# From Fractional Brownian Motion to Multifractional Brownian Motion

Antoine Ayache

USTL (Lille)

Antoine.Ayache@math.univ-lille1.fr

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# Main parts of the seminar

**Fractional Brownian Motion (FBM) is a quite classical example of a fractal process.** It is denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$ since it mainly depends on a unique parameter $H \in (0,1)$, called **the Hurst parameter.** The covariance kernel of this centered Gaussian process is given, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}$, by

$$\mathbb{E}\left(B_H(s)B_H(t)\right) = \frac{c_H}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\}, \tag{1}$$

where $c_H > 0$ is a constant only depending on $H$.
$\rightarrow$ FBM is **a natural extension of Brownian Motion**: $\{B_{1/2}(t)\}_{t\in\mathbb{R}^N}$ is the usual Brownian motion.
$\rightarrow$ FBM is **important in Probability and in Statistics** since it has many nice properties.

It is **the only centered Gaussian, self-similar process with stationary increments.**

- **Self-similar** means that: $\forall\ a > 0$

$$\{B_H(at)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \{a^H B_H(t)\}_{t\in\mathbb{R}}.$$

- **Stationary increments** means that: $\forall\ h, s, t \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\Big(\big(B_H(t+h) - B_H(t)\big)\big(B_H(s+h) - B_H(t)\big)\Big)\\
&= \mathbb{E}\Big(\big(B_H(t-s+h) - B_H(t-s)\big)\big(B_H(h) - B_H(0)\big)\Big)\\
&= \rho_h(t-s).
\end{aligned}$$

- In contrast with Brownian Motion, the increments of FBM are **correlated**. They even display **long range dependence or long memory** when $H > 1/2$: $\forall\ h \neq 0$,

$$\sum_{k\in\mathbb{Z}} \big|\rho_h(k)\big| = +\infty.$$

- The variance of the increments of FBM can be easily controled: $\forall\ s, t \in \mathbb{R}$,

$$\mathbb{E}\left(\big|B_H(s) - B_H(t)\big|^2\right) = c_H|s-t|^{2H}.$$

FBM was introduced in 1940 by Kolmogorov and made popular in 1968 by Mandelbrot and Van Ness. **It has turned out to be a powerful tool in modeling** and has been applied in many areas such as:

- **Hydrology**
  The levels of some rivers (for example the Nil river) do not satisfy a standard Central Limit Theorem.
- **Finance**
  FBM seems to be a more realistic model than Brownian Motion because of its long memory property and other nice properties. Since several years, there is a considerable interest in the problem of constructing **stochastic integrals and stochastic calculus with respect to FBM** (see for instance some works of Decreusefond, Nualart, Pipiras, Russo, Taqqu, Tindel, Tudor, Ustunel, Vallois,...).

- **Signals and images processing**
  For example FBM has been used as a model for some time series (physiological time series,...) or some images (brain images,...).
- **In telecommunications**
  In 1994, Leland, Taqqu, Willinger and Wilson gave numerical evidences that some traces of data traffic (Ethernet traffic) display long range dependence.

In spite of its usefulness, FBM model has some limitations, an important one of them is that **the roughness of its path remains everywhere the same.**

