Lattice coding for the Gaussian one- and two-way relay channels with correlated noises

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Abstract—This paper investigates two classes of relay channels, the Gaussian relay channel and the Gaussian two-way relay channel, when additive noises at the relay and destination(s) are correlated. Lattice codes are used to achieve the rate region for Compress-and-Forward (relay channel) and Compress/Decode-and-Forward (two-way relay channel). Numerical calculations show that there exist particular values of the correlation coefficient such that the gap between the Cut-Set Bound (CSB) and the proposed schemes is minimal.

I. Introduction

Lattice coding is interesting for AWGN channels as it achieves capacity with a low decoding complexity [1]. It is also a powerful tool for multi-user AWGN channels, like the relay channel, the two-way relay channel, the multi-way relay channel etc. In this paper we focus on the Gaussian relay channel and two-way relay channel, when noises at the relay and the destination(s) are correlated. The goal is to show that lattices can be used for this two setups by proving first the achievability of the rate region for the Compress-and-Forward (CF) for the Gaussian relay channel (proposition 5 in [2]) and then studying a Compress/Decode-and-Forward (CDF) protocol for the Gaussian two-way relay channel. For the latter, we assume that the relay is close to one of the users and far away from the other, thus neither the Decode-and-Forward (DF) protocol nor the CF protocol alone are efficient, but we can combine the two to build a more suitable protocol.

The rest of the paper is organized as follows: in Section II we briefly present lattice coding. In Section III we prove proposition 5 of [2] (achievable rate region using CF for the Gaussian relay channel with correlated noise) using nested lattices. We then use this result in Section IV to obtain the achievable rate-region of the CDF protocol for the Gaussian two-way relay channel with correlated noises.

II. Lattice coding

A *lattice* $\Lambda \subset \mathbb{R}^n$ is a discrete additive subgroup of $\mathbb{R}^n$. If $\lambda_1, \lambda_2 \in \Lambda$ then $\lambda_1 + \lambda_2 \in \Lambda$ and $\lambda_1 - \lambda_2 \in \Lambda$.

The *lattice quantizer* Q_Λ maps any point $x \in \mathbb{R}^n$ to the closest lattice point:

$$Q_\Lambda(x) = \arg\min_{\lambda \in \Lambda} ||x - \lambda||.$$

The *fundamental Voronoi region* $\mathcal{V}$ of Λ is the set of points that are closer to the origin than to any other lattice point:

$$\mathcal{V} = \{x \in \mathbb{R}^n | Q_\Lambda(x) = 0\}.$$

The *modulo Λ operation* yields the quantization error:

$$x \bmod \Lambda = x - Q_\Lambda(x).$$

This operation always returns a point within the fundamental Voronoi region of Λ.

The *second moment per dimension* $\sigma^2(\Lambda)$ defines the average power of the lattice Λ: $\sigma^2(\Lambda) = \frac{1}{nV} \int_{\mathcal{V}} ||x||^2 dx$ where V is the volume of $\mathcal{V}$, the fundamental Voronoi region of Λ.

The *covering radius* r_{cov} is the radius of the smallest sphere that covers $\mathcal{V}$, the *effective radius* r_{eff} is the radius of a sphere with the same volume as $\mathcal{V}$.

A lattice Λ is said *Rogers-good* if $\lim_{n \to \infty} \frac{r_{cov}}{r_{eff}} = 1$.

A lattice Λ is said *Poltyrev-good* (good for AWGN coding) if, for $z \sim \mathcal{N}(0, \sigma I)$, $Pr(z \notin \mathcal{V}) \leq e^{-nE_p(\mu)}$ where E_p is the Poltyrev exponent and μ is the volume-to-noise ratio defined as $\mu = \frac{V^{2/n}}{2\pi e \sigma}$.

For the single-user AWGN channel, where the average transmit power is P and the average noise power is N, capacity can be achieved using nested lattices and random dithers. To construct good lattice codebook, we intersect a coding lattice Λ_c (of volume V_c) with the fundamental Voronoi region $\mathcal{V}$ of a lattice Λ (of volume V). In order to satisfy the power constraint, we set $\sigma^2(\Lambda) = P$.

