Power Control in Parallel Symmetric α-Stable Noise Channels

Mauro de Freitas, Malcolm Egan, Laurent Clavier, Anne Savard and Jean-Marie Gorce

***Abstract*—Parallel channels form a basic building block for communication systems, including those based on OFDM and CDMA. While parallel Gaussian noise channels have been widely studied, parallel impulsive noise channels have received significantly less attention despite their importance in a range of modern communication systems. In this paper, this problem is addressed and a power allocation strategy is developed for parallel symmetric α-stable noise channels—a key class of impulsive noise channels. We show that our strategy can improve achievable rates by up to a factor of 1.5 over the standard waterfilling algorithm that assumes the noise is Gaussian.**

I. Introduction

Impulsive noise arises in a range of communication systems and is often modeled via the α-stable distribution. In particular, the memoryless additive α-stable noise channel models interference in wireless communications for the Internet of Things (IoT) [1]. In other wireless network scenarios, α-stable noise models have also been developed [2]. Unlike Gaussian models, the α-stable distribution is characterized by heavy tails, which accounts for a high probability of large amplitude noise. However, due to the lack of closed-form expressions for the noise probability density function, characterizations of achievable rates in the presence of α-stable noise are challenging to obtain. A consequence is that the problem of resource allocation has not been addressed.

Despite these challenges, there have recently been several new results characterizing the capacity of real additive α-stable noise channels subject to a range of constraints. In [3], the capacity of the Cauchy noise channel ($\alpha = 1$) was derived subject to a logarithmic constraint. More generally, capacity bounds for $\alpha > 1$ subject to absolute moments constraints were obtained in [4]. The extension to the complex isotropic α-stable channel was also studied in [5] and vector α-stable channels in [6]. These capacity bounds now provide a basis to optimize systems that experience additive symmetric α-stable noise.

In this paper, we consider the problem of power allocation in parallel symmetric α-stable noise channels ($1 < \alpha < 2$) subject to a power constraint. In the case of a Gaussian input in the presence of additive Gaussian noise, the rate-optimal solution is the well-known waterfilling algorithm. However, the waterfilling algorithm is tailored to the Gaussian noise channel. As such, it is highly desirable to develop alternative power control strategies that do not rely on the Gaussian noise assumption.

We adopt a two-step approach to the design of power control for the symmetric α-stable noise channel. The first step is to select the input distribution. We note that the optimal input distribution for this channel is discrete and can only be obtained via numerical optimization. Therefore, we evaluate achievable rates via Monte Carlo estimates of the symmetric α-stable noise channel with Gaussian and truncated symmetric α-stable inputs. We show that Gaussian inputs perform comparably or outperform truncated symmetric α-stable inputs, and also nearly achieve a numerical approximation of the capacity obtained via the Blahut-Arimoto algorithm [7, 8]. This is despite the fact that the truncated symmetric α-stable inputs approximately match the input with the noise distribution, and are known to be a good choice with fractional moment constraints [4].

The second step is to optimize the power control for the selected input distribution and develop a new power allocation scheme for Gaussian inputs. This is a challenging problem due to the fact that the achievable rate is not known in closed-form for symmetric α-stable noise channels with Gaussian inputs. To this end, we introduce an approximation via a scale parameter matching method (detailed in Section IV-A) by exploiting the achievable rate with symmetric α-stable inputs obtained in [4].

The resulting optimization problem is convex and therefore readily solved numerically, but differs from the waterfilling solution arising in Gaussian noise models. In particular, the KKT conditions lead to a system of non-linear equations unlike the linear equations arising from parallel Gaussian noise channels.

In summary, we make two key contributions:

1) We show that Gaussian inputs perform well on scalar additive symmetric α-stable noise channels subject to a power constraint, which is compatible with existing coding schemes.
2) We introduce a new power control scheme for parallel α-stable noise channel with Gaussian inputs based on scale parameter matching (detailed in Section IV-A). This leads to a convex optimization problem with numerical results demonstrating that our power control schemes can outperform by up to a factor of 1.5 the rate achieved by waterfilling for Gaussian inputs, where the α-stable noise is assumed to be Gaussian.

