Erratic random functions: the concept of local nondeterminism and the study of Harmonizable Fractional Stable Field

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Organization of the talk

In all this talk, g always denotes an arbitrary continuous and deterministic function from $\mathbb{R}^N$ to $\mathbb{R}$.

Definition 1.1

Let $\tau \in \mathbb{R}^N$ be an arbitrary fixed point. Let $B(\tau, \rho)$ be the closed ball centered at τ and of arbitrary radius $\rho > 0$. The oscillation of g on $B(\tau, \rho)$, is denoted by $\mathrm{osc}_{g,\tau}(\rho)$, and defined as:

$$\mathrm{osc}_{g,\tau}(\rho) := \sup_{t^1, t^2 \in B(\tau,\rho)} |g(t^1) - g(t^2)| = \max_{t \in B(\tau,\rho)} g(t) - \min_{t \in B(\tau,\rho)} g(t). \tag{1.1}$$

The continuity of g at τ clearly implies that

$$\mathrm{osc}_{g,\tau}(\rho) \xrightarrow[\rho \to 0_+]{} 0_+ \,. \tag{1.2}$$

The rate of convergence depends on the degree of smoothness/roughness of g at τ. When it is differentiable at τ, then the rate is ρ; when it is rough at τ, then the rate can be much slower.

The function g is said to be erratic when the convergence in (1.2) holds slowly (slower than ρ) for all the points τ in $\mathbb{R}^N$, or at least for most of them.

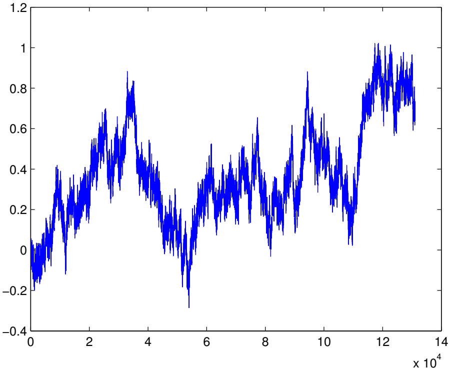

Figure: Graph of an erratic function

Finely studying the local behavior of an erratic function g consists in finding, for any $\tau \in \mathbb{R}^N$ and small ρ, an optimal explicit "nice" upper bound $M_{g,\tau}(\rho)$ for $\mathrm{osc}_{g,\tau}(\rho)$.

Usually, one looks for an optimal upper bound $M_{g,\tau}(\rho)$ of the form:

$$M_{g,\tau}(\rho) = (\text{power function in } \rho) \times (\text{logarithmic factor in } \rho).$$

In this issue, the most difficult task is to show the optimality of $M_{g,\tau}(\rho)$, that is:

$$\limsup_{\rho \to 0_+} \frac{\mathrm{osc}_{g,\tau}(\rho)}{M_{g,\tau}(\rho)} > 0. \tag{1.3}$$

A closely related, but less general, issue is whether or not g is a nowhere differentiable function.
In such kind of problems, loosely speaking, one of the major difficulties comes from the fact that in Real Analysis, usually, more effort is required in order to derive lower bounds (even in the sense of some sequence) than for deriving upper bounds. One reason is that in the case of lower bounds, the use of the triangle inequality is, in general, "less natural" than in that of upper bounds.

How can one overcome this difficulty?

Organization of the talk

The article by Geman and Horowitz "Occupation densities" (in *Annals of Probability 1980*) is a classical and excellent survey on the notions to be presented in this section and the next one. Throughout the present section, we denote by g an arbitrary continuous function from $\mathbb{R}^N$ to $\mathbb{R}$, depending on the multidimensional variable $t \in \mathbb{R}^N$, which is nevertheless viewed as a time variable.

Definition 2.1 (occupation measure)

Let T be an arbitrary compact subset of $\mathbb{R}^N$. The occupation measure *associated with g on T, is the deterministic positive finite measure $\mu_{g,T}$ on $\mathcal{B}(\mathbb{R})$, the Borel σ-field over $\mathbb{R}$, defined as*

$$\mu_{g,T}(A) := \lambda_N\big(\{t \in T : g(t) \in A\}\big) = \int_T \mathbf{1}_A\big(g(t)\big)\,dt, \text{ for all } A \in \mathcal{B}(\mathbb{R}), \quad (2.1)$$

where λ_N denotes the Lebesgue measure on $\mathbb{R}^N$. Observe that the quantity $\mu_{g,T}(A)$ can be viewed as a measure of "the amout of time t spent by the function g in the Borel set A during the time period T".

Remarks 2.2

1. *The finite measure $\mu_{g,T}$ is supported on the compact $g(T) := \{g(t) : t \in T\}$.*
2. *Assume that f is either a Borel measurable non-negative function on $\mathbb{R}$, or a Borel measurable bounded complex-valued function on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} f(x)\, d\mu_{g,T}(x) = \int_T f\big(g(t)\big)\, dt. \tag{2.2}$$

3. *The occupation measure $\mu_{g,T}$ is completely determined by its Fourier transform $\widehat{\mu}_{g,T}$ defined as*

$$\widehat{\mu}_{g,T}(\xi) := \int_{\mathbb{R}} e^{-i\xi x}\, d\mu_{g,T}(x), \quad \textit{for each } \xi \in \mathbb{R}; \tag{2.3}$$

also, in view of (2.2), one has

$$\widehat{\mu}_{g,T}(\xi) = \int_T e^{-i\xi g(t)}\, dt. \tag{2.4}$$

The notion of local time "mesure du voisinage" was first introduced by Paul Lévy ($\simeq$ 1940) in some his pioneering works on Brownian Motion.

