Continuous Gaussian multifractional processes with random pointwise Hölder regularity

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Main parts of the seminar

1. Introduction and motivation
2. Multifractional Brownian motion (mBm)
3. Our main result and a sketch of its proof

1-Introduction and motivation

Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be an arbitrary Gaussian process whose trajectories are, with probability 1, **continuous nowhere differentiable** functions.
$\rightarrow$ **The global Hölder regularity** of a trajectory $t \mapsto X(t,\omega)$, over a compact interval $J \subset \mathbb{R}$, can be measured through **the uniform Hölder exponent**:

$$\beta_X(J,\omega) = \sup\left\{\beta \geq 0 \,:\, \sup_{t',t''\in J} \frac{|X(t',\omega) - X(t'',\omega)|}{|t'-t''|^{\beta}} < \infty\right\}. \tag{1}$$

$\rightarrow$ **The local Hölder regularity** of a trajectory $t \mapsto X(t,\omega)$, in a neighborhood of some fixed point $s \in \mathbb{R}$, can be measured through **the pointwise Hölder exponent at** s:

$$\alpha_X(s,\omega) = \sup\left\{\alpha \geq 0 \,:\, \limsup_{h\to 0} \frac{|X(s+h,\omega) - X(s,\omega)|}{|h|^{\alpha}} = 0\right\}. \tag{2}$$

It follows from **zero-one law** that, **the random variables** $\beta_X(J)$ **and** $\alpha_X(s)$ **are in fact deterministic**; namely there exist deterministic quantities $b_X(J), a_X(s) \in [0,1]$ such that

$$\mathbb{P}\big\{\beta_X(J) = b_X(J)\big\} = 1 \tag{3}$$

and

$$\mathbb{P}\big\{\alpha_X(s) = a_X(s)\big\} = 1. \tag{4}$$

Thus, the deterministic function a_X is a modification of the stochastic process α_X. We call this function **the deterministic modification of the pointwise Hölder exponent of** X**.**

A natural question: for any arbitrary Gaussian process $X = \{X(t)\}_{t\in\mathbb{R}}$ whose trajectories are, with probability 1, continuous nowhere differentiable functions; is it true that a_X and α_X are **indistinguishable**?

Indistinguishable means that: **there is an event of probability** 1**, non depending on** s, denoted by $\widetilde{\Omega}$, such that,

$$\text{for all } \omega \in \widetilde{\Omega} \text{ and all } s \in \mathbb{R} \text{ one has, } \alpha_X(s,\omega) = a_X(s). \tag{5}$$

The question is non-trivial, since **the intersection of the non-countable family of the events of probability** 1**,** $\{\alpha_X(s) = a(s)\}$**,** $s \in \mathbb{R}$**,** namely:

$$\bigcap_{s\in\mathbb{R}} \big\{\omega \in \Omega \,:\, \alpha_X(s,\omega) = a(s)\big\},$$

is not necessarily an event of probability 1 **and may even not be an event** (i.e. a mesurable set).

$\rightarrow$ **The goal of our talk** is to show that the answer to this question is not always positive.

More precisely, we construct a family of Gaussian **multifractional Brownian motions** (mBm's) $\{X(t)\}_{t\in\mathbb{R}}$, whose trajectories are, with probability 1, continuous nowhere differentiable functions satisfying **the following property**: there exists **an event** D **of strictly positive probability**, such that for all $\omega \in D$, one has for some $s_0(\omega) \in \mathbb{R}$,

$$\alpha_X(s_0(\omega),\omega) \neq a_X(s_0(\omega)). \tag{6}$$

In other words, **though the deterministic function** a_X **is a modification of the stochastic process** α_X**, they are not indistinguishable.**

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2-1-The field B generating mBm

From now on, u **and** v **denotes two fixed reals satisfying:** $0 < u < v < 1$. Let $B = \{B(t,\theta)\}_{(t,\theta)\in\mathbb{R}\times[u,v]}$ be the Gaussian field defined for all $(t,\theta) \in \mathbb{R} \times [u,v]$ as **the Wiener integral**:

$$B(t,\theta) = \int_{\mathbb{R}} \left\{ (t-x)_+^{\theta-1/2} - (-x)_+^{\theta-1/2} \right\} dW(x), \tag{7}$$

with the convention that for every $(y,\theta) \in \mathbb{R} \times [u,v]$, $(y)_+^{\theta-1/2} = y^{\theta-1/2}$ if $y > 0$ and $(y)_+^{\theta-1/2} = 0$ else.

