



Equilibrium Problems for Vector Potentials with Semidefinite Interaction Matrices and Constrained Masses

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Abstract We prove existence and uniqueness of a solution to the problem of minimizing the logarithmic energy of vector potentials associated to a d -tuple of positive measures supported on closed subsets of the complex plane. The assumptions we make on the interaction matrix are weaker than the usual ones, and we also let the masses of the measures vary in a compact subset of \mathbb{R}_+^d . The solution is characterized in terms of variational inequalities. Finally, we review a few examples taken from the recent literature that are related to our results.

Keywords Weighted energy minimization problems · Vector potentials · External fields · Equilibrium conditions · Graph theory · Nikishin systems.

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1 Introduction

Vector equilibrium problems in logarithmic potential theory have been studied for a few decades and have been shown to be crucial in the investigation of many problems in approximation theory, like those involving multiple orthogonal polynomials (e.g., Hermite-Padé approximants, in particular Angelesco and Nikishin systems). This approach has been very fruitful in the analysis of numerous questions in numerical and applied mathematics, e.g., the eigenvalue distribution of Toeplitz matrices, models in random matrix theory, determinantal processes or nonintersecting random paths. Vector equilibrium problems were first considered in [14, 15]. The book [20] contains a nice introduction to the subject. Equilibrium problems on general locally compact spaces are studied in [21, 28, 29].

We first introduce some notation. Let μ be a (positive) Borel measure with closed support in \mathbb{C} , and set

$$U^\mu(z) := \int \log \frac{1}{|z-x|} d\mu(x) \quad (1.1)$$

for its logarithmic potential. Assume that μ has not too much mass at infinity (in a sense to be specified later), so that the above integral converges for $|z-x|$ large. Then, the logarithmic potential is a superharmonic function from \mathbb{C} to $(-\infty, \infty]$, and the energy of μ is defined as

$$I(\mu) := \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) = \int U^\mu(x) d\mu(x) > -\infty.$$

For a subset Σ of \mathbb{C} , let

$$\mathcal{M}(\Sigma) := \{\mu \text{ Borel measure, of finite mass, supported in } \Sigma, \text{ and } I(\mu) < \infty\}, \quad (1.2)$$

and

$$\mathcal{M}_t(\Sigma) := \{\mu \in \mathcal{M}(\Sigma), \|\mu\| = t\},$$

where $\|\mu\|$ denotes the total mass of the measure μ . For two measures $\mu, \nu \in \mathcal{M}(\Sigma)$, we define the so-called mutual energy

$$I(\mu, \nu) := \iint \log \frac{1}{|x-y|} d\mu(x) d\nu(y). \quad (1.3)$$

Again, if μ and ν do not have too much mass at infinity, this integral converges for $|x-y|$ large, and is well defined in $(-\infty, +\infty]$.

Throughout, we let

$$\Delta = (\Delta_1, \dots, \Delta_d), \quad \bigcup_{i=1}^d \Delta_i \subsetneq \mathbb{C}, \quad (1.4)$$

be a d -tuple of closed nonpolar sets of \mathbb{C} , i.e., of positive logarithmic capacities

$$\text{cap}(\Delta_i) > 0, \quad i = 1, \dots, d,$$

and we define the Cartesian products

$$\mathcal{M}^d(\Delta) := \mathcal{M}(\Delta_1) \times \cdots \times \mathcal{M}(\Delta_d), \quad \mathcal{M}_1^d(\Delta) := \mathcal{M}_1(\Delta_1) \times \cdots \times \mathcal{M}_1(\Delta_d).$$

Assume for the moment that the Δ_i , $i = 1, \dots, d$, are compact sets. For two d -tuples of measures

$$\mu = (\mu_1, \dots, \mu_d)^t \in \mathcal{M}^d(\Delta), \quad \nu = (\nu_1, \dots, \nu_d)^t \in \mathcal{M}^d(\Delta),$$

we define the mutual energy of μ and ν as

$$J(\mu, \nu) := \sum_{j=1}^d I(\mu_j, \nu_j),$$

which is finite. Actually, the compactness of the Δ_i implies that the mutual energy of two measures of finite energies is also finite.

Let $C = (c_{i,j})$ be a real symmetric positive definite matrix of order d such that

$$\forall(i, j), \quad \text{if } \Delta_i \cap \Delta_j \neq \emptyset \text{ then } c_{i,j} \geq 0. \quad (1.5)$$

The energy of μ with respect to the interaction matrix C is defined as

$$J(\mu) := J(C\mu, \mu) = \sum_{i,j=1}^d c_{i,j} I(\mu_i, \mu_j).$$

Note that because of (1.5), $J(\mu)$ is always well defined (even if some of the components of μ have infinite energies). Now, the extremal problem is the following:

find

$$J^* = \inf\{J(\mu), \mu \in \mathcal{M}_1^d(\Delta)\},$$

and characterize the extremal tuple of measures μ^ in $\mathcal{M}_1^d(\Delta)$ for which the infimum is attained.*

As the sets Δ_i are assumed to be of positive capacity, a solution μ^* to this problem, with $J^* = J(\mu^*) < \infty$, exists, and it is unique. The proof of existence is based on the fact that the mutual energy (1.3) is lower semi-continuous, which implies, together with (1.5), that $\mu \mapsto J(\mu)$ is also lower semi-continuous. Moreover, the map is strictly convex on the set $\mathcal{M}_1^d(\Delta)$, from which uniqueness follows, see [20, Propositions 5.4.1 and 5.4.2].

A characterization of the solution can be given via the so-called equilibrium conditions. For that, we introduce the partial potentials

$$U_i^\mu(x) = \sum_{j=1}^d c_{i,j} U^{\mu_j}(x), \quad i = 1, \dots, d,$$

where the scalar potentials $U^{\mu_j}(x)$ have been defined in (1.1). Then d -tuple of measures μ solves the minimization problem if and only if there exist constants w_i such that, for $i = 1, \dots, d$,

$$U_i^\mu(x) \geq w_i, \quad \text{quasi-everywhere on } \Delta_i, \quad (1.6)$$

$$U_i^\mu(x) \leq w_i, \quad \text{everywhere on } \text{supp}(\mu_i), \quad (1.7)$$

where quasi-everywhere means everywhere up to a set of capacity zero. Proofs of these results can be found in [20, Chap. 5].

Remark 1.1 For some $x \in \mathbb{C}$, it may happen that $U^{\mu_j}(x) = +\infty$ for several indices j . However, the partial potential U_i^μ is well defined quasi-everywhere since positive measures of finite mass and compact support have a finite potential quasi-everywhere, see [27, Theorem III.16].

Regarding applications, it is also very useful to consider an additional external field in equilibrium problems. The main reference for the study of equilibrium problems in the presence of an external field is the book [24].

Let $Q = (Q_j)_{j=1,\dots,d}$ be a vector of lower semi-continuous functions,

$$Q_j : \Delta_j \rightarrow (-\infty, \infty], \quad j = 1, \dots, d,$$

and define the weighted energy of a tuple of measures $\mu \in \mathcal{M}^d(\Delta)$ in the presence of the external field Q as

$$J_Q(\mu) := J(\mu) + 2 \sum_{j=1}^d \int Q_j d\mu_j. \quad (1.8)$$

For $\mu \in \mathcal{M}^d(\Delta)$, we have mentioned that $J(\mu) = J(C\mu, \mu)$ is finite. By lower-semicontinuity, each Q_j is bounded from below on Δ_j , $j = 1, \dots, d$. Hence, the integrals in (1.8) are well defined, and $J_Q(\mu) > -\infty$. It can also be checked that, in $\mathcal{M}_1^d(\Delta)$, there exists at least one measure μ with $J_Q(\mu) < \infty$, see the proof of Theorem 1.7(i).

Then, the extremal problem of minimizing the weighted energies

$$\{J_Q(\mu), \mu \in \mathcal{M}_1^d(\Delta)\}, \quad (1.9)$$

is solved by a unique d -tuple of measures $\mu^* \in \mathcal{M}_1^d(\Delta)$, with $J_Q(\mu^*) < \infty$, and it is characterized by the existence of constants w_i^Q , such that, for $i = 1, \dots, d$,

$$U_i^{\mu^*}(x) + Q_i(x) \geq w_i^Q, \quad \text{quasi-everywhere on } \Delta_i, \quad (1.10)$$

$$U_i^{\mu^*}(x) + Q_i(x) \leq w_i^Q, \quad \text{everywhere on } \text{supp}(\mu_i). \quad (1.11)$$

For a proof in the scalar case $d = 1$, we refer to [15] and [24, Theorem I.1.3]. The vector problem with external fields is considered in [15], see also [13].

In the past few years, generalizations of the above vector equilibrium problems have appeared repeatedly in the literature. By generalizations, we mean that the assumptions on the interaction matrix or on the masses were relaxed in various ways. For instance, [3, 4] allow for sets which no longer satisfy the compatibility condition (1.5), since some Δ_j are intervals with a common endpoint. In [1–3], interaction matrices are considered which are only positive semidefinite. In these papers, the authors also minimize J not over the set $\mathcal{M}_1^d(\Delta)$ of tuples of probability measures but over the set

$$\mathcal{M}_K^d(\Delta) = \{\mu = (\mu_1, \dots, \mu_d)^t \in \mathcal{M}^d(\Delta), \|\mu\| = (\|\mu_1\|, \dots, \|\mu_d\|)^t \in K\},$$

where K is a nonempty compact subset of the set \mathbb{R}_+^d of d -tuples of nonnegative real numbers. In addition, in [3, 4, 7, 9, 10, 12, 24, 26] extremal problems with not necessarily compact sets Δ_j are considered. In the papers [3, 9, 10, 12, 26], a solution satisfying the extremal properties (1.6)–(1.7) or (1.10)–(1.11) could be exhibited directly through some algebraic equation, hence settling the problem of existence of a minimizer.

The goal of this paper is to provide a more systematic approach, by showing existence, uniqueness, and characterization of the extremal solution for a large class of generalized equilibrium problems. At this point, the following simple examples are instructive, since they show that some care has to be taken when weakening the assumptions of the minimization problem.

Example 1.2 Consider the data

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \Delta_1 = [-1/2, 0], \quad \Delta_2 = [0, 1/2],$$

where C is positive semidefinite, and the problem of finding the minimum J^* of the corresponding energy

$$J(\mu) = I(\mu_1 - \mu_2) \geq 0, \quad \mu = (\mu_1, \mu_2)^t \in \mathcal{M}_1^2(\Delta).$$

It is known that the same problem on the pair of subsets $\Delta_{1,n} = [-1/2, -1/n]$ and $\Delta_{2,n} = [1/n, 1/2]$, $n \geq 1$, admits the minimal energy J_n^* with

$$J_n^* = \frac{1}{\text{cap}(\Delta_{1,n}, \Delta_{2,n})} = \frac{2\pi K(2/n)}{K'(2/n)},$$

where $\text{cap}(\Delta_{1,n}, \Delta_{2,n})$ denotes the capacity of the condenser with plates $\Delta_{1,n}$ and $\Delta_{2,n}$. The explicit value given in the second equality, in terms of the complete and complementary elliptic integrals of the first kind K and K' , can be found in [19], and may also be derived from Example II.5.14 in [24, pp. 133–134]. Since

$$K(k) = \frac{\pi}{2} + \mathcal{O}(k^2), \quad K'(k) = -\log k + \mathcal{O}(1), \quad \text{as } k \rightarrow 0,$$

we obtain, by letting n tend to infinity, that $J^* = 0$. However, this value cannot be reached by a couple of measures (μ_1, μ_2) of finite energy since $I(\mu_1 - \mu_2) = 0$ would imply $\mu_1 = \mu_2$, see Lemma 2.1 below.