Definition 2.3 (local time)

For having the existence of $L_g(\bullet, T)$, the local time on T of the function g, the occupation measure $\mu_{g,T}(\bullet)$ needs to be absolutely continuous with respect to λ, the Lebesgue measure on $\mathbb{R}$. *When this condition is fulfilled, then $L_g(\bullet, T)$ is defined as the Radon-Nikodým derivative of $\mu_{g,T}(\bullet)$ with respect to λ. In other words, $L_g(\bullet, T)$ is defined as the unique (up to a Lebesgue negligible set) non-negative function in the Lebesgue space $\mathbf{L}^1(\mathbb{R})$, such that the equality*

$$\int_{\mathbb{R}} f(x)\, d\mu_{g,T}(x) = \int_{\mathbb{R}} f(x) L_g(x, T)\, dx, \tag{2.5}$$

holds for any f, which is either a Borel measurable non-negative function on $\mathbb{R}$, or a Borel measurable bounded complex-valued function on $\mathbb{R}$.

Notice that the fact that $\mu_{g,T}$ is supported on $g(T)$ implies that $L_g(\bullet, T)$ is supported on $g(T)$ as well.

Proposition 2.4

A sufficient condition for the local time $L_g(\bullet, T)$ to exist is that the Fourier transform $\widehat{\mu}_{g,T}$ belongs to the Lebesgue Hilbert space $\mathbf{L^2}(\mathbb{R})$. Notice that, under this condition, one also has $L_g(\bullet, T) \in \mathbf{L^2}(\mathbb{R})$.

Proof: We denote by $\mathcal{F}(\cdot)$ the Fourier transform map. It is a bijection from the Schwartz class $\mathbf{S}(\mathbb{R})$ into itself; also it is a bijective "isometry" (Plancherel's formula) from $\mathbf{L^2}(\mathbb{R})$ into itself. Let $\varphi \in \mathbf{S}(\mathbb{R})$ be arbitrary; we have, for all $x \in \mathbb{R}$, $\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \widehat{\varphi}(\xi)\, d\xi$. Thus, using Fubini-Tonelli's theorem, we get that

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(x)\, d\mu_{g,T}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \Big(\int_{\mathbb{R}} e^{i\xi x}\, d\mu_{g,T}(x) \Big)\, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\xi) \overline{\widehat{\mu}_{g,T}(\xi)}\, d\xi \\
&= \int_{\mathbb{R}} \varphi(x) \overline{\mathcal{F}^{-1}(\widehat{\mu}_{g,T})(x)}\, dx \quad \text{(Plancherel's formula)}
\end{aligned}$$

which means that $\overline{\mathcal{F}^{-1}(\widehat{\mu}_{g,T})}$ is the local time we are looking for. $\square$

Notice that when $L_g(\bullet, T)$ exists then $L_g(\bullet, S)$ also exists for any compact $S \subseteq T$; this a consequence of the Radon-Nikodým Theorem (see for instance the well-known Rudin's book). Thus, L_g can be viewed as a function of 2 variables: the space variable $x \in \mathbb{R}$ and the set variable (or time variable) $S \subseteq T$.

The Berman's principle ($\simeq$ 1970): The more regular (smooth) is the local time L_g the more irregular (rough) is the associated function g. This principle is somehow clarified by the following lemma.

Lemma 2.5

Let $\tau \in \mathbb{R}^N$ be a fixed point such that, for some $\rho > 0$ small enough (or equivalently for all ρ small enough), the local time of g on the closed ball $B(\tau, \rho)$, that is $L_g(\bullet, B(\tau, \rho))$, exists. One sets

$$L_g^*(B(\tau, \rho)) := \sup_{x \in \mathbb{R}} L_g(x, B(\tau, \rho)) = \sup_{x \in g(B(\tau, \rho))} L_g(x, B(\tau, \rho)). \tag{2.6}$$

Then, assuming that the constant $c > 0$ is the Lebesgue measure of the unit ball of $\mathbb{R}^N$, one has

$$c\rho^N \le \mathrm{osc}_{g,\tau}(\rho) L_g^*(B(\tau, \rho)). \tag{2.7}$$

Thus, it turns out that in order to show that $M_{g,\tau}(\rho)$ is "a lower bound" for the oscillation $\mathrm{osc}_{g,\tau}(\rho)$, it is enough to prove that the ratio $\rho^N/M_{g,\tau}(\rho)$ is "an upper bound" for $L_g^*(B(\tau,\rho))$, more precisely:

$$\limsup_{\rho\to 0_+} \frac{\rho^N}{M_{g,\tau}(\rho)L_g^*(B(\tau,\rho))} > 0 \Longrightarrow \limsup_{\rho\to 0_+} \frac{\mathrm{osc}_{g,\tau}(\rho)}{M_{g,\tau}(\rho)} > 0.$$

Proof of Lemma 2.5: On one hand, one has

$$\mu_{g,B(\tau,\rho)}(\mathbb{R}) := \lambda_N\big(\{t\in B(\tau,\rho) : g(t)\in\mathbb{R}\}\big) = \lambda_N\big(B(\tau,\rho)\big) = c\rho^N. \qquad (2.8)$$

