On the monofractality of many stationary continuous Gaussian fields

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Organization of the talk

All the stochastic fields we will consider will be real-valued and indexed by $\mathbb{R}^N$.

A stochastic field $\{X(t)\}_{t\in\mathbb{R}^N}$ is stationary iff it has the same finite-dimensional distributions as the field $\{X(\tau+t)\}_{t\in\mathbb{R}^N}$, for any fixed $\tau\in\mathbb{R}^N$.

A stochastic field $\{Y(t)\}_{t\in\mathbb{R}^N}$ is with stationary increments iff the two fields $\{Y(t)-Y(0)\}_{t\in\mathbb{R}^N}$ and $\{Y(\tau+t)-Y(\tau)\}_{t\in\mathbb{R}^N}$ have the same finite-dimensional distributions, for any fixed $\tau\in\mathbb{R}^N$.

If $\{X(t)\}_{t\in\mathbb{R}^N}$ is stationary then it is with stationary increments.

Let $\{Z(t)\}_{t\in\mathbb{R}^N}$ be an arbitrary stochastic field with continuous (or even less regular) paths. Their local behaviors in neighborhoods of all points of $\mathbb{R}^N$ are usually described by the pointwise Hölder exponent random field $\{\alpha_Z(\tau)\}_{\tau\in\mathbb{R}^N}$ defined, for all $\tau\in\mathbb{R}^N$, as:

$$\alpha_Z(\tau) := \sup\left\{\alpha\in[0,1] \,:\, \limsup_{r>0,\, r\to 0}\left\{r^{-\alpha}\,\mathrm{Osc}_Z(\tau,r)\right\} < +\infty\right\}, \tag{1.1}$$

where, for any real number $r\in(0,1]$,

$$\mathrm{Osc}_Z(\tau,r) := \sup\left\{\left|Z(t')-Z(t'')\right| \,:\, (t',t'')\in\mathcal{B}(\tau,r)^2\right\} \tag{1.2}$$

is the oscillation of the field Z on the ball $\mathcal{B}(\tau,r) := \{t\in\mathbb{R}^N : |t-\tau|\le r\}$.

Remark 1.1

When $\{Z(t)\}_{t\in\mathbb{R}^N}$ is a Gaussian and more generally a stable stochastic field then it follows form zero-one law that, for each fixed $\tau\in\mathbb{R}^N$, the exponent $\alpha_Z(\tau)$ is almost surely equal to a deterministic quantity $a_Z(\tau)$. Yet the equality holds on a event of probability 1 which a priori depends on τ.

Thus a question naturally arises:

Question: Is it true that $\mathbb{P}(\forall\, \tau \in \mathbb{R}^N,\ \alpha_Z(\tau) = a_Z(\tau)) = 1$?

Three types of counter-examples

(a) Multifractal Lévy processes (Jaffard 1999); they have stationary increments, but they are non-Gaussian with discontinuous càdlàg paths.

(b) Multifractional Brownian Motions (MBMs) with sufficiently irregular continuous Hurst function (A. 2013); they are Gaussian with continuous paths but their increments are non-stationary.

(c) Multifractal Linear Fractional Stable Motions (LFSMs) with large enough Hurst parameter (Balança 2014); they have stationary increments and continuous paths but they are non-Gaussian.

Remark 1.2

The type (c) of counter-examples is in some sense more surprising than the two others since the almost sure deterministic value of the pointwise Hölder exponent $\alpha_{LFSM}(\tau)$ does not even depend on τ. The latter fact is due to continuity of paths of LFSMs and more importantly to stationarity of their increments.

Proposition 1.1 (A. (2018))

Let $Y = \{Y(t)\}_{t\in\mathbb{R}^N}$ be an arbitrary stochastic field with stationary increments and continuous paths. Then the associated pointwise Hölder exponent random field $\{\alpha_Y(\tau)\}_{\tau\in\mathbb{R}^N}$ is stationary. Thus, when Y satisfies zero-one law (this is for instance the case when it is Gaussian or non-Gaussian with stable distribution), then there exists a deterministic constant $a_Y \in [0,1]$ not depending on τ such that

$$\mathbb{P}\big(\alpha_Y(\tau) = a_Y\big) = 1\,, \quad \text{for all } \tau \in \mathbb{R}^N. \tag{1.3}$$

In view of (1.3) it is natural to wonder whether Y is monofractal or multifractal.

Definition 1.1

The singularity spectrum $\rho_Y = \{\rho_Y(x)\}_{x\in[0,1]}$ of Y is defined as:

$$\rho_Y(x) = \dim_{\mathcal{H}}\big(\{\tau \in \mathbb{R}^N : \alpha_Y(\tau) = x\}\big)\,. \tag{1.4}$$

Y is monofractal when the topological support of the random function ρ_Y almost surely reduces to one deterministic point; typically this is the case when one has

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}^N,\ \alpha_Y(\tau) = a_Y\big) = 1\,. \tag{1.5}$$

Organization of the talk

Theorem 2.1 (Bochner (1932))

Let $X = \{X(t)\}_{t\in\mathbb{R}^N}$ be a centred stationary Gaussian field having a continuous covariance function on $\mathbb{R}^N \times \mathbb{R}^N$. Then, it can be expressed, for all $(t^1, t^2) \in \mathbb{R}^N \times \mathbb{R}^N$, as:

$$\mathrm{Cov}\big(X(t^1), X(t^2)\big) = \mathbb{E}\big(X(t^1)X(t^2)\big) = \int_{\mathbb{R}^N} e^{-i(t^1-t^2)\cdot\xi}\, d\mu(\xi)\,, \qquad (2.1)$$

where μ is a symmetric nonnegative finite Borel measure on $\mathbb{R}^N$ called the spectral measure of X.