Λ is called the coarse lattice and Λ_c the fine lattice ($\Lambda \subseteq \Lambda_c$). Those lattices are chosen such that Λ_c is Poltyrev-good and Λ is both Rogers- and Poltyrev-good. The codebook is given by $\mathcal{C} = \Lambda_c \cap \mathcal{V}$. The rate of this codebook is $R = \frac{1}{n} \log_2 \frac{V}{V_c}$.

To transmit the codeword $c \in \mathcal{C}$, the source transmits

$$X = [c + u] \bmod \Lambda,$$

where u is a random dither uniformly distributed over $\mathcal{V}$ and is known to the source and the destination.

The destination receives $Y = X + Z$, where $Z \sim \mathcal{N}(0, N)$. To decode c, the destination scales its received signal by the MMSE coefficient $\beta = \frac{P}{P+N}$, subtracts the dither and takes the result modulo Λ.

$$\tilde{Y} = [\beta(c + Z) + (\beta - 1)u] \bmod \Lambda$$

The destination estimates c by quantization: $\hat{c} = Q_{\Lambda_c}(\tilde{Y})$.

Erez and Zamir showed [1] that averaging over the dither, perfect decoding is possible if $R \leq C\left(\frac{P}{N}\right)$, where $C(x) = \frac{1}{2}\log_2(1+x)$.

The notation $\bar{x} = 1 - x$ is used throughout the paper.

III. CF USING LATTICES FOR THE GAUSSIAN RELAY CHANNEL WITH CORRELATED NOISES

The relay channel, depicted on Fig. 1, is a major building block for wireless communications. The source wishes to send its message to a destination with the help of a relay. In the AWGN case, the source sends X_1 of power P_1 and the relay X_R of power P_R. The received signals are given by

$$Y_R = g_{r1}X_1 + Z_R,$$
$$Y_2 = g_{21}X_1 + g_{2r}X_R + Z_2,$$

where Z_2 and Z_R are Gaussian noises of variance N_2 and N_R, respectively. We also assume full duplex nodes (a node can receive and transmit at the same time) and correlated noises. The correlation coefficient is defined as $\rho_z = \frac{E[Z_2 Z_R]}{\sqrt{N_2 N_R}}$. The relay channel with correlated noises is a generalization that occurs when for instance a common interference signal contributes to the noises at both receivers.
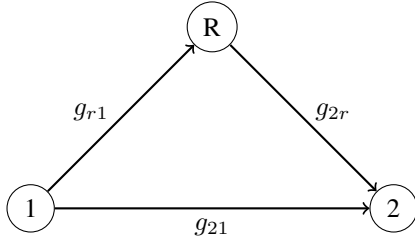

Fig. 1.   Relay channel

Proposition 1 *Using CF based on lattice coding on the Gaussian relay channel with correlated noises, the following rate region is achievable:*

$$R_{CF} \leq C\left(\frac{P_1}{N_2} \frac{g_{21}^2(N_R+D)+g_{r1}^2 N_2 - 2g_{21}g_{r1}\rho_z\sqrt{N_2 N_R}}{N_R(1-\rho_z^2)+D}\right)$$

with

$$D = \frac{(g_{r1}^2 N_2 + g_{21}^2 N_R)P_1 + N_2 N_R(1-\rho_z^2) - 2g_{r1}g_{21}P_1\rho_z\sqrt{N_2 N_R}}{g_{2r}^2 P_R}.$$

Remark: Note that this rate region is the same as obtained theoretically in Proposition 5 of [2].

Proof: The encoding and decoding procedure is based on block Markov coding and follows the ideas of [3].

A. Encoding

1) Source: The codebook for the source is given by $c_1 \in \mathcal{C}_1 = \Lambda_{c1} \cap \mathcal{V}_1$ where $\Lambda_1 \subset \Lambda_{c1}$ and Λ_1 is both Rogers- and Poltyrev-good and Λ_{c1} is Poltyrev-good. To ensure the power constraints, we choose $\sigma^2(\Lambda_1) = P_1$ and Λ_{c1} such that $|\mathcal{C}_1| = R_{CF}$.

During block b, the source sends $X_1(b) = [c_1(b) + u_1] \bmod \Lambda_1$.

