi(2^j x, H_2)\big)$$

$$+ \sum_{j=0}^{+\infty}\sum_{k\in 0}^{2^j-1} \epsilon_{j,k}\big(2^{-jH_1}\Psi_l(2^j x - k, H_1) - 2^{-jH_2}\Psi(2^j x - k, H_2)\big). \tag{14}$$

It follows from the Mean Value Theorem That there is a constant $c_1 > 0$ such that for all $j < 0$, $(H_1, H_2) \in [a,b]^2$ and $x \in [0,1]$,

$$\big|2^{-jH_1}\Psi_l(2^j x, H_1) - 2^{-jH_2}\Psi_l(2^j x, H_2)\big| \le c_1 |j| 2^{j(1-b)} |H_1 - H_2|. \tag{15}$$

By using again the Mean Value Theorem and our heuristical assumption that for all $H \in (0,1)$, $\operatorname{supp} \Psi(\cdot, H) \subseteq [0,1]$, one can show that there is a constant $c_2 > 0$ such that for all $j \geq 0$, $k \in \{0, \ldots, 2^j - 1\}$, $(H_1, H_2) \in [a,b]^2$ and $x \in [0,1]$,

$$\begin{aligned} &\left|2^{-jH_1}\Psi_l(2^j x - k, H_1) - 2^{-jH_2}\Psi_l(2^j x - k, H_2)\right| \\ &\qquad \leq c_2 |j| 2^{-ja} |H_1 - H_2| \mathbb{1}_{[2^{-j}k, 2^{-j}(k+1)]}(x). \end{aligned} \tag{16}$$

On the other hand, Borel-Cantelli Lemma allows to show that, there is a random variable C such that a.s. for all $(j,k) \in \mathbb{Z}^2$, one has,

$$|\epsilon_{j,k}| \leq C_3 \sqrt{\log(2 + |j| + |k|)}. \tag{17}$$

Putting together (14), (15), (16) and (17), one obtains (11). $\square$

Now, our aim will be to show that Lemma 4 holds. To this end, let us set for each $x \in \mathbb{R}$ and $H \in (0,1)$,

$$\widetilde{\Psi}(x,H) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\cdot\xi} |\xi|^{H+1/2} \widehat{\psi}(\xi)\, d\xi. \tag{18}$$

The function $\widetilde{\Psi}$ is C^∞ over $\mathbb{R} \times (0,1)$, moreover,

(i) it is well-localized in the first variable uniformly in the second that is, for all $(n,p) \in \mathbb{N}^2$,

$$\sup \Big\{ \big(1+|x|\big)^p \big| (\partial_x^n \widetilde{\Psi})(x,H) \big| \;:\; x \in \mathbb{R} \text{ and } H \in (0,1) \Big\} < +\infty.$$

(ii) For every $H \in (0,1)$, the first moment of $\widetilde{\Psi}(\cdot,H)$ vanishes i.e.

$$\int_{\mathbb{R}} \widetilde{\Psi}(x,H)\, dx = 0.$$

The following proposition allows us to understand the motivation behind the introduction of the function $\widetilde{\Psi}$

Proposition 1

Recall that,

$$B(x,H) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \epsilon_{j,k} \big(\Psi(2^j x - k, H) - \Psi(-k, H) \big).$$

One has, almost surely for all $H \in (0,1)$ *and* $(j,k) \in \mathbb{Z} \times \mathbb{Z}$*,*

$$2^{(1+H)j} \int_{\mathbb{R}} B(x,H) \widetilde{\Psi}(2^j x - k, H)\, dx = \epsilon_{j,k}.$$

Proposition 1 is a straightforward consequence of the following lemma.

Lemma 3

For every $H \in (0,1)$ the sequences of functions

$$\left\{2^{j/2}\widetilde{\Psi}(2^j\cdot -k, H)\;:\; j\in\mathbb{Z},\, k\in\mathbb{Z}\right\},$$

and

$$\left\{2^{j/2}\Psi(2^j\cdot -k, H)\;:\; j\in\mathbb{Z},\, k\in\mathbb{Z}^N\right\},$$

are biorthogonal i.e.

$$2^{(j+j')/2}\int_{\mathbb{R}}\Psi(2^jx-k,H)\widetilde{\Psi}(2^{j'}x-k',H)\,dx = \begin{cases} 1 \text{ if } (j,k)=(j',k') \\ 0 \text{ else.}\end{cases}$$

Proof of Lemma 3: It follows from Plancherel formula and from the definitions of Ψ and $\widetilde{\Psi}$ that

$$
\begin{aligned}
&2^{(j+j')/2}\int_{\mathbb{R}} \Psi(2^j x-k,H)\widetilde{\Psi}(2^{j'}x-k',H)\,dx = \\
&2^{-(j+j')/2}\int_{\mathbb{R}} \left(e^{-i2^{-j}k\xi}\frac{\widehat{\psi}(2^{-j}\xi)}{|2^{-j}\xi|^{H+1/2}}\right)\left(e^{i2^{-j'}k'\xi}|2^{-j'}\xi|^{H+1/2}\overline{\widehat{\psi}(2^{-j'}\xi)}\right)d\xi \\
&= 2^{(H+1/2)(j-j')-(j+j')/2}\int_{\mathbb{R}} \left(e^{-i2^{-j}k\cdot\xi}\widehat{\psi}(2^{-j}\xi)\right)\left(e^{i2^{-j'}k'\cdot\xi}\overline{\widehat{\psi}(2^{-j}\xi)}\right)\,d\xi \\
&= 2^{(H+1/2)(j-j')-(j+j')/2}\int_{\mathbb{R}} \psi_l(2^j x-k)\psi_l(2^{j'}x-k')\,dt.
\end{aligned}
$$