When μ is absolutely continuous with a spectral density $(g)^2$. Then, denoting by $\mathcal{F}^{-1}(g)$ the inverse Fourier transform of g (the square root of the spectral density), one has, for all $t \in \mathbb{R}^N$,

$$X(t) = \int_{\mathbb{R}^N} e^{-it\cdot\xi} g(\xi)\, d\widehat{W}(\xi) = (2\pi)^{N/2} \int_{\mathbb{R}^N} \mathcal{F}^{-1}(g)(x-t)\, dW(x)\,, \qquad (2.2)$$

where $\int_{\mathbb{R}^N}(\cdot)\, dW$ is the classical Wiener stochastic integral on $\mathbb{R}^N$.

Theorem 2.2 (Yaglom (1954))

Let $Y = \{Y(t)\}_{t\in\mathbb{R}^N}$ be a centred Gaussian field with stationary increments such that $Y(0) = 0$ and the covariance function of Y is continuous on $\mathbb{R}^N \times \mathbb{R}^N$. Then, one has

$$\mathrm{Cov}\big(Y(t^1), Y(t^2)\big) = \int_{\mathbb{R}^N} (e^{-it^1\cdot\xi} - 1)(e^{it^2\cdot\xi} - 1)\, d\widetilde{\mu}(\xi)\,, \tag{2.3}$$

where $\widetilde{\mu}$ is a symmetric nonnegative Borel measure on $\mathbb{R}^N$ (called the spectral measure) s.t.

$$\int_{\mathbb{R}^N} \big(1 \wedge |\xi|^2\big)\, d\widetilde{\mu}(\xi) < +\infty\,. \tag{2.4}$$

Let $\mathcal{B}_1 := \{|\xi| \le 1\}$ and $\overline{\mathcal{B}}_1 := \{|\xi| > 1\}$. Then, for all $(t^1, t^2) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\begin{aligned} &\mathrm{Cov}\big(Y(t^1), Y(t^2)\big) \\ &= \int_{\mathcal{B}_1} (e^{-it^1\cdot\xi} - 1)(e^{it^2\cdot\xi} - 1)\, d\widetilde{\mu}(\xi) + \int_{\overline{\mathcal{B}}_1} (e^{-it^1\cdot\xi} - 1)(e^{it^2\cdot\xi} - 1)\, d\widetilde{\mu}(\xi) \\ &= \mathrm{Cov}\big(R(t^1), R(t^2)\big) + \mathrm{Cov}\big(X(t^1) - X(0), X(t^2) - X(0)\big)\,, \end{aligned} \tag{2.5}$$

where:

- $\{R(t)\}_{t\in\mathbb{R}^N}$ is a smooth centred Gaussian field with stationary increments having the spectral measure $\widetilde{\mu}(\bullet \cap \mathcal{B}_1)$.
- $\{X(t)\}_{t\in\mathbb{R}^N}$ is a centred stationary Gaussian field independent on $\{R(t)\}_{t\in\mathbb{R}^N}$ and having the spectral measure $\widetilde{\mu}(\bullet \cap \overline{\mathcal{B}}_1)$.

It follows from (2.5) that $\{Y(t)\}_{t\in\mathbb{R}^N}$ is equal in distribution to the sum of the two fields $\{R(t)\}_{t\in\mathbb{R}^N}$ and $\{X(t)-X(0)\}_{t\in\mathbb{R}^N}$. Then, it results from the smoothness of $\{R(t)\}_{t\in\mathbb{R}^N}$ that $\{Y(t)\}_{t\in\mathbb{R}^N}$ and $\{X(t)\}_{t\in\mathbb{R}^N}$ have exactly the same local path behavior.

Thus, for proving that the random function $\tau \mapsto \alpha_Y(\tau)$ reduces to a deterministic constant, it is enough to prove that this is the case for the random function $\tau \mapsto \alpha_X(\tau)$.

From now on, we focus on the centred real-valued stationary Gaussian field $\{X(t)\}_{t\in\mathbb{R}^N}$ of the general form:

$$X(t) = \int_{\mathbb{R}^N} e^{-it\cdot\xi} g(\xi)\, d\widehat{W}(\xi) = (2\pi)^{N/2} \int_{\mathbb{R}^N} \mathcal{F}^{-1}(g)(x-t)\, dW(x)\,, \qquad (2.6)$$

where g is any arbitrary deterministic even and nonnegative function belonging to $L^2(\mathbb{R}^N)$; thus, its inverse Fourier transform $\mathcal{F}^{-1}(g)$ is even, real-valued and belongs to $L^2(\mathbb{R}^N)$.

One knows from e.g. Ledoux and Talagrand (1991) that path properties of any real-valued Gaussian process $Z = \{Z(t)\}_{t\in T}$ (the set T is arbitrary) are closely connected with d_Z, the natural pseudo-metric on T associated with $\{Z(t)\}_{t\in T}$ defined as: $d_Z(t',t'')^2 := \mathbb{E}\big(|Z(t')-Z(t'')|^2\big)$, for all $(t',t'')\in T\times T$.

In the case of $\{X(t)\}_{t\in\mathbb{R}^N}$, $d_X(t',t'')^2 = \mathbb{E}\big(|X(t'-t'')-X(0)|^2\big) = V_X(t'-t'')$; the bounded function V_X is called the variogram and defined, for all $h\in\mathbb{R}^N$, as:

$$V_X(h) := \mathbb{E}\big(|X(h)-X(0)|^2\big) = (2\pi)^N \int_{\mathbb{R}^N} \big|\mathcal{F}^{-1}(g)(x+h)-\mathcal{F}^{-1}(g)(x)\big|^2\, dx\,. \quad (2.7)$$

The last equality in (2.7) follows from the isometry property of the Wiener integral in (2.6).

1st reason for which our framework is convenient: the asymptotic behavior of $V_X(h)$, when $|h| \to 0$, is closely related to Besov regularity of $\mathcal{F}^{-1}(g)$.