2) Relay : The quantization codebook is given by $c_q \in \mathcal{C}_q = \{\Lambda_{cQ} \cap \mathcal{V}_Q\}$ where $\Lambda_Q \subseteq \Lambda_{cQ}$ and Λ_{cQ} is Rogers-good and Λ_Q is Poltyrev-good. We choose $\sigma^2(\Lambda_{cQ}) = D$ and

$$\sigma^2(\Lambda_Q) = g_{r1}^2 P_1 + N_R + D - \frac{(g_{21}g_{r1}P_1 + \rho_z\sqrt{N_2 N_R})^2}{g_{21}^2 P_1 + N_2}.$$

Thus, $R_q = \frac{1}{2}\log_2\left(\frac{\sigma^2(\Lambda_Q)}{D}\right)$.

The codebook for the relay is given by $c_R \in \mathcal{C}_R = \{\Lambda_{cR} \cap \mathcal{V}_R\}$ where $\Lambda_R \subseteq \Lambda_{cR}$ and Λ_R is both Rogers- and Poltyrev-good and Λ_{cR} is Poltyrev-good. To ensure the power constraints, we choose $\sigma^2(\Lambda_R) = P_R$. Each compression index i is mapped to one codeword c_R, that is Λ_R is chosen s.t. $|\mathcal{C}_R| = R_q$.

During block b, the relay sends

$$X_R(b) = [c_R(i(b-1)) + u_R] \bmod \Lambda_R.$$

B. Decoding

1) At the relay: During block b, the relay receives $Y_R(b) = g_{r1}X_1(b) + Z_R(b)$ and quantizes it to

$$I(b) = [Q_{cQ}(g_{r1}X_1(b) + Z_R(b) + u_{cQ})] \bmod \Lambda_Q$$
$$= [g_{r1}X_1(b) + Z_R(b) + u_{cQ} - E_{cQ}(b)] \bmod \Lambda_Q,$$

where E_{cQ} is the quantization error.

2) Decoding at the destination: During block b, the destination receives

$$Y_2(b) = g_{21}X_1(b) + g_{2r}[c_R(i(b-1)) + u_R] \bmod \Lambda_R + Z_2(b).$$

It starts by decoding the quantization index, considering the source signal as noise, which is possible if

$$R_q \leq C\left(\frac{g_{2r}^2 P_R}{g_{21}^2 P_1 + N_2}\right).$$

Then, it forms $\tilde{Y}_2(b) = g_{21}X_1(b) + Z_2(b)$.

The decoding of $X_1(b-1)$ is performed using Wyner-Ziv coding. During the previous block, the destination formed $\tilde{Y}_2(b-1)$ which is used in block b as side information to estimate $\hat{Y}_R(b-1)$:

$$\hat{Y}_R(b-1) = \left[g_{r1}X_1(b-1) + Z_R(b-1) - E_{cQ}(b-1)\right.$$
$$\left. -\beta\left(g_{21}X_1(b-1) + Z_2(b-1)\right)\right] \bmod \Lambda_Q$$
$$+\beta\left(g_{21}X_1(b-1) + Z_2(b-1)\right)$$
$$= g_{r1}X_1(b-1) + Z_R(b-1) - E_{cQ}(b-1).$$

The last equation is valid under perfect decoding, implying

$$\sigma^2(\Lambda) = (g_{r1} - \beta g_{21})^2 P_1 + N_R + D + \beta^2 N_2 - 2\beta\rho_z\sqrt{N_2 N_R}.$$

β is chosen as $\beta = \frac{g_{21}g_{r1}P_1 + \rho_z\sqrt{N_2 N_R}}{g_{21}^2 P_1 + N_2}$ to ensure that $g_{21}X_1(b-1) + Z_2(b-1)$ is orthogonal to $(g_{r1} - \beta g_{21})X_1(b-1) + Z_R(b-1) - \beta Z_2(b-1)$.

Thus, $\sigma^2(\Lambda) = g_{r1}^2 P_1 + N_R + D - \frac{(g_{21}g_{r1}P_1+\rho_z\sqrt{N_2 N_R})^2}{g_{21}^2 P_1 + N_2}$. Combining this with the quantization rate constraint, the distortion is

$$D = \frac{(g_{r1}^2 N_2 + g_{21}^2 N_R)P_1 + N_2 N_R(1-\rho_z^2) - 2g_{r1}g_{21}P_1\rho_z\sqrt{N_2 N_R}}{g_{2r}^2 P_R}.$$