Remarks:

(a) For all fixed $\theta \in [u,v]$, the stochastic process $B_\theta = \{B(t,\theta)\}_{t\in\mathbb{R}}$, is the usual **fractional Brownian motion** (fBm) of Hurst parameter θ.

(b) By using Kolmogorov criterion, one can prove that there is a modification of B whose trajectories are with probability 1, continuous functions; **in all the sequel B will be identified with the latter modification.**

Let $\gamma : \mathbb{R} \to [u, v]$ be **a continuous deterministic function**, $\{X(t)\}_{t\in\mathbb{R}}$ the **multifractional Brownian motion** (mBm) of functional parameter γ, is defined for all $t \in \mathbb{R}$, as,

$$X(t) = B(t, \gamma(t)). \tag{8}$$

$\to$ MBm has been introduced in Peltier and Lévy-Véhel (1995) and in Benassi, Jaffard and Roux (1997).
$\to$ With probability 1, the trajectories of $\{X(t)\}_{t\in\mathbb{R}}$ are continuous functions.
$\to$ When γ is a constant, then $\{X(t)\}_{t\in\mathbb{R}}$ reduces to the usual fBm.

2-2-Pointwise Hölder regularity of mBm: known results

→ Peltier and Lévy-Véhel (1995) and Benassi, Jaffard and Roux (1997): if $J \subset \mathbb{R}$ is a compact interval such that,

$$\max_{t \in J} \gamma(t) < \beta_\gamma(J), \tag{$*$}$$

$\beta_\gamma(J)$ being the uniform Hölder exponent of γ over J; then, for all $s \in \mathring{J}$,

$$\mathbb{P}\{\alpha_X(s) = \gamma(s)\} = 1. \tag{9}$$

→ Jaffard, Taqqu and Ayache (2007): when the Condition $(*)$ is satisfied, then the process $\{\alpha_X(s)\}_{s \in \mathring{J}}$ and its deterministic modification $\{\gamma(s)\}_{s \in \mathring{J}}$ are indistinguishable.

→ Herbin (2006): for all $s \in \mathbb{R}$ such that $\gamma(s) \neq \alpha_\gamma(s)$, one has,

$$\mathbb{P}\left\{\alpha_X(s) = \min\left(\gamma(s), \alpha_\gamma(s)\right)\right\} = 1. \tag{10}$$

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3-1-Statement of the main result

Before stating our main result, it is important to point out that wavelet methods (see Ayache and Taqqu (2005)) allow to show that:

- there is an event Ω^* of probability 1, such that for all fixed $\omega \in \Omega^*$ and $t \in \mathbb{R}$, the function $B(t, \cdot, \omega) : \theta \mapsto B(t, \theta, \omega)$ is continuously differentiable over $[u, v]$;
- the Gaussian field $\partial_\theta B = \big\{(\partial_\theta B)(t, \theta)\big\}_{(t,\theta) \in \mathbb{R} \times [u,v]}$ can be represented, for all (t, θ), almost surely, as the Wiener integral:

$$
\begin{aligned}
&\big(\partial_\theta B\big)(t,\theta) \\
&= \int_{\mathbb{R}} \Big\{(t-x)_+^{\theta-1/2} \log\big[(t-x)_+\big] - (-x)_+^{\theta-1/2} \log\big[(-x)_+\big]\Big\}\, dW(x),
\end{aligned}
\tag{11}
$$

with the convention that $\log 0 = -\infty$ and $0 \times (-\infty) = 0$.