More generally, for a rank 1 interaction matrix $C = yy^t$ with $y \in \{-1, 1\}^d$, our vector equilibrium problem corresponds to the electrostatics of a condenser with external field, see, e.g., [24, Chap. VIII]. Here one usually assumes disjoint Δ_j in order to ensure existence and uniqueness of an extremal tuple of measures, though, as we will see below, we may somewhat relax this condition.

Next, we present three simple examples where existence of an extremal tuple of measures holds but not uniqueness.

Example 1.3 Consider the data

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \Delta_1 = \Delta_2 = [-1, 1],$$

$$K = \{(x, y) \in \mathbb{R}^2, x + y = 1, x \geq 0, y \geq 0\};$$

then $J(\mu_1, \mu_2) = I(\mu_1 + \mu_2)$ is minimal over $\mathcal{M}_K^2(\Delta)$ for any couples $(x\omega_{[-1,1]}, y\omega_{[-1,1]})$, $x + y = 1$, where $\omega_{[-1,1]}$ denotes the equilibrium measure of $[-1, 1]$.

Here, one may show that J is convex but not strictly convex over $\mathcal{M}_K^2(\Delta)$. Notice also that there is even not a unique minimizer over $\mathcal{M}_K^2(\Delta)$.

Example 1.4 Consider the data

$$C = I_2, \quad \Delta_1 = \Delta_2 = [-1, 1], \quad K = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 = 1, x \geq 0, y \geq 0\}.$$

Then,

$$J(\mu_1, \mu_2) = I(\mu_1) + I(\mu_2),$$

which is minimal when both measures μ_1 and μ_2 are multiples of the equilibrium measure $\omega_{[-1,1]}$ of $[-1, 1]$. Hence, any couple $(x\omega_{[-1,1]}, y\omega_{[-1,1]})$ with $x^2 + y^2 = 1$ belongs to $\mathcal{M}_K^2(\Delta)$ and gives the minimum value $\log 2$ of the energy J .

For this example, it is not difficult to show that J is strictly convex over $\mathcal{M}^2(\Delta)$, but the nonuniqueness of the extremal tuple of measures comes from the lack of convexity of K . The next example shows that even convexity of K does not allow us to conclude.

Example 1.5 Consider the data

$$C = I_2, \quad \Delta_1 = \Delta_2 = [-4, 4], \quad K = \{(x, y) \in \mathbb{R}^2, x + y = 1, x \geq 0, y \geq 0\};$$

then $J(\mu_1, \mu_2) = I(\mu_1) + I(\mu_2)$ is minimal when both measures μ_1 and μ_2 are multiples of the equilibrium measure $\omega_{[-4,4]}$ of $[-4, 4]$, and in this case $J(x\omega_{[-4,4]}, y\omega_{[-4,4]}) = (x^2 + y^2)I(\omega_{[-4,4]})$. Since $I(\omega_{[-4,4]}) = -\log(2) < 0$, we get the minimal value $-\log 2$ both for $(\omega_{[-4,4]}, 0)$ and $(0, \omega_{[-4,4]})$ (and J is no longer convex).

In this work, we want to extend the aforementioned results about the minimization of (1.9) to the following situation:

- (i) The sets Δ_i , $i = 1, \dots, d$, are closed sets of \mathbb{C} (instead of compact sets).
- (ii) The interaction matrix $C \in \mathbb{R}^{d \times d}$, of rank r say, is positive semi-definite (instead of definite).
- (iii) The compatibility condition (1.5) is not necessarily satisfied.
- (iv) The minimization of J_Q is performed over $\mathcal{M}_K^d(\Delta)$ instead of $\mathcal{M}_1^d(\Delta)$.

To cope with the noncompactness of the sets Δ_i we need to add to the defining properties of the set $\mathcal{M}(\Sigma)$, see (1.2), a growth condition at infinity.

Hence, from now on, the set $\mathcal{M}(\Sigma)$ will consist of Borel measures μ of finite mass, supported on Σ , of finite energy, and such that

$$\int \log(1 + |t|) d\mu(t) < \infty. \quad (1.12)$$

The set of d -tuples of measures $\mathcal{M}_K^d(\Delta)$ is redefined accordingly; i.e., we assume that condition (1.12) is satisfied componentwise.

For a positive measure μ of finite mass, satisfying (1.12), we have

$$U^\mu(z) \geq -\|\mu\| \log(1 + |z|) - \int \log(1 + |t|) d\mu(t) > -\infty, \quad z \in \mathbb{C}. \quad (1.13)$$

The question raised in Remark 1.1 about the well-definedness of partial potentials can be answered in the same manner since the assertion given there still holds true for measures in $\mathcal{M}(\Sigma)$, see Lemma 2.3. For two measures μ and ν of finite masses, satisfying (1.12), we have

$$I(\mu, \nu) \geq -\|\mu\| \int \log(1 + |t|) d\nu(t) - \|\nu\| \int \log(1 + |t|) d\mu(t) > -\infty,$$

and in particular $I(\mu) > -\infty$. Moreover, denoting by $\tilde{\mu}$ the normalized measure $\mu/\|\mu\|$ for a nonzero $\mu \in \mathcal{M}(\Sigma)$, it is known that the inequality

$$I(\tilde{\mu} - \tilde{\nu}) \geq 0, \quad \mu, \nu \in \mathcal{M}(\Sigma),$$

holds true, see Lemma 2.1. In particular, we have $2I(\tilde{\mu}, \tilde{\nu}) \leq I(\tilde{\mu}) + I(\tilde{\nu})$, and since, by definition of $\mathcal{M}(\Sigma)$, the energies of μ and ν are finite, it then follows that the mutual energy $I(\mu, \nu)$ is finite as well. As a consequence, for $\mu \in \mathcal{M}_K^d(\Delta)$, the energy $J(\mu)$ is always well defined in \mathbb{R} .

For the external fields, we also need some growth condition at infinity. Throughout, we assume that $Q = (Q_j)_j$ is a vector of *admissible* functions, in the sense¹ of [24, Chap. VIII.1]:

¹ Compare with the slightly weaker growth condition at infinity given in [24, Definition I.1.1] for scalar extremal problems.

Definition 1.6 Let Σ be a closed subset of \mathbb{C} of positive capacity. A function $f : \Sigma \rightarrow (-\infty, \infty]$ is said to be *admissible* if it satisfies the following three conditions:

- (i) f is lower semi-continuous,
- (ii) f is finite on a set of positive capacity,
- (iii) $f(x)/\log|x| \rightarrow \infty$ as $|x| \rightarrow \infty$ (in case Σ is unbounded).

In view of the preceding examples, we also have to add assumptions² linking the matrix of interaction C to the topology of the sets Δ_j . For the proof of the existence of an extremal tuple of measures, we will assume that

$$\exists y \in \text{Im}(C), \forall (i, j), \quad \text{if } \text{dist}(\Delta_i, \Delta_j) = 0, \text{ then } y_i y_j > 0, \quad (1.14)$$

whereas, for uniqueness, we will also impose that, for any subset of indices $I \subset \{1, 2, \dots, d\}$, different from a singleton,

$$\text{if the columns } (C_i)_{i \in I} \text{ of } C \text{ are linearly dependent, then } \text{cap}\left(\bigcap_{i \in I} \Delta_i\right) = 0. \quad (1.15)$$

Notice that both conditions (1.14) and (1.15) are trivially true for positive definite interaction matrices C (for condition (1.14), take $y = (1, \dots, 1)^t$). Such interaction matrices appear, e.g., when studying the asymptotic behavior of Angelesco or Nikishin systems in approximation theory.

It is instructive to have a closer look at vector equilibrium problems corresponding to condensers, namely with interaction matrices $C = yy^t$ of rank 1, $y \in \{-1, 1\}^d$. In this case, (1.14) is equivalent to (1.5); it tells us that any two plates Δ_j with charges of opposite sign have positive distance, and (1.15) requires in addition that any two plates Δ_j with charges of the same sign have an intersection of capacity zero. Finally, notice that condition (1.14) fails to hold for Example 1.2, whereas condition (1.15) fails to hold for Example 1.3. For the other two examples, conditions (1.14) and (1.15) hold, indicating that there should be additional restrictions on the set K .

We now state the two main results of our paper. The first result shows, under assumption (1.14), the existence of a solution to our minimization problem.

Theorem 1.7 Consider some nonempty compact set $K \subset \mathbb{R}_+^d$, and assume that the positive semidefinite interaction matrix C satisfies (1.14). Let

$$J_Q^* := \inf\{J_Q(\mu), \mu \in \mathcal{M}_K^d(\Delta)\}. \quad (1.16)$$

Then, the following assertions hold:

- (a) J_Q^* is finite.
- (b) There exists a d -tuple of measures $\mu^* \in \mathcal{M}_K^d(\Delta)$ such that $J_Q(\mu^*) = J_Q^*$.

²In particular, Example 1.3 tells us that the classical condition (1.5) only ensures strict convexity in case of invertible interaction matrices.

Our second result is about uniqueness of a minimizer of the extremal problem (1.16) and about its characterization by equilibrium conditions, the so-called Euler–Lagrange inequalities. Here we restrict ourselves to measures μ whose vector of masses $(\|\mu_1\|, \dots, \|\mu_d\|)$ lies in a nonempty compact polyhedron K of \mathbb{R}_+^d .

Theorem 1.8 *Assume that the positive semidefinite interaction matrix C satisfies the assumptions (1.14) and (1.15), and that the set of masses K consists of a nonempty compact polyhedron of the form*

$$K = \{x \in \mathbb{R}_+^d, Ax = a\}, \quad (1.17)$$

with $A \in \mathbb{R}^{m \times d}$ and $a \in \mathbb{R}^m$, where we suppose in addition that

$$\text{Ker}(A) \subset \text{Ker}(C). \quad (1.18)$$

Then, the following assertions hold true:

- (a) *There exists a unique d -tuple of measures $\mu^* \in \mathcal{M}_K^d(\Delta)$, of finite energy $J_Q(\mu^*) < \infty$, such that*

$$J_Q(\mu^*) = \inf\{J_Q(\mu), \mu \in \mathcal{M}_K^d(\Delta)\}.$$

- (b) *The d -tuple of measures*

$$\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_K^d(\Delta)$$

is the minimizer of J_Q over $\mathcal{M}_K^d(\Delta)$ if and only if there exists $F \in \mathbb{R}^m$ such that, for $i = 1, \dots, d$,

$$U_i^\mu(x) + Q_i(x) \geq (A^t F)_i \quad \text{quasi-everywhere on } \Delta_i, \quad (1.19)$$

$$U_i^\mu(x) + Q_i(x) \leq (A^t F)_i \quad \mu_i\text{-almost everywhere on } \Delta_i. \quad (1.20)$$

Remark 1.9 Notice that Theorem 1.8 includes the particular case $A = I_d$ of a singleton K , where we prescribe the mass of all components of our tuple of measures. Nonsingleton K of the form (1.17) have been considered first in [1–3], where the authors impose equality in (1.18). From Example 1.5, we learn that in general the condition (1.18) cannot be dropped for establishing uniqueness.

As said before, in case of invertible C , all our (somehow technical) assumptions are trivially true for any configuration of sets Δ_j as in (1.4).