The remainder of this paper is organized as follows. In Section II, we detail the parallel symmetric α-stable noise

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model. In Section III, we study the effect of different input distributions. In Section IV, we develop our power control strategy. In Section V, we conclude the paper.

II. System Description

Consider the real-valued memoryless additive symmetric α-stable noise ($AS\alpha SN$) channel

$$Y = hX + N, \tag{1}$$

where $h \in \mathbb{R}$ is a constant, X is the channel input, and N is symmetric α-stable noise with $1 < \alpha < 2$. We focus on the range $1 < \alpha < 2$ as in wireless communications this corresponds to a path loss exponent in the range $(2, 4)$ [1], which captures a range of realistic electromagnetic environments.

The α-stable random variables ($\alpha < 2$) are an important class of random variables with heavy-tailed probability density functions and infinite second moments, which have been widely used to model impulsive signals [4]. The probability density function of an α-stable random variable is parameterized by four parameters: the exponent $0 < \alpha \leq 2$; the scale parameter $\sigma \in \mathbb{R}_+$; the skew parameter $\beta \in [-1, 1]$; and the shift parameter $\delta \in \mathbb{R}$. As such, a common notation for an α-stable distributed random variable is $X \sim S_\alpha(\sigma, \beta, \delta)$. In the case $\beta = \delta = 0$, the random variable X is said to be symmetric.

In general, symmetric α-stable random variables do not have closed-form probability density functions. Instead, they are usually represented by their characteristic function, given by

$$\mathbb{E}[e^{i\theta X}] = e^{-\sigma^\alpha |\theta|^\alpha} \tag{2}$$

As a consequence of the lack of a closed-form probability density function, there are few closed-form characterizations of the capacity for symmetric α-stable noise channels.

Nevertheless, it is possible to derive closed-form expressions for achievable rates. In particular, the rate of the $AS\alpha SN$ channel with a symmetric α-stable input was derived in [4], given by

$$R = \frac{1}{\alpha} \log \left(1 + |h|^\alpha \frac{\sigma_X^\alpha}{\sigma_N^\alpha} \right), \tag{3}$$

where σ_X is the scale parameter of the symmetric α-stable input and σ_N is the scale parameter of the symmetric α-stable noise.

The main problem we consider in this paper is power control for K parallel $AS\alpha SN$ channels with a sum power contraint. In this case, the system consists of K channels defined by

$$Y_i = h_i X_i + N_i, \; i = 1, 2, \ldots, K, \tag{4}$$

where $h_i \in \mathbb{R}$, X_i is the real-valued input to the i-th channel and N_i is real symmetric α-stable noise, independent for each i but not necessarily identically distributed. At present, there are no known closed-form expressions of achievable rates for power-constrained inputs in channels with symmetric α-stable noise. Nevertheless, if the power constraint is relaxed, it follows from (3) that the sum-rate for parallel channels achieved using a symmetric α-stable input for each channel is given by

$$R_S = \sum_{k=1}^{K} \frac{1}{\alpha} \log \left(1 + |h_k|^\alpha \frac{\sigma_{X,k}^\alpha}{\sigma_{N,k}^\alpha} \right). \tag{5}$$

III. Input Distribution Selection

Although the optimal input distribution for the power constrained additive Gaussian noise channel is well-known to be Gaussian, this is not the case for symmetric α-stable noise channels. As such, it is challenging to optimize the power control in the case of parallel channels and it is highly desirable to obtain input distributions that yield a high achievable rate with a simple parametric form.

We note that the optimal input distribution for an $AS\alpha SN$ channel is discrete [9] and does not have a simple parametric form. While it is possible to obtain numerical approximations of this optimal input, even for a single channel it is time consuming [10]. Since the optimal input distribution depends on the power constraint, this means that it is also difficult to implement such an input within the context of a power control algorithm as both the input and power need to be jointly optimized. Using existing numerical optimization methods, this leads to very slow power control, which is not useful when, for example, fading is time-varying.

