# Almost sure approximations in Hölder norms of a general stochastic process defined by a Young integral

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# Organization of the talk

Let $\{\sigma(s)\}_{s\in\mathbb{R}}$ and $\{X(s)\}_{s\in\mathbb{R}}$ be two real-valued stochastic processes indexed by $\mathbb{R}$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We always assume that they satisfy the following assumption:

Fundamental condition: There are two exponents $\alpha, \beta \in (0,1)$ satisfying $\alpha + \beta > 1$ such that, for all compact interval $\mathcal{K} \subset \mathbb{R}$ the restrictions of the paths of $\sigma$ (resp. $X$) to $\mathcal{K}$ belong to the Hölder space $C^{\alpha}(\mathcal{K})$ (resp. $C^{\beta}(\mathcal{K})$).

This is a very classical condition under which, for each $t \in \mathbb{R}$, the pathwise Young integral

$$Y(t) := \int_0^t \sigma(s)\,\mathrm{d}X(s), \tag{1.1}$$

is a well-defined real-valued random variable. Such kind of process $\{Y(t)\}_{t\in\mathbb{R}}$ is closely connected to (stochastic) differential equations driven by fractional Brownian motions and more generally by (random) Hölder functions with correlated increments (see e.g. *Gubinelli, Lejay and Tindel (2007); Gubinelli, Imkeller and Perkowski (2016); Lejay (2010); Lyons, Caruana and Lévy (2004)*).

$\Longrightarrow$ It is useful to find approximation procedures for $Y$ paths which converge at the fastest possible rate.

# A brief recall on the definition of pathwise Young integral

When $t > 0$, let $(\mathcal{D}_n)_{n\in\mathbb{Z}_+} := \big(\{\delta_0^n, \delta_1^n, \ldots, \delta_{r_n}^n :\ 0 = \delta_0^n < \delta_1^n < \ldots < \delta_{r_n}^n = t\}\big)_{n\in\mathbb{Z}_+}$ be an arbitrary sequence of partitions of $[0, t]$, such that $|\mathcal{D}_n| := \max_{1\le i\le r_n}(\delta_i^n - \delta_{i-1}^n) \xrightarrow[n\to+\infty]{} 0$.
Then, under the previous condition, for all $\omega \in \Omega$, the Riemann-Stieltjes sum:

$$\sum_{i=1}^{r_n} \sigma(\delta_{i-1}^n, \omega)(X(\delta_i^n, \omega) - X(\delta_{i-1}^n, \omega)) \tag{1.2}$$

converges, when $n \to +\infty$, to a random finite limit not depending on the choice of the sequence of partitions. The pathwise Young integral $\int_0^t \sigma(s,\omega)\, \mathrm{d}X(s,\omega)$ is simply defined to be this limit. The case where $t < 0$ is similar except that $[0, t]$ has to be replaced by $[t, 0]$ and $\int_0^t$ means $-\int_t^0$.
When a compactly supported function $f(\bullet, \omega)$ from $\mathbb{R}$ to $\mathbb{R}$ is $\alpha$-Hölder continuous with a support included in a compact interval $J$, then by convention one sets

$$\int_{\mathbb{R}} f(s,\omega)\, \mathrm{d}X(s,\omega) := \int_J f(s,\omega)\, \mathrm{d}X(s,\omega).$$

Of course, Young integral has much less "good" properties than the usual Lebesgue integral. Yet, it satisfies the following fundamental inequality, whose proof can for instance be found in the well-known book *Lyons, Caruana and Lévy (2004)* on rough paths.

**Proposition 1.1 (Young-Loeve inequality)**

*There exists a positive finite constant $\Lambda_{\alpha+\beta}$, depending only on $\alpha+\beta>1$, such that the inequality*

$$\Big| \int_{t_1}^{t_2} \sigma(s)\,\mathrm{d}X(s) - \sigma(\tilde{s})\big(X(t_2)-X(t_1)\big)\Big| \le \Lambda_{\alpha+\beta}\|\sigma\|_{C^{\alpha}([t_1,t_2])}\|X\|_{C^{\beta}([t_1,t_2])}(t_2-t_1)^{\alpha+\beta} \quad (1.3)$$

*holds for all real numbers $t_1 \le \tilde{s} \le t_2$.*

$\Longrightarrow$ The restrictions of $Y$ paths to any compact interval $\mathcal{K}\subset\mathbb{R}$ belong to the Hölder space $C^{\beta}(\mathcal{K})$.

# Appoximation of $Y$ paths through Euler scheme

Let $I := [0, T]$, where $T \in \mathbb{N}$ is fixed once and for all. The classical Euler scheme, corresponding to Riemann-Stieltjes sums associated with dyadic intervals of fixed length $2^{-J}$, $J \in \mathbb{N}$, provides a natural method for approximating paths of the process $\{Y(t)\}_{t\in I}$, which can be connected to Haar basis. More precisely, for every $m \in \{1, \ldots, 2^J T\}$, one approximates $Y(m/2^J) := \int_0^{m/2^J} \sigma(s)\,\mathrm{d}X(s)$ by

$$Y_J(m/2^J) := \int_0^{m/2^J} \Big( \sum_{l=0}^{m-1} \sigma(\tilde{s}_{J,l}) \mathbb{1}_{[l/2^J,(l+1)/2^J)}(s) \Big)\,\mathrm{d}X(s) = \sum_{l=0}^{m-1} \sigma(\tilde{s}_{J,l}) \Delta_{J,l}(X), \tag{1.4}$$

where $\Delta_{J,l}(X) := X((l+1)/2^J) - X(l/2^J)$, and $\tilde{s}_{J,l} \in [l/2^J, (l+1)/2^J]$ can be chosen arbitrarily. Thus, taking it such that $\sigma(\tilde{s}_{J,l}) = 2^J \int_{l/2^J}^{(l+1)/2^J} \sigma(s)\,\mathrm{d}s$, one has

$$Y_J(m/2^J) = \int_{\mathbb{R}} \mathrm{Proj}_{V_J^{\mathrm{H}}}(\sigma \mathbb{1}_{[0,m/2^J]})(s)\,\mathrm{d}X(s), \tag{1.5}$$

where $V_J^{\mathrm{H}} := \overline{\mathrm{span}\{\mathbb{1}_{[l/2^J,(l+1)/2^J)} : l \in \mathbb{Z}\}}$ is the closed subspace of $L^2(\mathbb{R})$ issued from the multiresolution analysis generating the Haar basis.