Corollary 1.10 *In case of a symmetric positive definite interaction matrix C and a singleton $K = \{a\}$, there exists one and only one minimizer of J_Q over $\mathcal{M}_K^d(\Delta)$, which is characterized by the equilibrium conditions (1.19) and (1.20) for $A = I_d$.*

Example 1.11 Let

$$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \Delta_2 \subset \Delta_1 \subset \mathbb{C}, \quad K = \{(a_1, a_2)\} \subset \mathbb{R}_+^2, \quad a_2 \leq 2a_1;$$

then, according to (1.19) and (1.20), the couple of measures

$$\mu_1 = \left(a_1 - \frac{a_2}{2}\right)\omega_{\Delta_1} + \frac{a_2}{2}\omega_{\Delta_2}, \quad \mu_2 = a_2\omega_{\Delta_2},$$

minimizes J over $\mathcal{M}_K^d(\Delta)$. As in Example 1.2, we can give an electrostatic interpretation in terms of a condenser with two plates Δ_1 and Δ_2 of opposite charge. However, here the Nikishin interaction matrix C translates some di-electric medium where particles of equal charge have stronger interaction than those of opposite sign. We observe the somehow surprising fact that there exists a unique electrostatic equilibrium even if the two plates overlap. Notice that Nikishin systems with touching intervals Δ_j have been considered before in the literature without addressing this issue.

The remainder of the paper is organized as follows. In Sect. 2, we gather several preliminary results that are needed later. In Sect. 3, we give the proof of Theorem 1.7. We also derive, under an additional condition, that the components of the solution have compact supports. In Sect. 4, we give the proof of Theorem 1.8. In Sect. 5, we review a few examples taken from the literature that are related to our results. Some open questions are discussed in Sect. 6.

2 Preliminary Results

Let us first recall the important fact that the energy of a signed measure of mass 0 is nonnegative.

Lemma 2.1 *Let $\mu, \nu \in \mathcal{M}(\mathbb{C})$ with $\|\mu\| = \|\nu\|$. Then*

$$I(\mu - \nu) \geq 0,$$

and $I(\mu - \nu) = 0$ if and only if $\mu = \nu$.

Proof See [24, Lemma I.1.8] for measures μ, ν with compact support and [7, Theorem 2.5] for the unbounded case, see also [25, Theorem 4.1]. \square

We proceed with a few results which are well known when the supports of the measures are compact, but for which we have not always found references in the noncompact case. We defer the proofs of these results to the [Appendix](#).

Lemma 2.2 *Let $\mu \in \mathcal{M}(\mathbb{C})$. Then $\mu(E) = 0$ for every Borel polar set E .*

Proof See [24, Remark I.1.7] for $\text{supp}(\mu)$ compact, and the [Appendix](#) for the general case. \square

Lemma 2.3 *Let μ be a positive measure of finite mass, satisfying (1.12). Then, the potential $U^\mu(z)$ can be $+\infty$ only on a Borel set of capacity 0.*

Proof It is well known that the assertion holds true for any super-harmonic function on \mathbb{C} , not identically $+\infty$, see [22, Theorem 3.5.1]. In particular, it holds true for the potential $U^\mu(z)$. \square

Throughout, we will use weak convergence of Borel measures. Let $(\mu_n)_n$ be a bounded sequence of Borel measures on \mathbb{C} ,

$$\|\mu_n\| \leq c < \infty, \quad n \in \mathbb{N}.$$

We recall that the sequence μ_n tends weakly to a measure μ , as $n \rightarrow \infty$, if

$$\int f d\mu_n \rightarrow \int f d\mu, \quad (2.1)$$

for every bounded, continuous, real-valued function f on \mathbb{C} . In the literature, the notion of vague convergence is also used, where it is assumed that (2.1) holds true only for continuous function f on \mathbb{C} with compact support. Clearly, vague convergence is weaker than the weak convergence. For example, the sequence δ_n of Dirac measures at $x = n$ converges vaguely to 0, although it does not converge weakly. For some comments on these two different notions of convergence of measures, one may have a look at [8, pp. 134–137].

Lemma 2.4 *Assume that the bounded sequence μ_n tends weakly to μ , and let Q be a lower bounded, lower semi-continuous function on \mathbb{C} . Then*

$$\int Q d\mu \leq \liminf_{n \rightarrow \infty} \int Q d\mu_n.$$

Proof See [24, Theorem 0.1.4] for μ_n all supported in a compact set, and the Appendix for the general case. \square

Definition 2.5 A bounded sequence of measures $(\mu_n)_{n \geq 0}$ in $\mathcal{M}(\mathbb{C})$ is said to be

(i) tight if:

$$\forall \epsilon > 0, \exists \text{ compact set } K \subset \mathbb{C}, \forall n \in \mathbb{N}, \quad \int_{\mathbb{C} \setminus K} d\mu_n(t) \leq \epsilon,$$

(ii) log-tight if:

$$\forall \epsilon > 0, \exists \text{ compact set } K \subset \mathbb{C}, \forall n \in \mathbb{N}, \quad \int_{\mathbb{C} \setminus K} \log(1 + |t|) d\mu_n(t) \leq \epsilon. \quad (2.2)$$

The notion of tightness of a bounded set of measures is classical, see, e.g., [5]. The notion of log-tightness is slightly stronger. Note that, from assumption (1.12), each individual measure $\mu \in \mathcal{M}(\mathbb{C})$ satisfies inequality (2.2). Here, for log-tightness of a sequence, we ask this condition to be satisfied uniformly with respect to n .

Theorem 2.6 (Prohorov) *Let $(\mu_n)_{n \geq 0}$ be a tight sequence of probability measures on \mathbb{C} . Then there is a subsequence of $(\mu_n)_{n \geq 0}$ which is weakly convergent to a probability measure on \mathbb{C} .*

Proof See Helly's selection theorem [24, Theorem 0.1.3] for μ_n all supported in some compact set; [5, Theorem 5.1] in a general metric space; and [11, Theorem 9.3.3] for the special case of the euclidean space \mathbb{R}^k . \square

Remark 2.7 The Prohorov theorem is actually stronger than Theorem 2.6, in that it also states, in the converse direction, that a weakly convergent sequence of measures is tight.

Lemma 2.8 *Let $(\mu_n)_{n \geq 0}$ and $(\nu_n)_{n \geq 0}$ be bounded log-tight sequences of measures in $\mathcal{M}(\mathbb{C})$. Assume μ and ν are two Borel measures such that $\mu_n \rightarrow \mu$ and $\nu_n \rightarrow \nu$ in the weak topology. Then*

$$I(\mu, \nu) \leq \liminf_{n \rightarrow \infty} I(\mu_n, \nu_n). \quad (2.3)$$

Proof See [20, Theorem 5.2.1] for all μ_n, ν_n supported in some compact set, and the Appendix for the general case. \square

Let us proceed with establishing four propositions, among which we prove the positiveness of J , the lower semi-continuity of J_Q , and an inequality relating the weighted energy $J_Q(\mu)$ with the scalar energies of the components of μ .

Throughout, we write the positive semidefinite matrix C of rank r as a full rank factorization of the form

$$C = B^t B, \quad B \text{ matrix of dimensions } (r, d), r \leq d, \text{ of rank } r. \quad (2.4)$$

Such a factorization is obtained, e.g., from the Jordan decomposition of C by recalling that there exists an orthonormal basis of eigenvectors of C . First, we generalize Lemma 2.1 to our vector setting.

Proposition 2.9 *Let $\mu, \nu \in \mathcal{M}_K^d(\Delta)$ with tuples of masses verifying $B\|\mu\| = B\|\nu\|$. Then,*

$$J(\mu - \nu) \geq 0. \quad (2.5)$$

Moreover, if condition (1.15) holds true, then

$$J(\mu - \nu) = 0 \quad \text{if and only if} \quad \mu = \nu. \quad (2.6)$$

Proof Let $\lambda = B(\mu - \nu)$. Then, we may write

$$J(\mu - \nu) = J(\lambda, \lambda) = \sum_{j=1}^r I(\lambda_j).$$

By assumption, each component λ_j of λ , with Hahn decomposition $\lambda_j = \lambda_{j,+} - \lambda_{j,-}$, is a signed measure of mass 0, whose absolute value $\lambda_{j,+} + \lambda_{j,-}$ is of finite energy. Hence Lemma 2.1 applies, showing that each $I(\lambda_j)$ is nonnegative, so that (2.5) holds true.

We also know from Lemma 2.1 that $J(\mu - \nu) > 0$ if $\lambda_j \neq 0$ for at least one index j . Hence, to establish (2.6) it only remains to show that $\mu \neq \nu$ implies $\lambda \neq 0$. This property is trivial for positive definite C and thus invertible B . In our setting with semidefinite C , we will need assumption (1.15).

Assuming $\mu - \nu \neq 0$, we deduce that there exists an index i_0 and a Borel set N such that $(\mu_{i_0} - \nu_{i_0})(N) \neq 0$. Now, we consider the partition

$$\bigcup_{j=1,\dots,d} \Delta_j = \bigcup_{I \subset \{1,\dots,d\}, I \neq \emptyset} E_I, \quad E_I = \left(\bigcap_{i \in I} \Delta_i \right) \cap \left(\bigcap_{i \notin I} \Delta_i^c \right),$$

where some of the E_I may be empty sets. This induces a partition of N ,

$$N = \bigcup_{I \subset \{1,\dots,d\}, I \neq \emptyset} N_I, \quad N_I = N \cap E_I,$$

so that $(\mu_{i_0} - \nu_{i_0})(N) = \sum_I (\mu_{i_0} - \nu_{i_0})(N_I)$. Therefore there exists a subset $I \subset \{1, \dots, d\}$ such that $(\mu_I - \nu_I)(N_I) \neq 0$, and

$$\forall i \notin I, \quad (\mu_i - \nu_i)(N_I) = (\mu_i - \nu_i)(N_I \setminus \Delta_i) = 0, \quad (2.7)$$

since $\text{supp}(\mu_i - \nu_i) \subset \Delta_i$. Note also that either $\mu_{i_0}(N_I)$ or $\nu_{i_0}(N_I)$ is nonzero, so that N_I is of positive capacity by Lemma 2.2. Denote by \tilde{B} and \tilde{C} , the submatrix of B and of C , respectively, obtained from selecting the columns of indices belonging to I . Since with N_I also $\bigcap_{i \in I} \Delta_i$ has positive capacity, we obtain from condition (1.15) that \tilde{C} and thus \tilde{B} has full column rank. By (2.7), the relation $B(\mu - \nu)(N_I) = \lambda(N_I)$ simplifies to $\tilde{B}(\mu - \nu)_{i \in I}(N_I) = \lambda(N_I)$, which cannot be zero. Thus $\lambda \neq 0$. \square

As in the classical case, the main ingredient in the proof of Theorem 1.7 will be the lower semi-continuity of the functional J_Q . We note that the proof does not use the compatibility condition (1.5).

Proposition 2.10 *Let $(\mu^{(n)})_{n \geq 0}$ be a sequence of d -tuples of measures in $\mathcal{M}_K^d(\Delta)$ which is log-tight (in the componentwise sense), and assume that $\mu^{(n)}$ tends to a d -tuple of measures $\mu \in \mathcal{M}_K^d(\Delta)$, again componentwise, as $n \rightarrow \infty$, in the weak topology. Then*

$$J_Q(\mu) \leq \liminf_{n \rightarrow \infty} J_Q(\mu^{(n)}).$$

Proof We first show the asserted inequality for the map $\mu \mapsto J(\mu)$. For that, we will use convolution of scalar finite Borel measures μ and ν , which, for a Borel set $B \subset \mathbb{C}$, is defined as follows:

$$(\mu * \nu)(B) := \int \nu(B - t) d\mu(t) = \int \mu(B - t) d\nu(t).$$

The convolution $\mu * \nu$ is a positive measure such that

$$\text{supp}(\mu * \nu) \subset \text{supp}(\mu) + \text{supp}(\nu), \quad (\mu * \nu)(\mathbb{C}) = \mu(\mathbb{C})\nu(\mathbb{C}).$$

From

$$(\mu * \nu)(B) = (\mu \times \nu)\{(x, y), x + y \in B\},$$

it is easy to see that convolution is a commutative and associative operation. We will also use the convolution of a function h with a measure μ ,

$$h * \mu(z) = \int h(z - t) d\mu(t),$$

so that the potential U^μ coincides with the convolution $-\log|\cdot| * \mu$.