On the other hand, one has

$$\begin{aligned} \mu_{g,B(\tau,\rho)}(\mathbb{R}) &= \int_{g(B(\tau,\rho))} L_g(x,B(\tau,\rho))\,dx \\ &\le \lambda\big(g(B(\tau,\rho)\big) \sup_{x\in g(B(\tau,\rho))} L_g(x,B(\tau,\rho)) \\ &= \mathrm{osc}_{g,\tau}(\rho)L_g^*(B(\tau,\rho)). \end{aligned} \qquad (2.9)$$

Combining (2.8) and (2.9) one gets the lemma. □

Yet, Lemma 2.5 is of no interest when $L_g^*(B(\tau,\rho)) = +\infty$! In fact, it becomes to be interesting, when one has, at least, that

$$\lim_{\rho\to 0_+} L_g^*(B(\tau,\rho)) = 0. \tag{2.10}$$

Notice that if $N = 1$ and (2.10) is satisfied, then one has $\lim_{\rho\to 0_+} \rho^{-1}\mathrm{osc}_{g,\tau}(\rho) = +\infty$, which entails that g is not differentiable at τ.

Let us now provide a sufficient condition on the local time L_g under which (2.10) is valid. To this end, we assume, for a while, that $N = 1$, thus the ball $B(\tau,\rho)$ reduces to the compact interval $[\tau-\rho,\tau+\rho] \subset \mathbb{R}$.

Condition (joint continuity (JC)):There exists $\rho_0 > 0$, such that the function from $\mathbb{R}\times[\tau-\rho_0,\tau+\rho_0]$ into $\mathbb{R}_+$, $(x,s) \mapsto L_g(x,[\tau-\rho_0,s])$ is continuous. The continuity is then uniform since the function is compactly supported.

The condition (JC) implies that (2.10) holds; since one has, for all $0 < \rho \le \rho_0$,

$$L_g^*(B(\tau,\rho)) = \sup_{x\in\mathbb{R}} \Big(L_g(x,[\tau-\rho_0,\tau+\rho]) - L_g(x,[\tau-\rho_0,\tau-\rho]) \Big). \tag{2.11}$$

This argumentation can be extended to the general case where $N \geq 1$ is arbitrary; the condition (JC) is then defined as follows:

Definition 2.6 (jointly continuous local time)

The function g is said to have a jointly continuous local time on a fixed compact rectangle $\prod_{l=1}^{N}[a_l, b_l]$ of $\mathbb{R}^N$, if $L_g(\bullet, \prod_{l=1}^{N}[a_l, b_l])$ exists, and the function from $\mathbb{R} \times \prod_{l=1}^{N}[a_l, b_l]$ into $\mathbb{R}_+$, $(x, s_1, \ldots, s_N) \mapsto L_g(x, \prod_{l=1}^{N}[a_l, s_l])$ is continuous.

Studying local behavior of erratic deterministic functions via their local times is very unusual. This strategy is much more adapted to the random framework of stochastic processes and fields.

Organization of the talk

$\mathcal{S}\alpha\mathcal{S}$ random variables

Let $(\Omega, \mathcal{A}, P)$ be a probability space; the integral $\int_\Omega (\cdot)\, dP$, with respect to the probability measure P, is called the expectation operator and denoted by $\mathbb{E}(\cdot)$.

A real-valued random variable Z on Ω is a measurable function from $(\Omega, \mathcal{A})$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Z is (almost) completely determined by its characteristic function:

$$\Phi_Z(\xi) := \mathbb{E}(e^{i\xi Z}), \quad \text{for all } \xi \in \mathbb{R}. \tag{3.1}$$

Let $\alpha \in (0, 2]$ be a parameter (called the stability parameter), one says that Z is Symmetric α-Stable ($\mathcal{S}\alpha\mathcal{S}$), if there exists $\sigma(Z) \geq 0$ (called the scale parameter of Z), such that

$$\Phi_Z(\xi) = \exp\big(- |\sigma(Z)\xi|^\alpha \big), \quad \text{for any } \xi \in \mathbb{R}. \tag{3.2}$$

→ When $\alpha = 2$, Z reduces to a centered Gaussian random variable with $\mathrm{Var}(Z) = \mathbb{E}(Z^2) = 2\sigma(Z)^2$; observe that $\mathbb{E}(|Z|^\gamma) < +\infty$, for every $\gamma \in \mathbb{R}_+$.
→ The situation is different when $\alpha \in (0, 2)$; Z has a heavy-tailed distribution, which, in particular implies that $\mathbb{E}(|Z|^\gamma) = +\infty$, as soon as $\gamma \geq \alpha$.

Local times of sample paths

A stochastic field Y on Ω is a collection $\{Y(t), t \in \mathbb{R}^N\}$ of real-valued random variables on Ω; when $N = 1$, then Y is called a stochastic process.

For all fixed $\omega \in \Omega$, the function $Y(\cdot, \omega)$, from $\mathbb{R}^N$ to $\mathbb{R}$, $t \mapsto Y(t, \omega)$ is called a sample path of the field Y. In the setting of this talk, any sample path is always a continuous on $\mathbb{R}^N$. Each $Y(\cdot, \omega)$ plays the same role as the function g in the previous section. The associated occupation measure, on an arbitrary compact $T \subset \mathbb{R}^N$, is denoted by $\mu_{Y,T}(\bullet, \omega)$, instead of $\mu_{Y(\cdot,\omega),T}(\bullet)$. Thus, when it exists, the corresponding local time is denoted by $L_Y(\bullet, T, \omega)$.