Once one has the $Y_J(m/2^J)$'s, using linear interpolation, one gets a random function, from $I$ to $\mathbb{R}$, $t \mapsto Y_J^{RS}(t)$ which approximates the whole path $t \mapsto Y(t)$. More precisely, one sets $Y_J^{RS}(0) := 0$, $Y_J^{RS}(T) := Y_J(T)$, and, for every $t \in \mathring{I} := (0, T)$,

$$Y_J^{RS}(t) \quad := \quad Y_J\left(\frac{[2^J t]}{2^J}\right) + (2^J t - [2^J t])\left(Y_J\left(\frac{[2^J t]+1}{2^J}\right) - Y_J\left(\frac{[2^J t]}{2^J}\right)\right).$$

The proof of the following proposition manly relies on the Young-Loeve inequality.

**Proposition 1.2**

*There exists a random finite constant $c > 0$ such that for all $\gamma \in [0, \beta)$ and $J \in \mathbb{N}$, one has*

$$\|Y - Y_J^{RS}\|_{C^\gamma(I)} \le c 2^{-J\min(\beta-\gamma, \alpha+\beta-1)}. \tag{1.6}$$

Question: Is it possible to find an approximation procedure for $\{Y(t)\}_{t\in I}$ paths allowing to have a better rate of convergence than the one provided by (1.6)?

Studying this issue is the main motivation of our talk.

# Organization of the talk

Our next goal is to provide a generalization of Proposition 1.2 in a wavelet-based framework (see *e.g. Meyer (1990, 1992); Daubechies (1992)*). The collection of functions, from $\mathbb{R}$ to itself,

$$\{\varphi(\bullet - l) : l \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \bullet - k) : (j,k) \in \mathbb{Z}_+ \times \mathbb{Z}\} \tag{2.1}$$

satisfies one of the following two hypotheses.

($\mathcal{H}_1$) This collection is simply the Haar basis of $L^2(\mathbb{R})$, in other words one has $\varphi := \mathbb{1}_{[0,1)}$ and $\psi := \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$.

($\mathcal{H}_2$) This collection is an arbitrary compactly supported orthonormal wavelet basis of $L^2(\mathbb{R})$ such that the scaling function $\varphi$ and the mother wavelet $\psi$ are $\alpha$-Hölder continuous on $\mathbb{R}$ with a support included in $[-N, N]$, where $N \in \mathbb{N}$. Thus, setting $N_{j,k}^- := (k-N)/2^j$ and $N_{j,k}^+ := (k+N)/2^j$, one gets that

$$\operatorname{supp} \psi(2^j \bullet - k) \subseteq [N_{j,k}^-, N_{j,k}^+]. \tag{2.2}$$

It is known that: $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$, $\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0$, and the integer translates of $\varphi$ form "a partition of unity" i.e. $\sum_{l\in\mathbb{Z}} \varphi(x-l) = 1$, for all $x \in \mathbb{R}$.

The increasing sequence $(V_J)_{J\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ denotes the multiresolution analysis associated to this basis, that is

$$V_J := \overline{\operatorname{span}\{\varphi(2^J \bullet - l) : l \in \mathbb{Z}\}}, \quad \text{for all } J \in \mathbb{Z}.$$

For all fixed $J \in \mathbb{N}$, let $\{Y_J^W(t)\}_{t\in I}$ be the stochastic process defined, for each $t \in I$, as

$$
\begin{aligned}
Y_J^W(t) &:= \int_{\mathbb{R}} \mathrm{Proj}_{V_J}(\sigma \mathbb{1}_{[0,t]})(s)\,\mathrm{d}X(s) \qquad (2.3)\\
&= \sum_{l\in\mathbb{L}_{J,t}\cup\partial\mathbb{L}_{J,t}} \left(2^J \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s\right)\left(\int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s)\right),
\end{aligned}
$$

where $\mathbb{L}_{J,t} := \left\{ l \in \mathbb{Z} : [N_{j,k}^-, N_{j,k}^+] \subseteq [0,t] \right\}$ and $\partial\mathbb{L}_{J,t} := \left\{ l \in \mathbb{Z} \setminus \mathbb{L}_{J,t} : [N_{j,k}^-, N_{j,k}^+] \cap [0,t] \neq \emptyset \right\}$. Observe that $\mathrm{card}(\mathbb{L}_{J,t}) \le c2^J$ and $\mathrm{card}(\partial\mathbb{L}_{J,t}) \le c$, where the deterministic constant $c > 0$ does not depend on $(J,t)$. The following theorem provides a generalization of Proposition 1.2.

## Theorem 2.1

*There is a finite random constant $c > 0$ such that, for all $\gamma \in [0,\beta)$ and $J \in \mathbb{N}$, one has*

$$\|Y - Y_J^W\|_{C^\gamma(I)} \le c\, 2^{-J\min(\beta-\gamma,\alpha+\beta-1)}.$$

The proof of Theorem 2.1 mainly relies on the following technical lemma.