To overcome this difficulty, we investigate the choice of the input distribution for the $AS\alpha SN$ channel in (1) subject to a power constraint. We search for the distribution that allows an achievable rate close to the capacity obtained through the mutual information optimization problem:

$$\begin{aligned} \underset{\mu \in \mathcal{P}}{\text{maximize}} \quad & I(X;Y) \\ \text{subject to} \quad & \mathbb{E}_\mu[X^2] \leq P, \end{aligned} \tag{6}$$

where $\mathcal{P}$ is the set of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

In order to investigate the choice of the input distribution, we consider the following four choices:

(i) Zero-mean Gaussian input X_G with probability density function

$$p_{X_G}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{\sigma^2} \right). \tag{7}$$

(ii) Truncated symmetric α-stable inputs: let X_S be a symmetric α-stable random variable, then the truncated symmetric α-stable input X_T with truncation level T is constructed via

$$X_T = \begin{cases} X_S, & |X_S| \leq T \\ \text{sign}(X_S)T & |X_S| > T. \end{cases} \tag{8}$$

The power of the truncated symmetric α-stable input is given by

$$\mathbb{E}[X_T^2] = \int_{-T}^{T} x^2 p_{X_S}(x) dx + 2 \int_{T}^{\infty} T^2 p_{X_S}(x) dx, \tag{9}$$

where p_{X_S} is the probability density function of the symmetric α-stable random variable X_S.

(iii) Truncated Gaussian inputs: let X_G be a Gaussian random variable. The truncated Gaussian input $X_{G,T}$ can be constructed as the truncated α-stable input based on (8), replacing X_S by X_G. The power of this input is given by (9), replacing $p_{X_S}(x)$ by $p_G(x)$, the probability density function of the Gaussian random variable X_G.

(iv) Numerical approximation of the capacity via the Blahut-Arimoto algorithm [7, 8].

We selected the Gaussian input due to the fact that existing codes are typically constructed to approximate Gaussian inputs. The truncated symmetric α-stable input is chosen because it approximately matches the noise distribution (nearly optimal for fractional moment constraints [4]) and also satisfies the finite power constraint. It is also an appropriate choice of input for the case where the channel is both power and amplitude constrained. The truncated Gaussian input is selected as it forms a natural choice in the case of both power and amplitude constraints. The numerical approximation of the capacity forms a baseline to assess the optimality of the other three schemes.

To understand how the choice of input distribution affects the achievable rate, Fig. 1 plots the achievable rates using a Gaussian input, a truncated symmetric α-stable input, and also a truncated Gaussian input. In each case, the power is constrained to be $P = 3$ and the figure shows the impact on the truncation level for each input distribution. As achievable rates of additive symmetric α-stable noise channels are not known, in the experiments they are estimated via Monte Carlo simulation. In particular, we use $5 \cdot 10^6$ input samples, the entropy of the output and the noise are obtained by estimating the corresponding probability density functions via the kernel method [11], which was performed by using a grid of 10^6 points and support $[-200, 200]$.

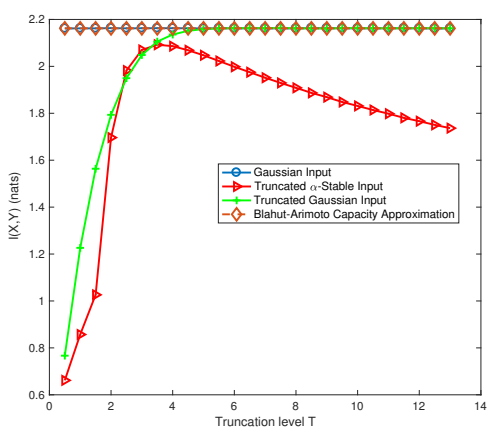

Fig. 1. Comparison of achievable rates using a truncated symmetric α-stable input ($\alpha = 1.4$, $\mathbb{E}[X_T^2] = 3$), a Gaussian input and a truncated Gaussian Input ($\mathbb{E}[X_G^2] = \mathbb{E}[X_{G,T}^2] = 3$) in the presence of symmetric α-stable noise ($\alpha = 1.4$, $\sigma_N = 0.1$).

We observe in Fig. 1 that the Gaussian input outperforms both the truncated Gaussian and truncated symmetric α-stable inputs. Similarly, for most choices of the truncation level, the truncated Gaussian input also outperforms the truncated symmetric α-stable input. Moreover, the truncation level rapidly has no effect on the achievable rate for the truncated Gaussian input. Importantly, the Gaussian input also yields a rate that is very close to a numerical approximation of the capacity obtained via the Blahut-Arimoto algorithm [7, 8].