In order to recover $X_1(b{-}1)$, the receiver combines $\hat{Y}_R(b{-}1)$ and $\tilde{Y}_2(b{-}1)$ as

$$\left(\frac{g_{r1}\sqrt{P_1}}{N_R+D} - \frac{g_{21}\sqrt{P_1}\rho_z\sqrt{N_2 N_R}}{N_2(N_R+D)}\right)\hat{Y}_R(b{-}1)$$
$$+\left(\frac{g_{21}\sqrt{P_1}}{N_2} - \frac{g_{r1}\sqrt{P_1}\rho_z\sqrt{N_2 N_R}}{N_2(N_R+D)}\right)\tilde{Y}_2(b{-}1)$$
$$= X_1(b{-}1)\left(\frac{\left(g_{21}^2(N_R+D)+g_{r1}^2 N_2 - 2g_{21}g_{r1}\rho_z\sqrt{N_2 N_R}\right)\sqrt{P_1}}{N(N_1+D)}\right)$$
$$+ Z_2(b{-}1)\left(\frac{\left(g_{21}(N_R+D)-g_{r1}\rho_z\sqrt{N_2 N_R}\right)\sqrt{P_1}}{N(N_1+D)}\right)$$
$$+ (Z_R(b{-}1)-E_{cQ}(b{-}1))\left(\frac{\left(g_{r1}N_2 - g_{21}\rho_z\sqrt{N_2 N_R}\right)\sqrt{P_1}}{N(N_1+D)}\right).$$

Thus, decoding succeeds if

$$R_{CF} \le C\left(\frac{P_1}{N_2}\frac{g_{21}^2(N_R+D)+g_{r1}^2 N_2 - 2g_{21}g_{r1}\rho_z\sqrt{N_2 N_R}}{N_R(1-\rho_z^2)+D}\right). \quad\blacksquare$$

In the next section, we combine this lattice-based scheme with the one proposed in [4] for Decode-and-Forward for the Gaussian relay channel to propose a Compress/Decode-and-Forward scheme using lattices for the Gaussian two-way relay channel with correlated noises.

IV. CDF FOR THE GAUSSIAN TWO-WAY RELAY CHANNEL, WITH CORRELATED NOISES, USING LATTICES

The two-way relay channel is a natural extension of the relay channel, in which two users wish to exchange their messages with the help of one relay. This channel is depicted on Fig. 2. In this paper we only consider *restricted* encoders, so that the signal sent by each user only depends on its own message and not on previously decoded ones. In the Gaussian case, user 1 sends X_1 of power P_1, user 2 X_2 of power P_2 and the relay X_R of power P_R. The received signals are

$$Y_R = g_{r1}X_1 + g_{r2}X_2 + Z_R,$$
$$Y_1 = g_{12}X_2 + g_{1r}X_R + Z_1,$$
$$Y_2 = g_{21}X_1 + g_{2r}X_R + Z_2,$$

where Z_1, Z_2 and Z_R are Gaussian noises of variance N_1, N_2 and N_R respectively. Again, we assume full-duplex nodes and correlation between the additive noises at the relay and destinations:

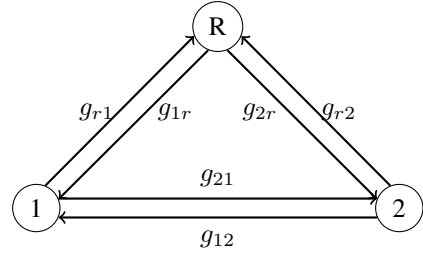$$\rho_{z1} = \frac{E[Z_1 Z_R]}{\sqrt{N_1 N_R}} \text{ and } \rho_{z2} = \frac{E[Z_2 Z_R]}{\sqrt{N_2 N_R}}.$$

Fig. 2.   Two-way relay channel

Proposition 2 *The cut-set bound (CSB) for the two-way relay channel with correlated noise is given by the convex closure of the cut-set region:*