## Lemma 2.1

*There is a random constant not depending on* $(J, l, t)$ *such that:*

$$\left|2^J \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s\right| \le c, \quad \textit{if } l \in \mathbb{Z}, \tag{2.4}$$

$$\left|2^J \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s - \sigma(l/2^j)\right| \le c2^{-J\alpha}, \quad \textit{if } l \in \mathbb{L}_{J,t}, \tag{2.5}$$

$$\left|\int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s)\right| \le c2^{-J\beta}, \quad \textit{if } l \in \mathbb{Z}, \tag{2.6}$$

$$\left|\int_0^t \left(\sigma(s) - \sigma(l/2^j)\right)\varphi(2^J s - l)\,\mathrm{d}X(s)\right| \le c2^{-J(\alpha+\beta)}, \quad \textit{if } l \in \mathbb{L}_{J,t}, \tag{2.7}$$

*and*

$$\left|\int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}X(s)\right| \le c2^{-J\beta}, \quad \textit{if } l \in \partial\mathbb{L}_{J,t}. \tag{2.8}$$

**The main ideas of the proof of Lemma 2.1:** The proofs of the first two inequalities are standard and easy. The proofs of the three others are rather similar and rely on the Young-Loeve inequality. So, we only give the one which concerns $\int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s)$. Using the Young-Loeve inequality, one gets

$$\Big| \int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s) - \varphi(2^J N_{J,l}^- - l)\big(X(N_{J,l}^+) - X(N_{J,l}^-)\big) \Big| \\ \leq \Lambda_{\alpha+\beta} \big\|\varphi(2^J \bullet - l)\big\|_{C^\alpha([N_{J,l}^-, N_{J,l}^+])} \|X\|_{C^\beta([N_{J,l}^-, N_{J,l}^+])} (N_{J,l}^+ - N_{J,l}^-)^{\alpha+\beta}. \quad (2.9)$$

Next, noticing that $\varphi(2^J N_{J,l}^- - l) = 0$ (since $\operatorname{supp} \varphi(2^J \bullet^- - l) \subseteq [N_{J,l}^-, N_{J,l}^+]$),

$$\big\|\varphi(2^J \bullet - l)\big\|_{C^\alpha([N_{J,l}^-, N_{J,l}^+])} \leq 2^{J\alpha} \|\varphi\|_{C^\alpha([-N,N])}$$

and $N_{J,l}^+ - N_{J,l}^- = 2^{1-J} N$, one obtains that

$$\Big| \int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s) \Big| \leq c 2^{-J\beta}. \qquad \square$$

**Sketch of the proof of Theorem 2.1 in the case $\gamma = 0$:** Using the fact that the integer translates of $\varphi$ form "a partition of unity", one gets

$$\begin{aligned} Y(t) &:= \int_0^t \sigma(s)\,\mathrm{d}s = \int_0^t \sigma(s)\Big(\sum_{l\in\mathbb{L}_{J,t}\cup\partial\mathbb{L}_{J,t}} \varphi(2^J s - l)\Big)\,\mathrm{d}s \\ &= \sum_{l\in\mathbb{L}_{J,t}\cup\partial\mathbb{L}_{J,t}} \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s \\ &\simeq \sum_{l\in\mathbb{L}_{J,t}} \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

On the other hand, one has

$$\begin{aligned} Y_J^W(t) &= \sum_{l\in\mathbb{L}_{J,t}\cup\partial\mathbb{L}_{J,t}} \Big(2^J \int_0^t \sigma(s)\varphi(2^J s - l)\,\mathrm{d}s\Big)\Big(\int_{N_{J,l}^-}^{N_{J,l}^+} \varphi(2^J s - l)\,\mathrm{d}X(s)\Big) \\ &\simeq \sum_{l\in\mathbb{L}_{J,t}} \int_0^t \sigma(l/2^j)\varphi(2^J s - l)\,\mathrm{d}X(s). \end{aligned} \tag{2.11}$$

One knows from Lemma 2.1 that the approximation errors in (2.10) and (2.11) for the norm $\|\cdot\|_{I,\infty}$ are less than $c_1\, 2^{-J\min(\beta,\alpha+\beta-1)}$. Finally, using the same lemma and the inequality $\mathrm{card}(\mathbb{L}_{J,T}) \le c_2\, 2^J$, it follows that

$$
\begin{aligned}
\|Y - Y_J^W\|_{I,\infty} &\simeq \Big\| \sum_{l\in\mathbb{L}_{J,\bullet}} \int_0^{\bullet} \big(\sigma(s)-\sigma(l/2^j)\big)\varphi(2^J s - l)\,\mathrm{d}X(s)\Big\|_{I,\infty} \\
&\le \sum_{l\in\mathbb{L}_{J,T}} \Big\| \int_0^{\bullet} \big(\sigma(s)-\sigma(l/2^j)\big)\varphi(2^J s - l)\,\mathrm{d}X(s)\Big\|_{I,\infty} \\
&\le c_3\,\mathrm{card}(\mathbb{L}_{J,T})\, 2^{-J(\alpha+\beta)} \le c_4\, 2^{-J(\alpha+\beta-1)}. \qquad \square
\end{aligned}
$$

One of the main advantages of the wavelet approach is that the difference $Y(t) - Y^W(t)$ can be expressed in Hölder spaces in an explicit exploitable way:

$$
Y(t) - Y^W(t) = \sum_{j=J}^{+\infty} \sum_{k\in\mathbb{L}_{j,t}\cup\partial\mathbb{L}_{j,t}} a_{j,k}(t)\lambda_{j,k} \tag{2.12}
$$

where $a_{j,k}(t) := 2^j \int_0^t \sigma(s)\psi(2^j s - k)\,\mathrm{d}s$ and $\lambda_{j,k} := \int_{N_{j,k}^-}^{N_{j,k}^+} \psi(2^J s - l)\,\mathrm{d}X(s)$.