$\rightarrow$ One of the main messages, we would like to deliver, is the following: "Basically, it is a less difficult problem to obtain a generic result on P-almost all local times $L_Y(\bullet, T, \omega)$ (that is an almost sure result on $L_Y(\bullet, T)$), than a result on a unique specified local time $L_g(\bullet, T)$". We mention that the Fubini-Tonelli's theorem plays an important role in this issue.

Existence and integrability of local times of $\mathcal{S}\alpha\mathcal{S}$ fields

One says that $\{Y(t), t \in \mathbb{R}^N\}$ is a $\mathcal{S}\alpha\mathcal{S}$ field, iff any linear combination of the random variables $Y(t)$'s is a $\mathcal{S}\alpha\mathcal{S}$ random variable: $\forall\, m \in \mathbb{N}$, $\forall\, a_1, \ldots, a_m \in \mathbb{R}$ and $\forall\, t^1, \ldots, t^m \in \mathbb{R}^N$, the random variable $\sum_{l=1}^m a_l Y(t^l)$ is $\mathcal{S}\alpha\mathcal{S}$.
$\sigma\big(\sum_{l=1}^m a_l Y(t^l)\big)$ denotes the scale parameter of $\sum_{l=1}^m a_l Y(t^l)$.

Theorem 3.1

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a $\mathcal{S}\alpha\mathcal{S}$ field and T a compact of $\mathbb{R}^N$. A "simple" sufficient condition for the local time $L_Y(\bullet, T)$ to exist almost surely is that

$$\int_{T^2} \sigma\big(Y(t^1) - Y(t^2)\big)^{-1} dt^1 dt^2 < +\infty. \tag{3.3}$$

Notice that, under this condition, one also has $(x,\omega) \mapsto L_Y(x, T, \omega) \in \mathbf{L}^2(\mathbb{R} \times \Omega)$.

Proof: For each $\omega \in \Omega$, the Fourier transform of the occupation measure $\mu_{Y,T}(\bullet, \omega)$, is given by

$$\widehat{\mu}_{Y,T}(\xi,\omega) = \int_T e^{-i\xi Y(t,\omega)}\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{3.4}$$

Therefore

$$\int_\Omega \int_\mathbb{R} \left|\widehat{\mu}_{Y,T}(\xi,\omega)\right|^2 d\xi dP(\omega) = \int_\Omega \int_\mathbb{R} \int_{T^2} e^{i\xi(Y(t^1,\omega)-(Y(t^2,\omega))}\, dt^1 dt^2 d\xi dP(\omega)$$
$$= \int_{T^2} \int_\mathbb{R} \int_\Omega e^{i\xi(Y(t^1,\omega)-(Y(t^2,\omega))}\, dP(\omega) d\xi dt^1 dt^2 \quad \text{(Fubini-Tonelli)}$$
$$= \int_{T^2} \int_\mathbb{R} \Phi_{Y(t^1)-Y(t^2)}(\xi)\, d\xi dt^1 dt^2$$
$$= \int_{T^2} \int_\mathbb{R} \exp\big(-|\sigma(Y(t^1)-Y(t^2))\xi|^\alpha\big)\, d\xi dt^1 dt^2 \ (\text{set } \eta = \sigma(Y(t^1)-Y(t^2))\xi)$$
$$= \Big(\int_\mathbb{R} e^{-|\eta|^\alpha}\, d\eta\Big)\Big(\int_{T^2} \sigma\big(Y(t^1)-Y(t^2)\big)^{-1}\, dt^1 dt^2\Big).$$

□

Remark 3.2

For all $f \in \mathbf{L}^\alpha(\mathbb{R}^N)$, the stochastic integral $\int_{\mathbb{R}^N} f(\xi)\, \widetilde{M}_\alpha(d\xi)$ is a complex-valued random variable whose real part is $\mathcal{S}\alpha\mathcal{S}$ with a scale parameter equals to the (quasi) norm $\|f\|_{L^\alpha(\mathbb{R})} := \Big(\int_{\mathbb{R}^N} \big|f(\xi)\big|^\alpha \, d\xi \Big)^{1/\alpha}$.

Definition 3.3 (Harmonizable Fractional Stable Field (HFSF))

The HFSF, of Hurst parameter $H \in (0,1)$ and of stability parameter $\alpha \in (0,2]$, is the $\mathcal{S}\alpha\mathcal{S}$ stochastic field denoted by $X = \{X(t), t \in \mathbb{R}^N\}$ and given by

$$X(t) := Re \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/\alpha}} \, \widetilde{M}_\alpha(d\xi), \quad \text{for all } t \in \mathbb{R}^N, \tag{3.5}$$

where $|\xi|$ is the Euclidian norm of ξ, and $t \cdot \xi$ the classical inner product of t and ξ.

HFSF generalizes classical processes and fields:

1. Brownian Motion ($\alpha = 2$, $N = 1$, and $H = 1/2$);
2. Fractional Brownian Motion ($\alpha = 2$, $N = 1$, and H arbitrary);
3. Fractional Brownian Field ($\alpha = 2$ and N and H arbitrary).

Proposition 3.4

Let X be a HFSF of arbitrary parameters, and T a compact of $\mathbb{R}^N$. Then the local time $L_X(\bullet, T)$ exists almost surely, moreover the function $(x,\omega) \mapsto L_X(x, T, \omega)$ belongs to $\mathbf{L^2}(\mathbb{R} \times \Omega)$.