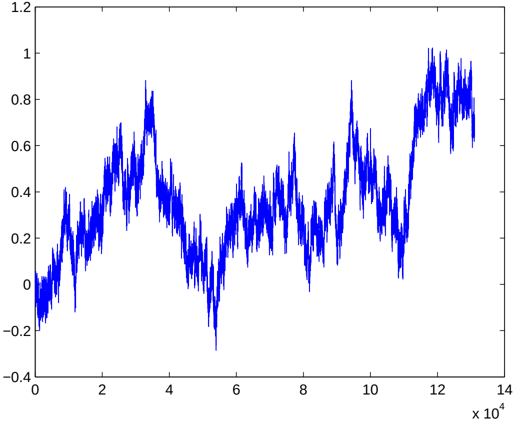


Figure: Simulation of a FBM path with Hurst parameter $H = 0.3$

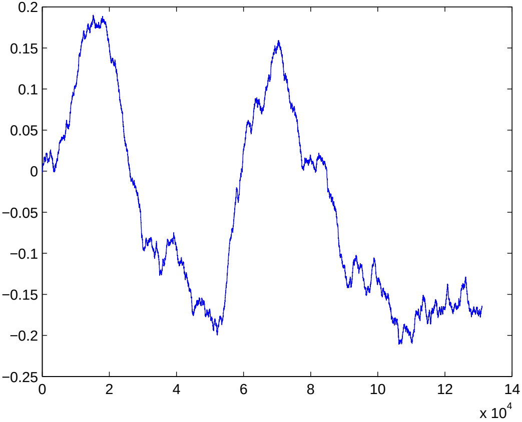


Figure: Simulation of a FBM path with Hurst parameter $H = 0.7$

In order to explain this important issue, we need to introduce the notion of **pointwise Hölder exponent** which provides a measure of the local Hölder regularity of a process path in neighborhood of some fixed point $t$.
Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stochastic process whose paths are with probability 1 continuous and nowhere differentiable functions (this is the case of FBM paths).
$\rightarrow$ Let $\alpha \in [0,1)$ and $t \in \mathbb{R}$ be fixed. One says that a path $X(\cdot,\omega)$ belongs to the **pointwise Hölder space** $\mathcal{C}^{\alpha}(t)$, if for all $s \in \mathbb{R}$ small enough, one has

$$\big|X(t+s,\omega) - X(t,\omega)\big| \le C(\omega)|s|^{\alpha}. \tag{2}$$

$\rightarrow$ The **pointwise Hölder exponent** of the path $X(\cdot,\omega)$ at the point $t$, is defined as

$$\alpha_X(t,\omega) = \sup\big\{\alpha \in [0,1) \;:\; X(\cdot,\omega) \in \mathcal{C}^{\alpha}(t)\big\}. \tag{3}$$

The roughness of FBM path remains everywhere the same since:

$$\mathbb{P}\Big\{\forall t \in \mathbb{R} : \alpha_{B_H}(t) = H\Big\} = 1. \tag{4}$$

$\rightarrow$ Basically, (4) is due to the fact that the increments of $B_H$ are stationary.
$\rightarrow$ This relation means that the pointwise Hölder exponent of FBM is not allowed to change from one place to another. **The constancy of the pointwise Hölder exponent of FBM is a strong limitation in many situations** (synthesis of artificial mountains, detection of cancer tumors in medical images, description of traces of Internet network traffic,...).
$\Rightarrow$ In order to overcome the latter drawback a more flexible model, called **Multifractional Brownian Motion (MBM)**, has been introduced independently in two articles (Peltier and Lévy Véhel, 1995) and (Benassi, Jaffard and Roux, 1997).

# Main parts of the seminar

1. Introduction and motivation
2. MBM: definition and properties
3. The field $B$ generating MBM
4. Wavelet series representation of $B$

# 2-MBM: definition and properties

**Harmonizable representation of FBM:** Up to a multiplicative constant, FBM can be represented, for all $t \in \mathbb{R}$, as

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+1/2}} \, d\widehat{W}(\xi), \tag{5}$$

where $d\widehat{W}$ is "the Fourier transform of the white noise", that is the unique complex-valued stochastic measure which satifies, for all $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(s)\, dW(s) = \int_{\mathbb{R}} \widehat{f}(\xi)\, d\widehat{W}(\xi), \tag{6}$$

$dW$ being the usual real-valued white noise (i.e. a Brownian measure). Observe that (6) implies that $B_H(t)$ is real-valued.

**non anticipative moving average representation of FBM:** Let $\{\widetilde{B}_H(t)\}_{t\in\mathbb{R}}$ be the process defined as

$$\widetilde{B}_H(t) = \int_{\mathbb{R}} \left( (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) dW(s), \qquad (7)$$

with the convention that for all reals $x$ and $\theta$, one has $(x)_+^\theta = x^\theta$ if $x > 0$ and $(x)_+^\theta = 0$ else.

$\rightarrow$ Up to a multiplicative constant, the process $\{\widetilde{B}_H(t)\}_{t\in\mathbb{R}}$ has the same distribution as $\{B_H(t)\}_{t\in\mathbb{R}}$.

Let $h : \mathbb{R} \to (0,1)$ be **a deterministic function.**
$\rightarrow$ Benassi, Jaffard and Roux have defined the MBM $\{X(t)\}_{t\in\mathbb{R}}$ by

$$X(t) = B_{h(t)}(t) = \int_{\mathbb{R}} \frac{e^{it\cdot\xi} - 1}{|\xi|^{h(t)+1/2}} \, d\widehat{W}(\xi). \tag{8}$$

$\rightarrow$ Peltier and Lévy Véhel have defined the MBM $\{\widetilde{X}(t)\}_{t\in\mathbb{R}}$ by

$$\widetilde{X}(t) = \widetilde{B}_{h(t)}(t) = \int_{\mathbb{R}} \left( (t-s)_+^{h(t)-1/2} - (-s)_+^{h(t)-1/2} \right) dW(s). \tag{9}$$

**Remark:** The distributions of the processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{\widetilde{X}(t)\}_{t\in\mathbb{R}}$ are not equal (Stoev and Taqqu 2005), however **they are nearly the same.** This is why both processes share similar properties. From now on we focus on the process $\{X(t)\}_{t\in\mathbb{R}}$.