# Organization of the talk

First, it is useful to make some brief recalls on the notion of Wiener chaos; our presentation of it is inspired by the one in the book *Janson (1997)*.

## Definition 3.1 (Wiener chaos)

*Let $G$ be an arbitrary fixed Gaussian subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, that is a closed subspace consisting of real-valued centred Gaussian random variables.*
*Let $n \in \mathbb{Z}_+$, the Wiener chaos of order $n$ associated with $G$ is denoted by $\overline{\mathcal{P}}_n(G)$, or more simply by $\overline{\mathcal{P}}_n$.*
*The space $\overline{\mathcal{P}}_0$ is defined to be the closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ consisting of all the constant random variables.*
*When $n \geq 1$, the space $\overline{\mathcal{P}}_n$ is defined as the closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ spanned by the following set of random variables:*

$$\left\{ \prod_{l=1}^{n} g_l^{m_l} \,:\, (g_1, \ldots, g_n) \in G^n \text{ and } (m_1, \ldots, m_n) \in \mathbb{Z}_+^n \text{ with } \sum_{l=1}^{n} m_l \leq n \right\}.$$

## Remark 3.1

(a) *One clearly has* $\overline{\mathcal{P}}_n \subseteq \overline{\mathcal{P}}_{n+1}$*, for every* $n \in \mathbb{Z}_+$*. Moreover, for all fixed* $p \in (0, +\infty)$*, the space* $\overline{\mathcal{P}}_* := \bigcup_{n \in \mathbb{Z}_+} \overline{\mathcal{P}}_n$ *is dense in* $L^p(\Omega, \mathcal{F}(G), \mathbb{P})$*.*

(b) *The* $L^p(\Omega)$*-norms are equivalent on* $\overline{\mathcal{P}}_n$*, for all fixed* $n \in \mathbb{Z}_+$*.*

(c) *For every fixed* $n \in \mathbb{Z}_+$ *and for each sequence of random variables in* $\overline{\mathcal{P}}_n$*, convergence in probability is equivalent to convergence in* $L^p(\Omega)$*-norm, for any fixed* $p \in (0, +\infty)$*.*

(d) *For all fixed integer* $n \geq 1$*, there exists a positive finite universal constant* $c(n)$*, depending only on* $n$*, such that, for every random variable* $\chi \in \overline{\mathcal{P}}_n$ *and for each real number* $y \geq 2$*, one has*

$$\mathbb{P}\Big(|\chi| > y\|\chi\|_{L^2(\Omega)}\Big) \leq \exp\big(-c(n)y^{2/n}\big)\,. \tag{3.13}$$

## Definition 3.2 (the Wiener chaos condition $(\mathcal{WC})$)

*One says that the stochastic process* $\{Y(t)\}_{t\in I}$ *satisfies* $(\mathcal{WC})$ *when, for some arbitrary integer* $n \geq 1$*, the integrand* $\{\sigma(s)\}_{s\in\mathbb{R}}$ *and the integrator* $\{X(s)\}_{s\in\mathbb{R}}$ *are two stochastic processes belonging to the Wiener chaos* $\overline{\mathcal{P}}_n$ *(i.e.* $\sigma(s) \in \overline{\mathcal{P}}_n$ *and* $X(s) \in \overline{\mathcal{P}}_n$*, for all* $s \in \mathbb{R}$*) and possessing the following two properties:*

$(\mathcal{C}_1)$ *There exist* $\alpha_0, \beta_0 \in (0,1]$*, satisfying* $\alpha_0 + \beta_0 > 1$*, such that, on any compact interval* $\mathcal{K}$*,* $\{\sigma(s)\}_{s\in\mathcal{K}}$ *and* $\{X(s)\}_{s\in\mathcal{K}}$ *are respectively* $\alpha_0$ *and* $\beta_0$ *Hölder continuous in the sense of the* $L^2(\Omega)$*-norm.*

$(\mathcal{C}_2)$ *The "wavelet coefficients"* $a_{j,k} := a_{j,k}(T)$ *and* $\lambda_{j,k}$ *have the following "short-range dependence" property: the inequality*

$$\sum_{k_1\in\mathbb{L}_{j,T}} \sum_{k_2\in\mathbb{L}_{j,T}} \left|\mathbb{E}\left[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\right]\right| \leq c2^{-2j(\alpha_0+\beta_0-1/2)} \tag{3.14}$$

*is satisfied, for some finite deterministic constant* $c > 0$ *and for all positive integer* $j$ *large enough.*

## Remark 3.2

(a) *The condition* $(\mathcal{C}_1)$ *implies that paths of* $\sigma$ *and* $X$ *respectively belong to the Hölder spaces* $C^{\alpha}(\mathcal{K})$ *and* $C^{\beta}(\mathcal{K})$ *for any* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (0, \beta_0)$ *(Kolmogorov-Čentsov Hölder continuity theorem).*

(b) *One has* $Y(t) \in \overline{\mathcal{P}}_{2n}$ *for all* $t$*, since* $\sigma(s) \in \overline{\mathcal{P}}_n$ *and* $X(s) \in \overline{\mathcal{P}}_n$ *for every* $s$*.*

(c) *The same argument shows that, for all* $(t, j, k)$*, the "wavelet coefficients"* $a_{j,k}(t)$ *and* $\lambda_{j,k}$ *are in* $\overline{\mathcal{P}}_n$*; therefore* $Y_J^W(t) \in \overline{\mathcal{P}}_{2n}$*.*

The main goal of the present section is to obtain the following theorem.