Proof: using the definition of X and elementary properties of the stochastic integral in it, one can show that, for some constant $c > 0$, one for has

$$\sigma\big(X(t^1) - X(t^2)\big) = c|t^1 - t^2|^H, \quad \text{for all } t^1, t^2 \in \mathbb{R}^N. \tag{3.6}$$

Therefore

$$\int_{T^2} \sigma\big(X(t^1) - X(t^2)\big)^{-1} dt^1 dt^2 < +\infty. \tag{3.7}$$

□

Organization of the talk

Theorem 4.1 (Kolmogorov's continuity theorem)

Let $Q \in \mathbb{N}$, J a compact rectangle of $\mathbb{R}^Q$, and $\{Z(\theta), \theta \in J\}$ a stochastic field. A sufficient condition for its sample paths to be, almost surely, continuous functions on J is the following: there exist 3 positive constants β, c, and ν such that

$$\mathbb{E}\big(|Z(\theta') - Z(\theta'')|^{\beta}\big) \leq c|\theta' - \theta''|^{Q+\nu}, \quad \text{for all } \theta', \theta'' \in J. \tag{4.1}$$

Let $I := \prod_{l=1}^{N}[a_l, b_l]$ be a compact rectangle of $\mathbb{R}^N$; for any $s = (s_1, \ldots, s_N) \in I$ we set $I(s) := \prod_{l=1}^{N}[a_l, s_l] \subseteq I$. Let $\{Y(t), t \in \mathbb{R}^N\}$ be a stochastic field such that the local time $L_Y(\bullet, I)$ exists almost surely, and belongs to $\mathbf{L}^2(\mathbb{R} \times \Omega)$. To say that L_Y is almost surely jointly continuous on I means that, for any fixed $r > 0$, the stochastic field $\big\{L_Y(x, I(s)),\ (x, s) \in [-r, r] \times I\big\}$ has, almost surely, continuous paths. To derive such a result, the classical strategy consists in trying to use Theorem 4.1, when $Q = N + 1$, $J = [-r, r] \times I$, $\theta = (x, s)$, and $Z(\theta) = L_Y(x, I(s))$. Thus, one has to find, for some fixed well-chosen even integer $k = \beta \geq 2$, an upper bound, of the same form as in (4.1), for the quantity

$$\Delta_k(x', x''; s', s'') := \mathbb{E}\Big(\big|L_Y(x', I(s')) - L_Y(x'', I(s''))\big|^k\Big), \tag{4.2}$$

where $(x', x'') \in [-r, r]^2$ and $(s', s'') \in I^2$ are arbitrary.

Let us "separate" the increment in s from that in x. One has

$$\Delta_k(x',x'';s',s'') \leq 2^{k-1}\big(\tilde{\Delta}_k(x';s',s'') + \check{\Delta}_k(x',x'';s'')\big). \qquad (4.3)$$

Notice that

$$\begin{aligned}
\tilde{\Delta}_k(x';s',s'') &:= \mathbb{E}\Big(\big|L_Y(x',I(s')) - L_Y(x',I(s''))\big|^k\Big) \\
&\leq \mathbb{E}\Big(L_Y\big(x',I(s')\setminus I(s'')\big)^k\Big) + \mathbb{E}\Big(L_Y\big(x',I(s'')\setminus I(s')\big)^k\Big) \\
&\leq 2\big(\mathrm{card}(\mathcal{S})\big)^{k-1}\sum_{j\in\mathcal{S}}\mathbb{E}\big(L_Y(x',B_j)^k\big)\,, \qquad (4.4)
\end{aligned}$$

where $\mathcal{S}$ is a finite set such that $\mathrm{card}(\mathcal{S}) \asymp |s'-s''|^{1-N}$, and each B_j is a cube included in I satisfying $\mathrm{diam}(B_j) \asymp |s'-s''|$.
Also, notice that

$$\check{\Delta}_k(x',x'';s'') := \mathbb{E}\Big(\big(L_Y(x',I(s'')) - L_Y(x'',I(s''))\big)^k\Big). \qquad (4.5)$$

Thus, it is useful to obtain convenient upper estimates for:

1. the kth moment of the local time $\mathcal{U}_k(x,B) := \mathbb{E}\big(L_Y(x,B)^k\big)$, where $x \in \mathbb{R}$ and the cube $B \subseteq I$ are abitrary;
2. the kth moment of its increments $\mathcal{W}_k(x,y,R) := \mathbb{E}\Big(\big(L_Y(x,R) - L_Y(y,R)\big)^k\Big)$, where $(x,y) \in \mathbb{R}^2$ and the rectangle $R \subseteq I$ are abitrary.