Then using the fact that

$$
\left\{2^{jN/2}\psi_l(2^j\cdot-k,H)\,:\,j\in\mathbb{Z},\,k\in\mathbb{Z}^N\right\},
$$

is an orthonormal sequence, we obtain the lemma. □

Proof of Lemma 2: We have to show that

$$\mathbb{P}\Big\{\alpha_{B_H}(t) \le H \ : \ \forall\, t \in \mathbb{R} \text{ and } \forall\, H \in (0,1)\Big\} = 1. \qquad (19)$$

Suppose ad absurdum that $\mathbb{P}(A) > 0$ where

$$A = \Big\{\omega \in \Omega \ : \ \text{there is } (t_0, H_0) \in \mathbb{R} \times (0,1) \text{ s.t. } \alpha_{B_{H_0}}(t_0, \omega) > H_0\Big\}.$$

Let $\omega \in A$, it follows from the definition of $\alpha_{B_{H_0}}(t_0)$, that there are two constants $\theta_0 > H_0$ and $c_0 > 0$ such that for all t close to t_0 one has

$$\big|B(t, H_0, \omega) - B(t_0, H_0, \omega)\big| \le c_0 |t - t_0|^{\theta_0}. \qquad (20)$$

Note that (20) remains valid for all $t \in \mathbb{R}$ because of the continuity of FBM $\{B(t, H_0)\}_{t \in \mathbb{R}}$ and the slowness of its increase at infinity.

By using Proposition 2, the fact that $\int_{\mathbb{R}} \widetilde{\Psi}(x, H_0)\,dx = 0$, (20), the change of variable $s = 2^j t - k$ and the triangle inequality, one has for all j, k,

$$
\begin{aligned}
|\epsilon_{j,k}(\omega)| &= 2^{(1+H_0)j}\Big|\int_{\mathbb{R}} B_{H_0}(t,\omega)\widetilde{\Psi}(2^j t - k, H_0)\,dt\Big| \\
&= 2^{(1+H_0)j}\Big|\int_{\mathbb{R}} \Big(B_{H_0}(t,\omega) - B_{H_0}(t_0,\omega)\Big)\widetilde{\Psi}(2^j t - k, H_0)\,dt\Big| \\
&\leq c_0 2^{(1+H_0)j}\int_{\mathbb{R}} |t - t_0|^{\theta_0}\Big|\widetilde{\Psi}(2^j t - k, H_0)\Big|dt \\
&= c_0 2^{jH_0}\int_{\mathbb{R}} |2^{-j}s + 2^{-j}k - t_0|^{\theta_0}\Big|\widetilde{\Psi}(s, H_0)\Big|\,ds \\
&\leq c_1 2^{-j(\theta_0 - H_0)}\Big(1 + |2^j t_0 - k|\Big)^{\theta_0}.
\end{aligned}
\tag{21}
$$

where $c_1 > 0$ is a constant non depending on j, k.

Finally (21) implies that

$$\lim_{j\to+\infty} \sup \left\{ |\epsilon_{j,k}(\omega)| \ : \ k \in \mathbb{Z} \text{ and } \left|2^j t_0 - k\right| \le j \right\} = 0,$$

but this is impossible (Borell-Cantelli Lemma) since $\{\epsilon_{1,j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is a sequence of independent standard Gaussian random variables. $\square$

3-Other properties of MPRE

Theorem 2 (Taqqu and Ayache)

Let $\{Z(t)\}_{t\in\mathbb{R}}$ be an MPRE whose parameter $\{S(t)\}_{t\in\mathbb{R}}$ is a stationary stochastic process independent of the white noise. Then $\{Z(t)\}_{t\in\mathbb{R}}$ satisfies the following self-similarity property. For any reals $a>0$ and t, one has

$$Z(at) \overset{(d_1)}{=} a^{S(t)} Z(t), \tag{22}$$

where $\overset{(d_1)}{=}$ means equality of the marginal distributions.

Theorem 3 (Taqqu and Ayache)

Let $\{Z(t)\}_{t\in[0,1]}$ *be an MPRE whose parameter* S *is a random variable independent of the white noise. Then the increments of* $\{Z(t)\}_{t\in[0,1]}$ *are stationary. Namely, for any* $t \in (0,1)$*, one has*

$$\{Z(t+h) - Z(t)\}_{h\in[0,1-t]} \overset{(d)}{=} \{Z(h) - Z(0)\}_{h\in[0,1-t]}, \tag{23}$$

where $\overset{(d)}{=}$ *means equality of the finite-dimensional distributions.*