## Theorem 3.1

*Under the condition* $(\mathcal{W}C)$*, for any fixed real numbers* $\alpha \in (0, \alpha_0)$*,* $\beta \in (0, \beta_0)$ *and* $\gamma \geq 0$ *satisfying* $\alpha + \beta > 1$ *and* $\gamma < \min(\beta, 1/2)$*, there is a finite random constant* $c > 0$ *such that the following inequality holds almost surely and for each* $J \in \mathbb{N}$*.*

$$\|Y - Y_J^W\|_{C^{\gamma}(I)} \leq c\, 2^{-J \min(\beta-\gamma, \alpha+\beta-1/2-\gamma)}. \tag{3.15}$$

We focus on the case $\gamma = 0$, that is:

$$\|Y - Y_J^W\|_{I,\infty} \le c\, 2^{-J\min(\beta,\alpha+\beta-1/2)}. \tag{3.16}$$

In the rest of the section, one presents the main lines of the strategy allowing to derive (3.16).

For each fixed $j \in \mathbb{N}$, one denotes by $Z_j := \{Z_j(t)\}_{t\in I}$ the stochastic process in $\overline{\mathcal{P}}_{2n}$ with Lipschitz continuous paths defined, for all $t \in I$, as

$$Z_j(t) := \sum_{k\in\mathbb{L}_{j,t}\cup\partial\mathbb{L}_{j,t}} a_{j,k}(t)\lambda_{j,k}. \tag{3.17}$$

One already knows from Theorem 2.1 that:

$$Y(\bullet,\omega) - Y_J^W(\bullet,\omega) = \sum_{j=J}^{+\infty} Z_j(\bullet,\omega),$$

where the convergence of the series holds in the Hölder space $C^\gamma(I)$, for any $\gamma \in [0,\beta)$.

Therefore, using the triangle inequality, one has

$$\big\|Y(\bullet,\omega) - Y_J^W(\bullet,\omega)\big\|_{I,\infty} \leq \sum_{j=J}^{+\infty} \big\|Z_j(\bullet,\omega)\big\|_{I,\infty}. \tag{3.18}$$

Thus, in order to derive $\|Y - Y_J^W\|_{I,\infty} \leq c\, 2^{-J\min(\beta,\alpha+\beta-1/2)}$, it is enough to obtain the following lemma.

**Lemma 3.1**

*One has almost surely*

$$\sup_{j\in\mathbb{N}} \Big\{ 2^{j\min(\beta,\alpha+\beta-1/2)} \|Z_j\|_{I,\infty} \Big\} < +\infty\,. \tag{3.19}$$

Next, let us point out that $\|Z_j\|_{I,\infty} := \sup_{t\in I} |Z_j(t)|$ is the supremum of infinitely many random variables. Actually, it is more convenient to work with a supremum of finite number of them; this can be done thanks to the following lemma.

## Lemma 3.2

*For each $j \in \mathbb{N}$, one sets*

$$\nu(Z_j) := \sup_{2^{-j} l \in I} |Z_j(2^{-j} l)|.$$

*Then, one has almost surely*

$$\sup_{j \in \mathbb{N}} \left\{ 2^{j\beta} \Big| \|Z_j\|_{I,\infty} - \nu(Z_j) \Big| \right\} < +\infty \,. \tag{3.20}$$

**Proof of Lemma 3.2:** For all $t \in I$, let $d_j(t)$ be the dyadic number of order $j$ such that $t \in [d_j(t), d_j(t) + 2^{-j})$.
Then, one has $\mathbb{L}_{j,t} = \mathbb{L}_{j,d_j(t)}$ and $a_{j,k}(t) = a_{j,k}(d_j(t))$ for any $k \in \mathbb{L}_{j,t}$. Therefore

$$\begin{aligned}
\Big| Z_j(t) - Z_j(d_j(t)) \Big| &\leq \sum_{k \in \partial \mathbb{L}_{j,t}} \Big| a_{j,k}(t) - a_{j,k}(d_j(t)) \Big| |\lambda_{j,k}| \\
&\leq c_1 2^{-j\beta} \sum_{k \in \partial \mathbb{L}_{j,t}} 2^j \int_{d_j(t)}^t \Big| \sigma(s) \psi(2^j s - k) \Big| \,\mathrm{d}s \\
&\leq c_1 2^{-j\beta} \|\sigma\|_{I,\infty} \|\psi\|_\infty \mathrm{card}(\partial \mathbb{L}_{j,t}) \leq c_2 2^{-j\beta}. \qquad \square
\end{aligned}$$

In view of the previous lemma, it turns out that for deriving the theorem it is enough to show that

$$\sup_{j\in\mathbb{N}}\left\{2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j)\right\}<+\infty\,. \tag{3.21}$$

Notice that if one shows that

$$\sum_{j=1}^{+\infty}\mathbb{P}\Big(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j)>1\Big)<+\infty\,, \tag{3.22}$$

then the Borel-Cantelli lemma entails that (3.21) holds.

Using the Markov inequality, one has, for every $j\in\mathbb{N}$,

$$\mathbb{P}\Big(2^{j\min(\beta,\alpha+\beta-1/2)}\nu(Z_j)>1\Big)\le 2^{j\min(\beta,\alpha+\beta-1/2)}\,\mathbb{E}\big(\nu(Z_j)\big)\,. \tag{3.23}$$

## Lemma 3.3

*There is a constant $c(n)$, only depending on $n$ the order of the chaos, such that for every $j \in \mathbb{N}$, one has*

$$\mathbb{E}\big(\nu(Z_j)\big) \leq c(n)\, j^{n/2} \sup_{2^{-j}l \in I} \Big(\mathbb{E}\big(|Z_j(2^{-j}l)|^2\big)\Big)^{1/2}. \tag{3.24}$$

Roughly speaking, the proof of the lemma mainly relies on the fact that one has, for some constant $c'$, not depending on $(j, l)$, and for all $\tau > 0$ large enough

$$\mathbb{P}\big(|Z_j(2^{-j}l)| > \tau\big) \leq \exp(-c'\tau^{2/n}). \tag{3.25}$$