## Theorem 1 (Lévy Vehel et al. and Benassi et al.)

*When $h$ satisfies, on each compact interval $K \subset \mathbb{R}$, a uniform Hölder condition of order $\beta = \beta(K)$, i.e. for all $s, t \in K$,*

$$|h(s) - h(t)| \leq c|s - t|^{\beta}$$

*and $\beta$ is such that $\max_{t \in K} h(t) < \beta$. Then for all $t \in \mathbb{R}$,*

$$\mathbb{P}\{\alpha_X(t) = h(t)\} = 1.$$

## Theorem 2 (Jaffard, Taqqu and Ayache)

*Under the same condition as in Theorem 1 one has*

$$\mathbb{P}\{\forall t \in \mathbb{R} : \alpha_X(t) = h(t)\} = 1.$$

**Remark:** Theorem 1 remains valid under the weaker condition that for all $t \in \mathbb{R}$, $h(t) \leq \alpha_h(t)$.

## Theorem 3 (Benassi, Jaffard and Roux)

*When $h$ satisfies, on each compact interval $K \subset \mathbb{R}$, a uniform Hölder condition of order $\beta = \beta(K) > \max_{t\in K} h(t)$. Then at each point $s \in \mathbb{R}$, the MBM $\{X(t)\}_{t\in\mathbb{R}}$ is* **locally asymptotically self-similar of order** *$h(s)$. More precisely,*

$$\lim_{\rho\to 0^+} law \left\{ \frac{X(s+\rho u) - X(s)}{\rho^{h(s)}} \right\}_{u\in\mathbb{R}} = law \left\{ B_{h(s)}(u) \right\}_{u\in\mathbb{R}}, \qquad (10)$$

*where $\left\{ B_{h(s)}(u) \right\}_u$ is the FBM of Hurst parameter $h(s)$ and where the convergence in distribution holds for the topology of uniform convergence on compact sets.*

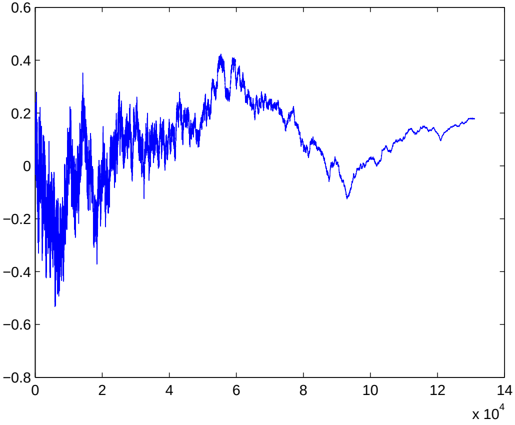


Figure: Simulation of a MBM path when $h(t) = 0,1 + 0,06t$ for all $t$.

# Main parts of the seminar

A useful tool for the study of MBM and other Multifractional processes is the field $\{B(x,H)\}_{(x,H)\in\mathbb{R}\times(0,1)}$ defined as,

$$B(x,H)=\int_{\mathbb{R}}\frac{e^{ix\cdot\xi}-1}{|\xi|^{H+1/2}}\,d\widehat{W}(\xi). \tag{11}$$

→ For every fixed $H\in(0,1)$, the process $x\mapsto B(x,H)$ is a FBM of Hurst parameter $H$.
→ One has for all $t\in\mathbb{R}$,

$$X(t)=B(t,h(t)), \tag{12}$$

$\{X(t)\}_{t\in\mathbb{R}}$ being the MBM of functional parameter $h$.