$$\bigcup_{\substack{0\le\rho_1,\rho_2\le 1 \\ 0\le\rho_{z1},\rho_{z2}\le 1}} (R_1, R_2):$$
$$R_1 \le \min\Big\{C\Big(\frac{P_1(1-\rho_1^2-\rho_2^2)(g_{21}^2 N_R + g_{r1}^2 N_2 - 2g_{21}g_{r1}\sqrt{N_2 N_R}\rho_{z2})}{(1-\rho_2^2)(1-\rho_{z2}^2)N_2 N_R}\Big),$$
$$C\Big(\frac{g_{2r}^2 P_R(1-\rho_2^2)+g_{21}^2 P_1 + 2g_{21}g_{2r}\rho_1\sqrt{P_1 P_R}}{N_2}\Big)\Big\}$$
$$R_2 \le \min\Big\{C\Big(\frac{P_2(1-\rho_2^2-\rho_1^2)(g_{12}^2 N_R + g_{r2}^2 N_1 - 2g_{12}g_{r2}\sqrt{N_1 N_R}\rho_{z1})}{(1-\rho_1^2)(1-\rho_{z1}^2)N_1 N_R}\Big),$$
$$C\Big(\frac{g_{1r}^2 P_R(1-\rho_1^2)+g_{12}^2 P_2 + 2g_{12}g_{1r}\rho_2\sqrt{P_2 P_R}}{N_1}\Big)\Big\}.$$

Proof: The cut-set region is given by

$$\begin{cases} R_1 \le \min[I(X_1;Y_R,Y_2|X_R,X_2), I(X_1,X_R;Y_2|X_2)] \\ R_2 \le \min[I(X_2;Y_R,Y_1|X_R,X_1), I(X_2,X_R;Y_1|X_1)] \end{cases}.$$

The proof extensively uses the fact that the mean-squared error of the linear MMSE estimate of Y given X is greater than or equal to the conditional variance $\text{Var}(Y|X)$. We introduced two correlation coefficients :

$$\rho_1 = \frac{E[X_1 X_R]}{\sqrt{P_1 P_R}} \text{ and } \rho_2 = \frac{E[X_2 X_R]}{\sqrt{P_2 P_R}}.$$

The detailed proof is omitted for space reasons. $\blacksquare$

In this part, we assume that the relay is very close to user 1 (and hence far from user 2), such that it can only decode the message from user 1 but not the one of user 2. Instead of only decoding the message from user 1, the relay will also use a part of its power to send a compressed version of the message of user 2.

Proposition 3 *Using Compress/Decode-and-Forward (CDF) on the Gaussian two-relay channel with correlated noises and lattices, the following rate region is achievable:*

$$\bigcup_{\substack{0\le\alpha\le 1 \\ 0<\gamma<1}} (R_1, R_2):$$
$$R_1 \le \min\Big\{C\Big(\frac{g_{r1}^2\bar{\alpha}P_1}{g_{r2}^2 P_2 + N_R}\Big),$$
$$\frac{1}{2}\log_2\Big(\frac{g_{21}^2 P_1 + g_{2r}^2 P_R + N_2 + 2g_{21}g_{2r}\sqrt{\alpha\bar{\gamma}P_1 P_R}}{g_{2r}^2\gamma P_R + N_2}\Big)\Big\}$$

$$R_2 \leq C\left(\frac{P_2}{N_1}\frac{g_{12}^2(N_R+D)+g_{r2}^2N_1-2g_{12}g_{r2}\rho_{z1}\sqrt{N_1N_R}}{N_R(1-\rho_{z1}^2)+D}\right)$$

$$D = \frac{(g_{r2}^2N_1+g_{12}^2N_R)P_2+N_1N_R(1-\rho_{z1}^2)}{g_{1r}^2\gamma P_R}$$
$$-\frac{2g_{r2}g_{12}P_2\rho_{z1}\sqrt{N_1N_R}}{g_{1r}^2\gamma P_R}.$$

At user 1, α allows to trade off power between repeating the message from the previous block and sending a new message. γ controls the power trade off at the relay between the decoded and the compressed part.

Proof: The encoding and decoding procedure is based on block Markov coding.

A. Encoding

1) User 1: For user 1, we use a doubly nested lattice coding scheme as proposed in [4] with $\Lambda_s \subseteq \Lambda_m \subseteq \Lambda_{c1}$. An example of doubly nested lattices is depicted on Fig. 3. Λ_s is, as in the standard nested lattice coding scheme, the shapping lattice that insures the power constraint and Λ_{c1} is the coding lattice. Λ_m is a meso lattice that groups codewords into clusters. Using these three lattices, we build the following three codebooks:

$$\mathcal{C}_1 = \Lambda_{c1}\cap\mathcal{V}_s \text{ of rate } R_1,$$
$$\mathcal{C}_{10} = \Lambda_{c1}\cap\mathcal{V}_m \text{ of rate } R_{10},$$
$$\mathcal{C}_{11} = \Lambda_m\cap\mathcal{V}_s \text{ of rate } R_{11},$$
$$(R_1 = R_{10}+R_{11}),$$

where Λ_s and Λ_m are both Rogers- and Poltyrev-good and Λ_{c1} is Poltyrev-good. We set $\sigma^2(\Lambda_s)=1$.