Let λ_N be the equilibrium measure of the circle centered at 0 of radius e^{-N} . Its potential is easily computed:

$$U^{\lambda_N}(x) = \min\left(N, \log \frac{1}{|x|}\right),$$

see, e.g., [24, Example 0.5.7]. It is a continuous function tending pointwise to $\log(1/|x|)$, $x \neq 0$, as N tends to ∞ . Then, by associativity and commutativity of the convolution, we get

$$U^{\mu * \lambda_N}(z) = -\log|\cdot| * (\mu * \lambda_N)(z) = (-\log|\cdot| * \lambda_N) * \mu(z) = \int U^{\lambda_N}(z - x) d\mu(x),$$

and for the mutual energies, we have

$$I(\mu * \lambda_N, \nu) = \int U^{\lambda_N}(x - y) d\mu(x) d\nu(y), \quad (2.8)$$

$$\begin{aligned} I(\mu * \lambda_N, \nu * \lambda_N) &= \int U^{\lambda_N}(x - y) d\mu(x) d(\nu * \lambda_N)(y) \\ &= \int (U^{\lambda_N} * (\nu * \lambda_N))(x) d\mu(x) \\ &= \int U^{\lambda_N * \lambda_N}(x - y) d\mu(x) d\nu(y). \end{aligned} \quad (2.9)$$

From the definition of U^{λ_N} , it follows that $I(\mu * \lambda_N, \nu * \lambda_N) \leq I(\mu, \nu)$. In particular, $I(\mu * \lambda_N) < \infty$ if $I(\mu) < \infty$. Moreover,

$$\begin{aligned} \int \log(1 + |x|) d(\mu * \lambda_N) &= \iint \log(1 + |x + y|) d\mu(x) d\lambda_N(y) \\ &\leq \int \log(1 + e^{-N} + |x|) d\mu(x) \\ &\leq \log(1 + e^{-N}) \|\mu\| + \int \log(1 + |x|) d\mu(x) < \infty. \end{aligned}$$

Hence, for any closed subset Σ of \mathbb{C} , the measure $\mu * \lambda_N$ lies in $\mathcal{M}(\Sigma + D(0, e^{-N}))$ if $\mu \in \mathcal{M}(\Sigma)$.

Now, consider a log-tight sequence $\mu^{(n)} \in \mathcal{M}_K^d(\Delta)$ such that

$$\mu^{(n)} \rightarrow \mu \in \mathcal{M}_K^d(\Delta),$$

in the weak sense. Let N be given. From the above remarks, the d -tuple of measures $\mu^{(n)} * \lambda_N$, where the convolution is taken componentwise, belongs to $\mathcal{M}_K^d(\Delta + D(0, e^{-N}))$, and the masses of $\mu^{(n)}$ and $\mu^{(n)} * \lambda_N$ are the same. Thus, from (2.5), we get

$$J(\mu^{(n)} - \mu^{(n)} * \lambda_N) \geq 0,$$

or equivalently,

$$\begin{aligned} J(\mu^{(n)}) &\geq \sum_{i,j=1}^d c_{i,j} (I(\mu_i^{(n)}, \mu_j^{(n)} * \lambda_N) + I(\mu_i^{(n)} * \lambda_N, \mu_j^{(n)} \\ &\quad - I(\mu_i^{(n)} * \lambda_N, \mu_j^{(n)} * \lambda_N)). \end{aligned}$$

Let us consider the first energy in the right-hand side of the above inequality. Since $\mu_i^{(n)} * \lambda_N$ is a log-tight family which tends weakly to $\mu_i * \lambda_N$, Lemma 2.8 tells us that

$$\liminf_{n \rightarrow \infty} I(\mu_i^{(n)}, \mu_j^{(n)} * \lambda_N) \geq I(\mu_i, \mu_j * \lambda_N). \quad (2.10)$$

Actually, we have more. Indeed, redoing the proof of Lemma 2.8 with the kernel $U^{\lambda_N}(x - y)$ instead of $\log(|x - y|^{-1})$, we now get an integrand in the first integral of (A.1) which is bounded and continuous. Hence, (2.10) can be strengthened to

$$\lim_{n \rightarrow \infty} I(\mu_i^{(n)}, \mu_j^{(n)} * \lambda_N) = I(\mu_i, \mu_j * \lambda_N).$$

The limits

$$\begin{aligned} \lim_{n \rightarrow \infty} I(\mu_i^{(n)} * \lambda_N, \mu_j^{(n)}) &= I(\mu_i * \lambda_N, \mu_j), \\ \lim_{n \rightarrow \infty} I(\mu_i^{(n)} * \lambda_N, \mu_j^{(n)} * \lambda_N) &= I(\mu_i * \lambda_N, \mu_j * \lambda_N) \end{aligned}$$

are proven in the same way. Consequently, we obtain that

$$\liminf_{n \rightarrow \infty} J(\mu^{(n)}) \geq \sum_{i,j=1}^d c_{i,j} (I(\mu_i, \mu_j * \lambda_N) + I(\mu_i * \lambda_N, \mu_j) - I(\mu_i * \lambda_N, \mu_j * \lambda_N)),$$

where the right-hand side is well defined since we assume that the limit measure $\mu \in \mathcal{M}_K^d(\Delta)$ (all its components have finite energy). Finally, both potentials U^{λ_N} and $U^{\lambda_N * \lambda_N}$ tend pointwise to $\log(1/|x|)$ for $x \neq 0$, as $N \rightarrow \infty$. They are dominated

by $|\log(1/|x|)|$, and moreover,

$$\begin{aligned} & \iint \left| \log \frac{1}{|x-y|} \right| d\mu_i(x) d\mu_j(y) \\ & \leq I(\mu_i, \mu_j) + 2\|\mu_j\| \int \log(1+|x|) d\mu_i(x) + 2\|\mu_i\| \int \log(1+|x|) d\mu_j(x), \end{aligned}$$

which is finite because $I(\mu_i, \mu_j)$ is, and we have (1.12). Hence, from the dominated convergence theorem, we get

$$\lim_{N \rightarrow \infty} I(\mu_i, \mu_j * \lambda_N) = I(\mu_i, \mu_j), \quad \lim_{N \rightarrow \infty} I(\mu_i * \lambda_N, \mu_j * \lambda_N) = I(\mu_i, \mu_j),$$

implying that

$$\liminf_{n \rightarrow \infty} J(\mu^{(n)}) \geq J(\mu).$$

Since the external fields Q_j are lower semi-continuous and lower bounded, the fact that

$$\liminf_{n \rightarrow \infty} \int Q_j d\mu_j^{(n)} \geq \int Q_j d\mu_j, \quad j = 1, \dots, d,$$

follows from Lemma 2.4. \square

The aim of the next proposition is to show an inequality which will be used in the proof of Proposition 2.12. It asserts that the scalar energy of a linear combination $\sum_j y_j \mu_j$ of bounded measures μ_j in $\mathcal{M}(\mathbb{C})$, with given coefficients y_j , is lower bounded, independently of the μ_j , as soon as it is weighted by a multiple γQ of the external field, with γ an arbitrary small positive number. Such a result is needed only to cope with unbounded Δ_j since, for compact Δ_j , it is not difficult to derive a lower bound for the energy of a signed measure which does not involve external fields.

Proposition 2.11 *Let $Q = (Q_1, \dots, Q_d)^t$ be an admissible external field, and let $y = (y_1, \dots, y_d)^t$ be a given vector in \mathbb{R}^d . Then,*

$$\forall \gamma > 0, \exists \Gamma \in \mathbb{R}, \forall \mu \in \mathcal{M}_K^d(\Delta), \quad \Gamma \leq I(y^t \mu) + \gamma \int Q^t d\mu. \quad (2.11)$$

Proof Since the union Σ of the sets $\Delta_i, i = 1, \dots, d$, is different from \mathbb{C} , recall (1.4), there exist some $z_0 \in \mathbb{C}$ and some $r < 1$, say, such that the disk $D(z_0, 2r)$ does not intersect Σ . Let ω_D be the equilibrium measure of the disk $D = D(z_0, r)$, and let

$$\tau = \lambda - \lambda(\mathbb{C})\omega_D,$$

where λ denotes the scalar signed measure $y^t \mu$. Since $I(\lambda)$ is finite, $I(\tau)$ is finite as well, and Lemma 2.1 applies: $I(\tau) \geq 0$, or, equivalently,

$$\begin{aligned} I(\lambda) &\geq 2\lambda(\mathbb{C})I(\lambda, \omega_D) + \lambda(\mathbb{C})^2 \log(r) \\ &= 2\lambda(\mathbb{C}) \sum_{j=1}^d y_j I(\mu_j, \omega_D) + \lambda(\mathbb{C})^2 \log(r). \end{aligned} \quad (2.12)$$

All the mutual energies $I(\mu_j, \omega_D)$ can be bounded above:

$$I(\mu_j, \omega_D) = \iint \log \frac{1}{|z-t|} d\mu_j d\omega_D \leq \log\left(\frac{1}{r}\right) \|\mu_j\| \leq \log\left(\frac{1}{r}\right) M_j(K), \quad (2.13)$$

with $M_j(K) = \sup_{\mu \in \mathcal{M}_K^d(\Delta)} \|\mu_j\|$. Moreover, the $I(\mu_j, \omega_D)$ can also be lower bounded. First, note that, in view of the third condition of admissibility in Definition 1.6 and the fact that Q_j is lower bounded on compact sets, we have

$$\forall \gamma_j > 0, \exists \Gamma_j \in \mathbb{R}, \forall z \in \Delta_j, \quad \log(1 + |z|) \leq \gamma_j Q_j(z) + \Gamma_j.$$

Then,

$$\begin{aligned} -I(\mu_j, \omega_D) &\leq \int \log(1 + |z|) d\mu_j(z) + \|\mu_j\| \int \log(1 + |t|) d\omega_D(t) \\ &\leq \gamma_j \int Q_j d\mu_j + \Gamma_j \|\mu_j\| + \|\mu_j\| \sup_{t \in D} (\log(1 + |t|)) \\ &\leq \gamma_j \int Q_j d\mu_j + M_j(K) \left(\Gamma_j + \sup_{t \in D} (\log(1 + |t|)) \right), \end{aligned} \quad (2.14)$$

and the proposition follows from plugging inequalities (2.13) or (2.14) into (2.12), according to the sign of $\lambda(\mathbb{C})y_j$, and noting that $\lambda(\mathbb{C})$ is bounded both above and below independently of μ . \square

Next, we show that the weighted energy of a tuple of measures $\mu \in \mathcal{M}_K^d(\Delta)$ dominates the energies of its components. This result requires the condition (1.14).