In view of Lemma 3.3 and of the fact that $\alpha \in (0, \alpha_0)$ and $\beta \in (0, \beta_0)$, it turns out that for proving the theorem it is enough that

$$\sup_{j \in \mathbb{N}} \sup_{2^{-j}l \in I} \Big\{ 2^{2j\min(\beta_0, \alpha_0+\beta_0-1/2)}\, \mathbb{E}\big(|Z_j(2^{-j}l)|^2\big) \Big\} < +\infty\,. \tag{3.26}$$

# End of the proof of Theorem 3.1

Finally, observe that

$$\begin{aligned}\mathbb{E}\left(|Z_j(2^{-j}l)|^2\right) &= \mathbb{E}\left(\Big|\sum_{k\in\mathbb{L}_{j,t}\cup\partial\mathbb{L}_{j,t}} a_{j,k}(t)\lambda_{j,k}\Big|^2\right)\\ &\simeq \sum_{k_1\in\mathbb{L}_{j,t}}\sum_{k_2\in\mathbb{L}_{j,t}} \mathbb{E}\left[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\right]\\ &\leq \sum_{k_1\in\mathbb{L}_{j,T}}\sum_{k_2\in\mathbb{L}_{j,T}} \Big|\mathbb{E}\left[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\right]\Big|. \end{aligned} \tag{3.27}$$

Using the inequality (3.27) and the condition

$$\sum_{k_1\in\mathbb{L}_{j,T}}\sum_{k_2\in\mathbb{L}_{j,T}} \Big|\mathbb{E}\left[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\right]\Big| \leq c2^{-2j(\alpha_0+\beta_0-1/2)} \tag{$\mathcal{C}_2$}$$

one gets the theorem.

# Organization of the talk

This section involves constructing wide classes of examples of real-valued stochastic processes $\sigma$ and $X$ satisfying the Wiener chaos condition $(\mathcal{W}C)$.

For the sake of convenience, one assumes that these two processes are independent, centred and given by multiple Itô-Wiener integrals. More precisely, for $\mu = \sigma$ or $\mu = X$, one has, for some $N_\mu \in \mathbb{N}$ and for all $s \in \mathbb{R}$,

$$\mu(s) = \int_{\mathbb{R}^{N_\mu}} \big(e^{is(\eta_1+\ldots+\eta_{N_\mu})} - 1\big) g_\mu(\eta_1,\ldots,\eta_{N_\mu})\, \mathrm{d}\widehat{\mathbb{W}}_\mu(\eta_1)\ldots \mathrm{d}\widehat{\mathbb{W}}_\mu(\eta_{N_\mu}), \quad (4.1)$$

where $\mathrm{d}\widehat{\mathbb{W}}_\mu$ is the "Fourier transform" of a Brownian measure $\mathrm{d}\mathbb{W}_\mu$ on $\mathbb{R}$, and $g_\mu$ is an arbitrary symmetric complex-valued Borel function such that $\overline{g(\eta)} = g(-\eta)$, for all $\eta \in \mathbb{R}^{N_\mu}$, and

$$\int_{\mathbb{R}^{N_\mu}} \min\big(1, (\eta_1+\ldots+\eta_{N_\mu})^2\big)\big|g(\eta_1,\ldots,\eta_{N_\mu})\big|^2 \mathrm{d}\eta_1\ldots \mathrm{d}\eta_{N_\mu} < +\infty\,. \quad (4.2)$$

These properties of $g_\mu$ guarantee the existence of the multiple Itô-Wiener integral in (4.1) and the fact that it is real-valued.

## Remark 4.1

- *It is worth mentioning that the well-known* Gaussian fractional Brownian motion *(fBm) of an arbitrary Hurst parameter* $H \in (0,1)$ *belongs to this class of processes* $\mu$*: in its case one has* $N_\mu = 1$ *and*

$$g_\mu^{\mathrm{fBm}}(\eta) = c|\eta|^{-H-1/2}, \quad \textit{for almost all } \eta \in \mathbb{R}, \tag{4.3}$$

*where* $c$ *is an arbitrary nonvanishing constant.*

- *Also we mention that the* non-Gaussian Rosenblatt process *of an arbitrary parameter* $d \in (1/4, 1/2)$ *belongs to this same class of processes: in its case one has* $N_\mu = 2$ *(second order chaos) and*

$$g_\mu^{\mathrm{Ros}}(\eta_1,\eta_2) = -i(\eta_1+\eta_2)^{-1}|\eta_1\eta_2|^{-d}, \quad \textit{for almost all } (\eta_1,\eta_2) \in \mathbb{R}^2. \tag{4.4}$$

Next, one denotes by $f_\mu$ the even and positive Borel function defined, for each $\xi \in \mathbb{R}$, as

$$f_\mu(\xi) := (N_\mu)! \int_{\mathbb{R}^{N_\mu - 1}} \left| g_\mu\left(\xi - \eta_2 - \ldots - \eta_{N_\mu}, \eta_2, \ldots, \eta_{N_\mu}\right)\right|^2 \mathrm{d}\eta_2 \ldots \mathrm{d}\eta_{N_\mu}\,, \quad (4.5)$$

with the convention that $f_\mu(\xi) := |g_\mu(\xi)|^2$, when $N_\mu = 1$. It can be derived from the properties of $g_\mu$, the change of variable $\xi = \eta_1 + \eta_2 + \ldots + \eta_{N_\mu}$ and the "isometry property" of the multiple Itô-Wiener integral that

$$\int_{\mathbb{R}} \min\left(1, \xi^2\right) f_\mu(\xi)\, d\xi < +\infty \quad (4.6)$$

and, for all $s_1, s_2 \in \mathbb{R}$

$$\mathbb{E}\left[\mu(s_1)\mu(s_2)\right] = \int_{-\infty}^{+\infty} (e^{is_1\xi} - 1)(e^{-is_2\xi} - 1) f_\mu(\xi)\, \mathrm{d}\xi\,. \quad (4.7)$$

Thus the function $f_\mu$ can be viewed as a spectral density.