Definition 2.1 (see e.g. Bergh and Löfström (1976))

Let $p \in [1, +\infty]$, $s \in [0, +\infty)$ and $m \in \mathbb{N} \cap (s, +\infty)$ be arbitrary and fixed. A real-valued (or more generally complex-valued) function f on $\mathbb{R}^N$ belongs to the Besov space $B^s_{p,\infty}(\mathbb{R}^N)$ if and only if f is in the space $L^p(\mathbb{R}^N)$ and satisfies

$$\sup_{h \in \mathbb{R}^N} \left\{ |h|^{-s} \left(\int_{\mathbb{R}^N} \left| \Delta_h^m f(x) \right|^p dx \right)^{1/p} \right\} < +\infty \,, \tag{2.8}$$

where, for all $h, x \in \mathbb{R}^N$, $\Delta_h^m f(x)$ is the m-th order increment of f defined as:

$$\Delta_h^m f(x) := \sum_{n=0}^{m} \binom{m}{n} (-1)^n f(x + nh) \,. \tag{2.9}$$

Remark 2.1

$B^0_{p,\infty}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ *and* $B^s_{\infty,\infty}(\mathbb{R}^N)$ *is the usual Hölder space of order s.*

Assumption on the asymptotic behavior of the variogram V_X at 0:

$(\mathcal{H}_0)$ There is a constant critical exponent $\overline{s} \in (0,1)$ such that one has

$$\sup_{h\in\mathbb{R}^N} \left\{ |h|^{-2s}\, V_X(h) \right\} < +\infty\,, \quad \text{for all } s \in (0,\overline{s}), \tag{2.10}$$

and

$$\sup_{h\in\mathbb{R}^N} \left\{ |h|^{-2s}\, V_X(h) \right\} = +\infty\,, \quad \text{for all } s \in (\overline{s},1). \tag{2.11}$$

Remark 2.2

The critical exponent $\overline{s}$ in the condition $(\mathcal{H}_0)$ can be expressed in terms of critical Besov regularity of the function $\mathcal{F}^{-1}(g)$ as follows:

$$\overline{s} = \sup\left\{ s \in [0,+\infty) : \mathcal{F}^{-1}(g) \in B^{s}_{2,\infty}(\mathbb{R}^N) \right\}. \tag{2.12}$$

One knows from (2.10) that, for any arbitrary fixed $s \in (0, \bar{s})$, there is a constant c_s such that, for all $t', t'' \in \mathbb{R}^N$,

$$\mathbb{E}\big(|X(t') - X(t'')|^2\big) \le c_s |t' - t''|^{2s}\,. \tag{2.13}$$

Thus (Kolmogorov Hölder continuity Theorem), paths of the field X are with probability 1 Hölder continuous functions of any arbitrary order $s' \in (0, \bar{s})$ on each compact subset of $\mathbb{R}^N$. This implies that

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}^N,\ \alpha_X(\tau) \ge \bar{s}\big) = 1\,. \tag{2.14}$$

On another hand (2.11) entails that

$$\forall\, \tau \in \mathbb{R}^N, \forall\, s \in (\bar{s}, 1) \quad \lim_{t \to \tau} \frac{\mathbb{E}\big(|X(t) - X(\tau)|^2\big)}{|t - \tau|^{2s}} = +\infty\,, \tag{2.15}$$

and consequently that

$$\forall\, \tau \in \mathbb{R}^N, \quad \mathbb{P}\big(\alpha_X(\tau) \le \bar{s}\big) = 1\,. \tag{2.16}$$

In some particular cases (essentially fractional Brownian fields and related fields) it was shown since a long time (see Berman (1972, 1973), Pitt (1978) and Xiao (1997)) that

$$\mathbb{P}\big(\forall\, \tau \in \mathbb{R}^N,\ \alpha_X(\tau) \le \overline{s}\big) = 1. \qquad (*)$$

The intuition is that everywhere irregularity of paths comes from some weak forms of independence of increments. Berman and Pitt introduced the concept of Local Non Determinism (LND) for proposing a formalization of this intuition. Thus, according to their point of view, a sufficient condition for $(*)$ holds is that X satisfies the LND property: for each fixed $s \in (\overline{s}, 1)$, compact rectangle $I \subset \mathbb{R}^N$ and integer $m \ge 2$, there exists a constant $c > 0$ such that one has

$$\mathrm{Var}\big(X(t^m)|X(t^1), \ldots, X(t^{m-1})\big) \ge c \min_{1 \le q < m} |t^m - t^q|^{2s}\,, \qquad (2.17)$$

for all distinct points $t^1, \ldots, t^m \in I$ sufficiently close together. The known proofs of (2.17) require to impose to g (the square root of the spectral density of X) restrictive conditions, as for instance:

$$\liminf_{|\xi| \to +\infty} \Big\{ |\xi|^{s+N/2}\, |g(\xi)| \Big\} > 0\,, \quad \text{for all } s \in (\overline{s}, 1). \qquad (2.18)$$

Our goal is to try to find the weakest possible conditions on g under which $(*)$ holds. Therefore, we will avoid to make use of the concept of LND and try to express the intuitive idea of weakly independent increments in two other different ways which will be presented in Sections 3 and 4.

Rather than working with usual increments of X we will work with its generalized increments consisting in its wavelet coefficients $\chi_{j,k}$, $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^N$, defined almost surely (a.s.) as the pathwise Lebesgue integral:

$$\chi_{j,k} := \int_{\mathbb{R}^N} 2^{jN} \psi_0(2^j t - k) X(t)\, dt\,. \tag{2.19}$$

j is called the dilation index and k the translation index. The real-valued mother wavelet ψ_0 belongs to the Schwartz class $S(\mathbb{R}^N)$. The following lemma shows that the pathwise Lebesgue integral in (2.19) is a.s. well-defined and finite.