Proposition 2.12 *Assume that the d -tuple of closed sets Δ and the interaction matrix C satisfy (1.14). Then, there exist positive constants a_0 and a_1 such that*

$$\forall \mu = (\mu_1, \dots, \mu_d)^t \in \mathcal{M}_K^d(\Delta), \quad \sum_{j=1}^d I(\mu_j) \leq a_1 J_Q(\mu) + a_0. \quad (2.15)$$

Proof Consider a vector y in the range of $C = B^t B$ that satisfies (1.14), and note that, since for all indices i , $y_i^2 > 0$, the minimum $m = \min(y_i^2)$ is positive. Let x be a nonzero vector in \mathbb{R}^r such that $y = B^t x$, and Q be an orthogonal matrix with $x/\|x\|$ as its first column. Then, the first row of $Q^t B$ is $y^t/\|x\|$, and

$$J(\mu) = J(C\mu, \mu) = J(Q^t B\mu, Q^t B\mu) = \frac{1}{\|x\|^2} I\left(\sum_{j=1}^d y_j \mu_j\right) + \sum_{k=2}^r I(\lambda_k),$$

where we have set $(\lambda_1, \dots, \lambda_r)^t = Q^t B\mu$. Next, we have the following lower bounds for the energies,

$$\begin{aligned} I(\mu_j, \mu_k) &\geq -\|\mu_k\| \int \log(1 + |t|) d\mu_j(t) - \|\mu_j\| \int \log(1 + |z|) d\mu_k(z) \\ &\geq -\gamma_{j,k} \left(\int Q_j d\mu_j + \int Q_k d\mu_k \right) - \Gamma_{j,k}, \end{aligned} \quad (2.16)$$

where, as in the proof of Proposition 2.11, the positive real number $\gamma_{j,k}$ can be arbitrarily small and $\Gamma_{j,k}$ is a sufficiently large number. Hence, from (2.16) applied with $j = k$, we deduce

$$\begin{aligned} m \sum_j \left(I(\mu_j) + 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right) \\ \leq \sum_j y_j^2 \left(I(\mu_j) + 2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right), \end{aligned}$$

so that

$$\begin{aligned} m \sum_j I(\mu_j) &\leq \sum_j y_j^2 I(\mu_j) + \sum_j (y_j^2 - m) \left(2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right) \\ &= I \left(\sum_j y_j \mu_j \right) - \sum_{j \neq k} y_j y_k I(\mu_j, \mu_k) \\ &\quad + \sum_j (y_j^2 - m) \left(2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right) \\ &= \|x\|^2 J_Q(\mu) - 2\|x\|^2 \sum_{j=1}^d \int Q_j d\mu_j - \|x\|^2 \sum_{k=2}^r I(\lambda_k) \\ &\quad - \sum_{j \neq k} y_j y_k I(\mu_j, \mu_k) \\ &\quad + \sum_j (y_j^2 - m) \left(2\gamma_{j,j} \int Q_j d\mu_j + \Gamma_{j,j} \right). \end{aligned} \quad (2.17)$$

For the signed measures λ_k , we have lower bounds provided by Proposition 2.11,

$$I(\lambda_k) \geq -\gamma_k \sum_{j=1}^d \int Q_j d\mu_j + \Gamma_k, \quad k = 1, \dots, r. \quad (2.18)$$

Finally, for indices j, k such that $y_j y_k < 0$, we know from (1.14) that $\text{dist}(\Delta_j, \Delta_k)$ is positive, so that in this case we also have the upper bound,

$$I(\mu_j, \mu_k) \leq \log\left(\frac{1}{\text{dist}(\Delta_j, \Delta_k)}\right) \|\mu_j\| \|\mu_k\|. \quad (2.19)$$

Making use of (2.16), (2.18), and (2.19) in (2.17) leads to

$$m \sum_{j=1}^d I(\mu_j) \leq \|x\|^2 J_Q(\mu) - c \sum_{j=1}^d \int Q_j d\mu_j - \Gamma,$$

where c is a positive real number since the constants γ_k , $\gamma_{j,j}$, and $\gamma_{j,k}$ are arbitrarily small. As the external fields Q_j are lower bounded, the above inequality implies (2.15) with two constants a_0 and a_1 that depend only on the tuple of sets Δ , the interaction matrix C , and the compact set of masses K . \square

3 Existence of a Solution

In this section, we give the proof of Theorem 1.7. We also prove, under an additional technical assumption, that the components of a solution have compact supports.

Proof of Theorem 1.7 We show that $J_Q^* < +\infty$ as in [24, Theorem I.1.3(a)]. For $\epsilon > 0$, the sets $\Delta_j(\epsilon) = \{x \in \Delta_j : Q_j(x) \leq 1/\epsilon\}$ are closed and thus compact by assumption on Q_j . Since Q_j is finite on a set of positive capacity, $\Delta_j(\epsilon)$ is of positive capacity for sufficiently small $\epsilon > 0$. Denoting by $\omega_{\Delta_j(\epsilon)}$ the equilibrium measure of such a $\Delta_j(\epsilon)$ of positive capacity, $I(\omega_{\Delta_j(\epsilon)}) < \infty$, we find for the d -tuple of measures $\mu \in \mathcal{M}_K^d(\Delta)$ with $\mu_j = b_j \omega_{\Delta_j(\epsilon)}$, $j = 1, \dots, d$, and $b = (b_j) \in K$, that $J_Q(\mu) < \infty$. Next, we prove that $J_Q^* > -\infty$. We have

$$J_Q(\mu) = \sum_{k=1}^r I(\lambda_k) + 2 \sum_{j=1}^d \int Q_j d\mu_j,$$

where $(\lambda_1, \dots, \lambda_r)^t = B\mu$, and the energies $I(\lambda_k)$ satisfy inequalities of the type (2.11) with arbitrarily small positive constants γ_k , the sum of which can be made less than 1. Hence, there exists a constant Γ such that

$$\forall \mu \in \mathcal{M}_K^d(\Delta), \quad J_Q(\mu) \geq \sum_{j=1}^d \int Q_j d\mu_j - \Gamma \geq - \sum_{j=1}^d |q_j| M_j(K) - \Gamma, \quad (3.1)$$

with

$$q_j = \inf_{z \in \mathbb{C}} Q_j(z) > -\infty, \quad M_j(K) = \sup_{\mu \in \mathcal{M}_K^d(\Delta)} \|\mu_j\| < \infty, \quad j = 1, \dots, d.$$

This finishes the proof of assertion (a).

The proof of assertion (b) follows usual lines, see, e.g., [20, Chap. 5], by constructing μ^* as a weak limit of a minimizing sequence of J_Q . We first note, in view of (3.1), that for minimizing the energy J_Q , it is sufficient to consider the subset \mathcal{T} of $\mathcal{M}_K^d(\Delta)$ consisting of d -tuples of measures μ such that

$$\sum_{j=1}^d \int Q_j d\mu_j \leq J_Q^* + \Gamma + 1. \quad (3.2)$$

Let us show that \mathcal{T} is a log-tight family. For $\mu \in \mathcal{T}$, we have

$$\sum_{j=1}^d \int (Q_j - q_j) d\mu_j \leq J_Q^* + \Gamma + 1 + \sum_{j=1}^d |q_j| M_j(K).$$

We simply denote by α the right-hand side of the above inequality. Let $\epsilon > 0$ be given. Since the Q_j are admissible, there exists a compact set $K \subset \mathbb{C}$ such that

$$\sum_j (Q_j(x) - q_j) \geq \frac{\alpha}{\epsilon} \log(1 + |x|), \quad x \in \mathbb{C} \setminus K.$$

Consequently, for any d -tuple of measures μ in \mathcal{T} ,

$$\begin{aligned} \sum_j \int_{\mathbb{C} \setminus K} \log(1 + |x|) d\mu_j &\leq \frac{\epsilon}{\alpha} \sum_j \int_{\mathbb{C} \setminus K} (Q_j(x) - q_j) d\mu_j \\ &\leq \frac{\epsilon}{\alpha} \sum_j \int_{\mathbb{C}} (Q_j(x) - q_j) d\mu_j \leq \epsilon, \end{aligned}$$

which shows that the set \mathcal{T} is indeed log-tight. Now, consider a minimizing sequence of d -tuples of measures $\mu^{(n)} \in \mathcal{T}$, namely

$$\lim_{n \rightarrow \infty} J_Q(\mu^{(n)}) = J_Q^*.$$

The family \mathcal{T} being log-tight, it is a fortiori tight, so that by Theorem 2.6, there exists a subsequence, that we still denote by $\mu^{(n)}$, having a weak limit μ^* . Its components μ_j^* are supported on Δ_j , and its d -tuple of masses belongs to K . Since $\log(1 + |x|)$ is a continuous and lower bounded function, we get from Lemma 2.4 that

$$\int \log(1 + |x|) d\mu_j^* \leq \liminf_{n \rightarrow \infty} \int \log(1 + |x|) d\mu_j^{(n)}, \quad j = 1, \dots, d.$$

Moreover, up to an additive constant, $\log(1 + |x|)$ is upper bounded by $Q_j(x)$, inequality (3.2) holds true for the sequence $\mu_j^{(n)}$, and

$$-|q_j| M_j(K) \leq q_j \|\mu_j^{(n)}\| \leq \int Q_j d\mu_j^{(n)}, \quad j = 1, \dots, d.$$

Therefore, we may deduce that

$$\int \log(1 + |x|) d\mu_j^*(x) < \infty, \quad j = 1, \dots, d.$$

Next, we show that each component μ_j^* is of finite energy. From Lemma 2.8, it follows that

$$I(\mu_k^*) \leq \liminf_{n \rightarrow \infty} I(\mu_k^{(n)}), \quad k = 1, \dots, d.$$

Adding these inequalities over k , and noting that, in view of Proposition 2.12, the sum obtained on the right-hand side is finite, we get

$$\begin{aligned} I(\mu_j^*) &\leq \liminf_{n \rightarrow \infty} \sum_{k=1}^d I(\mu_k^{(n)}) - \sum_{k \neq j} I(\mu_k^*) \leq a_1 \liminf_{n \rightarrow \infty} J_Q(\mu^{(n)}) + a_0 - \sum_{k \neq j} I(\mu_k^*) \\ &= a_1 J_Q^* + a_0 - \sum_{k \neq j} I(\mu_k^*) < \infty, \end{aligned} \quad (3.3)$$

where the last inequality comes from

$$I(\mu_k^*) \geq -2 \|\mu_k^*\| \int \log(1 + |x|) d\mu_k^*(x) > -\infty, \quad k = 1, \dots, d.$$

Consequently, $\mu^* \in \mathcal{M}_K^d(\Delta)$. From the lower semi-continuity of J_Q established in Proposition 2.10, we conclude that $J_Q^* \geq J_Q(\mu^*)$, and thus $J_Q^* = J_Q(\mu^*)$, showing that μ^* is a minimizer of the extremal problem (1.16). \square

We now turn to the question of whether the supports of the components of an extremal tuple of measures as in Theorem 1.7 are compact sets. This property was shown to hold true under more restrictive conditions on the matrix C and the tuple of sets Δ in [4, 24]. In our generalized setting, we have the following result.

Theorem 3.1 *Let $\mu \in \mathcal{M}_K^d(\Delta)$ be a solution to the minimization problem (1.16). Then, the components μ_i , $i = 1, \dots, d$, of μ , have compact supports if and only if the following assertion holds true:*

there exists a real α and a number $M > 0$ such that, for all pair (i, j) with Δ_i and Δ_j unbounded and $c_{i,j} < 0$, there holds

$$c_{i,j} U^{\mu_j}(z) + \frac{1}{d} Q_i(z) \geq \alpha, \quad \mu_i\text{-almost everywhere on } \Delta_i \setminus D_M, \quad (3.4)$$

where D_M denotes the closed disk of radius M centered at zero.