From now on:

- we restrict to the interval $(0,1)$;
- we assume that the function γ and its pointwise Hölder exponent α_γ satisfy the following inequalities: for all $s \in (0,1)$,

$$0 < \alpha_\gamma(s) < \gamma(s) < 2\alpha_\gamma(s) < 1. \qquad (\mathcal{K})$$

Then, it follows from Herbin's result, that $\{\alpha_\gamma(s)\}_{s\in(0,1)}$ **is the deterministic modification of** $\{\alpha_X(s)\}_{s\in(0,1)}$**, the pointwise Hölder exponent of the mBm** $\{X(t)\}_{t\in(0,1)}$**.**

$\rightarrow$ Our main result, **draws a connection between** $\{\alpha_X(s)\}_{s\in(0,1)}$ **and the zero-level set:**

$$\mathcal{L}_Y = \big\{ s \in (0,1) \,:\, Y(s) = 0 \big\},$$

where the Gaussian process $\{Y(s)\}_{s\in(0,1)}$ is defined as:

$$\begin{aligned} &Y(s) \\ &:= \big(\partial_\theta B\big)(s,\gamma(s)) \\ &= \int_{\mathbb{R}} \Big\{ (s-x)_+^{\gamma(s)-1/2} \log\big[(s-x)_+\big] - (-x)_+^{\gamma(s)-1/2} \log\big[(-x)_+\big] \Big\}\, dW(x). \end{aligned}$$

$\rightarrow$ It implies that $\{\alpha_\gamma(s)\}_{s\in(0,1)}$ **and** $\{\alpha_X(s)\}_{s\in(0,1)}$ **are not indistinguishable**.

Theorem (main result)

There exists an event Ω_0 of probability 1 satisfying the following property: for all $\omega \in \Omega_0$ and all $s \in (0,1)$, one has,

$$\alpha_s(s,\omega) = \begin{cases} \gamma(s) \text{ if } s \in \mathcal{L}_Y(\omega), \\ \\ \alpha_\gamma(s) \text{ else.} \end{cases} \tag{12}$$

Moreover, there is an event of strictly positive probability $D \subset \Omega_0$, such that for all $\omega \in D$,

$$\dim_{\mathcal{H}}\left(\mathcal{L}_Y(\omega)\right) \geq 1 - v > 0, \tag{13}$$

where $\dim_{\mathcal{H}}(\cdot)$ denotes the Hausdorff dimension.

3-2-Scketch of the proof

The increment at s of the mBm X:

$$\Delta_s X(h) = X(s+h) - X(s),$$

can be bounded as follows:

$$\Big| \big|\Delta_s B_{\gamma(s)}(h)\big| - \big|M_s(h)\big| \Big| \le \big|\Delta_s X(h)\big| \le \Big| \big|\Delta_s B_{\gamma(s)}(h)\big| + \big|M_s(h)\big| \Big|, \quad (14)$$

where:

$$\Delta_s B_{\gamma(s)}(h) = B_{\gamma(s)}(s+h) - B_{\gamma(s)}(s),$$

is **the increment at s of $B_{\gamma(s)}$, the fBm of Hurst parameter** $\gamma(s)$, and

$$M_s(h) = B(s+h, \gamma(s+h)) - B(s+h, \gamma(s)).$$

A useful notation

Let f and g be two nonnegative functions defined on a neighborhood of 0, which do not vanish except on 0, the notation:

$$f(h) \sim g(h),$$

means that: for all arbitrarily small $\epsilon > 0$, one has,

$$\limsup_{h \to 0} \frac{f(h)}{|h|^{-\epsilon} g(h)} = 0$$

and

$$\limsup_{h \to 0} \frac{f(h)}{|h|^{\epsilon} g(h)} = +\infty.$$

The Mean Value Theorem entails that

$$M_s(h) = \big(\partial_\theta B\big)(s+h, \kappa(s,h)) \times \big(\gamma(s+h) - \gamma(s)\big), \tag{15}$$

where

$$\kappa(s,h) \in \Big(\min\{\gamma(s), \gamma(s+h)\}, \max\{\gamma(s), \gamma(s+h)\} \Big).$$

Next, (15) implies that,

$$\big|M_s(h)\big| \sim \big|\big(\partial_\theta B\big)(s+h, \kappa(s,h))\big| \times \big|h\big|^{\alpha_\gamma(s)}. \tag{16}$$

On the other hand, one has,

$$\big|\Delta_s B_{\gamma(s)}(h)\big| \sim |h|^{\gamma(s)}. \tag{17}$$

Combining (16) with (17), it follows that,

$$\big|\Delta_s X(h)\big| \sim \max\Big\{|h|^{\gamma(s)}, \big|\big(\partial_\theta B\big)(s+h, \kappa(s,h))\big| \times \big|h\big|^{\alpha_\gamma(s)}\Big\}. \tag{18}$$

It remains to estimate

$$\left|(\partial_\theta B)(s+h,\kappa(s,h))\right|.$$

To this end, we need the following lemma, which can be proved by using wavelet methods.