Lemma 2.1

One has a.s.

$$\sup_{t \in \mathbb{R}^N} \left\{ \log^{-1/2}(3 + |t|) |X(t)| \right\} < +\infty\,. \tag{2.20}$$

Proof of Lemma 2.1: Let $I := [0,1]^N$, For all $k \in \mathbb{Z}^N$, one sets

$$\mu_k := \sup_{s \in I} |X(k+s)| \quad \text{and} \quad \sigma_k^2 := \sup_{s \in I} \mathbb{E}\big(|X(k+s)|^2\big)\,. \tag{2.21}$$

Using a Borell type inequality one gets, for all $k \in \mathbb{Z}^N$ and $z > 0$ that

$$\mathbb{P}\big(\big|\mu_k - \mathbb{E}(\mu_k)\big| > z\big) \le 2\exp\big(-2z^2/\pi^2\sigma_k^2\big). \tag{2.22}$$

The stationarity of X implies that $\mu_k \overset{d}{=} \mu_0$ and $\sigma_k^2 = \sigma_0^2 = \mathbb{E}\big(|X(0)|^2\big)$. Thus, one can derive from (2.22) that

$$\mathbb{P}\left(\big|\mu_k - \mathbb{E}(\mu_0)\big| > \pi\sigma_X\sqrt{N\log\big(3+|k|\big)}\right) \le 2\big(3+|k|\big)^{-2N}, \tag{2.23}$$

and consequently that

$$\sum_{k \in \mathbb{Z}^N} \mathbb{P}\left(\big|\mu_k - \mathbb{E}(\mu_0)\big| > \pi\sigma_X\sqrt{N\log\big(3+|k|\big)}\right) < +\infty. \tag{2.24}$$

Thus Lemma 2.1 results from Borel-Cantelli Lemma. □

Remark 2.3

Let $I \subset \mathbb{R}^N$ *be an arbitrary dyadic cube, that is there exists a unique couple* $(j_I, k_I) = (j_I, k_{I,1}, \dots, k_{I,N}) \in \mathbb{Z}_+ \times \mathbb{Z}^N$ *such that*

$$I = I_{j_I,k_I} := \prod_{n=1}^{N} \left[2^{-j_I} k_{I,n}, 2^{-j_I}(k_{I,n}+1)\right] \overset{abusive}{=} [2^{-j_I} k_I, 2^{-j_I}(k_I+1)]; \tag{2.25}$$

for simplicity the random variable χ_{j_I,k_I} *will sometimes be denoted by* χ_I*.*

Lemma 2.2

For all $\tau \in \mathbb{R}^N$ *and* $\rho \in (0,1]$*. One sets* $\mathcal{B}(\tau,\rho) := \{t \in \mathbb{R}^N : |t-\tau| \leq \rho\}$ *and*

$$\mathcal{M}_\chi(\tau,\rho) := \sup\left\{|\chi_I| : I \text{ dyadic cube s.t. } I \subseteq \mathcal{B}(\tau,\rho)\right\}. \tag{2.26}$$

Then, for every fixed $\alpha \in [0,1]$ *and* $\gamma > 1$*, the following inequality holds:*

$$\|\psi_0\|_{L^1(\mathbb{R}^N)} \limsup_{r>0,\, r\to 0} \left\{r^{-\alpha}\, \mathrm{Osc}_X(\tau,r)\right\} \geq \limsup_{r>0,\, r\to 0} \left\{r^{-\alpha}\, \mathcal{M}_\chi(\tau,r^\gamma)\right\}. \tag{2.27}$$

2nd reason for which our framework is convenient: the wavelet coefficients

$$\chi_{j,k} := \int_{\mathbb{R}^N} 2^{jN}\psi_0(2^j t-k)X(t)\,dt = \int_{\mathbb{R}^N} 2^{jN}\psi_0(2^j t-k)\Big(\int_{\mathbb{R}^N} e^{-it\cdot\xi} g(\xi)\,d\widehat{W}(\xi)\Big)\,dt \tag{2.28}$$

can be very conveniently expressed. By interchanging the integrals $\int_{\mathbb{R}^N}(\cdot)\,dt$ and $\int_{\mathbb{R}^N}(\cdot)\,d\widehat{W}(\xi)$ one obtains the following lemma:

Lemma 2.3

For all $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^N$, one has a.s.

$$\chi_{j,k} = \int_{\mathbb{R}^N} e^{-i2^{-j}k\cdot\xi}\,\widehat{\psi}_j(\xi)g(\xi)\,d\widehat{W}(\xi)\,, \tag{2.29}$$

where $\psi_j(\cdot) = 2^{jN}\psi_0(2^j\cdot)$. Thus, $\{\chi_{j,k}\}_{(j,k)\in\mathbb{Z}_+\times\mathbb{Z}^N}$ is a centred real-valued Gaussian process. Moreover, it follows from (2.29) and the isometry property of the stochastic integral $\int_{\mathbb{R}^N}(\cdot)\,d\widehat{W}$ that, for all $(j,k)\in\mathbb{Z}_+\times\mathbb{Z}^N$,

$$\mathrm{Var}(\chi_{j,k}) = \mathcal{V}_j = \big\|g\cdot\widehat{\psi}_j\big\|^2_{L^2(\mathbb{R}^N)} = (2\pi)^N\big\|\mathcal{F}^{-1}(g)*\psi_j\big\|^2_{L^2(\mathbb{R}^N)}\,. \tag{2.30}$$

Organization of the talk

Our next goal is to connect, when $j \to +\infty$, the rate of convergence to zero of $\mathcal{V}_j = (2\pi)^N \big\| \mathcal{F}^{-1}(g) * \psi_j \big\|^2_{L^2(\mathbb{R}^N)}$ (the common variance of wavelet coefficients at level j) to $\overline{s} = \sup \big\{ s \in (0,1) : \mathcal{F}^{-1}(g) \in B^s_{2,\infty}(\mathbb{R}^N) \big\}$. To this end, we will use characterization of Besov spaces through Littlewood-Paley decomposition for tempered distributions.