## Lemma 1

*Let* $K = [-M, M] \times [a, b] \subset \mathbb{R} \times (0,1)$ *be an arbitrary non-empty compact rectangle. One has for all* $(x_1, H_1) \in K$ *and* $(x_2, H_2) \in K$*,*

$$\mathbb{E}\left(\left|B(x_1, H_1) - B(x_2, H_2)\right|^2\right) \le c\left(|x_1 - x_2|^2 + |H_1 - H_2|^2\right)^a. \qquad (13)$$

*As a consequence, with probability* 1*, the paths of the field* $B$ *satisfy, on the rectangle* $K$*, a uniform Hölder condition of any order* $\gamma < a$*; that is*

$$\mathbb{P}\left(\sup_{(x_1,H_1)\in K;(x_2,H_2)\in K} \frac{\left|B(x_1, H_1) - B(x_2, H_2)\right|}{\left(|x_1 - x_2|^2 + |H_1 - H_2|^2\right)^{\gamma/2}} < \infty\right) = 1.$$

**Proof:** It follows from the isometry property of the integral $\int_{\mathbb{R}} (\cdot)\, d\widehat{W}$ and from the formula $(u+v)^2 \le 2(u^2+v^2)$ for all reals $u, v$, that,

$$
\begin{aligned}
&\mathbb{E}\left(\left|B(x_1,H_1) - B(x_2,H_2)\right|^2\right) \\
&= \int_{\mathbb{R}} \left| \frac{e^{ix_1\xi}-1}{|\xi|^{H_1+1/2}} - \frac{e^{ix_2\xi}-1}{|\xi|^{H_2+1/2}} \right|^2 d\xi \\
&\le 2\int_{\mathbb{R}} \frac{|e^{i(x_1-x_2)\xi}-1|^2}{|\xi|^{2H_1+1}}\, d\xi + 2\int_{\mathbb{R}} \frac{|e^{ix_2\xi}-1|^2}{|\xi|}\Big(\frac{1}{|\xi|^{H_1}} - \frac{1}{|\xi|^{H_2}}\Big)^2 d\xi.
\end{aligned}
$$

We shall now provide an upper bound of each integral in this last inequality. For simplicity we assume that $M \le 1/2$.

Setting $\eta = (x_1 - x_2)\xi$ in the first integral and using the inequalities $a \le H_1 \le b$, one obtains that

$$\begin{aligned} \int_{\mathbb{R}} \frac{|e^{i(x_1-x_2)\xi} - 1|^2}{|\xi|^{2H_1+1}}\, d\xi &= \Big(\int_{\mathbb{R}} \frac{|e^{i\eta} - 1|^2}{|\eta|^{2H_1+1}}\, d\eta\Big)|x_1 - x_2|^{2H_1} \\ &\le c_1|x_1 - x_2|^{2a}, \end{aligned}$$

where the constant $c_1 = \int_{\mathbb{R}} \frac{|e^{i\eta}-1|^2}{|\eta|^{2a+1}}\, d\eta + \int_{\mathbb{R}} \frac{|e^{i\eta}-1|^2}{|\eta|^{2b+1}}\, d\eta$.

Applying the Mean Value Theorem to the function $H \mapsto |\xi|^{-H-1/2} = e^{-(H+1/2)\log|\xi|}$ and using the inequalities $a \le H_1 \le b$, $a \le H_2 \le b$ and $|x_2| \le 1/2$, one gets that

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{|e^{ix_2\xi}-1|^2}{|\xi|}\Big(\frac{1}{|\xi|^{H_1}} - \frac{1}{|\xi|^{H_2}}\Big)^2 \, d\xi \\
&\le 2\Big(\int_1^{+\infty} \frac{|e^{ix_2\xi}-1|^2}{\xi^{2a+1}}(\log \xi)^2 \, d\xi \\
&\qquad\qquad + \int_0^1 \frac{|e^{ix_2\xi}-1|^2}{\xi^{2b+1}}(\log \xi)^2 \, d\xi\Big)|H_1 - H_2|^2 \\
&\le c_2 |H_1 - H_2|^2,
\end{aligned}$$

where the constant $c_2 = 8\int_1^{+\infty} \frac{\log^2 \xi}{\xi^{2a+1}}\, d\xi + 2\int_0^1 \frac{\log^2 \xi}{\xi^{2b-1}}\, d\xi$.

□

Proposition 1 is an easy consequence of Lemma 1.