Remark 3.2 The assumption (3.4) bears some similarity to assumption [A3] in [4, Definition 2.1], where it is assumed that the functions

$$c_{i,j} \log \frac{1}{|z-t|} + \frac{Q_i(z) + Q_j(t)}{d}, \quad i, j = 1, \dots, d,$$

are uniformly lower bounded on $\Delta_i \times \Delta_j$.

Remark 3.3 Assumption (3.4) is trivially satisfied if

$$\forall i, j, \text{ if } \Delta_i \text{ and } \Delta_j \text{ are unbounded then } c_{i,j} \geq 0.$$

This condition can be seen as an analog of (1.5), where we only consider the point at infinity in the intersection of Δ_i and Δ_j (in the Riemann sphere). Of course, it is more restrictive than the condition (3.4) but it has the advantage that, for a given extremal problem, whether it holds true or not can be checked at once from the data.

Remark 3.4 Condition (3.4) follows from (1.14) in the case of a matrix C of rank 1, for instance when considering a condenser as in [24, Chap. VIII]. Indeed, here necessarily C is a positive multiple of yy^t with the vector y as in (1.14). Thus $c_{i,j} < 0$ implies that $y_i y_j < 0$, and hence for all $z \in \Delta_i$,

$$U^{\mu_j}(z) \leq \|\mu_j\| \log \left(\frac{1}{\text{dist}(\Delta_i, \Delta_j)} \right).$$

Consequently, (3.4) follows by recalling that Q_i is lower bounded. Hence, as in [24, Theorem VIII.1.4], we may conclude that the components of an extremal tuple of measures in (1.16) in the case $\text{rank}(C) = 1$ have compact support.

Proof Suppose first that the support of the measures μ_i are compact. Then, for M sufficiently large, the sets $\text{supp}(\mu_i) \setminus D_M$ are empty sets so that (3.4) is trivially true.

Conversely, let us show that μ_i has a compact support if (3.4) holds. We first establish a property of μ_i similar to [20, Lemma 5.4.1], namely,

$$\forall v_i \in \mathcal{M}_{\|\mu_i\|}(\Delta_i) : \int (U_i^\mu + Q_i) d(v_i - \mu_i) \geq 0. \quad (3.5)$$

For a proof of (3.5), we define $v \in \mathcal{M}_K^d(\Delta)$ by $v_j = \mu_j$ for $j \neq i$. Notice that $\mu + t(v - \mu) \in \mathcal{M}_K^d(\Delta)$ for any $0 < t < 1$, and hence by the definition of μ ,

$$\begin{aligned} 0 &\leq J_Q(\mu + t(v - \mu)) - J_Q(\mu) \\ &= 2t \int (U_i^\mu + Q_i) d(v_i - \mu_i) + t^2 J(v - \mu). \end{aligned}$$

Dividing by t and letting $t \rightarrow 0$ gives the desired inequality (3.5).

For our proof of compactness of $\text{supp}(\mu_i)$, we may suppose without loss of generality that Δ_i is unbounded, $\|\mu_i\| > 0$, and that $\mu_i(D_M) > 0$, where for the last property we possibly choose a larger M . We consider

$$v_i := \frac{\|\mu_i\|}{\mu_i(D_M)} \mu_i \Big|_{D_M}$$

being clearly an element of $\mathcal{M}_{\|\mu_i\|}(\Delta_i)$. Then we may rewrite condition (3.5) as

$$\left(\frac{\|\mu_i\|}{\mu_i(D_M)} - 1\right) \int_{|z| \leq M} (U_i^\mu + Q_i) d\mu_i - \int_{|z| > M} (U_i^\mu + Q_i) d\mu_i \geq 0,$$

or

$$(\|\mu_i\| - \mu_i(D_M))\alpha_0 \geq \|\mu_i\| \int_{|z| > M} (U_i^\mu(z) + Q_i(z)) d\mu_i(z), \quad (3.6)$$

with the finite constant

$$\alpha_0 := \int (U_i^\mu(z) + Q_i(z)) d\mu_i(z).$$

It remains to show that $U_i^\mu(z) + Q_i(z)$ is sufficiently large μ_i -almost everywhere on $\Delta_i \setminus D_M$.

For this, notice first that, by possibly making α smaller and M larger, (3.4) also holds for all indices j with $c_{i,j} < 0$ and compact Δ_j since then $\text{supp}(\mu_j)$ is compact. In case $c_{i,j} \geq 0$, we use (1.13) to conclude that, for μ_i -almost all $z \in \Delta_i \setminus D_M$,

$$\begin{aligned} U_i^\mu(z) + Q_i(z) &\geq \sum_{j, c_{i,j} \geq 0} c_{i,j} U^{\mu_j}(z) + \sum_{j, c_{i,j} < 0} \left(\alpha - \frac{1}{d} Q_i(z) \right) + Q_i(z) \\ &\geq - \sum_{j, c_{i,j} \geq 0} c_{i,j} \|\mu_j\| \log(1 + |z|) + \frac{1}{d} Q_i(z) + \alpha_1 \end{aligned}$$

for some constant α_1 . Here we have used the fact that $c_{i,i} \geq 0$. According to the third condition of admissibility in Definition 1.6, i.e., the behavior of Q_i at infinity, we may now possibly choose a larger M such that $U_i^\mu(z) + Q_i(z) \geq (\alpha_0 + 1)/\|\mu_i\|$ for μ_i -almost all $z \in \Delta_i \setminus D_M$. Hence inequality (3.6) becomes

$$(\|\mu_i\| - \mu_i(D_M))\alpha_0 \geq (\|\mu_i\| - \mu_i(D_M))(\alpha_0 + 1),$$

implying that $\|\mu_i\| = \mu_i(D_M)$, and the fact that μ_i has compact support. \square

4 Uniqueness and Equilibrium Conditions

Proof of Theorem 1.8(a) Our proof relies on Proposition 2.9, but otherwise the arguments of [20] or [2, Proof of Theorem 1.1] carry over to our more general setting.

It is sufficient to show that the application $\mu \mapsto J_Q(\mu)$ is strictly convex³ in the convex subset of $\mathcal{M}_K^d(\Delta)$ consisting of d -tuples of measures μ with finite J_Q -energy.

³More precisely, we only show strict midpoint convexity, which is sufficient for our purposes. However, together with the lower semi-continuity established in Proposition 2.10, one may deduce strict convexity.

By finiteness of the J -energy on $\mathcal{M}_K^d(\Delta)$, that simply boils down to

$$\int Q_j d\mu_j < \infty, \quad j = 1, \dots, d.$$

For two distinct d -tuples of measures μ and ν of finite J_Q -energies, we have

$$\begin{aligned} & \frac{1}{2}(J_Q(\mu) + J_Q(\nu)) - J_Q\left(\frac{\mu + \nu}{2}\right) \\ &= \frac{1}{2}(J(\mu) + J(\nu)) - J\left(\frac{\mu + \nu}{2}\right) \\ &= \sum_{i,j=1}^d c_{i,j} \left(\frac{1}{2}(I(\mu_i, \mu_j) + I(\nu_i, \nu_j)) - I\left(\frac{\mu_i + \nu_i}{2}, \frac{\mu_j + \nu_j}{2}\right) \right) = J\left(\frac{\mu - \nu}{2}\right), \end{aligned}$$

and it only remains to show that the last term is positive. By the definition (1.17) of K , the vector of masses $\|\mu\| - \|\nu\|$ is an element of the kernel of the matrix A , which by (2.4) and (1.18) is a subset of the kernel of C and thus of B . Hence the strict convexity follows from Proposition 2.9. \square

Remark 4.1 There are other sufficient conditions to ensure strict convexity of the map $\mu \mapsto J_Q(\mu)$ on d -tuples of measures of $\mathcal{M}_K^d(\Delta)$ of finite J_Q -energy, for instance we may replace (1.18) by the requirement that the union of the Δ_j is compact, with capacity less than 1. Another sufficient condition for strict convexity, namely

$$\forall i \neq j, \quad \text{if } \Delta_i \cap \Delta_j \neq \emptyset \text{ then } c_{ij} = 0, \quad (4.1)$$

has been considered in [2, 16]. Notice that (4.1) is stronger than (1.5), and that (1.5) alone does not imply strict convexity, see Example 1.3.

Remark 4.2 We claim that if there is equality in assumption (1.18), then (1.14) holds. To see this, notice that from the full rank decomposition $C = B^t B$ and from the assumption $\text{Im}(C) = \text{Im}(A^t)$, we conclude that there exists a matrix E such that $A = EB$, implying that we may rewrite the nonempty compact K as $K = \{x \in \mathbb{R}_+^d, Bx = b\}$ for a suitable vector $b \in \mathbb{R}^r$. Writing $e = (1, \dots, 1)^t \in \mathbb{R}^d$, we conclude that the linear optimization problem $\max\{e^t x, Bx = b, x \geq 0\}$ has an optimal solution. In particular [6, Theorem 19.12], there is a Lagrange multiplier $\lambda \in \mathbb{R}^r$ with $B^t \lambda \geq e$. Hence $y := B^t \lambda$ is an element of $\text{Im}(C) = \text{Im}(B^t)$ with strictly positive components, implying (1.14).

Before entering the details of the proof of assertion (b) of Theorem 1.8, we shortly comment on the equilibrium conditions (1.19) and (1.20). First recall from Lemma 2.3 that the potentials U^{μ_j} for $j = 1, \dots, d$ are finite and hence $U_i^\mu + Q_i$ is well defined in $\Delta_i \setminus \Delta_{i,\infty}$ with $\Delta_{i,\infty} \subset \Delta_i$ some polar Borel set. Also, $U_i^\mu + Q_i$ as a sum of measurable functions is measurable, and hence both sets

$$\begin{aligned}\Delta_{i,+} &= \{z \in \Delta_i \setminus \Delta_{i,\infty}, U_i^\mu(z) + Q_i(z) > (A^t F)_i\}, \\ \Delta_{i,-} &= \{z \in \Delta_i \setminus \Delta_{i,\infty}, U_i^\mu(z) + Q_i(z) < (A^t F)_i\},\end{aligned}$$

are Borel sets. Hence (1.19) means that $\Delta_{i,\infty} \cup \Delta_{i,-}$ is polar, whereas (1.20) can be equivalently rewritten as $\mu_i(\Delta_{i,\infty} \cup \Delta_{i,+}) = 0$.

As in [20, Lemma 5.4.2], we have to establish a different characterization of an extremal tuple of measures which generalizes (3.5).