So, from now on, we assume that $\widehat{\psi}_0(\xi) = \widehat{\varphi}(2^{-1}\xi) - \widehat{\varphi}(\xi)$, for all $\xi \in \mathbb{R}^N$, where $\widehat{\varphi}$ is an even function, infinitely differentiable on $\mathbb{R}^N$, with values in $[0,1]$, which satisfies: $\mathrm{Supp}\, \widehat{\varphi} \subseteq \mathcal{B}(0,1) := \{\xi \in \mathbb{R}^N : |\xi| \le 1\}$ and $\widehat{\varphi}(\xi) = 1$, for every $\xi \in \mathcal{B}(0, 2^{-1}) := \{\xi \in \mathbb{R}^N : |\xi| \le 2^{-1}\}$. Thus, for each $j \in \mathbb{Z}_+$, $\widehat{\psi}_j(\cdot) = \widehat{\psi}_0(2^{-j}\cdot)$ is with values in $[0,1]$, $\mathrm{Supp}\, \widehat{\psi}_j \subseteq \mathcal{R}_j := \big\{ \xi \in \mathbb{R}^N : 2^{j-1} \le |\xi| \le 2^{j+1} \big\}$, and $\widehat{\varphi}(\xi) + \sum_{j=0}^{+\infty} \widehat{\psi}_j(\xi) = 1$, for all $\xi \in \mathbb{R}^N$. Therefore, each tempered distribution f can be expressed in $S'(\mathbb{R}^N)$ as:

$$f = f * \varphi + \sum_{j=0}^{+\infty} f * \psi_j \,. \tag{3.1}$$

The Littlewood-Paley components $f * \varphi$ and $f * \psi_j$, $j \in \mathbb{Z}_+$, are infinitely differentiable functions on $\mathbb{R}^N$ bounded by some polynomial functions.

Remark 3.1 (very important)

The nice framework of Littlewood-Paley decomposition gives us the opportunity to interpret the intuitive idea of weakly independent increments in two different ways which are of another nature than the rather constraining notion of local nondeterminism. The 1st one is an independence property of wavelet coefficients with respect to dilation index: for $l = 0$ or $l = 1$, the sequences of random variables $\left\{\chi_{2p+l,k}; k \in \mathbb{Z}^N\right\}$, $p \in \mathbb{Z}_+$, are independent. The 2nd interpretation concerns their translation index and will be presented in the next section.

Proof of Remark 3.1: Using the equality $\chi_{j,k} = \int_{\mathbb{R}^N} e^{-i2^{-j}k\cdot\xi}\, \widehat{\psi}_j(\xi) g(\xi)\, d\widehat{W}(\xi)$ and the isometry property of $\int_{\mathbb{R}^N}(\cdot)\, d\widehat{W}$, one gets, for all $p', p'' \in \mathbb{Z}_+$ with $p' \neq p''$ and for every $k', k'' \in \mathbb{Z}^N$, that

$$\operatorname{Cov}(\chi_{2p'+l,k'}, \chi_{2p''+l,k''}) = \int_{\mathbb{R}^N} e^{i(2^{-2p''-l}k'' - 2^{-2p'-l}k')\cdot\xi}\, \widehat{\psi}_{2p'+l}(\xi)\widehat{\psi}_{2p''+l}(\xi) g(\xi)\, d\xi = 0 \tag{3.2}$$

since $\operatorname{Supp}\widehat{\psi}_{2p'+l} \cap \operatorname{Supp}\widehat{\psi}_{2p''+l}$ is included in $\left\{2^{2p'+l-1} \leq |\xi| \leq 2^{2p'+l+1}\right\} \cap \left\{2^{2p''+l-1} \leq |\xi| \leq 2^{2p''+l+1}\right\}$ which has a vanishing Lebesgue measure. □

Characterization of Besov spaces through Littlewood-Paley decomposition:

Theorem 3.1 (see e.g. Bergh and Löfström (1976))

Let $p \in [1, +\infty]$ and $s \in (0, +\infty)$ be arbitrary and fixed. A necessary and sufficient condition for a tempered distribution f to be a function belonging to the Besov space $B^s_{p,\infty}(\mathbb{R}^N)$ is that its Littlewood-Paley components $f * \varphi$ and $f * \psi_j$, $j \in \mathbb{Z}_+$, belong to $L^p(\mathbb{R}^N)$ and satisfy

$$\limsup_{j \to +\infty} \left\{ 2^{js} \| f * \psi_j \|_{L^p(\mathbb{R}^N)} \right\} < +\infty . \tag{3.3}$$

Theorem 3.1 implies that $\mathcal{V}_j = (2\pi)^N \big\| \mathcal{F}^{-1}(g) * \psi_j \big\|^2_{L^2(\mathbb{R}^N)} = \mathcal{O}(2^{-2js})$, for any fixed $s \in (0, \overline{s})$.

More importantly, it also entails that $2^{-2j(m)s} = \mathcal{O}(\mathcal{V}_{j(m)})$, for any fixed $s \in (\overline{s}, 1)$, where $(j(m))_{m \in \mathbb{N}}$ is an increasing sequence of positive integers which a priori depends on the choice of s.

The fact that one has no information on the sequence $(j(m))_{m \in \mathbb{N}}$ is a difficulty.

One way to overcome this difficulty is to impose to g the condition:

($\mathcal{H}_1$) There are a constant $a_0 \in (0,1]$ and an integer $j_0 \geq 1$ such that, for every $j \geq j_0$ and a.a. $\eta \in \mathcal{R}_0 := \{\eta \in \mathbb{R}^N : 2^{-1} \leq |\eta| \leq 2\}$, $a_0\, g(2^{j+1}\eta) \leq g(2^j\eta)$.

Under ($\mathcal{H}_1$) the sequence of variances $(\mathcal{V}_j)_{j \geq j_0}$ becomes "non-increasing":

$$2^{-N}\, a_0^2\, \mathcal{V}_{j+1} \leq \mathcal{V}_j\,, \quad \text{for all } j \geq j_0. \tag{3.4}$$