## Proposition 1

*When the function h is continous on $\mathbb{R}$, then, with probability 1, the paths of the MBM $\{X(t)\}_{t\in\mathbb{R}}$ are continuous on $\mathbb{R}$.*

The reciprocal result is the following:

## Proposition 2

*When h is discontinuous at some point $t_0 \in \mathbb{R} \setminus \{0\}$, then with probability 1, the paths of the MBM $\{X(t)\}_{t\in\mathbb{R}}$ are discontinuous at $t_0$.*

**Proof of Proposition 2:** Using the fact that $h$ is discontinuous at $t_0$ and with values in $(0,1)$, there exists a sequence $(t_n)_{n\geq 1}$ such that

$$(t_0,\theta_0)=\lim_{n\to+\infty}(t_n,h(t_n))\neq(t_0,h(t_0)). \tag{14}$$

For simplicity we assume that $\theta_0\in(0,1)$. Then it follows from the continuity of the paths of the field $\{B(x,H)\}_{(x,H)\in\mathbb{R}\times(0,1)}$, that a.s.,

$$B(t_0,\theta_0)=\lim_{n\to+\infty}B(t_n,h(t_n))=\lim_{n\to+\infty}X(t_n). \tag{15}$$

Finally, one has, a.s.,

$$B(t_0,\theta_0)\neq X(t_0)=B(t_0,h(t_0)), \tag{16}$$

since $\int_{\mathbb{R}}|e^{it_0\xi}-1|^2\big(|\xi|^{-\theta_0-1/2}-|\xi|^{-h(t_0)-1/2}\big)^2\,d\xi$, the variance of the centered Gaussian random variable $B(t_0,\theta_0)-B(t_0,h(t_0))$ does not vanish.

# Main parts of the seminar

# 4-Wavelet series representation of $B$

**An orthonormal wavelet basis of** $L^2(\mathbb{R})$ is a Hilbertian basis of the form:

$$\left\{2^{j/2}\psi(2^j x - k)\ :\ j \in \mathbb{Z},\ k \in \mathbb{Z}\right\}.$$

The function $\psi$ is called **the mother wavelet**; we always assume that:

(i) It belongs to **the Schwartz class** $S(\mathbb{R})$ i.e. $\psi$ is an infinitely differentiable function and $\psi^{(n)}$, its derivative of any order $n \in \mathbb{N}$, is a **well-localized function**, which means that, for all $p \in \mathbb{N}$,

$$\sup\left\{(1+|x|)^p |\psi^{(n)}(x)|\ :\ x \in \mathbb{R}\right\} < +\infty.$$

(ii) $\widehat{\psi}(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\psi(x)\,dx$, the Fourier transform of $\psi$, is **compactly supported and vanishes in a neighborhood of the origin.**

The first construction of mother wavelets satisfying $(i)$ and $(ii)$ was given in 1986 by Meyer and Lemarié.

The isometry property of Fourier transform implies that:

$$\Big\{2^{-j/2} e^{i2^{-j}k\cdot\xi}\overline{\widehat{\psi}_l(2^{-j}\xi)}\ :\ j\in\mathbb{Z},\, k\in\mathbb{Z}\Big\},$$

forms **an orthonormal basis of** $L^2(\mathbb{R})$. By expanding for every fixed $x\in\mathbb{R}$ and $H\in(0,1)$, **the kernel corresponding to** $B(x,H)$, namely the function $\xi\mapsto\frac{e^{ix\xi}-1}{|\xi|^{H+1/2}}$, in the latter basis, it follows that

$$\frac{e^{ix\xi}-1}{|\xi|^{H+1/2}} = \sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}} c_{j,k}(x,H) 2^{-j/2} e^{i2^{-j}k\cdot\xi}\overline{\widehat{\psi}(2^{-j}\xi)}, \qquad (17)$$

where the series converges in $L^2(\mathbb{R})$ and each of its coefficients is given by

$$c_{j,k}(x,H) = 2^{-j/2}\int_{\mathbb{R}} \frac{(e^{ix\xi}-1)}{|\xi|^{H+1/2}} e^{-i2^{-j}k\cdot\xi}\widehat{\psi}(2^{-j}\xi)\,d\xi. \qquad (18)$$

Setting in (18), $\eta = 2^{-j}\xi$ one gets that

$$c_{j,k}(t,H) = 2^{-jH}\big(\Psi(2^j t - k, H) - \Psi(-k,H)\big), \tag{19}$$

where

$$\Psi(x,H) = \int_{\mathbb{R}^N} e^{ix\cdot\eta}\,\frac{\widehat{\psi}(\eta)}{|\eta|^{H+1/2}}\,d\eta. \tag{20}$$