A codeword $c_1 \in \mathcal{C}_1$ can be written as

$$c_1 = [c_{10}+c_{11}] \bmod \Lambda_s, \text{ where}$$

$c_{10} = c_1 \bmod \Lambda_m \in \mathcal{C}_{10}$ and $c_{11} = [c_1-c_{10}] \bmod \Lambda_s \in \mathcal{C}_{11}$.

To simplify the notation, we scale $\mathcal{C}_{10}$ to a unit power $\mathcal{C}_{10}^* = \Lambda_c^* \cap \mathcal{V}_m^*$ where $\sigma^2(\Lambda_m^*)=1$.
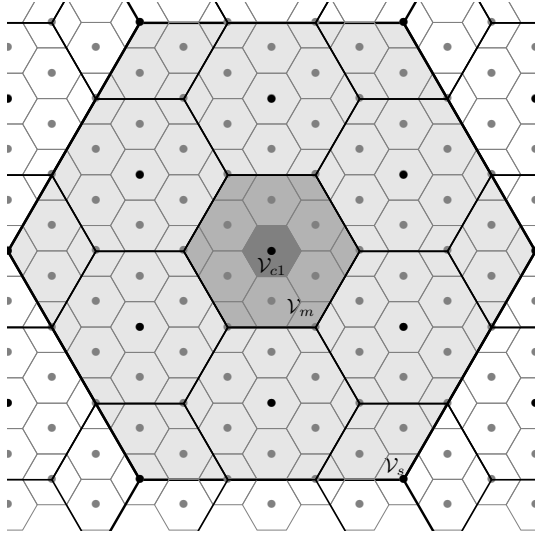

Fig. 3.   Doubly-nested lattices used for user 1

During block b, user 1 sends

$$X_1(b) = \sqrt{\alpha P_1}[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$$
$$+\sqrt{\bar{\alpha}P_1}[c_1(b)+u_s] \bmod \Lambda_s.$$

2) User 2 : The codebook for the user 2 is given by $c_2 \in \mathcal{C}_2 = \Lambda_{c2}\cap\mathcal{V}_2$ where $\Lambda_2 \subset \Lambda_{c2}$ and Λ_2 is both Rogers- and Poltyrev-good and Λ_{c2} is Poltyrev-good. To ensure the power constraint, we choose $\sigma^2(\Lambda_2) = P_2$ and Λ_{c2} such that $|\mathcal{C}_2| = R_2$.

During block b, user 2 sends $X_2(b) = [c_2(b)+u_2] \bmod \Lambda_2$.

3) Relay : The quantization codebook is given by $c_q \in \mathcal{C}_q = \{\Lambda_{cQ}\cap\mathcal{V}_Q\}$ where $\Lambda_Q \subseteq \Lambda_{cQ}$ and Λ_{cQ} is Rogers-good and Λ_Q is Poltyrev-good. We choose $\sigma^2(\Lambda_{cQ}) = D$, where D is given in Proposition 3 and

$$\sigma^2(\Lambda_Q) = g_{r2}^2P_2+N_R+D-\frac{(g_{12}g_{r2}P_2+\rho_{z1}N_1N_R)^2}{g_{12}^2P_2+N_1}.$$

Thus, $R_q = \frac{1}{2}\log_2\left(\frac{\sigma^2(\Lambda_Q)}{D}\right)$.

The codebook for the relay is given by $c_R \in \mathcal{C}_R = \{\Lambda_{cR}\cap\mathcal{V}_R\}$ where $\Lambda_R \subseteq \Lambda_{cR}$ and Λ_R is both Rogers- and Poltyrev-good and Λ_{cR} is Poltyrev-good. To ensure the power constraints, we choose $\sigma^2(\Lambda_R) = P_R$. Each compression index i is mapped to one codeword c_R, that is Λ_R is chosen s.t. $|\mathcal{C}_R| = R_q$.