We remark that based on extensive numerical experiments, we have observed that these trends hold for a wide range of channel parameters. This suggests that as in the Gaussian noise channel, a Gaussian input is a reasonable choice for the symmetric α-stable noise channel.

IV. Power Control Strategy

In this section, we develop a power control strategy for Gaussian inputs in parallel $AS\alpha SN$ channels. The lack of analytical expression for capacity or even achievable rates in the case of power constrained input makes the problem complex. Motivated by the results obtained in Section III, we propose to view the Gaussian inputs as approximations of symmetric α-stable inputs. This is possible since both of these inputs lie in the α-stable family and a scale parameter matching is used to approximate the Gaussian distribution via an α-stable distributions. This justifies the use of (5) to approximate the sum-rate with Gaussian inputs and a simple analytical form for the power optimisation problem. We verify the performance of our strategy via numerical simulation.

A. Scale Parameter Matching

Due to the fact that the input is Gaussian, no tractable closed-form expressions are known for the rate over α-stable noise channels. As a consequence it is highly desirable to approximate the Gaussian statistics by another α-stable random variable so that the sum-rate in (5) can be exploited. In order to do this, we propose a scale parameter matching method which selects the power of the Gaussian input by optimizing the scale parameter of another α-stable input.

Any zero-mean Gaussian random variable $X \sim \mathcal{N}(0, \mathrm{Var}(X))$ is also a stable random variable $\mathcal{S}_\alpha(\sigma_X, 0, 0)$, where the link between the scale parameter of the stable distribution σ_X and the variance of the Gaussian notation is given via

$$\sigma_X^2 = \frac{1}{2}\mathrm{Var}(X). \tag{10}$$

The idea of scale parameter matching is to approximate the Gaussian input distribution ($X \sim \mathcal{N}(0, 2\sigma_Z^2)$) by a symmetric α-stable random variable ($Z \sim \mathcal{S}_\alpha(\sigma_Z, 0, 0)$).

For our power control problem, this scale parameter matching method allows for the approximation of the sum-rate with Gaussian inputs by the sum-rate with symmetric α-stable inputs. As the rate of symmetric α-stable inputs for the additive symmetric α-stable noise channel is known, a tractable power control problem can be formulated as detailed in Section IV-B.

To justify the scale parameter matching approximation, it is necessary to establish how close the rate with Gaussian inputs

is to the rate with symmetric α-stable inputs. To this end, we have the following bound.

Theorem 1. *Let* $X_G \sim \mathcal{N}(0, 2\sigma_X^2)$, $X_S \sim \mathcal{S}_\alpha(\sigma_X, 0, 0)$, $Y_G = X_G + N$ *and* $Y_S = X_S + N$. *Then,*

$$\begin{aligned}
&|I(X_G;Y_G) - I(X_S;Y_S)| \\
&\leq \max\left\{\left|h_{\min} - \frac{1}{\alpha}\log\left(1+\left(\frac{\sqrt{P}}{\sqrt{2}\sigma_N}\right)^\alpha\right)\right|,\right. \\
&\qquad \left.\left|\frac{\log(2eb)}{\log 2} - h(N) - \frac{1}{\alpha}\log\left(1+\left(\frac{\sqrt{P}}{\sqrt{2}\sigma_N}\right)^\alpha\right)\right|\right\} \\
&=: E_B(P), \qquad (11)
\end{aligned}$$

where $b = \sqrt{\frac{2P}{\pi}} + \frac{2}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)\sigma_N$, $h_{\min} = \max\{h(X_G) - h(N), 0\}$ *and* $h(X_G) = \frac{1}{2}\log(2\pi e P)$.

Proof. Omitted due to space constraints. □

An observation from (11) is that as $P \to \infty$, $E_B(P)$ tends to a constant. This provides evidence that the scale matching approach gives consistent results.