Lemma 1

For every arbitrarily small $\epsilon > 0$, there is a random variable C of finite moment of any order (only depending on u, v and ϵ) such that one has, almost surely, for all (t_1,θ_1) and (t_2,θ_2) in $[0,1]\times[u,v]$,

$$\Big|(\partial_\theta B)(t_1,\theta_1)-(\partial_\theta B)(t_2,\theta_2)\Big| \le C\Big(|t_1-t_2|^{\max\{\theta_1,\theta_2\}-\epsilon}+|\theta_1-\theta_2|\Big).$$

Thus, using the lemma, $|h| \leq 1$ and $Y(s) = \big(\partial_\theta B\big)(s, \gamma(s))$ one has, a.s.

$$\Big|\big(\partial_\theta B\big)(s+h, \kappa(s,h)) - Y(s)\Big| \leq C\Big(|h|^{\max\{\kappa(s,h),\gamma(s)\}-\epsilon} + |\kappa(s,h) - \gamma(s)|\Big)$$
$$\leq C\Big(|h|^{\gamma(s)-\epsilon} + |\gamma(s+h) - \gamma(s)|\Big) \leq C'\Big(|h|^{\gamma(s)-\epsilon} + |h|^{\alpha_\gamma(s)-\epsilon}\Big)$$
$$\leq C''|h|^{\gamma(s)-\epsilon}, \tag{19}$$

where:

- the 2-nd inequality, follows from

$$\kappa(s,h) \in \big(\min\{\gamma(s), \gamma(s+h)\}, \max\{\gamma(s), \gamma(s+h)\}\big);$$

- while, the 4-th inequality, results from the assumtion, $\gamma(s) < \alpha_\gamma(s)$.

Then (19) implies that, a.s.,

$$\Big|\big(\partial_\theta B\big)(s+h, \kappa(s,h))\Big| = \big|Y(s)\big| + \mathcal{O}\big(|h|^{\gamma(s)-\epsilon}\big). \tag{20}$$

Putting together,

$$\left|\big(\partial_\theta B\big)(s+h,\kappa(s,h))\right| = \left|Y(s)\right| + \mathcal{O}\big(|h|^{\gamma(s)-\epsilon}\big),$$

$$\left|\Delta_s X(h)\right| \sim \max\left\{|h|^{\gamma(s)},\ \left|\big(\partial_\theta B\big)(s+h,\kappa(s,h))\right| \times \left|h\right|^{\alpha_\gamma(s)}\right\},$$

and the assumption $\alpha_\gamma(s) < \gamma(s) < 2\alpha_\gamma(s)$, we get:

- when $Y(s) \neq 0$,

$$\left|\Delta_s X(h)\right| \sim \max\{|h|^{\gamma(s)}, \left|h\right|^{\alpha_\gamma(s)}\} = |h|^{\alpha_\gamma(s)}; \tag{21}$$

- else,

$$\left|\Delta_s X(h)\right| \sim \max\{|h|^{\gamma(s)}, \left|h\right|^{\alpha_\gamma(s)}\} = |h|^{\gamma(s)}. \tag{22}$$

At last, the fact that with **a strictly positive probability**,

$$\dim_{\mathcal{H}}\left(\left\{s\in(0,1)\,:\,Y(s)=0\right\}\right)\geq 1-v,$$

follows from **capacity arguments (Frostman Theorem)**, as well as from the fact that the process Y satisfies on $\mathbb{R}_+$, **the property of one sided strong local nondeterminism**: there exists a constant $c>0$ which only depends on v, such that for all integer $n\geq 2$, and all real numbers $s^1,\ldots,s^n$ satisfying

$$0\leq s^1<\ldots<s^n, \tag{23}$$

one has,

$$\mathrm{Var}\big(Y(s^n)/Y(s^1),\ldots,Y(s^{n-1})\big)\geq c\big(s^n-s^{n-1}\big)^{2v}, \tag{24}$$

where $\mathrm{Var}\big(Y(s^n)/Y(s^1),\ldots,Y(s^{n-1})\big)$ denotes the conditional variance of $Y(s^n)$ given $Y(s^1),\ldots,Y(s^{n-1})$.