Notice that (4.7) is equivalent to

$$\mathbb{E}\Big[\big|\mu(s_1)-\mu(s_2)\big|^2\Big]=\mathbb{E}\Big[\big|\mu(|s_1-s_2|)\big|^2\Big]=4\int_{-\infty}^{+\infty}\sin^2\left(\frac{|s_1-s_2|\xi}{2}\right)f_\mu(\xi)\,\mathrm{d}\xi\,. \tag{4.8}$$

## Remark 4.2

*Assume that the process $\{\mu(s)\}_{s\in\mathbb{R}}$ is self-similar of order $\gamma_0\in(0,1)$, that is the processes $\{\mu(as)\}_{s\in\mathbb{R}}$ and $\{a^{\gamma_0}\mu(s)\}_{s\in\mathbb{R}}$ have the same finite-dimensional distributions, for any fixed positive real number a. Then, the corresponding spectral density $f_\mu$ satisfies*

$$f_\mu(\xi)=c|\xi|^{-2\gamma_0-1},\quad \text{for almost all } \xi\in\mathbb{R}, \tag{4.9}$$

*where c is some positive constant. We recall in passing that the Gaussian fractional Brownian motion of Hurst parameter $H\in(0,1)$ is self-similar of order $\gamma_0=H$. Also, we recall that the non-Gaussian Rosenblatt process of parameter $d\in(1/4,1/2)$ is self-similar of order $\gamma_0=2d$.*

## Remark 4.3

*A sufficient condition for the processes $\sigma$ and $X$ to satisfy $(\mathcal{C}_1)$ i.e. to be, on any compact interval $\mathcal{K}$, $\alpha_0$ and $\beta_0$ Hölder continuous for the norm $\|\cdot\|_{L^2(\Omega)}$ is the following: there exist two positive finite deterministic constants $c$ and $\xi_0$, such that the inequalities*

$$f_\sigma(\xi) \le c|\xi|^{-2\alpha_0-1} \quad \textit{and} \quad f_X(\xi) \le c|\xi|^{-2\beta_0-1} \tag{4.10}$$

*hold for almost all real number $\xi$ satisfying $|\xi| \ge \xi_0$.*

## Remark 4.4

*Suppose there are a finite constant $c > 0$ and two nonnegative integers $U_0$ and $V_0$ satisfying $U_0 + V_0 = 2$, such that, for every $j \in \mathbb{N}$ and for all $k_1, k_2 \in \mathbb{L}_{j,T}$, one has*

$$\left|\mathbb{E}\left[a_{j,k_1} a_{j,k_2}\right]\right| \le c 2^{-2j\alpha_0}\big(1 + |k_1 - k_2|\big)^{-U_0} \tag{4.11}$$

*and*

$$\left|\mathbb{E}\left[\lambda_{j,k_1} \lambda_{j,k_2}\right]\right| \le c 2^{-2j\beta_0}\big(1 + |k_1 - k_2|\big)^{-V_0}. \tag{4.12}$$

*Then $(\mathcal{C}2)$ is satisfied.*

**Proof of Remark 4.4:** The fact that $\sigma$ and $X$ are independent implies that the associated sequences of "wavelet coefficients" $(a_{j,k})_{j,k}$ and $(\lambda_{j,k})_{j,k}$ are independent as well. This together with the inequalities (4.11), (4.12) and the fact that $U_0 + V_0 = 2$ yields

$$\begin{aligned} &\Big|\mathbb{E}\big[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\big]\Big| = \Big|\mathbb{E}\big[a_{j,k_1}a_{j,k_2}\big]\Big|\Big|\mathbb{E}\big[\lambda_{j,k_1}\lambda_{j,k_2}\big]\Big| \\ &\leq \Big(c2^{-2j\alpha_0}\big(1+|k_1-k_2|\big)^{-U_0}\Big)\Big(c2^{-2j\beta_0}\big(1+|k_1-k_2|\big)^{-V_0}\Big) \\ &\leq c^2 2^{-2j(\alpha_0+\beta_0)}\big(1+|k_1-k_2|\big)^{-2}, \end{aligned} \tag{4.13}$$

for all $k_1, k_2 \in \mathbb{L}_{j,T}$. Then setting $c_1 := 2c^2 \sum_{q\in\mathbb{N}} q^{-2}$ and using (4.13) and the inequality $\mathrm{card}(\mathbb{L}_{j,T}) \leq c_2 2^j$, one gets:

$$\begin{aligned} &\sum_{k_1\in\mathbb{L}_{j,T}}\sum_{k_2\in\mathbb{L}_{j,T}}\Big|\mathbb{E}\big[a_{j,k_1}\lambda_{j,k_1}a_{j,k_2}\lambda_{j,k_2}\big]\Big| \\ &\leq c^2 2^{-2j(\alpha_0+\beta_0)}\sum_{k_1\in\mathbb{L}_{j,T}}\sum_{k_2\in\mathbb{L}_{j,T}}\big(1+|k_1-k_2|\big)^{-2} \\ &\leq c_1\mathrm{card}(\mathbb{L}_{j,T})2^{-2j(\alpha_0+\beta_0)} \leq c_1 c_2 2^{-2j(\alpha_0+\beta_0-1/2)}. \qquad \square \end{aligned}$$