Indeed, it results from the Plancherel formula, the change of variable $\eta = 2^{-j}\xi$, ($\mathcal{H}_1$) and the change of variable $\xi' = 2^{j+1}\eta$ that:

$$\begin{aligned}
&\mathcal{V}_j := (2\pi)^N \left\|\mathcal{F}^{-1}(g) * \psi_j\right\|^2_{L^2(\mathbb{R}^N)} = \left\|\mathcal{F}(\mathcal{F}^{-1}(g) * \psi_j)\right\|^2_{L^2(\mathbb{R}^N)} \\
&= \int_{\mathbb{R}^N} \left|g(\xi)\widehat{\psi}_0(2^{-j}\xi)\right|^2 d\xi = 2^{jN} \int_{\mathcal{R}_0} \left|g(2^j\eta)\widehat{\psi}_0(\eta)\right|^2 d\eta \\
&\geq 2^{jN} a_0^2 \int_{\mathcal{R}_0} \left|g(2^{j+1}\eta)\widehat{\psi}_0(\eta)\right|^2 d\eta = 2^{-N} a_0^2 \int_{\mathbb{R}^N} \left|g(\xi')\widehat{\psi}_0(2^{-j-1}\xi')\right|^2 d\xi' \\
&= 2^{-N}\, a_0^2\, (2\pi)^N \left\|\mathcal{F}^{-1}(g) * \psi_{j+1}\right\|^2_{L^2(\mathbb{R}^N)} = 2^{-N}\, a_0^2\, \mathcal{V}_{j+1}\,.
\end{aligned}$$

Theorem 3.2

Assume that the conditions $(\mathcal{H}_0)$ *and* $(\mathcal{H}_1)$ *hold then one has*

$$\mathbb{P}\big(\forall\,\tau \in \mathbb{R}^N,\ \alpha_X(\tau) = \overline{s}\big) = 1\,. \tag{3.5}$$

The following lemma, inspired from Kahane (1985), which results from Borel-Cantelli Lemma and independence property of wavelet coefficients with respect to dilation index, is the main ingredient of the proof of the theorem.

Lemma 3.1

For any $\varepsilon > 0$*, there exists* $n_0(\varepsilon) \in \mathbb{N}$ *such that one has a.s. for all* $p \in \mathbb{Z}^N$

$$\liminf_{j\to+\infty} \left\{ 2^{2\varepsilon j}\, \mathcal{V}_j^{-1/2} \inf_{k\in\mathcal{K}_{p,j}} \sup \left\{ |\chi_{\mathcal{I}}|\,;\, \mathcal{I} \in \mathcal{A}_{j,k}^{n_0(\varepsilon)} \right\} \right\} = +\infty\,, \tag{3.6}$$

where $\mathcal{K}_{p,j} := 2^j p + \{0,\ldots,2^j-1\}^N$ *and* $\mathcal{A}_{j,k}^{n_0(\varepsilon)}$ *is the set of "ancestors" of* $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$ *consisting in the dyadic cubes* $\mathcal{I}_n$, $0 \le n < n_0(\varepsilon)$*, such that* $I_{j,k} = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \ldots \subset \mathcal{I}_{n_0(\varepsilon)-1}$ *and* $\lambda(\mathcal{I}_n) = 2^{-(j-2n)N}$*, for all* n.

Organization of the talk

In the previous section we presented our 1st interpretation of the intuitive idea of weakly independent increments of the stationary Gaussian field X; this 1st interpretation was related with the dilation index j of its wavelet coefficients $\chi_{j,k}$. Let us now present our 2nd interpretation of this intuitive idea; this 2nd interpretation will be related with the translation index k of $\chi_{j,k}$'s.
Recall that in frameworks of limit theorems in statistics a stationary sequence of Gaussian random variables $(\mathcal{G}_k)_{k\in\mathbb{Z}^N}$ is said to have a short-range dependence property when the sequence $\big(\mathrm{Cov}(\mathcal{G}_k,\mathcal{G}_0)\big)_{k\in\mathbb{Z}^N}$ belongs to the space $l^1(\mathbb{Z}^N)$ i.e.

$$\sum_{k\in\mathbb{Z}^N}\big|\mathrm{Cov}(\mathcal{G}_k,\mathcal{G}_0)\big|<+\infty\,. \tag{4.1}$$

In the same spirit, our 2nd interpretation of this intuitive idea will be to say that increments of X are weakly independent if, for some $\theta\in[2,+\infty)$ and for all fixed j big enough, the sequence $\big(\mathrm{Cov}(\chi_{j,k},\chi_{j,0})\big)_{k\in\mathbb{Z}^N}$ belongs to the space $l^\theta(\mathbb{Z}^N)$ i.e.

$$\sum_{k\in\mathbb{Z}^N}\big|\mathrm{Cov}(\chi_{j,k},\chi_{j,0})\big|^\theta<+\infty\,. \tag{4.2}$$

The condition (4.2) is weaker than the condition (4.1) since $l^1(\mathbb{Z}^N)\subset l^\theta(\mathbb{Z}^N)$.

Our next goal is to connect the condition (4.2) to an L^p norm of Littlewood-Paley component $\mathcal{F}^{-1}(g) * \psi_j$.

In fact, up to a multiplicative factor, $\mathrm{Cov}(\chi_{j,k}, \chi_{j,0})$ can be interpreted as a Fourier coefficient.

Indeed, using the equality $\chi_{j,k} = \int_{\mathbb{R}^N} e^{-i2^{-j}k\cdot\xi}\, \widehat{\psi}_0(2^{-j}\xi) g(\xi)\, d\widehat{W}(\xi)$, the isometry property of the stochastic integral $\int_{\mathbb{R}^N}(\cdot)\, d\widehat{W}$, the change of variable $\eta = 2^j\xi$, and the fact that $\eta \mapsto e^{-ik\cdot\eta}$ is a $(2\pi\mathbb{Z})^N$-periodic function, one gets that

$$\begin{aligned}\mathrm{Cov}(\chi_{j,k}, \chi_{j,0}) &= \int_{\mathbb{R}^N} e^{-i2^{-j}k\cdot\xi} \left|\widehat{\psi}_0(2^{-j}\xi) g(\xi)\right|^2 d\xi \\ &= 2^{jN} \int_{\mathbb{R}^N} e^{-ik\cdot\eta} \left|\widehat{\psi}_0(\eta) g(2^j\eta)\right|^2 d\eta \\ &= 2^{jN} \int_{\mathbb{T}^N} e^{-ik\cdot\eta}\, G_j(\eta)\, d\eta =: 2^{jN} \mathcal{C}_k(G_j)\,, \qquad (4.3)\end{aligned}$$

where $\mathbb{T}^N := [0, 2\pi]^N$ is the N-dimensional torus and $G_j \in L^1(\mathbb{T}^N)$ is s.t.