During block b, the relay sends

$$X_R(b) = \sqrt{\bar{\gamma}P_R}[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$$
$$+\sqrt{\gamma}[c_R(i(b-1))+u_R] \bmod \Lambda_R$$

where i is the quantization index.

B. Decoding

1) Decoding at the relay: During block b, the relay receives

$$Y_R(b) = g_{r2}X_2(b)+g_{r1}(\sqrt{\alpha P_1}[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$$
$$+\sqrt{\bar{\alpha}P_1}[c_1(b)+u_s] \bmod \Lambda_s)+Z_R(b).$$

It first starts by removing $[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$ (the part of the message it has already decoded in the previous block) and decodes $c_1(b)$ which is possible if

$$R_1 \leq C\left(\frac{g_{r1}^2\bar{\alpha}P_1}{g_{r2}^2P_2+N_R}\right).$$

Thus it can form $\tilde{Y}_R(b) = g_{r2}X_2(b)+Z_R(b)$ and compress it to $I(b) = [Q_{cQ}(g_{r2}X_2(b)+Z_R(b)+u_{cQ})] \bmod \Lambda_Q$.

2) Decoding at user 1: At block b, user 1 receives

$$Y_1(b) = g_{12}X_2(b)+g_{1r}\sqrt{\bar{\gamma}P_R}[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$$
$$+g_{1r}\sqrt{\gamma}[c_R(i(b-1))+u_R] \bmod \Lambda_R+Z_1(b).$$

It subtracts $[c_{10}^*(b-1)+u_m^*] \bmod \Lambda_m^*$ and then the decoding of $X_2(b-1)$ is performed as in Section III and succeeds if

$$R_2 \leq C\left(\frac{P_2}{N_1}\frac{g_{12}^2(N_R+D)+g_{r2}^2N_1-2g_{12}g_{r2}\rho_{z1}\sqrt{N_1N_R}}{N_R(1-\rho_{z1}^2)+D}\right).$$

3) Decoding at user 2: At block b, user 2 receives

$$Y_2(b) = \left(g_{21}\sqrt{\alpha P_1} + g_{2r}\sqrt{\bar{\gamma}P_R}\right)[c_{10}^*(b-1) + u_m^*] \bmod \Lambda_m^*$$
$$+ g_{21}\sqrt{\bar{\alpha}P_1}[c_1(b) + u_s] \bmod \Lambda_s$$
$$+ g_{2r}\sqrt{\gamma}[c_R(i(b-1)) + u_R] \bmod \Lambda_R + Z_2(b).$$

The user won't decode the quantization index but instead consider it as noise. It starts by decoding $c_{10}^*(b-1)$ which is possible if

$$R_{10} \leq C\left(\frac{\alpha g_{21}^2 P_1 + g_{2r}^2 \bar{\gamma}P_R + 2g_{21}g_{2r}\sqrt{\alpha\bar{\gamma}P_1 P_R}}{g_{21}^2 \bar{\alpha}P_1 + g_{2r}^2 \gamma P_R + N_2}\right).$$

Then, it decodes $c_{11}(b-1)$ from the previous block which is possible if

$$R_{11} \leq C\left(\frac{g_{21}^2 \bar{\alpha}P}{g_{2r}^2 \gamma P_R + N_2}\right).$$

Thus, the decoding of $c_1(b-1)$ succeeds if

$$R_1 \leq \frac{1}{2}\log_2\left(\frac{g_{21}^2 P_1 + g_{2r}^2 P_R + N_2 + 2g_{21}g_{2r}\sqrt{\alpha\bar{\gamma}P_1 P_R}}{g_{2r}^2 \gamma P_R + N_2}\right). \blacksquare$$

We now compare the sum-rate achieved with CDF, DF and the direct links only.

Recall that, in case of correlated noise, the rate-region achieved with DF is the same as for the standard two-way relay channel, without correlated noises, since the relay decodes everything. The achievable rate region for this protocol can be found for example in [5].

For our numerical examples, we consider the following way to assign the channel gains: $g_{r1} = g_{1r} = d$ and $g_{r2} = g_{2r} = (1-d)$ ($0 \leq d \leq 1$).

One major result of [2] is that if $\rho_{z1} = \frac{g_{2r}}{g_{21}}$ for the Gaussian relay channel with $N_1 = N_R$, then CF can achieve the CSB.

