Identification of the pointwise Hölder exponent of Generalized Multifractional Brownian Motion

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$\rightarrow$ Let $(h_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of Lipschitz functions defined on $[0,1]$ and with values in $[a,b]\subset(0,1)$ which satisfies $\|h_n\|_{\mathrm{Lip}}=O(n)$. We set $h(t)=\liminf_{n\to+\infty}h_n(t)$.

$\rightarrow$ Let $(widehatf_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{C}^d(\mathbb{R})$ with values in $[0,1]$, such that:

- $\widehat{f}_0(\xi)=1$ when $|\xi|\leq 2\pi/3$ and $\widehat{f}_0(\xi)=0$ when $|\xi|\geq\pi$;
- for all $n\geq 1$ and ξ, $\widehat{f}_n(\xi)=\widehat{f}(2^{-n}\xi)-\widehat{f}(2^{-(n-1)}\xi)$.

$\{X(t)\}_{t\in[0,1]}$ the GMBM of parameter $(h_n)_{n\in\mathbb{N}}$ is defined as

$$X(t)=\int_{\mathbb{R}}\Big(\sum_{n=0}^{+\infty}\frac{e^{it\xi}-1}{|\xi|^{h_n(t)+1/2}}\widehat{f}_n(\xi)\Big)\,d\widehat{W}(\xi). \quad (1)$$

Recall that it has been shown by Jaffard, Taqqu and Ayache that

$$\mathbb{P}\Big\{\forall t \in [0,1] \,:\, \alpha_X(t) = h(t)\Big\} = 1, \tag{2}$$

which means that $\{\alpha_X(t)\}_{t\in[0,1]}$ the pointwise Hölder exponent of $\{X(t)\}_{t\in[0,1]}$ may exhibit **very irregular behavior**.

$\Rightarrow$ A natural question which can be addressed is that, **whether or not, in this case, it is possible to construct $\widehat{\alpha}_{X,N}(t)$ a statistical estimator of** $\alpha_X(t)$ at a given point t, starting from the observation of $X(p\Delta_N)$, $p = 0, \ldots, [\Delta_N^{-1}]$ a discretized path of $\{X(t)\}_{t\in[0,1]}$; here Δ_N denotes the discretization mesh, of course $\lim_{N\to+\infty} \Delta_N = 0$.

$\to$ In this seminar, we will see that **under the mild condition that** $(h_n(t))_{n\in\mathbb{N}}$ **be a convergent sequence, the answer to this question is (theoretically speaking) positive.**

In the case of FBM several approaches have already been proposed in order to estimate the Hurst parameter H (recall that this parameter equals to the pointwise Hölder exponent of FBM). Let us mention some of these approaches.

- Hall and Wood (1994): Box dimension;
- Hall, Wood and Feuerverger (1994): number of crossings;
- Lévy Véhel and Peltier (1994): maximum likelihood;
- Abry, Flandrin, Taqqu and Veitch (1998): Wavelets.

The approach developed by **Guyon and Léon (1987)** and that developed by **Istas and Land (1994)** will be of particular interest to us. The first of these two approaches is base on **Quadratic Variations** and the second one on **Generalized Quadratic Variations**.

One observes $\{B_H(p/N)\}_{p=0,\dots,N}$ a discretized of FBM.
$\rightarrow$ **Quadratic Variation (VQ):**

$$\widetilde{V}_N = \sum_{p=0}^{N-1} \left(B_H\Big(\frac{k+1}{N}\Big) - B_H\Big(\frac{k}{N}\Big) \right)^2 . \tag{3}$$

Guyon and Léon have shown that $N^{2H-1}\widetilde{V}_N$ coverges a.s. to a strictly positive deterministic constant c when $N \rightarrow +\infty$; therefore

$$\widetilde{H}_N = \frac{1}{2}\Big(1 + \frac{\log \widetilde{V}_N}{\log N}\Big), \tag{4}$$

converges, a.s., to H when $N \rightarrow +\infty$.

Speed of convergence:

- when $0 < H < 3/4$, a standard Central Limit Theorem (i.e. with a rate of convergence $\sqrt{N}$ and a Gaussian limit) holds for the Quadratic Variations;
- when $3/4 < H < 1$, a non standard Central Limit Theorem (i.e. with a rate of convergence N^{2-2H} and a non Gaussian limit) holds for the Quadratic Variations.