$$G_j(\eta) := \sum_{\kappa\in\mathbb{Z}^N} \left|\widehat{\psi}_0(\eta + 2\pi\kappa) g\big(2^j(\eta + 2\pi\kappa)\big)\right|^2, \quad \text{for a.a. } \eta \in \mathbb{R}^N. \qquad (4.4)$$

Proposition 4.1

For each $\theta \in [2, +\infty)$, *the exponent* $p(\theta) \in [4/3, +\infty)$ *is s.t.*

$$p(\theta) := 2\theta/(\theta + 1)\,. \tag{4.5}$$

Then, one has, for some constant c *and for all* $j \in \mathbb{Z}_+$,

$$\Big\| \big(\mathrm{Cov}(\chi_{j,\bullet}, \chi_{j,0}) \big) \Big\|_{l^\theta(\mathbb{Z}^N)} = 2^{jN} \big\| \mathcal{C}(G_j) \big\|_{l^\theta(\mathbb{Z}^N)} \leq c\, 2^{-jN/\theta} \, \big\| \mathcal{F}^{-1}(g) * \psi_j \big\|^2_{L^{p(\theta)}(\mathbb{R}^N)} \,. \tag{4.6}$$

The main two ingredients of the proof of Proposition 4.1 are "discrete and continuous versions" of the Hausdorff-Young inequality which are recalled in the sequel.

These two crucial inequalities can be derived from the Riesz-Thorin interpolation (see e.g. Bergh and Löfström (1976)).

Proposition 4.2 (Hausdorff-Young inequality)

Let $\beta \in [2, +\infty)$ *and* $\beta' \in (1, 2]$ *be arbitrary and s.t.* $1/\beta + 1/\beta' = 1$.

- *Discrete version: for all function* $F \in L^{\beta'}(\mathbb{T}^N)$, *the sequence of the Fourier coefficients* $\mathcal{C}(F) = (\mathcal{C}_k(F))_{k \in \mathbb{Z}^N}$ *belongs to* $l^{\beta}(\mathbb{Z}^N)$ *and satisfies:*

$$\left\|\mathcal{C}(F)\right\|_{l^{\beta}(\mathbb{Z}^N)} \le (2\pi)^{N/\beta} \left\|F\right\|_{L^{\beta'}(\mathbb{T}^N)} . \tag{4.7}$$

- *Continuous version: for all function* $f \in L^{\beta'}(\mathbb{R}^N)$, *the Fourier transform* $\widehat{f}$ *belongs to* $L^{\beta}(\mathbb{R}^N)$ *and satisfies:*

$$\|\widehat{f}\,\|_{L^{\beta}(\mathbb{R}^N)} \le (2\pi)^{N/\beta} \|f\|_{L^{\beta'}(\mathbb{R}^N)} . \tag{4.8}$$

Lemma 4.1

One assumes that $\theta \in [2, +\infty)$ *and* $j \in \mathbb{Z}_+$ *are s.t.* $\left\| \mathcal{F}^{-1}(g) * \psi_j \right\|_{L^{p(\theta)}(\mathbb{R}^N)} < +\infty$; *recall that* $p(\theta) := 2\theta/(\theta+1)$. *Let* $\Gamma \subset \mathbb{R}^N$ *be an arbitrary non-degenerate compact rectangle. One sets* $\nu_j(\Gamma) := \{k \in \mathbb{Z}^N;\ 2^{-j}k \in \Gamma\}$, $n_{j,\Gamma} := \mathrm{Card}\big(\nu_j(\Gamma)\big)$ *and* $A_j(\Gamma) := \sup_{k \in \nu_j(\Gamma)} |\chi_{j,k}|^2$. *Then, for any fixed* $q \in \mathbb{N}$, *there is a constant* $c(q)$, *not depending on* Γ *and* j, *s.t.*

$$\mathbb{P}\big(A_j(\Gamma) \le 2^{-1}\mathcal{V}_j\big) \le c(q) \left(n_{j,\Gamma}^{-1/2\theta} \times \frac{2^{j\nu_{N,p(\theta)}} \left\| \mathcal{F}^{-1}(g) * \psi_j \right\|_{L^{p(\theta)}(\mathbb{R}^N)}}{\left\| \mathcal{F}^{-1}(g) * \psi_j \right\|_{L^2(\mathbb{R}^N)}} \right)^{4q}, \quad (4.9)$$

where $\mathcal{V}_j = (2\pi)^N \left\| \mathcal{F}^{-1}(g) * \psi_j \right\|^2_{L^2(\mathbb{R}^N)}$ *is the common variance of the wavelet coefficients* $\chi_{j,k}$, $k \in \mathbb{Z}^N$, *and where* $\nu_{N,p(\theta)} := N(2 - p(\theta))/2p(\theta)$.

If for some $\xi_0 > 0$, $\mathcal{F}^{-1}(g \mathbb{1}_{\{|\xi| \ge \xi_0\}}) \in \bigcup_{p \in [4/3, 2)} \bigcap_{s \in (0, \bar{s})} B^{s+\nu_{N,p}}_{p,\infty}(\mathbb{R}^N)$. Then for any fixed $\varepsilon > 0$ and $q \in \mathbb{N}$, there are 2 positive constants b_ε and $c_{\varepsilon,q}$ s.t. for all Γ with $\lambda(\Gamma) \asymp 2^{-j(m)(N-\varepsilon)}$ one has $\mathbb{P}\big(A_{j(m)}(\Gamma) \le 2^{-1}\mathcal{V}_{j(m)}\big) \le c_{\varepsilon,q} 2^{-j(m) q b_\varepsilon}$.
Thus, when $m \to +\infty$, the latter probability converges to 0, uniformly in Γ, in a very fact way!