## Proposition 4.1

*Assume that the wavelet $\psi$ is the Haar function, that is $\psi := \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$. Then (4.12) holds as soon as $f_X$ is $V_0$ times continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfies the following condition: There exist two finite deterministic constants $\beta_0' \in [\beta_0, 1)$ and $c > 0$ such that, for all $n \in \{0, \ldots, V_0\}$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has*

$$\left| f_X^{(n)}(\xi) \right| \leq c \max\left( |\xi|^{-2\beta_0 - n - 1}, |\xi|^{-2\beta_0' - n - 1} \right). \qquad (\mathcal{D}_{1,X})$$

## Proposition 4.2

*Let $M \in \mathbb{N}$ be arbitrary and fixed. Assume that the wavelet $\psi$ is continuously differentiable on the real line and has at least $M$ vanishing moments. Then (4.12) holds as soon as $f_X$ is $V_0$ times continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfies the following condition $(\mathcal{D}_{M,X})$, which is weaker than $(\mathcal{D}_{1,X})$: There exist two finite deterministic constants $\beta_0' \in [\beta_0, 1)$ and $c > 0$ such that, for all $n \in \{0, \ldots, V_0\}$ and $\xi \in \mathbb{R} \setminus \{0\}$, one has*

$$\left| f_X^{(n)}(\xi) \right| \leq c \max\left( |\xi|^{-2\beta_0 - n - 1}, |\xi|^{-2\beta_0' - nM - 1} \right). \qquad (\mathcal{D}_{M,X})$$

## Proposition 4.3

*Assume that the wavelet $\psi$ is the Haar function. Also assume that the integer $U_0$ in (4.11) belongs to the set $\{0,1\}$. Then (4.11) holds as soon as $f_\sigma$ is $U_0$ times continuously differentiable on $\mathbb{R}\setminus\{0\}$ and satisfies the following condition: There exist two finite deterministic constants $\alpha_0' \in [\alpha_0, 1)$ and $c>0$ such that, for all $n \in \{0,\dots,U_0\}$ and $\xi \in \mathbb{R}\setminus\{0\}$, one has*

$$\left|f_\sigma^{(n)}(\xi)\right| \le c \max\left(|\xi|^{-2\alpha_0-n-1}, |\xi|^{-2\alpha_0'-n-1}\right). \qquad (\mathcal{D}_{1,\sigma})$$

## Proposition 4.4

*Let $M \in \mathbb{N}$ be arbitrary and fixed. Assume that the wavelet $\psi$ is continuously differentiable on the real line and has at least $M+1$ vanishing moments. Then (4.11) holds as soon as $f_\sigma$ is $U_0$ times continuously differentiable on $\mathbb{R}\setminus\{0\}$ and satisfies the following condition ($\mathcal{D}_{M,\sigma}$), which is weaker than ($\mathcal{D}_{1,\sigma}$): There exist two finite deterministic constants $\alpha_0' \in [\alpha_0, 1)$ and $c>0$ such that, for all $n \in \{0,\dots,U_0\}$ and $\xi \in \mathbb{R}\setminus\{0\}$, one has*

$$\left|f_\sigma^{(n)}(\xi)\right| \le c \max\left(|\xi|^{-2\alpha_0-n-1}, |\xi|^{-2\alpha_0'-nM-1}\right). \qquad (\mathcal{D}_{M,\sigma})$$

## Remark 4.5

*For $\mu = \sigma$ or $\mu = X$, it is clear that $(\mathcal{D}_{1,\mu})$ holds when the process $\mu$ is self-similar of order $\gamma_0 = \alpha_0$ or $\gamma_0 = \beta_0$.*

## Remark 4.6

*A major motivation for weakening the condition $(\mathcal{D}_{1,\mu})$ to the condition $(\mathcal{D}_{M,\mu})$ is the following: the behavior of $f_\mu$ in the neighborhood of 0 can then be much more singular, namely $f_\mu$ can have infinitely many oscillations in the vicinity of 0. This is for instance the case, when $f_\mu$ is the "chirp function": for all $\xi \in \mathbb{R} \setminus \{0\}$,*

$$f_\mu(\xi) = |\xi|^{-2u-1} + |\xi|^{-2v-1} \sin^2\left(|\xi|^{-w}\right), \qquad (4.14)$$

*where the three parameters $u$, $v$ and $w$ are arbitrary real numbers such that $0 < u \leq v < 1$ and $w > 0$. Observe that the larger is $w$ the more oscillating is this function $f_\mu$ in the neighborhood of 0. Also observe that this function fails to satisfy $(\mathcal{D}_{1,\mu})$; yet, for any integer $M \geq 1 + w$, it satisfies $(\mathcal{D}_{M,\mu})$, with $\beta_0 = u$ and $\beta_0' = v$.*

# Conclusion

## Remark 4.7

*Assume for instance that the integrand $\sigma$ and the integrator $X$ are two independent Gaussian fractional Brownian motions whose Hurst parameters satisfy $H_1 \geq 1/2$ and $H_1 + H_2 > 1$. Then, one has $\alpha_0 = H_1$, $\beta_0 = H_2$ and $H_2 = \min(\beta_0, \alpha_0 + \beta_0 - 1/2)$. Thus, it results from Theorem 3.1 that, for all fixed $\gamma \in [0, \min(H_2, 1/2))$ and for every $\varepsilon > 0$ small enough, one has almost surely*

$$\limsup_{J \to +\infty} 2^{J(H_2 - \gamma - \varepsilon)} \| Y - Y_J^W \|_{C^\gamma(I)} < +\infty \,. \tag{4.15}$$

## Proposition 4.5

*The rate of convergence provided by (4.15) is optimal, namely: for all fixed $\gamma \in [0, \min(H_2, 1/2))$ and for every $\varepsilon > 0$ small enough, one has almost surely*

$$\limsup_{J \to +\infty} 2^{J(H_2 - \gamma + \varepsilon)} \| Y - Y_J^W \|_{C^\gamma(I)} = +\infty \,. \tag{4.16}$$

Proposition 4.5 can be extended to a general framework.