We only study $\mathcal{U}_k(x,B)$ since $\mathcal{W}_k(x,y,R)$ can be studied in a rather similar way. Using the heuristic computations

$$L_Y(x,B) = \overline{\mathcal{F}^{-1}(\widehat{\mu}_{Y,B})(x)} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\,\overline{\widehat{\mu}_{Y,B}(\xi)}\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\int_B e^{i\xi Y(t)}\,dtd\xi$$

and Fubini-Tonelli's theorem, we get (see Geman and Horowitz, 1980), that

$$\mathcal{U}_k(x,B) = \frac{1}{(2\pi)^k}\int_{B^k}\int_{\mathbb{R}^k} \exp\Big(-ix\sum_{l=1}^k \xi_l\Big)\mathbb{E}\Big(\exp\Big(i\sum_{l=1}^k \xi_l Y(t^l)\Big)\Big)\,d\overline{\xi}d\overline{t}\,, \quad (4.6)$$

where $\overline{\xi} = (\xi_1,\ldots,\xi_k)$ and $\overline{t} = (t^1,\ldots,t^k)$. The quantity $\mathbb{E}\Big(e^{i\sum_{l=1}^k \xi_l Y(t^l)}\Big)$ is the value at 1 of the characteristic function of the random variable $\sum_{l=1}^k \xi_l Y(t^l)$.

Thus, when Y is a centered Gaussian field, one has that

$$\mathbb{E}\left(\exp\left(i\sum_{l=1}^{k}\xi_l Y(t^l)\right)\right) = \exp\left(-2^{-1}\operatorname{Var}\left(\sum_{l=1}^{k}\xi_l Y(t^l)\right)\right) \tag{4.7}$$

and consequently that

$$\int_{\mathbb{R}^k}\mathbb{E}\left(\exp\left(i\sum_{l=1}^{k}\xi_l Y(t^l)\right)\right)d\overline{\xi} = \sqrt{\frac{(2\pi)^k}{\det\big(\mathrm{CovMat}(Y(t^1),\ldots,Y(t^k))\big)}}. \tag{4.8}$$

Thus, the problem of finding a convenient upper estimate for the kth moment of the local time $\mathcal{U}_k(x,B)$, reduces to that of finding a convenient lower estimate for $\det\big(\mathrm{CovMat}(Y(t^1),\ldots,Y(t^k))\big)$. This is in fact a Gram determinant. Therefore

$$\det\big(\mathrm{CovMat}(Y(t^1),\ldots,Y(t^k))\big) = \operatorname{Var}\big(Y(t^1)\big)\prod_{m=2}^{k}\operatorname{Var}\big(Y(t^m)|Y(t^1),\ldots,Y(t^{m-1})\big) \tag{4.9}$$

where

$$\sqrt{\operatorname{Var}\big(Y(t^m)|Y(t^1),\ldots,Y(t^{m-1})\big)} := \mathrm{dist}_{\mathbf{L}^2(\Omega)}\Big(Y(t^m),\mathrm{span}\big\{Y(t^1),\ldots,Y(t^{m-1})\big\}\Big) \tag{4.10}$$

The concept of local nondeterminism (LND) was first introduced by Berman (1973) in the framework of Gaussian processes. Pitt (1978) extended it to the framework of Gaussian fields, and Nolan (1989) to that of $\mathcal{S}\alpha\mathcal{S}$ fields.

Definition 4.2 (LND for Gaussian stochastic fields)

A centered Gaussian field $\{Y(t), t \in \mathbb{R}^N\}$ is said to be LND on a compact rectangle $I \subset \mathbb{R}^N$, if for any fixed integer $m \geq 2$, and for all points $t^1, \ldots, t^m \in I$ sufficiently close together, one has

$$\mathrm{Var}\big(Y(t^m) | Y(t^1), \ldots, Y(t^{m-1})\big) \geq c \min_{1 \leq q < m} \mathrm{Var}\big(Y(t^m) - Y(t^q)\big) , \qquad (4.11)$$

where $c > 0$ is a constant which may only depend on I and m.

Remark 4.3

When the centered Gaussian field $\{Y(t), t \in \mathbb{R}^N\}$ has a stochastic integral representation of the form: for all $t \in \mathbb{R}^N$,

$$Y(t) = Re \int_{\mathbb{R}^N} \mathcal{K}(t,\xi)\, \widetilde{M}_2(d\xi), \quad \text{where } \mathcal{K}(t,\cdot) \in \mathbf{L}^2_\xi(\mathbb{R}^N). \tag{4.12}$$

Then, thanks to the isometry property (from $\mathbf{L}^2_\xi(\mathbb{R}^N)$ into $\mathbf{L}^2(\Omega)$) of this stochastic integral, the inequality

$$\mathrm{Var}\Big(Y(t^m) | Y(t^1), \ldots, Y(t^{m-1}\Big) \geq c \min_{1 \leq q < m} \mathrm{Var}\Big(Y(t^m) - Y(t^q)\Big),$$

can be expressed in terms of the deterministic kernel function $\mathcal{K}$:

$$\begin{aligned} \mathrm{dist}_{\mathbf{L}^2_\xi(\mathbb{R}^N)}\Big(\mathcal{K}(t^m,\cdot), span\big\{\mathcal{K}(t^1,\cdot), \ldots, \mathcal{K}(t^{m-1},\cdot)\big\}\Big) \\ \geq \sqrt{c} \min_{1 \leq q < m} \big\|\mathcal{K}(t^m,\cdot) - \mathcal{K}(t^q,\cdot)\big\|_{\mathbf{L}^2_\xi(\mathbb{R}^N)}. \end{aligned} \tag{4.13}$$

Definition 4.4 (LND for $\mathcal{S}\alpha\mathcal{S}$ stochastic fields)