Lemma 4.3 *The d -tuple of measures $\mu = (\mu_1, \dots, \mu_d) \in \mathcal{M}_K^d(\Delta)$ with $J_Q(\mu) < \infty$ is extremal for (1.16) if and only if for any d -tuple of measures $\nu = (\nu_1, \dots, \nu_d) \in \mathcal{M}_K^d(\Delta)$ with $J_Q(\nu) < \infty$, we have*

$$\sum_{i=1}^d \int (U_i^\mu + Q_i) d\nu_i \geq \sum_{i=1}^d \int (U_i^\mu + Q_i) d\mu_i. \quad (4.2)$$

Proof In order to see that (4.2) is necessary for optimality, notice that, for all $0 < t \leq 1$, we have $\mu + t(\nu - \mu) \in \mathcal{M}_K^d(\Delta)$, with

$$J_Q(\mu + t(\nu - \mu)) - J_Q(\mu) = 2t \sum_{i=1}^d \int (U_i^\mu + Q_i) d(\nu_i - \mu_i) + t^2 J(\nu - \mu) \quad (4.3)$$

being nonnegative. Dividing by t and letting $t \rightarrow 0$ gives (4.2). Conversely, we recall from Proposition 2.9 that $J(\nu - \mu) \geq 0$. Inserting (4.2) into (4.3) for $t = 1$, we conclude as required that μ is extremal. \square

Proof of Theorem 1.8(b) Suppose first that $\mu \in \mathcal{M}_K^d(\Delta)$ satisfies (1.19) and (1.20). Then $\mu_i(\Delta_{i,\infty} \cup \Delta_{i,+}) = 0$, and integrating (1.20) with respect to μ_i shows that $J_Q(\mu) < \infty$. Now let $\nu \in \mathcal{M}_K^d(\Delta)$ with $J_Q(\nu) < \infty$. Then $\nu_i(\Delta_{i,\infty} \cup \Delta_{i,-}) = 0$ by Lemma 2.2. Hence integrating (1.19) with respect to ν_i and (1.20) with respect to μ_i gives

$$\sum_{i=1}^d \int (U_i^\mu + Q_i) d\nu_i - \sum_{i=1}^d \int (U_i^\mu + Q_i) d\mu_i \geq \sum_{i=1}^d (\|\nu_i\| - \|\mu_i\|) (A^t F)_i = 0,$$

the last equality following from the definition of the polyhedron of masses K . Hence μ is extremal according to Lemma 4.3.

Suppose now that $\mu \in \mathcal{M}_K^d(\Delta)$ is extremal. Consider the set of indices $I = \{i \in \{1, \dots, d\} : \|\mu_i\| > 0\}$; set for $i \in I$,

$$w_i := \frac{1}{\|\mu_i\|} \int (U_i^\mu + Q_i) d\mu_i;$$

and consider as before the Borel sets $\Delta_{i,+} = \{z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) > w_i\}$ and $\Delta_{i,-} = \{z \in \Delta_i \setminus \Delta_{i,\infty} : U_i^\mu(z) + Q_i(z) < w_i\}$. Following [20, Proposition 5.4.4], we claim that, for $i \in I$,

$$U_i^\mu(x) + Q_i(x) \geq w_i, \quad \text{quasi-everywhere on } \Delta_i. \quad (4.4)$$

Suppose the contrary for some $i \in I$. Since $\Delta_{i,\infty}$ is polar, we conclude that $\Delta_{i,-}$ is of positive capacity. Thus there exists a compact set $E \subset \Delta_i$ with U_i^μ well defined and finite on E , $\text{cap}(E) > 0$, and $U_i^\mu(x) + Q_i(x) < w_i$ for all $x \in E$. Taking any $v_i \in \mathcal{M}_{\|\mu_i\|}(E)$, then with $v_j = \mu_j$ for $j \neq i$, we get $v \in \mathcal{M}_K^d(\Delta)$ and, by Lemma 4.3,

$$0 \leq \sum_{\ell=1}^d \int (U_\ell^\mu + Q_\ell) d(v_\ell - \mu_\ell) = \int (U_i^\mu + Q_i) dv_i - \|v_i\| w_i,$$

but the term on the right is negative by construction of E and v_i , a contradiction. Thus (4.4) holds.

Following [20, Proposition 5.4.5], we now claim that, for $i \in I$,

$$U_i^\mu(x) + Q_i(x) \leq w_i, \quad \mu_i\text{-almost everywhere.} \quad (4.5)$$

Suppose the contrary for some $i \in I$. Since $\mu_i(\Delta_{i,\infty}) = 0$ by Lemma 2.2, we get $\mu_i(\Delta_{i,+}) > 0$. Applying, e.g., [23, Theorem 2.18], we conclude that there exists a compact set $E \subset \Delta_i$ with U_i^μ well defined and finite on E , $\mu_i(E) > 0$, and $U_i^\mu(x) + Q_i(x) > w_i$ for all $x \in E$. A combination of Lemma 2.2 with (4.4) tells us that

$$\int_{\Delta_i \setminus E} (U_i^\mu + Q_i) d\mu_i \geq w_i \mu_i(\Delta_i \setminus E),$$

and thus

$$\|\mu_i\| w_i \geq \int_E (U_i^\mu + Q_i) d\mu_i + w_i \mu_i(\Delta_i \setminus E) > w_i \mu_i(E) + w_i \mu_i(\Delta_i \setminus E),$$

a contradiction. Hence (4.5) is also true. Thus we have shown so far that, for indices i with $\|\mu_i\| > 0$, (1.19) and (1.20) hold true if we replace $(A^t F)_i$ by a suitable constant $w_i \in \mathbb{R}$. It remains to relate these constants w_i with A and also to discuss the partial potentials U_i^μ for indices i such that $\|\mu_i\| = 0$. For this purpose, similar to [2, Part 3 of proof of Theorem 1.2], we consider the quadratic optimization problem in \mathbb{R}^d ,

$$\min \{x^t H x + 2h^t x, \quad x \in K\},$$

where $H \in \mathbb{R}^{d \times d}$ and $h \in \mathbb{R}^d$, with

$$H_{i,j} = c_{i,j} I(v_i, v_j), \quad h_i = \int Q_i dv_i, \quad i, j = 1, \dots, d,$$

and the probability measures $v_i \in \mathcal{M}_1(\Delta_i)$ are defined by $v_i = \mu_i / \|\mu_i\|$ if $\|\mu_i\| \neq 0$, and else arbitrary but fixed. Then, by Theorem 1.8(a), $\|\mu\| \in K$ is the unique solution of the above quadratic problem. From [6, Theorem 19.12], we know that there exist Lagrange multipliers $F \in \mathbb{R}^m$ and $G \in \mathbb{R}^d$ such that

$$H\|\mu\| + h = A^t F + G, \quad \forall i, \quad G_i \geq 0, \quad \|\mu_i\| G_i = 0. \quad (4.6)$$

In case $\|\mu_i\| \neq 0$, we find from (4.4) and (4.5) that

$$(H\|\mu\| + h)_i = \sum_{j=1}^d c_{i,j} I\left(\frac{\mu_i}{\|\mu_i\|}, \mu_j\right) + \int Q_i \frac{d\mu_i}{\|\mu_i\|} = \int (U_i^\mu + Q_i) \frac{d\mu_i}{\|\mu_i\|} = w_i.$$

Also, $G_i = 0$, and hence $(H\|\mu\| + h)_i = w_i = (A^t F)_i$. In particular, relations (4.4) and (4.5) imply the desired relations (1.19) and (1.20). In case $\|\mu_i\| = 0$, we learn from (4.6) that

$$\forall v_i \in \mathcal{M}_1(\Delta_i), \quad \int (U_i^\mu + Q_i) dv_i \geq (A^t F)_i,$$

implying (1.19), and assertion (1.20) is trivially true. \square

5 A Review of Some Examples

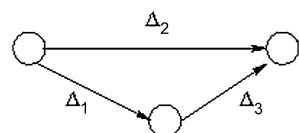
Many recently studied problems, such as, e.g., the behavior of Hermite-Padé approximants, the limit eigenvalue distribution of banded Toeplitz matrices, or the limit distribution of nonintersecting Brownian paths, translate into vector equilibrium problems with external fields. Existence and uniqueness of the solution were shown under conditions that are actually covered by the results of the previous sections. The above-mentioned equilibrium problems can be stated in terms of graphs. We recall that for a graph $G = (\mathcal{V}, \mathcal{E})$, the set of edges \mathcal{E} is a subset of the Cartesian product $\mathcal{V} \times \mathcal{V}$, where \mathcal{V} denotes the set of vertices. For multigraphs, we allow for repeated edges between two given vertices. We also remind the reader that the incidence matrix A is labeled in rows by vertices and in columns by edges, with a column corresponding to an edge from the vertex u to the vertex v having entry -1 at row u , 1 at row v and 0 elsewhere.

In what follows, we always suppose that a graph or a multigraph $G = (\mathcal{V}, \mathcal{E})$ is given. We denote its incidence matrix by A , and we consider as interaction matrix the matrix $C = A^t A$, together with the polyhedron of masses $K = \{x \in \mathbb{R}_+^d, Ax = a\}$. In what follows, K is supposed to contain at least one element with strictly positive components. For instance, for the graph of Fig. 1, we have

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}. \quad (5.1)$$

As a consequence, the interaction matrix C is indexed in rows and columns by the edges, and it can be checked that its entries are $-2, -1, 0, 1, 2$ with the following

Fig. 1 A graph with undirected cycle but no directed cycle



interpretation:

$$C_{\alpha,\beta} = \begin{cases} 2 & \text{if } \alpha = \beta \text{ or } \begin{array}{c} \xrightarrow{\alpha} \bullet \\ \xleftarrow{\beta} \bullet \end{array} \\ 1 & \text{if } \begin{array}{c} \bullet \xrightarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \\ \bullet \xleftarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \end{array} \text{ or } \begin{array}{c} \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \\ \bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \end{array} \\ -1 & \text{if } \begin{array}{c} \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \\ \bullet \xleftarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \end{array} \text{ or } \begin{array}{c} \bullet \xrightarrow{\alpha} \bullet \xleftarrow{\beta} \bullet \\ \bullet \xleftarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \end{array} \\ -2 & \text{if } \begin{array}{c} \bullet \xrightarrow{\alpha} \bullet \\ \bullet \xleftarrow{\beta} \bullet \end{array} \\ 0 & \text{elsewhere (i.e., } \alpha, \beta \text{ do not have any vertex in common)} \end{cases}$$

By construction, the matrix C is always positive semi-definite. With each edge i , we associate a closed set Δ_i and a measure μ_i supported on Δ_i .

We can interpret the different assumptions we made in the previous sections about the matrix C and the supports Δ_i in terms of graph theory.

Proposition 5.1 *The following assertions hold true:*

- The following three statements are equivalent: (i) matrix C is invertible; (ii) G has no undirected cycle; (iii) the polyhedron of masses $K = \{x \in \mathbb{R}_+^d; Ax = a\}$ is a singleton.
- The polyhedron of masses K is compact if and only if G has no directed cycle.
- Condition (1.5), is equivalent to the fact that any two edges which follow each other correspond to nonintersecting sets Δ_i and Δ_j .
- Condition (4.1) is equivalent to the fact that any two distinct edges corresponding to intersecting sets Δ_i do not have any vertex in common.
- Condition (1.15) is equivalent to:

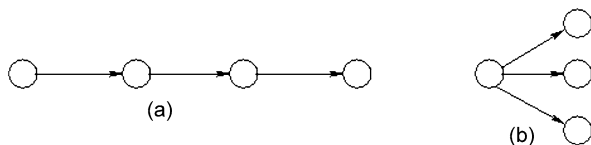
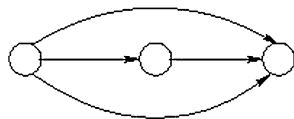
$$\forall \text{ set } I \text{ of edges of } \mathcal{E} \text{ forming an undirected cycle in } G, \text{cap}\left(\bigcap_{\alpha \in I} \Delta_\alpha\right) = 0.$$

- Let G^* be the undirected intersection graph of the sets $\{\Delta_i\}_{i=1}^d$; that is, the vertices of G^* are the edges of G and there is an edge in G^* between i and j if the corresponding sets Δ_i and Δ_j are intersecting. Condition (1.14) is equivalent to:
each connected component of G^* corresponds to a subgraph in G that does not contain a directed cycle.

We do not present here complete proofs for these assertions, which follow from graph theory. Notice however that (a) is based on the classical fact that the rank of an incidence matrix is given by the number of its columns if and only if the underlying graph has no undirected cycle. Assertions (c) and (d) immediately follow from the above graph interpretation of the entries of C .