For the two-way relay channel, numerical evaluations show that the gap between the CSB and CDF is also minimal for this value of ρ_{z1} (i.e. $\rho_{z1} = \frac{(1-d)}{g_{21}}$).

On Fig. 4 and Fig. 5, we represent the sum-rate as a function of ρ_{z1}. In both cases, the value $\rho_{z1} = \frac{(1-d)}{g_{21}}$ minimizes the gap between the CSB and CDF. In both cases, DF achieves low sum-rate since the relayed links are very asymmetric. We can note that on Fig. 4, using only the direct links achieves a higher sum-rate than using the relay, since the direct link is better than at least on of the relayed links, but on Fig. 5 we see that when the direct link is worse than both the relayed links, using the relay achieves higher sum-rate.

V. CONCLUSION

We have shown that lattice coding can achieve the CF rates for the Gaussian relay channel when noises at the relay and the destination are correlated. We then used this achievability scheme to propose a CDF protocol for the two-way relay channel when noises at the relay and the destinations are correlated. This protocol combines doubly-nested lattice coding at one user and standard lattice coding at the other. Numerical examples show that, given the channel gains, CDF
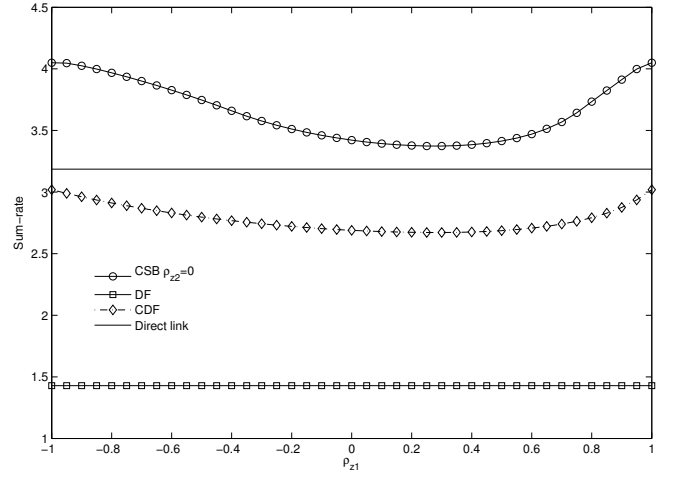

Fig. 4.   Sum-rate as a function of ρ_{z1}, $g_{21} = 0.9$, $P_1 = P_2 = P_R = 10$, $N_1 = N_2 = N_R = 1$ $d = 0.75$, $g_{r1} = d$, $g_{r2} = 1-d$, minimal gap between the cut-set bound and CDF is achieved for $\rho_{z1} = 0.27$.
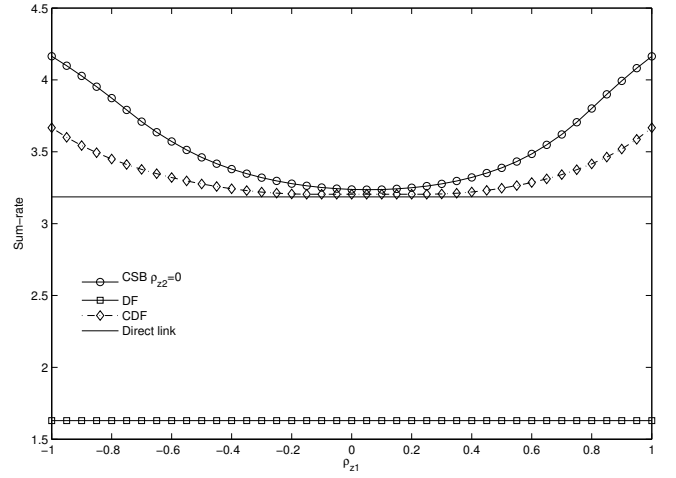

Fig. 5.   Sum-rate as a function of ρ_{z1}, $g_{21} = 0.9$, $P_1 = P_2 = P_R = 10$, $N_1 = N_2 = N_R = 1$ $d = 0.95$, $g_{r1} = d$, $g_{r2} = 1-d$, minimal gap between the cut-set bound and CDF is achieved for $\rho_{z1} = 0.05$.

can outperform a transmission only over the direct links or two-way DF. We also noted that a particular value of the correlation coefficient minimizes the gap between the CDF and the CSB.

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