In order to obtain a standard Central Limit Theorem for all the values of $H \in (0,1)$, Istas and Lang have proposed to replace the Quadratic Variation $\widetilde{V}_N$ by a Generalized Quadratic Variation, for example,

$$V_N = \sum_{p=0}^{N-2} \left(B_H\Big(\frac{p+2}{N}\Big) - 2B_H\Big(\frac{p+1}{N}\Big) + B_H\Big(\frac{p}{N}\Big) \right)^2 . \qquad (5)$$

$\rightarrow$ By making use of Generalized Quadratic Variations, Istas and Lang have constructed asymptotically normal estimators of Hölder exponents of a wide class of stationary increments Gaussian processes, which includes FBM.
$\rightarrow$ Later, in 1998, by localizing Generalized Quadratic Variations, Benassi, Cohen and Istas have extended this estimation method to the non stationary increments setting of Multifractional Brownian Motion (MBM).

Estimation of pointwise Hölder exponent of MBM

Let $t \in [0,1]$ be an arbitrary fixed point.

Assume that one observe $\{X(p/N)\}_{p=0,\dots N}$ a discretized path of $\{X(s)\}_{s\in[0,1]}$ the MBM of functional parameter h.

$\rightarrow$ By localizing Generalized Quadratic Variations of MBM around t, one can build an estimator of its pointwise Hölder exponent $h(t)$. More precisely, let $\gamma \in (0,1)$ be fixed let $\mathcal{W}_N(t)$ be set defined as,

$$\mathcal{W}_N(t) = \left\{ p \in \{0,\dots,N-2\} \,:\, \left| t - \frac{p}{N} \right| \le N^{-\gamma} \right\}. \tag{6}$$

In fact $\mathcal{W}_N(t)$ plays the role of a localizing window whose size can be controled via the parameter γ.

The Generalized Quadratic Variations of MBM loaclized around t, are defined as

$$V_N(t) = \sum_{p \in \mathcal{W}_N(t)} \left(X\left(\frac{p+2}{N}\right) - 2X\left(\frac{p+1}{N}\right) + X\left(\frac{p}{N}\right) \right)^2 . \qquad (7)$$

Theorem 1 (Benassi, Cohen and Istas)

When h is a continuously differentiable function, then

$$\widehat{h}_N(t) = \frac{1}{2}\left(1 - \gamma - \frac{\log V_N(t)}{\log N}\right), \qquad (8)$$

is a strongly consistent and asymptotically normal estimator of $h(t)$.

→ We denote by $\{X(s)\}_{s\in[0,1]}$ a GMBM of parameter $(h_n)_{n\in\mathbb{N}}$ (recall that the functions h_n are with values in $[a,b]\subset(0,1)$).
→ Let $t\in[0,1]$ be a fixed point, we denote by $V_{N^\delta}(t)$ the localized Generalized Quadratic Variations of GMBM, defined as

$$V_{N^\delta}(t)=\sum_{p\in\mathcal{W}_{N^\delta}(t)}\left(X\Big(\frac{p+2}{N^\delta}\Big)-2X\Big(\frac{p+1}{N^\delta}\Big)+X\Big(\frac{p}{N^\delta}\Big)\right)^2, \tag{9}$$

where

$$\mathcal{W}_{N^\delta}(t)=\left\{p\in\{0,\ldots,[N^\delta]-2\}\;:\;\left|t-\frac{p}{N^\delta}\right|\le N^{-\gamma}\right\}. \tag{10}$$

Here $\delta\ge 1$ and $\gamma\in(0,1)$ are two fixed real numbers.

Theorem 2 (Lévy Véhel and Ayache)

Assume that the sequence $(h_n(t))_{n\in\mathbb{N}}$ is convergent to $h(t)$. Also assume that δ and γ satisfy $0 < \delta b < \gamma < \delta - 1/2$. Then

$$\widehat{h}_{N^\delta}(t) = \frac{1}{2\delta}\left(\delta - \gamma - \frac{\log V_{N^\delta}(t)}{\log N}\right), \tag{11}$$

is a strongly consistent estimator $h(t)$. It is worth noticing on can take $\delta = 1$ when $h(t) \in (0, 1/2)$.