In order to gain further insights into the behavior of the approximation, Fig. 2 plots both the actual relative error and the relative errors using the bound in (11) for varying power levels P. A key observation is that the relative error in both cases is approximately constant. This implies that the bound and actual relative error have the same qualitative behavior. Moreover, despite the fact that the relative error is greater than 10%, we note that for a wide range of values of P

$$\frac{|I(X_G;Y_G) - I(X_S;Y_S)|}{\frac{1}{\alpha}\log\left(1+\left(\frac{\sqrt{P}}{\sqrt{2}\sigma_N}\right)^\alpha\right)} \approx \kappa, \qquad (12)$$

where κ is a non-negative constant. It then follows that

$$I(X_G;Y_G) \approx (\kappa+1)\frac{1}{\alpha}\log\left(1+\left(\frac{\sqrt{P}}{\sqrt{2}\sigma_N}\right)^\alpha\right). \qquad (13)$$

For the purpose of optimizing P, the factor of $(1+\kappa)$ does not affect the solution. This provides a justification of the scale parameter matching method, which is further validated by numerical results in Section IV.

B. Optimization Problem Formulation

The scale parameter matching method provides a means of approximating the rate of a symmetric α-stable noise channel with a Gaussian input by a symmetric α-stable noise channel with an α-stable input. This method yields a power control optimization problem given by

$$\begin{aligned}
\underset{\sigma_{X,k},\ k=1,2,\ldots,n}{\text{maximize}} \quad & \sum_{k=1}^{n}\frac{1}{\alpha}\log\left(1+|h_k|^\alpha\frac{\sigma_{X,k}^\alpha}{\sigma_{N,k}^\alpha}\right) \qquad (14)\\
\text{subject to} \quad & \sum_{k=1}^{n} 2\sigma_{X,k}^2 \leq P_{\max} \\
& \sigma_{X,k} \geq 0,\ k=1,2,\ldots,n.
\end{aligned}$$

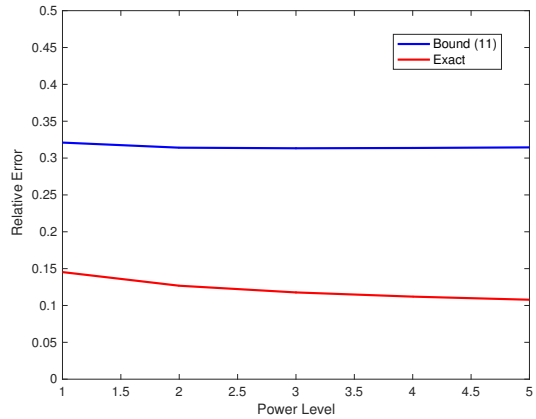

Fig. 2. Relative error of bound for $\alpha = 1.5$ and $\sigma_N = 0.1$.

In particular, the parameter $\sigma_{X,k}$ is the parameter for a symmetric α-stable input. Using the scale parameter matching method, the Gaussian inputs are assumed to have the same parameters $\sigma_{X,k}$ and as such, the power levels of the inputs are obtained via (10). Note that this relationship is consistent with the fact that the constraint in (14) is in fact a sum power constraint for Gaussian inputs.

To solve (14), we apply the transformation $\rho_k = \sigma_{X,k}^2$, which yields the problem

$$\begin{aligned}
\underset{\rho_k,\ k=1,2,\ldots,n}{\text{maximize}} \quad & \sum_{k=1}^{n}\frac{1}{\alpha}\log\left(1+|h_k|^\alpha\frac{\rho_k^{\alpha/2}}{\sigma_{N,k}^\alpha}\right) \qquad (15)\\
\text{subject to} \quad & \sum_{k=1}^{n} 2\rho_k \leq P_{\max} \\
& \rho_k \geq 0,\ k=1,2,\ldots,n.
\end{aligned}$$

We observe that the problem in (15) is convex, which follows from the fact that the function $\rho_k^{\alpha/2}$ is concave for $0 < \alpha < 2$ and the linearity of the constraints or from the computation of the Hessian for the objective in (14).

Observe that unlike the Gaussian noise model, the system of equations arising from the KKT conditions cannot be reduced to simple function parameterized by λ, able to be found using the bisection method. That is, either the non-linear system of equations must be solved numerically or a general purpose convex optimization solver must be applied to (14).