Putting together (17), (19) and the isometry property of the stochastic integral $\int_{\mathbb{R}} (\,\cdot\,)\, d\widehat{W}$, it follows that the random variable $B(x,H) = \int_{\mathbb{R}} \frac{e^{ix\cdot\xi}-1}{|\xi|^{H+1/2}}\, d\widehat{W}(\xi)$, can be expressed as,

$$B(x,H) = \sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}} 2^{-jH}\epsilon_{j,k}\big(\Psi(2^j x - k, H) - \Psi(-k,H)\big),$$

where the $\epsilon_{j,k}$'s are the **independent** $\mathcal{N}(0,1)$ real-valued Gaussian random variables defined as

$$\epsilon_{j,k} = 2^{-j/2}\int_{\mathbb{R}^N} e^{i2^{-j}k\cdot\xi}\,\overline{\widehat{\psi}(2^{-j}\xi)}\, d\widehat{W}(\xi) = 2^{j/2}\int_{\mathbb{R}} \psi(-2^j s - k)\, dW(s).$$

## Definition 1

*The representation of the field* $\{B(x,H)\}_{(x,H)\in\mathbb{R}\times(0,1)}$ *as,*

$$B(x,H)=\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}}2^{-jH}\epsilon_{j,k}\big(\Psi(2^jx-k,H)-\Psi(-k,H)\big), \tag{21}$$

*is called the* **(standard) random wavelet series representation of** $B$**.**

→ This representation can also be viewed as **a random wavelet series representation of the FBM** $B_H=B(\cdot,H)$**.**

→ A priori, the random series (21), is for every fixed $(x,H)\in\mathbb{R}\times(0,1)$, convergent in $L^2(\Omega)$, $\Omega$ being the underlying probability space. In fact, as shown by the following theorem, **it is also convergent in a much stronger sense.**

## Theorem 4 (Jaffard, Taqqu and Ayache)

*Let $K$ be arbitrary compact subset of $\mathbb{R} \times (0,1)$. The random wavelet series representation of the field $B$* **is almost surely convergent in $\mathcal{C}(K)$, the Banach space of continuous functions over $K$, equipped with the uniform norm**.

**Proof:** In view of **Itô-Nisio Theorem**, it is sufficient to prove that $\left(\{B^n(x,H)\}_{(x,H)\in K}\right)_{n\geq 1}$, the sequence of the partial sums of the series, is **weakly relatively compact in $\mathcal{C}(K)$**.

This compactness can be obtained (see e.g. Billingsley) by showing that, for all $(x_1, H_1) \in K$ and $(x_2, H_2) \in K$,

$$\mathbb{E}\left|B^n(x_1,H_1) - B^n(x_2,H_2)\right|^2 \leq c\left(|x_1 - x_2|^2 + |H_1 - H_2|^2\right)^a. \tag{22}$$

where $c > 0$ is a constant **non depending on** $n$.

$B^n(x,H)$ can be expressed as

$$B^n(x,H) = \sum_{(j,k)\in\mathcal{D}_n} 2^{-jH}\epsilon_{j,k}\big(\Psi(2^j x - k, H) - \Psi(-k,H)\big),$$

where $\mathcal{D}_n$ is **a finite set whose cardinality depends on** $n$. By using the fact that **the** $\epsilon_{j,k}$**'s are independent** $\mathcal{N}(0,1)$ **Gaussian variables**, one obtains that

$$\begin{aligned}
&\mathbb{E}\left|B^n(x_1,H_1) - B^n(x_2,H_2)\right|^2 \\
&= \sum_{(j,k)\in\mathcal{D}_n} \Big|2^{-jH_1}\big(\Psi(2^j x_1 - k, H_1) - \Psi(-k,H_1)\big) \\
&\qquad\qquad - 2^{-jH_2}\big(\Psi(2^j x_2 - k, H_2) - \Psi(-k,H_2)\big)\Big|^2 \\
&\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \Big|2^{-jH_1}\big(\Psi(2^j x_1 - k, H_1) - \Psi(-k,H_1)\big) \\
&\qquad\qquad - 2^{-jH_2}\big(\Psi(2^j x_2 - k, H_2) - \Psi(-k,H_2)\big)\Big|^2 \\
&= \mathbb{E}\left|B(x_1,H_1) - B(x_2,H_2)\right|^2
\end{aligned}$$

Finally combining the latter inequality with Lemma 1, one has that,

$$\begin{aligned}\mathbb{E}\left|B^n(x_1,H_1)-B^n(x_2,H_2)\right|^2 &\le \mathbb{E}\left|B(x_1,H_1)-B(x_2,H_2)\right|^2\\ &\le c\left(|t_1-t_2|^2+|H_1-H_2|^2\right)^a.\end{aligned}$$

□