Sketch of the proof of Lemma 4.1: Let the centred empirical mean

$$\mathcal{S}_j(\Gamma) := n_{j,\Gamma}^{-1} \sum_{k \in \nu_j(\Gamma)} \left(|\chi_{j,k}|^2 - \mathcal{V}_j \right) = -\mathcal{V}_j + n_{j,\Gamma}^{-1} \sum_{k \in \nu_j(\Gamma)} |\chi_{j,k}|^2 . \tag{4.10}$$

Thus, one has

$$\mathbb{P}\big(A_j(\Gamma) \le 2^{-1}\mathcal{V}_j\big) \le \mathbb{P}\big(\mathcal{S}_j(\Gamma) \le -2^{-1}\mathcal{V}_j\big) \le \mathbb{P}\big(\big|\mathcal{S}_j(\Gamma)\big| \ge 2^{-1}\mathcal{V}_j\big) . \tag{4.11}$$

One reason for which $\mathcal{S}_j(\Gamma)$ is more convenient than $A_j(\Gamma)$ is that $\mathcal{S}_j(\Gamma)$ belongs to the 2nd order Wiener chaos; thus it has some properties reminiscent to that of a Gaussian random variable. In particular, for any fixed $q \in \mathbb{N}$, one has for some constant $c_0(q)$, only depending on q, that

$$\mathbb{E}\Big(\big|\mathcal{S}_j(\Gamma)\big|^{2q}\Big) \le c_0(q)\bigg(\mathbb{E}\Big(\big|\mathcal{S}_j(\Gamma)\big|^2\Big)\bigg)^q . \tag{4.12}$$

Combining (4.11) and (4.12) with Markov inequality one gets that

$$\mathbb{P}\big(A_j(\Gamma) \le 2^{-1}\mathcal{V}_j\big) \le c_1(q)\bigg(\mathcal{V}_j^{-2}\,\mathbb{E}\Big(\big|\mathcal{S}_j(\Gamma)\big|^2\Big)\bigg)^q . \tag{4.13}$$

Moreover, using the definition of $\mathcal{S}_j(\Gamma)$, the stationarity of the sequence $\{\chi_{j,k}\}_{k\in\mathbb{Z}^N}$, $\mathrm{Cov}(\chi_{j,k'-k''}^2,\chi_{j,0}^2)=2\big|\mathrm{Cov}(\chi_{j,k'-k''},\chi_{j,0})\big|^2$, Hölder inequality, Proposition 4.1, $\theta=p(\theta)/(2-p(\theta))$ and $\nu_{N,p(\theta)}:=N(2-p(\theta))/2p(\theta)$, one has that

$$
\begin{aligned}
\mathbb{E}\Big(\big|\mathcal{S}_j(\Gamma)\big|^2\Big) &= 2n_{j,\Gamma}^{-1}\sum_{k'\in\nu_j(\Gamma)}\Big(n_{j,\Gamma}^{-1}\sum_{k''\in\nu_j(\Gamma)}\big|\mathrm{Cov}(\chi_{j,k'-k''},\chi_{j,0})\big|^2\Big)\\
&\le 2n_{j,\Gamma}^{-1}\sum_{k'\in\nu_j(\Gamma)}\Big(n_{j,\Gamma}^{-1}\sum_{k''\in\nu_j(\Gamma)}\big|\mathrm{Cov}(\chi_{j,k'-k''},\chi_{j,0})\big|^\theta\Big)^{2/\theta}\\
&\le 2n_{j,\Gamma}^{-2/\theta}\Big(\sum_{k\in\mathbb{Z}^N}\big|\mathrm{Cov}(\chi_{j,k},\chi_{j,0})\big|^\theta\Big)^{2/\theta}\\
&\le c_2 n_{j,\Gamma}^{-2/\theta}\Big(2^{jN/2\theta}\,\big\|\mathcal{F}^{-1}(g)*\psi_j\big\|_{L^{p(\theta)}(\mathbb{R}^N)}\Big)^4\\
&\le c_2 n_{j,\Gamma}^{-2/\theta}\Big(2^{j\nu_{N,p(\theta)}}\,\big\|\mathcal{F}^{-1}(g)*\psi_j\big\|_{L^{p(\theta)}(\mathbb{R}^N)}\Big)^4. \qquad (4.14)
\end{aligned}
$$

Finally, combining (4.13) and (4.14) one obtains the lemma. □

One can derive from Lemma 4.1 and Borel-Cantelli Lemma that:

Theorem 4.1

Assume that the condition $(\mathcal{H}_0)$ *holds i.e. the critical Besov exponent of* $\mathcal{F}^{-1}(g)$ $\overline{s} = \sup\left\{s \in [0,+\infty) : \mathcal{F}^{-1}(g) \in B^s_{2,\infty}(\mathbb{R}^N)\right\}$ *belongs to the open interval* $(0,1)$*. Also, assume that:*

$$\text{for some } \xi_0 > 0,\ \mathcal{F}^{-1}(g\mathbb{1}_{\{|\xi|\geq\xi_0\}}) \in \bigcup_{p\in[4/3,2)} \bigcap_{s\in(0,\overline{s})} B^{s+\nu_{N,p}}_{p,\infty}(\mathbb{R}^N)\,, \qquad (\mathcal{H}_2)$$

where $\nu_{N,p} := N(2-p)/2p$*. Then one has*

$$\mathbb{P}\big(\forall\,\tau \in \mathbb{R}^N,\ \alpha_X(\tau) = \overline{s}\big) = 1\,. \qquad (4.15)$$

One knows (see e.g. Bergh and Löfström (1976)) that, for all $s > 0$ and $3/4 \leq p_1 \leq p_2 < 2$, $B^{s+\nu_{N,p_1}}_{p_1,\infty}(\mathbb{R}^N) \subseteq B^{s+\nu_{N,p_2}}_{p_2,\infty}(\mathbb{R}^N) \subset B^s_{2,\infty}(\mathbb{R}^N)$, and consequently that $\bigcup_{p\in[4/3,2)} \bigcap_{s\in(0,\overline{s})} B^{s+\nu_{N,p}}_{p,\infty}(\mathbb{R}^N) \subset \bigcap_{s\in(0,\overline{s})} B^s_{2,\infty}(\mathbb{R}^N)$.

It remains an open question to know whether (4.15) is true under the sole condition $(\mathcal{H}_0)$.