C. Numerical Results

We compare the proposed approach with the waterfilling algorithm designed for Gaussian noise in the case of two parallel channels. In applying the waterfilling algorithm, we assume that the system does not know the noise is non-Gaussian. As such, the variance of the noise is estimated by observing $N_S = 5 \cdot 10^6$ samples and applying the estimator

$$\hat{\sigma}_{G,k}^2 = \frac{1}{N_S - 1}\sum_{i=1}^{N_S} n_{i,k}^2,\ k=1,2, \qquad (16)$$

where $n_{i,k}$ is the i-th noise sample on the k-th channel. Note that since the variance of α-stable noise is infinite, it follows that the variance estimate in (16) does not converge. Nevertheless, (16) provides a means of systematically choosing the noise variance parameter required for the waterfilling algorithm, corresponding to the behavior of a system that does not know the noise is non-Gaussian.

In order to provide a fair comparison with power allocation based on our proposed method, the exponent α and the dispersion are also estimated based on $N_S = 5 \cdot 10^6$ samples, using the characteristic function method [12]. This is to ensure that noise parameters are estimated rather than assumed known.

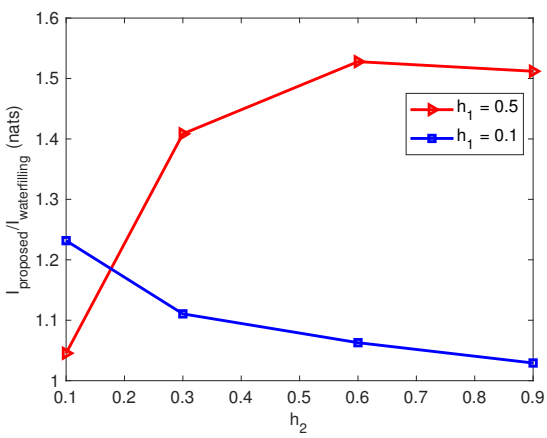

Fig. 3. Throughput gain, where $\mathbf{h} = [h_1 \ h_2]$ for $h_1 \in \{0.1, 0.5\}$

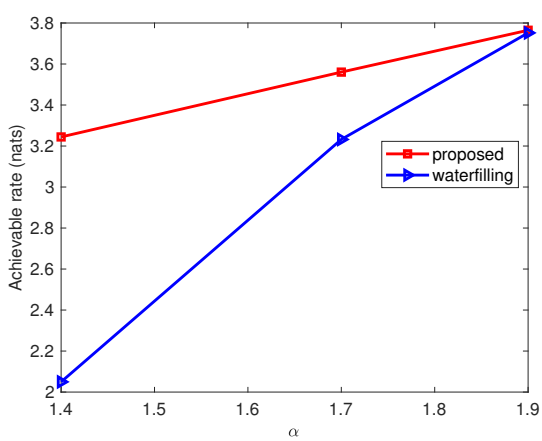

Fig. 4. Achievable rates varying α and $\mathbf{h} = [0.9 \ 0.7]$

In the experiments, the scale parameter of the symmetric α-stable noise is $\sigma_{N,k} = 0.1, \ k = 1, 2$ and $5 \cdot 10^6$ Gaussian input samples are generated. Fig. 3 shows the estimated achievable rate for each choice of α, channel $\mathbf{h}$ and power allocation method. The rates are estimated for several choices of the channel vector $\mathbf{h}$ using the same procedure as for Fig. 1 with 50 Monte Carlo iterations. Observe that our proposed strategy implemented in CVX [13] outperforms the waterfilling algorithm for each choice of parameters. In particular, for $\alpha = 1.4$ and $h_1 \in \{0.1, 0.5\}$ and h_2 varying between 0.1 and 0.9 an increase by up to a factor of 1.5 is achieved.

Fig. 4 plots the achievable rates for varying α and a fixed channel $\mathbf{h} = [0.9 \ 0.7]$. Observe that the proposed strategy outperforms waterfilling in all cases. However for $\alpha \approx 2$, there is negligible difference between the two approaches and our proposal closely matches the optimal waterfilling.

V. Conclusion

We have considered the problem of power control for parallel symmetric α-stable noise channels. We have shown that Gaussian inputs are a good choice, consistent with the Gaussian noise case. We then developed a new power control strategy for Gaussian inputs tailored to symmetric α-stable noise. This strategy significantly outperforms the rate achieved when the impulsive nature of the noise is ignored. A natural extension of our model is to MIMO systems, which is a target for future work.

Acknowledgements

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