Let $\alpha \in (0,2]$ and $\{Y(t), t \in \mathbb{R}^N\}$ a $\mathcal{S}\alpha\mathcal{S}$ field having a stochastic integral representation of the form: for all $t \in \mathbb{R}^N$,

$$Y(t) = Re \int_{\mathbb{R}^N} \mathcal{K}(t,\xi)\, \widetilde{M}_\alpha(d\xi), \quad \text{where } \mathcal{K}(t,\cdot) \in \mathbf{L}^\alpha_\xi(\mathbb{R}^N). \tag{4.14}$$

Such a field is said to be LND on a compact rectangle $I \subset \mathbb{R}^N$, if for any fixed integer $m \geq 2$, and for all points $t^1, \ldots, t^m \in I$ sufficiently close together, one has

$$\begin{aligned} &\mathrm{dist}_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)}\Big(\mathcal{K}(t^m,\cdot), span\big\{\mathcal{K}(t^1,\cdot),\ldots,\mathcal{K}(t^{m-1},\cdot)\big\}\Big) \\ &\qquad\qquad \geq c \min_{1\leq q<m} \big\|\mathcal{K}(t^m,\cdot) - \mathcal{K}(t^q,\cdot)\big\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)}, \end{aligned} \tag{4.15}$$

where $c > 0$ is a constant which may only depend on I and m.

Definition 4.5 (Nolan (1989), locally appoximately independent increments)

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a $\mathcal{S}\alpha\mathcal{S}$ field which has the same stochastic integral representation, through a kernel function $\mathcal{K}$, as in Definition 4.4. Such a field is said to have locally appoximately independent increments on a compact rectangle $I \subset \mathbb{R}^N$, if for any fixed integer $m \geq 2$, for every real numbers $b_1, \ldots, b_m$, and for all points $t^1, \ldots, t^m \in I$ sufficiently close together, one has

$$\begin{aligned} & c^{-1}\left(\left\|b_1\mathcal{K}(t^1,\cdot)\right\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)} + \sum_{j=2}^m \left\|b_j\big(\mathcal{K}(t^j,\cdot)-\mathcal{K}(t^{j-1},\cdot)\big)\right\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)}\right) \\ & \leq \left\|b_1\mathcal{K}(t^1,\cdot)+\sum_{j=2}^m b_j\big(\mathcal{K}(t^j,\cdot)-\mathcal{K}(t^{j-1},\cdot)\big)\right\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)} \qquad (4.16) \\ & \qquad \leq c\left(\left\|b_1\mathcal{K}(t^1,\cdot)\right\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)} + \sum_{j=2}^m \left\|b_j\big(\mathcal{K}(t^j,\cdot)-\mathcal{K}(t^{j-1},\cdot)\big)\right\|_{\mathbf{L}^\alpha_\xi(\mathbb{R}^N)}\right), \end{aligned}$$

where $c > 0$ is a constant which may only depend on I and m.

Theorem 4.6 (Nolan (1989))

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a $\mathcal{S}\alpha\mathcal{S}$ field which has the same stochastic integral representation as in Definition 4.4. Then this field is LND on a compact rectangle $I \subset \mathbb{R}^N$ if and only if it has locally appoximately independent increments on I.

For proving this theorem Nolan made use of arguments from Linear Algebra relying on a generalization of the notion of Gram determinant.

Organization of the talk

Recall that the Harmonizable Fractional Stable Field (HFSF), of Hurst parameter $H \in (0,1)$ and of stability parameter $\alpha \in (0,2]$, is the $\mathcal{S}\alpha\mathcal{S}$ stochastic field denoted by $\{X(t), t \in \mathbb{R}^N\}$ and given by

$$X(t) := Re \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/\alpha}} \widetilde{M}_\alpha(d\xi), \quad \text{for all } t \in \mathbb{R}^N, \tag{5.1}$$

where $|\xi|$ is the Euclidian norm of ξ, and $t \cdot \xi$ the classical inner product of t and ξ.

Theorem 5.1

Let I be a compact rectangle of $\mathbb{R}^N$ such that $0 \notin I$. Then the HFSF $\{X(t), t \in \mathbb{R}^N\}$ is LND on I.

This theorem was obtained by:

- Pitt (1978) in the Gaussian case $\alpha = 2$;
- Nolan (1989) in the $\mathcal{S}\alpha\mathcal{S}$ case, where $\alpha \in [1,2)$;
- Ayache and Xiao (recently) in the $\mathcal{S}\alpha\mathcal{S}$ case, where $\alpha \in (0,1)$.

Notice that, in the Gaussian case, an alternative proof for Theorem 5.1 was proposed by Kahane. Actually, the method used by Ayache and Xiao, in their $\mathcal{S}\alpha\mathcal{S}$ setting, is, to a certain extent, inspired by this Kahane's proof.

Proof when $\alpha = 2$ (Kahane): The integer $m \geq 2$ is arbitrary an fixed. One has to show that for every real numbers $a_1, \ldots, a_{m-1}$, and for all points $t^1, \ldots, t^m \in I$ (sufficiently close together), the following inequality holds

$$\int_{\mathbb{R}^N} \Big| e^{it^m \cdot \xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l \cdot \xi} - 1 \big) \Big|^2 \frac{d\xi}{|\xi|^{2H+N}} \geq c \min_{1 \leq q < m} |t^m - t^q|^{2H}, \qquad (5.2)$$

where $c > 0$ is a constant which may only depend on I and m.