Condition (4.1) is obviously stronger than (1.5). From the graph theory interpretation given in assertions (d) and (e), we see that (4.1) implies (1.15). From assertions (b) and (f), we see that the compactness of the polyhedron K implies (1.14), as noticed already in Remark 4.2.

The first vector equilibrium problems using the terminology of graphs were studied in [16], where systems of Markov functions generated by a rooted tree

Fig. 2 Tree graphs

Fig. 3 Rooted multigraph with undirected cycle


$G = (\mathcal{V}, \mathcal{E})$, the so-called *generalized Nikishin systems*, were considered. Recall that a tree is a connected graph without undirected cycles. In particular, by properties (a), (e), and (f) of Proposition 5.1, C is invertible and conditions (1.14) and (1.15) are satisfied. So the result [16, Theorem 1] also follows from our work, and we may drop in [16, Theorem 1] any further requirements on the sets Δ_j like (4.1) or (1.5). The authors associate to each vertex in \mathcal{V} a Markov function, and to each edge α in \mathcal{E} a measure with support in an interval Δ_α . This class includes the well-known Nikishin systems, see Fig. 2(a), and the Angelesco systems, see Fig. 2(b), with interaction matrices C respectively given by

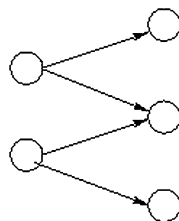
$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The solution of their extremal problem is related to the limit distributions of the zeros of the polynomial denominators of the Hermite-Padé approximants to the generalized Nikishin systems.

In [2], the results of [16] were generalized to rooted multigraphs $G = (\mathcal{V}, \mathcal{E}, \mathcal{O})$ with a root \mathcal{O} , that is, multigraphs which have no directed cycles but do have directed paths from \mathcal{O} to any other vertex. An example of such a graph with undirected cycles is shown in Fig. 3. By generalizing the ideas of [16], the graph is associated to a system of Markov functions with intersecting supports. According to assertion (f) of Proposition 5.1, condition (1.14) holds since there are no directed cycles. Also, as said before, the condition (4.1) imposed in [2] implies (1.15). Thus [2, Theorem 1.1] dealing with K as in (1.17) is covered by our work as well. The Hermite-Padé approximants to specific systems of Markov functions related to graphs with cycles were also investigated in [26] in connection with applications to number theory.

Another vector equilibrium problem appears in [1] and [3] in the study of the asymptotics of diagonal simultaneous Hermite-Padé approximants to two analytic functions with separated pairs of branch points. The authors define the class $\mathcal{H}(\mathbb{C} \setminus \Gamma)$ of holomorphic functions in $\mathbb{C} \setminus \Gamma$, where Γ is a piecewise analytic arc joining two points a and b in \mathbb{C} . A typical example of such a function is

$$f(z) = \log \left(\frac{z-a}{z-b} \right).$$

Fig. 4 bipartite directed graph

For $f_1 \in H(\mathbb{C} \setminus \Gamma_1)$, $f_2 \in H(\mathbb{C} \setminus \Gamma_2)$, with

$$\Delta_1 = \Gamma_1, \quad \Delta_2 = \text{Clos}(\Gamma_2 \setminus \Gamma_1),$$

and Δ_3 a piecewise analytic arc containing the intersection $\Delta_1 \cap \Delta_2$, they show the existence and uniqueness of a triple of measures

$$\mu = (\mu_1, \mu_2, \mu_3) \quad \text{with } \text{supp}(\mu_i) \subset \Delta_i, \quad i = 1, 2, 3,$$

minimizing the energy $J(\mu)$, where the interaction matrix C is given in (5.1), corresponding to the graph in Fig. 1, and the set of masses is given by

$$K = \left\{ x \in \mathbb{R}_+^3, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} = \left\{ x \in \mathbb{R}_+^3, Ax = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Notice that this graph contains an undirected cycle but, since $\text{cap}(\Delta_1 \cap \Delta_2) = 0$, we are again in the settings of our theorems. The measure $\mu_1 + \mu_2$ is the limit distribution of the poles of the diagonal simultaneous Hermite-Padé approximants of the functions (f_1, f_2) , and the measure μ_3 describes the limit distribution of the extra interpolation points to f_1 .

In [10], the limit distribution of nonintersecting one-dimensional Brownian paths with prescribed starting and ending points is also characterized by a vector equilibrium problem. As explained in [10], there is an underlying undirected graph G_u whose edges connect vertices in the set of starting points with vertices in the set of ending points, that is, a bipartite graph. The authors show, in addition, that their graph is a tree, see [10, Proposition 2.1]. In [10, Corollary 2.9.], they establish existence and uniqueness of a solution to an extremal vector equilibrium problem with interaction matrix $C = (B^t B)/2$, B being the incidence matrix of G_u , with quadratic external fields, fixed masses, and sets $\Delta_j = \mathbb{R}$. The supports of the extremal measures are compact and describe the limiting behavior of such nonintersecting one-dimensional Brownian paths. In order to relate [10, Corollary 2.9.] to our findings, notice that, by considering the natural orientation of edges from starting points to ending points, we get a directed graph G which is both a tree and a bipartite graph, see the example in Fig. 4. Using this last property, it is not difficult to see that $B^t B = A^t A$ with A the incidence matrix of G . Thus, we learn from assertion (a) of Proposition 5.1 that C is invertible, see also [2, Proposition 2.8], and that K is a singleton. In particular, both conditions (1.14) and (1.15) are true and even the condition (1.5) holds. Nevertheless, [10, Corollary 2.9] is not a consequence of [20, Chap. 5] since the sets Δ_j are not

compact. However, the quadratic external fields of [10] are admissible in the sense of our Definition 1.6, and thus existence, uniqueness, and equilibrium conditions for an extremal tuple of measures also follow from our general findings. Note also that the compactness of the supports of these extremal measures follows from Remark 3.3 since all entries of C are nonnegative.

6 Conclusion

In this paper, we have shown existence and uniqueness of an extremal tuple of measures for a vector generalization of a weighted energy problem in logarithmic potential theory with a polyhedron of masses, substantially weakening the assumptions typically assumed in other papers on this subject. We have also derived a characterization of such an extremal tuple of measure in terms of equilibrium conditions for the vector potentials.

We have not been able to prove in our general setting that the supports of the components of the extremal tuple of measure are always compact. We conjecture that, because of the growth of the external field at infinity and condition (1.14), it should be true. In any case, we note that the variational inequality (1.20) implies that the potentials U^{μ_j} such that $c_{i,j} < 0$ satisfy $U^{\mu_j}(z)/\log|z| \rightarrow \infty$ as $z \in \Delta_i$ tends to infinity (up to a set of μ_i -measure zero). Hence, in view of assertion (ii) of [18, Theorem 5.7.15], we may at least conclude that the support of μ_i is the union of a set of μ_i -measure zero and a set thin at infinity.

There are also examples of vector-valued extremal problems in logarithmic potential theory where the external fields have a slow increase near ∞ , or are even not present. For instance, in [12], the authors describe the limiting eigenvalue distribution of banded Toeplitz matrices. It is obtained as a component of the solution of a vector equilibrium problem with a positive definite interaction matrix C (namely the one of a Nikishin system), without any external field at all. Also, in [9], these results have been extended to Toeplitz matrices with rational symbol, and in this case the vector equilibrium problem includes external fields of the form $Q(z) = C \log(|z|)$. In these examples, it may happen that the extremal measures do not have a bounded support. For a general analysis of such examples, one should work on the Riemann sphere instead of the complex plane, see the recent contribution [17] in case of positive definite C .

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Appendix

Proof of Lemma 2.2 Assume E is a Borel set such that $\mu(E) > 0$. By regularity of μ , there exists a compact subset K of E with $\mu(K) > 0$. Set $\tilde{\mu} = \mu|_K$. Then,

$$\begin{aligned}
I(\tilde{\mu}) &= I(\mu) + \int_{\mathbb{C} \setminus K} \int_{\mathbb{C}} \log(|z - t|) d\mu(z) d\mu(t) \\
&\quad + \int_K \int_{\mathbb{C} \setminus K} \log(|z - t|) d\mu(z) d\mu(t) \\
&\leq I(\mu) + 4 \int_{\mathbb{C}} \int_{\mathbb{C}} \log(1 + |t|) d\mu(t) d\mu(z),
\end{aligned}$$

which shows that $I(\tilde{\mu}) < \infty$ and thus $\text{cap}(E) > 0$. \square

Proof of Lemma 2.4 By [22, Theorem 2.1.3], there exists an increasing sequence of continuous functions h_m which converges pointwise to Q . Assume Q is lower bounded by $c \in \mathbb{R}$. Set

$$\tilde{h}_m = \min(c + m, \max(c, h_m)).$$

Then, $(\tilde{h}_m)_m$ is an increasing sequence of continuous bounded functions that still tends pointwise to Q , and we have

$$\liminf_{n \rightarrow \infty} \int Q d\mu_n \geq \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \int \tilde{h}_m d\mu_n = \lim_{m \rightarrow \infty} \int \tilde{h}_m d\mu = \int Q d\mu,$$

where in the last equality we use the monotone convergence theorem. \square

Proof of Lemma 2.8 Let $\epsilon > 0$ be given, and let $M > 1$ be such that

$$\forall n \geq 0, \quad \iint_{|x-y| \geq M} \log(|x-y|) d\mu_n(x) dv_n(y) \leq \epsilon.$$

Note that the existence of M follows from the simple inequalities

$$0 \leq \log(|x-y|) \leq \log(1+|x|) + \log(1+|y|),$$

satisfied for $|x-y| \geq 1$, the fact that the masses of the measures are uniformly bounded, and the log-tightness of the sequences. We also set $h(t)$ for a continuous function on \mathbb{R}_+ such that

$$0 \leq h(t) \leq 1, \quad \forall t \in \mathbb{R}_+, \quad h(t) = 1 \quad \text{for } t \leq M, \quad h(t) = 0 \quad \text{for } t \geq M+1.$$

Then, we have

$$\begin{aligned}
I(\mu_n, \nu_n) &= \iint \log(|x-y|^{-1}) h(|x-y|) d\mu_n(x) dv_n(y) \\
&\quad + \iint \log(|x-y|^{-1}) (1-h(|x-y|)) d\mu_n(x) dv_n(y). \quad (\text{A.1})
\end{aligned}$$

On one hand, the Cartesian product measure $\mu_n \times \nu_n$ tends weakly to $\mu \times \nu$, see [5, Theorem 2.8] or [11, Theorem 9.5.9], and the integrand in the first integral is lower semi-continuous and lower bounded on \mathbb{C} . Hence, Lemma 2.4 applies (more precisely, a version of it on \mathbb{C}^2 which holds true as well). On the other hand, the second integral has a modulus less than ϵ uniformly in n . Consequently,

$$\begin{aligned}\liminf_{n \rightarrow \infty} I(\mu_n, \nu_n) &\geq \iint \log(|x - y|^{-1}) h(|x - y|) d\mu(x) d\nu(y) - \epsilon \\ &= I(\mu, \nu) - \iint \log(|x - y|) (1 - h(|x - y|)) d\mu(x) d\nu(y) - \epsilon.\end{aligned}$$

The integrand in the last integral is continuous and lower bounded on \mathbb{C} . Hence, again by Lemma 2.4, this integral is less than

$$\liminf_{n \rightarrow \infty} \iint \log(|x - y|) (1 - h(|x - y|)) d\mu_n(x) d\nu_n(y) \leq \epsilon,$$

which implies

$$\liminf_{n \rightarrow \infty} I(\mu_n, \nu_n) \geq I(\mu, \nu) - 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, (2.3) follows. \square

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