Shetch of the proof: Let $\{Y(u,v)\}_{(u,v)\in[0,1]^2}$ generating the GMBM $\{X(s)\}_{t\in[0,1]}$, namely the field defined as,

$$Y(u,v) = \int_{\mathbb{R}} \Big(\sum_{n=0}^{+\infty} \frac{e^{iu\xi} - 1}{|\xi|^{h_n(v)+1/2}} \widehat{f}_n(\xi) \Big)\, d\widehat{W}(\xi). \tag{12}$$

One has for all $s \in [0,1]$,

$$X(t) = Y(s,s). \tag{13}$$

Recall that there is a random variable C of finite moment of any order, such that one has, a.s., for all $v_1, v_2 \in [0,1]$,

$$\sup_{u\in[0,1]} \left| Y(u,v_1) - Y(u,v_2) \right| \leq C|v_1 - v_2|. \tag{14}$$

Let $\{Z(u)\}_{u\in[0,1]}$ be the stationary increments process defined for all $u \in [0,1]$ as

$$Z(u) = Y(u,t), \tag{15}$$

By using, (13) and (14), the Generalized Quadratic Variation of GMBM, $V_{N^\delta}(t)$, can be expressed as,

$$V_{N^\delta}(t) = T_{N^\delta}(t) + R_{N^\delta}(t), \tag{16}$$

where $T_{N^\delta}(t)$ is the Generalized Quadratic Variation of $\{Z(u)\}_{u\in[0,1]}$ and $R_{N^\delta}(t)$ a negligible term. In fact, $\{Z(u)\}_{u\in[0,1]}$ is almost an FBM of Hurst parameter $h(t)$; therefore $N^{2\delta H+\gamma-\delta}T_{N^\delta}(t)$ a.s. converges to a positivie constant, when $N\to+\infty$.

$\square$

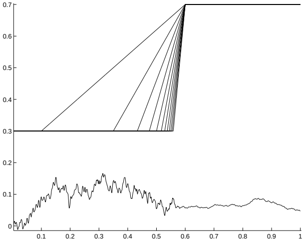


Figure: simulated path of a GMBM such that $h = 0.3\mathbb{1}_{[0,0.6)} + 0.7\mathbb{1}_{[0.6,1]}$

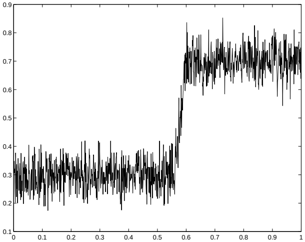


Figure: Estimated pointwise Hölder exponents

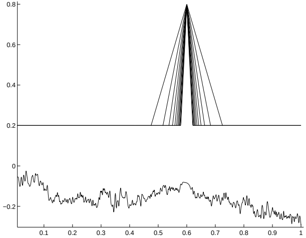
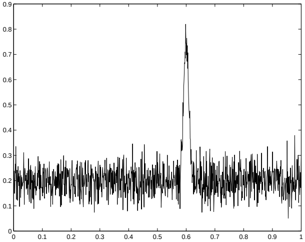


Figure: Simulated path of a GMBM such that $h = 0.2\mathbb{1}_{[0,1]\setminus\{0.6\}} + 0.8\mathbb{1}_{\{0.6\}}$



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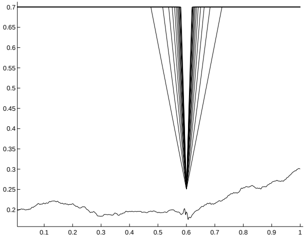
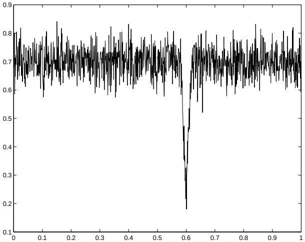


Figure: Simulated path of a GMBM such that $h = 0.7\mathbb{1}_{[0,1]\setminus\{0.6\}} + 0.25\mathbb{1}_{\{0.6\}}$



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