We suppose that $I := [\varepsilon, 1]^N$, $\varepsilon \in (0,1)$ being arbitrary. We set

$$r := \min_{1 \leq q < m} |t^m - t^q| \quad \text{and} \quad \rho = \rho(r, |t^m|) := \min \big(r, |t^m| \big). \qquad (5.3)$$

The inequality $|t^m| \geq \varepsilon \sqrt{N}$ implies that one has, for some constant $c_1 > 0$, only depending on ε,

$$\rho(r, |t^m|) \geq c_1 \, r. \qquad (5.4)$$

Next, let φ be an arbitrary fixed infinitely differentiable function from $\mathbb{R}^N$ into the compact interval $[0,1]$ of the real line, such that

$$\varphi(0) = 1 \text{ and } \operatorname{Supp}\varphi \subseteq \big\{x \in \mathbb{R}^N : \, |x| \leq 1\big\}. \tag{5.5}$$

We denote by φ_ρ the function from $\mathbb{R}^N$ into the compact interval $[0,1]$, defined as

$$\varphi_\rho(s) = \rho^{-N}\varphi\big(\rho^{-1}s\big) \text{ for all } s \in \mathbb{R}^N. \tag{5.6}$$

Observe that (5.5) and (5.6) entail that

$$\varphi_\rho(0) = \rho^{-N} \text{ and } \operatorname{Supp}\varphi_\rho \subseteq \big\{s \in \mathbb{R}^N : \, |s| \leq \rho\big\}. \tag{5.7}$$

Next, let $\Lambda = \Lambda\big(a_1, \ldots, a_{m-1}; t^1, \ldots, t^m\big)$ be the integral defined as

$$\Lambda := \int_{\mathbb{R}^N} e^{-it^m\cdot\xi}\widehat{\varphi}_\rho(\xi)\Big(e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l\big(e^{it^l\cdot\xi} - 1\big)\Big)\, d\xi, \tag{5.8}$$

where $\widehat{\varphi}_\rho$ denotes the Fourier transform of φ_ρ.

Using (5.8) and the equality

$$\varphi_\rho(s) := (2\pi)^{-N} \int_{\mathbb{R}^N} e^{is\cdot\xi} \widehat{\varphi}_\rho(\xi)\, ds, \quad \text{for all } s \in \mathbb{R}^N,$$

one gets that

$$\begin{aligned}\Lambda &= \int_{\mathbb{R}^N} \widehat{\varphi}_\rho(\xi)\, d\xi + \Big(-1 + \sum_{l=1}^{m-1} a_l \Big) \int_{\mathbb{R}^N} e^{-it^m\cdot\xi} \widehat{\varphi}_\rho(\xi)\, d\xi \\ &\qquad - \Big(\sum_{l=1}^{m-1} a_l \Big) \int_{\mathbb{R}^N} e^{-i(t^l - t^m)\cdot\xi} \widehat{\varphi}_\rho(\xi)\, d\xi \\ &= (2\pi)^{-N} \varphi_\rho(0) + (2\pi)^{-N/2} \Big(-1 + \sum_{l=1}^{m-1} a_l \Big) \varphi_\rho(-t^m) + (2\pi)^{-N} \varphi_\rho\big(t^m - t^l\big).\end{aligned}$$

Therefore, the equality $\varphi_\rho(0) = \rho^{-N}$ and the inequalities $|t^m| \geq \rho$ and $\min_{1\leq l<m} |t^m - t^l| \geq \rho$ imply that $\Lambda = (2\pi)^{-N}\rho^{-N}$.

On the other hand, using Cauchy-Schwarz inequality, one obtains that

$$
\begin{aligned}
|\Lambda|^2 &= \left| \int_{\mathbb{R}^N} e^{-it^m\cdot\xi} \widehat{\varphi}_\rho(\xi) |\xi|^{H+N/2} \Big(e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l\cdot\xi} - 1 \big) \Big) \frac{d\xi}{|\xi|^{H+N/2}} \right|^2 \\
&\leq \int_{\mathbb{R}^N} \big|\widehat{\varphi}_\rho(\xi)\big|^2 |\xi|^{2H+N}\, d\xi \times \int_{\mathbb{R}^N} \Big| e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l\cdot\xi} - 1 \big) \Big|^2 \frac{d\xi}{|\xi|^{2H+N}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(2\pi)^{-2N} \rho^{-2N} \\
&\leq \int_{\mathbb{R}^N} \big|\widehat{\varphi}(\rho\xi)\big|^2 |\xi|^{2H+N}\, d\xi \times \int_{\mathbb{R}^N} \Big| e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l\cdot\xi} - 1 \big) \Big|^2 \frac{d\xi}{|\xi|^{2H+N}} \\
&= \rho^{-2(H+N)} \int_{\mathbb{R}^N} \big|\widehat{\varphi}(\eta)\big|^2 |\eta|^{2H+N}\, d\eta \\
&\qquad\qquad \times \int_{\mathbb{R}^N} \Big| e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l\cdot\xi} - 1 \big) \Big|^2 \frac{d\xi}{|\xi|^{2H+N}}.
\end{aligned}
$$

Finally, one gets that

$$\int_{\mathbb{R}^N} \Big| e^{it^m\cdot\xi} - 1 - \sum_{l=1}^{m-1} a_l \big(e^{it^l\cdot\xi} - 1 \big) \Big|^2 \frac{d\xi}{|\xi|^{2H+N}} \geq c_2 \rho^{2H} \geq c_3 r^{2H}, \tag{5.9}$$

where $c_2 > 0$ and $c_3 > 0$ are two constants only depending on H and N.
□